Strictly equivalent a-posteriori error bounds for quasi-optimal nonconforming finite element methods

Applications to Poisson, biharmonic and pressure-robust Stokes discretizations

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Strictly equivalent a-posteriori error bounds for quasi-optimal nonconforming finite element methods
Applications to Poisson, biharmonic and pressure-robust Stokes discretizations

Fakultät für Mathematik
Technische Universität Dortmund

Erstgutachter: Prof. Dr. Christian Kreuzer

Zweitgutachter: JProf. Dr. Pietro Zanotti, Mailand

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Tables of spaces and methods

Table of finite element spaces

Name	Dimensions	Degree	Symbol	$\subset H^1$	$\subset H^2$	Definition
conforming Lagrange	2,3	$\ell \in \mathbb{N}$	$S^1_\ell(\mathcal{T})$	yes	no	page 21
Crouzeix-Raviart first order	2,3	1	CR_1	no	no	page 22
Crouzeix-Raviart higher order	2	$\ell \in \mathbb{N}$	CR_ℓ	no	no	page 23
piecewise polynomials	2,3	$\ell \in \mathbb{N}$	$S^0_\ell(\mathcal{T})$	no	no	(1.26)
Morley	2	2	MR	no	no	page 33
HCT	2	macro 3	HCT	yes	yes	page 35

Table 1: Table of finite element spaces

Table of methods

Equation	Name	Abbrv.	FEM space	Dim.	Def.
Poisson	Lagrange	P-LAG	$\mathring{S}^1_\ell(\mathcal{T})$	2,3	p. 21
	Crouzeix-Raviart first order	P-CR1	CR_1	2,3	p. 22
	Crouzeix-Raviart higher order	P-CRL	CR_ℓ	2	p. 23
	Symmetric interior penalty	P-SIP	$S^0_\ell(\mathcal{T})$	2,3	p. 28
	Non symmetric interior penalty	P-NSIP	$S^0_\ell(\mathcal{T})$	2,3	p. 28
	reduced symmetric interior penalty	P-RSIP	$S^0_\ell(\mathcal{T})$	2,3	p. 30
	red. non-sym. int. penalty	P-RNSIP	$S^0_\ell(\mathcal{T})$	2,3	p. 30
Elasticity	Crouzeix-Raviart first order	E-CR1	CR^d_1	2,3	sec. 1.4.3
Biharmonic	Morley	B-MR	MR	2	sec. 1.4.4.1
	C_0 Symmetric interior penalty	B-SCZ	$\mathring{S}^1_2(\mathcal{T})$	2	sec. 1.4.4.2
	C_0 non symmetric interior penalty	B-NCZ	$\mathring{S}^1_2(\mathcal{T})$	2	-
Stokes	augmented $P^\ell - P^{\ell-d}$	S-AP	$(\mathring{S}^1_\ell)^d - \hat{S}^0_{\ell-d}$	2,3	sec. 3.2.1
	augmented CR_ℓ	S-ACRL	$(\mathrm{CR}_\ell)^2 - \hat{S}^0_{\ell-1}$	2	p. 100
	augmented Taylor Hood	S-ATH	$(\mathring{S}^1_\ell)^2 - \hat{S}^1_{\ell-1}$	2	sec. 3.3.2.3
	sym. int. penalty for Stokes	S-SIP	$(S^0_\ell)^d - \hat{S}^0_{\ell-1}$	2, 3 only $\ell = 1$	sec. 3.3.2.4
	non-augmented CR_ℓ	S-CRL	$(\mathrm{CR}_\ell)^d - \hat{S}^0_{\ell-1}$	2, 3 only $\ell = 1$	Rem. 3.22

Table 2: Table of discrete methods

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Abstract

We construct several a-posteriori error bounds for nonconforming finite element methods that are strictly equivalent to the error. The estimators are based upon ideas developed by Kreuzer and Veeser and employ their idea of an error-dominated oscillation.

Classical a-posteriori estimators might overestimate the error and the involved oscillation term might become arbitrary larger than the error. This problem is overcome by an error-dominated oscillation. Since the oscillation term is bounded by the error, an error estimator with an error-dominated oscillation cannot overestimate. The resulting a-posteriori bounds are therefore strictly equivalent to the error.

The thesis is split into two parts. In the first part, we consider symmetric, elliptic problems. After introducing an abstract a-posteriori framework, we apply it to nonconforming, quasi-optimal finite element discretizations for the Poisson and biharmonic problem.

In the second part, we construct a-posteriori bounds for discretizations of the stationary Stokes problem. We adapt the framework of the first part to pressure-robust, quasi-optimal finite element discretizations to obtain bounds for the velocity error. The resulting estimators are pressure-robust and have an error-dominated oscillation term. Therefore, they are strictly equivalent to the velocity error and are not affected by the pressure or the pressure approximations. Two of these estimators are constructed for two dimensions and one is built for a three dimensional setting.

Introduction

In the first part of this thesis, we consider numerical discretizations of linear problems which can be formulated as following:

$$\text{Given } f \in V', \text{ find } u \in V \text{ such that} \qquad a(u,w) = \langle f, w\rangle \quad \forall w \in V. \tag{0.1}$$

Here, V is a Hilbert space and a is the associated scalar product. In this setting, we consider the Poisson (section 1.4.2), the biharmonic (section 1.4.4) and the linear elasticity (section 1.4.3) problem as examples. In the second part of the thesis, we consider the computation of the velocity field of the Stokes equation, which also can be expressed in the form (0.1).

In this thesis, we consider the finite element method to discretize problem (0.1). The finite element method is a widely applied discretization method and is the topic of many textbooks, see e.g. [BS08, EG04]. A common approach are the so called conforming finite element methods. These methods discretize problem (0.1) by choosing a finite-dimensional vector space $V_h \subset V$. The discrete solution is then obtained by the problem

$$\text{Given } f \in V', \text{ find } u_h \in V_h \text{ such that} \qquad a(u_h,w_h) = \langle f, w_h\rangle \quad \forall w \in V_h.$$

It is well-known that the error of the discrete solution $u_h \in V_h$ fulfills the following relation, commonly referred to as *Cea's Lemma*, see e.g. [BS08, Theorem 2.8.1]:

$$\|u - u_h\| = \inf_{v_h \in V_h} \|u - v_h\|. \tag{0.2}$$

The norm $\|\cdot\|$ is the norm corresponding to the Hilbert space V, i.e. $\|\cdot\|^2 := a(\cdot,\cdot)$.

However, there are situations in which it is desirable to weaken the constraint $V_h \subset V$. We recall some of the reasons given in [BS08, Bra07, CKS00]. The solutions of some equations, such as the biharmonic equation, are in H_0^2. Building a space $V_h \subset H_0^2$ (called a H^2-conforming finite element space) is involved, see [GH09]. Common examples of H^2-conforming finite element spaces are based upon Argyris or HCT element.

The Argyris element (introduced in [AFS68]) uses polynomials of 5th or 7th order with 21 resp. 36 degrees of freedom, see [BS08, p. 77]. The HCT element uses a subtriangulation with polynomials of 3rd order with 12 degrees of freedom, see [CT65]. The high polynomial degrees increase on the one hand the memory usage and computing time, (see [Bra07]), on the other hand they might improve the accuracy of the solution, but only if the analytic solution has enough regularity, c.f. [Cia78, chapter 6] and [DS96, section III.2].

At this point nonconforming methods can help to overcome these problems, see [Cia78, p. 334], [Bra07] and [Str72]. In this thesis, we consider the Morley method, introduced in [Mor68], and the C^0-interior penalty method, based upon [BS05], as nonconforming methods for the biharmonic equation, see section 1.4.4. Both methods only use polynomials of 2nd order which reduces the computational effort, see section 1.4.4.

Also for the discretization of the Stokes equation, nonconforming finite element spaces are helpful. We comment on this situation in more detail below.

A drawback of nonconforming methods is that a result similar to (0.2) is not always available. Instead, techniques such as Strang's Lemma [BS08, section 10.1] or Gudi's medius analysis [Gud10] are used to derive a-priori estimates. These estimates, however, often suppose additional regularity on the data or analytic solution and involve additional terms that cannot be bounded by the best-error $\inf_{v_h \in V_h} \|u - v_h\|$, c.f. the discussion in [VZ18a, section 1]. Often, under additional assumptions, these additional terms can be shown to converge with the asymptotic rate of the best-error, see e.g. [Gud10, p. 2171]. However, a bound by the best-error is stronger than an asymptotic bound, see [TV16].

This problem of the a-priori analysis of nonconforming, linear, finite element methods is resolved by a recent development boost in [VZ18a, VZ19a, VZ18b]. With minor modifications to the original nonconforming methods, the authors obtain improved a-priori estimates and show that the error of the modified methods is equivalent to the best error on a fixed mesh:

$$\|u - u_h\| \eqsim \inf_{v_h \in V_h} \|u - v_h\|. \tag{0.3}$$

Because V_h is no longer a subset of V, $\|\cdot\|$ denotes an extension to $\tilde{V} := V + V_h$. Methods that fulfill (0.3) are called *quasi-optimal*. These new estimates do not expose possibly unstable terms and do not impose additional regularity assumptions, neither on the analytic solution nor the data. In [VZ18a], it is shown that within the proposed framework, a discrete method is quasi-optimal if and only if it is fully stable and algebraic consistent, see [VZ18a, Theorem 3.2] and Proposition 1.12 below. A further discussion of the advantages of quasi-optimal methods can be found in [VZ18a, Zan17] and [TV16].

Naturally, an a-posteriori analysis for these new methods should not spoil the a-priori achievements but uphold the method's design principles. The a-posteriori analysis should avoid to dictate neither the regularity of the solutions nor the data or reintroduce possibly unstable data approximation terms.

Yet, standard a-posteriori analysis, as described e.g. in [Ver13, AO00, Car05], provides an upper bound for the error but in the lower bound a term called oscillation appears. It is denoted by $\operatorname{osc}_{c,h}(f)$ and depends on the data f used in (0.1). Also the a-posteriori analysis for nonconforming methods exposes the oscillation terms, see [CH07, CHO07, CGJ09]. As discussed in [CDN12, KV21] and reviewed in section 2.1.3, on a fixed mesh this oscillation term $\operatorname{osc}_{c,h}(f)$ might be arbitrary larger than the error, which might result in overestimation. The goal of this thesis is to construct a suitable a-posteriori analysis for the nonconforming methods of [VZ18a, VZ19a, VZ18b] that does not have this problem.

The foundation of the new a-posteriori analysis is the concept of a new *error-dominated* oscillation $\operatorname{osc}_{d,h}(f)$, which is introduced in [KV21] for conforming finite element methods. Also, the survey [BCNV24] is based upon this idea. As the name suggests, this new oscillation $\operatorname{osc}_{d,h}(f)$ is up to a constant bounded by the error $\operatorname{osc}_{d,h}(f) \lesssim \|u - u_h\|$. Therefore, overestimation cannot occur. In combination with the estimator $\eta_h(u_h, f)$, the new a-posteriori analysis for conforming methods provides the equivalence

$$\|u - u_h\|^2 \eqsim \eta_h(u_h, f)^2 + \operatorname{osc}_{d,h}^2(f), \tag{0.4}$$

see [KV21, (1.6)].

We transfer the concept of an estimator with an *error-dominated* oscillation to the quasi-optimal nonconforming finite element methods of the framework introduced in [VZ18a, VZ19a, VZ18b]. Beyond an estimator $\eta_h(u_h, f)$ and the oscillation $\operatorname{osc}_{d,h}(f)$, we have an additional term $\operatorname{ncf}_{\mathrm{h}}(u_h)$ called conformity error. This quantity measures the distance between the discrete solution $u_h \in V_h$ to the space V within $\tilde{V}$. The equivalence corresponding to (0.4) reads in the setting of nonconforming finite element methods as

$$\|u - u_h\|^2 \eqsim \eta_h(u_h, f)^2 + \operatorname{osc}_{d,h}^2(f) + \operatorname{ncf}_h(u_h)^2, \tag{0.5}$$

see [KRVZ25, (1.2)]. The terms $\eta_h(u_h, f)$ and $\operatorname{osc}_{d,h}^2(f)$ are sums of local quantities, c.f. [KRVZ25, (1.3)] and section 2.1.2. The local quantities can be used for adaptive mesh refinement, see [BCNV24].

We discuss the construction of the estimators first in an abstract manner and then construct estimators for the examples provided in [VZ19a, VZ18b]. In the abstract part, we construct a framework for a-posteriori estimators for the nonconforming methods fitting into the setting of [VZ18a]. After that, we apply the framework to the Poisson and biharmonic problem.

These results have already been made available as a preprint, see [KRVZ25]. The first part of the preprint has been submitted to a scientific journal, the second part is currently under preparation for submission as a separate paper.

As already mentioned, another setting where nonconforming methods are useful is the Stokes equation. The incompressible Stokes equation is a model for the flow of fluids with a high viscosity, and reads in its distributional form as (see [BF13, Lio13])

Given $f \in H^{-1}(\Omega)^d$, find a velocity field $u \in H_0^1(\Omega)^d$ and a pressure field $p \in L_0^2(\Omega)$ such that

$$\begin{aligned} -\nu\Delta u + \nabla p &= f \quad \text{in } H^{-1}(\Omega)^d, \\ \operatorname{div} u &= 0 \quad \text{in } L_0^2(\Omega). \end{aligned} \tag{0.6}$$

The variable u is the velocity field and p is the pressure field. The remaining standard definitions are recalled in section 3.1.2.

We only consider discretizations of the Stokes equation with the mixed finite element methods, c.f. [BBF13]. These methods require two finite element spaces: a space V_h used for the approximation of the velocity and a space Q_h used for the approximation of the pressure. With these spaces, the discrete method is formulated with two discrete bilinear forms; $a_h : V_h \times V_h \to \mathbb{R}$ and $b_h : V_h \times Q_h \to \mathbb{R}$. The discretization is well-posed if it fulfills the *inf-sup* condition, see [BBF13] and Lemma 3.11. To fulfill this condition, it is often helpful to use a finite element space as V_h that is not contained in $H_0^1(\Omega)^d$, thus nonconforming.

In this thesis, we are mainly concerned with the computation of the velocity field. The analytic velocity field can be computed via the following equation, c.f. (3.9)

$$\text{Given } f \in H^{-1}(\Omega)^d, \text{ find } u \in Z(\Omega) \text{ such that} \quad a(u,v) = \nu \int_\Omega \nabla u : \nabla v = \langle f, v\rangle \text{ for all } v \in Z(\Omega). \tag{0.7}$$

Here $Z := Z(\Omega) := \{v \in H_0^1(\Omega)^d \mid \operatorname{div} v = 0\}$ is the space of divergence-free functions. We see that the computation of the velocity field fits into the setting of (0.1) after relabeling the spaces.

The corresponding problem for the discrete velocity reads

$$a_h(u_h, z_h) = \langle f, z_h\rangle \quad \forall z_h \in Z_h \tag{0.8}$$

where a_h is a suitable discretization of the bilinear form $a(u,v)$ from (0.7) and Z_h contains the discretely divergence-free functions, the discrete counterpart of Z. In general, we have $Z_h \not\subset Z$.

If one does not handle $Z_h \not\subset Z$ properly, the discrete solution u_h might be influenced by the analytic pressure. This may lead to unphysical phenomena. The resulting problems are analyzed in e.g. [Lin09, Lin14, Lin12, JLM+17]. We exemplify some problems.

In [GLBB97] and [JLM+17, Example 1.1], examples are constructed, in which the analytical velocity field vanishes but the discrete velocity field does not vanish since it is influenced by the analytic pressure. Consequently, a-priori bounds for the velocity field error of these methods have to involve the approximability of the pressure. In the case of methods with $V_h \subset H_0^1(\Omega)^d$, the a-priori bounds often read as (c.f. see [KZ20, 2.4] and [BBF13, Proposition 8.2.1]):

$$\|\nabla(u-u_h)\|_{L^2(\Omega)} \lesssim \inf_{v_h \in V_h} \|\nabla(u-v_h)\|_{L^2(\Omega)} + \frac{1}{\nu} \inf_{q_h \in Q_h} \|p-q_h\|_{L^2(\Omega)}. \tag{0.9}$$

Therefore, the publications [Lin14, LMT16, LMNN18, LLMS17] propose modifications of the classical methods for the Stokes equation which avoid these problems. The resulting methods are called pressure-robust. However, the a-priori analysis is based upon the classical tools for nonconforming methods. For instance, in [LMNN18, Theorem 5.1] shows the a-priori estimate

$$\|\nabla_h(u-u_h)\|_{L^2(\Omega)} \lesssim \inf_{v_h \in Z_h} \|\nabla_h(u-v_h)\|_{L^2(\Omega)} + \frac{1}{\nu} R(\mathbb{P}(f)).$$

where $\mathbb{P}(f)$ is the Helmholtz projection ([LMNN18, Definition 2.2], see also Lemma 4.17) and R is an oscillation-like term. For the problems of the oscillation term, we refer to section 2.1.3. Although these pressure-robust methods produce discrete velocity fields which are independent of the pressure, the a-priori estimates exhibit the same problems as the estimates for nonconforming finite element methods for elliptic equations.

To overcome these drawbacks and still have a pressure-robust method, in the series [KZ20, VZ19b, KVZ21] Kreuzer, Verfürth and Zanotti developed new methods based upon the framework in [VZ18a] which are both: quasi-optimal and pressure-robust. The new methods fulfill the following quasi-optimal, a-priori velocity estimate

$$\|u-u_h\| \eqsim \inf_{z_h \in Z_h} \|u-z_h\| \eqsim \inf_{z_h \in V_h} \|u-z_h\|.$$

(The latter equivalence is a consequence of [BBF13, Proposition 5.1.3].)

To construct the pressure-robust methods, they enhanced the smoothing operators of [VZ19a, VZ18b] with a suitable divergence correction, for details see section 3.3.2.

For a pressure-robust method, an a-posteriori velocity error estimator should be pressure-robust as well. This means the velocity estimator should not depend on the pressure. A pressure-robust velocity estimator has been constructed in [LMS19]. The estimator uses the classical a-posteriori techniques involving classical oscillation term. The problems of such terms were already laid out. In particular, in the lower bound of their estimator, the term $\operatorname{osc}_{c,h}(f - \nabla p)$ appears. Additionally, this term has the problem that one needs a good approximation of the exact pressure. Further details are provided in section 2.1.3 and chapter 4.

Moreover, to apply the estimator of [LMS19] the right hand side f of (0.6) needs to be in L^2 and elementwisely in $H(\operatorname{curl})$, which are even more severe assumptions. Therefore, it seems inconsistent to apply this estimator to the new, quasi-optimal and pressure-robust methods.

In chapter 4, we construct new pressure-robust estimators with an error-dominated oscillation term in the spirit of [KV21, BCNV24, KVZ]. Hence, the new estimators cannot overestimate. Additionally, these new estimators work with the natural data regularity $f \in H^{-1}(\Omega)^d$ of the Stokes equation.

To ensure that the estimator is unaffected by the pressure, we test the residual only with exactly divergence-free test functions.

In consequence, we obtain a-posteriori estimates of a form similar to (0.5):

$$\|u - u_h\|^2 \eqsim \eta_h(u_h, f_{|Z})^2 + \operatorname{osc}_{d,h}^2(f_{|Z}) + \operatorname{ncf}_h(u_h)^2,$$

The conformity error $\operatorname{ncf}_h(u_h)^2$ characterizes the distance of $u_h \in Z_h$ to the space Z in $\tilde{Z} := Z + Z_h$, c.f. section 4.1.1. Because u and u_h depend through the corresponding problems (the discrete method is assumed to be pressure-robust) only on $f_{|Z}$, the whole estimator is independent of the analytic pressure, thus it is pressure-robust. Again $\eta_h(u_h, f_{|Z})$, $\operatorname{osc}_{d,h}^2(f_{|Z})$ and also $\operatorname{ncf}_h(u_h)$ can be expressed as sums of local quantities, which can be used to adapt the mesh, see [BCNV24].

Beyond the a-posteriori analysis, we deal with two minor topics. In chapter 5, we derive a-priori estimates in weaker norms within the framework of [VZ18a]. This is commonly referred to as the *Aubin-Nitsche trick*, c.f. [EG04, section 2.3.4]. These estimates are available if the method fulfills an additional condition, which was already proposed in [CN22]. We explore when this condition is true. All methods with a symmetric discrete bilinear form in [VZ19a, VZ18b] fulfill this condition. Nevertheless, we also provide an example, in which this condition is not true although the methods are quasi-optimal.

In the last chapter 6, we consider local estimators for the difference of the discrete solutions between quasi-optimal and non-quasi-optimal methods. With these local estimators, we can apply the estimators derived in chapter 2 also to non-quasi-optimal methods.

The structure of the thesis is as follows: At first we recall the framework for linear variation problems of [VZ18a] together with the examples in [VZ19a, VZ18b] in chapter 1. Then we construct error-estimators with an error-dominated oscillation for these methods in chapter 2. The main results of this chapter have already been published in the preprint [KRVZ25].

After that we turn to the Stokes equation in chapter 3. We recall the pressure-robust setting of [KZ20, KVZ21, VZ19b] and then construct the pressure-robust estimators with an error-dominated oscillation in chapter 4.

After that we present the two minor results. The a-priori estimates in weaker norms are found in chapter 5 and the comparison estimators between quasi-optimal and classical methods are found in chapter 6.

1 A framework for the discretization of symmetric elliptic problems

In the series [VZ18a, VZ19a, VZ18b], Veeser and Zanotti discuss a quite general class of discretizations for linear symmetric elliptic problems. Within their abstract setting, they completely characterize the quasi-optimality of a discretization. In this chapter, we review at first some functional analytic results and then recall the relevant aspects of framework in [VZ18a].

1.1 The analytic framework

In this section, we fix some notation and recall standard functional analytical results, also covered in textbooks, e.g. [Aub00]. After that, we introduce the analytic parts of the framework built by Veeser, Zanotti in [VZ18a].

Let $\mathbb{N}$ denote the natural numbers without 0 and set $\mathbb{N}_0 := \{0\} \cup \mathbb{N}$. Let $\mathbb{Z}$ be the integers and $\mathbb{R}$ the real numbers. We only consider vector spaces which are based upon the real numbers $\mathbb{R}$. A normed vector space V is a vector space equipped with a norm $\|\cdot\|_V$. The norm defines the standard notions of open and closed sets: For any $\omega \subset V$, we write $\mathring{\omega}$ for its interior and $\overline{\omega}$ for its closure. If V is complete, we call it a *Banach space*. If in addition the norm is induced by a scalar product via $\|\cdot\|_V^2 := (\cdot,\cdot)_V$ then V is also a *Hilbert space*.

Definition 1.1. *The* dual space *of a normed vector space V with norm $\|\cdot\|_V$ is defined as*

$$V' = \{\phi : V \to \mathbb{R} \mid \phi \text{ is linear and continuous}\}.$$

V' is a vector space and becomes a Banach space if equipped with the canonical norm

$$\|\phi\|_{V'} := \sup_{v \in V} \frac{\phi(v)}{\|v\|_V}, \quad \forall \phi \in V'. \tag{1.1}$$

The dual space of the dual space is called bi-dual space.

For readability purposes, we do not write the condition $v \neq 0$ in (1.1).

The application of a functional $\phi \in V'$ to $v \in V$ is conveniently denoted by the dual pairing.

Definition 1.2. *Given a normed vector space V and its dual V', we define the* dual pairing *as*

$$\langle \phi, v \rangle := \phi(v), \qquad \phi \in V', v \in V.$$

To avoid ambiguities, we sometimes write

$$\langle \phi, v \rangle_{V',V} = \phi(v).$$

Definition 1.3. *A linear map $A : V \to W$ between two normed vector spaces V, W is called* bounded *if there is a constant $C_A > 0$ such that $\|Av\|_W \leqslant C_A \|v\|_V$ for all $v \in V$. We define the vector space of all linear, bounded maps from V to W as*

$$\mathbb{L}(V,W) := \{A : V \to W \mid A \text{ is linear and bounded}\}$$

and the corresponding operator norm

$$\|A\|_{\mathbb{L}(V,W)} := \sup_{v \in V} \frac{\|Av\|_W}{\|v\|_V}, \quad A \in \mathbb{L}(V,W). \tag{1.2}$$

For convenience, we introduce the abbreviation $\mathbb{L}(V) := \mathbb{L}(V,V)$.

We define the *image* and the *kernel* of a map $A : V \to W$ as

$$\operatorname{im}(A) := \{w \in W \mid \exists v \in V \text{ with } A(v) = w\} \quad \text{and} \quad \operatorname{kern}(A) := \{v \in V \mid A(v) = 0\}.$$

The properties of a bounded, linear map A can be further characterized in terms of its adjoint.

Definition 1.4. *Let V, W be normed vector spaces. The* adjoint *of a bounded, linear map $A : V \to W$ is the unique linear map $A' : W' \to V'$ defined for each $\psi \in W'$ via the linear system:*

$$\langle A'(\psi), v\rangle = \langle \psi, A(v)\rangle \quad \forall v \in V.$$

Now, we discuss the analytic parts of the framework created by Veeser and Zanotti in [VZ18a]. We assume that V is a Hilbert space with scalar-product $a(\cdot,\cdot)$ and $\|v\|_V^2 := a(v,v)$ for all $v \in V$.

We are interested in problems of the following form:

$$\text{Given } f \in V', \text{ find } u \in V \text{ such that} \qquad a(u,w) = \langle f, w\rangle \quad \forall w \in V. \tag{1.3}$$

Proposition 1.5. *The problem* (1.3) *is well posed, which means it is uniquely solvable for all $f \in V'$ and the solution depends continuously on the data, i.e. we have $\|u\|_V = \|f\|_{V'}$.*

The result is a direct consequence of Riesz representation theorem. The Proposition shows that problem (1.3) can be rewritten with the bijective map $A : V \to V'$; $A(v) := a(v,\cdot)$ in the form $A(v) = f$, see [VZ18a, section 2.1]

1.2 The numerical setting

Generally, the space V is infinite-dimensional, which makes the computation of the solution impossible in practice. Therefore, we want to approximate the solution numerically. To approximate problem (1.3), we follow the approach of [VZ18a]. To this end, we select a finite-dimensional space V_h as a discretization of V. This space does *not* need to be contained in its analytic counterpart V.

We define the space $\tilde{V} := V + V_h$. In addition to the previous section, we assume that the form a is extendable to a scalar product $\tilde{a} : \tilde{V} \times \tilde{V} \to \mathbb{R}$. This defines an operator $\tilde{A} : \tilde{V} \to \tilde{V}'$ with $\tilde{A}(\tilde{v}) := \tilde{a}(\tilde{v},\cdot), \tilde{v} \in \tilde{V}$. On $\tilde{V}$, the scalar product $\tilde{a}$ induces the norm $\|\tilde{v}\|^2 := \|\tilde{v}\|_{\tilde{V}}^2 := \tilde{a}(\tilde{v},\tilde{v})$, $\tilde{v} \in \tilde{V}$.

We follow the setting of [VZ18a] and define the method with yet another bilinear form $a_h : V_h \times V_h \to \mathbb{R}$ to gain flexibility. We assume $a_h(\cdot,\cdot)$ to be non-degenerate, meaning:

$$\forall v_h \in V_h : \quad (\forall w_h \in V_h : \quad a_h(v_h, w_h) = 0) \implies v_h = 0. \tag{1.4}$$

Analogously to the analytic setting with the operators $A, \tilde{A}$, we define $A_h : V_h \to V_h'$ as $A_h(v_h) = a_h(v_h,\cdot), v_h \in V_h$. The inverse A_h^{-1} is well-defined due to (1.4).

To circumvent the problem that a functional in V' is in general not defined on V_h, we introduce a map $\mathrm{L} : D(\mathrm{L}) \subseteq V' \to V_h'$ with $D(\mathrm{L})$ a dense subspace in V'. With the form a_h and L, we define a discrete method $\mathrm{M} : D(\mathrm{L}) \subseteq V' \to V_h$ for each $f \in D(\mathrm{L})$ as:

$$a_h(\mathrm{M}\, f, w_h) = \langle \mathrm{L}\, f, w_h\rangle \quad \forall w_h \in V_h. \tag{1.5}$$

We can also write the method as $\mathrm{M} = A_h^{-1} \circ \mathrm{L}$. At this stage, we do not assume that M or L are bounded. Finally, the method defines an operator

$$\begin{gathered} \mathrm{P_M} : D(\mathrm{P_M}) \subseteq V \to V_h \quad \text{with} \quad D(\mathrm{P_M}) := A^{-1}\big(D(\mathrm{L})\big) \subset V \\ \mathrm{P_M} := \mathrm{M} \circ A = A_h^{-1} \circ \mathrm{L} \circ A \end{gathered} \tag{1.6}$$

Definition 1.6 ([VZ18a])**.** *The method $\mathrm{M} : D(\mathrm{L}) \subseteq V' \to V_h$ is called* conforming *if $V_h \subset V$, otherwise it is* nonconforming*. The method is called* entire *if $D(\mathrm{L}) = V'$.*

In the conforming case, we obtain an entire method by selecting $a_h = a_{|V_h \times V_h}$ and $\mathrm{L} : V' \to V_h'$ as $\mathrm{L}(f) = f_{|V_h}$, $f \in V'$.

An overview of the operators in the case of an entire method is given by the commutative diagram in Figure 1 (which is adapted from [VZ18a, Figure 1]).

As in [VZ19a, Lemma 4.2], we define the norm induced by a_h on V_h as

$$\|w_h\|_{V_h} := \sup_{v_h \in V_h} \frac{a_h(v_h, w_h)}{\|v_h\|_{\tilde{V}}}, \quad w_h \in V_h. \tag{1.7}$$

This defines a norm because a_h is assumed to be non-degenerate, see (1.4). With [VZ18a, Lemma 4.2], we have

$$\|v_h\|_{\tilde{V}} = \sup_{w_h \in V_h} \frac{a_h(v_h, w_h)}{\|w_h\|_{V_h}}, \quad v_h \in V_h. \tag{1.8}$$

Within this framework, we are interested in quasi-optimal methods. Their advantages were laid out in the introduction of the chapter, see also [VZ18a], [TV16].

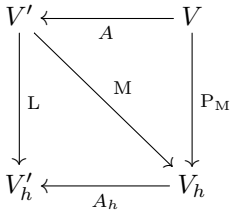


Figure 1.1: Commutative diagram of entire nonconforming methods adapted from [VZ18a].

Definition 1.7 ([VZ18a, Definition 2.3]). *A method* $\mathrm{M} : V' \supset D(\mathrm{M}) \to V_h$ *is* quasi-optimal *if there exists* $C \geqslant 1$ *such that*

$$\|u - \mathrm{P_M}\, u\| \leqslant C \inf_{v_h \in V_h} \|u - v_h\| \text{ for all } u \in D(\mathrm{P_M}). \tag{1.9}$$

The smallest of these constants is denoted by C_{qopt}.

The simplest examples of quasi-optimal methods are well-posed, conforming methods with $a_h = a_{|V_h \times V_h}$ and $\mathrm{L} : V' \to V_h'$ defined as $\mathrm{L}(f) = f_{|V_h}$, $f \in V'$, see [VZ18a, Remark 4.15]. Nonconforming methods, however, are in general not quasi-optimal. To characterize the quasi-optimality of nonconforming methods, the authors of [VZ18a] introduce several new concepts.

Definition 1.8 (Full stability ([VZ18a, Definition 2.5])). *A method* M *is* fully stable *if it is entire and there is a constant* $C \geqslant 0$ *such that*

$$\|\mathrm{M}\, f\|_{\mathbb{L}(V', \tilde{V})} \leqslant C \|f\|_{V'} \text{ for all } f \in V'.$$

The smallest possible constant is denoted by C_{stab}.

The relevance of full stability is set forth by the following proposition.

Proposition 1.9 ([VZ18a, Remarks 2.4 and 2.6]). *A quasi-optimal method is fully stable thus entire.*

If a method is fully stable, problem (1.5) can be rewritten with the adjoint of L.

Proposition 1.10 ([VZ18a, Theorem 4.7]). *A method* M *is fully stable if and only if there is a* smoothing operator *or* smoother $\mathrm{E_M} : V_h \to V$ *such that the discrete problem* (1.5) *can be written as*

$$a_h(\mathrm{M}\, f, w_h) = \langle f, \mathrm{E_M}\, w_h \rangle \quad \forall w_h \in V_h. \tag{1.10}$$

The smoother fulfills the stability bound

$$C_{\mathsf{stab}} = \|\mathrm{M}\|_{\mathbb{L}(V', \tilde{V})} = \sup_{v_h \in V_h} \frac{\|\mathrm{E_M}\, v_h\|_V}{\|v_h\|_{V_h}},$$

where $\|\cdot\|_{V_h}$ *is defined in* (1.7).

Full stability is necessary for a quasi-optimal method but not sufficient. The method additionally needs to be *algebraic consistent*.

Definition 1.11 (Full algebraic consistency, [VZ18a, Definition 2.7]). *A method* $\mathrm{M} : V' \to V_h$ *is fully algebraically consistent if* M *is entire and fulfills for all* $u \in V \cap V_h$

$$a_h(u, w_h) = \langle (\mathrm{L} \circ A)u, w_h \rangle \quad \forall w_h \in V_h. \tag{1.11}$$

If M is fully stable, we may rewrite condition (1.11) *using the smoother* $\mathrm{E_M}$ *from Proposition 1.10 as*

$$a_h(u, w_h) = a(u, \mathrm{E_M}\, w_h) \quad \forall w_h \in V_h, \tag{1.12}$$

for all $u \in V \cap V_h$.

These definitions allow a characterization of quasi-optimal methods.

Proposition 1.12 ([VZ18a, Remark 4.5]). *Given a method* $\mathrm{M} : V' \supset D(\mathrm{M}) \to V_h$ *with* $\mathrm{P_M} : V \supset D(\mathrm{P_M}) \to V_h$ *as in (1.6), the following statements are equivalent:*

(i) M *is a quasi-optimal method,*

(ii) $\mathrm{P_M}$ *fulfills* $\mathrm{P_{M|}}_{V\cap V_h} = \mathrm{id}_{|V\cap V_h}$ *and* $\|\,\mathrm{P_M}\,\|_{\mathbb{L}(V,\tilde{V})} < \infty$,

(iii) the method is fully algebraic-consistent and fully stable.

If these conditions are satisfied, the method M *is entire, meaning* $D(\mathrm{M}) = V'$*, and there is a bounded smoother* $\mathrm{E_M} : V_h \to V$ *such that for each* $f \in V'$ *as* $\mathrm{M}\,f$ *is determined via*

$$a_h(\mathrm{M}\,f, w_h) = \langle f, \mathrm{E_M}\,w_h\rangle, \quad \forall w_h \in V_h.$$

If the method M is quasi-optimal, the operator $\mathrm{P_M}$ is extensible to $\tilde{V}$:

Lemma 1.13 ([VZ18a, Lemma 3.1, Theorem 3.2])**.** *Assume the entire method* $\mathrm{M} : V' \to V_h$ *is quasi-optimal. The operator* $\mathrm{P_M} : V \to V_h$*, defined in (1.6) can be uniquely extended to a bounded, linear map* $\tilde{\mathrm{P}}_\mathrm{M} : \tilde{V} \to V_h$ *with* $\tilde{\mathrm{P}}_{\mathrm{M}|V_h} = \mathrm{id}_{|V_h}$.

The next Lemma quantifies the constants C_{stab} and C_{qopt}.

Lemma 1.14 ([VZ18a, Remark 3.8, Theorems 3.2 and 4.7], [KVZ])**.** *For an entire quasi-optimal method* $\mathrm{M} : V' \to V_h$ *of the above form, the associated operator* $\mathrm{P_M} : V \to V_h$*, defined in (1.6), and the smoother* $\mathrm{E_M}$ *of Proposition 1.10 fulfill*

$$C_{\mathsf{stab}} = \|\,\mathrm{M}\,\|_{\mathbb{L}(V',\tilde{V})} = \|\,\mathrm{P_M}\,\|_{\mathbb{L}(V,\tilde{V})} = \|\,\mathrm{E_M}\,\|_{\mathbb{L}(V_h,V)} = \sup_{v_h\in V_h} \frac{\|\,\mathrm{E_M}\,v_h\|_V}{\|v_h\|_{V_h}},$$

with $C_{\mathsf{stab}} \geqslant 0$ *as in Definition 1.8. Additionally, the quasi-optimality constant* $C_{\mathsf{qopt}} \geqslant 1$ *in Definition 1.7 is quantified by*

$$C_{\mathsf{qopt}} = \|\,\tilde{\mathrm{P}}_\mathrm{M}\,\|_{\mathbb{L}(\tilde{V},\tilde{V})}.$$

The constants $C_{\mathsf{stab}}, C_{\mathsf{qopt}}$ can also be quantified in terms of some consistency measures introduced in [VZ18a, Propositions 3.6, 3.9]. These consistency measures will reappear in the a-posteriori analysis in chapter 2.

Definition 1.15 ([VZ18a, Propositions 3.6, 3.9])**.** *Let* M *be a quasi-optimal method and* $\mathrm{P_M}$ *the associated operator given in (1.6). Let* $\Pi_V : \tilde{V} \to V$*, and* $\Pi_{V_h} : \tilde{V} \to V_h$ *be the orthogonal projections with respect to the scalar product* $\tilde{a}(\cdot,\cdot)$ *onto* V *resp.* V_h*. We define* $\delta_V \geqslant 0$ *to be the smallest constant such that*

$$\forall v \in V \qquad \|\Pi_{V_h} v - \mathrm{P_M}\,v\|_{\tilde{V}} \leqslant \delta_V \|v - \Pi_{V_h} v\|_{\tilde{V}},$$

and $\delta_{V_h} \geqslant 0$ *to be the smallest constant such that*

$$\forall v_h \in V_h \qquad \|v_h - \mathrm{P_M}\,\Pi_V(v_h)\|_{\tilde{V}} \leqslant \delta_{V_h} \|v_h - \Pi_V(v_h)\|_{\tilde{V}}.$$

If $\delta_{V_h} = 0$*, the method is called* overconsistent*.*

The consistency measures δ_V and δ_{V_h} provide bounds on C_{stab} and C_{qopt}.

Proposition 1.16 ([VZ18a, Propositions 3.6, 3.9, Theorem 4.19])**.** *Given a quasi-optimal method* M *with corresponding smoother* $\mathrm{E_M}$*, we have the estimate*

$$\max\{C_{\mathsf{stab}}, \delta_{V_h}\} \leqslant C_{\mathsf{qopt}} = \sqrt{1+\delta_V^2} \leqslant \sqrt{C_{\mathsf{stab}}^2 + \delta_{V_h}^2}$$

with the consistency measures from Definition 1.15.

Additionally, $\delta_V \geqslant 0$ *and* $\delta_{V_h} \geqslant 0$ *are the smallest constants fulfilling*

$$\forall v \in V, w_h \in V_h \qquad a(v, \mathrm{E_M}\,w_h) - a_h(\Pi_{V_h} v, w_h) \leqslant \delta_V \|v - \Pi_{V_h} v\|_{\tilde{V}} \|w_h\|_{V_h}$$

and

$$\forall v_h, w_h \in V_h \qquad a(\Pi_V v_h, \mathrm{E_M}\,w_h) - a_h(v_h, w_h) \leqslant \delta_{V_h} \|v_h - \Pi_V v_h\|_{\tilde{V}} \|w_h\|_{V_h}. \tag{1.13}$$

The estimate (1.13) is a replacement for the missing Galerkin orthogonality, which is usually not available owing to the smoother. More on this topic will be discussed in section 2.1.2.

1.3 Quasi-optimal smoothers

A quasi-optimal method is algebraic consistent, which implies the invariance of $\mathrm{P_M}$ on $V \cap V_h$ and thus the extendability of $\mathrm{P_M}$ to $\tilde{V}$. Looking at the examples in [VZ19a, VZ18b], many smoothers are invariant on $V \cap V_h$ as well, thus can be extended to $\tilde{V}$. This similarity leads us to the notion of a quasi-optimal smoother.

The results about quasi-optimal smoothers will help us in the estimates of chapter 5 and in the a-posteriori analysis in chapters 2 and 4. In both situations, we often use the estimates involving quasi-optimal smoother in Corollary 1.24. As this Corollary is applied repeatedly and uses techniques just discussed, we derive it at this central point.

At first, we provide the definition for a quasi-optimal smoother.

Definition 1.17 ([CN22, Definition 2.1])**.** *A smoothing operator $E : V_h \to V$ is called* quasi-optimal *if there exists a constant $C \geqslant 0$ such that*

$$\|v_h - Ev_h\|_{\tilde{V}} \leqslant C \inf_{v \in V} \|v - v_h\|_{\tilde{V}} \tag{1.14}$$

for all $v_h \in V_h$. The smallest possible constant C is denoted by $C_{\mathsf{qopt}}(E)$.

Lemma 1.18. *Assume that the bounded smoother $E : V_h \to V$ satisfies* (1.15)*. Then an unique, bounded operator $\tilde{E} : \tilde{V} \to V$ with $\tilde{E}_{|V_h} = E$ and $\tilde{E}_{|V} = \mathrm{id}$ exists.*

Proof. We follow the arguments of [VZ18a, Lemmas 2.10, 3.1]. For any $\tilde{v} \in \tilde{V}$, we write $\tilde{v} = v + v_h$ with appropriate $v \in V$ and $v_h \in V_h$. We define the operator $\tilde{E} : \tilde{V} \to V$ as $\tilde{E}(\tilde{v}) := v + E(v_h)$.

If we have another splitting $\tilde{v} = \hat{v} + \hat{v}_h$, $\hat{v} \in V$ and $\hat{v}_h \in V_h$, then $v - \hat{v} = \hat{v}_h - v_h \in V \cap V_h$ on which E and hence $\tilde{E}$ is invariant:

$$\tilde{E}(v + v_h) = v + E(v_h) = v + v_h - \hat{v}_h + E(\hat{v}_h) = v - v + \hat{v} + E(\hat{v}_h) = \tilde{E}(\hat{v} + \hat{v}_h).$$

As in [VZ18a, Lemma 3.1], we define two projections: Let $\cdot^\perp$ denote the $\tilde{a}(\cdot,\cdot)$-orthogonal complement within $\tilde{V}$. The space $Y := (V_h \cap V)^\perp$ gives rise to another bounded $\tilde{a}(\cdot,\cdot)$-orthogonal projection $\Pi_Y : \tilde{V} \to Y$. We yet define another projection $Q : \tilde{V} \to V_h \cap Y$ with kernel $V \cap Y$ and image $V_h \cap Y$. These projections allow us to rewrite the operator $\tilde{E}$ in the following way:

$$\tilde{E} = \tilde{E}Q\Pi_Y + (\mathrm{id}_Y - Q)\Pi_Y + (\mathrm{id}_{\tilde{V}} - \Pi_Y) = EQ\Pi_Y + \mathrm{id}_{\tilde{V}} - Q\Pi_Y.$$

The operator $\tilde{E}$ is a linear combination of compositions of bounded operators, hence it is bounded itself. □

Despite many example of quasi-optimal methods using a quasi-optimal smoother, this is not necessary for quasi-optimality. An example is provided in [VZ18b, p. 2885] and will be recalled on page 30. Nevertheless, we can reuse the quantitative results for quasi-optimal methods in [VZ18a] to obtain estimates for $C_{\mathsf{qopt}}(E)$ of a quasi-optimal smoother E.

Theorem 1.19 ([CN22, Lemma 2.2])**.** *A bounded smoother $E : V_h \to V$ is quasi-optimal if and only if it satisfies*

$$E_{|V \cap V_h} = \mathrm{id}_{|V \cap V_h} \tag{1.15}$$

The constant fulfills $C_{\mathsf{qopt}}(E) = \|\tilde{E}\|_{\mathbb{L}(\tilde{V},V)}$ and the operator E is extendable to $\tilde{V}$.

Proof. Although a proof is given in [CN22, Appendix A], we follow the arguments of [VZ18a, Theorem 3.2], as they are more consistent with our notation. Given a quasi-optimal smoother, the condition (1.14) directly yields (1.15).

Showing the other direction, we assume that we are given a smoother $E : V_h \to V$ that fulfills (1.15). Then Lemma 1.18 ensures the existence of an unique, bounded extension $\tilde{E} : \tilde{V} \to V$. Hence, for any $v \in V$ and $v_h \in V_h$, we calculate

$$\|v_h - Ev_h\|_{\tilde{V}} = \|v_h - v + \tilde{E}(v - v_h)\|_{\tilde{V}} \leqslant \|\mathrm{id}_{\tilde{V}} - \tilde{E}\|_{\mathbb{L}(\tilde{V},\tilde{V})} \|v - v_h\|_{\tilde{V}}. \tag{1.16}$$

This shows that E is a quasi-optimal smoother.

Now, we turn to the quantification of $C_{\mathsf{qopt}}(E)$ for a quasi-optimal, bounded smoother $E : V_h \to V$. From (1.16), we deduce $C_{\mathsf{qopt}}(E) \leqslant \|\mathrm{id}_{\tilde{V}} - \tilde{E}\|_{\mathbb{L}(\tilde{V},\tilde{V})}$. The quasi-optimality of E now implies for $v \in V$ and $v_h \in V_h$

$$\|(\mathrm{id}_{\tilde{V}} - \tilde{E})(v + v_h)\|_{\tilde{V}} = \|v_h - Ev_h\|_{\tilde{V}} \leqslant C_{\mathsf{qopt}}(E) \inf_{v \in V} \|v_h - v\|_{\tilde{V}} \leqslant C_{\mathsf{qopt}}(E) \|v + v_h\|_{\tilde{V}}.$$

The well-known projection theorem

$$C_{\mathsf{qopt}}(E) = \|\operatorname{id}_{\tilde{V}} - \tilde{E}\|_{\mathbb{L}(\tilde{V})} = \|\tilde{E}\|_{\mathbb{L}(\tilde{V},V)}, \tag{1.17}$$

shown for example in [Buc99, Theorem 2], completes the proof. □

Remark 1.20. In [CN22, Lemma 2.2], Carstensen and Nataraj proved Theorem 1.19 with the help of compactness arguments and the Petree-Tartar Lemma.

Theorem 1.19 shows that the quasi-optimality of a smoothing operator closely resembles the situation for the operator $\tilde{\mathrm{P}}_{\mathrm{M}}$ of a quasi-optimal method. Also, the analysis in [VZ18a, Proposition 3.6 and 3.9], which we recalled in Proposition 1.16, can be transferred to the setting of smoothing operators. To best of our knowledge, this analysis has not been performed yet.

For the reader's convenience we recall [VZ18a, Lemma 3.4].

Lemma 1.21 ([VZ18a, Lemma 3.4])**.** *Given a Hilbert space H with the norm $\|\cdot\|_H$ induced by the scalar product and Y a closed, linear subspace of H then the operator norm of $T \in \mathbb{L}(H)$ is estimated by*

$$\max\{C, \delta\} \leqslant \|T\|_{\mathbb{L}(H)} \leqslant \sqrt{C^2 + \delta^2},$$

where

$$C = \|T_{|Y}\|_{\mathbb{L}(Y,H)}, \qquad \delta = \|T_{|Y^\perp}\|_{\mathbb{L}(Y^\perp,H)}. \tag{1.18}$$

Here, $\cdot^\perp$ is the orthogonal complement in H.

With this Lemma, we turn to the adaption of [VZ18a, Propositions 3.6 and 3.9] to quasi-optimal smoothing operators in the next two Propositions.

Proposition 1.22. *Let $E : V_h \to V$ be a bounded, quasi-optimal smoother and $\Pi_{V_h} : \tilde{V} \to V_h$ be the $\tilde{a}(\cdot,\cdot)$-orthogonal projection onto V_h. We define $\gamma_V \geqslant 0$ to be the smallest constant fulfilling*

$$\forall v \in V \qquad \|v - E\Pi_{V_h} v\|_V \leqslant \gamma_V \|v - \Pi_{V_h} v\|_{\tilde{V}}.$$

Then we have

$$\max\{\|E\|_{\mathbb{L}(V_h,V)}, \gamma_V\} \leqslant C_{\mathsf{qopt}}(E) \leqslant \sqrt{\|E\|^2_{\mathbb{L}(V_h,V)} + \gamma_V^2}.$$

Proof. Following the proof of [VZ18a, Proposition 3.9], we apply Lemma 1.21 with $H = \tilde{V}$ and $Y = V_h$. Let $V_h^\perp$ be the $\tilde{a}(\cdot,\cdot)$-orthogonal complement of V_h in $\tilde{V}$. We show that $\gamma_V = \|\tilde{E}\|_{\mathbb{L}(V_h^\perp,\tilde{V})}$. For this, we take $v_h^\perp \in V_h^\perp \subset \tilde{V}$ and decompose it into $v_h^\perp = v_h + v$ with $v_h \in V_h$ and $v \in V$. This splitting yields

$$v_h^\perp = v_h^\perp - \Pi_{V_h} v_h^\perp = v - \Pi_{V_h} v \qquad \text{and} \qquad \tilde{E} v_h^\perp = v - E(\Pi_{V_h} v).$$

We see that γ_V is the desired quantity. □

The other possible application of Lemma 1.21 is investigated in the next Proposition, which is similar to [VZ18a, Proposition 3.6].

Proposition 1.23. *Let $E : V_h \to V$ be a bounded, quasi-optimal smoother and $\Pi_V : \tilde{V} \to V$ be the $\tilde{a}(\cdot,\cdot)$-orthogonal projection onto V. We define $\gamma_{V_h} \geqslant 0$ to be the smallest constant fulfilling*

$$\forall v_h \in V_h \qquad \|\Pi_V v_h - E v_h\|_{\tilde{V}} \leqslant \gamma_{V_h} \|v_h - \Pi_V v_h\|_{\tilde{V}}. \tag{1.19}$$

The quasi-optimality constant of E is characterized as $C_{\mathsf{qopt}}(E) = \sqrt{1 + \gamma_{V_h}^2}$.

Proof. In contrast to [VZ18a, Proposition 3.6], we can simplify the proof in this situation. We have for any $v_h \in V_h$

$$\|v_h - E v_h\|^2_{\tilde{V}} = \|v_h - \Pi_V v_h\|^2_{\tilde{V}} + \|\Pi_V v_h - E v_h\|^2_{\tilde{V}} \leqslant \big(1 + \gamma^2_{V_h}\big)\|v_h - \Pi_V v_h\|^2_{\tilde{V}}.$$

This shows $C_{\mathsf{qopt}}(E)^2 \leqslant 1 + \gamma^2_{V_h}$. Because V_h is finite-dimensional, there exists a specific $v_h \in V_h$ such that (1.19) is an equality. Thus, $C_{\mathsf{qopt}}(E)^2 = 1 + \gamma^2_{V_h}$. □

The simplest and optimal example of a quasi-optimal smoother is the $\tilde{a}(\cdot,\cdot)$-orthogonal projection $\Pi_V : \tilde{V} \to V$ with $C_{\mathsf{qopt}}(\Pi_V) = 1$. But to compute this operator, we need to test with all $v \in V$, which is in general an infinite-dimensional space. As this is prohibitive on current computers, the quasi-optimal smoother Π_V is more of theoretical interest.

In particular, Π_V will be useful in the a-posteriori analysis. The term $\|v_h - \Pi_V v_h\|_{\tilde{V}}$, will often appear in the a-posteriori analysis and is also called the *conformity error*. As the orthogonal projection Π_V is not computable in general, we employ other computable, quasi-optimal smoothers to quantify the conformity error.

Corollary 1.24. *Given a quasi-optimal smoother $E : V_h \to V$, the $\tilde{a}(\cdot,\cdot)$-orthogonal projection $\Pi_V : \tilde{V} \to V$ and $v_h \in V_h$, we have*

$$\inf_{v\in V} \|v_h - v\|_{\tilde{V}} = \|v_h - \Pi_V v_h\|_{\tilde{V}} \leqslant \|v_h - Ev_h\|_{\tilde{V}} \leqslant C_{\mathsf{qopt}}(E)\|v_h - \Pi_V v_h\|_V. \tag{1.20}$$

The Corollary 1.24 allows for literally *any* quasi-optimal smoother, definitely it does not have to be the one used in the method.

1.4 Applications to finite element methods

In this section, we introduce examples of quasi-optimal, nonconforming methods from [VZ19a, VZ18b], which we will discuss throughout this thesis. For this purpose, we introduce concrete function spaces and the finite element setting.

1.4.1 Functions spaces and the mesh

To discuss the examples of partial differential equations, we introduce the Lebesgue and Sobolev spaces. For the discretization we introduce the mesh together with piecewise polynomial spaces. We recall some standard notations.

Let $\Omega \subset \mathbb{R}^d, d \in \{2,3\}$, be an open, connected and bounded domain with polyhedral, Lipschitz continuous boundary. If a subset $\omega \subset \Omega$ is Lebesgue measurable, we denote its Lebesgue measure as $|\omega|$. We write $L^p(\omega)$, $1 \leqslant p \leqslant \infty$, for the Lebesgue spaces on the Lebesgue measurable set $\omega \subset \Omega$, while their norms are written as $\|\cdot\|_{L^p(\omega)}$, $1 \leqslant p \leqslant \infty$.

Given a vector $\vec{\nu} \in \mathbb{R}^d$, we write the distributional derivative of a function $v : \Omega \to \mathbb{R}$ in the direction $\vec{\nu}$ as $\partial_{\vec{\nu}} v$. We denote the canonical basis of $\mathbb{R}^d$ by the vectors $e_1, ..., e_d$. For simplicity, we set $\partial_j := \partial_{e_j}$ for all $1 \leqslant j \leqslant d$. We define the gradient as $\nabla := (\partial_1, ..., \partial_d)$.

For any $k \in \mathbb{N}_0$, we define $C^k(\omega)$ to be the space of all k-times continuously differentiable real-valued functions on $\omega \subset \Omega$, with the convention that $C^0(\omega)$ are the continuous functions on ω. The function space $C^\infty(\omega)$ shall contain all real-valued functions on $\omega \subset \Omega$ which are infinitely times continuously differentiable. The subspace $C_0^\infty(\omega) \subset C^\infty(\omega)$ contains all functions which additionally have compact support.

Similar as in [Eva10, p.260], for a measurable, open, non-empty $\omega \subset \Omega$, we write $W^{k,p}(\omega)$, $k \in \mathbb{N}_0$, $1 \leqslant p \leqslant \infty$ for the standard Sobolev spaces with the norms $\|\cdot\|_{W^{k,p}(\omega)}$ and the seminorms $|\cdot|_{W^{k,p}(\omega)}$. In the case $p = 2$, we abbreviate $H^k(\omega) := W^{k,2}(\omega)$. The spaces $H^k(\omega)$ are Hilbert spaces, whose norms are induced by the scalar-products $(\cdot,\cdot)_{H^k(\omega)}$. We also write $W^{0,p}(\omega) := L^p(\omega)$ and $H^0(\omega) := L^2(\omega)$.

We define $W_0^{k,p}(\omega)$, $k \in \mathbb{N}_0$, $1 \leqslant p \leqslant \infty$, as the closure of $C_0^\infty(\omega)$ in the $\|\cdot\|_{W^{k,p}(\omega)}$-topology. Again, we abbreviate $H_0^k(\omega) := W_0^{k,2}(\omega)$, $k \in \mathbb{N}_0$. The preceding definitions assumed that ω is open. If we write the spaces with a non-open set ω with non-empty interior we implicitly take the interior of ω.

All spaces introduced so far are infinitely dimensional. To built finite-dimensional spaces, we use a mesh of simplices, as e.g. in [VZ19a, section 3.1].

Definition 1.25 ([BS08, Definition 3.3.8, Definition 3.3.11], [EG04, section 1.2.3])**.** *Let $x_0, \dots, x_d \in \mathbb{R}^d$ be $d+1$ points in $\mathbb{R}^d$ such that $x_1 - x_0, \dots, x_d - x_0$ are linearly independent. The convex hull of these $d+1$ points is the* d-simplex $T \subset \mathbb{R}^d$ *with the* vertices $x_0, \dots, x_d$.

A (simplicial) subdivision $\mathcal{T}$ of the domain Ω is a finite set of d-simplices T fulfilling

$$\overline{\Omega} = \bigcup_{T\in\mathcal{T}} T \qquad \text{and} \qquad \mathring{T} \cap \mathring{T}' = \varnothing \text{ for all } T, T' \in \mathcal{T},\ T \neq T'.$$

We write $\mathcal{V}(\mathcal{T}) \subset \overline{\Omega}$ for the set of all vertices of all simplices in $\mathcal{T}$. Furthermore, each simplex has $d+1$ subsimplices that are $d-1$-dimensional, which we call faces. The set $\mathcal{F}(\mathcal{T})$ contains all faces of the subdivision $\mathcal{T}$. The subset $\mathcal{F}^i(\mathcal{T})$ are all faces which are not contained in $\partial\Omega$. We define the skeleton as $\Sigma := \bigcup_{F\in\mathcal{F}} F$ and the interior skeleton as $\Sigma^i := \bigcup_{F\in\mathcal{F}^i} F$

A subdivision $\mathcal{T}(\Omega)$ is called a mesh if $x \notin \mathring{F}$ for all $x \in \mathcal{V}(\Omega)$ and $F \in \mathcal{F}$. Here, $\mathring{F}$ is the interior of F relative to the face F itself.

For each $T \in \mathcal{T}$, we write $h_T > 0$ for its diameter and $\rho_T > 0$ for the radius of the largest ball inscribed in T. Using these two numbers, then we define $\sigma(T) = \frac{h_T}{\rho_T}$ for each element $T \in \mathcal{T}$. A regularity property of a mesh $\mathcal{T}$ is the so called shape constant *defined as $\sigma(\mathcal{T}) := \max_{T\in\mathcal{T}} \sigma(T) = \max_{T\in\mathcal{T}} \frac{h_T}{\rho_T}$.*

We define the mesh-size function $h : \overline{\Omega} \to [0,\infty)$ as

$$h_{|\mathring{T}} := h_T \quad \text{and} \quad h_{|F} := h_F \quad \text{for all } T \in \mathcal{T},\ F \in \mathcal{F}.$$

Here and below, we fix a mesh $\mathcal{T}$ of Ω with vertices $\mathcal{V} := \mathcal{V}(\mathcal{T})$, faces $\mathcal{F} := \mathcal{F}(\mathcal{T})$ and interior faces $\mathcal{F}^i := \mathcal{F}^i(\mathcal{T})$.

We provide here all the notations for subsets of the domain, elements and faces, so they are easily accessible for future reference.

Definition 1.26. *We define the subgrids for $T \in \mathcal{T}, F \in \mathcal{F}$ and $z \in \mathcal{V}$ as*

$$\begin{aligned} \mathcal{T}_z &:= \{T' \in \mathcal{T} \mid z \in T'\}, \quad & \mathcal{T}_z^+ &:= \{T' \in \mathcal{T} \mid T' \cap T \neq \varnothing, \quad T \in \mathcal{T}_z\}, \\ \mathcal{T}_F &:= \{T' \in \mathcal{T} \mid F \subset T'\}, \quad & \mathcal{T}_F^+ &:= \{T' \in \mathcal{T} \mid T' \cap T \neq \varnothing, \quad T \in \mathcal{T}_F\} \quad \textit{and} \\ & & \mathcal{T}_T^+ &:= \{T' \in \mathcal{T} \mid T' \cap T \neq \varnothing\}. \end{aligned}$$

With the notation $\omega(\mathcal{T}') := \bigcup_{T \in \mathcal{T}'} T$ for $\mathcal{T}' \subset \mathcal{T}$, we define the patches

$$\omega_z := \omega(\mathcal{T}_z), \qquad \omega_z^+ := \omega(\mathcal{T}_z^+), \qquad \omega_F := \omega(\mathcal{T}_F), \qquad \omega_F^+ := \omega(\mathcal{T}_F^+) \quad \textit{and} \quad \omega_T^+ := \omega(\mathcal{T}_T^+).$$

We set $\mathcal{F}(\mathcal{T}') := \{F \in \mathcal{F} \mid F \subset \omega(\mathcal{T}')\}$, $\mathcal{T}' \subset \mathcal{T}$ and define

$$\mathcal{F}_z := \mathcal{F}(\mathcal{T}_z), \qquad \mathcal{F}_z^+ := \mathcal{F}(\mathcal{T}_z^+), \qquad \mathcal{F}_F^+ := \mathcal{F}(\mathcal{T}_F^+), \qquad \mathcal{F}_T := \mathcal{F}(\{T\}), \qquad \mathcal{F}_T^+ := \mathcal{F}(\mathcal{T}_T^+).$$

Furthermore, we set

$$\begin{aligned} \mathcal{F}_z^i &:= \{F \in \mathcal{F}^i \mid z \in F\}, & \mathcal{F}_z^{i,+} &:= \{F \in \mathcal{F} \mid F \cap \omega_z \neq \varnothing\}, \\ \mathcal{F}_F^{i,+} &:= \{F \in \mathcal{F} \mid F \cap \omega_F \neq \varnothing\}, & \mathcal{F}_T^{i,+} &:= \{F \in \mathcal{F} \mid F \cap T \neq \varnothing\}. \end{aligned}$$

For each subgrids $\hat{\mathcal{T}} \in \{\mathcal{T}_z, \mathcal{T}_z^+, \mathcal{T}_F, \mathcal{T}_F^+, \mathcal{T}_T^+\}$, the number of elements is bounded by a constant depending only on the dimension and the shape constant:

$$\#\hat{\mathcal{T}} \leqslant C(d, \sigma(\mathcal{T})). \tag{1.22}$$

We refer to this as the *finite overlap property*.

We extend the definition of h_T to the sides and patches. For each $F \in \mathcal{F}$ and $z \in \mathcal{V}$, we define $h_F > 0$ as the diameter of F and $h_z > 0$ as the diameter of ω_z. For any $z \in \mathcal{V}$, $T \in \mathcal{T}_z$, and $F \in \mathcal{F}_z$ we have $h_z \eqsim h_T \eqsim h_F$ with constants depending on $\sigma(\mathcal{T})$.

The fixed mesh $\mathcal{T}$ helps us to build global function spaces from local ones. One useful local space is the space of piecewise polynomials. The space of polynomials is defined on each subset $\omega \subset \Omega$ as

$$\mathbb{P}_\ell(\omega) := \{p : \omega \to \mathbb{R} \mid p \text{ is a polynomial of degree } \ell\} \quad \forall \ell \in \mathbb{N}_0 \quad \text{and} \quad \mathbb{P}_\ell := \varnothing \quad \forall \ell \in \mathbb{Z} \backslash \mathbb{N}_0.$$

In particular, we write $\mathbb{P}_\ell(T)$ and $\mathbb{P}_\ell(F)$ for the spaces of ℓ-degree polynomials on $T \in \mathcal{T}$ resp. $F \in \mathcal{F}$.

To have more control over a local function space, we introduce a *finite element*. We present the form provided in [BS08, Definition 3.1.1].

Definition 1.27 ([BS08, Definition 3.1.1])**.** *A finite element is a triple $(T, \mathbb{P}, \mathcal{N})$ with*

(i) an element $T \in \mathcal{T}$,

(ii) a k-dimensional vector space $\mathbb{P}$ of functions on T, (called shape functions*) and*

(iii) nodal variables *or* degrees of freedom *$\mathcal{N} = \{N_1, \ldots, N_k\}$ forming a basis of the dual space $\mathbb{P}'$.*

In this thesis, the element T is always a simplex.

The shape functions and degrees of freedom are often specified in terms of the *barycentric coordinates*. This coordinate system is used to determine points within a simplex $T \in \mathcal{T}$. For their definition, we introduce for each $0 \leqslant i \leqslant d$ the unique linear polynomials $\lambda_i^T : \mathbb{R}^d \to \mathbb{R}$ specified through $\lambda_i^T(x_j) = \delta_{x_j, x_i}, \quad \forall j = 0, 1, \ldots, d$, where $x_0, ..., x_d \in T \cap \mathcal{V}$. The ith barycentric coordinate of $x \in T \in \mathcal{T}$ with respect to $T \in \mathcal{T}$ is defined as $\lambda_i^T(x)$. An explicit representation of λ_i^T is given in [EG04, (1.37)]. All $d+1$ barycentric coordinates of a point $x \in T$, $T \in \mathcal{T}$, are provided by the map $\texttt{Bary}_T : T \to \mathbb{R}^{d+1}$ defined as $\texttt{Bary}_T(x) = (\lambda_i^T(x))_{0 \leqslant i \leqslant d}, \quad x \in T$.

We are now in the position to define an important finite element, the Lagrange element of degree $\ell \in \mathbb{N}_0$. Its degrees of freedom are defined via evaluations at specific points, the *Lagrange basis points*. The barycentric coordinates of these points are collected in the following set ([EG04, Proposition 1.34])

$$\begin{aligned} \mathcal{L}_\ell^{\texttt{Bary}} &:= \left\{ (\lambda_j)_{0 \leqslant j \leqslant d} \in \mathbb{R}^{d+1} \,\middle|\, \lambda_j = \frac{i_j}{\ell}, \quad i_j \in \{0, \ldots \ell\}, \quad \sum_{j=0}^{d} i_j = \ell \right\}, \quad \text{for} \quad 1 \leqslant \ell < \infty \quad \text{and} \\ \mathcal{L}_0^{\texttt{Bary}} &:= \left\{ \left(\frac{1}{d}\right)_{0 \leqslant j \leqslant d} \right\} \quad \text{for } \ell = 0. \end{aligned} \tag{1.23}$$

For convenience, we define

$$\mathcal{L}_\ell(T) := \mathtt{Bary}_T^{-1}(\mathcal{L}_\ell^{\mathtt{Bary}}), \quad \mathcal{L}_\ell(T,F) := \mathtt{Bary}_T^{-1}(\mathcal{L}_\ell^{\mathtt{Bary}}) \cap F \text{ for } T \in \mathcal{T} \text{ with } F \subset T. \tag{1.24}$$

The Lagrange finite element of degree $\ell \in \mathbb{N}_0$ on $T \in \mathcal{T}$ is now defined with the shape functions $\mathbb{P} := \mathbb{P}_\ell(T)$ and the degrees of freedom $\mathcal{N}_\mathcal{L}^\ell := \{N_y(v) := v(y) \mid v \in \mathbb{P}(T), y \in \mathcal{L}_\ell(T)\}$, which is a basis of the dual space of $\mathbb{P}(T)$; see for example [BS08, chapter 3]. The degrees of freedom also induce a basis on $\mathbb{P}_\ell(T)$. This basis consists of the function $\Psi_{T,z}^\ell \in \mathbb{P}_\ell(T)$, $z \in \mathcal{L}_\ell(T)$, defined via

$$\Psi_{T,z}^\ell(y) := \delta_{z,y} \quad \text{for all } y \in \mathcal{L}_\ell(T). \tag{1.25}$$

Finite elements defined for each $T \in \mathcal{M}$ can now be combined to form a finite element space on some subset $\mathcal{M}$ of the mesh $\mathcal{T}$. Additionally, we might impose some regularity. Given $\ell \in \mathbb{N}_0$ and $k \in \mathbb{N}_0$, we define finite element spaces of piecewise polynomials for a subset $\mathcal{M} \subset \mathcal{T}$ as

$$S_\ell^k(\mathcal{M}) := \big\{v \in H^k\big(\omega(\mathcal{M})\big) \;\big|\; \forall T \in \mathcal{M} \;\; v_{|T} \in \mathbb{P}_\ell(T)\big\} \quad \text{and} \quad \mathring{S}_\ell^k(\mathcal{M}) := S_\ell^k(\mathcal{M}) \cap H_0^k\big(\omega(\mathcal{M})\big). \tag{1.26}$$

Recall that we write $H^0(\omega) = H_0^0(\omega) = L^2(\omega)$ for measurable $\omega \subset \Omega$.

With the Lagrange basis functions $\Psi_{T,z}^\ell$, $z \in \mathcal{L}_\ell^{\mathtt{Bary}}$, defined on $T \in \mathcal{T}$, we can build bases for the spaces $S_\ell^0(\mathcal{M})$ and $S_\ell^1(\mathcal{M})$, $\ell \in \mathbb{N}$. We start with the former, simpler case, following [VZ18b, p. 2879].

With a slight abuse of notation, given $T \in \mathcal{M}$ and $z \in \mathcal{L}_\ell(T)$, we write $\Psi_{T,z}^\ell : \omega(\mathcal{M}) \to \mathbb{R}$, for the extension of $\Psi_{T,z}^\ell$ from T to $\omega(\mathcal{M})$ by 0. Then, a basis of $S_\ell^0(\mathcal{M})$ is given by $\{\Psi_{T,z}^\ell\}_{T\in\mathcal{M}, z\in\mathcal{L}_\ell(T)}$.

We turn to the construction of a basis of $S_\ell^1(\mathcal{M})$ and follow again [VZ18b, p. 2879] but see also [EG04, section 1.4.5]. The difficulty in this situation is that the functions need to be H^1-conforming. Hence, we have to identify coinciding Lagrange nodes and define for this purpose

$$\mathcal{L}_\ell(\mathcal{M}) := \bigcup_{T\in\mathcal{M}} \mathcal{L}_\ell(T), \qquad \mathcal{L}_\ell := \mathcal{L}_\ell(\mathcal{T}), \qquad \mathcal{L}_\ell(F) := \mathcal{L}_\ell \cap F, \quad \forall F \in \mathcal{F} \tag{1.27}$$

The reader finds more details on this issue in [EG04, section 1.4.5]. For each $z \in \mathcal{L}_\ell(\mathcal{M})$, we elementwisely define a function Φ_z^ℓ of $S_\ell^1(\mathcal{M})$. This function Φ_z^ℓ is defined on $T \in \mathcal{M}$ as

$$\Phi_{z|T}^\ell = \begin{cases} \Psi_{T,z}^\ell & \text{if } z \in T, \\ 0 & \text{otherwise.} \end{cases} \tag{1.28}$$

The function Φ_z^ℓ fulfills the identity

$$\Phi_z^\ell(y) = \delta_{z,y} \quad \forall y, z \in \mathcal{L}_\ell. \tag{1.29}$$

According to [EG04, example 1.79] and also [VZ18b, p. 2879], $\{\Phi_z^\ell\}_{z\in\mathcal{L}_\ell}$ is a basis of $S_\ell^1(\mathcal{M})$.

Next, we discuss the special case $\mathring{S}_\ell^1(\mathcal{M})$. The functions in this space have to vanish at the points $\mathcal{L}_\ell \cap \partial(\omega(\mathcal{M}))$ and only the degrees of freedom $\mathcal{L}_\ell^i(\mathcal{M}) := \mathcal{L}_\ell \cap \mathring{\omega}(\mathcal{M})$ are relevant. Then $\{\Phi_z^\ell\}_{z\in\mathcal{L}_\ell^i(\mathcal{M})}$ is a basis of the space $\mathring{S}_\ell^1(\mathcal{M})$, see [VZ18b, p. 2879], [EG04, section 1.4.5].

Later, we will introduce additional finite element spaces when needed. At this point, we turn to a different kind of function spaces, which are not finite-dimensional, although they are defined with the help of the mesh.

Given $\mathcal{M} \subset \mathcal{T}$, $k \in \mathbb{N}_0$, and $1 \leqslant p \leqslant \infty$, we define the piecewise Sobolev spaces as

$$W^{k,p}(\mathcal{M}) := \{v : \omega(\mathcal{M}) \to \mathbb{R} \mid v_{|\mathring{T}} \in W^{k,p}(\mathring{T}) \quad \forall T \in \mathcal{M}\}.$$

We set $H^k(\mathcal{M}) := W^{k,2}(\mathcal{M})$.

On the piecewise defined Sobolev spaces, we will repeatedly apply Gauß theorem for which we need to have normal vectors on the skeleton.

Definition 1.28. *For any open set $\omega \subset \Omega$ with Lipschitz boundary, we define almost everywhere the normal vector function $\mathsf{n}_\omega : \partial\omega \to \mathbb{R}^d$ such that at each $x \in \partial\omega$ the vector $\mathsf{n}_\omega(x)$ is the normal vector at x pointing outwards of ω.*

For each $F \in \mathcal{F}$, we select a normal vector $\mathsf{n}_F \in \mathbb{R}^d$ which points outwards of Ω if $F \subset \partial\Omega$ and otherwise the direction is chosen arbitrary but fixed. This defines another normal vector function as

$$\mathsf{n} : \Sigma \to \mathbb{R}^d, \quad \mathsf{n}_{|F} := \mathsf{n}_F.$$

Remark 1.29. The function n_ω is well-defined because the set ω is assumed to have Lipschitz boundary therefore the normal is well-defined except a Lebesgue null set of finitely many points.

For $F \in \mathcal{F}^i$ and $\mathcal{T}_F = \{T_1, T_2\} \subset \mathcal{T}$ with n_F pointing from T_1 to T_2, we define the jump and mean values of a function $v \in W^{1,1}(\mathcal{T}_F)$ as

$$[\![v]\!]_{|F} := v_{|T_1} - v_{|T_2} \quad \text{and} \quad \{\{v\}\}_{|F} := \frac{v_{|T_1} + v_{|T_2}}{2}.$$

For $F \in \mathcal{F} \backslash \mathcal{F}^i$, we define

$$[\![v]\!]_{|F} := \{\{v\}\}_{|F} := v_{|F}.$$

We define the jump and mean values in literally the same way, if $v \in W^{1,1}(\mathcal{T}_F)^d$, i.e. v is a vector field where each individual component is in $W^{1,1}(\mathcal{T}_F)$.

Sometimes, we need to evaluate the jump of a product for which we recall an often used integration by parts formula for jump terms. It is a standard result in the analysis of discontinuous Galerkin finite element methods and can be found in e.g. [ABCM02, (3.3)].

Lemma 1.30. *In the case $F \in \mathcal{F}^i$ with $\mathcal{T}_F = \{T_1, T_2\} \subset \mathcal{T}$, we have for functions $v, w \in H^1(\mathcal{T}_F)$*

$$[\![vw]\!]_{|F} = [\![v]\!]_{|F} \{\{w\}\}_{|F} + \{\{v\}\}_{|F} [\![w]\!]_{|F}.$$

In the case $F \in \mathcal{F} \backslash \mathcal{F}^i$ with $\mathcal{T}_F = \{T\} \subset \mathcal{T}$, we have for $v, w \in H^1(\mathcal{T}_F) \to \mathbb{R}$

$$[\![vw]\!]_{|F} = [\![v]\!]_{|F} \{\{w\}\}_{|F} = \{\{v\}\}_{|F} [\![w]\!]_{|F}.$$

The same equations are true if $v, w \in H^1(\mathcal{T}_F)^d$ and the scalar multiplications are replaced by a scalar product.

From now on, we use the convention that C is a positive constant, which might change from line to line. The constant may depend on $\mathcal{T}$ only by its shape regularity $\sigma(\mathcal{T})$. It may depend on the dimension d and the polynomial degree ℓ. In some situations, we explicitly write the variables on which C depends, i.e. $C_{d,\ell}$ depends only on the dimension and the polynomial degree. For $A, B \in \mathbb{R}$, we write $A \lesssim B$ for $A \leqslant CB$ and $A \gtrsim B$ for $CA \geqslant B$. By $A \eqsim B$, we mean $A \lesssim B$ and $B \lesssim A$.

1.4.2 Discretizations for the Poisson problem

In the first examples, we consider the Poisson problem which reads in its distributional form as:

$$\begin{aligned} &\text{Given } f \in H^{-1}(\Omega), \text{ find } u \in H^1_0(\Omega) \text{ such that} \\ &-\Delta u = f \quad \text{in } \Omega \qquad \text{and} \qquad u = 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{1.30}$$

To fit (1.30) into the framework of section 1.1, we choose the Hilbert space $V := H^1_0(\Omega)$ with

$$a = a_{\mathsf{P}} : V \times V \to \mathbb{R} \qquad a(v, w) := a_{\mathsf{P}}(v, w) := \int_\Omega \nabla v \cdot \nabla w. \tag{1.31}$$

For this analytic setting, we introduce several discretizations. Each discrete method is labeled by an identifier. For reference purposes, an overview of these identifiers is provided in Table 2 on page 5.

1.4.2.1 The conforming Lagrange finite method

For completeness, we recall the method using conforming Lagrange finite elements. We label this method with P-LAG. For this method, we choose $V_h = \mathring{S}^1_\ell(\mathcal{T})$ with $\ell \in \mathbb{N}$.

To obtain a discrete method, we select $a_h = a_{\mathsf{P}|V_h \times V_h}$ together with $\mathrm{E_M} = \mathrm{id}_{V_h}$. This is a conforming method which is quasi-optimal. This setting is thoroughly discussed in the literature ([BS08, EG04]).

$$\begin{aligned} \tilde{V} = H^1_0(\Omega) + \mathring{S}^1_\ell(\mathcal{T}) = H^1_0(\Omega) \quad \text{with} \quad V = H^1_0(\Omega), \quad V_h = \mathring{S}^1_\ell(\mathcal{T}), \\ \tilde{a} = a_{\mathsf{P}}, \quad \|\cdot\| = \|\nabla \cdot\|_{L^2(\Omega)}, \\ a_h = a_{\mathsf{P}|V_h \times V_h}; \quad \text{and} \quad E = \mathrm{id}_{|\mathring{S}^1_\ell(\mathcal{T})}. \end{aligned} \tag{P-LAG}$$

Given $f \in H^{-1}(\Omega)$, seek $u_h \in \mathring{S}^1_\ell(\mathcal{T})$ such that $a_{\mathsf{P}}(u_h, v_h) = \langle f, v_h \rangle, \quad \forall v_h \in \mathring{S}^1_\ell(\mathcal{T})$.

The framework of [VZ18a], however, is primarily intended for the analysis of nonconforming methods, a first one is discussed in the following section.

1.4.2.2 A first-order Crouzeix-Raviart method

The first-order Crouzeix-Raviart element is introduced in [CR73]. We write $m_F = \frac{1}{d}\sum_{z\in\mathcal{V}\cap F} z$ for the barycenter of $F \in \mathcal{F}$. On a single element $T \in \mathcal{T}$, we define the shape functions $\mathbb{P} := \mathbb{P}_1(T)$ and the degrees of freedom

$$\mathcal{N}_{\mathrm{CR}_1}(T) := \left\{ N_F(v) := v(m_F) = \frac{1}{|F|}\int_F v \,\middle|\, F \in \mathcal{F}(T) \right\}. \tag{1.32}$$

We combine these finite elements with a constraint to obtain the global CR_1 finite element space

$$\mathrm{CR}_1 := \left\{ v \in S_1^0(\mathcal{T}) \,\middle|\, \forall F \in \mathcal{F} \ \int_F [\![v]\!] = 0 \right\}. \tag{1.33}$$

An overview of classical methods for the CR_1-element without quasi-optimal a-priori estimates can be found in [Bre15]. In general, those methods require additional regularity assumptions either for the analytic solution or the data beyond $u \in H_0^1(\Omega)$ and $f \in H^{-1}(\Omega)$, see the discussion in the introduction and [VZ18a, VZ19a]. We, therefore, recall only the quasi-optimal method of [VZ19a], which we label with P-CR1.

Within the framework of section 1.1, we select the discrete space $V_h = \mathrm{CR}_1$. This implies $\tilde{V} = H_0^1(\Omega) + \mathrm{CR}_1$. We introduce the piecewise gradient defined as $(\nabla_h v)_{|T} := \nabla(v_{|T})$, $v \in H^1(\mathcal{T})$, for all $T \in \mathcal{T}$ and define the extended bilinear form $\tilde{a}^{\mathsf{CR}} : \tilde{V} \times \tilde{V} \to \mathbb{R}$ via

$$\tilde{a}^{\mathsf{CR}}(v, w) := \int_\Omega \nabla_h v \cdot \nabla_h w \quad \forall v, w \in \tilde{V}.$$

Thus, we have in this section $\|\cdot\| := \|\cdot\|_{\tilde{V}} := \|\cdot\|_{\mathsf{CR}} := \|\nabla_h \cdot\|_{L^2(\Omega)}$. We define the form $a_h(v_h, w_h) := a_h^{\mathsf{CR}}(v_h, w_h) := \tilde{a}^{\mathsf{CR}}(v_h, w_h)$, $v_h, w_h \in V_h$, which fulfills (1.4).

In section 1.1, we learned that a quasi-optimal method needs to be fully stable and fully algebraic consistent, c.f. Proposition 1.12. Then, [VZ18a, Proposition 1.2] shows that a smoothing operator must be involved. Therefore, to be quasi-optimal, the method P-CR1 requires a smoothing operator.

Smoothing operators for the CR_1-element have already been discussed in the literature before the publication of [VZ18a], see for example [Bre15, section 2.2.1]. These operators map a discrete function into the conforming Lagrange finite element space $\mathring{S}_1^1(\mathcal{T})$ by defining the values at the points $\mathcal{L}_1$. One example is the operator $\mathcal{A}_1 : \mathrm{CR}_1 \to \mathring{S}_1^1(\mathcal{T})$ with

$$(\mathcal{A}_1 v_h)(z) = \begin{cases} v_{h|T_z}(z) \text{ for a fixed } T_z \in \mathcal{T} \text{ with } z \in T_z, & \text{if } z \in \Omega, \\ 0, & \text{otherwise.} \end{cases} \tag{1.34}$$

In general, however, the operator $\mathcal{A}_1$ does not ensure the algebraic consistency condition (1.12), thus it can not serve as a smoother for a quasi-optimal method. To build a quasi-optimal method, [VZ19a] enhances this operator $\mathcal{A}_1$.

Remark 1.31. In other contributions such as [Bre15, section 2.2.1] or [Bre96, (3.3)], the operator is defined via an average of all values in the adjacent elements. Both operators, the one using averages and $\mathcal{A}_1$ as defined in (1.34), satisfy similar estimates, hence we could use either one for our purposes.

To understand how to enhance the operator $\mathcal{A}_1$, we follow [VZ19a, Lemma 3.2] and elementwisely integrate (1.12) by parts. For an abstract smoother $E : \mathrm{CR}_1 \to H_0^1(\Omega)$, $u \in H_0^1(\Omega) \cap \mathrm{CR}_1$ and $v_h \in \mathrm{CR}_1$ we get

$$\begin{aligned} \tilde{a}^{\mathsf{CR}}(u, v_h - Ev_h) &= \int_\Omega \nabla_h u \cdot \nabla_h (v_h - Ev_h) \\ &= -\sum_{T\in\mathcal{T}} \int_T \Delta u (v_h - Ev_h) + \sum_{F\in\mathcal{F}} \int_F [\![\nabla_h u \cdot \mathsf{n}_F (v_h - Ev_h)]\!] \\ &= \sum_{F\in\mathcal{F}^i} \int_F [\![\nabla_h u \cdot \mathsf{n}_F]\!] (\{\!\{v_h\}\!\} - Ev_h) + \sum_{F\in\mathcal{F}} \int_F \{\!\{\nabla_h u \cdot \mathsf{n}_F\}\!\} [\![v_h]\!]. \end{aligned} \tag{1.35}$$

We used that $\Delta_h u_{|T} = 0$, for all $T \in \mathcal{T}$ because of $u \in S_1^0(\mathcal{T})$. Then we applied Lemma 1.30. Since $\nabla_h u$ is piecewise constant, the definition of CR_1 implies that the last summand vanishes. This calculation suggests using an operator E that fulfills a *momentum condition*

$$\int_F \{\!\{v_h\}\!\} = \int_F Ev_h \quad \forall F \in \mathcal{F}^i \tag{1.36}$$

to make the method algebraic consistent, c.f. [VZ19a, Lemma 3.2].

To fulfill this momentum condition, we construct a bubble operator $\mathcal{B}_1$ as in [VZ19a, (3.13)]:

$$\mathcal{B}_1 v_h = \sum_{F\in\mathcal{F}^i} \left(\int_F v_h\right) \Psi_F, \quad \text{with} \quad \Psi_F := \frac{(2d-1)!}{(d-1)!\,|F|} \prod_{z\in\mathcal{V}, z\in F} \Phi_z^1, \quad F\in\mathcal{F}^i. \tag{1.37}$$

As discussed in [VZ19a, Remark 3.5], the bubble operator is unstable under mesh refinement with respect to the $\|\nabla_h \cdot\|_{L^2(\Omega)}$ norm and needs stabilization. This stabilization can be provided by the operator $\mathcal{A}_1$.

Therefore, in [VZ19a, VZ18b], the authors complement the bubble operator $\mathcal{B}_1$ with the operator $\mathcal{A}_1$ and define:

$$\mathcal{E}_1 v_h := \mathcal{A}_1 v_h + \mathcal{B}_1(v_h - \mathcal{A}_1 v_h), \quad v_h \in V_h. \tag{1.38}$$

This operator is stable with respect to $\|\nabla_h \cdot\|_{L^2(\Omega)}$, see [VZ19a, Proposition 3.3], while $\|\mathcal{E}_1\|_{\mathbb{L}(V_h,V)}$ depends only on the dimension d and the shape constant $\sigma(\mathcal{T})$. The operator $\mathcal{E}_1$ is invariant on $H_0^1(\Omega)\cap \mathrm{CR}_1$, making this operator a quasi-optimal smoother. Finally, the operator $\mathcal{E}_1$ fulfills the momentum condition (1.36), therefore we can define a quasi-optimal Crouzeix-Raviart method

$$\begin{aligned} \tilde{V} = H_0^1(\Omega) + \mathrm{CR}_1 \quad \text{with} \quad V = H_0^1(\Omega), \quad V_h = \mathrm{CR}_1,\\ \tilde{a} = \tilde{a}^{\mathsf{CR}}, \quad \|\cdot\| := \|\cdot\|_{\mathsf{CR}} := \|\nabla_h \cdot\|_{L^2(\Omega)},\\ a_h = a_h^{\mathsf{CR}}; \quad \text{and} \quad E = \mathcal{E}_1. \end{aligned}$$

Given $f\in H^{-1}(\Omega)$, find $u_h\in \mathrm{CR}_1$ such that $\tilde{a}^{\mathsf{CR}}(u_h,w_h) = a_h^{\mathsf{CR}}(u_h,w_h) = \langle f, \mathcal{E}_1 w_h\rangle, \quad \forall w_h\in \mathrm{CR}_1$. (P-CR1)

This is the method introduced in [VZ19a, section 3.2].

The result in [VZ19a, Theorem 3.4] shows that this method (P-CR1) is quasi-optimal with a quasi-optimality constant $C_{\mathsf{qopt}} \geqslant 0$ depending only on the dimension d and the shape-constant $\sigma(\mathcal{T})$. We have that $C_{\mathsf{stab}} = C_{\mathsf{qopt}} = \|\mathcal{E}_1\|_{\mathbb{L}(V_h,V)}$ holds ([VZ19a, Corollary 2.7]) because the method is overconsistent, which means $\delta_{V_h} = 0$. The overconsistency is achieved because the smoother $\mathcal{E}_1$ is a right-inverse of the canonical CR_1-projection, for details see [VZ19a, Proposition 3.3].

1.4.2.3 Higher order Crouzeix-Raviart methods

We turn to higher order Crouzeix-Raviart methods. Generalization of the Crouzeix-Raviart element were considered in [FS83] for second order and [CF89] for third order. An arbitrary order generalization is discussed in the two publications [SB06, BS07]. A basis in the general 3-dimensional case is constructed in [CDS18]. We recall the exposition of [VZ19a] for a quasi-optimal CR-method of higher order.

Definition 1.32. *We define for $\ell\in\mathbb{N}$ the Crouzeix-Raviart space of order $\ell\in\mathbb{N}$ as*

$$\mathrm{CR}_\ell := \left\{ v\in S_\ell^0(\mathcal{T}) \,\middle|\, \forall F\in\mathcal{F} \text{ and } \forall r\in\mathbb{P}_{\ell-1}(F) \quad \int_F [\![v]\!]\, r = 0 \right\}. \tag{1.39}$$

The degrees of freedom for this spaces are needed later, see Proposition 2.79.

To construct a quasi-optimal method for the polynomial degree $\ell\in\mathbb{N}$, we proceed similarly as before. We reuse the scalar product $\tilde{a} = \tilde{a}^{\mathsf{CR}}$, bilinear form $a_h = a_h^{\mathsf{CR}}$ and norm $\|\cdot\|_{\tilde{V}} = \|\cdot\|_{\mathsf{CR}}$. In contrast to (1.35), the element contributions do not vanish for $\ell > 1$, hence we have for $u\in H_0^1(\Omega)\cap \mathrm{CR}_\ell$, $v_h\in\mathrm{CR}_\ell$ and an abstract smoother $E:\mathrm{CR}_\ell\to H_0^1(\Omega)$ that

$$\begin{aligned} \tilde{a}^{\mathsf{CR}}(u, v_h - Ev_h) &= \int_\Omega \nabla_h u\cdot\nabla_h(v_h - Ev_h)\\ &= -\sum_{T\in\mathcal{T}} \int_T \Delta u (v_h - Ev_h) + \sum_{F\in\mathcal{F}} \int_F [\![\nabla_h u\cdot \mathsf{n}_F (v_h - Ev_h)]\!]\\ &= -\sum_{T\in\mathcal{T}} \int_T \Delta u (v_h - Ev_h) + \sum_{F\in\mathcal{F}^i} \int_F [\![\nabla_h u\cdot \mathsf{n}_F]\!] \left(\{\{v_h\}\} - Ev_h\right)\\ &\quad + \underbrace{\sum_{F\in\mathcal{F}} \int_F \{\{\nabla_h u\cdot \mathsf{n}_F\}\} [\![v_h]\!]}_{=0}. \end{aligned} \tag{1.40}$$

Similar to the lowest order case, the computation (1.40) suggests using a smoothing operator that preserves higher order moments on the faces and elements.

Definition 1.33. *A smoothing operator $E : S^0_\ell(\mathcal{T}) \to H^1_0(\Omega)$ fulfills the* momentum conditions *of order $\ell \in \mathbb{N}$ if for every $v_h \in S^0_\ell(\mathcal{T})$*

$$\begin{aligned} \forall F \in \mathcal{F}^i, p_F \in \mathbb{P}_{\ell-1}(F) \quad & \int_F E(v_h) p_F = \int_F \{\{v_h\}\}\, p_F \quad \text{and} \\ \forall T \in \mathcal{T}, p_T \in \mathbb{P}_{\ell-2}(T) \quad & \int_T E(v_h) p_T = \int_T v_h p_T. \end{aligned} \tag{1.41}$$

The construction of a smoothing operator satisfying the momentum conditions is similar to the first order setting. At first, we construct a bubble operator $\mathcal{B}_\ell$ fulfilling the momentum conditions in Definition 1.33 and then add a stabilization.

We only sketch the construction performed in [VZ19a, pp. 281], but present it directly for a more general situation, so we can reuse the smoother in the next section. For each $F \in \mathcal{F}^i$ with $T_1, T_2 \in \mathcal{T}$ and $T_1 \cap T_2 = F$, we define the local weighted L^2-projection $Q_F : L^2(F) \to \mathbb{P}_{\ell-1}(F)$ for $v \in L^2(F)$ implicitly via the system of equations (c.f. [VZ18a, (3.19)]):

$$\int_F (Q_F v) q \Psi_F = \int_F vq \quad \forall q \in \mathbb{P}_{\ell-1}(F).$$

Here, $\Psi_F \in H^1_0(\omega_F)$ is the bubble defined in (1.37).

For an element $T \in \mathcal{T}$, we analogously define the weighted L^2-projection $Q_T : L^2(T) \to \mathbb{P}_{\ell-2}(T)$ (c.f. [VZ19a, (3.20)] and [VZ18b, p. 2880]). If $\ell = 1$, we set $Q_T = 0$ for all $T \in \mathcal{T}$. Otherwise, we define Q_T via the system

$$\int_T (Q_T v) r \Psi_T = \int_T vr \quad \forall r \in \mathbb{P}_{\ell-2}(T), v \in L^2(T)$$

The function $\Psi_T \in H^1_0(T)$ is a bubble function defined as

$$\Psi_T := \frac{(2d+1)!}{d!|T|} \prod_{z \in \mathcal{V}, z \in T} \Phi^1_z. \tag{1.42}$$

Both weighted L^2-projections are used to define two bubble operators as (c.f. [VZ18b, section 3.2])

$$\mathcal{B}_{\mathcal{T},\ell} v := \sum_{T \in \mathcal{T}} (Q_T v) \Psi_T, \quad \mathcal{B}_{\mathcal{F},\ell} v := \sum_{F \in \mathcal{F}^i} \sum_{z \in \mathcal{L}_{\ell-1}(F)} (Q_F \{\{v\}\})(z) \Phi^{\ell-1}_z \Psi_F, \quad v \in H^1(\mathcal{T}). \tag{1.43}$$

Here $\Phi^{\ell-1}_z$ are the Lagrange basis functions of order $\ell - 1$, which are defined in (1.28). If $\ell = 1$, we set $\mathcal{L}_0(F) = \{m_F\}$ and $\Phi^0_{m_F} = 1$ as in [VZ18b, p. 2881].

The final bubble operator $\mathcal{B}_\ell : H^1(\mathcal{T}) \to H^1_0(\Omega)$ is defined as (see [VZ18b, Lemma 3.2])

$$\mathcal{B}_\ell v := \mathcal{B}_{\mathcal{F},\ell} v + \mathcal{B}_{\mathcal{T},\ell}\big(v - \mathcal{B}_{\mathcal{F},\ell} v\big), \quad v \in H^1(\mathcal{T}). \tag{1.44}$$

For $\ell = 1$, we obtain the bubble operator of the previous section. The bubble operator $\mathcal{B}_\ell$ is, similar to its lowest order counterpart $\mathcal{B}_1$, not $\|\nabla_h \cdot\|_{L^2(\Omega)}$-stable under mesh refinement, c.f. [VZ19a, Remark 3.5] and [VZ18b, p. 2881]. Thus, an appropriate stabilization is needed again, which could be achieved by any approximation operator with suitable estimates. A possible choice would be to reuse the operator $\mathcal{A}_1$ from above. However, this would result in a non quasi-optimal smoother. Nevertheless, this option was considered in [VZ18b, (3.25)] and we discuss such a method in section 1.4.2.5.

For the moment, we would like to construct a quasi-optimal smoother. Thus, we present a generalization called $\mathcal{A}_\ell$ of the operator $\mathcal{A}_1$ which is invariant on piecewise polynomials of degree ℓ. The definition involves the Lagrange basis for the space $\mathring{S}^1_\ell(\mathcal{T})$, which has the points of the set $\mathcal{L}^i_\ell$ as degrees of freedom, see (1.27).

Definition 1.34 ([VZ19a, (3.5)])**.** *We define $\mathcal{A}_\ell : S^0_\ell(\mathcal{T}) \to \mathring{S}^1_\ell(\mathcal{T})$ at each $z \in \mathcal{L}_\ell$ as*

$$\mathcal{A}_\ell v_h(z) = \begin{cases} v_{h\,|T_z}(z) \text{ for a fixed } T_z \in \mathcal{T} \text{ with } z \in T_z, & \text{if } z \in \Omega, \\ 0 & \text{otherwise.} \end{cases} \tag{1.45}$$

With this new operator $\mathcal{A}_\ell$, we define a higher order, quasi-optimal smoothing operator $\mathcal{E}_\ell$.

Definition 1.35 ([VZ19a, Proposition 3.9])**.** *We define the smoothing operator $\mathcal{E}_\ell : S^0_\ell(\mathcal{T}) \to H^1_0(\Omega)$ as*

$$\mathcal{E}_\ell(v_h) := \mathcal{A}_\ell v_h + \mathcal{B}_\ell(v_h - \mathcal{A}_\ell v_h), \quad v_h \in S^0_\ell(\mathcal{T}). \tag{1.46}$$

If $\ell = 1$, (1.46) coincides with (1.38). The smoother $\mathcal{E}_\ell$ is quasi-optimal because it is invariant on $\mathring{S}^0_\ell(\mathcal{T}) \cap V = \mathring{S}^1_\ell(\mathcal{T})$.

Next, we derive appropriate estimates for the smoothing operator $\mathcal{E}_\ell$. The estimates are primarily determined by the averaging operator $\mathcal{A}_\ell$. As in [VZ19a, (3.5)], the operator $\mathcal{A}_\ell$ can be rewritten in the form

$$\mathcal{A}_\ell v_h = \sum_{z \in \mathcal{L}_\ell \cap \Omega} v_{h|T_z}(z) \Phi^\ell_z, \quad v_h \in S^0_\ell(\mathcal{T}), \tag{1.47}$$

where T_z is the element chosen in (1.45) for the respective node z. The function Φ^ℓ_z is the Lagrange basis function corresponding to $z \in \mathcal{L}_\ell$ as defined in (1.28).

With this representation of $\mathcal{A}_\ell$, we can derive the following basic estimate.

Lemma 1.36 ([VZ19a, Lemma 3.1, (3.6)])**.** *Given $T \in \mathcal{T}$, the operator $\mathcal{A}_\ell$ fulfills for each $z \in \mathcal{L}_\ell(T)$ and $v_h \in S^0_\ell(\mathcal{T})$*

$$\big|v_{h|T}(z) - (\mathcal{A}_\ell v_h)(z)\big| \leqslant \sum_{\substack{F \in \mathcal{F} \\ F \ni z}} \Big|[\![v_h]\!]_{|F}(z)\Big|. \tag{1.48}$$

Proof. The proof can be found in [VZ19a, Lemma 3.1]. □

The estimate (1.48) can be continued in two ways. One is suitable for the Crouzeix-Raviart method and another is useful for the discontinuous Galerkin methods of the next section.

Lemma 1.37. *There exist constants $C_{d,\ell} > 0$, depending only on d and ℓ, such that for all $v_h \in S^0_\ell(\mathcal{T})$, $v \in H^1_0(\omega_F)$, $F \in \mathcal{F}$ and $z \in \mathcal{L}_\ell \cap F$, we have ([VZ18b, Lemma 3.3]):*

$$\Big|[\![v_h]\!]_{|F}(z)\Big| \leqslant \|\,[\![v_h]\!]_{|F}\,\|_{L^\infty(F)} \leqslant \frac{C_{d,\ell}}{|F|^{\frac12}} \|\,[\![v_h - v]\!]_{|F}\,\|_{L^2(F)}. \tag{1.49a}$$

Let $F \in \mathcal{F}^i$ and $T_1, T_2 \in \mathcal{T}$ such that $T_1 \cap T_2 = F$, then we have ([VZ19a, (3.8a)]):

$$\Big|[\![v_h]\!]_{|F}(z)\Big| \leqslant \frac{C_{d,\ell}}{|F|}\left|\int_F [\![v_h - v]\!]_{|F}\right| + \sum_{i=1,2} C_{d,\ell} \frac{h_{T_i}}{|T_i|^{\frac12}} \|\nabla(v_h - v)\|_{L^2(T_i)}. \tag{1.49b}$$

If, however, $F \subset \partial\Omega$ with $F \subset T \in \mathcal{T}$, we have ([VZ19a, (3.8b)])

$$\Big|[\![v_h]\!]_{|F}(z)\Big| \leqslant \frac{C_{d,\ell}}{|F|}\left|\int_F [\![v_h]\!]_{|F}\right| + C_{d,\ell} \frac{h_T}{|T|^{\frac12}} \|\nabla(v_h - v)\|_{L^2(T)}. \tag{1.49c}$$

Since (1.49b) and (1.49c) slightly differ from [VZ19a, Lemma 3.1], we provide their proof.

Proof. We closely follow the arguments of [VZ19a, Lemma 3.1]. Without loss of generality, we assume that the normal vector n_F points from T_1 to T_2. We define $f_j := |F|^{-1} \int_F (v_h - v)_{|T_j}$ and $k_j := |T_j|^{-1} \int_{T_j} (v_h - v)$, $j = 1, 2$. With an inverse estimate, we obtain for $F \in \mathcal{F}^i$

$$\begin{aligned}
\Big|[\![v_h]\!]_{|F}(z)\Big| &\leqslant \frac{C_{d,\ell}}{|F|^{\frac12}} \|\,[\![v_h]\!]_{|F}\,\|_{L^2(F)} = \frac{C_{d,\ell}}{|F|^{\frac12}} \|v_{h|T_1} - v - v_{h|T_2} + v\|_{L^2(F)} \\
&= \frac{C_{d,\ell}}{|F|^{\frac12}} \|v_{h|T_1} - v - k_1 + k_1 - f_1 + f_1 - f_2 + f_2 - k_2 + k_2 + v - v_{h|T_2}\|_{L^2(F)} \\
&\leqslant \frac{C_{d,\ell}}{|F|^{\frac12}} \big(\|v_{h|T_1} - v - k_1\|_{L^2(F)} + \|k_1 - f_1\|_{L^2(F)} + \|f_1 - f_2\|_{L^2(F)} + \\
&\quad \|f_2 - k_2\|_{L^2(F)} + \|k_2 - v_{h|T_2} + v\|_{L^2(F)}\big) \\
&= \frac{C_{d,\ell}}{|F|^{\frac12}} \big(\|v_{h|T_1} - v - k_1\|_{L^2(F)} + \|k_2 - v_{h|T_2} + v\|_{L^2(F)}\big) + \\
&\quad C_{d,\ell}(|k_1 - f_1| + |f_1 - f_2| + |f_2 - k_2|).
\end{aligned}$$

Now, we have $|f_1 - f_2| = \left|\frac{1}{|F|}\int_F [\![v_h - v]\!]_{|F}\right|$. The trace identity [VV09, Proposition 4.2] provides the estimate $|k_j - f_j| \leqslant \frac{h_{T_j}}{d|T_j|^{\frac12}} \|\nabla(v_h - v)\|_{L^2(T_j)}$ for $j = 1, 2$. The remaining terms $\|v_{h|T_j} - v - k_j\|_{L^2(F)}$ can be estimated with [Vee16, Lemma 3] as $|F|^{-\frac12} \|v_{h|T_j} - v - k_j\|_{L^2(F)} \leqslant C_d \frac{h_{T_j}}{|T_j|^{\frac12}} \|\nabla(v_h - v)\|_{L^2(T_j)}$, for $j = 1, 2$. This proves (1.49b) and (1.49c) follows suit. □

In the CR_ℓ case, the definition of the CR_ℓ-element shows $\int_F [\![v_h - v]\!]_{|F} = 0$ thus the jump terms in (1.49b), (1.49c) vanish.

Combining the last two Lemmas provides us with the following stability estimates (see [VZ19a, Proposition 3.3] and [VZ18b, Proposition 3.4]).

Proposition 1.38. *There exist constants $C_{d,\ell} > 0$ only depending on d and ℓ such that the operator $\mathcal{A}_\ell : S^0_\ell(\mathcal{T}) \to \mathring{S}^1_\ell(\mathcal{T})$ fulfills for all $v_h \in S^0_\ell(\mathcal{T})$, $v \in H^1_0(\Omega)$ and $T \in \mathcal{T}$ the estimates ([VZ19a, (3.15)]):*

$$\|v_h - \mathcal{A}_\ell v_h\|_{L^2(T)} \leqslant C_{d,\ell}\Big(\sum_{F\in\mathcal{F}_T^{i,+}} \frac{|T|^{\frac12}}{|F|}\Big|\int_F [\![v_h - v]\!]\Big| + \sum_{T'\in\mathcal{T}_T^+} \frac{|T|^{\frac12} h_{T'}}{|T'|^{\frac12}} \|\nabla_h(v_h - v)\|_{L^2(T')}\Big), \tag{1.50a}$$

$$\|\nabla_h(v_h - \mathcal{A}_\ell v_h)\|_{L^2(T)} \leqslant C_{d,\ell}\Big(\sum_{F\in\mathcal{F}_T^{i,+}} \frac{|T|^{\frac12}}{\rho_T|F|}\Big|\int_F [\![v_h - v]\!]\Big| + \sum_{T'\in\mathcal{T}_T^+} \frac{|T|^{\frac12} h_{T'}}{|T'|^{\frac12}\rho_T} \|\nabla_h(v_h - v)\|_{L^2(T')}\Big). \tag{1.50b}$$

Furthermore, we have ([VZ18b, (3.16)]):

$$\|v_h - \mathcal{A}_\ell v_h\|_{L^2(T)} \leqslant C_{d,\ell} \sum_{F\in\mathcal{F}_T^{i,+}} \frac{|T|^{\frac12}}{|F|^{\frac12}} \| [\![v_h - v]\!] \|_{L^2(F)}, \tag{1.51a}$$

$$\|\nabla_h(v_h - \mathcal{A}_\ell v_h)\|_{L^2(T)} \leqslant C_{d,\ell} \sum_{F\in\mathcal{F}_T^{i,+}} \frac{|T|^{\frac12}}{\rho_T|F|^{\frac12}} \| [\![v_h - v]\!] \|_{L^2(F)}. \tag{1.51b}$$

In addition, for each element $T \in \mathcal{T}$ and face $F \in \mathcal{F}_T$, we have the estimates

$$\|v_{h|T} - \mathcal{A}_\ell v_h\|_{L^2(F)} \leqslant C_{d,\ell}\Big(\sum_{F'\in\mathcal{F}_F^{i,+}} \underbrace{\frac{|F|^{\frac12}}{|F'|}}_{\eqsim |F|^{-\frac12}} \Big|\int_{F'} [\![v_h - v]\!]\Big| + \sum_{T'\in\mathcal{T}_F^+} \underbrace{\frac{|F|^{\frac12} h_{T'}}{|T'|^{\frac12}}}_{\eqsim h_T^{\frac12}} \|\nabla_h(v_h - v)\|_{L^2(T')}\Big), \tag{1.52a}$$

$$\|v_{h|T} - \mathcal{A}_\ell v_h\|_{L^2(F)} \leqslant C_{d,\ell} \sum_{F'\in\mathcal{F}_F^{i,+}} \| [\![v_h - v]\!] \|_{L^2(F')}. \tag{1.52b}$$

Proof. First, we show the estimates (1.50) and (1.51). To bound the L^2-norms, we have for $v_h \in S^0_\ell(\mathcal{T})$, similar to [VZ19a, (3.15)], the estimate:

$$\begin{aligned} \|\mathcal{A}_\ell v_h - v_h\|_{L^2(T)} &\leqslant \sum_{z\in\mathcal{L}_\ell(T)} \big|\mathcal{A}_\ell v_h(z) - v_{h|T}(z)\big| \|\Phi^\ell_z\|_{L^2(T)} \\ &\leqslant C_{d,\ell}|T|^{\frac12} \sum_{z\in\mathcal{L}_\ell(T)} \big|\mathcal{A}_\ell v_h(z) - v_{h|T}(z)\big| \end{aligned} \tag{1.53}$$

with $C_{d,\ell}$ depending only on the dimension and the polynomial degree ℓ, c.f. [VZ19a, (3.4)]. Next, we can apply Lemma 1.36 and together with (1.49b) or (1.49c), we obtain (1.50a) resp. (1.51a). To obtain (1.50b) and (1.51b), we use an estimate analogous to (1.53) but employ the bound $\|\nabla\Phi^\ell_z\|_{L^2(T)} \leqslant C_{d,\ell}\frac{|T|^{\frac12}}{\rho_T}$, as in [VZ19a, (3.15)].

Second, we turn to the face estimates (1.52). Fix $F \in \mathcal{F}$ and choose $T \in \mathcal{T}_F$. We compute

$$\begin{aligned} \|v_{h|T} - \mathcal{A}_\ell v_h\|_{L^2(F)} &\leqslant \sum_{z\in\mathcal{L}_\ell(F)} \big|v_{h|T}(z) - \mathcal{A}_\ell v_h(z)\big| \|\Phi^\ell_z\|_{L^2(F)} \\ &\leqslant C_{d,\ell} \sum_{z\in\mathcal{L}_\ell(F)} \big|v_{h|T}(z) - \mathcal{A}_\ell v_h(z)\big| |F|^{\frac12} \end{aligned}$$

The rest follows as above with Lemmas 1.36 and 1.37. □

To show the corresponding results for the complete smoothing operator $\mathcal{E}_\ell$, we additionally derive estimates for the bubble operator $\mathcal{B}_\ell$. These results can be found in [VZ19a, Lemma 3.8] for the special case of the CR_ℓ-element. We directly provide a more general version, which is reusable for the methods in the next section.

Lemma 1.39 (Stability of the bubble operator [VZ19a, Lemma 3.8], [VZ18b, Lemma 3.2])**.** *For any $v_h \in S^0_\ell(\mathcal{T})$ and $T \in \mathcal{T}$, we have the bound*

$$\begin{aligned}\|\nabla \mathcal{B}_\ell v_h\|_{L^2(T)} &\leqslant \frac{C_{d,\ell}}{\rho_T}\|\mathcal{B}_\ell v_h\|_{L^2(T)} \\ &\leqslant \frac{C_{d,\ell}}{\rho_T}\Big(\sup_{r\in\mathbb{P}_{\ell-2}(T)}\frac{\int_T v_h r}{\|r\|_{L^2(T)}} + \sum_{F\in\mathcal{F}_T}\frac{|T|^{\frac12}}{|F|^{\frac12}}\sup_{q\in\mathbb{P}_{\ell-1}(F)}\frac{\int_F \{\{v_h\}\}\, q}{\|q\|_{L^2(F)}}\Big) \\ &\leqslant \frac{C_{d,\ell}}{\rho_T}\Big(\|v_h\|_{L^2(T)} + \sum_{F\in\mathcal{F}_T}\frac{|T|^{\frac12}}{|F|^{\frac12}}\|\{\{v_h\}\}\|_{L^2(F)}\Big)\end{aligned}$$

for some constant $C_{d,\ell}$ depending only on the dimension d and the polynomial degree $\ell \in \mathbb{N}$. Again, ρ_T is the radius of the largest inner ball of T.

The next Proposition derives estimates for the expression $\mathcal{B}_\ell(v_h - \mathcal{A}_\ell v_h)$.

Proposition 1.40 ([VZ19a, Proposition 3.9], [VZ18b, Proposition 3.4])**.** *For any $v_h \in S^0_\ell(\mathcal{T})$, $v \in H^1_0(\Omega)$ and $T \in \mathcal{T}$, there exists $C_{d,\ell} > 0$, only depending on d and ℓ such that*

$$\begin{aligned}\|\nabla(\mathcal{B}_\ell(v_h - \mathcal{A}_\ell v_h))\|_{L^2(T)} &\leqslant \frac{C_{d,\ell}}{\rho_T}\|(\mathcal{B}_\ell(v_h - \mathcal{A}_\ell v_h))\|_{L^2(T)} \\ &\leqslant \frac{C_{d,\ell}}{\rho_T}\Big(\sum_{F\in\mathcal{F}_T^{i,+}}\frac{|T|^{\frac12}}{|F|}\Big|\int_F [\![v_h - v]\!]\Big| + \sum_{T'\in\mathcal{T}_T^+}\frac{|T|^{\frac12}h_{T'}}{|T'|^{\frac12}}\|\nabla_h(v_h - v)\|_{L^2(T')}\Big)\end{aligned} \tag{1.54a}$$

and

$$\|\nabla(\mathcal{B}_\ell(v_h - \mathcal{A}_\ell v_h))\|_{L^2(T)} \leqslant \frac{C_{d,\ell}}{\rho_T}\|(\mathcal{B}_\ell(v_h - \mathcal{A}_\ell v_h))\|_{L^2(T)} \leqslant \frac{C_{d,\ell}}{\rho_T}\sum_{F\in\mathcal{F}_T^{i,+}}\frac{|T|^{\frac12}}{|F|^{\frac12}}\|[\![v_h - v]\!]\|_{L^2(F)}. \tag{1.54b}$$

Proof. First, we show (1.54a). The first inequality in (1.54a) follows by an inverse estimate. Then, the bubble operator estimate in Lemma 1.39 shows in combination with (1.50a) and (1.52a):

$$\begin{aligned}\|\mathcal{B}_\ell(v_h - \mathcal{A}_\ell v_h)\|_{L^2(T)} &\leqslant C_{d,\ell}\Big(\|v_h - \mathcal{A}_\ell v_h\|_{L^2(T)} + \sum_{F\in\mathcal{F}_T}\frac{|T|^{\frac12}}{|F|^{\frac12}}\|\{\{v_h\}\} - \mathcal{A}_\ell v_h\|_{L^2(F)}\Big) \\ &\leqslant C_{d,\ell}\Big(\sum_{F\in\mathcal{F}_T^{i,+}}\frac{|T|^{\frac12}}{|F|}\Big|\int_F [\![v_h - v]\!]\Big| + \sum_{T'\in\mathcal{T}_T^+}\frac{|T|^{\frac12}h_{T'}}{|T'|^{\frac12}}\|\nabla_h(v_h - v)\|_{L^2(T')}\Big).\end{aligned}$$

Here, we exploited that if $T_1, T_2 \in \mathcal{T}$ with $T_1 \cap T_2 = F$ we have

$$\|\{\{v_h\}\} - \mathcal{A}_\ell v_h\|_{L^2(F)} \leqslant \frac12(\|v_{h|T_1} - \mathcal{A}_\ell v_h\|_{L^2(F)} + \|v_{h|T_2} - \mathcal{A}_\ell v_h\|_{L^2(F)})$$

and that for each $F \in \mathcal{F}_T$, we have $\mathcal{T}_F^+ \subset \mathcal{T}_T^+$.

Second, we show (1.54b). We again apply an inverse estimate to show the first inequality. The rest of (1.54b) is derived with (1.51a) and (1.52b):

$$\begin{aligned}\|\mathcal{B}_\ell(v_h - \mathcal{A}_\ell v_h)\|_{L^2(T)} &\leqslant C_{d,\ell}\Big(\|v_h - \mathcal{A}_\ell v_h\|_{L^2(T)} + \sum_{F\in\mathcal{F}_T}\frac{|T|^{\frac12}}{|F|^{\frac12}}\|\{\{v_h\}\} - \mathcal{A}_\ell v_h\|_{L^2(F)}\Big) \\ &\leqslant C_{d,\ell}\sum_{F\in\mathcal{F}_T^{i,+}}\frac{|T|^{\frac12}}{|F|^{\frac12}}\|[\![v_h - v]\!]\|_{L^2(F)}.\end{aligned}$$

□

Corollary 1.41 ([VZ19a, Proposition 3.9], [VZ18b, Proposition 3.4])**.** *The estimates* (1.50) *and* (1.51) *are also true if we replace $\mathcal{A}_\ell$ by $\mathcal{E}_\ell : S^0_\ell(\mathcal{T}) \to \mathring{S}^1_\ell(\mathcal{T})$, i.e. for each $v_h \in S^0_\ell(\mathcal{T})$ and $v \in V$, we have*

$$\begin{aligned}\rho_T\|\nabla_h(v_h - \mathcal{E}_\ell v_h)\|_{L^2(T)} &\leqslant C_{d,\ell}\|v_h - \mathcal{E}_\ell v_h\|_{L^2(T)} \\ &\leqslant C_{d,\ell}\Big(\sum_{F\in\mathcal{F}_T^{i,+}}\frac{|T|^{\frac12}}{|F|}\Big|\int_F [\![v_h - v]\!]\Big| + \sum_{T'\in\mathcal{T}_T^+}\frac{|T|^{\frac12}h_{T'}}{|T'|^{\frac12}}\|\nabla_h(v_h - v)\|_{L^2(T')}\Big),\end{aligned} \tag{1.55a}$$

$$\rho_T\|\nabla_h(v_h-\mathcal{E}_\ell v_h)\|_{L^2(T)}\leqslant C_{d,\ell}\|v_h-\mathcal{E}_\ell v_h\|_{L^2(T)}\leqslant C_{d,\ell}\sum_{F\in\mathcal{F}_T^{i,+}}\frac{|T|^{\frac12}}{|F|^{\frac12}}\|\,[\![v_h-v]\!]\,\|_{L^2(F)};\tag{1.55b}$$

and

$$\|v_{h|T}-\mathcal{E}_\ell v_h\|_{L^2(F)}\leqslant C_{d,\ell}\Big(\sum_{F'\in\mathcal{F}_F^{i,+}}\frac{|F|^{\frac12}}{|F'|}\Big|\int_{F'}[\![v_h-v]\!]\Big|+\sum_{T'\in\mathcal{T}_F^{+}}\frac{|F|^{\frac12}h_{T'}}{|T'|^{\frac12}}\|\nabla_h(v_h-v)\|_{L^2(T')}\Big),\tag{1.56a}$$

$$\|v_{h|T}-\mathcal{E}_\ell v_h\|_{L^2(F)}\leqslant C_{d,\ell}\sum_{F'\in\mathcal{F}_F^{i,+}}\|\,[\![v_h-v]\!]\,\|_{L^2(F')}.\tag{1.56b}$$

Proof. After applying the triangle inequality to $v_h-\mathcal{E}_\ell v_h=v_h-\mathcal{A}_\ell v_h+\mathcal{B}_\ell(v_h-\mathcal{A}_\ell v_h)$, we use the respective inequalities of Propositions 1.38 and 1.40. □

Corollary 1.42 ([VZ19a, Proposition 3.9])**.** *The smoothing operator $\mathcal{E}_\ell:\mathrm{CR}_\ell\to\mathring{S}^1_\ell(\mathcal{T})$ fulfills the momentum conditions in Definition 1.33 and is stable in the $\|\nabla_h\cdot\|_{L^2(\Omega)}$-norm.*

Proposition 1.43 ([VZ19a, Theorem 3.10])**.** *For $\ell\geqslant1$ and $d=2$, we define the CR_ℓ-method defined via*

$$\begin{aligned}\tilde V=H^1_0(\Omega)+\mathrm{CR}_\ell\quad&\text{with}\quad V=H^1_0(\Omega),\quad V_h=\mathrm{CR}_\ell,\\ \tilde a=\tilde a^{\mathsf{CR}},\quad&\|\cdot\|:=\|\cdot\|_{\mathsf{CR}}:=\|\nabla_h\cdot\|_{L^2(\Omega)},\\ a_h=a_h^{\mathsf{CR}};\quad&\text{and}\quad E=\mathcal{E}_\ell.\end{aligned}\tag{P-CRL}$$

Given $f\in H^{-1}(\Omega)$, seek $u_h\in\mathrm{CR}_\ell$ such that $a_h^{\mathsf{CR}}(u_h,v_h)=\langle f,\mathcal{E}_\ell v_h\rangle,\quad\forall v_h\in\mathrm{CR}_\ell$.

The method is well posed and quasi-optimal with $C_{\mathsf{qopt}}<C_{d,\ell,\sigma(\mathcal{T})}$. Moreover, it is overconsistent, i.e. $\delta_{V_h}=0$.

1.4.2.4 Discontinuous Galerkin methods

As another discretization for the Poisson problem (1.30), we discuss two methods which use the space $S^0_\ell(\mathcal{T})$ for discretization. Many different methods using this space were already investigated in the literature. An overview is provided by the works [ABCM02, DPE12].

We are interested in two specific methods. One method, which is reviewed in [ABCM02, DPE12] and found in the original literature [Whe78, Arn82], uses a symmetric bilinear form. The other one uses a non-symmetric bilinear form. This method is treated in the review [ABCM02] and the papers [RWG99, RWG01]. These two methods form the basis for the improved quasi-optimal methods devised in [VZ18b].

We set $V_h=S^0_\ell(\mathcal{T})$. The bilinear form $\tilde a^{\mathsf{CR}}$ cannot be used as $\tilde a$ in the framework of section 1.2 in this case because it is not definite on $S^0_\ell(\mathcal{T})$. For this reason, we select the form

$$\tilde a^{\mathsf{dG}}(\tilde v,\tilde w):=\sum_{T\in\mathcal{T}}\int_T\nabla\tilde v\cdot\nabla\tilde w+\sum_{F\in\mathcal{F}}\frac{\gamma}{h_F}\int_F[\![\tilde v]\!]\,[\![\tilde w]\!]\,,$$

for all $\tilde v,\tilde w\in\tilde V=H^1_0(\Omega)+S^0_\ell(\mathcal{T})$ and $\gamma>0$ is a fixed stabilization parameter. The form induces the norm

$$\|\cdot\|^2_{\tilde V}=\|\cdot\|^2:=\|\cdot\|^2_{\mathsf{dG}}:=\|\nabla_h\cdot\|^2_{L^2(\Omega)}+\sum_{F\in\mathcal{F}}\int_F\frac{\gamma}{h_F}[\![\cdot]\!]^2\quad\text{on }\tilde V,$$

c.f. [VZ18b, (3.7)]. We consider two different discretizations, which only differ in a sign in the discrete bilinear form a_h. To simplify matters, we introduce a parameter $\theta\in\{1,-1\}$ for this purpose. Setting $\theta=1$ in a_h results in the non-symmetric interior penalty method (NSIP) and $\theta=-1$ in the symmetric interior penalty method (SIP). The discrete bilinear form $a_h:S^0_\ell(\mathcal{T})\times S^0_\ell(\mathcal{T})\to\mathbb{R}$ is defined as in [VZ18b, (3.19)] (but see also [Arn82, section 3] and [RWG01, section 2.2]):

$$\begin{aligned}a_h^{\mathsf{dG}}(v_h,w_h)=&\int_\Omega\nabla_hv_h\cdot\nabla_hw_h+\sum_{F\in\mathcal{F}}\int_F\frac{\gamma}{h_F}[\![v_h]\!]\,[\![w_h]\!]\\&+\sum_{F\in\mathcal{F}}\Big\{\theta\int_F[\![v_h]\!]\,\{\{\nabla_hw_h\cdot\mathsf{n}_F\}\}-\int_F\{\{\nabla_hv_h\cdot\mathsf{n}_F\}\}\,[\![w_h]\!]\Big\}\end{aligned}\tag{1.57}$$

for $v_h,w_h\in S^0_\ell(\mathcal{T})$ with the stabilization parameter γ chosen as in the form $\tilde a^{\mathsf{dG}}$ above.

The definiteness of the form a_h is crucial for the method. In the NSIP-case ($\theta = 1$), [VZ18b, Lemma 3.5] shows the coercivity of a_h, implying its definiteness, but see also [RWG01, Lemma 2.3] and [ABCM02, section 4.2]. In the SIP-case ($\theta = -1$), the situation is more involved. In [VZ18b, Lemma 3.5] the authors show the existence of a constant $\eta_* > 0$, depending on the dimension, the shape constant and the polynomial degree ℓ (see also [Whe78, Lemma 3], [Arn82, section 3] and [ABCM02, section 4.2]). Then, [VZ18b, (3.23)] ensures the definiteness of a_h if the stabilization constant $\gamma > 0$ in $\tilde{a}$ and a_h is chosen strictly larger than η_*.

In [VZ18b], the authors show that if the smoothing operator fulfills the momentum conditions in Definition 1.33 then the discrete method is fully algebraic consistent. To see this, we elementwisely calculate for an abstract smoother $E : S^0_\ell(\mathcal{T}) \to H^1_0(\Omega)$, a function $u \in H^1_0(\Omega) \cap S^0_\ell(\mathcal{T})$, a test function $w_h \in S^0_\ell(\mathcal{T})$ and the help of Lemma 1.30:

$$\begin{aligned} a_{\mathsf{P}}(u, Ew_h) &= \int_\Omega \nabla_h u \cdot \nabla_h E w_h = \int_\Sigma [\![\nabla_h u \cdot \mathsf{n}\, E w_h]\!] - \int_\Omega \Delta_h u E w_h \\ &= \int_{\Sigma^i} [\![\nabla_h u \cdot \mathsf{n}]\!] E w_h - \int_\Omega \Delta_h u E w_h. \end{aligned}$$

A similar computation for a_h provides

$$\begin{aligned} a_h^{\mathsf{dG}}(u, w_h) &= \int_\Omega \nabla_h u \cdot \nabla_h w_h - \int_\Sigma \{\!\{ \nabla_h u \cdot \mathsf{n} \}\!\} [\![w_h]\!] \\ &= \int_\Sigma [\![\nabla_h u \cdot \mathsf{n} w_h]\!] - \int_\Omega \Delta_h u w_h - \int_\Sigma \{\!\{ \nabla_h u \cdot \mathsf{n} \}\!\} [\![w_h]\!] \\ &= \int_{\Sigma^i} [\![\nabla_h u \cdot \mathsf{n}]\!] \{\!\{ w_h \}\!\} - \int_\Omega \Delta_h u w_h. \end{aligned}$$

A comparison of the last two displays shows that the algebraic consistency condition is fulfilled if the smoothing operator E satisfies the momentum conditions in Definition 1.33.

The operator $\mathcal{E}_\ell$ is stable with respect to the $\|\cdot\|_{\tilde{V}}$-norm because of Corollary 1.41. For completeness, we summarize the result in the following Proposition.

Proposition 1.44 ([VZ18b, Proposition 3.4])**.** *For the linear operator $\mathcal{E}_\ell$, the following approximation properties hold*

$$\|v_h - \mathcal{E}_\ell v_h\|_{L^2(\Omega)} \leqslant C_{d,\sigma(\mathcal{T}),\ell} \|h^{\frac{1}{2}} [\![v_h]\!] \|_{L^2(\Sigma)}, \tag{1.58a}$$

$$\|\nabla_h (v_h - \mathcal{E}_\ell v_h)\|_{L^2(\Omega)} \leqslant C_{d,\sigma(\mathcal{T}),\ell} \|h^{-\frac{1}{2}} [\![v_h]\!] \|_{L^2(\Sigma)}. \tag{1.58b}$$

We label the symmetric dG method with P-SIP and the non-symmetric with P-NSIP. With these insights, we can define the two methods as follows. Within the framework of chapter 1, we have in the symmetric case

$$\begin{gathered} \tilde{V} = H^1_0(\Omega) + S^0_\ell(\mathcal{T}) \quad \text{with} \quad V = H^1_0(\Omega), \quad V_h = S^0_\ell(\mathcal{T}), \\ \tilde{a} = \tilde{a}^{\mathsf{dG}}, \quad \|\cdot\| = \|\cdot\|_{\mathsf{dG}}, \\ a_h = a_h^{\mathsf{dG}} \text{ with } \theta = -1 \quad \text{and} \quad E = \mathcal{E}_\ell. \\ \text{Given } f \in H^{-1}(\Omega), \text{ seek } u_h \in S^0_\ell(\mathcal{T}) \text{ such that } a_h^{\mathsf{dG}}(u_h, v_h) = \langle f, \mathcal{E}_\ell v_h \rangle, \quad \forall v_h \in S^0_\ell(\mathcal{T}). \end{gathered} \tag{P-SIP}$$

In the non-symmetric case, we have:

$$\begin{gathered} \tilde{V} = H^1_0(\Omega) + S^0_\ell(\mathcal{T}) \quad \text{with} \quad V = H^1_0(\Omega), \quad V_h = S^0_\ell(\mathcal{T}), \\ \tilde{a} = \tilde{a}^{\mathsf{dG}}, \quad \|\cdot\| = \|\cdot\|_{\mathsf{dG}}, \\ a_h = a_h^{\mathsf{dG}} \text{ with } \theta = 1 \quad \text{and} \quad E = \mathcal{E}_\ell. \\ \text{Given } f \in H^{-1}(\Omega), \text{ seek } u_h \in S^0_\ell(\mathcal{T}) \text{ such that } a_h^{\mathsf{dG}}(u_h, v_h) = \langle f, \mathcal{E}_\ell v_h \rangle, \quad \forall v_h \in S^0_\ell(\mathcal{T}). \end{gathered} \tag{P-NSIP}$$

The quasi-optimality of the methods is shown [VZ18b, Theorem 3.6] and [VZ18b, Theorem 3.7]:

Proposition 1.45. *The P-NSIP-method is quasi-optimal with δ_V and C_{qopt} depending on $d, \sigma(\mathcal{T}), \ell, \gamma$. The P-SIP-method is quasi-optimal if $\gamma > \eta_*$ and δ_V and C_{qopt} depend in addition on η_**

It is discussed in [VZ18b, Lemma 2.3] that the methods P-SIP and P-NSIP are not overconsistent, which means $\delta_{V_h} > 0$. Consequently, (1.13) bounds $a(\Pi_V v_h, \mathrm{E_M}\, w_h) - a_h(v_h, w_h)$ in general not by 0 but with the global quantity $\|v_h - \Pi_V v_h\|_{\tilde{V}} \|w_h\|_{V_h}$, $v_h, w_h \in S^0_\ell(\mathcal{T})$. In some cases, we would like to have a more local control of these quantities. This is provided in the next Lemma, which is specific to the discontinuous Galerkin methods. We will later see the usefulness of the result in the Lemmas 2.68 and 5.7.

Lemma 1.46. *We have for all $v_h, w_h \in S^\ell_0(\mathcal{T})$:*

$$\tilde{a}^{\mathsf{dG}}(v_h, \mathcal{E}_\ell w_h - w_h) = -\int_\Sigma \{\{\nabla_h v_h \cdot \mathsf{n}\}\} \, [\![w_h]\!] - \int_\Sigma \frac{\gamma}{h} \, [\![v_h]\!] \, [\![w_h]\!] \,,$$
$$\tilde{a}^{\mathsf{dG}}(\mathcal{E}_\ell v_h - v_h, w_h) = -\int_\Sigma [\![v_h]\!] \, \{\{\nabla_h w_h \cdot \mathsf{n}\}\} - \int_\Sigma \frac{\gamma}{h} \, [\![v_h]\!] \, [\![w_h]\!] \,.$$

Proof. The result is an application of an integration by parts and Lemma 1.30:

$$\begin{aligned}
&\tilde{a}^{\mathsf{dG}}(v_h, \mathcal{E}_\ell w_h - w_h) + \int_\Sigma \frac{\gamma}{h} \, [\![v_h]\!] \, [\![w_h]\!] \\
&\quad = \sum_{T\in\mathcal{T}} \int_T \nabla v_h \cdot \nabla(\mathcal{E}_\ell w_h - w_h) \\
&\quad = \sum_{F\in\mathcal{F}} \int_F [\![\nabla v_h \cdot \mathsf{n}_F (\mathcal{E}_\ell w_h - w_h)]\!] - \sum_{T\in\mathcal{T}} \int_T \Delta v_h (\mathcal{E}_\ell w_h - w_h) \\
&\quad = \sum_{F\in\mathcal{F}^i} \int_F [\![\nabla v_h \cdot \mathsf{n}_F]\!] \, \{\{\mathcal{E}_\ell w_h - w_h\}\} - \sum_{F\in\mathcal{F}} \int_F \{\{\nabla v_h \cdot \mathsf{n}_F\}\} \, [\![w_h]\!] - \sum_{T\in\mathcal{T}} \int_T \Delta v_h (\mathcal{E}_\ell w_h - w_h).
\end{aligned}$$

The first and last summand vanishes due to the momentum condition of $\mathcal{E}_\ell$. Exploiting the symmetry of $\tilde{a}^{\mathsf{dG}}$ yields the second result as well. □

1.4.2.5 Discontinuous Galerkin methods with non quasi-optimal smoother

Looking at the examples discussed so far, one could get the impression that a quasi-optimal method requires a quasi-optimal smoother. To show that this is not the case, we recall an example given in [VZ18b, section 3.2, pp.2885].

Using the bubble operator $\mathcal{B}_\ell$ defined in (1.44) and the *first order* smoothing operator $\mathcal{A}_1$, we define an alternative smoothing operator:

$$\begin{aligned}
&\mathcal{E}_{\mathsf{red},\ell} : S^0_\ell(\mathcal{T}) \to H^1_0(\Omega) \\
&\mathcal{E}_{\mathsf{red},\ell}(v_h) = \mathcal{A}_1 v_h + \mathcal{B}_\ell(v_h - \mathcal{A}_1 v_h), \quad v_h \in S^0_\ell(\mathcal{T}).
\end{aligned}$$

We see that the bubble operator still ensures that $\mathcal{E}_{\mathsf{red},\ell}$ satisfies the momentum conditions in Definition 1.33, which makes the method algebraic consistent (Definition 1.11). The first-order operator $\mathcal{A}_1$, however, is only invariant on $\mathring{S}^1_1(\mathcal{T})$ and not on the conforming subspace of $S^0_\ell(\mathcal{T})$, which is $\mathring{S}^1_\ell$. Consequently, $\mathcal{E}_{\mathsf{red},\ell}$ cannot be a quasi-optimal smoother in the sense of Definition 1.17. Still, the resulting method is quasi-optimal [VZ18b, Theorem 3.10].

The estimates for the smoothing operator have to be adapted to this setting. We observe the bounds in Proposition 1.44 cannot be true in this setting because they imply that the smoother is invariant on $\mathring{S}^1_\ell(\mathcal{T})$. The result [VZ18b, Proposition 3.9] provides an adapted estimate.

Proposition 1.47 ([VZ18b, Proposition 3.9])**.** *The operator $\mathcal{E}_{\mathsf{red},\ell} : S^0_\ell(\mathcal{T}) \to H^1_0(\Omega)$ fulfills the momentum conditions of Definition 1.33 and has for all $v_h \in S^0_\ell(\mathcal{T})$ the property:*

$$\|\nabla_h(v_h - \mathcal{E}_{\mathsf{red},\ell} v_h)\|^2_{L^2(\Omega)} \leqslant C_{d,\sigma(\mathcal{T}),\ell} \left(\sum_{F\in\mathcal{F}} h_F^{-d} \left| \int_F [\![v_h]\!] \right|^2 + \|\nabla_h v_h\|^2_{L^2(\Omega)} \right)$$

We again select

$$\tilde{a}(\tilde{v}, \tilde{w}) = \tilde{a}^{\mathsf{dG}}(\tilde{v}, \tilde{w}) = \int_\Omega \nabla_h \tilde{v} \cdot \nabla_h \tilde{w} + \int_\Sigma \frac{\gamma}{h_F} \, [\![\tilde{v}]\!] \, [\![\tilde{w}]\!] \quad \forall \tilde{v}, \tilde{w} \in \tilde{V} = H^1_0(\Omega) + S^0_\ell(\mathcal{T}),$$

with a suitable stabilization parameter $\gamma > 0$, as before. Then, the induced norms $\|\cdot\|_{\mathsf{dG}}$ is the same, as well. We consider the two variants of the form a_h^{dG} defined in (1.57), the one with $\theta = 1$ (non-symmetric), labeled as P-RNSIP, and the one with $\theta = -1$ (symmetric), labeled as P-RSIP.

The resulting methods possess the same stiffness matrices as the P-NSIP resp. P-RSIP methods discussed before. The only difference is that the right hand side is handled by the alternative non quasi-optimal smoother $\mathcal{E}_{\mathsf{red},\ell}$. The result [VZ18b, Theorem 3.10] proves the quasi-optimality of the new methods with a quasi-optimality constant depending only on the dimension d, shape-regularity $\sigma(\mathcal{T})$, the polynomial degree ℓ and the stabilization constant $\gamma > 0$. In the symmetric case, we need (as before) the additional assumption that $\gamma > \eta_*$ for a specific $\eta_* > 0$ holds.

We summarize the two methods for a polynomial degree $\ell \in \mathbb{N}$ in the next two displays. The symmetric case reads:

$$\begin{gathered}\tilde{V} = H^1_0(\Omega) + S^0_\ell(\mathcal{T}) \quad \text{with} \quad V = H^1_0(\Omega), \quad V_h = S^0_\ell(\mathcal{T}),\\ \tilde{a} = \tilde{a}^{\mathsf{dG}}, \quad \|\cdot\| = \|\cdot\|_{\mathsf{dG}},\\ a_h = a_h^{\mathsf{dG}} \text{ with } \theta = -1 \quad \text{and} \quad E = \mathcal{E}_{\mathsf{red},\ell}.\end{gathered} \tag{P-RSIP}$$

Given $f \in H^{-1}(\Omega)$, seek $u_h \in S^0_\ell(\mathcal{T})$ such that $a_h^{\mathsf{dG}}(u_h, v_h) = \langle f, \mathcal{E}_{\mathsf{red},\ell} v_h\rangle, \quad \forall v_h \in S^0_\ell(\mathcal{T})$.

The non-symmetric case reads:

$$\begin{gathered}\tilde{V} = H^1_0(\Omega) + S^0_\ell(\mathcal{T}) \quad \text{with} \quad V = H^1_0(\Omega), \quad V_h = S^0_\ell(\mathcal{T}),\\ \tilde{a} = \tilde{a}^{\mathsf{dG}}, \quad \|\cdot\| = \|\cdot\|_{\mathsf{dG}},\\ a_h = a_h^{\mathsf{dG}} \text{ with } \theta = 1 \quad \text{and} \quad E = \mathcal{E}_{\mathsf{red},\ell}.\\ \text{Given } f \in H^{-1}(\Omega), \text{ seek } u_h \in S^0_\ell(\mathcal{T}) \text{ such that } a_h^{\mathsf{dG}}(u_h, v_h) = \langle f, \mathcal{E}_{\mathsf{red},\ell} v_h\rangle, \quad \forall v_h \in S^0_\ell(\mathcal{T}).\end{gathered} \tag{P-RNSIP}$$

This is the last discretization of Poisson problem we will consider. We have presented the quasi-optimal P-LAG, P-CR1, P-CRL, P-SIP, P-NSIP, P-RSIP and P-RNSIP methods for the Poisson problem. In the next section, we turn to a new equation.

1.4.3 A Crouzeix-Raviart method for linear elasticity

In this section, we present the discretization of a linear elasticity problem involving the symmetric gradient as presented in [VZ18b, section 3.3]. The equation models deformations of elastic materials and describes the bending of beams or plates. More details can be found in [Cia22a, Cia22b, Cia22c].

As before, let $\Omega \subset \mathbb{R}^d$ be an open, bounded and connected domain with Lipschitz continuous, polyhedral boundary. Let $v : \Omega \to \mathbb{R}^d$ be a vector field. The notation v_i, $1 \leqslant i \leqslant d$, refers to the i th component of the vector field.

On vector fields, we have several differential operators. The definition of the gradient ∇ is extended to vector fields $v = (v_1, \dots, v_d)$ by applying it to each component $\nabla v := \big((\partial_j v_i)_{i,j}\big)_{1\leqslant i,j\leqslant d}$. Another operator is the divergence operator, which is defined as $\operatorname{div} v := \sum_{i=1}^d \partial_i v_i$. When we apply div to a matrix-valued function $v := (v_{i,j})_{1\leqslant i,j\leqslant d}$, we apply it row-wise and obtain the vector field $(\operatorname{div} v)_i := \sum_{j=1}^d \partial_j v_{i,j}$. Finally, we introduce the symmetric gradient $\varepsilon(v) := \nabla v + (\nabla v)^T$ where $\cdot^T$ is the transposed of a matrix.

For a function space V, the space V^d contains vector-fields whose individual components are contained in V. In particular $H^1_0(\Omega)^d$ is the space of the d-vector-valued functions, where each component is in $H^1_0(\Omega)$. In this section, we equip $H^1_0(\Omega)^d$ with a non-standard norm, induced by the scalar product

$$a_{\mathsf{LE}}(v, w) = \int_\Omega 2\mu\varepsilon(v) : \varepsilon(w) + \lambda \operatorname{div} v \operatorname{div} w, \quad v, w \in H^1_0(\Omega)^d. \tag{1.59}$$

Correspondingly, we define the norm $\|\cdot\|_V^2 = \|\cdot\|_{\mathsf{LE}}^2 := a_{\mathsf{LE}}(\cdot,\cdot)$.

The elasticity problem describes the elastic deformations effects of forces on a solid body. The physical properties of its material are modeled by two material parameters $\mu, \lambda > 0$, which are called Lamé constants. With these, we consider the following elasticity problem ([VZ18b, (3.29)]):

$$\begin{gathered}\text{Given } f \in H^{-1}(\Omega)^d, \text{ find } u \in H^1_0(\Omega)^d \text{ such that}\\ -\operatorname{div}\big(2\mu\varepsilon(u) + \lambda \operatorname{div}(u)\,\mathrm{id}_d\big) = f \text{ in } \Omega \quad \text{and} \quad u = 0 \text{ on } \partial\Omega.\end{gathered} \tag{1.60}$$

Here, $\mathrm{id}_d \in \mathbb{R}^{d\times d}$ is the identity matrix. For a precise physical motivation behind this model, we refer to [Cia22a, Cia22b, Cia22c].

We discretize this problem with the first order Crouzeix-Raviart element, defined in (1.33). Thus, we set $V_h = \mathrm{CR}^d_1$ and $\tilde{V} = H^1_0(\Omega)^d + \mathrm{CR}^d_1$. We define the bilinear form $\tilde{a}^{\mathsf{LE}} : \tilde{V} \times \tilde{V} \to \mathbb{R}$ as

$$\tilde{a}^{\mathsf{LE}}(\tilde{v}, \tilde{w}) = \int_\Omega 2\mu\varepsilon_h(\tilde{v}) : \varepsilon_h(\tilde{w}) + \lambda \operatorname{div}_h \tilde{v} \operatorname{div}_h \tilde{w} + \int_\Sigma \frac{\mu\gamma}{h} [\![\tilde{v}]\!] \cdot [\![\tilde{w}]\!], \quad \tilde{v}, \tilde{w} \in \tilde{V}. \tag{1.61}$$

The operators ε_h and div_h are the elementwisely applied versions of ε and div. The Frobenius product : is defined for $d \times d$-matrices A, B as $A : B := \sum_{i,j=1}^{d} A_{i,j} B_{i,j}$. The parameter $\gamma > 0$ is a stabilization parameter. The stabilization term is necessary because the paper [Arn93] shows that otherwise the form is not definite on CR_1^d.

As in [VZ18b, section 3.3], the two other bilinear forms of the framework are obtained via $a_{\mathsf{LE}} = \tilde{a}^{\mathsf{LE}}_{|V\times V}$ and $a_h^{\mathsf{LE}} = \tilde{a}^{\mathsf{LE}}_{|V_h\times V_h}$. The next step is to construct a smoother such that the algebraic consistency in Definition 1.11 is fulfilled. We follow [VZ18b, pp. 2887].

To derive suitable conditions for algebraic consistency, we perform an element-wise integration by parts. Let $v \in V \cap V_h$, $w_h \in V_h$ and $E : V_h \to V$ an abstract smoother. We compute:

$$\begin{aligned}\tilde{a}^{\mathsf{LE}}(v, w_h - Ew_h) &= \sum_{T\in\mathcal{T}} \int_T 2\mu\varepsilon(v) : \varepsilon_h(w_h - Ew_h) + \lambda \operatorname{div} v \operatorname{div}_h(w_h - Ew_h) \\ &= \sum_{F\in\mathcal{F}^i} \int_F 2\mu \left[\!\left[(\varepsilon(v)\mathsf{n}_F)\cdot(w_h - Ew_h)\right]\!\right] + \lambda \left[\!\left[\operatorname{div}(v)(w_h - Ew_h)\cdot\mathsf{n}_F\right]\!\right] \\ &\quad - \sum_{T\in\mathcal{T}} \int_T 2\mu \operatorname{div}(\varepsilon(v))(w_h - Ew_h) + \lambda\nabla(\operatorname{div}(v))\cdot(w_h - Ew_h).\end{aligned} \tag{1.62}$$

The latter element contributions vanish because $V_h \subset S_1^0(\mathcal{T})^d$. It remains to consider the face-contributions. We apply Lemma 1.30 to obtain:

$$\begin{aligned}&\tilde{a}^{\mathsf{LE}}(v, w_h - Ew_h) \\ &= \sum_{F\in\mathcal{F}^i} \int_F 2\mu \left[\!\left[(\varepsilon(v)\mathsf{n}_F)\right]\!\right] \cdot \{\!\{(w_h - Ew_h)\}\!\} + 2\mu \{\!\{(\varepsilon(v)\mathsf{n}_F)\}\!\} \cdot \left[\!\left[(w_h - Ew_h)\right]\!\right] \\ &\quad + \lambda \left[\!\left[\operatorname{div}(v)\right]\!\right] \{\!\{(w_h - Ew_h)\cdot\mathsf{n}_F\}\!\} + \lambda \{\!\{\operatorname{div}(v)\}\!\} \left[\!\left[(w_h - Ew_h)\cdot\mathsf{n}_F\right]\!\right].\end{aligned} \tag{1.63}$$

The terms where the conformity error $w_h - Ew_h$ appears in the jump part vanish because of the definition of the Crouzeix-Raviart element. For the two remaining terms, we see that the algebraic consistency is fulfilled if the smoothing operator $E : V_h \to V$ satisfies for all $F \in \mathcal{F}^i$ and $w_h \in V_h = \mathrm{CR}_1^d$

$$\int_F Ew_h - \{\!\{w_h\}\!\} = 0, \tag{1.64}$$

understood componentwisely, compare with [VZ18b, Lemma 3.11].

To fulfill (1.64), we can reuse the smoothing operator $\mathcal{E}_1$ from (1.38) in a componentwise form. These ingredients provide a quasi-optimal method to approximate the elasticity problem, which we label with E-CR1. The framework of chapter 1 is applied with the following settings:

$$\begin{gathered}\tilde{V} = H_0^1(\Omega)^d + \mathrm{CR}_1(\Omega)^d \quad \text{with} \quad V = H_0^1(\Omega)^d, \quad V_h = \mathrm{CR}_1^d, \\ \tilde{a} = \tilde{a}^{\mathsf{LE}}, \quad \|\cdot\| = \|\cdot\|_{\mathsf{LE}}, \\ a_h = a_h^{\mathsf{LE}} \quad \text{and} \quad E = \mathcal{E}_1^d\end{gathered} \tag{E-CR1}$$

Given $f \in H^{-1}(\Omega)^d$, find $u_h \in \mathrm{CR}_1^d$ such that $a_h^{\mathsf{LE}}(u_h, w_h) = \langle f, \mathcal{E}_1^d w_h\rangle, \quad \forall w_h \in \mathrm{CR}_1^d$.

This method is quasi-optimal.

Proposition 1.48 ([VZ18b, Theorem 3.12])**.** *The method E-CR1 is quasi-optimal with quasi-optimality constant satisfying* $C_{\mathsf{qopt}} \leqslant \sqrt{1 + C_{d,\sigma(\mathcal{T})}(2\mu + \lambda)\gamma^{-1}}$.

Remark 1.49. This result is not satisfying because the bound on C_{qopt} deteriorates as λ goes to ∞. Therefore, the method might be affected by a phenomena called *locking*, see [BS08, section 11.3]. A discussion of this problem in the context of this method can be found in [VZ18b, Remarks 3.13, 3.14 and Lemma 3.15].

For later purposes, we prove the following Lemma.

Lemma 1.50. *The bilinear form* $\tilde{a}^{\mathsf{LE}}$ *for the elasticity problem in* (1.61) *fulfills for all* $v_h, w_h \in \mathrm{CR}_1^d$*:*

$$\tilde{a}^{\mathsf{LE}}(v_h, \mathcal{E}_1^d w_h - w_h) = \tilde{a}^{\mathsf{LE}}(\mathcal{E}_1^d v_h - v_h, w_h) = -\int_\Sigma \frac{\mu\gamma}{h} \left[\!\left[v_h\right]\!\right] \cdot \left[\!\left[w_h\right]\!\right]. \tag{1.65}$$

Proof. We use (1.62) and (1.63) to compute

$$\tilde{a}^{\mathsf{LE}}(v_h, w_h - \mathcal{E}_1^d w_h) = \sum_{F\in\mathcal{F}^i} \int_F 2\mu \left[\!\left[(\varepsilon(v_h)\cdot \mathsf{n}_F)\right]\!\right] \cdot \left\{\!\left\{(w_h - \mathcal{E}_1^d w_h)\right\}\!\right\} + \int_F \lambda \left[\!\left[\operatorname{div}(v_h)\right]\!\right] \left\{\!\left\{(w_h - \mathcal{E}_1^d w_h)\cdot \mathsf{n}_F\right\}\!\right\} + \int_\Sigma \frac{\mu\gamma}{h} \left[\!\left[v_h\right]\!\right] \cdot \left[\!\left[w_h\right]\!\right].$$

The momentum condition of $\mathcal{E}_1^d$ in (1.64) and the symmetry of $\tilde{a}^{\mathsf{LE}}$ provide the result. □

1.4.4 Discretizations for the biharmonic problem

Next, we consider a fourth order problem, the biharmonic problem. In this thesis, the biharmonic equation is only considered in two dimensions. The biharmonic equation is a model for fluid flows in terms of stream functions, c.f. [GR79, section 5.2] and plate bending problems, c.f. [Cia22c].

The biharmonic problem reads in the distributional sense as ([VZ18b, (3.40)]):

$$\begin{aligned} &\text{Given } f \in H^{-2}(\Omega), \text{ find } u \in H_0^2(\Omega) \text{ such that} \\ &\Delta\Delta u = f \text{ in } \Omega \quad \text{and} \quad u = \partial_{\mathsf{n}_\Omega} u = 0 \text{ on } \partial\Omega. \end{aligned} \tag{1.66}$$

Here, Δ is the standard Laplace operator and the zero boundary values are understood in the trace sense. The differential operator $\partial_{\mathsf{n}_\Omega}$ is the derivative in the direction of the outward pointing normal vector of Ω. The derivative is well-defined on the boundary owing to $u \in H_0^2(\Omega)$.

Definition 1.51. *For any function $v \in H^2(\Omega)$, we define the* Hessian $D^2 v \in \mathbb{R}^{d,d}$ *as*

$$(D^2 v)_{i,j} = \partial_i \partial_j v$$

where ∂_i and ∂_j are the directional derivatives in the directions of the ith and jth standard basis vector, respectively.

To apply the framework of chapter 1 to the biharmonic equation, we choose $V = H_0^2(\Omega)$ and

$$a(v,w) := a_{\mathsf{B}}(v,w) := \int_\Omega D^2 v : D^2 w, \quad v, w \in H_0^2(\Omega). \tag{1.67}$$

The weak form of the biharmonic problem reads:

$$\begin{aligned} &\text{Given } f \in H^{-2}(\Omega), \text{ find } u \in H_0^2(\Omega) \text{ such that} \\ &a(u,w) = \int_\Omega D^2 u : D^2 w = \langle f, w\rangle \quad \forall w \in H_0^2(\Omega). \end{aligned}$$

Remark 1.52. The weak formulation can be found in two variants in the literature. Some authors use the bilinear form $a(v,w) = \int \Delta v \Delta w$ as [BBF13, Example 1.2.4], while others (e.g. [VZ19a, VZ18b, BS05]) use $a(v,w) = \int D^2 v : D^2 w$, $v, w \in H_0^2(\Omega)$. Although the two forms are equivalent in the continuous setting the resulting methods usually are different. This can be seen by an integration by parts, see [Cia13].

1.4.4.1 A Morley method

The non-affine Morley element was introduced in [Mor68], see also [LL75]. We define the Morley space as (c.f. [VZ18b, (3.28)]):

$$V_h = \mathrm{MR} := \left\{ v_h \in S_2^0(\mathcal{T}) \;\middle|\; v_h \text{ is cont. in } \mathcal{V}\cap\Omega,\ v_h = 0 \text{ in } \mathcal{V}\cap\partial\Omega \text{ and } \forall F \in \mathcal{F} \int_F \left[\!\left[\partial_{\mathsf{n}} v_h\right]\!\right] = 0 \right\}. \tag{1.68}$$

Remark 1.53. The conforming part $H_0^2(\Omega)\cap \mathrm{MR}$ can be $\{0\}$, see [VZ18b, Remark 3.12] and [dBD83]. Moreover, the Morley element is not affine-equivalent, see [Cia78, section 2.3]. This complicates the derivation of estimates. For this reason, we derive estimates by expressing the MR-functions in terms of an affine element, see below.

We apply the framework of chapter 1 by setting $\tilde{V} = H_0^2(\Omega) + \mathrm{MR}$. For $\tilde{a}$ we choose the bilinear form $\tilde{a}^{\mathsf{MR}} : \tilde{V}\times\tilde{V} \to \mathbb{R}$ defined as

$$\tilde{a}^{\mathsf{MR}}(\tilde{v},\tilde{w}) = \int_\Omega D_h^2 \tilde{v} : D_h^2 \tilde{w}, \qquad \tilde{v}, \tilde{w} \in \tilde{V}.$$

The form $a = a_{\mathsf{B}} : H_0^2(\Omega) \times H_0^2(\Omega) \to \mathbb{R}$ and $a_h = a_h^{\mathsf{MR}} : \mathrm{MR} \times \mathrm{MR} \to \mathbb{R}$ are the restrictions of $\tilde{a}^{\mathsf{MR}}$ to $H_0^2(\Omega)$ and MR respectively. In other words, $a_{\mathsf{B}}(v,w) = \tilde{a}^{\mathsf{MR}}(v,w)$, $v,w \in H_0^2(\Omega)$, and $a_h^{\mathsf{MR}}(v_h,w_h) = \tilde{a}^{\mathsf{MR}}(v_h,w_h)$ $v_h, w_h \in \mathrm{MR}$.

To obtain a quasi-optimal method for the MR-space, we need a smoothing operator mapping MR into $H_0^2(\Omega)$, Moreover, the smoothing operator has to make the method algebraic consistent, see Definition 1.11.

Similar to the previous examples, we perform integration by parts to find sufficient conditions. Beforehand, the next Lemma provides a general integration by parts result, which we often need in what follows.

Lemma 1.54. *For $v_h \in S_\ell^0(\mathcal{T})$ and $\tilde{w} \in H_0^2(\Omega) + S_\ell^0(\mathcal{T})$ we have*

$$\int_\Omega D_h^2 v_h : D_h^2 \tilde{w} = -\sum_{T\in\mathcal{T}} \int_T \nabla\Delta v_h \cdot \nabla\tilde{w} + \sum_{F\in\mathcal{F}^i} \int_F \left[\!\left[D_h^2 v_h \mathsf{n}_F \right]\!\right] \cdot \{\!\{\nabla_h \tilde{w}\}\!\} + \sum_{F\in\mathcal{F}} \int_F \{\!\{ D_h^2 v_h \mathsf{n}_F \}\!\} \cdot \left[\!\left[\nabla_h \tilde{w} \right]\!\right]. \tag{1.69}$$

Proof. As introduced in Definition 1.28, we write n for a function defined on the skeleton Σ of $\mathcal{T}$ which assigns each point in $\Sigma\backslash\mathcal{V}$ to its normal vector. We extend this notation and write n_i for the ith component of the normal vector function. With this, we can write Gauß Theorem on $T \in \mathcal{T}$ as

$$\int_T \partial_i v_h = \int_{\partial T} v_h \mathsf{n}_i, \quad 1 \leqslant i \leqslant d.$$

This helps us to compute for $T \in \mathcal{T}$, $v_h \in S_\ell^0(\mathcal{T})$ and $\tilde{w} \in H_0^2(\Omega) + S_\ell^0(\mathcal{T})$ to compute

$$\int_T D_h^2 v_h : D^2 \tilde{w} = \int_T \sum_{i,j=1}^2 \partial_i\partial_j v_h \partial_i\partial_j \tilde{w} = -\int_T \sum_{i,j=1}^2 \partial_i\partial_i\partial_j v_h \partial_j \tilde{w} + \int_{\partial T} \sum_{i,j=1}^2 \partial_i\partial_j v_h \partial_j \tilde{w} \mathsf{n}_i.$$

We rephrase this with the Δ and ∇ operators and Schwarz's Theorem to

$$\int_T D^2 v_h : D^2 \tilde{w} = -\int_T \nabla\Delta v_h \cdot \nabla\tilde{w} + \int_{\partial T} D^2 v_h \mathsf{n} \cdot \nabla\tilde{w}.$$

Summing over all $T \in \mathcal{T}$, we have

$$\sum_{T\in\mathcal{T}} \int_T D^2 v_h : D^2 \tilde{w} = -\sum_{T\in\mathcal{T}} \int_T \nabla\Delta v_h \cdot \nabla\tilde{w} + \sum_{F\in\mathcal{F}} \int_F \left[\!\left[D_h^2 v_h \mathsf{n}_F \cdot \nabla_h \tilde{w} \right]\!\right].$$

The latter integral expression is rewritten with Lemma 1.30. For a fixed $F \in \mathcal{F}^i$, we have

$$\int_F \left[\!\left[D_h^2 v_h \mathsf{n}_F \cdot \nabla_h \tilde{w} \right]\!\right] = \int_F \left[\!\left[D_h^2 v_h \mathsf{n}_F \right]\!\right] \cdot \{\!\{\nabla_h \tilde{w}\}\!\} + \int_F \{\!\{ D_h^2 v_h \mathsf{n}_F \}\!\} \cdot \left[\!\left[\nabla_h \tilde{w} \right]\!\right]$$

and if $F \in \mathcal{F}\backslash\mathcal{F}^i$, we have

$$\int_F \left[\!\left[D_h^2 v_h \mathsf{n}_F \cdot \nabla_h \tilde{w} \right]\!\right] = \int_F \{\!\{ D_h^2 v_h \mathsf{n}_F \}\!\} \cdot \left[\!\left[\nabla_h \tilde{w} \right]\!\right]. \qquad \square$$

To derive conditions for the algebraic consistency, we integrate $\tilde{a}^{\mathsf{MR}}(v, Ew_h - w_h)$, $v \in H_0^2(\Omega) \cap \mathrm{MR}$, $w_h \in \mathrm{MR}$, by parts for some abstract smoother $E : \mathrm{MR} \to H_0^2(\Omega)$. For this purpose, we utilize Lemma 1.54 and get

$$\begin{aligned}\tilde{a}^{\mathsf{MR}}(v, Ew_h - w_h) = \int_\Omega D^2 v : D_h^2(Ew_h - w_h) &= -\sum_{T\in\mathcal{T}} \int_T \nabla\Delta v \cdot \nabla(Ew_h - w_h) \\ &+ \sum_{F\in\mathcal{F}^i} \int_F \left[\!\left[D_h^2 v \mathsf{n}_F \right]\!\right] \cdot \{\!\{\nabla_h(Ew_h - w_h)\}\!\} + \sum_{F\in\mathcal{F}} \int_F \{\!\{ D_h^2 v \mathsf{n}_F \}\!\} \cdot \left[\!\left[\nabla_h(Ew_h - w_h) \right]\!\right].\end{aligned}$$

The structure of the MR-method allows some simplifications. The element contributions vanish due to $v \in S_2^0(\mathcal{T})$ and the contributions on each $F \in \mathcal{F}$ can be expressed through two constant vectors, say $C_1^F, C_2^F \in \mathbb{R}^2$. Consequently, we have:

$$\tilde{a}^{\mathsf{MR}}(v, Ew_h - w_h) = \sum_{F\in\mathcal{F}^i} \int_F C_1^F \cdot \{\!\{\nabla_h(Ew_h - w_h)\}\!\} + \sum_{F\in\mathcal{F}} \int_F C_2^F \cdot \left[\!\left[\nabla_h(Ew_h - w_h) \right]\!\right]. \tag{1.70}$$

We analyze this expression further using the next Lemma. For its formulation, we introduce the tangent vector

Definition 1.55 (Tangential vector of an edge)**.** *On each $F \in \mathcal{F}$, we select a normalized* tangential vector $\mathtt{t}_F$ *with arbitrary but fixed orientation.*

Lemma 1.56. *Let $F \in \mathcal{F}$ with endpoints $z_0, z_1 \in \mathcal{V}$ and $\{z_0, z_1\} \subset F$ such that $z_0 + |z_1 - z_0|\mathtt{t}_F = z_1$. Then for all $\tilde{w} \in H^2_0(\Omega) + S^0_2(\mathcal{T})$ and $C^F_1, C^F_2 \in \mathbb{R}^2$ we have*

$$\begin{aligned}\int_F C^F_1 \cdot \{\{\nabla_h \tilde{w}\}\} &= \int_F \{\{\partial_{\mathtt{n}_F} \tilde{w}\}\}\, \mathtt{n}_F \cdot C^F_1 + \int_F \{\{\partial_{\mathtt{t}_F} \tilde{w}\}\}\, \mathtt{t}_F \cdot C^F_1 \\ &= \int_F \{\{\partial_{\mathtt{n}_F} \tilde{w}\}\}\, \mathtt{n}_F \cdot C^F_1 + \big(\{\{\tilde{w}\}\}_{|F}(z_1) - \{\{\tilde{w}\}\}_{|F}(z_0)\big)\mathtt{t}_F \cdot C^F_1\end{aligned} \tag{1.71a}$$

and

$$\begin{aligned}\int_F C^F_2 \cdot [\![\nabla_h \tilde{w}]\!] &= \int_F [\![\partial_{\mathtt{n}_F} \tilde{w}]\!]\, \mathtt{n}_F \cdot C^F_2 + \int_F [\![\partial_{\mathtt{t}_F} \tilde{w}]\!]\, \mathtt{t}_F \cdot C^F_2 \\ &= \int_F [\![\partial_{\mathtt{n}_F} \tilde{w}]\!]\, \mathtt{n}_F \cdot C^F_2 + \big([\![\tilde{w}]\!]_{|F}(z_1) - [\![\tilde{w}]\!]_{|F}(z_0)\big)\mathtt{t}_F \cdot C^F_2 .\end{aligned} \tag{1.71b}$$

Proof. On F, we have $\nabla_h \tilde{w} = \partial_{\mathtt{n}_F} \tilde{w} \mathtt{n}_F + \partial_{\mathtt{t}_F} \tilde{w} \mathtt{t}_F$. Applying the fundamental theorem of calculus by integrating in the direction of the vector $\mathtt{t}_F$ yields

$$\int_F \{\{\partial_{\mathtt{t}_F} \tilde{w}\}\}\, \mathtt{t}_F = \big(\{\{\tilde{w}\}\}_{|F}(z_1) - \{\{\tilde{w}\}\}_{|F}(z_0)\big)\mathtt{t}_F.$$

The second relation (1.71b) follows suit by replacing $\{\{\cdot\}\}$ with $[\![\cdot]\!]$. □

We can now use the continuity assumptions of MR to simplify (1.70) further. Since every $w_h \in \mathrm{MR}$ is continuous at all points $z \in \mathcal{V} \cap \Omega$ and zero at all $z \in \mathcal{V} \cap \partial\Omega$, we have after applying Lemma 1.56 to (1.70) on one particular $F \in \mathcal{F}$ that

$$\big([\![(Ew_h - w_h)]\!]_{|F}(z_1) - [\![(Ew_h - w_h)]\!]_{|F}(z_0)\big)\mathtt{t}_F = 0.$$

Here, $z_1, z_0 \in \mathcal{V}$ are chosen as in Lemma 1.56.

Exploiting that $\int_F [\![\partial_{\mathtt{n}_F}(w_h - Ew_h)]\!] = 0$ for each $F \in \mathcal{F}$ and $w_h \in \mathrm{MR}$ due to the definition of MR and $E(w_h) \in H^2_0(\Omega)$, we are left with

$$\begin{aligned}\tilde{a}^{\mathsf{MR}}(v, Ew_h - w_h) = \sum_{F \in \mathcal{F}^i} \int_F C^F_1 \cdot \{\{\nabla_h (Ew_h - w_h)\}\} &= \sum_{F \in \mathcal{F}^i} \int_F \{\{\partial_{\mathtt{n}_F}(Ew_h - w_h)\}\}\, C^F_1 \cdot \mathtt{n}_F \\ &+ \Big(\{\{Ew_h - w_h\}\}_{|F}(z_1) - \{\{Ew_h - w_h\}\}_{|F}(z_0)\Big) C^F_1 \cdot \mathtt{t}_F.\end{aligned}$$

This calculation shows that the algebraic consistency is fulfilled if for all $w_h \in \mathrm{MR}$ the abstract smoother satisfies

$$Ew_h(z) = w_h(z) \quad \forall z \in \mathcal{V} \cap \Omega \quad \text{and} \quad \int_F \partial_{\mathtt{n}_F} Ew_h = \int_F \{\{\partial_{\mathtt{n}_F} w_h\}\} \quad \forall F \in \mathcal{F}^i \tag{1.72}$$

compare with ([VZ18b, (3.47)]).

Based upon this, we recall the construction of a smoothing operator $\mathcal{E}_{\mathsf{hct}} : S^0_2(\mathcal{T}) \to H^2_0(\Omega)$ with these properties. We follow the constructions offered by [VZ19a, section 3.4], [VZ18b, section 3.4]. A slightly different operator is build in [Gal15, Proposition 2.6].

The operator $\mathcal{E}_{\mathsf{hct}} : S^0_2(\mathcal{T}) \to H^2_0(\Omega)$ uses a H^2-conforming finite element space called HCT. The finite element space $\mathrm{HCT} \subset H^2(\Omega)$ was first introduced in [CT65]. The corresponding HCT-element uses a subtriangulation, which is obtained by refining each $T \in \mathcal{T}$. The subtriangulation in each $T \in \mathcal{T}$ contains the three triangles created by connecting each pair of the three vertices of T with the barycenter m_T of T. The elements of the subtriangulation of an element $T \in \mathcal{T}$ are contained in the set $\mathcal{T}_T$.

On the subtriangulation $\mathcal{T}_T$ of a $T \in \mathcal{T}$, we introduce the analogous piecewise function spaces as on the original triangulation. Thus, $S^2_3(\mathcal{T}_T)$ is the space of piecewise polynomials of degree 3 with respect to the subtriangulation $\mathcal{T}_T$, which are in addition in $H^2(T)$. With these spaces, we define the HCT-space as ([VZ18b, section 3.4])

$$\mathrm{HCT} := \left\{ w_h \in C^1(\overline{\Omega}) \mid w_{h|T} \in S^2_3(\mathcal{T}_T) \quad \forall T \in \mathcal{T} \right\}. \tag{HCT}$$

On a single element $T \in \mathcal{T}$, a set of degrees of freedom is given by (c.f. [Cia78, p. 340f.])

$$\begin{aligned}\mathcal{N}_{\mathrm{HCT}} = \{N^0_y(v) = v(y) \mid y \in \mathcal{V}\} \oplus \{N^1_y(v) = \partial_1 v(y) \mid y \in \mathcal{V}\} \oplus \\ \{N^2_y(v) = \partial_2 v(y) \mid y \in \mathcal{V}\} \oplus \{N_F(v) = \partial_{\mathtt{n}_F} v(m_F) \mid F \in \mathcal{F}\}.\end{aligned}$$

These degrees of freedom induce the corresponding basis functions, where we use the notation introduced in [VZ19a, p. 285].

(i) For N_z^0, the corresponding basis functions is $\Upsilon_z^0 \in \mathrm{HCT}$, $z \in \mathcal{V}$.

(ii) For N_z^1 and N_z^2, the corresponding basis functions are $\Upsilon_z^1 \in \mathrm{HCT}$ resp. $\Upsilon_z^2 \in \mathrm{HCT}$, $z \in \mathcal{V}$.

(iii) For N_F, $F \in \mathcal{F}$ the corresponding basis function is written as $\Upsilon_F \in \mathrm{HCT}$.

The smoothing operator $\mathcal{E}_{\mathsf{hct}}$ uses the following variant of the HCT-space that has zero boundary values.

$$\mathring{\mathrm{HCT}} := \left\{ w_h \in C^1(\overline{\Omega}) \cap H_0^2(\Omega) \mid w_{h|T} \in S_3^2(\mathcal{T}_T) \quad \forall T \in \mathcal{T} \right\} \subset H_0^2(\Omega). \tag{1.73}$$

This space has the following set of degrees of freedom on a given element $T \in \mathcal{T}$

$$\begin{aligned}\mathcal{N}_{\mathring{\mathrm{HCT}}} = \{N_y^0(v) = v(y) \mid y \in \mathcal{V} \cap \Omega\} \oplus \{N_y^1(v) = \partial_1 v(y) \mid y \in \mathcal{V} \cap \Omega\} \oplus \\ \{N_y^2(v) = \partial_2 v(y) \mid y \in \mathcal{V} \cap \Omega\} \oplus \{N_F(v) = \partial_{\mathsf{n}_F} v(m_F) \mid F \in \mathcal{F}^i\}.\end{aligned}$$

As $\mathcal{N}_{\mathring{\mathrm{HCT}}} \subset \mathcal{N}_{\mathrm{HCT}}$, we can reuse the basis functions of HCT that correspond to the degrees of freedom in $\mathcal{N}_{\mathring{\mathrm{HCT}}}$.

As the MR-element, the finite element space $\mathring{\mathrm{HCT}}$ is not affine equivalent. Suitable estimates can be found in [VZ19a, Lemma 3.14] and [KRVZ25, Lemma 5.2].

Lemma 1.57 ([VZ19a, Lemma 3.14])**.** *For the basis functions Υ_z^i with $z \in \mathcal{V} \cap \Omega$, $i \in \{1,2\}$ we have for each $T \in \mathcal{T}$ and $F \in \mathcal{F}^i$*

$$\begin{aligned}\frac{1}{\rho_T^2}\|\Upsilon_z^i\|_{L^2(T)} \leqslant C\frac{1}{\rho_T}\|\nabla\Upsilon_z^i\|_{L^2(T)} \leqslant C\|D^2\Upsilon_z^i\|_{L^2(T)} \leqslant C\frac{|T|^{\frac{1}{2}}}{\rho_T}, \\ \left|\int_F \nabla\Upsilon_z^i \cdot \mathsf{n}_F\right| \leqslant C|F|\end{aligned} \tag{1.74}$$

For the basis functions Υ_F with $F \in \mathcal{F}^i$, we have for all $T \in \mathcal{T}$ and $F' \in \mathcal{F}^i$

$$\begin{aligned}\frac{1}{\rho_T^2}\|\Upsilon_F\|_{L^2(T)} \leqslant C\frac{1}{\rho_T}\|\nabla\Upsilon_F\|_{L^2(T)} \leqslant C\|D^2\Upsilon_F\|_{L^2(T)} \leqslant C\frac{|T|^{\frac{1}{2}}}{\rho_T}, \\ \left|\int_{F'} \nabla\Upsilon_F \cdot \mathsf{n}_{F'}\right| \leqslant C\delta_{F,F'}\big|F'\big|.\end{aligned} \tag{1.75}$$

The constants $C > 0$ depend on the shape parameter $\sigma(\mathcal{T})$.

A proof is found in [VZ19a].

With this preparatory work, we provide the construction of the smoothing operator $\mathcal{E}_{\mathsf{hct}}$ from [VZ19a, section 3.4]. The second condition in (1.72) requires to preserve the mean values of the normal-derivatives on each $F \in \mathcal{F}^i$. To achieve this, we employ a bubble operator of the form

$$\mathcal{B}_{\partial_{\mathsf{n}}} : S_2^0(\mathcal{T}) \to H_0^2(\Omega) \qquad \mathcal{B}_{\partial_{\mathsf{n}}}(v_h) = \sum_{F \in \mathcal{F}^i} \Big(\int_F \{\{\partial_{\mathsf{n}_F} v_h\}\} \Big) \overline{\Phi}_{\mathsf{n}_F}. \tag{1.76}$$

The function $\overline{\Phi}_{\mathsf{n}_F}$ is a product of an H_0^2-face-bubble $\Psi_{2,F}$ and a ramp-like function $\eta_F : \mathbb{R}^2 \to \mathbb{R}$. First, we discuss the bubble $\Psi_{2,F}$ then we construct η_F.

The face bubble $\Psi_{2,F}$ can be defined in several ways. One approach is found in [Gal15], another is presented in [VZ19a, p. 287]. We present the latter one. Anyway, both bubbles entail additional notation.

Recall that for each $T \in \mathcal{T}$ the barycentric coordinate $\lambda_z^T : T \to \mathbb{R}$, $z \in \mathcal{V} \cap T$, is the uniquely defined linear function which fulfills for each $z' \in \mathcal{V} \cap T$: $\lambda_z^T(z') = \delta_{z,z'}$, page 19.

The barycentric coordinates allow us to define the bubble $\Psi_{2,F} \in H_0^2(\omega_F)$, $F \in \mathcal{F}^i$. Given some $F \in \mathcal{F}^i$ with $T_1, T_2 \in \mathcal{T}$, $T_1 \cap T_2 = F$, we define as in [VZ19a, p. 287]

$$\Psi_{2,F} := \begin{cases} \frac{630}{|F|} \prod_{z \in \mathcal{V} \cap F} (\lambda_z^{T_1} \lambda_z^{T_2})^2 & \text{in } T_1 \cup T_2, \\ 0 & \text{else.} \end{cases} \tag{1.77}$$

This bubble is normalized to $\int_F \Psi_{2,F} = 1$.

The normal derivative of the bubble $\Psi_{2,F}$ is not constant 1 on F. This is fixed by the second function η_F, which is a kind of ramp.

Lemma 1.58. *For each $F \in \mathcal{F}^i$, there exists a linear function $\eta_F : \mathbb{R}^2 \to \mathbb{R}$ satisfying:*

1. $\partial_{\mathsf{n}_F} \eta_F = 1$ *on* F,
2. $\partial_{\mathsf{t}_F} \eta_F = 0$ *on* F,
3. $\eta_F = 0$ *on* F.

Such a function is constructed in [VZ19a, (3.33)]: For a face $F \in \mathcal{F}^i$ with barycenter $m_F := \frac{1}{d}\sum_{z \in \mathcal{V} \cap F} z$, we set $\eta_F(x) := (x - m_F) \cdot \mathsf{n}_F$ for all $x \in \mathbb{R}^2$.

With these two ingredients, the $H_0^2(\omega_F)$-bubble and the ramp η_F, we define the bubble $\overline{\Phi}_{\mathsf{n}_F} := \Psi_{2,F}\eta_F$, which fulfills the desired properties. Besides [VZ19a, (3.33)], this construction can also be found in [Gal15, Proposition 2.6].

Corollary 1.59. *For $F \in \mathcal{F}^i$, the function $\overline{\Phi}_{\mathsf{n}_F} = \Psi_{2,F}\eta_F \in H^2(\Omega)$ fulfills similar identities as η_F*

1. $\int_F \partial_{\mathsf{n}_F} \overline{\Phi}_{\mathsf{n}_F} = 1$ *on* F,
2. $\partial_{\mathsf{t}_F} \overline{\Phi}_{\mathsf{n}_F} = 0$ *on* F,
3. $\overline{\Phi}_{\mathsf{n}_F} = 0$ *on* F.

For the scaling of the bubble, we recall [VZ19a, Lemma 3.16].

Lemma 1.60 ([VZ19a, Lemma 3.16])**.** *The function $\overline{\Phi}_{\mathsf{n}_F}$, $F \in \mathcal{F}^i$, satisfies for each $T \in \mathcal{T}_F$*

$$\|D^2 \overline{\Phi}_{\mathsf{n}_F}\|_{L^2(T)} \leqslant C\rho_T^{-1} \|\nabla \overline{\Phi}_{\mathsf{n}_F}\|_{L^2(T)} \leqslant C\rho_T^{-2} \|\overline{\Phi}_{\mathsf{n}_F}\|_{L^2(T)} \leqslant C \frac{|T|^{\frac{1}{2}}}{\rho_T |F|}.$$

Proof. We follow the lines of [VZ19a, Lemma 3.16]. At first, we note that η_F can be rewritten on $T \in \mathcal{T}_F$ as $\eta_F = \operatorname{dist}(F, z)\mathsf{n}_F \cdot \mathsf{n}_T \lambda_z^T$, where $z \in \mathcal{V} \cap T$ is the vertex opposite of F, and $\operatorname{dist}(F, z)$ is the distance between F and z; the idea of the rewrite stems from [Gal15, Proposition 2.6]. Then, the first inequalities follow from inverse estimates with appropriate scaling. Using that $\eta_F \leqslant h_T$ on T, we establish the estimate with the help of affine equivalence

$$\|\overline{\Phi}_{\mathsf{n}_F}\|_{L^2(T)} \leqslant C h_T \|\Psi_{2,F}\|_{L^2(T)} \leqslant C h_T \frac{|T|^{\frac{1}{2}}}{|F|}. \qquad \square$$

The bubble operator preserves the normal derivative on each $F \in \mathcal{F}^i$. The preservation of the point values at each $z \in \mathcal{V}$ is still lacking to establish algebraic consistency. Moreover, the bubble operator needs stabilization. These issues are resolved with an averaging operator built with the HCT element.

This averaging operator, stemming from [Zan17, (2.2.10)], reads as

$$\mathcal{A}_{\mathsf{hct}} v_h := \sum_{z \in \mathcal{V} \cap \Omega} \left(v_{h|T_z}(z)\Upsilon_z^0 + \sum_{j=1}^{2} \frac{\partial}{\partial x_j}(v_{h|T_z})(z)\Upsilon_z^j \right) + \sum_{F \in \mathcal{F}^i} \nabla_h v_{h|T_F}(m_F) \cdot \mathsf{n}_F \Upsilon_F,$$

where $T_z \in \mathcal{T}_z$, $z \in \mathcal{V}$ and $T_F \in \mathcal{T}_F$, $F \in \mathcal{F}$, are arbitrary but fixed elements. For later purposes, this operator does not require a continuous normal derivative at m_F, although this is unnecessary for the MR-element.

With these ingredients, we are able to define the final smoothing operator $\mathcal{E}_{\mathsf{hct}} : S_2^0(\mathcal{T}) \to H_0^2(\Omega)$ as (see also [VZ18b, (3.48)])

$$\mathcal{E}_{\mathsf{hct}} v_h := \mathcal{A}_{\mathsf{hct}} v_h + \mathcal{B}_{\partial_{\mathsf{n}}}(v_h - \mathcal{A}_{\mathsf{hct}} v_h). \tag{1.78}$$

It remains to derive suitable estimates for $\mathcal{E}_{\mathsf{hct}}$.

First, we establish counterparts for the estimates of Lemma 1.36 and 1.37, c.f. [Zan17, Lemma 2.2.5]:

Lemma 1.61 ([Zan17, Lemma 2.2.5])**.** *For each $T \in \mathcal{T}$, $z \in \mathcal{V} \cap T$ and $v_h \in S_2^0(\mathcal{T})$ we have*

$$\left|\nabla v_{h|T}(z) - \nabla(\mathcal{A}_{hct} v_h)(z)\right| \leqslant \sum_{F \in \mathcal{F}_z^i} \left| [\![\nabla v_h]\!]_{|F}(z) \right|.$$

For each $F \in \mathcal{F}$ with barycenter m_F, we have

$$\left|\partial_{\mathsf{n}_F} v_{h|T}(m_F) - \partial_{\mathsf{n}_F} \mathcal{A}_{hct} v_h(m_F)\right| \leqslant |[\![\partial_{\mathsf{n}_F}(v_h)]\!](m_F)|.$$

Proof. The result follows by applying Lemma 1.36 for the Laplace case componentwisely to $\nabla_h v_h \in S_\ell^0(\mathcal{T})^2$. Further details are found in [Zan17, Lemma 2.2.5]. □

The estimates of Lemma 1.61 can be continued by the analogous ones of Lemma 1.37 by replacing in that Lemma v_h by ∇v_h and ∇v_h by $D^2 v_h$.

Lemma 1.62 ([Zan17, Lemma 2.2.5])**.** *For all $v_h \in S_2^0(\mathcal{T})$, $v \in H_0^2(\omega_F)$, $F \in \mathcal{F}$ and $z \in \mathcal{L}_2 \cap F$, we have as in [Zan17, (2.2.11a)]*

$$\left|[\![\nabla v_h]\!]_{|F}(z)\right| \leqslant \|\,[\![\nabla v_h]\!]_{|F}\,\|_{L^\infty(F)} \leqslant \frac{C}{|F|^{\frac{1}{2}}}\|\,[\![\nabla(v_h - v)]\!]_{|F}\,\|_{L^2(F)}. \tag{1.80a}$$

Let $F \in \mathcal{F}^i$ and $T_1, T_2 \in \mathcal{T}$ such that $T_1 \cap T_2 = F$, then we have as in [Zan17, (2.2.11b)]

$$\left|[\![\nabla v_h]\!]_{|F}(z)\right| \leqslant \frac{C}{|F|}\left|\int_F [\![\nabla(v_h - v)]\!]_{|F}\right| + \sum_{i=1,2}\frac{Ch_{T_i}}{|T_i|^{\frac{1}{2}}}\|D^2(v_h - v)\|_{L^2(T_i)}. \tag{1.80b}$$

If, however, $F \subset \partial\Omega$ with $F \subset T \in \mathcal{T}$, we have as a special case of [Zan17, (2.2.11b)]

$$\left|[\![\nabla v_h]\!]_{|F}(z)\right| \leqslant \frac{C}{|F|}\left|\int_F [\![\nabla(v_h - v)]\!]_{|F}\right| + \frac{Ch_T}{|T|^{\frac{1}{2}}}\|D^2(v_h - v)\|_{L^2(T)}. \tag{1.80c}$$

Finally, we have as in [Zan17, (2.2.11c)]

$$|[\![\partial_{\mathsf{n}_F}(v_h)]\!](m_F)| \leqslant \frac{1}{|F|}\left|\int_F [\![\partial_{\mathsf{n}_F} v_h]\!]\right|. \tag{1.80d}$$

The Lemma is proven in [Zan17, Lemma 2.2.5] and is essentially a rewriting of Lemma 1.37. The last bound (1.80d) follows by observing that $\partial_{\mathsf{n}_F} v_h$ is a linear function and an application of the corresponding quadrature formula.

The next Proposition combines Lemmas 1.61 and 1.62 to obtain bounds for the operator $\mathcal{E}_{\mathsf{hct}}$. To simplify the notation, we define $D^1 := \nabla$ and D^0 is just a placeholder.

Proposition 1.63 ([Zan17, Proposition 3.3.15 and 4.2.24])**.** *The operator $\mathcal{E}_{\mathsf{hct}} : \mathring{S}_2^1(\mathcal{T}) + \mathrm{MR} \to H_0^2(\Omega)$ fulfills for each $T \in \mathcal{T}$, $k \in \{0,1,2\}$, $v_h \in \mathring{S}_2^1(\mathcal{T}) + \mathrm{MR}$ and $v \in H_0^2(\Omega)$ the estimates*

$$\begin{aligned}&\|D^k(v_h - \mathcal{E}_{\mathsf{hct}} v_h)\|_{L^2(T)}\\ &\quad\leqslant C\Big(\sum_{F\in\mathcal{F}_T^{i,+}}\frac{|T|^{\frac{1}{2}}}{\rho_T^{k-1}|F|}\left|\int_F [\![\nabla_h(v_h - v)]\!]\right| + \sum_{T'\in\mathcal{T}_T^+}\frac{|T|^{\frac{1}{2}}h_{T'}}{\rho_T^{k-1}|T'|^{\frac{1}{2}}}\|D_h^2(v_h - v)\|_{L^2(T')}\Big)\end{aligned} \tag{1.81a}$$

and

$$\|D^k(v_h - \mathcal{E}_{\mathsf{hct}} v_h)\|_{L^2(T)} \leqslant C\Big(\sum_{F\in\mathcal{F}_T^{i,+}}\frac{|T|^{\frac{1}{2}}}{\rho_T^{k-1}|F|^{\frac{1}{2}}}\|\,[\![\nabla_h(v_h - v)]\!]\,\|_{L^2(F)}\Big). \tag{1.81b}$$

Proof. We follow the exposition of [Zan17, p. 85]. For any $T \in \mathcal{T}$ and $v_h \in \mathring{S}_2^1(\mathcal{T}) + \mathrm{MR}$, we split the operator $\mathcal{E}_{\mathsf{hct}}$ into $\mathcal{A}_{\mathsf{hct}}$ and $\mathcal{B}_{\partial_{\mathsf{n}}}$:

$$\|D^k(v_h - \mathcal{E}_{\mathsf{hct}} v_h)\|_{L^2(T)} \leqslant \|D^k(v_h - \mathcal{A}_{\mathsf{hct}} v_h)\|_{L^2(T)} + \|D^k\big(\mathcal{B}_{\partial_{\mathsf{n}}}(v_h - \mathcal{A}_{\mathsf{hct}} v_h)\big)\|_{L^2(T)}. \tag{1.82}$$

We estimate the first part in the following way:

$$\begin{aligned}\|D^k(v_h - \mathcal{A}_{\mathsf{hct}} v_h)\|_{L^2(T)} \leqslant &\sum_{z\in\mathcal{V}\cap T}\sum_{j\in\{1,2\}}\big(\partial_j(v_{h|T} - \mathcal{A}_{\mathsf{hct}} v_h)(z)\big)\|D^k\Upsilon_z^j\|_{L^2(T)}\\ &+ \sum_{F\in\mathcal{F}_T}\big(\partial_{\mathsf{n}_F}(v_{h|T} - \mathcal{A}_{\mathsf{hct}} v_h)(m_F)\big)\|D^k\Upsilon_F\|_{L^2(T)}.\end{aligned}$$

We can estimate this expression in two ways. First, we can use Lemmas 1.61 and 1.57 and then apply (1.80a) to get the term on the right-hand side of (1.81b):

$$\|D^k(v_h - \mathcal{A}_{\mathsf{hct}} v_h)\|_{L^2(T)} \lesssim \sum_{F\in\mathcal{F}_T^{i,+}}\frac{|T|^{\frac{1}{2}}}{\rho_T^{k-1}|F|^{\frac{1}{2}}}\|\,[\![\nabla_h(v_h - v)]\!]\,\|_{L^2(F)}.$$

Second, we can again use Lemmas 1.61 and 1.57 but then apply (1.80b) or (1.80c). In this case, we get the right-hand side of (1.81a)

$$\|D^k(v_h - \mathcal{A}_{\mathsf{hct}} v_h)\|_{L^2(T)} \lesssim \sum_{F\in\mathcal{F}_T^{i,+}} \frac{|T|^{\frac12}}{\rho_T^{k-1}|F|}\left|\int_F [\![\nabla_h(v_h - v)]\!]\right| + \sum_{T'\in\mathcal{T}_T^+} \frac{|T|^{\frac12} h_{T'}}{\rho_T^{k-1}|T'|^{\frac12}} \|D_h^2(v_h - v)\|_{L^2(T')}.$$

For the second term in (1.82), we have with $\mathcal{F}_T^i := \mathcal{F}_T \cap \mathcal{F}^i$ that

$$\begin{aligned}\|D^k \mathcal{B}_{\partial_{\mathsf{n}}}(v_h - \mathcal{E}_{\mathsf{hct}} v_h)\|_{L^2(T)} \leqslant \sum_{F\in\mathcal{F}_T^i} \Bigg(&\sum_{z\in\mathcal{V}\cap T} \sum_{j\in\{1,2\}} \big(\partial_j(v_{h|T} - \mathcal{A}_{\mathsf{hct}} v_h)(z)\big) \int_F \partial_{\mathsf{n}_F} \Upsilon_z^j \\ &+ \big(\partial_n(v_{h|T} - \mathcal{A}_{\mathsf{hct}} v_h)(m_F)\big) \int_F \partial_{\mathsf{n}_F} \Upsilon_F \Bigg) \|D^k \overline{\Phi}_{\mathsf{n}_F}\|_{L^2(T)}.\end{aligned}$$

We estimate the bubble $\overline{\Phi}_{\mathsf{n}_F}$ with the help of Lemma 1.60. Then we complete the proof with Lemmas 1.57, 1.61 and 1.62 as before. □

Corollary 1.64 ([VZ18b, Proposition 3.17])**.** *The smoothing operator $\mathcal{E}_{hct} : \mathring{S}_2^1(\mathcal{T}) + \mathrm{MR} \to \mathrm{HCT} \subset H_0^2(\Omega)$ is a quasi-optimal smoother with $C_{\mathsf{qopt}}(\mathcal{E}_{hct})$ depending only on the shape-constant $\sigma(\mathcal{T})$.*

The quasi-optimality of the method is proven in [VZ19a, Theorem 3.18].

Proposition 1.65 (Quasi-Optimality of the Morley method ([VZ19a, Theorem 3.18]))**.** *Let $d = 2$. The method defined via*

$$\begin{gathered}\tilde{V} = H_0^2(\Omega) + \mathrm{MR} \quad \textit{with} \quad V = H_0^2(\Omega), \quad V_h = \mathrm{MR}, \\ \tilde{a} = \tilde{a}^{\mathsf{MR}}, \quad \|\cdot\|^2 = \tilde{a}(\cdot,\cdot), \\ a_h = \tilde{a}^{\mathsf{MR}} \quad \textit{and} \quad E = \mathcal{E}_{hct}.\end{gathered} \tag{B-MR}$$

For $f \in H^{-2}(\Omega)$, find $u_h \in \mathrm{MR}$ such that $\displaystyle\int_\Omega D_h^2 u_h : D_h^2 v_h = \langle f, \mathcal{E}_{hct} v_h\rangle, \quad \forall v_h \in \mathrm{MR}$

is a well-posed, quasi-optimal method with a quasi-optimality constant C_{qopt} only depending on the shape-coefficient of the mesh $\sigma(\mathcal{T})$. Furthermore, the method is overconsistent , i.e. $\delta_{V_h} = 0$ and we have

$$C_{\mathsf{stab}} = C_{\mathsf{qopt}} = \|\mathcal{E}_{hct}\|_{\mathbb{L}(\mathrm{MR}, H_0^2(\Omega))}.$$

We label this method with B-MR.

1.4.4.2 The C^0-interior penalty method

In this section, we present another quasi-optimal but non-overconsistent method for the biharmonic equation (1.66). We follow the exposition of [VZ18b, section 3.4]. For this, we select the discrete space $V_h = \mathring{S}_2^1(\mathcal{T})$ which are the piecewise quadratic functions contained in $H_0^1(\Omega)$. We equip the space $\tilde{V} = S_2^1(\mathcal{T}) + H_0^2(\Omega)$ with the scalar product

$$\tilde{a}^{\mathsf{C}_0}(\tilde{v},\tilde{w}) = \sum_{T\in\mathcal{T}} \int_T D^2\tilde{v} : D^2\tilde{w} + \sum_{F\in\mathcal{F}} \int_F \frac{\gamma}{h_F} [\![\partial_{\mathsf{n}_F}\tilde{v}]\!] [\![\partial_{\mathsf{n}_F}\tilde{w}]\!], \quad \tilde{v},\tilde{w} \in \tilde{V}$$

which induces a norm $\|\cdot\|_{\tilde{V}}^2 = \|\cdot\|_{\mathsf{C}_0}^2$ on $\tilde{V}$. Here, $\gamma > 0$ is a stabilization parameter which will be chosen later, see the proof of Proposition 1.66.

The discrete bilinear form $a_h = a_h^{\mathsf{C}_0} : V_h \times V_h = \mathring{S}_2^1(\mathcal{T}) \times \mathring{S}_2^1(\mathcal{T}) \to \mathbb{R}$ is defined as:

$$\begin{aligned}a_h^{\mathsf{C}_0}(v_h, w_h) := &\int_\Omega D_h^2 v_h : D_h^2 w_h + \sum_{F\in\mathcal{F}} \int_F \frac{\gamma}{h_F} [\![\partial_{\mathsf{n}_F} v_h]\!] [\![\partial_{\mathsf{n}_F} w_h]\!] \\ &+ \sum_{F\in\mathcal{F}} \left\{ \theta \int_F [\![\partial_{\mathsf{n}_F} v_h]\!] \{\{\partial_{\mathsf{n}_F}^2 w_h\}\} - \int_F \{\{\partial_{\mathsf{n}_F}^2 v_h\}\} [\![\partial_{\mathsf{n}_F} w_h]\!] \right\},\end{aligned} \tag{1.83}$$

with $\theta \in \{1,-1\}$. For $\theta = -1$ this is the symmetric method, $\theta = 1$ the non-symmetric method. We mention the non-symmetric case only for completeness and do not consider it further.

The algebraic consistency condition (1.12) reads in this case $a_{\mathsf{B}}(u, Ew_h) - a_h^{\mathsf{C}_0}(u, w_h) = 0$ for all $u \in H_0^2(\Omega) \cap \mathring{S}_2^1(\mathcal{T})$, $w_h \in \mathring{S}_2^1(\mathcal{T})$ and an abstract smoothing operator $E : \mathring{S}_2^1(\mathcal{T}) \to H_0^2(\Omega)$. To see

which properties a smoother needs, we perform an integration by parts with the formula given in Lemma 1.54:

$$\begin{aligned}\tilde{a}^{\mathsf{C}_0}(u, Ew_h) = \sum_{T\in\mathcal{T}} -\int_T (\nabla\Delta u)\cdot\nabla Ew_h \\ + \sum_{F\in\mathcal{F}^i}\int_F \left[\!\left[D^2u\mathsf{n}_F\right]\!\right]\cdot\{\{\nabla Ew_h\}\} + \sum_{F\in\mathcal{F}}\int_F \{\{D^2u\mathsf{n}_F\}\}\cdot\left[\!\left[\nabla Ew_h\right]\!\right].\end{aligned}$$

The discrete form can also be rewritten by an integration by parts:

$$\begin{aligned}a_h^{\mathsf{C}_0}(u, w_h) = \sum_{T\in\mathcal{T}} -\int_T (\nabla\Delta u)\cdot\nabla w_h + \sum_{F\in\mathcal{F}^i}\int_F \left[\!\left[D^2u\mathsf{n}_F\right]\!\right]\cdot\{\{\nabla w_h\}\} \\ + \sum_{F\in\mathcal{F}}\int_F \{\{D^2u\mathsf{n}_F\}\}\cdot\left[\!\left[\nabla w_h\right]\!\right] - \sum_{F\in\mathcal{F}}\int_F \{\{\partial^2_{\mathsf{n}_F}u\}\}\cdot\left[\!\left[\partial_{\mathsf{n}_F}w_h\right]\!\right].\end{aligned}$$

As u is a piecewise polynomial of second order, the element integrals vanish. The last two terms are equal owing to equation (1.71b) and the continuity of $w_h \in \mathring{S}^1_2(\mathcal{T})$. The remaining terms are equal if the operator E fulfills (c.f. [VZ18b, Lemma 3.16])

$$\int_F \nabla Ew_h = \int_F \{\{\nabla w_h\}\} \quad \forall F \in \mathcal{F}^i. \tag{1.84}$$

This condition can be rephrased with Lemma 1.56 to the conditions in (1.72), which are already fulfilled by the $\mathcal{E}_{\mathsf{hct}}$-operator. Therefore, we can use the operator $\mathcal{E}_{\mathsf{hct}}$ as a smoother to obtain a quasi-optimal method.

Proposition 1.66 ([VZ18b, Theorem 3.18])**.** *Let $d = 2$. There is a $\eta_* > 0$ such that for all $\gamma > \eta_*$, the method given by*

$$\begin{aligned}\tilde{V} = H^2_0(\Omega) + \mathring{S}^1_2(\mathcal{T}) \quad \textit{with} \quad V = H^2_0(\Omega), \quad V_h = \mathring{S}^1_2(\mathcal{T}), \\ \tilde{a} = \tilde{a}^{\mathsf{C}_0}, \quad \|\cdot\| = \|\cdot\|_{\mathsf{C}_0}, \\ a_h = a_h^{\mathsf{C}_0} \quad \textit{and} \quad E = \mathcal{E}_{\mathit{hct}}.\end{aligned} \tag{B-SCZ}$$

Given $f \in H^{-2}(\Omega)$, find $u_h \in \mathring{S}^1_2(\mathcal{T})$ such that $a_h^{\mathsf{C}_0}(u_h, w_h) = \langle f, \mathcal{E}_{\mathit{hct}} w_h\rangle \quad \forall w_h \in \mathring{S}^1_2(\mathcal{T})$,

using the norm $\|\cdot\|_{\tilde{V}} = \|\cdot\|_{\mathsf{C}_0}$ and $a_h^{\mathsf{C}_0}$ with the chosen γ, is a well-posed, quasi-optimal method with a quasi-optimality constant C_{qopt} depending on the shape constant $\sigma(\mathcal{T})$ and γ and η_*.

Proof. In [VZ18b, (3.50)], the existence of a constant $\eta_* > 0$ is shown, which ensures the coercivity of the form $a_h^{\mathsf{C}_0}$ with respect to the norm $\|\cdot\|_{\mathsf{C}_0}$. As a consequence, if the stability parameter γ in $\tilde{a}^{\mathsf{C}_0}$ and $a_h^{\mathsf{C}_0}$ is larger than η_* then problem (B-SCZ) is well-posed and quasi-optimal. The quasi-optimality constant C_{qopt} is estimated in the proof of Theorem 3.18 in [VZ18b]. □

Finally, we formulate a counterpart of the estimates of Lemma 1.46 in the case of the B-SCZ method.

Lemma 1.67. *For all $v_h, w_h \in \mathring{S}^1_2(\mathcal{T})$ holds*

$$\begin{aligned}\tilde{a}^{\mathsf{C}_0}(v_h, \mathcal{E}_{\mathit{hct}}w_h - w_h) = -\int_\Sigma \{\{\partial^2_{\mathsf{n}}v_h\}\}\left[\!\left[\partial_{\mathsf{n}}w_h\right]\!\right] - \int_\Sigma \frac{\gamma}{h}\left[\!\left[\partial_{\mathsf{n}}v_h\right]\!\right]\left[\!\left[\partial_{\mathsf{n}}w_h\right]\!\right], \\ \tilde{a}^{\mathsf{C}_0}(\mathcal{E}_{\mathit{hct}}v_h - v_h, w_h) = -\int_\Sigma \left[\!\left[\partial_{\mathsf{n}}v_h\right]\!\right]\{\{\partial^2_{\mathsf{n}}w_h\}\} - \int_\Sigma \frac{\gamma}{h}\left[\!\left[\partial_{\mathsf{n}}v_h\right]\!\right]\left[\!\left[\partial_{\mathsf{n}}w_h\right]\!\right].\end{aligned}$$

Proof. Apply Lemma 1.54 and use the momentum condition of $\mathcal{E}_{\mathsf{hct}}$ in combination with Lemma 1.56 together with the symmetry of $\tilde{a}^{\mathsf{C}_0}$. □

2 Error estimators for symmetric elliptic linear problems

Most often, one computes a numerical approximation because the analytic solution is unavailable. Up to now, we estimated the approximation error in expressions involving this unavailable solution. To gain knowledge about the error without referring to the analytic solution, a-posteriori error-estimators are commonly used. Introductions to this topic can be found in [Ver13, AO00, NV12, BCNV24], where the latter already uses the new error-dominated oscillation technique of [KV21], which is also the subject of this thesis.

The error-estimators provide means to quantify the error in terms of the given data and the computed, discrete solution. Consequently, we can compute the estimator only after the numerical calculation of the approximation (a-posteriori), in contrast to the a-priori error analysis.

To obtain the estimators, several ways are described in the literature. This thesis deals with residual based estimators, which, in a second step, may be rewritten in forms resembling other types. The interested reader finds introductions to other flavours of estimators in [AO00] and [EG04, Chapter 10].

2.1 Abstract setting

In this section, we derive an abstract a-posteriori analysis for the quasi-optimal nonconforming methods of the framework of chapter 1. We fix $f \in V'$ and the corresponding analytic solution $u \in V$ of (1.3). To discretize, we consider one quasi-optimal method M with a smoother $\mathrm{E_M}$ given by (1.10) and denote the discrete solution by $u_h \in V_h$. The goal is to quantify the error $\|u - u_h\|_{\tilde{V}}$ in terms of u_h and f.

In a simple, conforming setting with $V_h \subset V$, one usually rewrites the error with the equations of the problem as (see also [KRVZ25, (3.1)])

$$\|u - u_h\|_V = \sup_{w \in V} \frac{a(u - u_h, w)}{\|w\|_V} = \sup_{w \in V} \frac{\langle f, w\rangle - a(u_h, w)}{\|w\|_V}.$$

The last expression is the residual $\langle f, \cdot\rangle - a(u_h, \cdot)$ measured in the dual norm $\|\cdot\|_{V'}$.

But in a nonconforming setting this does not work. If we rewrite the error as (c.f. [KRVZ25, (3.2)])

$$\|u - u_h\|_{\tilde{V}} = \sup_{\tilde{w} \in \tilde{V}} \frac{\tilde{a}(u - u_h, \tilde{w})}{\|\tilde{w}\|_{\tilde{V}}} \tag{2.1}$$

then neither (1.3) nor (1.10) are directly applicable.

But we have $\tilde{a}(u, \tilde{w}) = \tilde{a}(u, \Pi_V \tilde{w}) = \langle f, \Pi_V \tilde{w}\rangle$, where $\Pi_V : \tilde{V} \to V$ is the $\tilde{a}$-orthogonal projection, see [KRVZ25, (3.3)]. This motivates to define the *residual* as in [KRVZ25, (3.4)]:

$$\mathrm{res_h} := \mathrm{res_h}(f, u_h) := \langle f, \Pi_V \cdot\rangle - \tilde{A} u_h \in \tilde{V}'.$$

With the decomposition $\tilde{V} = V \oplus V^\perp$ (where $\cdot^\perp$ is the orthogonal complement in $\tilde{V}$ with respect to $\tilde{a}(\cdot, \cdot)$), we write as in [KRVZ25, (3.5)]

$$\mathrm{res_h^{c}} := \mathrm{res_h^{c}}(f, u_h) := f - \tilde{A} u_h \in V', \tag{2.2a}$$

$$\mathrm{res_h^{nc}} := \mathrm{res_h^{nc}}(f, u_h) := -\tilde{A} u_h \in (V^\perp)'. \tag{2.2b}$$

Here, $\mathrm{res_h^{c}}$ is the *conforming residual* and $\mathrm{res_h^{nc}}$ is the *nonconforming residual*. They are connected via the relation $\langle \mathrm{res_h}, \cdot\rangle = \langle \mathrm{res_h^{c}}, \Pi_V \cdot\rangle + \langle \mathrm{res_h^{nc}}, (\mathrm{id}_{\tilde{V}} - \Pi_V)\cdot\rangle$, c.f. [KRVZ25].

The relation of these three residuals is clarified by the next Lemma.

Lemma 2.1 ([KRVZ25, Lemma 3.1])**.** *Let $u \in V$ and $u_h \in V_h$ be the solutions of* (1.3) *resp.* (1.10)*. Then the following relations are true:*

$$\|u - u_h\|_{\tilde{V}}^2 = \|\mathrm{res_h}(f, u_h)\|_{\tilde{V}'}^2 = \|\mathrm{res_h^{c}}(f, u_h)\|_{V'}^2 + \|\mathrm{res_h^{nc}}(f, u_h)\|_{(V^\perp)'}^2. \tag{2.3}$$

Moreover, we have

$$\|\operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f,u_h)\|_{V'} = \|u-\Pi_V u_h\|_V \quad \textit{and} \quad \|\operatorname{res}_{\mathrm{h}}^{\mathsf{nc}}(f,u_h)\|_{(V^\perp)'} = \|u_h-\Pi_V u_h\|_{\tilde{V}}. \tag{2.4}$$

Here, $V^\perp$ is again the orthogonal complement within $\tilde{V}$ with respect to $\tilde{a}$. We also have

$$\|u-u_h\|_{\tilde{V}}^2 = \|u-\Pi_V u_h\|_{\tilde{V}}^2 + \|\Pi_V u_h - u_h\|_{\tilde{V}}^2. \tag{2.5}$$

Proof. To prove (2.4), we note that

$$\|\operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f,u_h)\|_{V'} = \sup_{w\in V}\frac{\tilde{a}(u-u_h,w)}{\|w\|_V} = \sup_{w\in V}\frac{\tilde{a}(u-\Pi_V u_h,w)}{\|w\|_V} = \|u-\Pi_V u_h\|_V.$$

Furthermore, we have

$$\|\operatorname{res}_{\mathrm{h}}^{\mathsf{nc}}(f,u_h)\|_{V'} = \sup_{w\in V^\perp}\frac{\tilde{a}(u_h,w)}{\|w\|_{\tilde{V}}} = \sup_{w\in \tilde{V}}\frac{\tilde{a}(u_h,w-\Pi_V w)}{\|w\|_{\tilde{V}}} = \sup_{w\in \tilde{V}}\frac{\tilde{a}(u_h-\Pi_V u_h,w)}{\|w\|_{\tilde{V}}} = \|u_h-\Pi_V u_h\|_{\tilde{V}}.$$

For the last relation (2.5), observe that $u-\Pi_V u_h$ and $u_h-\Pi_V u_h$ are orthogonal. Then, the relation (2.3) follows directly. □

We analyze the conforming and nonconforming residuals separately. In the next section, we discuss estimates for the nonconforming residual that is the term $\|\Pi_V u_h - u_h\|_{\tilde{V}}$, which we also call the *conformity error*.

2.1.1 The nonconforming residual and quasi-optimal smoothers

In this section, we analyze the nonconforming residual $\operatorname{ncf}_{\mathrm{h}} := \|\operatorname{res}_{\mathrm{h}}^{\mathsf{nc}}\|_{(V^\perp)'} = \|u_h-\Pi_V u_h\|_{\tilde{V}}$ The expression $\|u_h-\Pi_V u_h\|_{\tilde{V}}$ is in general not computable because V is infinite-dimensional. Hence, we provide an equivalent quantity for the conformity error with the help of quasi-optimal smoothers, see section 1.3 and also the discussion in [KRVZ25, section 3.2].

Choosing an arbitrary quasi-optimal smoother $E_l : V_h \to V$, we get with (1.20) an estimate for the conformity error:

$$\|\Pi_V u_h - u_h\|_{\tilde{V}} \leqslant \|E_l u_h - u_h\|_{\tilde{V}} \leqslant C_{\mathsf{qopt}}(E_l)\|\Pi_V u_h - u_h\|_{\tilde{V}}. \tag{2.6}$$

We explicitly require the smoother to be quasi-optimal because otherwise the upper bound does not holds. The upper bound guarantees the sharpness of our estimates.

The smoother E_l can be any quasi-optimal, computable smoother, especially it does not have to be the one used in the method. Therefore, we can choose a smoother which is easy to evaluate, for example in terms of jumps. We have seen in section 1.4 that many quasi-optimal smoothers presented there allow for an easy evaluation of $\|E_l u_h - u_h\|_{\tilde{V}}$.

If the method, however, already uses a quasi-optimal smoother, it might be beneficial to use that one because one has already a working implementation. Using a different smoother can nevertheless be useful for methods without a quasi-optimal smoother such as P-RSIP and P-RNSIP introduced in section 1.4.2.5.

Now, we have a global, computable equivalent quantity for the conformity error $\|\Pi_V u_h - u_h\|_{\tilde{V}}$. In many applications, this global quantity can be split into local quantities such that one can refine elements accordingly to reduce the conformity error.

2.1.2 Equivalence of local and global error estimators

The conforming residual $\operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f,u_h)$ can be used to assess the global error $\|u-\Pi_V u_h\|_{\tilde{V}}$ a-posteriori, i.e. after the solution u_h has been computed. The goals of the a-posteriori analysis are "to quantify the error of the numerical solution" and "to gain information for adapting the discretization to the exact solution" ([BCNV24, p. 54], but see also [Ver13, section 1.1]).

The techniques to adapt the mesh presented in [BCNV24, Ver13] require *local* error indicators. Therefore, we split $\operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f,u_h)$ into local contributions and the global estimator $\|\operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f,u_h)\|_{V'}$ should be equivalent to local error indicators, similar to [BCNV24, section 4.1].

For expressing the localized residual, we need appropriate local spaces together with a condition on the norms such that the globalization works.

Assumption 2.2. *Let $\mathcal{I}$ be a finite index set. For each $z \in \mathcal{I}$, let $V_z \subset V$ be a linear subspace of V. The scalar product $a(\cdot,\cdot)$ provides a topology on all V_z, $z \in \mathcal{I}$, via restriction. For each $z \in \mathcal{I}$, we write V_z' for the dual space of V_z.*

Moreover, we assume that there is a constant $C_{\mathsf{loc}} > 0$ such that: If we select for each $z \in \mathcal{I}$ an arbitrary $w_z \in V_z$ then

$$\Big\| \sum_{z\in\mathcal{I}} w_z \Big\|_V^2 \leqslant C_{\mathsf{loc}} \sum_{z\in\mathcal{I}} \|w_z\|_V^2, \tag{2.7}$$

c.f. [KRVZ25, (3.4b)].

Remark 2.3. All functionals $R \in V'$ are also functionals in V_z', but the concept of equality changes. Two functionals R, R' in V' are equal if $\langle R, w\rangle = \langle R', w\rangle$ for all $w \in V$ but within V_z' they are equal if $\langle R, w_z\rangle = \langle R', w_z\rangle$ for all $w_z \in V_z$, where the latter is the weaker condition. This means that two functions might be different in V' but are the same in V_z'.

Within this setting, we obtain a general globalization result, see also [KV21, Lemma 4] and [KRVZ25, Lemma 3.7].

Theorem 2.4. *Assume the spaces V and V_z, $z \in \mathcal{I}$, fulfill Assumption 2.2. Let $R \in V'$ be a continuous linear functional. Then*

$$\sum_{z\in\mathcal{I}} \|R\|_{V_z'}^2 \leqslant C_{\mathsf{loc}} \|R\|_{V'}^2.$$

Proof. For each $z \in \mathcal{I}$, we select a $w_z \in V_z$ with $\|w_z\|_V = 1$ and define the global function $w := \sum_{z\in\mathcal{I}} \langle R, w_z\rangle w_z \in V$.

We have

$$\sum_{z\in\mathcal{I}} \langle R, w_z\rangle^2 = \langle R, \sum_{z\in\mathcal{I}} \langle R, w_z\rangle w_z\rangle = \langle R, w\rangle \leqslant \|R\|_{V'} \|w\|_V.$$

Together with (2.7) in Assumption 2.2, this provides:

$$\|w\|_V^2 \leqslant C_{\mathsf{loc}} \sum_{z\in\mathcal{I}} \langle R, w_z\rangle^2 \leqslant C_{\mathsf{loc}} \|R\|_{V'} \|w\|_V.$$

We combine the last two inequalities and get:

$$\sum_{z\in\mathcal{I}} \langle R, w_z\rangle^2 \leqslant C_{\mathsf{loc}} \|R\|_{V'}^2.$$

Taking the supremum over all $w_z \in V_z$, $z \in \mathcal{I}$, with $\|w_z\|_V = 1$, yields the result. □

The reverse direction is usually derived with the Galerkin orthogonality, see [KV21] and [Ver13]. Within the framework of Chapter 1, a Galerkin orthogonality is not available in general. But (1.13) allows us to derive a quasi-Galerkin orthogonality result.

Theorem 2.5 ([KRVZ25, Lemma 3.6])**.** *Let $u \in V$ be the solution of* (1.3) *and $u_h \in V_h$ the solution of* (1.10) *with $\mathrm{E_M} : V_h \to V$ the smoother of the method. For all $w_h \in V_h$, we have*

$$\langle \mathrm{res_h^c}(f, u_h), \mathrm{E_M}\, w_h\rangle \leqslant \delta_{V_h} \|u_h - \Pi_V u_h\|_{\tilde{V}} \|w_h\|_{V_h}.$$

Here, δ_{V_h} is defined in (1.13).

Proof. We employ inequality (1.13) with the analytic and discrete equation and obtain:

$$\begin{aligned}\langle \mathrm{res_h^c}(f, u_h), \mathrm{E_M}\, w_h\rangle &= a(u - \Pi_V u_h, \mathrm{E_M}\, w_h)\\ &= a_h(u_h, w_h) - a(\Pi_V u_h, \mathrm{E_M}\, w_h) \leqslant \delta_{V_h} \|u_h - \Pi_V u_h\|_{\tilde{V}} \|w_h\|_{V_h}.\end{aligned}$$

□

If $\delta_{V_h} = 0$ the method is called *overconsistent*. This property is thoroughly discussed in [VZ19a]. For an overconsistent method, the quasi-Galerkin orthogonality becomes a Galerkin orthogonality:

$$\langle \mathrm{res_h^c}(f, u_h), \mathrm{E_M}\, w_h\rangle = 0 \quad \text{for all } w_h \in V_h.$$

Using the quasi-Galerkin orthogonality, we can localize $\mathrm{res_h^c}(f, u_h)$. For this, we usually multiply the argument of the residual with a partition of unity. But the multiplication of two elements in V

is beyond the algebraic structure of the space V assumed in the abstract framework of [VZ18a]. We circumvent this problem by introducing localization operators $\Theta_z : V \to V_z$ with appropriate subspaces $V_z \subset V$, $z \in \mathcal{I}$, from Assumption 2.2. These operators Θ_z act from the right, which minimizes the notational differences to a partition of unity. Thus, we write $w\Theta_z \in V_z$, $w \in V$. Similarly to a partition of unity, the localization operators fulfill for all $w \in V$

$$\sum_{z\in\mathcal{I}} w\Theta_z = w. \tag{2.8}$$

With these general tools, we present two localization procedures. At first, we recall the one discussed in [KRVZ25, (H2), Lemma 3.7]. This localization requires the following Assumptions.

Assumption 2.6. *Assume that Assumption 2.2 is true. There exist localization operators $\Theta_z : V \to V_z$ satisfying (2.8). For each $w \in V$ with $\|w\|_V = 1$ the estimate*

$$\inf_{\substack{w_h\in V_h\\ \|w_h\|_{V_h}\leqslant C_\mathsf{l}}} \sum_{z\in\mathcal{I}} \|(w - \mathrm{E_M}\, w_h)\Theta_z\|_V^2 \leqslant C_\mathsf{apx}^2 \tag{2.9}$$

is true with constants $C_\mathsf{l} > 0$ and $C_\mathsf{apx} > 0$, both independent of w. Here, $\mathrm{E_M}$ is the smoothing operator of the method.

This assumption is quite abstract, but we immediately obtain the following more concrete version.

Corollary 2.7 ([KRVZ25, (H2)])**.** *Assume that Assumption 2.2 is true. Moreover, assume that there exist localization operators $\Theta_z : V \to V_z$ satisfying (2.8) and that for each $w \in V$ with $\|w\|_V = 1$ there exists an approximation operator $\mathcal{I}_h : V \to V_h$ fulfilling*

$$\|\mathcal{I}_h w\|_{V_h} \leqslant C_\mathsf{l} \quad \text{and} \quad \sum_{z\in\mathcal{I}} \|(w - \mathrm{E_M}\,\mathcal{I}_h w)\Theta_z\|_V^2 \leqslant C_\mathsf{apx}^2 \tag{2.10}$$

with constants $C_\mathsf{apx} > 0$ and $C_\mathsf{l} > 0$ both independent of w. Here, $\mathrm{E_M}$ is the smoothing operator of the method. Then, Assumption 2.6 is true.

Remark 2.8. The operator $\mathcal{I}_h$ might be chosen differently for each $w \in V$. For the localization, however, it is important that C_apx and C_l are independent of w. Despite this freedom of choice, in most cases one will choose one approximation operator $\mathcal{I}_h$.

Remark 2.9. For each $w \in V$, we can immediately replace the smoother of the method $\mathrm{E_M}$ in (2.10) by another smoother $\mathrm{E_R}$ which coincides with $\mathrm{E_M}$ on the one-dimensional vector space $\operatorname{span}(\mathcal{I}_h w)$.

Remark 2.10. If the operator $\mathcal{I}_h$ maps into a subspace of V_h on which $\mathrm{E_M}$ is invariant, the second condition of (2.10) simplifies to $\sum_{z\in\mathcal{I}} \|(w - \mathcal{I}_h w)\Theta_z\|_V^2 \leqslant C_\mathsf{apx}^2$. This is for example the case when $\mathrm{E_M}$ is a quasi-optimal smoother and $\mathcal{I}_h$ maps into the conforming part $V \cap V_h$.

Theorem 2.11. *Let $u_h \in V_h$ be the discrete solution of (1.10). If Assumption 2.6 is fulfilled, we have*

$$\|\operatorname{res}_\mathrm{h}^\mathsf{c}(f, u_h)\|_{V'} \leqslant C_\mathsf{apx}\Big(\sum_{z\in\mathcal{I}} \|\operatorname{res}_\mathrm{h}^\mathsf{c}(f, u_h)\|_{V_z'}^2\Big)^{\frac{1}{2}} + \delta_{V_h} C_\mathsf{l} \|\Pi_V u_h - u_h\|_{\tilde{V}}. \tag{2.11}$$

Here $\Pi_V : \tilde{V} \to V$ is the $\tilde{a}$-orthogonal projection onto V, C_apx and C_l are the constants defined in Assumption 2.6 and finally δ_{V_h} is the consistency measure of the method as introduced in Definition 1.15.

Proof. For any $w \in V$ with $\|w\|_V = 1$, we obtain

$$\langle \operatorname{res}_\mathrm{h}^\mathsf{c}(f, u_h), w\rangle = \inf_{\substack{w_h\in V_h\\ \|w_h\|_{V_h}\leqslant C_\mathsf{l}}} \langle \operatorname{res}_\mathrm{h}^\mathsf{c}(f, u_h), w - \mathrm{E_M}\, w_h\rangle + \langle \operatorname{res}_\mathrm{h}^\mathsf{c}(f, u_h), \mathrm{E_M}\, w_h\rangle. \tag{2.12}$$

The last summand is bounded with Theorem 2.5

$$\langle \operatorname{res}_\mathrm{h}^\mathsf{c}(f, u_h), \mathrm{E_M}\, w_h\rangle \leqslant \delta_{V_h} \|u_h - \Pi_V u_h\|_{\tilde{V}} \|w_h\|_{V_h} \leqslant \delta_{V_h} C_\mathsf{l} \|u_h - \Pi_V u_h\|_{\tilde{V}}.$$

The remaining term in (2.12) is estimated as

$$\begin{aligned}
\inf_{\substack{w_h\in V_h\\ \|w_h\|_{V_h}\leqslant C_\mathsf{l}}} \langle \operatorname{res}_\mathrm{h}^\mathsf{c}(f, u_h), w - \mathrm{E_M}\, w_h\rangle &= \inf_{\substack{w_h\in V_h\\ \|w_h\|_{V_h}\leqslant C_\mathsf{l}}} \sum_{z\in\mathcal{I}} \langle \operatorname{res}_\mathrm{h}^\mathsf{c}(f, u_h), (w - \mathrm{E_M}\, w_h)\Theta_z\rangle \\
&\leqslant \inf_{\substack{w_h\in V_h\\ \|w_h\|_{V_h}\leqslant C_\mathsf{l}}} \sum_{z\in\mathcal{I}} \|\operatorname{res}_\mathrm{h}^\mathsf{c}(f, u_h)\|_{V_z'} \|(w - \mathrm{E_M}\, w_h)\Theta_z\|_V \\
&\leqslant \inf_{\substack{w_h\in V_h\\ \|w_h\|_{V_h}\leqslant C_\mathsf{l}}} \Big(\sum_{z\in\mathcal{I}} \|\operatorname{res}_\mathrm{h}^\mathsf{c}(f, u_h)\|_{V_z'}^2\Big)^{\frac{1}{2}} \Big(\sum_{z\in\mathcal{I}} \|(w - \mathrm{E_M}\, w_h)\Theta_z\|_V^2\Big)^{\frac{1}{2}}.
\end{aligned}$$

Applying (2.9) to the last factor, we get the final result. □

In the applications, presented in section 2.2, we usually show Assumption 2.6 via Corollary 2.7 with the help of an interpolation operator. The operator is often the Scott-Zhang operator ([SZ90]) or the canonical one for the discrete space V_h. Some operators eliminate the smoother $\mathrm{E_M}$ and to show (2.10), we only need to consider the discrete approximation operator, see also Remark 2.10.

For example, in the case of the discretizations of the Poisson problem, Proposition 2.31 shows that the Scott-Zhang operator fulfills the assumptions in Corollary 2.7. There, the estimate is simplified because the Scott-Zhang operator maps into the conforming space $H_0^1(\Omega)$ and the smoothing operator can be directly omitted.

If the smoother cannot be eliminated (as in Theorem 2.54, for example), one often needs additional properties of the smoother $\mathrm{E_M}$. These are often approximability properties in weaker norms. For this reason, we also derived the estimates in the L^2- resp. H^1-norms in chapter 1.4. Despite these examples, from a theoretical standpoint there is no reason to assume that a smoother fulfills such estimates.

Moreover, it is not clear if the kind of localization based upon Assumption 2.6 or Corollary 2.7 can be generalized to a pressure-robust localization in the Stokes setting, which we will discuss in section 4.1. The problems concerning such a generalization are discussed in Remark 4.12.

Therefore, we derive another kind of localization which usually uses other properties of $\mathrm{E_M}$. This localization procedure is also the basis for the localization in the setting of the Stokes equation, see Assumption 4.10 and Theorem 4.11. This setting is inspired by the localizations in [KV21] and [CDN12].

The assumptions in this setting are the following.

Assumption 2.12. *Assume that Assumption 2.2 is true. In addition, for each local space V_z, $z \in \mathcal{I}$, there shall be a larger space $V_z \subset V_z^+ \subset V$. We suppose that for each $w \in V$ with $\|w\|_V = 1$, there exist*

(i) for each $z \in \mathcal{I}$ a localization operator $\Theta_z^a : V \to V_z$ such that $w\Theta_z^a \in V_z$ and $\sum_{z\in\mathcal{I}} w\Theta_z^a = w$,

(ii) for each $z \in \mathcal{I}$ a vector space $\mathbb{P}_z$, a $p_z \in \mathbb{P}_z$ and another operator $\Theta_z^b : \mathbb{P}_z \to V_h$,

(iii) and an operator $\mathrm{E_R} : V_h \to V$ that fulfills $\mathrm{E_R}(\sum_{z\in\mathcal{I}} p_z\Theta_z^b) = \mathrm{E_M}(\sum_{z\in\mathcal{I}} p_z\Theta_z^b)$, where $\mathrm{E_M}$ is the smoothing operator of the method, and $\mathrm{E_R}(p_z\Theta_z^b) \in V_z^+$.

These objects shall satisfy the bounds

$$\Big\| \sum_{z\in\mathcal{I}} p_z\Theta_z^b \Big\|_{V_h} \leqslant C_{\mathsf{I}} \quad \text{and} \tag{2.13a}$$

$$\sum_{z\in\mathcal{I}} \|w\Theta_z^a - \mathrm{E_R}(p_z\Theta_z^b)\|_V^2 \leqslant C_{\mathsf{apx}}^2 \tag{2.13b}$$

with constants $C_{\mathsf{apx}} > 0$ and $C_{\mathsf{I}} > 0$ both independent of w.

Remark 2.13. If the smoother $\mathrm{E_R}$, and therefore also the smoother $\mathrm{E_M}$, are invariant under $\sum_{z\in\mathcal{I}} p_z\Theta_z^b$, meaning $\mathrm{E_R}(\sum_{z\in\mathcal{I}} p_z\Theta_z^b) = \sum_{z\in\mathcal{I}} p_z\Theta_z^b$, then (2.13b) is often followed by the Bramble-Hilbert Lemma, see [BS08, Lemma 4.3.8]. Otherwise, one commonly extends $\mathrm{E_R}$ to the whole space $\tilde{V} = V + V_h$, for which $\mathrm{E_R}$ needs to be quasi-optimal, and then exploits the stability of the extended $\mathrm{E_R}$ in combination with the Bramble-Hilbert Lemma.

Since the smoothing operator has to be quasi-optimal for the extension, we do not use $\mathrm{E_M}$ but $\mathrm{E_R}$ to gain flexibility. There are methods, which do not employ a quasi-optimal smoother (e.g. P-RSIP and P-RNSIP for $\ell > 1$) but to which this localization method is still applicable because another quasi-optimal smoother is available and coincides with the smoother of the method on a sufficiently larger subspace of its domain to fulfill Assumption 2.12. Another difficulty in the application of Assumption 2.12 is to find the vector space $\mathbb{P}_z$ and suitable $p_z \in \mathbb{P}_z$ for all $z \in \mathcal{I}$ such that $p_z\Theta_z^b \in V_h$.

The localization Theorem corresponding to Assumption 2.12 is similar to Theorem 2.11 but differs in some aspects.

Theorem 2.14. *Let Assumption 2.12 be satisfied. Then the localization*

$$\|\operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f,u_h)\|_{V'} \leqslant C_{\mathsf{apx}}\Big(\sum_{z\in\mathcal{I}} \|\operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f,u_h)\|_{V_z^{+\prime}}^2\Big)^{\frac{1}{2}} + \delta_{V_h} C_{\mathsf{I}} \|\Pi_V u_h - u_h\|_{\tilde{V}} \tag{2.14}$$

holds, with the notation as in Theorem 2.11 except that C_{I} and C_{apx} are provided by Assumption 2.12.

Proof. We start in the same spirit as in the proof of Theorem 2.11. Let $w \in V$, $\|w\|_V = 1$ be arbitrary. For this particular w, Assumption 2.12 provides vector spaces $\mathbb{P}_z$ and $p_z \in \mathbb{P}_z$ for all $z \in \mathcal{I}$. Combining these with the localization operators $\Theta_z^b : \mathbb{P}_z \to V_h$, we define the function $\sum_{z\in\mathcal{I}} p_z\Theta_z^b \in V_h$. Applying the smoother $\mathrm{E_R}$ onto this function, we conclude that

$$\langle \mathrm{res}_{\mathrm{h}}^{\mathsf{c}}(f,u_h), w\rangle = \langle \mathrm{res}_{\mathrm{h}}^{\mathsf{c}}(f,u_h), w - \mathrm{E_R}\big(\sum_{z\in\mathcal{I}} p_z\Theta_z^b\big)\rangle + \langle \mathrm{res}_{\mathrm{h}}^{\mathsf{c}}(f,u_h), \mathrm{E_R}\big(\sum_{z\in\mathcal{I}} p_z\Theta_z^b\big)\rangle. \tag{2.15}$$

The latter summand is bounded by Theorem 2.5 because of $\mathrm{E_R}(\sum_{z\in\mathcal{I}} p_z\Theta_z^b) = \mathrm{E_M}(\sum_{z\in\mathcal{I}} p_z\Theta_z^b)$ by Assumption 2.12 and (2.13a):

$$\langle \mathrm{res}_{\mathrm{h}}^{\mathsf{c}}(f,u_h), \mathrm{E_R}\big(\sum_{z\in\mathcal{I}} p_z\Theta_z^b\big)\rangle \leqslant \delta_{V_h} C_{\mathsf{I}} \|\Pi_V u_h - u_h\|_{\tilde{V}}.$$

Turning to the other part in (2.15), we estimate:

$$\begin{aligned}
\langle \mathrm{res}_{\mathrm{h}}^{\mathsf{c}}(f,u_h), w - \mathrm{E_R}\big(\sum_{z\in\mathcal{I}} p_z\Theta_z^b\big)\rangle &= \sum_{z\in\mathcal{I}} \langle \mathrm{res}_{\mathrm{h}}^{\mathsf{c}}(f,u_h), w\Theta_z^a - \mathrm{E_R}(p_z\Theta_z^b)\rangle \\
&\leqslant \sum_{z\in\mathcal{I}} \|\mathrm{res}_{\mathrm{h}}^{\mathsf{c}}(f,u_h)\|_{V_z^{+\prime}} \|w\Theta_z^a - \mathrm{E_R}(p_z\Theta_z^b)\|_V \\
&\leqslant \big(\sum_{z\in\mathcal{I}} \|\mathrm{res}_{\mathrm{h}}^{\mathsf{c}}(f,u_h)\|_{V_z^{+\prime}}^2\big)^{\frac{1}{2}} \big(\sum_{z\in\mathcal{I}} \|w\Theta_z^a - \mathrm{E_R}(p_z\Theta_z^b)\|_V^2\big)^{\frac{1}{2}}.
\end{aligned}$$

Applying (2.13b) yields the result. □

The introduction of V_z^+ is necessary because most smoothers usually enlarge the support of their arguments, hence we need a space larger than V_z. This seems to be a disadvantage, missing in the previous result Theorem 2.14. But this is not the case. The most problematic step is to localize the global residual $\|\mathrm{res}_{\mathrm{h}}^{\mathsf{c}}(f,u_h)\|_{V'}$ to something local. In many examples, another localization, which we call *local localization*, can localize from $\|\mathrm{res}_{\mathrm{h}}^{\mathsf{c}}(f,u_h)\|_{V_z^{+\prime}}$ to $\|\mathrm{res}_{\mathrm{h}}^{\mathsf{c}}(f,u_h)\|_{V_z'}$, which is the topic of the next Theorem. In [BCNV24, Lemma 4.8], a related but different result is presented.

For the local localization, we need the following assumptions.

Assumption 2.15. *Assume that Assumption 2.2 is true. From this Assumption, we recall the finite index-set $\mathcal{I}$ and the spaces V_y, $y \in \mathcal{I}$.*

For each $z \in \mathcal{I}$, there is a subspace $V_z^+ \subset V$ and an index-set $\mathcal{I}_z$ satisfying $\mathcal{I}_z \subset \mathcal{I}$ and $\#\mathcal{I}_z \leqslant C_{\mathsf{N}}$ with a constant $C_{\mathsf{N}} \in \mathbb{N}$ independent of z. For each $y \in \mathcal{I}_z$, there exists a localization operator $\Theta_y : V_z^+ \to V_y \cap V_z^+$. These operators shall fulfill for all $w \in V_z^+$

$$\sum_{y\in\mathcal{I}_z} w\Theta_y = w \quad \text{and} \quad \sum_{y\in\mathcal{I}_z} \|w\Theta_y\|_V^2 \leqslant C_{\mathsf{P}}^2 \|w\|_V^2 \tag{2.16}$$

for a constant $C_{\mathsf{P}} > 0$ independent of z.

Assumption 2.15 enables us to prove the local localization Theorem.

Theorem 2.16. *Let Assumption 2.15 be true with $C_{\mathsf{P}} > 0$. Given any functional $R \in V'$, we have for all $z \in \mathcal{I}$*

$$\|R\|_{V_z^{+\prime}}^2 \leqslant C_{\mathsf{P}}^2 \sum_{y\in\mathcal{I}_z} \|R\|_{V_y'}^2.$$

Proof. Fix $z \in \mathcal{I}$. Let $w \in V_z^+$ and compute with (2.16)

$$\begin{aligned}
\langle R, w\rangle &= \sum_{y\in\mathcal{I}_z} \langle R, w\Theta_y\rangle \leqslant \sum_{y\in\mathcal{I}_z} \|R\|_{V_y'} \|w\Theta_y\|_V \\
&\leqslant \big(\sum_{y\in\mathcal{I}_z} \|R\|_{V_y'}^2\big)^{\frac{1}{2}} \big(\sum_{y\in\mathcal{I}_z} \|w\Theta_y\|_V^2\big)^{\frac{1}{2}} \leqslant C_{\mathsf{P}} \big(\sum_{y\in\mathcal{I}_z} \|R\|_{V_y'}^2\big)^{\frac{1}{2}} \|w\|_V.
\end{aligned}$$

□

2.1.3 Construction of a computational bound

In this section, we assume that Assumption 2.2 is true. The previous section discussed the localization of the global, conforming residual $\|\operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f,u_h)\|_{V'}$ to local quantities $\|\operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f,u_h)\|_{V_z'}$. In many applications, one can bound $\|\operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f,u_h)\|_{V_z'}$ again by the local error. This means: if $\|\operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f,u_h)\|_{V_z'}$ is large, the local error is large. With this information, in many applications involving meshes, one could heuristically try around z to reduce the error.

But a major hindrance remains: the crucial expression $\|\operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f,u_h)\|_{V_z'}$ is in general not computable because the data f usually stems from the infinite-dimensional space V'. Without any further a-priori information on f, the expression $\|\operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f,u_h)\|_{V_z'}$ cannot be bounded by only finitely many evaluations of f, see [KV21, Lemma 2].

The problem can be overcome by replacing f by an approximation at the cost of an additional error term. The latter is usually called *oscillation*. This approach is popular for error indicators, see e.g. [Ver13].

To illustrate this problem with an example, let us write Π_T^k for the L^2 projection of $L^2(T)$ onto $\mathbb{P}_k(T)$, $T\in\mathcal{T}$, $k\in\mathbb{N}_0$. Then the classical oscillation is defined for $f\in L^2(\Omega)$ as

$$\operatorname{osc}_k^{L^2}(f)^2 := \sum_{T\in\mathcal{T}} h_T^2 \|f-\Pi_T^k f\|_{L^2(T)}^2, \tag{2.17}$$

c.f. [BCNV24, (4.13)].

The classical approach in [Ver13] with the oscillation terms defined in (2.18a) has several disadvantages. These disadvantages are laid out in [CDN12, KV21, BCNV24]. We exemplify these problems with three results from [KV21] in the case of the discretization of the Poisson problem with first order Lagrange elements (P-LAG). In this case, the classical L^2 and H^{-1} oscillations are defined as

$$\operatorname{osc}_0^{L^2}(f)^2 := \sum_{T\in\mathcal{T}} h_T^2 \int_T \left(f - \frac{1}{|T|}\int_T f\right)^2 \quad \text{for } f\in L^2(\Omega) \tag{2.18a}$$

and

$$\operatorname{osc}_0^{H^{-1}}(f)^2 := \min_{q\in S_0^0(\mathcal{T})} \sum_{z\in\mathcal{V}} \|f-q\|_{H^{-1}(\omega_z)}^2 \qquad \text{for } f\in H^{-1}(\Omega), \tag{2.18b}$$

c.f. [Ver13, Remark 1.8] and [CDN12, p. 685].

The first problem of these oscillations is that the oscillation $\operatorname{osc}_0^{L^2}$ might become arbitrary larger than the error on a fixed mesh.

Lemma 2.17 ([KV21, Lemma 20])**.** *One can construct a sequence $(f_k)_k \subset L^2(\Omega)$ such that for the corresponding solutions $u_k\in H_0^1(\Omega)$ of (1.30) and the discrete solutions obtained with P-LAG, $\ell=1$, denoted by $u_{h,k}\in \mathring{S}_1^1(\mathcal{T})$, we have:*

$$\frac{\operatorname{osc}_0^{L^2}(f_k)}{\|\nabla(u_k-u_{h,k})\|_{L^2(\Omega)}} \to \infty \quad \text{as} \quad k\to\infty.$$

The next two examples deal with the classical H^{-1}-oscillation, [KV21, p. 66]. For a fixed $0\neq v_h\in \mathring{S}_1^1(\mathcal{T})$, we define $f=-\Delta v_h\in H^{-1}(\Omega)$ in the distributional sense. For this f, the Lagrange finite element method (P-LAG), $\ell=1$, computes the exact analytic solution, hence the error is zero. But still the H^{-1}-oscillation is strictly larger than zero.

The last example shows that even the asymptotic convergence rate of the error might be higher than the rate of the classical H^{-1}-oscillation.

Lemma 2.18 ([KV21, Lemma 21])**.** *For this Lemma let $\Omega := (0,1)^2$. We consider all conforming meshes $\mathcal{T}$ produced by a recursive newest vertex bisection (the algorithm is described in [BCNV24, Lemma 8.4]). One can construct a functional $f\in H^{-1}(\Omega)$ and a sequence of real numbers $(L_n)_n$, which satisfy $\log(n)\gtrsim L_n\to\infty$ as $n\to\infty$, such that the analytic solution $u\in H_0^1(\Omega)$ and the corresponding discrete solutions $u_{\mathcal{T}}\in S_1^1(\mathcal{T})$ of the method P-LAG using the mesh $\mathcal{T}$ fulfill:*

$$\min_{\#\mathcal{T}\leqslant n} \|\nabla(u-u_{\mathcal{T}})\|_{L^2(\Omega)} \lesssim n^{-\frac{1}{2}} \quad \text{and} \quad \min_{\#\mathcal{T}\leqslant n}\min_{q\in S_0^0(\mathcal{T})} \Big(\sum_{z\in\mathcal{V}(\mathcal{T})} \|f-q\|_{H^{-1}(\omega_z)}^2\Big)^{\frac{1}{2}} \geqslant L_n n^{-\frac{1}{2}}$$

where $\mathcal{V}(\mathcal{T})$ are the vertices of $\mathcal{T}$.

Within the situation of Lemma 2.18, [KV21, Remark 22] and [CDN12] remark that the classical H^{-1}-residual based estimator

$$\sum_{z\in\mathcal{V}} \|\Delta u_{\mathcal{T}}\|^2_{H^{-1}(\omega_z)} + \|f\|^2_{H^{-1}(\omega_z)}$$

overestimates the error.

In view of these problems and because an oscillation term is unavoidable for computability, [KV21] suggest another kind of oscillation term. Their construction uses a global operator but because we are working with local operators, we directly adapt their construction to our setting. We define for each $z \in \mathcal{I}$ a local, finite-dimensional space $D_z \subset V_z'$ together with a bounded, linear operator $\mathcal{P}_z : V_z' \to D_z$. If the operator $\mathcal{P}_z$ is invariant under $(\tilde{A}V_h)_{|V_z}$, we get

$$\|\operatorname{res}^{\mathsf{c}}_{\mathrm{h}}(f,u_h)\|_{V_z'} \leqslant \underbrace{\|\mathcal{P}_z f - \tilde{A}u_h\|_{V_z'}}_{\text{estimator}} + \underbrace{\|\mathcal{P}_z f - f\|_{V_z'}}_{\text{oscillation}} \lesssim \|\operatorname{res}^{\mathsf{c}}_{\mathrm{h}}(f,u_h)\|_{V_z'}. \tag{2.20}$$

The new oscillation term is bounded by $\operatorname{res}^{\mathsf{c}}_{\mathrm{h}}(f,u_h)$ thus also by the error. Hence, we have an error-dominated oscillation. This new oscillation does not have the problems described in Lemmas 2.17, 2.18. The resulting estimator cannot overestimate. For further details, see [KV21].

After we have motivated, why an error-dominated oscillation is beneficial, we discuss the construction of the projection $\mathcal{P}_z$. In the original paper [KV21], a projection for $\ell = 1$ and P-LAG is explicitly constructed by using a biorthogonal system. But in the case of P-LAG and $\ell > 1$, the situation becomes more difficult. Therefore, the authors of [KVZ] suggest to define the projection implicitly via a system of linear equations. In particular, after selecting a finite-dimensional space $D_z \subset V_z'$, they choose a vector space of test functions (also finite-dimensional) $S_z \subset V_z$ and define the projection $\mathcal{P}_z : V_z' \to D_z$ for each $\chi \in V_z'$ via the equation:

$$\langle \mathcal{P}_z\chi, w\rangle = \langle \chi, w\rangle, \quad \forall w \in S_z.$$

The spaces D_z and S_z have to be chosen such that the equation is well-posed.

We, however, have to use a more complicated setup, for the following two reasons. One reason is that the condition $S_z \subset V_z$ is too restrictive. We will see that using test functions from the space $\mathrm{E}_{\mathrm{M}}(V_h)$ has several advantages. Nonetheless, the relevant functions are usually not contained in V_z but in a larger space V_z^+ because many smoothers enlarge the support of their arguments. Thus, the condition $S_z \subset V_z$ is too restrictive.

The other reason is that in the second part of this thesis, where we deal with the Stokes equation, the precise definition of equality of two functionals becomes important (c.f. Remark 2.3). Two functionals $\chi, \chi' \in D_z \subset V_z'$ are equal if they coincide on V_z:

$$\langle \chi, w\rangle = \langle \chi', w\rangle \quad \forall w \in V_z.$$

In the situations of [KV21, KVZ], the question of equality is easier to decide, but we have to be more careful. Therefore, we directly develop a more general framework for the construction of the projection.

For the construction of the projection $\mathcal{P}_z$, $z \in \mathcal{I}$, we suppose following Assumptions, cf. [KRVZ25, (H3)]. To simplify our presentation, we postpone condition (ii) of [KRVZ25, (H3)] to Proposition 2.22.

Assumption 2.19. *Assume that Assumption 2.2 is true. Fix $z \in \mathcal{I}$. By Assumption 2.2 there is a local vector space $V_z \subset V$.*

We select a finite-dimensional vector space $S_z \subset V$ of simple (test) functions *and another finite-dimensional vector space $D_z \subset \tilde{V}_z' := (V_z + S_z)'$ of* simple functionals *such that*

$$\dim(S_z) \leqslant \dim(D_z) \quad \text{and} \quad \|\chi\|_{V_z'} \leqslant C_{\overline{\mathtt{com}}}\|\chi\|_{S_z'} \quad \forall \chi \in D_z \tag{2.21}$$

for a constant $C_{\overline{\mathtt{com}}} > 0$. Here, equality of two functionals $\chi, \chi' \in \tilde{V}_z'$ means

$$\langle \chi, w\rangle = \langle \chi', w\rangle \quad \forall w \in \tilde{V}_z.$$

The next Proposition employs Assumption 2.19 to construct $\mathcal{P}_z : S_z' \to D_z$ for a fixed $z \in \mathcal{I}$. The result is a generalization of [KV21] and [KVZ] (compare also with [KRVZ25, section 3.4]).

Proposition 2.20. *Under Assumption 2.19, the projection $\mathcal{P}_z : S_z' \to D_z$ determined via*

$$\forall s_z \in S_z \quad \langle \mathcal{P}_z f, s_z\rangle = \langle f, s_z\rangle \tag{2.22}$$

for all $f \in S_z'$ is well-defined. Its operator norm is given by

$$\|\mathcal{P}_z\|_{\mathbb{L}(S_z',V_z')} = \sup_{\chi\in D_z} \frac{\|\chi\|_{V_z'}}{\|\chi\|_{S_z'}} < \infty. \tag{2.23}$$

Proof. We show that (2.22) is uniquely solvable. For a given $\chi \in D_z$, the second condition in (2.21) ensures that $\|\chi\|_{S_z'} = 0$ implies $\|\chi\|_{V_z'} = 0$ and thus $\|\chi\|_{\tilde{V}_z'} = 0$. Consequently, the system (2.22) has at most one solution. In other words, if we choose a basis $s_z^i \in S_z$, $1 \leqslant i \leqslant \dim(S_z)$, of S_z, then the map $\chi \mapsto (\langle \chi, s_z^i \rangle)_i \in \mathbb{R}^{\dim(S_z)}$ is injective. Thus, $\dim(S_z) = \dim(D_z)$ and with standard results from linear algebra, we conclude that (2.22) has a unique solution.

The expression (2.23) for the operator norm follows from [TV16, Theorem 2.1], see also [BCNV24, Lemma 4.28]. For the readers' convenience, we recall the proof in our setting. For $f \in S_z'$, we have

$$\|\mathcal{P}_z f\|_{V_z'} = \frac{\|\mathcal{P}_z f\|_{V_z'}}{\|\mathcal{P}_z f\|_{S_z'}} \|\mathcal{P}_z f\|_{S_z'} \leqslant \sup_{\chi \in D_z} \frac{\|\chi\|_{V_z'}}{\|\chi\|_{S_z'}} \|\mathcal{P}_z f\|_{S_z'}. \tag{2.24}$$

With (2.22), the last factor satisfies

$$\|\mathcal{P}_z f\|_{S_z'} = \sup_{s_z \in S_z} \frac{\langle \mathcal{P}_z f, s_z \rangle}{\|s_z\|_V} = \sup_{s_z \in S_z} \frac{\langle f, s_z \rangle}{\|s_z\|_V} = \|f\|_{S_z'}. \tag{2.25}$$

Combining (2.24) and (2.25), we get

$$\|\mathcal{P}_z\|_{\mathbb{L}(S_z', V_z')} = \sup_{f \in S_z'} \frac{\|\mathcal{P}_z f\|_{V_z'}}{\|f\|_{S_z'}} \leqslant \sup_{\chi \in D_z} \frac{\|\chi\|_{V_z'}}{\|\chi\|_{S_z'}}. \tag{2.26}$$

Now, we prove the other direction. Let $s_z \in S_z$ be arbitrary. We have with [Lax02, section 8.2, Theorem 6] and (2.22) that

$$\|s_z\|_V = \sup_{f \in S_z'} \frac{\langle f, s_z \rangle}{\|f\|_{S_z'}} = \sup_{f \in S_z'} \frac{\langle \mathcal{P}_z f, s_z \rangle}{\|f\|_{S_z'}} = \sup_{f \in S_z'} \frac{\|\mathcal{P}_z f\|_{V_z'}}{\|f\|_{S_z'}} \frac{\langle \mathcal{P}_z f, s_z \rangle}{\|\mathcal{P}_z f\|_{V_z'}} \leqslant \sup_{f \in S_z'} \frac{\|\mathcal{P}_z f\|_{V_z'}}{\|f\|_{S_z'}} \sup_{\chi \in D_z} \frac{\langle \chi, s_z \rangle}{\|\chi\|_{V_z'}}.$$

After taking the infimum over all $s_z \in S_z$, we can apply Nečas' theorem (see [BCNV24, Theorem 2.8]) because of Assumption 2.19:

$$\begin{aligned} 1 &\leqslant \sup_{f \in S_z'} \frac{\|\mathcal{P}_z f\|_{V_z'}}{\|f\|_{S_z'}} \inf_{s_z \in S_z} \sup_{\chi \in D_z} \frac{\langle \chi, s_z \rangle}{\|\chi\|_{V_z'} \|s_z\|_V} = \sup_{f \in S_z'} \frac{\|\mathcal{P}_z f\|_{V_z'}}{\|f\|_{S_z'}} \inf_{\chi \in D_z} \sup_{s_z \in S_z} \frac{\langle \chi, s_z \rangle}{\|\chi\|_{V_z'} \|s_z\|_V} \\ &= \|\mathcal{P}_z\|_{\mathbb{L}(S_z', V_z')} \inf_{\chi \in D_z} \frac{\|\chi\|_{S_z'}}{\|\chi\|_{V_z'}}. \end{aligned}$$

Therefore, we have shown

$$\|\mathcal{P}_z\|_{\mathbb{L}(S_z', V_z')} = \sup_{f \in S_z'} \frac{\|\mathcal{P}_z f\|_{V_z'}}{\|f\|_{S_z'}} \geqslant \sup_{\chi \in D_z} \frac{\|\chi\|_{V_z'}}{\|\chi\|_{S_z'}}. \tag{2.27}$$

Combining (2.26) and (2.27) yields the result. □

Remark 2.21. The projection $\mathcal{P}_z$ is computable. Since D_z and S_z are assumed to be finite-dimensional, the equation (2.22) is a finite-dimensional linear system. Hence, if $f \in V'$ is exactly evaluable on S_z and the matrix components $\langle \chi, s \rangle$ with $\chi \in D_z$, $s \in S_z$ can be exactly computed, the whole projection operator is computable. Furthermore, for all $\chi \in D_z$ the norm $\|\chi\|_{V_z'}$ is bounded by $\|\mathcal{P}_z\|_{\mathbb{L}(S_z', V_z')} \|\chi\|_{S_z'}$ which is evaluable due to the finite-dimensionality of S_z. Compare also with [KRVZ25, Remark 3.9].

This projection allows us to split the localized residual into two parts: the computable error estimator and the oscillation, which is only computable with suitable a-priori information on $f \in V'$:

$$\|\operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f, u_h)\|_{V_z'} \leqslant \underbrace{\|\mathcal{P}_z f - \tilde{A} u_h\|_{V_z'}}_{\text{estimator}} + \underbrace{\|\mathcal{P}_z f - f\|_{V_z'}}_{\text{oscillation}}. \tag{2.28}$$

Recall that we write $\tilde{A} : \tilde{V} \to \tilde{V}' \subset V'$ for $\tilde{A}(\tilde{v}) = \tilde{a}(\tilde{v}, \cdot)$, $\tilde{v} \in \tilde{V}$.

The crucial insight of [KV21] is that the oscillation term shall be again bounded by the original residual $\|\operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f, u_h)\|_{V_z'}$ to obtain an error-dominated oscillation. As already pointed out, an estimator with this new kind of oscillation cannot overestimate. Such an estimator is constructed in the next Proposition under some additional assumptions. The conditions are more elaborated than the ones in [KV21, KVZ] because we have to handle the case that $S_z \subset V_z^+$ but $S_z \not\subset V_z$.

Proposition 2.22 ([KRVZ25, (H3)])**.** *If the projection $\mathcal{P}_z$ from Proposition 2.20 in addition fulfills*

$$(\tilde{A}V_h)_{|\tilde{V}_z} \subset (D_z)_{|\tilde{V}_z}, \tag{2.29}$$

where $\tilde{A} : \tilde{V} \to \tilde{V}' \subset V'$ is defined as $\tilde{A}(\tilde{v}) := \tilde{a}(\tilde{v}, \cdot)$, then for each $f \in V'$

$$\|\mathcal{P}_z f - f\|_{V_z'} \leqslant (1 + \|\mathcal{P}_z\|_{\mathbb{L}(S_z', V_z')}) \inf_{v_h \in V_h} \|\tilde{A}v_h - f\|_{\tilde{V}_z'}, \tag{2.30a}$$

$$\|\mathcal{P}_z f - f\|_{V_z'} \leqslant (1 + \|\mathcal{P}_z\|_{\mathbb{L}(S_z', V_z')}) \|\operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f, u_h)\|_{\tilde{V}_z'} \tag{2.30b}$$

and combined

$$\|\mathcal{P}_z \operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f, u_h)\|_{V_z'} + \|\mathcal{P}_z f - f\|_{V_z'} \leqslant (1 + 2\|\mathcal{P}_z\|_{\mathbb{L}(S_z', V_z')}) \|\operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f, u_h)\|_{\tilde{V}_z'}. \tag{2.30c}$$

Proof. Together with the invariance (2.29), we obtain for all $v_h \in V_h$

$$\|\mathcal{P}_z f - f\|_{V_z'} = \|(\mathrm{id} - \mathcal{P}_z) f\|_{V_z'} = \|(\mathrm{id} - \mathcal{P}_z)\big(f - \tilde{A}v_h\big)\|_{V_z'} \leqslant (1 + \|\mathcal{P}_z\|_{\mathbb{L}(S_z', V_z')}) \|f - \tilde{A}v_h\|_{\tilde{V}_z'}. \qquad \square$$

If S_z' is not contained in V_z, the estimator and oscillation are only bounded by $\|\operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f, u_h)\|_{\tilde{V}_z'}$ and not by $\|\operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f, u_h)\|_{V_z'}$, which prohibits an equivalence. But the next Corollary provides a mitigation.

Corollary 2.23 ([KRVZ25, (3.8)])**.** *Assume that Assumption 2.2 is true, i.e. there is a finite index set $\mathcal{I}$ and for each $z \in \mathcal{I}$ there is a linear subspace V_z of V. Furthermore, let S_z, $z \in \mathcal{I}$, be linear subspaces of V. Define $\tilde{V}_z := V_z + S_z$. If, in addition, the remaining parts of Assumption 2.15 are fulfilled with $V_z^+ := \tilde{V}_z$, then we have for all $z \in \mathcal{I}$*

$$\|\operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f, u_h)\|_{\tilde{V}_z'}^2 \leqslant C_{\mathsf{P}}^2 \sum_{y \in \mathcal{I}_z} \|\operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f, u_h)\|_{V_y'}^2,$$

where $\mathcal{I}_z \subset \mathcal{I}$, $z \in \mathcal{I}$, and $C_{\mathsf{P}} > 0$ are provided by Assumption 2.15. The constant C_{P} is independent of z.

Proof. The result is a direct consequence of Theorem 2.16. $\square$

If, $S_z \subset V_z$, we can simplify Propositions 2.20 and 2.22.

Corollary 2.24. *If we assume $S_z \subset V_z$ in addition to the setting of Propositions 2.20 and 2.22 then condition* (2.29) *simplifies to*

$$(\tilde{A}V_h)_{|V_z} \subset (D_z)_{|V_z}. \tag{2.31}$$

and we can replace $\tilde{V}_z$ by V_z in the three estimates (2.30)*.*

Proof. As $S_z \subset V_z$, we have $V_z \subset \tilde{V}_z \subset V_z$ and thus $\tilde{V}_z = V_z$. $\square$

Remark 2.25. Up to now, we considered the estimator-oscillation-splitting only for one particular z. In practical computations, however, one has to do this for each $z \in \mathcal{I}$. Accordingly, the stability constant shall be uniformly bounded within $z \in \mathcal{I}$. Hence, it is reasonable to write the second condition in (2.21) as

$$\sup_{z \in \mathcal{I}} \sup_{\chi \in D_z} \frac{\|\chi\|_{V_z'}}{\|\chi\|_{S_z'}} = \sup_{z \in \mathcal{I}} \|\mathcal{P}_z\|_{\mathbb{L}(S_z', V_z')} < \infty. \tag{2.32}$$

We turn to the computation of the estimator $\|\mathcal{P}_z f - \tilde{A}u_h\|_{V_z'}$. Since V_z' is an infinite-dimensional space, we switch to the S_z' norm:

$$\|\mathcal{P}_z f - \tilde{A}u_h\|_{V_z'} \leqslant \|\mathcal{P}_z\|_{\mathbb{L}(S_z', V_z')} \|\mathcal{P}_z f - \tilde{A}u_h\|_{S_z'} \leqslant \|\mathcal{P}_z\|_{\mathbb{L}(S_z', V_z')} \|\mathcal{P}_z f - \tilde{A}u_h\|_{\tilde{V}_z'}.$$

To assess the value of $\|\mathcal{P}_z f - \tilde{A}u_h\|_{S_z'}$, we rewrite it as a sum of test functions evaluations, as suggested in [KVZ].

Lemma 2.26. *Assume the setting of the Assumption 2.19. Let $s_1, \dots, s_n$, $n \in \mathbb{N}$, be a basis of S_z. Let $G = \big(\frac{(s_i, s_j)}{\|s_i\|_{\tilde{V}_z} \|s_j\|_{\tilde{V}_z}}\big)_{1 \leqslant i,j \leqslant n} \in \mathbb{R}^{n \times n}$ be the Gram matrix of this basis with $(\cdot, \cdot)$ the scalar product on S_z. For each $\chi \in D_z$ we have*

$$\|\chi\|_{S_z'}^2 \|G^{-1}\|_{\mathbb{R}^{n \times n}}^2 \leqslant \sum_{i=1}^{n} \frac{\langle \chi, s_i \rangle^2}{\|s_i\|_{\tilde{V}_z}^2} \leqslant \|\chi\|_{S_z'}^2 \|G\|_{\mathbb{R}^{n \times n}}^2.$$

Here $\|\cdot\|_{\mathbb{R}^{n \times n}}$ is the 2-norm of the matrix.

Proof. We can consider the statement as an equivalence of norms on finite-dimensional vector spaces because $\|\cdot\|_{S_z'}$ is a norm on S_z' due to (2.21). The precise constants are a consequence of in [Chr08, Proposition 3.4.5]. $\square$

2.1.4 Summary of the abstract setting

We summarize the abstract construction of the error estimators. The construction consists of three steps, presented in sections 2.1.1, 2.1.2 and 2.1.3, respectively.

(i) We split the error into the conformity error $\|\Pi_V u_h - u_h\|_{\tilde{V}}^2$ and the smoothed error $\|u - \Pi_V u_h\|_{\tilde{V}}^2$. The conformity error is estimated with a quasi-optimal smoother as shown in (2.6) and becomes computable. The smoothed error is rewritten as the conforming residual $\operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f, u_h)$, which is treated in the next two steps.

(ii) To obtain local estimators for the global quantity $\|\operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f, u_h)\|_{V'}$, we want to derive an equivalence to local quantities $\|\operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f, u_h)\|_{V_z'}$, based upon local spaces V_z, $z \in \mathcal{I}$, as introduced in Assumption 2.2. The local space setup is strong enough that Theorem 2.4 directly bounds the local quantities in terms of the global one.

For the other direction, the localization, we need further assumptions: either Assumption 2.6 or 2.12. The latter Assumption only localizes to larger spaces V_z^+. In this case, we apply a local localization, which is provided in Assumption 2.15 and Theorem 2.16. With this additional procedure, we again get the local quantities $\|\operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f, u_h)\|_{V_z'}$.

Regardless which method we apply, the localization uses the quasi-Galerkin orthogonality from Theorem 2.5. If the method is not overconsistent, this Theorem produces an additional conformity error. But we already dealt with the conformity error in step (i), thus it is no problem.

(iii) In general, the local quantities $\|\operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f, u_h)\|_{V_z'}$ are not computable because no further information on $f \in V'$ are available. To overcome this, we project f into a finite-dimensional space called D_z. We call the functionals in D_z *simple functionals* in correspondence with [KRVZ25]. The projection on D_z is implicitly defined via equation (2.22) with the help of the finite-dimensional space S_z of test functions. In [KRVZ25], the functions in S_z are termed *simple functions*. If Assumption 2.19 is fulfilled, then Proposition 2.20 shows that the equation (2.22) is well-posed, and hence the projection $\mathcal{P}_z : S_z' \to D_z$ exists.

The projection $\mathcal{P}_z$ splits the local quantities $\|\operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f, u_h)\|_{V_z'}$ into the estimator and oscillation, see (2.28). To revert this splitting and have an oscillation which is dominated by the error (an error-dominated oscillation), the projection has to satisfy the invariant condition (2.29).

In this case, (2.28), (2.30c) can be combined to an equivalence if $\tilde{V}_z = V_z$. However, if $V_z \subsetneq \tilde{V}_z$, we do not obtain an equivalence, see Theorem 2.16 and Corollary 2.23.

This completes the abstract construction of an error estimator, and we turn to the applications in the next section.

2.2 Localization of the examples

In this section, we apply the abstract localization results of section 2.1.2 to the Poisson problem, the symmetric gradient setting and the biharmonic problem.

2.2.1 Localizations for the Poisson problem

In this subsection, we present localizations for the discretizations of the Poisson problem, which is introduced in section 1.4.2. At first, we consider localizations that rely on $\mathring{S}_1^1(\mathcal{T}) \subset V_h$. Later, we discuss localizations for nonconforming methods which do not use a conforming subspace.

Recall that we choose for the Poisson problem $V := H_0^1(\Omega)$ and $a : V \times V \to \mathbb{R}$ as $a(v, w) = a_{\mathsf{P}}(v, w) = \int_\Omega \nabla v \cdot \nabla w$, $v, w \in V$ together with $\|v\|_V^2 = a(v, v)$ for all $v \in V$. Furthermore, we fix a shape-regular mesh $\mathcal{T}$ as in Definition 1.25 and use the notations introduced in section 1.4. Let $u \in H_0^1(\Omega)$ be the analytic solution of (1.30) and $u_h \in V_h$ the discrete solution of one quasi-optimal method, we presented in section 1.4.2, that are P-LAG, P-CR1, P-CRL, P-SIP, P-NSIP, P-RSIP, P-RNSIP.

The first step in the a-posteriori analysis is to split the error into the conformity error and the smoothed error as depicted in (2.5). In the case of the conforming Lagrange method, we have $\Pi_V u_h = u_h$ for $u_h \in \mathring{S}_\ell^1(\mathcal{T})$, hence the conformity error vanishes.

In the nonconforming cases, we estimate the conformity error with the following proposition.

Proposition 2.27. *If $u_h \in S^0_\ell(\mathcal{T})$ for $\ell \geqslant 1$, we can use $\mathcal{E}_\ell : S^0_\ell(\mathcal{T}) \to H^1_0(\Omega)$ from Definition 1.35 to rewrite the conformity error* $\operatorname{ncf}_{\mathrm{h}} := \|\Pi_V u_h - u_h\|_{\tilde{V}}$ *in the cases P-CR1, P-CRL, P-SIP, P-NSIP, P-RSIP, P-RNSIP as*

$$\|\Pi_V u_h - u_h\|_{\tilde{V}} \leqslant \|\mathcal{E}_\ell u_h - u_h\|_{\tilde{V}} \leqslant C^{\tilde{V}}_{\mathsf{qopt}}(\mathcal{E}_\ell)\|\Pi_V u_h - u_h\|_{\tilde{V}}. \tag{2.33}$$

The quasi-optimality constant $C^{\tilde{V}}_{\mathsf{qopt}}(\mathcal{E}_\ell)$ of the smoother $\mathcal{E}_\ell$ potentially depends on the norm used on $\tilde{V}$. In the cases P-CR1 and P-CRL, we have

$$\|\nabla_h(\mathcal{E}_\ell u_h - u_h)\|^2_{L^2(\Omega)} \lesssim \sum_{F\in\mathcal{F}} \frac{1}{h_F}\int_F [\![u_h]\!]^2 \lesssim \|\nabla_h(\mathcal{E}_\ell u_h - u_h)\|^2_{L^2(\Omega)}. \tag{2.34}$$

In the cases P-SIP, P-NSIP, P-RSIP, P-RNSIP, we have the equivalence

$$\frac{\gamma}{1+\gamma}\|\mathcal{E}_\ell u_h - u_h\|^2_{\mathsf{dG}} \lesssim \sum_{F\in\mathcal{F}} \frac{\gamma}{h_F}\int_F [\![u_h]\!]^2 \leqslant \|\mathcal{E}_\ell u_h - u_h\|^2_{\mathsf{dG}}. \tag{2.35}$$

Proof. Estimate (2.33) is shown in (1.20) because (1.55b) shows that $\mathcal{E}_\ell$ is a quasi-optimal smoother.

In the cases P-CR1 and P-CRL, the estimate (2.34) is obtained by Corollary 1.41, which shows that (1.56a) holds. Since $\int_F [\![v_h]\!] = 0$ for all $v_h \in \mathrm{CR}_\ell$ and $F \in \mathcal{F}$, $F \in \mathcal{F}^i$, we obtain with (1.56a):

$$\begin{aligned}\frac{1}{h_F}\int_F [\![u_h - \mathcal{E}_\ell u_h]\!]^2 &\lesssim \frac{1}{h_F}\big(\|u_{h|T_1} - (\mathcal{E}_\ell u_h)_{|T_1}\|^2_{L^2(F)} + \|u_{h|T_2} - (\mathcal{E}_\ell u_h)_{|T_2}\|^2_{L^2(F)}\big)\\ &\lesssim \|\nabla_h(u_h - v)\|^2_{L^2(\omega_F^+)}\end{aligned}$$

with $\mathcal{T}_F = \{T_1, T_2\} \subset \mathcal{T}$ for any $v \in H^1_0(\Omega)$. If $F \in \mathcal{F}\backslash\mathcal{F}^i$ with $\mathcal{T}_F = \{T\} \subset \mathcal{T}$, a similar calculation yields again with (1.56b):

$$\frac{1}{h_F}\int_F [\![u_h - \mathcal{E}_\ell u_h]\!]^2 \lesssim \frac{1}{h_F}\|u_{h|T} - (\mathcal{E}_\ell u_h)_{|T}\|^2_{L^2(F)} \lesssim \|\nabla_h(u_h - v)\|^2_{L^2(\omega_F^+)}. \tag{2.36}$$

for any $v \in H^1_0(\Omega)$. With $v = \mathcal{E}_\ell u_h$ and summing over all $F \in \mathcal{F}$ finishes the proof of (2.34).

The first part of estimate (2.35) also follows from Corollary 1.41 using (1.55b). The second bound follows from the fact that the jump term is already contained in $\|\cdot\|_{\mathsf{dG}}$. □

Remark 2.28. Proposition 2.27 is still true if we replace $\mathcal{E}_\ell$ with $\mathcal{A}_\ell$.

Having discussed the estimates of the conformity error, we start with the localization of the residual $\langle \operatorname{res}^{\mathrm{c}}_{\mathrm{h}}(f, u_h), w\rangle = \langle f, w\rangle - a(\Pi_V u_h, w)$, $w \in V$. At first, we present methods which are applicable if V_h contains the conforming subspace $\mathring{S}^1_1(\mathcal{T})$. After that, we consider methods whose discrete spaces contain CR_1. Finally, we present localization methods which need $S^0_0(\mathcal{T}) \subset V_h$.

Remark 2.29. In the case of Poisson discretizations, the approximability of the space $S^0_\ell(\mathcal{T})$ is up to a constant the same as $\mathring{S}^1_\ell(\mathcal{T})$, which is the main result of [Vee16]. This means the practical benefits of these nonconforming methods for the Poisson problem are limited. We should see these methods more in the light of a model problem and study the properties of nonconforming methods in this easy setting to learn how to deal with more complicated situations.

There are examples where the nonconformity is essential. Certain finite element pairs for the Stokes equation become unstable if the velocity space is restricted to the H^1-subspace. In the biharmonic setting H^2-conforming spaces are difficult to construct and the conforming subspace in nonconforming settings might be trivial, see sections 1.4.4 and 3.2.

Since these considerations for the Poisson problem shall be a model for more general settings, it is doubtful if the reliance on a conforming subspace is actually sensible as this does not generalize to more sophisticated settings. This motivates us to present also localization methods which do not rely on this.

But the reader shall keep in mind that the approximations (2.9) and (2.13b) actually take place in the conforming space. Therefore, the question whether the conforming discrete functions are still powerful enough for approximation is less relevant. It is more relevant if the image of the smoothing operator $\mathrm{E_M}$ is rich enough to approximate any conforming function and if the inverse of $\mathrm{E_M}$ is bounded in an acceptable manner.

We set $\mathcal{I} := \mathcal{V}$ and define the local spaces as $V_z = H^1_0(\omega_z)$. The next Proposition shows that these spaces fulfill Assumption 2.2.

Proposition 2.30. *The spaces $V_z = H_0^1(\omega_z)$, $z \in \mathcal{I} = \mathcal{V}$, fulfill Assumption 2.2 thus relation* (2.7) *is true. In particular, Theorem 2.4 holds for the linear functional* $\operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f, u_h)$ *with $u_h \in V_h$ the discrete solution of the considered method for the Poisson problem.*

Proof. Choose for each $z \in \mathcal{I} = \mathcal{V}$ a $w_z \in V_z = H_0^1(\omega_z)$. Define $C(\sigma(\mathcal{T})) := \sup_{x\in\Omega} \#\{z \in \mathcal{I} | x \in \omega_z\}$. The constant $C(\sigma(\mathcal{T}))$ is finite and depends only on $\sigma(\mathcal{T})$ owing to the finite overlap property (1.22) of the patches ω_z. Thus, by Jensen's inequality we obtain

$$\Big\| \sum_{z\in\mathcal{I}} w_z \Big\|_V^2 = \int_\Omega \Big(\sum_{z\in\mathcal{I}} |\nabla w_z| \Big)^2 \leqslant C(\sigma(\mathcal{T})) \int_\Omega \sum_{z\in\mathcal{I}} |\nabla w_z|^2 = C(\sigma(\mathcal{T})) \sum_{z\in\mathcal{I}} \|w_z\|_V^2. \tag{2.37}$$

Hence, Assumption 2.2 is fulfilled with a C_{loc} depending only on $\sigma(\mathcal{T})$ and Theorem 2.4 is applicable. □

After the globalization is settled, we show that $\|\operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f, u_h)\|_{V'}^2$ is localizable to $\sum_{z\in\mathcal{V}} \|\operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f, u_h)\|_{\tilde{V}_z'}^2$. The abstract setting provides two localization procedures in Theorems 2.11 and 2.14; we start with the former. Theorem 2.11 requires that Assumption 2.6 is fulfilled which we will show now with the help of Corollary 2.7.

Corollary 2.7 requires a localization operator. We build one with a partition of unity, for which we choose the first order Lagrange hat functions $\Phi_z^1 \in S_1^1(\mathcal{T})$, $z \in \mathcal{V}$, satisfying $\operatorname{supp}(\Phi_z^1) = \omega_z$. The localization operators $\Theta_z : V \to V_z$ are then defined as

$$\Theta_z : H_0^1(\Omega) \to H_0^1(\omega_z) \qquad w\Theta_z := w\Phi_z^1, \quad w \in V = H_0^1(\Omega),\ z \in \mathcal{V}. \tag{2.38}$$

The next part of Corollary 2.7 supposes an interpolation operator $\mathcal{I}_h : V \to V_h$. Since all discrete spaces, which we consider for the Laplace problem, contain the conforming Lagrange space $\mathring{S}_1^1(\mathcal{T})$, we choose an approximation operator $\mathcal{I}_h : H_0^1(\Omega) \to S_1^1(\mathcal{T})$ mapping into that space. Then, we can benefit from the simplification in Remark 2.10. We do not use the Lagrange interpolation since it is not well-defined on $w \in H_0^1(\Omega)$. Thus, we choose the Scott-Zhang interpolation operator, originally introduced in [SZ90].

Proposition 2.31. *The Scott-Zhang operator $\Pi_{\mathsf{SZ}} : H_0^1(\Omega) \to \mathring{S}_1^1(\mathcal{T})$ fulfills the assumptions in Corollary 2.7 with the localization operator defined in* (2.38)*, which are*

$$\begin{aligned} \|\Pi_{\mathsf{SZ}} w\|_{V_h} &\leqslant C_{\mathsf{I}} \|\nabla w\|_{L^2(\Omega)} \qquad \forall w \in V \quad \textit{and} \\ \sum_{z\in\mathcal{I}} \|\nabla\big((w - \mathrm{E_M}\,\Pi_{\mathsf{SZ}} w)\Phi_z^1\big)\|_{L^2(\Omega)}^2 &\leqslant C_{\mathsf{apx}}^2 \|\nabla w\|_{L^2(\Omega)}^2 \qquad \forall w \in V \end{aligned}$$

for appropriate $C_{\mathsf{I}} > 0$, $C_{\mathsf{apx}} > 0$. Here, $\mathrm{E_M}$ is the smoother of one of the methods for the Poisson problem.

Proof. By the definition of $\|\cdot\|_{V_h}$ and the continuity of a_h we have $\|\Pi_{\mathsf{SZ}} w\|_{V_h} \lesssim \|\nabla \Pi_{\mathsf{SZ}} w\|_{L^2(\Omega)}$. Then the first estimate follows from [SZ90, (4.5)]. For the second estimate, we need the following property of the Lagrange hat functions with a constant depending on $\sigma(\mathcal{T})$ (see also [KV21])

$$h_z \|\nabla \Phi_z^1\|_{L^\infty(\omega_z)} \leqslant C \|\Phi_z^1\|_{L^\infty(\omega_z)} \leqslant C. \tag{2.40}$$

All methods for the Poisson problem use smoothers $\mathrm{E_M}$ which are invariant under $\mathring{S}_1^1(\mathcal{T})$. Combining this with the approximability of the Scott-Zhang operator ([SZ90, (4.3)]), we have the estimate

$$\begin{aligned} &\|\nabla\big((\mathrm{E_M}(\Pi_{\mathsf{SZ}} w) - w)\Phi_z^1\big)\|_{L^2(\Omega)} = \|\nabla\big((\Pi_{\mathsf{SZ}} w - w)\Phi_z^1\big)\|_{L^2(\Omega)} \\ &\quad \leqslant C\big(\|\Pi_{\mathsf{SZ}} w - w\|_{L^2(\omega_z)} \|\nabla \Phi_z^1\|_{L^\infty(\omega_z)} + \|\nabla(\Pi_{\mathsf{SZ}} w - w)\|_{L^2(\omega_z)} \|\Phi_z^1\|_{L^\infty(\omega_z)}\big) \leqslant C \|\nabla w\|_{L^2(\omega_z^+)}. \end{aligned}$$

The enlarged patches ω_z^+ are defined in Definition 1.26. The number of patches ω_z^+ which contain a fixed $T \in \mathcal{T}$ is bounded by a constant depending only on the shape regularity of $\mathcal{T}$, see (1.22). Therefore, we obtain the result by taking the squared sum over all $z \in \mathcal{I}$. □

Proposition 2.31 finalizes the proof of the Assumptions of Corollary 2.7 such that Theorem 2.11 is applicable. Thus, the residual $\operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f, u_h)$ fulfills (2.11). As already mentioned, [VZ19a] shows that the P-LAG, P-CR1 and P-CRL are overconsistent, meaning $\delta_{V_h} = 0$. Consequently, the conformity term in the localization vanishes in these cases.

We can combine the localization and the globalization.

Proposition 2.32. *The solution $u_h \in V_h$ of one of the methods P-LAG, P-CR1, P-CRL, P-SIP, P-NSIP, P-RSIP, P-RNSIP, fulfills*

$$\begin{aligned}\|\operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f,u_h)\|_{H^{-1}(\Omega)}^2 &\lesssim \sum_{z\in\mathcal{I}} \|\operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f,u_h)\|_{H^{-1}(\omega_z)}^2 + \delta_{V_h}^2 \|u_h - \Pi_V u_h\|_{\tilde{V}}^2 \\ &\lesssim \|\operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f,u_h)\|_{H^{-1}(\Omega)}^2 + \delta_{V_h}^2 \|u_h - \Pi_V u_h\|_{\tilde{V}}^2.\end{aligned}$$

The first hidden constant depends on the constants C_{I} and C_{apx} provided by Proposition 2.31; the second hidden constant depends only on the shape regularity $\sigma(\mathcal{T})$ of the mesh. The conformity error $\operatorname{ncf}_{\mathrm{h}} := \|u_h - \Pi_V u_h\|_{\tilde{V}}^2$ *can be estimated with Proposition 2.27.*

Proof. Because of Proposition 2.31, we can apply Theorem 2.11 and get the first inequality. The second bound follows since Proposition 2.30 shows that Assumption 2.2 is valid in the current setting. Therefore, Theorem 2.4 shows $\sum_{z\in\mathcal{I}} \|\operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f,u_h)\|_{H^{-1}(\omega_z)}^2 \lesssim \|\operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f,u_h)\|_{H^{-1}(\Omega)}^2$. The rest follows by combining the local quantities to global ones and using the finite overlap property of the grid. □

Next, we present an example of the second approach of localization provided by Theorem 2.14. This way has been used in [KV21] and [CDN12]. As already pointed out, this method has the drawback that we do not localize to $H^{-1}(\omega_z)$ but to $H^{-1}(\omega_z^+)$. This, however, can be mitigated by the local localization in Theorem 2.16, which is applicable in this setting. More on this later.

To apply the corresponding Theorem 2.14, we need to verify Assumption 2.12. We again select the partition of unity provided by the first order Lagrange hat functions Φ_z^1, $z \in \mathcal{V}$. We do not need the full power of the abstract setting as we can choose $\mathrm{E_R} := \mathrm{E_M}$ and independent of the actual $w \in V$.

Theorem 2.33 ([KV21, Lemma 4])**.** *All considered discretizations for the Laplace equation fulfill Assumption 2.12 with $V_z := H_0^1(\omega_z)$, $V_z^+ := H_0^1(\omega_z^+)$ $z \in \mathcal{I} = \mathcal{V}$. Hence, Theorem 2.14 is applicable.*

Proof. Fix a $w \in V$ with $\|w\|_V = 1$. We choose $\mathcal{I} = \mathcal{V}$ and $\mathbb{P}_z = \mathbb{P}_0(\Omega)$ (constant functions over Ω) for all $z \in \mathcal{V}$. The Lagrange hat functions Φ_z^1, $z \in \mathcal{V}$, define the localization operators:

$$\Theta_z^a : V \to V_z, \quad v\Theta_z^a := v\Phi_z^1 \in H_0^1(\omega_z), \qquad \text{for all } z \in \mathcal{V} \text{ and } v \in H_0^1(\Omega), \tag{2.41a}$$

$$\Theta_z^b : \mathbb{P}_0(\Omega) \to V_h, \quad p_z\Theta_z^b = p_z\Phi_z^1 \in S_1^1(\mathcal{T}_z), \qquad \text{for all } z \in \mathcal{V} \text{ and } p_z \in \mathbb{P}_0(\Omega). \tag{2.41b}$$

The localization operator Θ_z^a directly fulfills point (i) of Assumption 2.12.

We define $p_z := \frac{1}{|\omega_z|}\int_{\omega_z} w \in \mathbb{P}_0(\Omega)$ for all $z \in \mathcal{V} \cap \Omega$ and $p_z = 0 \in \mathbb{P}_0(\Omega)$ for all $z \in \mathcal{V} \cap \partial\Omega$. With these choices, point (ii) of Assumption 2.12 is satisfied. After setting $\mathrm{E_R} := \mathrm{E_M}$, also (iii) is valid. It remains to show (2.13a) and (2.13b).

To show (2.13a), we bound the V_h-norm by the $\tilde{V}$-norm via

$$\|w_h\|_{V_h} = \sup_{v_h \in V_h} \frac{a_h(v_h, w_h)}{\|v_h\|_{\tilde{V}}} \lesssim \|w_h\|_{\tilde{V}}, \tag{2.42}$$

where we have exploited that a_h is continuous in all considered methods. Therefore,

$$\Big\| \sum_{z\in\mathcal{I}} p_z\Phi_z^1 \Big\|_{V_h} \lesssim \Big\| \sum_{z\in\mathcal{I}} w\Phi_z^1 - p_z\Phi_z^1 \Big\|_{\tilde{V}} + \|w\|_{\tilde{V}}. \tag{2.43}$$

Since at each point only finitely many summands are non-zero (1.22), we can employ Jensen's inequality on the first term to show

$$\begin{aligned}\Big\|\nabla\Big(\sum_{z\in\mathcal{I}} w\Phi_z^1 - p_z\Phi_z^1\Big)\Big\|_{L^2(\Omega)}^2 &\lesssim \sum_{z\in\mathcal{I}} \|\nabla\big((w-p_z)\Phi_z^1\big)\|_{L^2(\omega_z)}^2 \\ &\lesssim \sum_{z\in\mathcal{I}} \big(\|\nabla w\|_{L^2(\omega_z)}^2 + \frac{1}{h_z^2}\|w - p_z\|_{L^2(\omega_z)}^2\big) \\ &\lesssim \sum_{z\in\mathcal{I}} \|\nabla w\|_{L^2(\omega_z)}^2 \lesssim \|\nabla w\|_{L^2(\Omega)}^2.\end{aligned} \tag{2.44}$$

We used the scaling properties in (2.40) and finished with Poincaré's inequality to show (2.13a).

It remains to show (2.13b). At first, we simplify the expression by using the invariance of $\mathrm{E_R} = \mathrm{E_M}$ under $\mathring{S}_1^1(\mathcal{T})$. Then, similar to (2.44), we obtain with Poincaré's inequality:

$$\sum_{z\in\mathcal{I}} \|\nabla\big(w\Phi_z^1 - p_z\Phi_z^1\big)\|_{L^2(\omega_z)}^2 \lesssim \sum_{z\in\mathcal{I}} 2\big(\|\nabla w\|_{L^2(\omega_z)}^2 + \frac{1}{h_z^2}\|w - p_z\|_{L^2(\omega_z)}^2\big) \lesssim C\|\nabla w\|_{L^2(\Omega)}^2. \qquad \square$$

Remark 2.34. In both localization methods, we only need a smoother which is invariant on $\mathring{S}^1_1(\mathcal{T})$ and coincides with the smoother of the method on this space. The smoothers of all considered examples of the Poisson problem in section 1.4.2 fulfill this property, even the one for P-RSIP, P-RNSIP.

The preceding Theorem 2.33 only localizes to the spaces $H^{-1}(\omega_z^+)$. The next Proposition shows that we can perform a local localization. We will also need this local localization in section 2.3. The argument is similar to [BCNV24, Lemma 4.8].

Proposition 2.35. *All discussed methods for the Poisson problem satisfy Assumption 2.15 with* (2.16) *fulfilled for a constant $C_{\mathsf{P}} > 0$. Consequently, for any $f \in H^{-1}(\Omega)$ and $z \in \mathcal{V}$, we have by Theorem 2.16 that*

$$\|f\|^2_{H^{-1}(\omega_z^+)} \leqslant C_{\mathsf{P}}^2 \sum_{y \in \mathcal{I}_z} \|f\|^2_{H^{-1}(\omega_y)},$$

where the local index sets $\mathcal{I}_z \subset \mathcal{I} := \mathcal{V}$ are defined as $\mathcal{I}_z := \mathcal{V} \cap \omega_z^+$, $z \in \mathcal{V}$.

Proof. We apply Assumption 2.15 with $V_z^+ := H^1_0(\omega_z^+)$. Due to the finite overlap property (1.22), there exists a $C_{\mathsf{N}} > 0$ such that $\#(\mathcal{V} \cap \omega_z^+) \leqslant C_{\mathsf{N}}$ for all $z \in \mathcal{V}$.

Now, fix a vertex $z \in \mathcal{V}$. For each $y \in \mathcal{I}_z := \mathcal{V} \cap \omega_z^+$, we define the localization operator $\Theta_y : V_z^+ \to V_y \cap V_z^+$ similar to (2.41a) as

$$\Theta_y : H^1_0(\omega_z^+) \to H^1_0(\omega_y) \cap H^1_0(\omega_z^+) \qquad w\Theta_y := w\Phi^1_y \quad \forall w \in H^1_0(\omega_z^+).$$

Then (2.16) follows for all $w \in H^1_0(\omega_z^+)$ by Poincaré's inequality:

$$\begin{aligned} \sum_{y \in \mathcal{I}_z} \|\nabla(w\Phi^1_y)\|^2_{L^2(\omega_y)} &\lesssim \sum_{y \in \mathcal{I}_z} \|\nabla w\|^2_{L^2(\omega_y)} + \frac{1}{h_z^2}\|w\|^2_{L^2(\omega_y)} \\ &\lesssim \|\nabla w\|^2_{L^2(\omega_z^+)} + \frac{1}{h_z^2}\|w\|^2_{L^2(\omega_z^+)} \lesssim \|\nabla w\|^2_{L^2(\omega_z^+)}. \end{aligned} \tag{2.45}$$

□

Both localization methods for the Poisson setting introduced so far can be also applied to conforming P-LAG. Next, we present localizations which are not applicable to P-LAG because they do not use the conforming space $V \cap V_h$, see Remark 2.29.

We start with a localization for P-CR1, using the CR_1-interpolation operator.

Definition 2.36 ([CR73, (5.6)])**.** *We define the CR_1-interpolation operator $\Pi_{\mathsf{CR1}} : H^1_0(\Omega) \to \mathrm{CR}_1$ via*

$$(\Pi_{\mathsf{CR1}} v)(m_F) = \frac{1}{|F|}\int_F v \quad \forall F \in \mathcal{F}^i, v \in H^1_0(\Omega),$$

where m_F are the barycenters of $F \in \mathcal{F}^i$, which form a set of degrees of freedom for CR_1, see (1.32)*.*

The properties of the operator are laid out in the next Proposition.

Proposition 2.37 ([Bre15, (2.14)/(2.15)], [CR73, (5.7)/(5.8)])**.** *The CR_1-interpolation operator satisfies for all $T \in \mathcal{T}$ and $w \in H^1_0(\Omega)$*

$$\|w - \Pi_{\mathsf{CR1}} w\|_{L^2(T)} + h_T\|\nabla(w - \Pi_{\mathsf{CR1}} w)\|_{L^2(T)} \lesssim h_T\|\nabla w\|_{L^2(T)}. \tag{2.46}$$

Moreover, we have for all $F \in \mathcal{F}$ and $w \in H^1_0(\Omega)$ with the trace theorem the bound

$$\int_F \frac{1}{h_F}|[\![w - \Pi_{\mathsf{CR1}} w]\!]|^2 \lesssim \sum_{T \in \mathcal{T}_F} \frac{1}{h_T^2}\|w - \Pi_{\mathsf{CR1}} w\|^2_{L^2(T)} + \|\nabla(w - \Pi_{\mathsf{CR1}} w)\|^2_{L^2(T)} \lesssim \|\nabla w\|^2_{L^2(\omega_F)}. \tag{2.47}$$

With these properties, we can use Π_{CR1} as a replacement for the Scott-Zhang operator to obtain an alternative proof for the assumptions in Corollary 2.7.

Corollary 2.38 ([KRVZ25, Lemma 4.3])**.** *The operator $\Pi_{\mathsf{CR1}} : H^1_0(\Omega) \to \mathrm{CR}_1$ fulfills the assumptions of Corollary 2.7 with the localization operator defined in* (2.38) *for the method P-CR1, P-SIP, P-NSIP, P-RSIP and P-RNSIP if $\ell = 1$, hence Theorem 2.11 is applicable for these methods.*

Proof. To show the stability in (2.10), we recall that Π_{CR1} fulfills in combination with the bilinear form $\tilde{a}^{\mathsf{CR}}(\cdot,\cdot)$ (see [Bre15, Remark 2.5]):

$$\tilde{a}^{\mathsf{CR}}(\Pi_{\mathsf{CR1}} w, v_h) = \tilde{a}^{\mathsf{CR}}(w, v_h), \quad w \in H^1_0(\Omega) \quad \text{and} \quad v_h \in \mathrm{CR}_1\,.$$

This property is shown by an integration by parts and Lemma 1.30. Now, one immediately has for any $T \in \mathcal{T}$ and $w \in H^1_0(\Omega)$ that

$$\|\nabla \Pi_{\mathsf{CR1}} w\|_{L^2(T)} \leqslant \|\nabla w\|_{L^2(T)}.$$

In the case of the dG-methods, the jump terms are bounded with the help of (2.47):

$$\int_F \frac{\gamma}{h_F} [\![\Pi_{\mathsf{CR1}} w]\!]^2 \lesssim \int_F \frac{\gamma}{h_F} [\![\Pi_{\mathsf{CR1}} w - w]\!]^2 \lesssim \gamma \|\nabla w\|^2_{L^2(\omega_F)}.$$

This shows the stability in (2.10) for all methods.

To show the approximation property in (2.10), we split the expression into two parts:

$$\|\nabla\big((\mathcal{E}_1(\Pi_{\mathsf{CR1}} w) - w)\Phi^1_z\big)\|_{L^2(\Omega)} \leqslant \|\nabla_h\big((\mathcal{E}_1(\Pi_{\mathsf{CR1}} w) - \Pi_{\mathsf{CR1}} w)\Phi^1_z\big)\|_{L^2(\Omega)} + \|\nabla_h\big((\Pi_{\mathsf{CR1}} w - w)\Phi^1_z\big)\|_{L^2(\Omega)}.$$

Note that the jump terms of the dG methods directly vanish because all involved functions on the left-hand side are in $H^1_0(\Omega)$. The first summand is estimated with Corollary 1.41 and the stability of Π_{CR1}:

$$\begin{aligned}
&\|\nabla_h\big((\mathcal{E}_1(\Pi_{\mathsf{CR1}} w) - \Pi_{\mathsf{CR1}} w)\Phi^1_z\big)\|_{L^2(\Omega)} \\
&\quad\lesssim \|\nabla_h(\mathcal{E}_1(\Pi_{\mathsf{CR1}} w) - \Pi_{\mathsf{CR1}} w)\|_{L^2(\omega_z)} + \frac{1}{h_z}\|\mathcal{E}_1(\Pi_{\mathsf{CR1}} w) - \Pi_{\mathsf{CR1}} w\|_{L^2(\omega_z)} \\
&\quad\lesssim \|\nabla_h \Pi_{\mathsf{CR1}} w\|_{L^2(\omega_z^+)} \leqslant \|\nabla w\|_{L^2(\omega_z^+)}.
\end{aligned}$$

The second summand is estimated with the approximability of Π_{CR1}:

$$\|\nabla_h\big((\Pi_{\mathsf{CR1}} w - w)\Phi^1_z\big)\|_{L^2(\Omega)} \lesssim \|\nabla_h(\Pi_{\mathsf{CR1}} w - w)\|_{L^2(\omega_z)} + \frac{1}{h_z}\|\Pi_{\mathsf{CR1}} w - w\|_{L^2(\omega_z)} \lesssim \|\nabla w\|_{L^2(\omega_z)}.$$

□

The next method of localization relies on approximation by discontinuous functions. Here, we directly prove Assumption 2.6 and do not rely on Corollary 2.7.

Theorem 2.39. *The methods P-SIP, P-NSIP, P-RSIP, P-RNSIP fulfill for any $\ell \geqslant 1$ Assumption 2.6 with the localization operator defined in* (2.38).

Proof. Given $w \in H^1_0(\Omega)$, we select $w_h \in S^0_0(\mathcal{T})$ with $w_{h|T} = \frac{1}{|T|}\int_T w$ for all $T \in \mathcal{T}$. The norm $\|w_{h|T}\|_{V_h}$ is bounded by $\|w\|_V$. To see this, we apply (2.42), the trace theorem and Poincaré's inequality:

$$\|w_h\|^2_{V_h} \lesssim \sum_{F\in\mathcal{F}} \int_F \frac{\gamma}{h_F} [\![w_h]\!]^2 \lesssim \sum_{T\in\mathcal{T}} \frac{1}{h_T^2}\|w - w_h\|^2_{L^2(T)} + \|\nabla w\|^2_{L^2(T)} \lesssim \|w\|^2_V.$$

To show (2.9), we compute:

$$\|\nabla\big((w - \mathcal{E}_\ell w_h)\Phi_z\big)\|^2_{L^2(\omega_z)} \leqslant \|\nabla(w - \mathcal{E}_\ell w_h)\|^2_{L^2(\omega_z)} + \frac{1}{h_z^2}\|w - \mathcal{E}_\ell w_h\|^2_{L^2(\omega_z)}.$$

We insert $w_h \in S^0_0(\mathcal{T})$ and get

$$\begin{aligned}
\|\nabla\big((w - \mathcal{E}_\ell w_h)\Phi_z\big)\|^2_{L^2(\omega_z)} &\leqslant \|\nabla_h(\mathcal{E}_\ell w_h - w_h)\|^2_{L^2(\omega_z)} + \frac{1}{h_z^2}\|\mathcal{E}_\ell w_h - w_h\|^2_{L^2(\omega_z)} \\
&\quad + \|\nabla_h(w - w_h)\|^2_{L^2(\omega_z)} + \frac{1}{h_z^2}\|w - w_h\|^2_{L^2(\omega_z)}.
\end{aligned}$$

An application of (1.55b) provides us with

$$\begin{aligned}
\|\nabla_h(\mathcal{E}_\ell w_h - w_h)\|^2_{L^2(\omega_z)} + \frac{1}{h_z^2}\|\mathcal{E}_\ell w_h - w_h\|^2_{L^2(\omega_z)} &\lesssim \sum_{F\in\mathcal{F}_z^{i,+}} \frac{1}{h_F}\int_F [\![w - w_h]\!]^2 \\
&\lesssim \sum_{T\in\mathcal{T}_z^+} \frac{1}{h_z^2}\|w - w_h\|^2_{L^2(T)} + \|\nabla_h(w - w_h)\|^2_{L^2(T)}.
\end{aligned}$$

We conclude with Poincaré's inequality:

$$\|\nabla\big((w - \mathcal{E}_\ell w_h)\Phi_z\big)\|^2_{L^2(\omega_z)} \lesssim \sum_{T\in\mathcal{T}_z^+} \frac{1}{h_z^2}\|w - w_h\|^2_{L^2(T)} + \|\nabla_h(w - w_h)\|^2_{L^2(T)} \lesssim \|\nabla w\|^2_{L^2(\omega_z^+)}.$$ □

2.2.2 Localization of the Crouzeix-Raviart method for linear elasticity

In this section we show that the residual of the symmetric gradient method E-CR1, presented in section 1.4.3, is localizable. For this, we recall the setting of E-CR1. The function spaces are $V = H^1_0(\Omega)^d$, $V_h = \mathrm{CR}^d_1$ and $\tilde{V} = H^1_0(\Omega)^d + \mathrm{CR}^d_1$, but the scalar product $\tilde{a}^{\mathsf{LE}}$ is given by

$$\tilde{a}^{\mathsf{LE}}(\tilde{v},\tilde{w}) = \int_\Omega 2\mu\varepsilon_h(\tilde{v}) : \varepsilon_h(\tilde{w}) + \lambda \operatorname{div}_h \tilde{v} \operatorname{div}_h \tilde{w} + \int_\Sigma \frac{\mu\gamma}{h} [\![\tilde{v}]\!] \cdot [\![\tilde{w}]\!], \quad \tilde{v},\tilde{w} \in \tilde{V},$$

with a stabilization parameter $\gamma > 0$ and material constants μ and λ. The scalar product $\tilde{a}^{\mathsf{LE}}$ induces the norm $\|\cdot\|_{\tilde{V}} = \|\cdot\|_{\mathsf{LE}}$.

Let $u \in H^1_0(\Omega)^d$ be the analytic solution of (1.60) and $u_h \in \mathrm{CR}^d_1$ the discrete solution of (E-CR1).

Remark 2.40. The hidden constants in the section may depend on the parameters μ, λ and γ. Although it would be beneficial if the constants did not depend on these parameters, we remark that already the quasi-optimality constant of the method deteriorates if λ tends to infinity. Compare with [VZ18b, Remark 3.14].

We split the error into the smoothed error and the conformity error with the $\tilde{a}$-orthogonal projection Π_V as in (2.5):

$$\|u - u_h\|^2_{\tilde{V}} = \|u - \Pi_V u_h\|^2_V + \|\Pi_V u_h - u_h\|^2_{\tilde{V}}.$$

The vector-valued smoothing operator $\mathcal{E}^d_1$ is quasi-optimal in the norm $\|\cdot\|_{\mathsf{LE}}$ which is shown by applying (1.55b) componentwisely:

$$\|\mathcal{E}^d_1 v_h - v_h\|^2_{\mathsf{LE}} \lesssim (2\mu+\lambda)\|\nabla_h(\mathcal{E}^d_1 v_h - v_h)\|^2_{L^2(\Omega)} + \sum_{F\in\mathcal{F}} \frac{\mu\gamma}{h_F}\int_F [\![v_h]\!]^2 \lesssim \sum_{F\in\mathcal{F}} \frac{2\mu+\lambda+\mu\gamma}{h_F}\int_F [\![v_h]\!]^2. \tag{2.48}$$

The jump term can now be estimated with (1.56a) and the properties of CR_1 as

$$\sum_{F\in\mathcal{F}}\int_F \frac{1}{h_F}[\![v_h]\!]^2 \lesssim \sum_{T\in\mathcal{T}} \|\nabla_h(\mathcal{E}^d_1 v_h - v_h)\|^2_{L^2(T)}.$$

We apply Korn's inequality for piecewise-functions with zero boundary values ([Bre04, Theorem 3.1]) and obtain $\|\nabla_h(\mathcal{E}^d_1 v_h - v_h)\|^2_{L^2(\Omega)} \lesssim \|\varepsilon_h(\mathcal{E}^d_1 v_h - v_h)\|^2_{L^2(\Omega)} + \sum_{F\in\mathcal{F}}\int_F \frac{1}{h_F}[\![\mathcal{E}^d_1 v_h - v_h]\!]^2$ with a constant only depending on $\sigma(\mathcal{T})$ and independent of the size of Ω. Adding the term $\|\operatorname{div}_h(\mathcal{E}^d_1 v_h - v_h)\|^2_{L^2(\Omega)}$ provides the quasi-optimality of $\mathcal{E}^d_1$ in this setting. The constant $C_{\mathsf{qopt}}(\mathcal{E}^d_1)$ nevertheless depends on μ, λ, γ.

Turning to the localization, we first check the general Assumption 2.2. The index set $\mathcal{I}$ is chosen as the vertices $\mathcal{V}$ of the mesh $\mathcal{T}$ and the local spaces are $H^1_0(\omega_z)^d$, $z \in \mathcal{V}$. Condition (2.7) is a consequence of Jensen's inequality, similar to (2.37) in Proposition 2.30 owing to the finite overlap of ω_z, $z \in \mathcal{V}$, see (1.22). Therefore, Theorem 2.4 is valid in this setting E-CR1.

The next Theorem shows that we can also localize the conforming residual in this setting.

Theorem 2.41. *The method E-CR1 fulfills the assumptions of Corollary 2.7 and thus Theorem 2.11 provides a localization for $\|\operatorname{res}^{\mathsf{c}}_{\mathrm{h}}(f,u_h)\|_{V'}$ in this setting.*

Proof. First, we recall the Scott-Zhang Π_{SZ} operator, c.f. [SZ90] and Proposition 2.31. We apply this operator in its vectorized version $\Pi^d_{\mathsf{SZ}} : H^1_0(\Omega)^d \to \mathring{S}^1_1(\mathcal{T})^d \subset \mathrm{CR}^d$ for which the analogous estimates of Proposition 2.31 are valid.

Moreover, we consider the localization operators $\Theta_z : H^1_0(\Omega)^d \to H^1_0(\omega_z)^d$, $z \in \mathcal{V}$. They are similar to the ones used in Proposition 2.31 but here the Lagrange basis functions are applied componentwisely to the vector valued function: For each $v \in H^1_0(\Omega)^d$ and $z \in \mathcal{V}$, we define $v\Theta_z := v\Phi^1_z$.

Now, we show the conditions (2.10) of Corollary 2.7. Let $w \in H^1_0(\Omega)^d$ with $\|w\|_V = 1$. We switch to the H^1_0-seminorm $\|(w - \mathrm{E_M}\,\Pi_{\mathsf{SZ}} w)\Phi^1_z\|^2_{\mathsf{LE}} \lesssim \|\nabla\big((w - \mathrm{E_M}\,\Pi_{\mathsf{SZ}} w)\Phi^1_z\big)\|^2_{L^2(\Omega)}$ with a constant depending on μ, λ. Then Proposition 2.31 bounds this expression by $\|\nabla w\|_{L^2(\Omega)}$. Using Korn's inequality of [Bre04, Theorem 3.1] and adding the divergence part of the norm completes the proof. □

In section 2.3.1.2, we discuss an error estimator for which we need the local localization. For that reason, we show that Assumption 2.15 is true in this setting.

Proposition 2.42. *The symmetric gradient setting E-CR1 fulfills Assumption 2.15 and thus Theorem 2.16 provides the local localization.*

Proof. We select $V_z := H^1_0(\omega_z)^d$ and $V_z^+ := H^1_0(\omega_z^+)^d$. We set $\mathcal{I}_z := \mathcal{V} \cap \omega_z^+$, $z \in \mathcal{V}$, for which the finite overlap property (1.22) ensures that there exists a $C_{\mathsf{N}} \in \mathbb{N}$ with $\#\mathcal{I}_z \leqslant C_{\mathsf{N}}$ for all $z \in \mathcal{V}$. Then, we use the Lagrange functions Φ^1_z, $z \in \mathcal{V}$, to construct a localization operator as in the proof of Theorem 2.41.

To show (2.16), we note that $\|w\Phi^1_z\|_{\mathsf{LE}} \lesssim \|\nabla(w\Phi^1_z)\|_{L^2(\Omega)}$, $w \in V$, with constants depending on μ, λ similar to (2.48). The jump terms vanish because $w\Phi^1_z \in H^1_0(\omega_z)^d$. The expression term $\|\nabla(w\Phi^1_z)\|_{L^2(\Omega)}$ can be estimated with a vectorized version of (2.45). The result $\|\nabla w\|_{L^2(\Omega)}$ is bounded with Korn's inequality [Bre04, Theorem 3.1]. After adding the divergence expression, the proof is completed. □

2.2.3 Localization for the biharmonic problems

In this section, we present several localization procedures for the Morley (B-MR) and the C^0-interior penalty (B-SCZ) method, which were introduced in section 1.4.4. The localizations were already published in [KRVZ25, section 5].

Recall that the biharmonic setting is defined with $V = H^2_0(\Omega)$ and only for $d = 2$. The analytic bilinear form $a_{\mathsf{B}} : H^2_0(\Omega) \times H^2_0(\Omega) \to \mathbb{R}$ is defined as $a_{\mathsf{B}}(v, w) = \int_\Omega D^2 v : D^2 w$.

To apply a localization method, we first check Assumption 2.2.

Lemma 2.43. *The setting of B-MR and B-SCZ fulfill Assumption 2.2 with $\mathcal{I} = \mathcal{V}$ and $V_z = H^2_0(\omega_z)$, $z \in \mathcal{V}$. In particular, condition* (2.7) *is satisfied in the two settings. Consequently, Theorem 2.4 is applicable.*

Proof. Condition (2.7) is shown similarly to (2.37) by using Jensen's inequality and the finite overlap property of the mesh (1.22). □

2.2.3.1 The Morley method

First, we localize the Morley method B-MR, presented in section 1.4.4.1. On the space $\tilde{V} := (H^2_0(\Omega) +$ MR), we define the extended bilinear form $\tilde{a}^{\mathsf{MR}} : \tilde{V} \times \tilde{V} \to \mathbb{R}$ as $\tilde{a}^{\mathsf{MR}}(\tilde{v}, \tilde{w}) = \int_\Omega D^2_h \tilde{v} : D^2_h \tilde{w}$, $\tilde{v}, \tilde{w} \in \tilde{V}$.

The splitting into conformity error and smoothed error reads in this case as

$$\|D^2_h(u - u_h)\|^2_{L^2(\Omega)} = \|D^2(u - \Pi_V u_h)\|^2_{L^2(\Omega)} + \|D^2_h(u_h - \Pi_V u_h)\|^2_{L^2(\Omega)},$$

with $u \in H^2_0(\Omega)$ the solution of (1.67) and $u_h \in$ MR the solution of the method B-MR, c.f. (2.5).

The next Proposition shows, how we can estimate the conformity error.

Proposition 2.44. *In the setting of B-MR, the conformity error $\|D^2_h(u_h - \Pi_V u_h)\|_{L^2(\Omega)}$ can be estimated as*

$$\|D^2_h(u_h - \Pi_V u_h)\|_{L^2(\Omega)} \leqslant \|D^2_h(u_h - \mathcal{E}_{\mathsf{hct}} u_h)\|_{L^2(\Omega)} \leqslant C^{\mathrm{MR}}_{\mathsf{qopt}}(\mathcal{E}_{\mathsf{hct}}) \|D^2_h(u_h - \Pi_V u_h)\|_{L^2(\Omega)}. \tag{2.49}$$

The middle term is equivalent to gradient jumps

$$\|D^2_h(u_h - \mathcal{E}_{\mathsf{hct}} u_h)\|^2_{L^2(\Omega)} \eqsim \sum_{F \in \mathcal{F}} \int_F \frac{1}{h_F} |[\![\nabla_h u_h]\!]|^2. \tag{2.50}$$

Proof. The equivalence in (2.49) follows from the quasi-optimality of $\mathcal{E}_{\mathsf{hct}}$, shown in Corollary 1.64. The second equivalence (2.50) follows by (1.81b) and that

$$\begin{aligned} \int_F \frac{1}{h_F} |[\![\nabla_h u_h]\!]|^2 &\lesssim \sum_{T \in \mathcal{T}_F} \frac{1}{h^2_F} \|\nabla_h(u_h - \mathcal{E}_{\mathsf{hct}} u_h)\|^2_{L^2(T)} + \|D^2_h(u_h - \mathcal{E}_{\mathsf{hct}} u_h)\|^2_{L^2(T)} \\ &\lesssim \sum_{T \in \mathcal{T}^+_F} \|D^2_h(u_h - \mathcal{E}_{\mathsf{hct}} u_h)\|^2_{L^2(T)}, \quad \forall F \in \mathcal{F}. \end{aligned} \tag{2.51}$$

In the last estimate, we used (1.81a) together with the Definition of the Morley element. □

Remark 2.45. In [KRVZ25, Lemma 5.10] an alternative proof of (2.50) is provided. That proof is based upon an integration by parts result described in [Gal15, Proposition 2.3].

We localize the method B-MR by using Theorem 2.11. We show the required Assumption 2.6, by Corollary 2.7. This Corollary needs a localization operator which we built with a H^2-partition of unity. Derivatives of this partition of unity shall be bounded in such a way that we can show (2.10).

The partition of unity is constructed with the H^2-conforming HCT-space, which is defined in (HCT). The space HCT contains the constant function 1. Thus, the constant 1 is expressible in terms of the

basis functions defined on page 35. Using the functions $\Upsilon_z^0 \in \mathrm{HCT}$, $z \in \mathcal{I} = \mathcal{V}$, defined on page 35, we write

$$1 = \sum_{z \in \mathcal{V}} \Upsilon_z^0. \tag{2.52}$$

After having an H^2-conforming partition of unity at hand, we still need estimates for it. They are challenging because the HCT-element is not affine-equivalent.

Lemma 2.46 ([KRVZ25, Lemma 5.3])**.** *Let $d = 2$ and $z \in \mathcal{V}$ be given, then* $\operatorname{supp}(\Upsilon_z^0) \subset \omega_z$ *and*

$$h_z^2 \|D^2 \Upsilon_z^0\|_{L^\infty(\Omega)} \eqsim h_z \|\nabla \Upsilon_z^0\|_{L^\infty(\Omega)} \eqsim \|\Upsilon_z^0\|_{L^\infty(\Omega)} \eqsim 1. \tag{2.53}$$

Proof. For the proof, we express the basis functions of HCT on one fixed element $T \in \mathcal{T}$ in terms of an alternative basis induced by degrees of freedom $\tilde{\mathcal{N}}$. This alternative basis has the advantage that it is affine equivalent.

Fix $T \in \mathcal{T}$. We write m_T for the barycenter of T and m_F for the barycenter of $F \in \mathcal{F}_T$. Furthermore, we write $\mathcal{V}_T := \mathcal{V} \cap T$.

Recall that the HCT-element is defined by splitting the element T into a subtriangulation $\mathcal{T}_T$ consisting of three simplices, see page 35. Accordingly, we write $S_3^2(\mathcal{T}_T)$ for functions in $H^2(T)$ that consist of piecewise polynomials of third order with respect to $\mathcal{T}_T$.

The degrees of freedom of the alternative, affine-equivalent basis of the HCT-element on T are given as (c.f. see [Cia78, Theorem 6.1.3])

$$\begin{aligned}\tilde{\mathcal{N}}_{\mathrm{HCT}}^T := \{\tilde{N}_z^0(v) := v(z) \mid z \in \mathcal{V}_T\} \oplus \{\tilde{N}_z^y(v) := \nabla v(z) \cdot (y - z) \mid y, z \in \mathcal{V}_T,\ y \neq z\} \\ \oplus \{\tilde{N}_F(v) := \nabla v(m_F) \cdot (m_T - m_F) \mid F \in \mathcal{F}_T\}.\end{aligned} \tag{2.54}$$

The corresponding basis functions are:

(i) For $\tilde{N}_z^0$, we write $\tilde{\Upsilon}_z \in S_3^2(\mathcal{T}_T)$, $z \in \mathcal{V}_T$,

(ii) for $\tilde{N}_z^y$, we write $\tilde{\Upsilon}_z^y \in S_3^2(\mathcal{T}_T)$, $z, y \in \mathcal{V}_T$ and

(iii) for $\tilde{N}_F$, we write $\tilde{\Upsilon}_F \in S_3^2(\mathcal{T}_T)$, $F \in \mathcal{F}_F$.

To derive for a fixed $z \in \mathcal{V}_T$ the estimates for Υ_z^0 on the fixed $T \in \mathcal{T}$, we express Υ_z^0 in terms of the new basis functions on T. We test Υ_z^0 with the functionals from the new set of degrees of freedom $\tilde{\mathcal{N}}_{\mathrm{HCT}}$ to obtain the coefficients to write Υ_z^0 as a linear combination of $\tilde{\Upsilon}_z$, $\tilde{\Upsilon}_z^y$, $z, y \in \mathcal{V}_T$, and $\tilde{\Upsilon}_F$, $F \in \mathcal{F}_T$.

We have $\tilde{N}_y^0(\Upsilon_z^0) = \delta_{y,z}$ for all $y \in \mathcal{V}_T$ and $\tilde{N}_y^x(\Upsilon_z^0) = 0$ for all $x, y \in \mathcal{V}_T$ because $\nabla \Upsilon_z^0(x) = 0$ for all $x \in \mathcal{V}$. The only non-trivial case is the functional $\tilde{N}_F$ for a $F \in \mathcal{F}_T$. Fixing $F \in \mathcal{F}_T$ and $y_F \in F_T \cap \mathcal{V}_T$ with $y_F \neq z$ the vectors n_F and $\frac{z - y_F}{|z - y_F|}$ form an orthonormal basis. This allows us to decompose $\nabla \Upsilon_z^0(m_F) \cdot (m_T - m_F)$ as:

$$\nabla \Upsilon_z^0(m_F) \cdot (m_T - m_F) = \nabla \Upsilon_z^0(m_F) \cdot n_F (m_T - m_F) \cdot n_F + \nabla \Upsilon_z^0(m_F) \cdot \frac{z - y_F}{|z - y_F|} (m_T - m_F) \cdot \frac{z - y_F}{|z - y_F|}. \tag{2.55}$$

As $\partial_{\mathfrak{n}_F} \Upsilon_z^0(m_F) = 0$ by the definition of the HCT element, we only have to compute $\nabla \Upsilon_z^0(m_F) \cdot \frac{z - y_F}{|z - y_F|}$.

To evaluate this derivative along F, we express the function Υ_z^0 on F as a polynomial. Since $\Upsilon_z^0(z) = 1$, $\Upsilon_z^0(y) = 0$, $\nabla \Upsilon_z^0(z) = 0$ and $\nabla \Upsilon_z^0(y) = 0$, we can determine the polynomial through a one-dimensional Hermite interpolation. This interpolation shows that $\Upsilon_z^0\big(y_F + t(z - y_F)\big) = 3t^2 - 2t^3$ for $t \in [0, 1]$.

Taking the derivative with respect to t provides $\nabla \Upsilon_z^0\big(y_F + t(z - y_F)\big) \cdot (z - y_F) = 6t - 6t^2$. Setting $t = \frac{1}{2}$, we obtain $\nabla \Upsilon_z^0(m_F) \cdot (z - y_F) = \frac{3}{2}$. Therefore, we can continue (2.55) as

$$\nabla \Upsilon_z^0(m_F) \cdot (m_T - m_F) = \nabla \Upsilon_z^0(m_F) \cdot \frac{z - y_F}{|z - y_F|} (m_T - m_F) \cdot \frac{z - y_F}{|z - y_F|} = \frac{3}{2|z - y_F|} (m_T - m_F) \cdot \frac{z - y_F}{|z - y_F|}.$$

For this reason, the function Υ_z^0 can be expressed on T as

$$\Upsilon_z^0 = \tilde{\Upsilon}_z + \frac{3}{2} \sum_{F \in \mathcal{F}_T \cap \mathcal{F}_z} \frac{z - y_F}{|z - y_F|^2} \cdot (m_T - m_F) \tilde{\Upsilon}_F.$$

Since $\left| \frac{z - y_F}{|z - y_F|^2} \cdot (m_T - m_F) \right| \leqslant 1$, and the alternative basis is affine equivalent, the estimates (2.53) follow by standard scaling arguments. □

With the partition of unity, we define for each $z \in \mathcal{V}$ the local spaces $V_z = H^2_0(\omega_z)$ and the localization operators $\Theta_z : H^2_0(\Omega) \to H^2_0(\omega_z)$ as $w\Theta_z := w\Upsilon_z$, $w \in V = H^2_0(\Omega)$. Beyond the partition of unity, a suitable interpolation operator $\mathcal{I}_h$ is needed.

In this case, a possible operator is the Morley interpolator, see [Gal15, section 2.2] and [HS09, (4.6)].

Definition 2.47. *The Morley interpolation* $\Pi_{\mathsf{MR}} : \tilde{V} \to \mathrm{MR}$ *is defined for* $v \in \tilde{V}$ *as*

$$\Pi_{\mathsf{MR}}(v)(z) = v(z) \quad \forall z \in \mathcal{V} \tag{2.56a}$$

$$\int_F \partial_{\mathsf{n}_F} \Pi_{\mathsf{MR}}(v) = \int_F \partial_{\mathsf{n}_F} v \quad \forall F \in \mathcal{F} \tag{2.56b}$$

The H^2-regularity of v ensures that point-evaluations and traces of the gradient are well-defined.

Properties of the Morley interpolation operator are summarized in the next Proposition, see [Gal15], [HS09] and [HSX12, Lemma 3.1].

Proposition 2.48. *The Morley interpolation operator fulfills for all* $v \in \tilde{V}$ *the stability property*

$$\|D^2_h \Pi_{\mathsf{MR}} v\|_{L^2(T)} \leqslant \|D^2_h v\|_{L^2(T)} \tag{2.57a}$$

and the approximation property

$$\frac{1}{h_T^2}\|\Pi_{\mathsf{MR}} v - v\|_{L^2(T)} + \frac{1}{h_T}\|\nabla_h(\Pi_{\mathsf{MR}} v - v)\|_{L^2(T)} + \|D^2_h(\Pi_{\mathsf{MR}} v - v)\|_{L^2(T)} \leqslant C\|D^2_h v\|_{L^2(T)}. \tag{2.57b}$$

Proof. The stability follows because Π_{MR} is a projection onto MR, which follows by an integration by parts, see [HSX12, Lemma 3.1]. The approximation property is proved in [Gal15, Proposition 2.3]. □

Proposition 2.49 ([KRVZ25, Lemma 5.5])**.** *Let* $u \in H^2_0(\Omega)$ *be the solution of* (1.66) *and* $u_h \in \mathrm{MR}$ *be the solution of the method B-MR. The Morley interpolation operator in combination with the localization operators induced by the H^2-partition of unity Υ^0_z, $z \in \mathcal{V}$, fulfill the assumptions of Corollary 2.7. Hence, Theorem 2.11 provides a localization of the form*

$$\|\operatorname{res}^{\mathsf{c}}_{\mathrm{h}}(f, u_h)\|_{V'} \leqslant C_{\mathsf{apx}}\Big(\sum_{z\in\mathcal{I}} \|\operatorname{res}^{\mathsf{c}}_{\mathrm{h}}(f, u_h)\|^2_{V'_z}\Big)^{\frac{1}{2}}. \tag{2.58}$$

Proof. Assumption 2.2 is shown in Lemma 2.43 for the biharmonic setting with the spaces $V_z = H^2_0(\omega_z)$ and $\mathcal{I} = \mathcal{V}$.

After choosing $\Theta_z : H^2_0(\Omega) \to H^2_0(\omega_z)$ as $w\Theta_z := w\Upsilon_z$, $w \in V = H^2_0(\Omega)$, $\mathrm{E_M} = \mathcal{E}_{\mathsf{hct}}$ and $\mathcal{I}_h = \Pi_{\mathsf{MR}}$, it remains to verify (2.10).

The stability of the Morley interpolation operator Π_{MR} is provided by Proposition 2.48. For the second condition in (2.10), we need estimates for the smoother $\mathcal{E}_{\mathsf{hct}}$ because Π_{MR} does not map into $V \cap V_h$, c.f. Remark 2.10. In the next section, where we discuss the C^0-interior penalty method, we present a more general method, but here we exploit that $\mathcal{E}_{\mathsf{hct}}$ is a right inverse of Π_{MR}, shown by [VZ19a, Lemma 3.13]. This simplifies the estimate as already presented in [KRVZ25, Lemma 5.12]. We recall the argument. First, we observe that $v - \mathcal{E}_{\mathsf{hct}}\Pi_{\mathsf{MR}} v = (\mathrm{id} - \Pi_{\mathsf{MR}})(v - \mathcal{E}_{\mathsf{hct}}\Pi_{\mathsf{MR}} v)$ for all $v \in H^2_0(\Omega)$ since $\Pi_{\mathsf{MR}}\mathcal{E}_{\mathsf{hct}} = \mathrm{id}_{\mathrm{MR}}$. We set $\hat{v} := (v - \mathcal{E}_{\mathsf{hct}}\Pi_{\mathsf{MR}} v)$ and compute:

$$\begin{aligned}
\big\|D^2\big((v - \mathcal{E}_{\mathsf{hct}}\Pi_{\mathsf{MR}} v)\Upsilon^0_z\big)\big\|_{L^2(\Omega)} &= \big\|D^2_h\big((\hat{v} - \Pi_{\mathsf{MR}}\hat{v})\Upsilon^0_z\big)\big\|_{L^2(\Omega)} \\
&\leqslant C\Big(\frac{1}{h_z^2}\|\hat{v} - \Pi_{\mathsf{MR}}\hat{v}\|_{L^2(\omega_z)} + \frac{1}{h_z}\|\nabla_h(\hat{v} - \Pi_{\mathsf{MR}}\hat{v})\|_{L^2(\omega_z)} + \|D^2_h(\hat{v} - \Pi_{\mathsf{MR}}\hat{v})\|_{L^2(\omega_z)}\Big) \\
&\leqslant C\|D^2\hat{v}\|_{L^2(\omega_z)} = C\|D^2(v - \mathcal{E}_{\mathsf{hct}}\Pi_{\mathsf{MR}} v)\|_{L^2(\omega_z)} \leqslant C\|D^2 v\|_{L^2(\omega_z^+)}.
\end{aligned}$$

At first, we used the product rule together with (2.53), the approximability of the Morley interpolator in Proposition 2.48 and finally the stability of Π_{MR} and $\mathcal{E}_{\mathsf{hct}}$. Recall that $\delta_{V_h} = 0$ for (B-MR), c.f. Proposition 1.65. □

Finally, the following localization result is valid (compare also with [KRVZ25, Lemma 5.5]):

Theorem 2.50. *Let* $u \in H^2_0(\Omega)$ *be the analytic solution of the biharmonic problem* (1.66) *and* $u_h \in \mathrm{MR}$ *the discrete solution of the quasi-optimal Morley method* (B-MR)*. Then the error fulfills*

$$\|D^2_h(u_h - u)\|^2_{L^2(\Omega)} \eqsim \sum_{z\in\mathcal{V}} \|\operatorname{res}^{\mathsf{c}}_{\mathrm{h}}(f, u_h)\|^2_{H^{-2}(\omega_z)} + \sum_{F\in\mathcal{F}} \int_F \frac{1}{h_F}|[\![\nabla_h u_h]\!]|^2.$$

Proof. At first, we use (2.49) to split the error into the conformity and smoothed error

$$\|D_h^2(u-u_h)\|_{L^2(\Omega)}^2 = \|D^2(u-\Pi_V u_h)\|_{L^2(\Omega)}^2 + \|D_h^2(\Pi_V u_h - u_h)\|_{L^2(\Omega)}^2.$$

The conformity error $\|D_h^2(\Pi_V u_h - u_h)\|_{L^2(\Omega)}^2$ is estimated by (2.50). The final result follows with Theorem 2.11 (applicable due to Proposition 2.49) and Theorem 2.4 (applicable due to Lemma 2.43). □

2.2.3.2 The C^0-interior penalty method

We now turn to the C^0-interior penalty method (B-SCZ) abbreviated as B-SCZ. At first, we provide a variant of Proposition 2.44 for B-SCZ.

Proposition 2.51 ([KRVZ25, Lemma 5.3, Corollary 5.4])**.** *In the setting of B-SCZ, the conformity error* $\|u_h - \Pi_V u_h\|_{\mathsf{C}_0}$ *can be estimated as*

$$\|u_h - \Pi_V u_h\|_{\mathsf{C}_0} \leqslant \|u_h - \mathcal{E}_{\mathsf{hct}} u_h\|_{\mathsf{C}_0} \leqslant C_{\mathsf{qopt}}^{\mathsf{C}^0}(\mathcal{E}_{\mathsf{hct}}) \|u_h - \Pi_V u_h\|_{\mathsf{C}_0}. \tag{2.59}$$

The middle term is equivalent to gradient jumps

$$\frac{\gamma}{\gamma+1}\|D_h^2(u_h - \mathcal{E}_{\mathsf{hct}} u_h)\|_{\mathsf{C}_0}^2 \lesssim \sum_{F\in\mathcal{F}} \int_F \frac{\gamma}{h_F} |[\![\nabla_h u_h]\!]|^2 \leqslant \|D_h^2(u_h - \mathcal{E}_{\mathsf{hct}} u_h)\|_{\mathsf{C}_0}. \tag{2.60}$$

Remark 2.52. The constant $C_{\mathsf{qopt}}^{\mathrm{MR}}(\mathcal{E}_{\mathsf{hct}})$ in (2.49) and $C_{\mathsf{qopt}}^{\mathsf{C}^0}(\mathcal{E}_{\mathsf{hct}})$ in (2.59) are potentially different.

Remark 2.53. Alternatively, we could use the operator $\mathcal{A}_{\mathsf{hct}}$ in (2.59) and (2.60) as done in [KRVZ25].

In contrast to the Morley method (B-MR), the Morley interpolation operator cannot be used as it does not map into $\mathring{S}_2^1(\mathcal{T})$. But the Lagrange nodal interpolation operator is applicable since point evaluations of functions in $H^2(\Omega)$ are well-defined for $d = 2$. Together with the HCT partition of unity in (2.52), the Lagrange interpolation operator is sufficient for the localization.

Theorem 2.54 ([KRVZ25, Lemma 5.5])**.** *Let* $u \in H_0^2(\Omega)$ *be the solution of* (1.66) *and* $u_h \in \mathring{S}_2^1(\mathcal{T})$ *be the solution of (B-SCZ). The method (B-SCZ) fulfills the assumptions in Corollary 2.7 with* $\mathcal{I}_h = \Pi_{\mathsf{Lag}} : H_0^2(\Omega) \to \mathring{S}_2^1(\mathcal{T})$ *and therefore Theorem 2.11 provides a localization of the form*

$$\|\operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f,u_h)\|_{V'} \leqslant C_{\mathsf{apx}}\Big(\sum_{z\in\mathcal{I}} \|\operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f,u_h)\|_{V_z'}^2\Big)^{\frac{1}{2}} + \delta_{V_h} C_{\mathsf{I}} \|\Pi_V u_h - u_h\|_{\tilde{V}}.$$

Proof. Assumption 2.2 is also fulfilled in the B-SCZ setting with $V_z = H_0^2(\omega_z)$, $z \in \mathcal{V}$, see Lemma 2.43. Furthermore, we use the H^2-partition of unity Υ_z^0 for the localization operators $v\Theta_z \mapsto v\Upsilon_z^0$, $v \in H_0^2(\Omega)$. The Lagrange nodal interpolation operator Π_{Lag} fulfills the stability requirement in (2.10), see [BS08, section 4.4]. We show the approximability property in (2.10). At first, we use the triangle inequality:

$$\|D_h^2\big((v - \mathcal{E}_{\mathsf{hct}}\Pi_{\mathsf{Lag}} v)\Upsilon_z^0\big)\|_{L^2(\Omega)} \leqslant \|D_h^2\big((v-\Pi_{\mathsf{Lag}} v)\Upsilon_z^0\big)\|_{L^2(\Omega)} + \|D_h^2\big((\Pi_{\mathsf{Lag}} v - \mathcal{E}_{\mathsf{hct}}\Pi_{\mathsf{Lag}} v)\Upsilon_z^0\big)\|_{L^2(\Omega)}. \tag{2.61}$$

The first term on the right-hand side is bounded by the approximability of Π_{Lag} ([BS08, Theorem 4.4.4]):

$$\begin{aligned}&\|D_h^2\big((v-\Pi_{\mathsf{Lag}} v)\Upsilon_z^0\big)\|_{L^2(\Omega)}\\ &\quad\lesssim \big(h_z^{-2}\|v-\Pi_{\mathsf{Lag}} v\|_{L^2(\omega_z)} + h_z^{-1}\|\nabla_h(v-\Pi_{\mathsf{Lag}} v)\|_{L^2(\omega_z)} + \|D_h^2(v-\Pi_{\mathsf{Lag}} v)\|_{L^2(\omega_z)}\big) \lesssim \|D^2 v\|_{L^2(\omega_z)}.\end{aligned}$$

The second part of (2.61) is estimated with (1.81b):

$$\begin{aligned}&\|D_h^2\big((\Pi_{\mathsf{Lag}} v - \mathcal{E}_{\mathsf{hct}}\Pi_{\mathsf{Lag}} v)\Upsilon_z^0\big)\|_{L^2(\Omega)}\\ &\quad\leqslant C\big(h_z^{-2}\|\Pi_{\mathsf{Lag}} v - \mathcal{E}_{\mathsf{hct}}(\Pi_{\mathsf{Lag}} v)\|_{L^2(\omega_z)} + h_z^{-1}\|\nabla_h(\Pi_{\mathsf{Lag}} v - \mathcal{E}_{\mathsf{hct}}(\Pi_{\mathsf{Lag}} v))\|_{L^2(\omega_z)}\\ &\qquad + \|D_h^2(\Pi_{\mathsf{Lag}} v - \mathcal{E}_{\mathsf{hct}}(\Pi_{\mathsf{Lag}} v))\|_{L^2(\omega_z)}\big)\\ &\quad\leqslant C\sum_{F\in\mathcal{F}_z^{i,+}} \left(\frac{1}{h_F}\int_F [\![\partial_{\mathsf{n}_F}(v-\Pi_{\mathsf{Lag}} v)]\!]^2\right)^{\frac{1}{2}}.\end{aligned}$$

The Trace-Theorem and another application of the approximability of Π_{Lag} yields for each $F \in \mathcal{F}_z^{i,+}$:

$$\begin{aligned}\int_F [\![\partial_{\mathsf{n}_F}(v - \Pi_{\mathsf{Lag}} v)]\!]^2 &\leqslant C \left(h_F \|D_h^2 (v - \Pi_{\mathsf{Lag}} v)\|^2_{L^2(\omega_F)} + h_F^{-1} \|\nabla_h (v - \Pi_{\mathsf{Lag}} v)\|^2_{L^2(\omega_F)} \right) \\ &\leqslant C h_F \|D_h^2 v\|^2_{L^2(\omega_F)}.\end{aligned}$$

Combining the last estimates and summing over all $z \in \mathcal{I}$ provides the desired result. □

Remark 2.55. Since the Scott-Zhang operator $\Pi_{\mathsf{SZ}} : H_0^2(\Omega) \to \mathring{S}_2^1(\Omega)$ defined in [SZ90] fulfills similar estimates as Π_{Lag}, we could use Π_{SZ} as well. The drawback is that for $v \in H_0^2(\Omega)$ in general $\operatorname{supp}(\Pi_{\mathsf{SZ}}(v))$ is larger than $\operatorname{supp}(v)$. In the previous cases with $V = H_0^1(\Omega)$, the Lagrange interpolator could not be utilized because it needs point evaluations, which are not well-defined for a function in $H_0^1(\Omega)$. In this setting, however, we have $H_0^2(\Omega) \hookrightarrow C^0(\overline{\Omega})$.

For later purposes, we show that the biharmonic setting is also local localizable by showing that Assumption 2.15 is fulfilled.

Proposition 2.56. *The biharmonic setting fulfills Assumption 2.15 with $V_z := H_0^2(\omega_z)$, $V_z^+ := H_0^2(\omega_z^+)$ and $\mathcal{I}_z := \mathcal{V} \cap \omega_z^+$, for all $z \in \mathcal{V}$. Therefore, Theorem 2.16 is applicable to the biharmonic setting with a constant $C_{\mathsf{P}} > 0$.*

Proof. The proof proceeds similar to Proposition 2.35. As pointed out in that Proposition, the sets $\mathcal{V} \cap \omega_z^+$ are uniformly bounded in $z \in \mathcal{V}$.

Fix $z \in \mathcal{V}$. The localization operator $\Theta_y : V_z^+ \to V_y \cap V_z^+$ for $y \in \mathcal{I}_z$ is again induced by the HCT partition of unity, see (2.52):

$$\Theta_y : H_0^2(\omega_z^+) \to H_0^2(\omega_y) \cap H_0^2(\omega_z^+), \qquad w\Theta_y := w\Upsilon_y^0 \quad \forall w \in H_0^2(\omega_z^+).$$

The product rule, the estimates in Lemma 2.46 and Poincaré's inequality show for all $w \in H_0^2(\omega_z^+)$

$$\sum_{y \in \mathcal{I}_z} \|D^2(w\Upsilon_y^0)\|_{L^2(\omega_y)} \lesssim \sum_{y \in \mathcal{I}_z} \|D^2 w\|_{L^2(\omega_y)} + \frac{1}{h_z}\|\nabla w\|_{L^2(\omega_y)} + \frac{1}{h_z^2}\|w\|_{L^2(\omega_y)} \lesssim \|D^2 w\|_{L^2(\omega_z^+)}. \tag{2.62}$$

□

With this, we close the section of applications of the abstract localization methods to the examples.

2.3 Construction of calculable estimators for the examples

In this section, we turn to the second part of the abstract framework described in section 2.1.3. We assume that the residual is localized as discussed in section 2.1.2. In this section, we apply the abstract framework of section 2.1.3 for the computation of the localized residual $\|\operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f, u_h)\|_{V_z'}$ to the examples. Here, $f \in V'$ and $u_h \in V_h$ the corresponding discrete solution. The space V_z is defined in Assumption 2.2, which shall be valid throughout this section. As discussed in the abstract setting, the localized residual is in general not computable, c.f. section 2.1.3. For this reason, we introduce a projection operator. With its help, we split the residual into two parts, the estimator and the oscillation, see (2.20). In this section, we will construct several projection operators for our examples.

To be more precise, we construct these operators $\mathcal{P}_z : S_z' \to D_z$ by the recipe provided in Proposition 2.20. We choose different pairs of spaces (D_z, S_z) with $S_z \subset V$ and $D_z \subset \tilde{V}_z' = (V_z + S_z)'$ which fulfill (2.21) and (2.29).

The condition (2.29) requires that certain functionals are contained in D_z. If we restrain ourselves to settings with $S_z \subset V_z$, determining a minimal space D_z is simpler. In this case, condition (2.29) reduces to

$$(\tilde{A} V_h)_{|V_z} \subset (D_z)_{|V_z}. \tag{2.63}$$

This condition does not involve the test function spaces S_z. Therefore, we can a choose D_z satisfying condition (2.63) first, and then start constructing test functions S_z.

If, however, we allow $S_z \subsetneq V_z$, then condition (2.29) involves both D_z and also S_z: They depend on each other. It seems that we cannot choose D_z first and then construct S_z, making the construction of the pair D_z and S_z more involved in this setting.

Nevertheless, we will see that our construction is not that involved. First, we will construct a pair (D_z, S_z) fulfilling (2.63). As $S_z \subset V_z$, we directly have (2.29) satisfied as well. After this, we will construct another space $\hat{S}_z$ for the same D_z and ignore the condition (2.29) for that moment. Finally, we will show with an argument that the new pair $(D_z, \hat{S}_z)$ also fulfills (2.29).

To be more precise, this means: Given one of our model problems (Poisson, linear elasticity, biharmonic and later in chapter 4 also the Stokes problem), we choose at first a space D_z such that (2.63) is fulfilled. As all the spaces V_h in the examples are embeddable into the space of piecewise polynomials $S^0_\ell(\mathcal{T})$, we can choose for each analytic problem and polynomial degree ℓ a single space D_z. After we have chosen a space D_z, we construct (sometimes several) spaces of test functions S_z. Then we can derive from (2.63) the condition (2.29) easily because either $S_z \subset V_z$ or the functions in S_z that are not contained in V_z are constructed in such a way that we can derive (2.29) in a straightforward manner.

After constructing the pairs (D_z, S_z), we define unique projections $\mathcal{P}_z : S_z' \to D_z$ via (2.22). Because all constructed pairs (D_z, S_z) fulfill (2.29), we can apply Proposition 2.22. This Proposition provides the bound of the oscillation and the estimator by the localized conforming residual in the $\tilde{V}_z'$-norm, i.e. $\|\tilde{A}u_h - f\|_{\tilde{V}_z'}$. The question is: Can we get back to $\|\tilde{A}u_h - f\|_{V_z'}$?

If $S_z \subset V_z$, we can obviously bound the estimator and oscillation term by the localized residual $\|\tilde{A}u_h - f\|_{V_z'}$ again. If, however, $S_z \subsetneq V_z$, to get back to $\|\operatorname{res}^{\mathsf{c}}_{\mathsf{h}}(f, u_h)\|_{V_z'} = \|\tilde{A}u_h - f\|_{V_z'}$, we need to verify Assumption 2.15 such that we can apply Theorem 2.16.

After this introduction, we start with the construction of the projections for our examples.

2.3.1 Projections for the second order problem

In this section, we build projections for the Poisson problem. Throughout this section, we assume the setting of section 1.4.2 for any method therein. Deviations of individual methods will be explicitly explained. We start with the projection introduced in [KV21] and [KRVZ25].

The goal is to build for a fixed $z \in \mathcal{V}$ a projection $\mathcal{P}_z : S_z' \to D_z$ with a suitable space of simple functionals D_z and a space of simple test functions S_z as required by Assumptions 2.19 and Proposition 2.20. The first step is to identify a suitable space D_z, which fulfills (2.63), the simplified version of (2.29). After that, we construct suitable spaces of simple test functions and verify (2.29) if necessary.

To identify the requirements on D_z to fulfill (2.63), we rewrite $\langle \tilde{A}(u_h), w\rangle$, $w \in V_z = H^1_0(\omega_z)$, $u_h \in S^0_\ell$, with an elementwise integration by parts

$$\langle \tilde{A}(u_h), w\rangle = \int_{\omega_z} \nabla_h u_h \cdot \nabla_h w = -\sum_{T\in\mathcal{T}_z} \int_T \Delta u_h w + \sum_{F\in\mathcal{F}_z^i} \int_F [\![\nabla u_h]\!] \cdot \mathsf{n}_F w. \tag{2.64}$$

where $\mathcal{F}_z^i = \mathcal{F}_z \cap \mathcal{F}^i$. Since in all our settings V_h consists of piecewise polynomial functions, we could choose the space D_z^ℓ for $\ell \in \mathbb{N}$ as

$$\begin{aligned} D_z^\ell := \Big\{ \chi \in H^{-1}(\Omega) \;\Big|\; \chi(\psi) = \sum_{T\in\mathcal{T}_z} \int_T r_T\psi + \sum_{F\in\mathcal{F}_z^i} \int_F r_F\psi \\ \text{for } \psi \in H^1_0(\Omega) \text{ with } r_T \in \mathbb{P}_{\ell-1}(T) \text{ and } r_F \in \mathbb{P}_{\ell-1}(F) \text{ for } T \in \mathcal{T}_z, F \in \mathcal{F}_z^i \Big\}. \end{aligned} \tag{2.65}$$

For the element contributions, we choose polynomials one degree higher than strictly necessary. The reason behind this is later explained in Corollary 2.63.

This space D_z^ℓ needs to be complemented with a test function space S_z. Within this thesis, this space is either contained in $H^1_0(\omega_z)$ or $H^1_0(\omega_z^+)$. Then the pair (D_z, S_z) induces a projection via (2.22).

Before we turn to the construction of spaces of simple functions S_z for the functional space D_z^ℓ, we formulate the following Lemma. It will be crucial for the estimates of the operator norms of the resulting projections $\mathcal{P}_z$.

Lemma 2.57. *Let $\chi \in D_z^\ell$ be given in the form*

$$\chi(\psi) = \sum_{T\in\mathcal{T}_z} \int_T r_T\psi + \sum_{F\in\mathcal{F}_z^i} \int_F r_F\psi, \quad \psi \in H^1_0(\omega_z),$$

with $r_T \in \mathbb{P}_{\ell-1}(T), r_F \in \mathbb{P}_{\ell-1}(F)$, $T \in \mathcal{T}_z, F \in \mathcal{F}_z^i$, then χ is bounded by

$$\|\chi\|_{H^{-1}(\omega_z)} \lesssim \Big(\sum_{T\in\mathcal{T}_z} h_z^2 \|r_T\|^2_{L^2(T)}\Big)^{\frac{1}{2}} + \Big(\sum_{F\in\mathcal{F}_z^i} h_z \|r_F\|^2_{L^2(F)}\Big)^{\frac{1}{2}}. \tag{2.66}$$

Proof. Let $\chi \in D_z^\ell$ and assume it can be written in the form $\chi(\psi) = \sum_{T\in\mathcal{T}_z} \int_T r_T\psi + \sum_{F\in\mathcal{F}_z^i} \int_F r_F\psi$

for $\psi \in H_0^1(\omega_z)$ and some $r_T \in \mathbb{P}_{\ell-1}(T), r_F \in \mathbb{P}_{\ell-1}(F)$, $T \in \mathcal{T}_z$ and $F \in \mathcal{F}_z^i$. Then we have

$$\begin{aligned}\chi(\psi) &= \sum_{T\in\mathcal{T}_z}\int_T r_T\psi + \sum_{F\in\mathcal{F}_z^i}\int_F r_F\psi \leqslant \sum_{T\in\mathcal{T}_z}\|r_T\|_{L^2(T)}\|\psi\|_{L^2(T)} + \sum_{F\in\mathcal{F}_z^i}\|r_F\|_{L^2(F)}\|\psi\|_{L^2(F)}\\ &\leqslant \Big(\sum_{T\in\mathcal{T}_z}\|r_T\|^2_{L^2(T)}\Big)^{\frac12}\Big(\sum_{T\in\mathcal{T}_z}\|\psi\|^2_{L^2(T)}\Big)^{\frac12} + \Big(\sum_{F\in\mathcal{F}_z^i}\|r_F\|^2_{L^2(F)}\Big)^{\frac12}\Big(\sum_{F\in\mathcal{F}_z^i}\|\psi\|^2_{L^2(F)}\Big)^{\frac12}.\end{aligned}$$

With Poincaré's inequality, we estimate

$$\sum_{T\in\mathcal{T}_z}\|\psi\|^2_{L^2(T)} = \|\psi\|^2_{L^2(\omega_z)} \lesssim h_z^2\|\nabla\psi\|^2_{L^2(\omega_z)}. \tag{2.67}$$

On a face $F \in \mathcal{F}_z^i$, we have with the Trace-Theorem

$$\|\psi\|^2_{L^2(F)} \lesssim h_F\|\nabla\psi\|^2_{L^2(\omega_F)} + h_F^{-1}\|\psi\|^2_{L^2(\omega_F)}.$$

Summing over all $F \in \mathcal{F}_z^i$ provides with another Poincaré's inequality

$$\sum_{F\in\mathcal{F}_z^i}\|\psi\|^2_{L^2(F)} \lesssim h_z\|\nabla\psi\|^2_{L^2(\omega_z)} + h_z^{-1}\|\psi\|^2_{L^2(\omega_z)} \lesssim h_z\|\nabla\psi\|^2_{L^2(\omega_z)}. \tag{2.68}$$

as $h_F \eqsim h_z$. Putting everything together, we obtain

$$\chi(\psi) \leqslant \left[\Big(\sum_{T\in\mathcal{T}_z} h_z^2\|r_T\|^2_{L^2(T)}\Big)^{\frac12} + \Big(\sum_{F\in\mathcal{F}_z^i} h_z\|r_F\|^2_{L^2(F)}\Big)^{\frac12}\right]\|\nabla\psi\|_{L^2(\omega_z)}. \qquad \square$$

2.3.1.1 Lowest order methods

After we have introduced the space of simple functionals for all polynomial degrees $\ell \in \mathbb{N}$, we have to construct associated test function spaces. For simplicity, we consider the case $\ell = 1$ at first before turning to the arbitrary order situation.

In the case $\ell = 1$ the space D_z^1 is

$$\begin{aligned}D_z^1 = \Big\{\chi \in H^{-1}(\Omega)\Big|\chi(\psi) = \sum_{T\in\mathcal{T}_z}\int_T r_T\psi + \sum_{F\in\mathcal{F}_z^i}\int_F r_F\psi\\ \text{for } \psi \in H_0^1(\Omega) \text{ with } r_T \in \mathbb{R} \text{ and } r_F \in \mathbb{R} \text{ for } T \in \mathcal{T}_z,\ F \in \mathcal{F}_z^i\Big\}.\end{aligned} \tag{2.69}$$

Lemma 2.57 is also true for D_z^1.

Corollary 2.58. *In the case $\ell = 1$ and $S_z \subset H_0^1(\omega_z)$ the space D_z^1 fulfills* (2.29) *for the methods P-LAG, P-CR1, P-SIP, P-NSIP, P-RSIP and P-RNSIP.*

Proof. The Corollary follows by the computation in (2.64). $\square$

The next step is to define for a fixed $z \in \mathcal{V}$ a test function space S_z for D_z^1 to obtain a projection $\mathcal{P}_z$ via the equation (2.22). There are several possibilities. At first, we recall the global construction in [KV21], which we adapt to our local setting. The construction is based upon bubble functions. The bubbles are in $H_0^1(\omega_z)$, thus we have $S_z \subset H_0^1(\omega_z)$ and $\tilde{V}_z = H_0^1(\omega_z)$.

Lemma 2.59. *The bubbles defined as (see* (1.37) *and* (1.42)*)*

$$\Psi_F := \frac{(2d-1)!}{(d-1)!|F|}\prod_{z\in\mathcal{V}, z\in F}\Phi_z^1 \qquad \text{and} \qquad \Psi_T := \frac{(2d+1)!}{d!|T|}\prod_{z\in\mathcal{V}, z\in T}\Phi_z^1 \tag{2.70}$$

satisfy $\int_T \Psi_T = 1$ and $\int_F \Psi_F = 1$ for all $T \in \mathcal{T}$ and $F \in \mathcal{F}$.

Proof. Use the integration formula for barycentric coordinates. $\square$

With these bubbles, we define the test function space

$$S_z^1 := \operatorname{span}\big(\{\Psi_F \mid F \in \mathcal{F}_z^i\} \cup \{\Psi_T \mid T \in \mathcal{T}_z\}\big) \subset H_0^1(\omega_z). \tag{2.71}$$

The next Proposition about the stability of the induced projection was already shown in [KV21], compare also [KRVZ25, Lemma 4.4, Theorem 4.5].

Proposition 2.60. *The spaces D_z^1 and S_z^1 fulfill Assumption 2.19 hence the projection $\mathcal{P}_z^1 : (S_z^1)' \to D_z^1$ is well-defined via (2.22). For $f \in H^{-1}(\omega_z)$, we have the representation*

$$\|\mathcal{P}_z^1(f)\|_{H^{-1}(\omega_z)} \eqsim \Big(\sum_{T\in\mathcal{T}_z} h_z^2 \|r_T\|_{L^2(T)}^2 \Big)^{\frac{1}{2}} + \Big(\sum_{F\in\mathcal{F}_z^i} h_z \|r_F\|_{L^2(F)}^2 \Big)^{\frac{1}{2}} \tag{2.72}$$

where $r_T \in \mathbb{R}$ and $r_F \in \mathbb{R}$ are determined as

$$r_T = \langle f, \Psi_T \rangle \quad \text{and} \quad r_F = \langle f, \Psi_F \rangle - \sum_{T\in\mathcal{T}_F} \int_T r_T \Psi_F,$$

for all $T \in \mathcal{T}_z$ and $F \in \mathcal{F}_z^i$. Furthermore, we have

$$\|\mathcal{P}_z^1(f)\|_{H^{-1}(\omega_z)}^2 \eqsim \sum_{T\in\mathcal{T}_z} \frac{\langle f, \Psi_T\rangle^2}{\|\nabla\Psi_T\|_{L^2(\omega_z)}^2} + \sum_{F\in\mathcal{F}_z^i} \frac{\langle f, \Psi_F\rangle^2}{\|\nabla\Psi_F\|_{L^2(\omega_z)}^2}. \tag{2.73}$$

Proof. We show that Assumption 2.19 is valid for the pair D_z^1 and S_z^1. Assumption 2.2 is true in the Poisson setting, see Proposition 2.30. As $S_z^1 \subset H_0^1(\omega_z)$, we have $\tilde{V}_z = H_0^1(\omega_z)$. It remains to validate (2.21).

For each $F \in \mathcal{F}_z^i$ and $T \in \mathcal{T}_z$, each space S_z and D_z contains one degree of freedom. This means $\dim(S_z) = \dim(D_z)$. It remains to verify the second condition in (2.21).

To deduce (2.21), we bound $\|\chi\|_{H^{-1}(\omega_z)}$, $\chi \in D_z^1$, by $\|\chi\|_{S_z^{1\prime}}$, using the L^2-representation of Lemma 2.57 as an intermediate step. Consequently, it remains to show that the L^2-representation supplied by Lemma 2.57 is bounded by $\|\chi\|_{S_z^{1\prime}}$.

For $\chi \in D_z^1$, we assume that the L^2-representation of Lemma 2.57 is given with the notation defined there. We bound the facial and element contributions of this representation by $\|\chi\|_{S_z^{1\prime}}$. The technique is similar to [Ver13, section 1.4.5].

The element contributions for all $T \in \mathcal{T}_z$ are estimated as

$$\|r_T\|_{L^2(T)}^2 \lesssim |T| \int_T r_T^2 \Psi_T = |T| \langle \chi, r_T \Psi_T \rangle \leqslant |T| \|\chi\|_{S_z^{1\prime}} \|r_T \nabla \Psi_T\|_{L^2(T)} \lesssim \|\chi\|_{S_z^{1\prime}} h_T^{-1} \|r_T\|_{L^2(T)}. \tag{2.74}$$

We used the equivalence of norms on finite-dimensional spaces and inverse estimates.

The face contributions $F \in \mathcal{F}_z^i$ are estimated similarly, but we have to consider the coupling between the face and element contributions, hence

$$\begin{aligned} \|r_F\|_{L^2(F)}^2 &\lesssim |F| \int_F r_F^2 \Psi_F = |F| \langle \chi, r_F \Psi_F \rangle - |F| \sum_{T\in\mathcal{T}_F} \int_T r_T r_F \Psi_F \\ &\lesssim \Big(\|\chi\|_{S_z^{1\prime}}^2 + \sum_{T\in\mathcal{T}_F} h_z^2 \|r_T\|_{L^2(T)}^2 \Big)^{\frac{1}{2}} |F| \|r_F \nabla \Psi_F\|_{L^2(\omega_z)}. \end{aligned} \tag{2.75}$$

An inverse estimate shows

$$|F| \|r_F \nabla \Psi_F\|_{L^2(\omega_z)} \lesssim h^{-\frac{1}{2}} \|r_F\|_{L^2(F)}. \tag{2.76}$$

Inserting (2.76) and (2.74) into (2.75) bounds the face contributions. All together, this gives the stability estimate

$$\|\chi\|_{H^{-1}(\omega_z)} \lesssim \|\chi\|_{S_z^{1\prime}}. \tag{2.77}$$

Consequently, the projection induced by D_z^1 and S_z^1 is well-posed.

The alternative representation (2.73) is obtained by Lemma 2.26 after an application of (2.77). The formulas for computing r_T and r_F are obtained through the representation of $\mathcal{P}_z^1(f)$ in D_z^1 and application of the test functions in (2.22). □

The preceding choice of S_z coincides with the space of test functions in [KV21]. Therefore, the projection operator $\mathcal{P}_z^1$ obtained here is the same as in [KV21], although [KV21, (3.21)] writes the projection more explicitly in a global manner.

We apply the projection to the residual to gain the error estimator. As $\tilde{V}_z = H_0^1(\omega_z)$, we do not need the local localization in Corollary 2.23.

Corollary 2.61 ([KRVZ25, Theorem 4.5])**.** *Let $u \in H^1_0(\Omega)$ be the solution of (1.30) and $u_h \in V_h$ the discrete solution of P-LAG, P-CR1, P-SIP, P-NSIP, P-RSIP or P-RNSIP for $\ell = 1$. Then we have*

$$\|\operatorname{res}^{\mathsf{c}}_{\mathrm{h}}(f,u_h)\|_{V_z'} \leqslant \underbrace{\|\mathcal{P}^1_z \operatorname{res}^{\mathsf{c}}_{\mathrm{h}}(f,u_h)\|_{V_z'}}_{\text{estimator}} + \underbrace{\|\mathcal{P}^1_z f - f\|_{V_z'}}_{\text{oscillation}} \leqslant (1+2\|\mathcal{P}_z\|_{\mathbb{L}(S^{1\prime}_z,V_z')})\|\operatorname{res}^{\mathsf{c}}_{\mathrm{h}}(f,u_h)\|_{V_z'}. \tag{2.78}$$

The estimator $\|\mathcal{P}^1_z \operatorname{res}^{\mathsf{c}}_{\mathrm{h}}(f,u_h)\|_{V_z'}$ can be evaluated with either (2.72) or (2.73) by replacing f with $\operatorname{res}^{\mathsf{c}}_{\mathrm{h}}(f,u_h)$.

Proof. Because of (2.64), (2.29) is valid in this setting and thus Proposition 2.22 is applicable and we obtain the result. Note, that in the case of the discontinuous Galerkin methods, the jump terms of the bilinear form $\tilde{a} = \tilde{a}^{\mathsf{dG}}$ vanish because in (2.29), we test only with functions contained in $H^1_0(\Omega)$. □

Summing up, we have the following estimator.

Theorem 2.62 ([KRVZ25, Theorem 4.5])**.** *Let $u \in H^1_0(\Omega)$ be the analytic solution of (1.30). Let $u_h \in V_h$ be the corresponding discrete solution of one of the methods P-LAG, P-CR1, P-SIP, P-NSIP, P-RSIP or P-RNSIP for $\ell = 1$. Let $\mathcal{P}^1_z$ be the projection defined by the spaces D^1_z and S^1_z via (2.22). Then we have the error estimate*

$$\|u-u_h\|^2_{\tilde{V}} \eqsim \sum_{z\in\mathcal{V}} \left\{\|\mathcal{P}^1_z(\operatorname{res}^{\mathsf{c}}_{\mathrm{h}}(f,u_h))\|^2_{H^{-1}(\omega_z)} + \|\mathcal{P}^1_z f - f\|^2_{H^{-1}(\omega_z)}\right\} + \|\Pi_V u_h - u_h\|^2_{\tilde{V}}. \tag{2.79}$$

The conformity error is estimated with Proposition 2.27.

Locally, we have the lower bound

$$\|\mathcal{P}^1_z(\operatorname{res}^{\mathsf{c}}_{\mathrm{h}}(f,u_h))\|^2_{H^{-1}(\omega_z)} + \|\mathcal{P}^1_z f - f\|^2_{H^{-1}(\omega_z)} \lesssim \|\nabla_h(u-u_h)\|^2_{L^2(\omega_z)}. \tag{2.80}$$

The hidden constants only depend on d, $\sigma(\mathcal{T})$ and in the case of the dG-methods also on γ.

Proof. We show (2.79). First, we switch to the conforming residual as in Lemma 2.1.

$$\|u-u_h\|^2_{\tilde{V}} = \|\operatorname{res}^{\mathsf{c}}_{\mathrm{h}}(f,u_h)\|^2_{V'} + \|\Pi_V u_h - u_h\|^2_{\tilde{V}}. \tag{2.81}$$

We apply one of the localizations in section 2.2.1 to the conforming residual, e.g. Proposition 2.32, to obtain

$$\|\operatorname{res}^{\mathsf{c}}_{\mathrm{h}}(f,u_h)\|^2_{H^{-1}(\Omega)} \lesssim \sum_{z\in\mathcal{I}} \|\operatorname{res}^{\mathsf{c}}_{\mathrm{h}}(f,u_h)\|^2_{H^{-1}(\omega_z)} + \delta^2_{V_h}\|u_h - \Pi_V u_h\|^2_{\tilde{V}}.$$

We continue to estimate $\sum_{z\in\mathcal{I}} \|\operatorname{res}^{\mathsf{c}}_{\mathrm{h}}(f,u_h)\|^2_{H^{-1}(\omega_z)}$ and use the projection $\mathcal{P}^1_z$ to split the local residual $\|\operatorname{res}^{\mathsf{c}}_{\mathrm{h}}(f,u_h)\|_{H^{-1}(\omega_z)}$ into the estimator and oscillation as presented in the first bound of (2.78) of Corollary 2.61. This completes the proof of the upper bound in (2.79).

To show the lower bound of (2.79), we use the second estimate of (2.78) in Corollary 2.61. This estimate bounds the estimator $\|\mathcal{P}^1_z \operatorname{res}^{\mathsf{c}}_{\mathrm{h}}(f,u_h)\|_{V_z'}$ and oscillation $\|\mathcal{P}^1_z f - f\|_{V_z'}$ by the local residual $\|\operatorname{res}^{\mathsf{c}}_{\mathrm{h}}(f,u_h)\|_{V_z'}$. The globalization result in Proposition 2.30 followed by (2.81) provides the lower bound.

The local estimate (2.80) follows via the bound

$$\|\operatorname{res}^{\mathsf{c}}_{\mathrm{h}}(f,u_h)\|_{H^{-1}(\omega_z)} = \sup_{v\in H^1_0(\omega_z)} \frac{\tilde{a}(u-u_h,v)}{\|\nabla v\|_{L^2(\omega_z)}} \leqslant \|\nabla_h(u-u_h)\|_{L^2(\omega_z)}. \qquad \square$$

The new estimator is computable if one can evaluate $\langle f, s\rangle$ for all $s \in S^1_z$ and one can quantify the oscillation term. In general one cannot quantify the oscillation term without further knowledge, see [KRVZ25, Remark 3.15]. We show how one obtains an upper bound (or *surrogate oscillation* as this is called in [KRVZ25, Remark 3.15] and [BCNV24, section 7.1]) of the oscillation term. This bound can already found in [KV21, (3.29)], [KRVZ25, Theorem 4.5] and [BCNV24, (4.47)].

To derive this estimate, we have already used in the element contributions polynomials that are one degree higher than strictly necessary for (2.63). This addition allows us to bound the error-dominated oscillation term by the classical oscillation if $\tilde{V}_z = H^1_0(\omega_z)$ as depicted in [KV21, section 3.8].

We interpret $g \in S^0_0(\mathcal{T}_z)$ as an $H^{-1}(\omega_z)$ functional via $w \mapsto \int_{\omega_z} gw$, $w \in H^1_0(\omega_z)$. This functional is contained in $D^1_{z|H^1_0(\omega_z)}$ and the projection property (2.29) of $\mathcal{P}^1_z$ yields for $f \in H^{-1}(\omega_z)$ that

$$\|f - \mathcal{P}^1_z f\|_{H^{-1}(\omega_z)} \leqslant \|f-g\|_{H^{-1}(\omega_z)} + \|g - \mathcal{P}^1_z f\|_{H^{-1}(\omega_z)} \lesssim \|f-g\|_{H^{-1}(\omega_z)}. \tag{2.82a}$$

Hence, we have:

Corollary 2.63. *For any $f \in H^{-1}(\omega_z)$, the oscillation term of $\mathcal{P}_z^1$ defined via D_z^1 and $S_z^1 \subset H_0^1(\omega_z)$ satisfies:*

$$\|f - \mathcal{P}_z^1 f\|_{H^{-1}(\omega_z)} \lesssim \inf_{g \in S_0^0(\mathcal{T}_z)} \|f - g\|_{H^{-1}(\omega_z)}. \tag{2.82b}$$

If $f \in L^2(\omega_z)$, then the last term in (2.82b) *can be further bounded as*

$$\inf_{g \in S_0^0(\mathcal{T}_z)} \|f - g\|_{H^{-1}(\omega_z)} \lesssim h_z \inf_{g \in S_0^0(\mathcal{T}_z)} \|f - g\|_{L^2(\omega_z)}. \tag{2.82c}$$

The last term in (2.82c) is the classical oscillation term, which is common in the standard a-posteriori analysis, c.f. [Ver13, BCNV24]. Hence, if $\tilde{V}_z = H_0^1(\omega_z)$ and $f \in L^2(\omega_z)$, the new oscillation terms are bounded up to a constant by the classical ones. Further possibilities to bound the new oscillation term are investigated in [KRVZ25, section 6] and [BCNV24, section 7.3].

As mentioned before, the choice of the space of simple functions S_z is not unique, and we present here another choice for $\ell = 1$ which simplifies the evaluation of the edge contributions for some methods. The idea is to replace the face-bubble Ψ_F by the CR_1 face bubble: For $F \in \mathcal{F}_z^i$, we replace the bubble Ψ_F in (2.70) by the Crouzeix-Raviart basis function

$$\Psi_F^{\mathsf{cr}} \in \mathrm{CR}_1 \text{ defined by } \Psi_F^{\mathsf{cr}}(m_{F'}) = \delta_{FF'}, \quad \forall F' \in \mathcal{F}^i. \tag{2.83}$$

Here $m_{F'} = \frac{1}{d}\sum_{z \in \mathcal{V} \cap F'} z$ is the barycenter of F'. The CR_1 bubble is not in $H_0^1(\Omega)$ but the smoother $\mathcal{E}_1$ transforms it into an H_0^1-function, which can be used as a test function. With these functions, we can define the alternative test function spaces for each $z \in \mathcal{V}$ as:

$$S_z^{\mathsf{cr},1} := \operatorname{span}\left(\left\{\mathcal{E}_1 \Psi_F^{\mathsf{cr}} \mid F \in \mathcal{F}_z^i\right\} \cup \left\{\Psi_T \mid T \in \mathcal{T}_z\right\}\right) \subset H_0^1(\omega_z^+). \tag{2.84}$$

Note, that $\mathcal{E}_1(\Psi_F^{\mathsf{cr}}) \subset H_0^1(\omega_z^+)$ because $\mathcal{E}_1(v_h)$ might have larger support than its argument $v_h \in \mathrm{CR}_1$ with $\operatorname{supp}(v_h) \subset \omega_z$.

In principle, we could use the space D_z^1 again to obtain a projection. But with this space of simple functionals, we cannot bound the new oscillation by the classical one as discussed above, see Corollary 2.63. Because $S_z^{\mathsf{cr},1} \subset H_0^1(\omega_z^+)$ such that $\tilde{V}_z \subset H_0^1(\omega_z^+)$ but $\tilde{V}_z \not\subset H_0^1(\omega_z)$, we only get the estimate

$$\|f - \mathcal{P}_z f\|_{H^{-1}(\omega_z)} \lesssim \inf_{g \in S_0^0(\mathcal{T}_z)} \|f - g\|_{H^{-1}(\omega_z^+)}, \tag{2.85}$$

where $\mathcal{P}_z$ is the projection induced by $S_z^{\mathsf{cr},1}$ and D_z^1. In the last expression of (2.85), on each $T \in \mathcal{T}_z^+ \backslash \mathcal{T}_z$, f is approximated only by 0. Thus, $\inf_{g \in S_0^0(\mathcal{T}_z)} \|f - g\|_{H^{-1}(\omega_z^+)}$ cannot be bounded by the classical oscillation.

To mitigate this problem, we introduce the augmented space $D_z^{+,1}$, see [KRVZ25, (4.18)], defined as

$$\begin{aligned} D_z^{+,1} := \Big\{ \chi \in H^{-1}(\Omega) \;\Big|\; \chi(\psi) = \sum_{T \in \mathcal{T}_z^+} \int_T r_T \psi + \sum_{F \in \mathcal{F}_z^i} \int_F r_F \psi \\ \text{for } \psi \in H_0^1(\Omega) \text{ with } r_T \in \mathbb{P}_0(T) \text{ and } r_F \in \mathbb{P}_0(F) \text{ for } T \in \mathcal{T}_z^+, F \in \mathcal{F}_z^i \Big\}. \end{aligned} \tag{2.86}$$

Then the projection $\mathcal{P}_z$ induced by the enlarged space of simple functionals $D_z^{+,1}$ and suitably enlarged space of simple functions $S_z \subset H_0^1(\omega_z^+)$ permits the following computation for all $f \in H^{-1}(\omega_z^+)$ and $g \in S_0^0(\mathcal{T}_z^+)$ (similar to (2.82) and [KRVZ25, Theorem 4.9]):

$$\|f - \mathcal{P}_z f\|_{H^{-1}(\omega_z)} \leqslant \|f - g\|_{H^{-1}(\omega_z)} + \|g - \mathcal{P}_z f\|_{H^{-1}(\omega_z)} \lesssim \|f - g\|_{H^{-1}(\omega_z^+)} \tag{2.87a}$$

since $w \mapsto \int_{\omega_z^+} gw \in D_z^{+,1}$, $w \in H_0^1(\omega_z^+)$. We also have

$$\|f - \mathcal{P}_z f\|_{H^{-1}(\omega_z)} \lesssim \inf_{g \in S_0^0(\mathcal{T}_z^+)} \|f - g\|_{H^{-1}(\omega_z^+)}. \tag{2.87b}$$

Again, if $f \in L^2(\omega_z^+)$, then the last term can be estimated as

$$\inf_{g \in S_0^0(\mathcal{T}_z^+)} \|f - g\|_{H^{-1}(\omega_z^+)} \lesssim h_z \inf_{g \in S_0^0(\mathcal{T}_z^+)} \|f - g\|_{L^2(\omega_z^+)}. \tag{2.87c}$$

As this is a beneficial property, we only consider the projection called $\mathcal{P}_z^{1+,\mathsf{cr}}$ induced by $D_z^{1,+}$ and

$$S_z^{\mathsf{cr},1+} = \operatorname{span}\left(\left\{\mathcal{E}_1 \Psi_F^{\mathsf{cr}} \mid F \in \mathcal{F}_z^i\right\} \cup \left\{\Psi_T \mid T \in \mathcal{T}_z^+\right\}\right) \subset H_0^1(\omega_z^+)$$

with $\tilde{V}_z = H_0^1(\omega_z) + S_z^{\mathsf{cr},1+}$. We provide the analogous result to Proposition 2.60 for this new pair of spaces.

Proposition 2.64 ([KRVZ25, Lemma 4.8]). *The spaces $D_z^{1,+}$ and $S_z^{cr,1+}$ with $\tilde{V}_z = H_0^1(\omega_z) + S_z^{cr,1+}$ fulfill Assumption 2.19, thus (2.22) defines a projection $\mathcal{P}_z^{1+,cr} : (S_z^{cr,1+})' \to D_z^{1,+}$, which satisfies for all $f \in (S_z^{cr,1+})'$*

$$\begin{aligned}\|\mathcal{P}_z^{1+,cr}(f)\|_{H^{-1}(\omega_z)} &\lesssim \big(\sum_{T\in\mathcal{T}_z} h_z^2\|r_T\|_{L^2(T)}^2\big)^{\frac12} + \big(\sum_{F\in\mathcal{F}_z^i} h_z\|r_F\|_{L^2(F)}^2\big)^{\frac12}\\ &\lesssim \|\mathcal{P}_z^{1+,cr}(f)\|_{(S_z^{cr,1+})'} \leqslant \|\mathcal{P}_z^{1+,cr}(f)\|_{H^{-1}(\omega_z^+)}.\end{aligned} \tag{2.88}$$

Here r_T, r_F are given by

$$r_T = \langle f, \Psi_T\rangle \qquad \text{and} \qquad \int_F r_F\mathcal{E}_1\Psi_F^{cr} = \langle f, \mathcal{E}_1\Psi_F^{cr}\rangle - \sum_{T\in\mathcal{T}_F}\int_T r_T\mathcal{E}_1\Psi_F^{cr}, \tag{2.89}$$

for $T \in \mathcal{T}_z$ and $F \in \mathcal{F}_z^i$. Additionally, we can write the $H^{-1}(\omega_z)$-norm of $\mathcal{P}_z^{1+,cr}(f)$, $f \in (S_z^{cr,1+})'$, as a sum of test function evaluations

$$\begin{aligned}\|\mathcal{P}_z^{1+,cr}(f))\|_{H^{-1}(\omega_z)}^2 &\lesssim \|\mathcal{P}_z^{1+,cr}(f)\|_{(S_z^{cr,1+})'} \lesssim \sum_{T\in\mathcal{T}_z^+}\frac{\langle f,\Psi_T\rangle^2}{\|\nabla\Psi_T\|_{L^2(\omega_z)}^2} + \sum_{F\in\mathcal{F}_z^i}\frac{\langle f,\mathcal{E}_1\Psi_F^{cr}\rangle^2}{\|\nabla\mathcal{E}_1\Psi_F^{cr}\|_{L^2(\omega_z^+)}^2}\\ &\lesssim \|\mathcal{P}_z^{1+,cr}(f)\|_{(S_z^{cr,1+})'} \leqslant \|\mathcal{P}_z^{1+,cr}(f)\|_{H^{-1}(\omega_z^+)}.\end{aligned} \tag{2.90}$$

Remark 2.65. The condition $\mathrm{CR}_1 \subset V_h$ is not needed for the well-posedness of the projection operator but only to simplify the representation of the estimator as discussed in Theorem 2.69. The projection can be applied to methods without these conditions, but then the representation cannot be simplified, see below.

Proof. As before, the space of simple functionals $D_z^{1,+}$ has for each element $T \in \mathcal{T}_z^+$ and each face $F \in \mathcal{F}_z^i$ one degree of freedom and correspondingly the space of test functions $S_z^{cr,1+}$ has for each element in $\mathcal{T}_z^+$ and side in $\mathcal{F}_z^i$ one degree of freedom, hence the dimensions of both vector spaces match: $\dim(D_z^{1,+}) = \dim(S_z^{cr,1+})$.

To show the bound (2.21), we fix a $\chi \in D_z^{1,+}$ and employ Lemma 2.57, which is also valid for functionals from $D_z^{1,+}$. Let us write $\langle\chi,\psi\rangle = \sum_{T\in\mathcal{T}_z}\int_T r_T\psi + \sum_{F\in\mathcal{F}_z^i}\int_F r_F\psi$ for $\psi \in H_0^1(\omega_z)$, $r_T \in \mathbb{R}$, $T \in \mathcal{T}_z$, and $r_F \in \mathbb{R}$, $F \in \mathcal{F}_z^i$. We as in Lemma 2.57

$$\|\chi\|_{H^{-1}(\omega_z)} \lesssim \big(\sum_{T\in\mathcal{T}_z} h_z^2\|r_T\|_{L^2(T)}^2\big)^{\frac12} + \big(\sum_{F\in\mathcal{F}_z^i} h_z\|r_F\|_{L^2(F)}^2\big)^{\frac12}. \tag{2.91}$$

For the element contributions, $\|r_T\|_{L^2(T)}^2$, we apply (2.74). For the face contributions, we perform a similar estimate as in (2.75) but use $\mathcal{E}_1\Psi_F^{\mathsf{cr}}$ instead of Ψ_F:

$$\begin{aligned}\|r_F\|_{L^2(F)}^2 &= \int_F r_F^2\mathcal{E}_1\Psi_F^{\mathsf{cr}} = \langle\chi, r_F\mathcal{E}_1\Psi_F^{\mathsf{cr}}\rangle - \sum_{T\in\mathcal{T}_z^+}\int_T r_T r_F\mathcal{E}_1\Psi_F^{\mathsf{cr}}\\ &\lesssim \Big(\|\chi\|_{(S_z^{cr,1+})'}^2 + \sum_{T\in\mathcal{T}_z^+} h_z^2\|r_T\|_{L^2(T)}^2\Big)^{1/2}\|r_F\nabla\mathcal{E}_1\Psi_F^{\mathsf{cr}}\|_{L^2(\omega_z^+)}.\end{aligned} \tag{2.92}$$

The last factor was estimated with Poincaré's inequality. The element contributions are again estimated as in (2.74). We apply the stability of $\mathcal{E}_1$ and an inverse estimate similar to (2.76) and obtain:

$$\|r_F\nabla\mathcal{E}_1\Psi_F^{\mathsf{cr}}\|_{L^2(\omega_z^+)} \lesssim \|r_F\nabla_h\Psi_F^{\mathsf{cr}}\|_{L^2(\omega_z)} \lesssim h_F^{-\frac12}\|r_F\|_{L^2(F)}. \tag{2.93}$$

This completes the proof of (2.21).

Accordingly, Assumption 2.19 is satisfied and by Proposition 2.20, we conclude that the spaces $D_z^{1,+}$ and $S_z^{\mathsf{cr},1+}$ induce via (2.22) a stable projection $\mathcal{P}_z^{1+,\mathsf{cr}} : (S_z^{\mathsf{cr},1+})' \to D_z^{1,+}$. The bound (2.88) follows by using (2.21) with $\chi = \mathcal{P}_z^{1+,\mathsf{cr}}(f)$, $f \in (S_z^{\mathsf{cr},1+})'$.

The hierarchical representation (2.90) follows by (2.88) and Lemma 2.26. □

To obtain the estimator, we apply the projection to the residual similar to Corollary 2.61.

Corollary 2.66 ([KRVZ25, Theorem 4.9]). *For $u \in H_0^1(\Omega)$ the solution of (1.30) with $f \in H^{-1}(\Omega)$ and $u_h \in V_h$ the discrete solution of P-LAG, P-CR1, P-SIP, P-NSIP, P-RSIP or P-RNSIP for $\ell = 1$, we have*

$$\begin{aligned}\|\operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f,u_h)\|_{V_z'} &\leqslant \|\mathcal{P}_z^{1+,cr}\operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f,u_h)\|_{V_z'} + \|\mathcal{P}_z^{1+,cr}f - f\|_{V_z'}\\ &\leqslant (1+2\|\mathcal{P}_z^{1+,cr}\|_{\mathbb{L}((S_z^{cr,1+})',V_z')})\|\operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f,u_h)\|_{H^{-1}(\omega_z^+)}\end{aligned}$$

and as in Corollary 2.23 the latter factor is bounded by

$$\|\operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f,u_h)\|_{H^{-1}(\omega_z^+)}^2 \leqslant C_{\mathsf{P}}^2 \sum_{y\in\mathcal{V}\cap\omega_z^+} \|\operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f,u_h)\|_{H^{-1}(\omega_y)}^2.$$

Here, C_{P} is the constant of the local localization provided in Proposition 2.35.

Proof. The Corollary is a direct consequence of Proposition 2.22 because Condition (2.29) is fulfilled. The condition $(\tilde{A}V_h)_{|H_0^1(\omega_z)} \subset (D_z^{1,+})_{|H_0^1(\omega_z)}$ has been shown in Corollary 2.58. Therefore, we have to show (2.29) only for the test functions in $S_z^{\mathsf{cr},1+}$ that are not contained in $H_0^1(\omega_z)$. Let $v_h \in V_h$ and due to (2.64), we can write

$$\langle \tilde{A}(v_h)_{|H_0^1(\omega_z^+)}, \psi\rangle = \sum_{F\in\mathcal{F}_z^{i,+}} \int_F r_F\psi, \quad \psi \in H_0^1(\omega_z^+),$$

for $r_F \in \mathbb{R}$, $F \in \mathcal{F}_z^{i,+}$.

First, there are the test functions Ψ_T for each $T \in \mathcal{T}_z^+\backslash\mathcal{T}_z$. Each of these fulfills

$$\langle \tilde{A}(v_h), \Psi_T\rangle = \int_T r_T\Psi_T = 0 \tag{2.94}$$

as $\tilde{A}(v_h)$ does not contain element contributions.

Second, in $S_z^{\mathsf{cr},1+}$ there are test functions of the form $\mathcal{E}_1\Psi, \Psi \in \mathrm{CR}_1$ with $\operatorname{supp}(\Psi) \subset \omega_z$. The momentum condition (1.36) of $\mathcal{E}_1$ implies for each $F \in \mathcal{F}_z^{i,+}\backslash\mathcal{F}_z^i$ and $r_F \in \mathbb{P}_0(F)$

$$\int_F r_F\mathcal{E}_1\Psi = \int_F r_F\Psi = 0$$

due to $\operatorname{supp}(\Psi) \subset \omega_z$. This insight allows us to rewrite (2.94) as

$$\langle \tilde{A}(v_h), s\rangle = \sum_{F\in\mathcal{F}_z^{i,+}} \int_F r_F s = \sum_{F\in\mathcal{F}_z^{i}} \int_F r_F s, \quad \forall v_h \in V_h, s \in S_z^{\mathsf{cr},1+},$$

with appropriately chosen r_F, $F \in \mathcal{F}_z^{i,+}$.

Thus, we have proven (2.29) in this setting. □

The construction of the test functions $S_z^{\mathsf{cr},1+}$ allows us to simplify the estimator for some methods. The simplification exploits that often the evaluation of the residual on a function that stems from the discrete space of the method can be bounded. As we use Crouzeix-Raviart functions to construct some functions in $S_z^{\mathsf{cr},1+}$, these functions are ideal candidates to bound their evaluation on the residual.

Still, we have to clarify how we obtain such bounds. In the case of methods without a smoothing operator, the canonical Galerkin orthogonality provides a bound by 0. In the case of our methods, the quasi-Galerkin orthogonality in Theorem 2.5, our remedy for the lacking Galerkin orthogonality, bounds the residual not by a local quantity but by the global conformity error $\|\Pi_V u_h - u_h\|_{\tilde{V}}$, where $u_h \in V_h$ is the solution of the method. Because the estimator works locally, this is undesirable. Nevertheless, in our examples, we can derive a local quasi-Galerkin orthogonality, which is bounded by local quantities. These bounds are specific to the methods. To argue independently of the method, we introduce the following notion.

Definition 2.67. *We define the* local conformity error *for a patch ω_z, $z \in \mathcal{V}$, as*

$$\mathbf{CE}_z(u_h) = \sup_{\substack{v_h\in V_h\\ \operatorname{supp}(v_h)\subset\omega_z}} \frac{\langle \operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f,u_h), \mathrm{E_M}\, v_h\rangle}{\|v_h\|_{\tilde{V}}}.$$

Here, $u \in V$ is solution of (1.30) and $u_h \in V_h$ is the corresponding discrete solution obtained by a method using the smoother $\mathrm{E_M}$.

For different methods, the next Lemma provides bounds for the local conformity error. We directly derive the bounds for the arbitrary order case.

Lemma 2.68. *In the cases of P-CR1 and P-CRL the local conformity error* $\mathbf{CE}_z(u_h)$ *vanishes for all* $z \in \mathcal{V}$. *In the cases of the P-SIP, P-NSIP, P-RSIP, P-RNSIP methods, we have for* $w_h \in V_h$ *with* $\operatorname{supp}(w_h) \subset \omega_z$, $z \in \mathcal{V}$, *that*

$$\begin{aligned}\langle \operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f,u_h), \mathcal{E}_\ell w_h\rangle &= \sum_{F'\in\mathcal{F}_z} \theta \int_{F'} [\![u_h]\!] \{\{\nabla_h w_h \cdot \mathsf{n}_{F'}\}\} + \sum_{F'\in\mathcal{F}_z} \int_{F'} \frac{\gamma}{h_{F'}} [\![u_h]\!] [\![w_h]\!] \\ &\lesssim \left(\sum_{F'\in\mathcal{F}_z} \int_{F'} \frac{1+\gamma}{h_{F'}} |[\![u_h]\!]|^2 \right)^{\frac{1}{2}} \|w_h\|_{\tilde{V}}.\end{aligned} \tag{2.95}$$

Summing over all $z \in \mathcal{V}$, *we obtain*

$$\sum_{z\in\mathcal{V}} \mathbf{CE}_z(u_h)^2 \lesssim \frac{1+\gamma}{\gamma} \|\Pi_V u_h - u_h\|_{\tilde{V}}^2. \tag{2.96}$$

Proof. For P-CR1, P-CRL methods, the result follows from the overconsistency, i.e. $\delta_{V_h} = 0$, see [VZ19a].

In the other cases, we have for $w_h \in S_\ell^0(\mathcal{T})$ with $\operatorname{supp}(w_h) \subset \omega_z$:

$$\tilde{a}^{\mathsf{dG}}(u-u_h, \mathcal{E}_\ell w_h) = \langle f, \mathcal{E}_\ell w_h\rangle - \tilde{a}^{\mathsf{dG}}(u_h, \mathcal{E}_\ell w_h).$$

To use the discrete equation, we perform the following calculation with the help of Lemma 1.46:

$$\begin{aligned}\tilde{a}^{\mathsf{dG}}(u_h, \mathcal{E}_\ell w_h) &= \tilde{a}^{\mathsf{dG}}(u_h, w_h) + \tilde{a}^{\mathsf{dG}}(u_h, \mathcal{E}_\ell w_h - w_h) \\ &= \tilde{a}^{\mathsf{dG}}(u_h, w_h) - \sum_{F'\in\mathcal{F}_z} \int_{F'} \{\{\nabla_h u_h \cdot \mathsf{n}_{F'}\}\} [\![w_h]\!] - \int_{F'} \frac{\gamma}{h_{F'}} [\![u_h]\!] [\![w_h]\!] \\ &= \underbrace{\tilde{a}^{\mathsf{dG}}(u_h, w_h) + \sum_{F'\in\mathcal{F}_z} \theta \int_{F'} [\![u_h]\!] \{\{\nabla_h w_h \cdot \mathsf{n}_{F'}\}\} - \int_{F'} \{\{\nabla_h u_h \cdot \mathsf{n}_{F'}\}\} [\![w_h]\!]}_{=a_h(u_h,w_h)} \\ &\quad - \int_{F'} \frac{\gamma}{h_{F'}} [\![u_h]\!] [\![w_h]\!] - \theta \int_{F'} [\![u_h]\!] \{\{\nabla_h w_h \cdot \mathsf{n}_{F'}\}\}.\end{aligned} \tag{2.97}$$

This yields the first identity in (2.95). The second estimate follows with Hölder's inequality and the estimate in [VZ18b, (3.20)]

$$\|h_F^{\frac{1}{2}} \{\{\nabla_h w_h\}\}\|_{L^2(F)}^2 \lesssim \|\nabla_h w_h\|_{L^2(\omega_F)}^2.$$

The estimate (2.96) follows by applying (2.35) using the operator $\mathcal{E}_\ell$ followed by (2.33). □

With the bounds on the local conformity error, we have the following global estimates.

Theorem 2.69 ([KRVZ25, Theorem 4.9])**.** *Let* $u \in H_0^1(\Omega)$ *be the solution of* (1.30). *We consider the methods P-CR1, P-SIP, P-NSIP, P-RSIP and P-RNSIP with* $\ell = 1$ *and let* $u_h \in V_h$ *be the discrete solution. Using the projection* $\mathcal{P}_z^{1+,\mathsf{cr}}$ *induced by* $D_z^{1,+}$ *and* $S_z^{\mathsf{cr},1+}$, $z \in \mathcal{V}$, *we have the representation*

$$\eta_z^2 \lesssim \|\mathcal{P}_z^{1+,\mathsf{cr}}(\operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f,u_h))\|_{(S_z^{\mathsf{cr},1+})'}^2 \lesssim \eta_z^2 + \mathbf{CE}_z^2(u_h), \tag{2.98}$$

where $\mathbf{CE}_z(u_h)$ *is the local conformity error, and*

$$\eta_z^2 := \sum_{T\in\mathcal{T}_z^+} h_z^2 \|r_T\|_{L^2(T)}^2 \quad \text{or} \quad \eta_z^2 := \sum_{T\in\mathcal{T}_z^+} \frac{\langle \mathcal{P}_z^{1+,\mathsf{cr}}(\operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f,u_h)), \Psi_T\rangle^2}{\|\nabla\Psi_T\|_{L^2(\omega_z)}^2}. \tag{2.99}$$

The variables $r_T \in \mathbb{R}$ *are obtained by replacing* f *through* $\operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f,u_h)$ *in* (2.89).

We have the local bound

$$\eta_z^2 + \|\mathcal{P}_z^{1+,\mathsf{cr}} f - f\|_{H^{-1}(\omega_z)}^2 \lesssim \|\nabla_h(u-u_h)\|_{L^2(\omega_z^+)}^2. \tag{2.100}$$

We have with

$$\operatorname{est}^2 := \sum_{z\in\mathcal{V}} \eta_z^2 + \|\mathcal{P}_z^{1+,\mathsf{cr}} f - f\|_{H^{-1}(\omega_z)}^2 + \|\Pi_V u_h - u_h\|_{\tilde{V}}^2$$

the global estimates

$$\operatorname{est}^2 \lesssim \|u-u_h\|_{\tilde{V}}^2 \lesssim \operatorname{est}^2 + \sum_{z\in\mathcal{V}} \mathbf{CE}_z^2(u_h) \lesssim \operatorname{est}^2. \tag{2.101}$$

The hidden constants only depend on d, $\sigma(\mathcal{T})$, *and in the case of the dG-methods also on* γ.

Proof. We discuss the option $\eta_z^2 := \sum_{T\in\mathcal{T}_z^+} h_z^2\|r_T\|_{L^2(T)}^2$ at first. Then, the first bound in (2.98) is a consequence of (2.74). To show the second bound in (2.98), we recall from (2.88) that

$$\|\mathcal{P}_z^{1+,\mathsf{cr}}(\operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f,u_h))\|_{H^{-1}(\omega_z)}^2 \lesssim \sum_{T\in\mathcal{T}_z} h_z^2\|r_T\|_{L^2(T)}^2 + \sum_{F\in\mathcal{F}_z^i} h_z\|r_F\|_{L^2(F)}^2.$$

The variables $r_T \in \mathbb{R}$, $T\in\mathcal{T}_z$, and $r_F\in\mathbb{R}$, $F\in\mathcal{F}_z^i$, are obtained in the analogous way as in (2.89). We intend to bound the terms $\|r_F\|_{L^2(F)}$ by $\mathbf{CE}_z(u_h)$. By testing the functional

$$\langle \mathcal{P}_z^{1+,\mathsf{cr}}\operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(u_h),\cdot\rangle = \sum_{T\in\mathcal{T}_z^+}\int_T r_T\cdot + \sum_{F'\in\mathcal{F}_z^i}\int_{F'} r_{F'}\cdot, \tag{2.102}$$

(the variables $r_T\in\mathbb{R}$, $T\in\mathcal{T}_z$ are obtained analogously to (2.89)) with the function $r_F\mathcal{E}_1\Psi_F^{\mathsf{cr}}$, Definition 2.67 leads to

$$\sum_{T\in\mathcal{T}_z^+}\int_T r_T r_F\mathcal{E}_1\Psi_F^{\mathsf{cr}} + \sum_{F'\in\mathcal{F}_z^i}\int_{F'} r_{F'}r_F\mathcal{E}_1\Psi_F^{\mathsf{cr}} = \langle \mathcal{P}_z^{1+,\mathsf{cr}}\operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(u_h), r_F\mathcal{E}_1\Psi_F^{\mathsf{cr}}\rangle \lesssim \mathbf{CE}_z(u_h)\|r_F\Psi_F^{\mathsf{cr}}\|_{\tilde V}. \tag{2.103}$$

We note that $\int_{F'}\mathcal{E}_1\Psi_F^{\mathsf{cr}} = \delta_{F',F}|F|$ for all $F',F\in\mathcal{F}_z^i$ and therefore $\|r_F\|_{L^2(F)}^2 = \sum_{F'\in\mathcal{F}_z^i}\int_{F'} r_{F'}r_F\mathcal{E}_1\Psi_F^{\mathsf{cr}}$. Inserting this insight into (2.103) and reordering the resulting terms provides

$$\|r_F\|_{L^2(F)}^2 \lesssim \int_F r_F^2\mathcal{E}_1\Psi_F^{\mathsf{cr}} \lesssim \mathbf{CE}_z(u_h)\|r_F\Psi_F^{\mathsf{cr}}\|_{\tilde V} + \left|\sum_{T\in\mathcal{T}_z^+}\int_T r_T r_F\mathcal{E}_1\Psi_F^{\mathsf{cr}}\right|. \tag{2.104}$$

We estimate the last summand, i.e. the element contributions as

$$\sum_{T\in\mathcal{T}_z^+}\int_T |r_T r_F\mathcal{E}_1\Psi_F^{\mathsf{cr}}| \leqslant \Big(\sum_{T\in\mathcal{T}_z^+}\|r_T\|_{L^2(T)}^2\Big)^{\frac12}\Big(\sum_{T\in\mathcal{T}_z^+}\|r_F\mathcal{E}_1\Psi_F^{\mathsf{cr}}\|_{L^2(T)}^2\Big)^{\frac12} \lesssim \Big(\sum_{T\in\mathcal{T}_z^+} h_z^2\|r_T\|_{L^2(T)}^2\Big)^{\frac12}\|r_F\Psi_F^{\mathsf{cr}}\|_{V_h},$$

where we used in last factor an inverse estimate and the stability of $\mathcal{E}_1$, c.f. Proposition 1.10 and also [VZ19a, Proposition 3.3]. An inverse estimate similar to the one in (2.76) provides

$$\|r_F\Psi_F^{\mathsf{cr}}\|_{V_h} \lesssim h_F^{-\frac12}\|r_F\|_{L^2(F)}.$$

Inserting the last two estimates into (2.104), we have

$$\|r_F\|_{L^2(F)}^2 \lesssim \Big(\mathbf{CE}_z(u_h) + \Big(\sum_{T\in\mathcal{T}_z^+} h_z^2\|r_T\|_{L^2(T)}^2\Big)^{\frac12}\Big)h_F^{-\frac12}\|r_F\|_{L^2(F)}.$$

This finishes the proof of (2.98) for the first choice of η_z^2.

We turn to the second choice of η_z^2 given in (2.99). The bound

$$\eta_z^2 \lesssim \|\mathcal{P}_z^{1+,\mathsf{cr}}(\operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f,u_h))\|_{H^{-1}(\omega_z^+)}^2 \text{ for the choice } \eta_z^2 := \sum_{T\in\mathcal{T}_z^+}\frac{\langle \mathcal{P}_z^{1+,\mathsf{cr}}(\operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f,u_h)),\Psi_T\rangle^2}{\|\nabla\Psi_T\|_{L^2(\omega_z)}^2} \tag{2.105}$$

follows directly from the finite overlap property (1.22). For the other direction in (2.98), we use (2.90) with f replaced by $\operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f,u_h)$. Then we apply the local quasi-Galerkin orthogonality of Definition 2.67 to the face contributions and obtain the desired result.

The local bound (2.100) follows for each $z\in\mathcal{V}$ from the first bound in (2.98) and (2.30a)

$$\begin{aligned}\eta_z^2 + \|\mathcal{P}_z^{1+,\mathsf{cr}}f - f\|_{H^{-1}(\omega_z)}^2 &\lesssim \|\mathcal{P}_z^{1+,\mathsf{cr}}(\operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f,u_h))\|_{(S_z^{\mathsf{cr},1+})'}^2 + \|\mathcal{P}_z^{1+,\mathsf{cr}}f - f\|_{H^{-1}(\omega_z)}^2\\ &\lesssim \|\operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f,u_h)\|_{H^{-1}(\omega_z^+)} \lesssim \|\nabla_h(u-u_h)\|_{L^2(\omega_z^+)}^2.\end{aligned} \tag{2.106}$$

Summing over all $z\in\mathcal{V}$ in (2.106) gives the first inequality in (2.101) because of the finite overlap property of the mesh, c.f (1.22). The second bound in (2.101) follows by splitting the error into the smoothed error and the conformity error as in (2.5). Then the conformity error is bounded by the jump expressions as depicted in Proposition 2.27. The conforming residual $\|\operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f,u_h)\|_{V'}$ can be localized since Assumption 2.6 is true for all considered methods, see also Proposition 2.32. Decomposing the localized residuals with $\mathcal{P}_z^{1+,\mathsf{cr}}$ as presented in Corollary 2.66 and using (2.98) completes the proof. □

The technique that some components of the estimator can be eliminated was already observed for the classical, non quasi-optimal predecessor P-CR1 in [DDPV96] and [CGH13, p. 317].

2.3.1.2 The linear elasticity example

Before turning to higher order methods, we discuss, how the projections $\mathcal{P}_z^1$ and $\mathcal{P}_z^{1+,\mathsf{cr}}$ can be applied to the linear elasticity example (E-CR1). In this section, we assume that $d = 2$ and that the spaces $H_0^1(\Omega)^2$ is equipped with the scalar product $\tilde{a}^{\mathsf{LE}}$, which was defined in section 1.4.3, In particular, this means, that $H_0^1(\Omega)^2$ is equipped with the norm $\|\cdot\|_V^2 = \|\cdot\|_{\mathsf{LE}}^2 := a_{\mathsf{LE}}(\cdot,\cdot)$. The space $H^{-1}(\Omega)^2$ is the corresponding dual space.

To derive a projection, we enhance the spaces D_z^1 and $D_z^{+,1}$ for each $z \in \mathcal{V}$ to the two-dimensional setting:

$$\begin{aligned}\left(D_z^1\right)^2 = \Big\{\chi \in H^{-1}(\Omega)^2 \;\Big|\; \chi(\psi) = \sum_{T\in\mathcal{T}_z}\int_T r_T\psi + \sum_{F\in\mathcal{F}_z^i}\int_F r_F\psi \\ \text{for } \psi \in H_0^1(\Omega)^2 \text{ with } r_T \in \mathbb{R}^2 \text{ and } r_F \in \mathbb{R}^2 \text{ for } T \in \mathcal{T}_z,\ F \in \mathcal{F}_z^i\Big\}\end{aligned}$$

and

$$\begin{aligned}\left(D_z^{+,1}\right)^2 := \Big\{\chi \in H^{-1}(\Omega)^2 \;\Big|\; \chi(\psi) = \sum_{T\in\mathcal{T}_z^+}\int_T r_T\psi + \sum_{F\in\mathcal{F}_z^i}\int_F r_F\psi \\ \text{for } \psi \in H_0^1(\Omega)^2 \text{ with } r_T \in \mathbb{R}^2 \text{ and } r_F \in \mathbb{R}^2 \text{ for } T \in \mathcal{T}_z^+, F \in \mathcal{F}_z^i\Big\}.\end{aligned}$$

The following modification of Lemma 2.57 provides the L^2-representations of functionals from the spaces $\left(D_z^1\right)^2$ and $\left(D_z^{+,1}\right)^2$:

Lemma 2.70. *For $z \in \mathcal{V}$, let $\chi \in \left(D_z^1\right)^2$ or $\chi \in \left(D_z^{+,1}\right)^2$ and assume that the restriction $\chi_{|H_0^1(\omega_z)^2}$ can be written as*

$$\chi_{|H_0^1(\omega_z)^2}(\psi) = \sum_{T\in\mathcal{T}_z}\int_T r_T\cdot\psi + \sum_{F\in\mathcal{F}_z^i}\int_F r_F\cdot\psi, \quad \psi \in H_0^1(\omega_z)^2,$$

with $r_T \in \mathbb{R}^2, r_F \in \mathbb{R}^2$, $T \in \mathcal{T}_z, F \in \mathcal{F}_z^i$, then χ is bounded by

$$\|\chi\|_{H^{-1}(\omega_z)^2} \lesssim \Big(\sum_{T\in\mathcal{T}_z}\frac{h_z^2}{2\mu}\|r_T\|_{L^2(T)}^2\Big)^{\frac{1}{2}} + \Big(\sum_{F\in\mathcal{F}_z^i}\frac{h_z}{2\mu}\|r_F\|_{L^2(F)}^2\Big)^{\frac{1}{2}}. \tag{2.107}$$

This hidden constant is independent of μ, λ and γ.

Proof. The proof works in the same way as the one of Lemma 2.57 except that we use in (2.67) and (2.68) Poincaré's and Korn's inequalities (see [Bre04, Theorem 3.1] for Korn's inequality) in the following way:

$$2\mu\|v\|_{L^2(\omega_z)}^2 \lesssim 2\mu h_z^2\|\nabla v\|_{L^2(\omega_z)}^2 \lesssim 2\mu h_z^2\|\varepsilon(v)\|_{L^2(\omega_z)}^2 + \lambda h_z^2\|\operatorname{div} v\|_{L^2(\omega_z)}^2$$

for $v \in H_0^1(\omega_z)^2$ and $z \in \mathcal{V}$. □

For each $z \in \mathcal{V}$, we complement these spaces of simple functionals with spaces of simple functions that are based upon S_z^1 and $S_z^{\mathsf{cr},1+}$:

$$(S_z^1)^2 := \operatorname{span}\left(\{\Psi_F e_i \mid i = 1,2;\ F \in \mathcal{F}_z^i\} \cup \{\Psi_T e_i \mid i = 1,2;\ T \in \mathcal{T}_z\}\right) \subset H_0^1(\omega_z)^2. \tag{2.108}$$

$$(S_z^{\mathsf{cr},1+})^2 := \operatorname{span}\left(\left\{\mathcal{E}_1^2(\Psi_F^{\mathsf{cr}} e_i) \mid i = 1,2;\ F \in \mathcal{F}_z^i\right\} \cup \left\{\Psi_T e_i \mid i = 1,2;\ T \in \mathcal{T}_z^+\right\}\right) \subset H_0^1(\omega_z^+)^2.$$

With these spaces, we define the vectorized projection $\widetilde{\mathcal{P}}_z^1 := (\mathcal{P}_z^1)^2$ through the pair $\left(\left(D_z^1\right)^2, (S_z^1)^2\right)$ and the projection $\widetilde{\mathcal{P}}_z^{1+,\mathsf{cr}} := (\mathcal{P}_z^{1+,\mathsf{cr}})^2$ through the pair $\left(\left(D_z^{+,1}\right)^2, (S_z^{\mathsf{cr},1+})^2\right)$. The next Theorem shows that these two projections are stable, and we can use them to split the conforming residual into an estimator and an oscillation.

Theorem 2.71. *Let $z \in \mathcal{V}$ be fixed. The vectorized projections $\widetilde{\mathcal{P}}_z^1$ and $\widetilde{\mathcal{P}}_z^{1+,\mathsf{cr}}$ are well-defined and stable in the setting of (E-CR1), described in section 1.4.3. Moreover, both projections fulfill condition* (2.29)*, hence we have for the discrete solution $u_h \in (\mathrm{CR}_1)^2$ and the exact solution $u \in H_0^1(\Omega)^2$ of* (1.60) *for a given $f \in H^{-1}(\Omega)^2$ that*

$$\begin{aligned}\|\operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f,u_h)\|_{H^{-1}(\omega_z)^2} &\lesssim \underbrace{\|\widetilde{\mathcal{P}}_z^1\operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f,u_h)\|_{H^{-1}(\omega_z)^2}}_{\eta_{LE}} + \underbrace{\|\widetilde{\mathcal{P}}_z^1 f - f\|_{H^{-1}(\omega_z)^2}}_{\operatorname{osc}_{LE}} \\ &\lesssim \|\operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f,u_h)\|_{H^{-1}(\omega_z)^2}\end{aligned} \tag{2.109a}$$

and

$$\begin{aligned}\|\operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f,u_h)\|_{H^{-1}(\omega_z)^2} &\leqslant \underbrace{\|\widetilde{\mathcal{P}}_z^{1+,\mathsf{cr}}\operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f,u_h)\|_{H^{-1}(\omega_z)^2}}_{\eta_{LE}^+} + \underbrace{\|\widetilde{\mathcal{P}}_z^{1+,\mathsf{cr}}f-f\|_{H^{-1}(\omega_z)^2}}_{\operatorname{osc}_{LE}^+}\\ &\lesssim \|\operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f,u_h)\|_{H^{-1}(\omega_z^+)^2}.\end{aligned} \tag{2.109b}$$

The last term in (2.109b) *can be bounded by*

$$\|\operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f,u_h)\|^2_{H^{-1}(\omega_z^+)^2} \lesssim \sum_{y\in\mathcal{V}\cap\omega_z^+}\|\operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f,u_h)\|^2_{H^{-1}(\omega_y)^2}. \tag{2.110}$$

The hidden constants in (2.109a), (2.109b) *depend on* μ,λ *and* γ. *To quantify the estimator* η_{LE}, *one applies either* (2.72) *or* (2.73) *with* f *replaced by* $\operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f,u_h)$ *in a two-dimensional setting. Analogously, to quantify the estimator* η_{LE}^+ *one applies either* (2.88) *or* (2.90).

Proof. At first, we establish the stability of the two projections by showing Assumption 2.19. This is done similarly to the proofs in Propositions 2.60 and 2.64.

We exemplify the proof for the projection $\widetilde{\mathcal{P}}_z^1$, the estimates for $\widetilde{\mathcal{P}}_z^{1+,\mathsf{cr}}$ are similar. As the first step, we write $\chi\in\left(D_z^1\right)^2$ of the form

$$\chi(\psi)=\sum_{T\in\mathcal{T}_z}\int_T r_T\cdot\psi+\sum_{F\in\mathcal{F}_z^i}\int_F r_F\cdot\psi,\quad \psi\in H_0^1(\omega_z)^2,$$

in its L^2-representation provided by Lemma 2.70

$$\|\chi\|_{H^{-1}(\omega_z)}\lesssim\Big(\sum_{T\in\mathcal{T}_z}\frac{h_z^2}{2\mu}\|r_T\|^2_{L^2(T)}\Big)^{\frac12}+\Big(\sum_{F\in\mathcal{F}_z^i}\frac{h_z}{2\mu}\|r_F\|^2_{L^2(F)}\Big)^{\frac12}$$

with the variables as defined there. For clarity, we write $\|\cdot\|_{S_z^{1\prime}}$ for the canonical norm of the dual space of $(S_z^1)^2$. Similar to Proposition 2.60, we estimate the element and face contributions separately. The element contributions are estimated in this setting analogous to (2.74):

$$\begin{aligned}\frac{1}{2\mu}\|r_T\|^2_{L^2(T)}&\lesssim\frac{|T|}{2\mu}\|\chi\|_{S_z^{1\prime}}\|r_T\Psi_T\|_{\mathsf{LE}}\lesssim\frac{|T|\sqrt{2\mu+\lambda}}{2\mu}\|\chi\|_{S_z^{1\prime}}\|r_T\nabla\Psi_T\|_{L^2(T)}\\&\lesssim\frac{\sqrt{2\mu+\lambda}}{2\mu}\|\chi\|_{S_z^{1\prime}}h_T^{-1}\|r_T\|_{L^2(T)}.\end{aligned}$$

Also, the face contributions can be estimated as in (2.75):

$$\frac{1}{2\mu}\|r_F\|^2_{L^2(F)}\lesssim\frac{|F|}{2\mu}\int_F r_F^2\Psi_F\quad\lesssim\Big(\frac{\sqrt{2\mu+\lambda}}{2\mu}\|\chi\|^2_{S_z^{1\prime}}+\frac{1}{2\mu}\sum_{T\in\mathcal{T}_F}h_z^2\|r_T\|^2_{L^2(T)}\Big)^{\frac12}|F|\|r_F\nabla\Psi_F\|_{L^2(\omega_z)}.$$

Using an inverse estimate as in (2.76) completes the stability proof.

We check for both projections the conditions (2.29). An integration by parts similar to (1.62) implies that

$$\begin{aligned}\tilde a_{\mathrm{LE}}(v_h,w)&=\sum_{T\in\mathcal{T}}\int_T 2\mu\varepsilon_h(v_h):\varepsilon(w)+\lambda\operatorname{div}_h v_h\operatorname{div}w\\&=\sum_{F\in\mathcal{F}_z^i}\int_F 2\mu\left[\!\left[\varepsilon(v_h)\cdot\mathsf{n}_F\right]\!\right]\cdot\{\!\{w\}\!\}+\lambda\left[\!\left[\operatorname{div}_h v_h\right]\!\right]\{\!\{w\cdot\mathsf{n}_F\}\!\}.\end{aligned}$$

for $v_h\in(\mathrm{CR}_1)^2$ and $w\in H_0^1(\omega_z)^2$. As the method utilizes only polynomials of first order, $(D_z^1)^2$ fulfills (2.29) and Proposition 2.22 is applicable in this setting with the induced projection $\widetilde{\mathcal{P}}_z^1$. Consequently, Proposition 2.22 shows (2.109a).

Turning to the projection induced by $(D_z^{1,+})^2$ with $\tilde V_z=H_0^1(\omega_z)^2+(S_z^{\mathsf{cr},1+})^2$, we see by exploiting the momentum conditions of $\mathcal{E}_1^2$ (similar to the proof of Corollary 2.66) that (2.29) is also fulfilled for $\widetilde{\mathcal{P}}_z^{1+,\mathsf{cr}}$. Again, Proposition 2.22 implies (2.109b). By Proposition 2.42, the additional bound (2.110) is obtained.

The alternative representations of the estimators are obtained as in the Poisson setting by using either the L^2-representation derived in Lemma 2.70 or using the hierarchical approach in Lemma 2.26. □

Summing over all $z \in \mathcal{I}$ in (2.109) leads to the global results.

Corollary 2.72. *Let $u_h \in (\mathrm{CR}_1)^2$ be the discrete solution* (E-CR1) *and $u \in H^1_0(\Omega)^2$ be the solution of* (1.60)*. For the projection $\widetilde{\mathcal{P}}^1_z$, we have*

$$\|u - u_h\|^2_{\mathsf{LE}} \eqsim \sum_{z\in\mathcal{V}} \left\{ \|\widetilde{\mathcal{P}}^1_z(\operatorname{res}^{\mathsf{c}}_{\mathrm{h}}(f,u_h))\|^2_{H^{-1}(\omega_z)^2} + \|f - \widetilde{\mathcal{P}}^1_z(f)\|^2_{H^{-1}(\omega_z)^2} \right\} + \|u - \Pi_V u_h\|^2_{\mathsf{LE}}. \tag{2.111}$$

Locally, the estimator is bounded as:

$$\begin{aligned} &\|\widetilde{\mathcal{P}}^1_z(\operatorname{res}^{\mathsf{c}}_{\mathrm{h}}(f,u_h))\|^2_{H^{-1}(\omega_z)^2} + \|f - \widetilde{\mathcal{P}}^1_z(f)\|^2_{H^{-1}(\omega_z)^2} \\ &\quad\lesssim \|\operatorname{res}^{\mathsf{c}}_{\mathrm{h}}(f,u_h)\|^2_{H^{-1}(\omega_z)^2} \leqslant 2\mu\|\varepsilon_h(u-u_h)\|^2_{L^2(\omega_z)} + \lambda\|\operatorname{div}_h(u-u_h)\|^2_{L^2(\omega_z)}. \end{aligned} \tag{2.112}$$

The hidden constants only depend on $\sigma(\mathcal{T})$, μ, λ and γ.

In the case of the projection $\widetilde{\mathcal{P}}^{1+,\mathsf{cr}}_z$, we can estimate the face contributions in the L^2-representation given in (2.107) or the hierarchical representation with the local conformity error introduced in Definition 2.67.

Corollary 2.73. *Fix $z \in \mathcal{V}$. Let $u_h \in (\mathrm{CR}_1)^2$ be the discrete solution* (E-CR1) *and $u \in H^1_0(\Omega)^2$ be the solution of* (1.60)*. The local conformity error is bounded as*

$$\mathbf{CE}_z(u_h) \leqslant \sum_{F\in\mathcal{F}_z} \int_F \frac{\mu\gamma}{h_F} |[\![u_h]\!]|^2. \tag{2.113}$$

The projection $\widetilde{\mathcal{P}}^{1+,\mathsf{cr}}_z$ fulfills the corresponding results in Theorem 2.69. In particular, we have

$$\eta_z^2 \lesssim \|\widetilde{\mathcal{P}}^{1+,\mathsf{cr}}_z(\operatorname{res}^{\mathsf{c}}_{\mathrm{h}}(f,u_h))\|^2_{\left((S^{\mathsf{cr},1+}_z)^2\right)'} \lesssim \eta_z^2 + \mathbf{CE}_z^2(u_h),$$

with

$$\eta_z^2 := \sum_{T\in\mathcal{T}_z^+} h_z^2 \|r_T\|^2_{L^2(T)} \quad \text{or} \quad \eta_z^2 := \sum_{T\in\mathcal{T}_z^+} \sum_{i=1,2} \frac{\langle \widetilde{\mathcal{P}}^{1+,\mathsf{cr}}_z(\operatorname{res}^{\mathsf{c}}_{\mathrm{h}}(f,u_h)), \Psi_T e_i\rangle^2}{\|\nabla\Psi_T\|^2_{L^2(\omega_z)}},$$

where $r_T \in \mathbb{R}^2$ is obtained for all $T \in \mathcal{T}_z^+$ via

$$r_T \cdot e_i = \langle \operatorname{res}^{\mathsf{c}}_{\mathrm{h}}(f,u_h), \Psi_T e_i\rangle, \quad i = 1,2.$$

We have the local bound

$$\begin{aligned} \eta_z^2 + \|\widetilde{\mathcal{P}}^{1+,\mathsf{cr}}_z f - f\|^2_{H^{-1}(\omega_z)^2} &\lesssim \|\operatorname{res}^{\mathsf{c}}_{\mathrm{h}}(f,u_h)\|^2_{H^{-1}(\omega_z^+)^2} \\ &\leqslant 2\mu\|\varepsilon_h(u-u_h)\|^2_{L^2(\omega_z^+)} + \lambda\|\operatorname{div}_h(u-u_h)\|^2_{L^2(\omega_z^+)}. \end{aligned}$$

Setting

$$\operatorname{est}^2 := \sum_{z\in\mathcal{V}} \eta_z^2 + \|\widetilde{\mathcal{P}}^{1+,\mathsf{cr}}_z f - f\|^2_{H^{-1}(\omega_z)^2} + \|\Pi_V u_h - u_h\|^2_{\tilde{V}},$$

we have the global estimate

$$\operatorname{est}^2 \lesssim \|u - u_h\|^2_{\tilde{V}} \lesssim \operatorname{est}^2 + \sum_{z\in\mathcal{V}} \mathbf{CE}_z^2(u_h) \lesssim \operatorname{est}^2.$$

The hidden constants only depend on $\sigma(\mathcal{T})$, μ, λ and γ.

Proof. From (E-CR1), we have for all $w_h \in (\mathrm{CR}_1)^2$ with $\operatorname{supp}(w_h) \subset \omega_z$ that

$$\langle \operatorname{res}^{\mathsf{c}}_{\mathrm{h}}(u_h), \mathcal{E}_1^2 w_h\rangle = \tilde{a}(u - u_h, \mathcal{E}_1^2 w_h) = \tilde{a}(u_h, w_h - \mathcal{E}_1^2 w_h).$$

A similar calculation as in (1.63) for $u_h \in V_h$ shows that

$$\langle \operatorname{res}^{\mathsf{c}}_{\mathrm{h}}(u_h), \mathcal{E}_1^2 w_h\rangle = \int_\Sigma \frac{\mu\gamma}{h} [\![u_h]\!] [\![w_h]\!] = \sum_{F\in\mathcal{F}_z} \int_F \frac{\mu\gamma}{h_F} [\![u_h]\!] [\![w_h]\!].$$

We conclude the proof (2.113) with Hölder's inequality. The other results follow as in Theorem 2.69. □

2.3.1.3 Higher order methods

After treating the lowest order case, we generalize the estimator to the higher order cases. For this, we reuse the space of functionals D_z^ℓ and construct a test function space S_z^ℓ, which consists of bubble functions multiplied with polynomials, as introduced in [KRVZ25, section 4.3].

To extend a polynomial $\mathbb{P}_{\ell-1}(F)$, $F \in \mathcal{F}$, to the patch ω_F, we recall the extension operator from [Ver94], see also [KRVZ25, p.31]. On $F = \operatorname{convexhull}\{z_0 \dots, z_{d-1}\} \in \mathcal{F}$, $z_0, \dots, z_{d-1} \in \mathcal{V}$, we define the extension operator $E_F : \mathbb{P}_{\ell-1}(F) \to H^1(\omega_F)$ as

$$E_F(v)(x) := v\left(z_0 + \sum_{i=1}^{d-1} \Phi^1_{z_i}(x)(z_i - z_0)\right), \qquad v \in \mathbb{P}_{\ell-1}(F). \tag{2.114}$$

Here, $\Phi^1_{z_i}$ are the first order Lagrange basis functions. Using $E_F(v)^2 = E_F(v^2)$, we get

$$\|E_F(v)\|^2_{L^2(\omega_F)} \leqslant \frac{|\omega_F|}{|F|}\|v\|^2_{L^2(F)},$$

for a complete proof, e.g. [BCNV24, Lemma 4.20].

Remark 2.74. The extension operator E_F cannot be confused with the smoothers E since the smoothers are never written with subscript $F \in \mathcal{F}$.

With the help of E_F, we define the space of test functions for the patch ω_z, $z \in \mathcal{V}$, c.f. [KRVZ25, 4.37b]:

$$S_z^\ell := \operatorname{span}\left(\{E_F(p)\Psi_F \mid p \in \mathbb{P}_{\ell-1}(F), F \in \mathcal{F}_z^i\} \cup \{p\Psi_T \mid p \in \mathbb{P}_{\ell-1}(T), T \in \mathcal{T}_z\}\right) \subset H^1_0(\omega_z). \tag{2.115}$$

The bubbles Ψ_F and Ψ_T are defined in (2.70).

This space of simple functions defines with D_z^ℓ, which was defined in (2.65), a stable projection.

Proposition 2.75 ([KRVZ25, Lemma 4.18])**.** *The pair of spaces D_z^ℓ and S_z^ℓ fulfills* (2.21) *and induces via* (2.22) *a stable projection $\mathcal{P}_z^\ell : S_z^{\ell\prime} \to D_z^\ell$. For each $f \in H^{-1}(\omega_z)$, we can write $\mathcal{P}_z^\ell(f) \in D_z^\ell$ as*

$$\langle \mathcal{P}_z^\ell(f), \psi\rangle = \sum_{T\in\mathcal{T}_z}\int_T r_T\psi + \sum_{F\in\mathcal{F}_z^i}\int_F r_F\psi, \quad \psi \in H^1_0(\omega_z),$$

for $r_T \in \mathbb{P}_{\ell-1}(T)$, $T \in \mathcal{T}_z$, and $r_F \in \mathbb{P}_{\ell-1}(F)$, $F \in \mathcal{F}_z^i$, that are determined via (2.22)*. Moreover, we can write*

$$\|\mathcal{P}_z^\ell(f)\|_{H^{-1}(\omega_z)} \eqsim \Big(\sum_{T\in\mathcal{T}_z} h_z^2\|r_T\|^2_{L^2(T)}\Big)^{\frac12} + \Big(\sum_{F\in\mathcal{F}_z^i} h_F\|r_F\|^2_{L^2(F)}\Big)^{\frac12}. \tag{2.116}$$

The hidden constants only depend on d, $\sigma(\mathcal{T})$ and ℓ.

We choose on a reference element $\hat{T}$ a basis $\hat{p}_i^T$ of $\mathbb{P}_{\ell-1}(\hat{T})$, $1 \leqslant i \leqslant \binom{d+\ell-1}{\ell-1} =: d_{\hat{T}}$. For each $T \in \mathcal{T}_z$, the affine map $F_T : \hat{T} \to T$ provides a basis p_i^T on $\mathbb{P}_{\ell-1}(T)$ with $1 \leqslant i \leqslant d_{\hat{T}}$. Similarly, on a reference face $\hat{F}$, we choose a basis $\hat{p}_j^F \in \mathbb{P}_{\ell-1}(\hat{F})$, $1 \leqslant j \leqslant \binom{d+\ell-2}{\ell-1} =: d_{\hat{F}}$, such that we obtain for each $F \in \mathcal{F}_z^i$ with the affine map $F_F : \hat{F} \to F$ a basis p_j^F, $1 \leqslant j \leqslant d_{\hat{F}}$, of $\mathbb{P}_{\ell-1}(F)$. Then, we have the equivalence:

$$\|\mathcal{P}_z^\ell(f)\|^2_{H^{-1}(\omega_z)} \eqsim \sum_{T\in\mathcal{T}_z}\sum_{i=1}^{d_T}\frac{\langle f, p_i^T\Psi_T\rangle^2}{\|\nabla(p_i^T\Psi_T)\|^2_{L^2(\omega_z)}} + \sum_{F\in\mathcal{F}_z^i}\sum_{j=1}^{d_F}\frac{\langle f, E_F(p_j^F)\Psi_F\rangle^2}{\|\nabla(E_F(p_j^F)\Psi_F)\|^2_{L^2(\omega_z)}}. \tag{2.117}$$

In addition to the above quantities, this estimate depends also on the bases chosen on $\hat{T}$ and $\hat{F}$.

Proof. Obviously, the dimensions of S_z^ℓ and D_z^ℓ match.

The upper bound is obtained by Lemma 2.57

$$\|\mathcal{P}_z^\ell(f)\|_{H^{-1}(\omega_z)} \lesssim \Big(\sum_{T\in\mathcal{T}_z} h_z^2\|r_T\|^2_{L^2(T)}\Big)^{\frac12} + \Big(\sum_{F\in F_z^i} h_F\|r_F\|^2_{L^2(F)}\Big)^{\frac12}. \tag{2.118}$$

The new aspect in this Proposition is the higher order reverse bound. For $T \in \mathcal{T}_z$ and $r_T \in \mathbb{P}_{\ell-1}(T)$, we have

$$\begin{aligned}\|r_T\|^2_{L^2(T)} \lesssim |T|\int_T r_T^2\Psi_T = |T|\langle \mathcal{P}_z^\ell(f), r_T\Psi_T\rangle &\lesssim |T|\|\mathcal{P}_z^\ell(f)\|_{S_z^{\ell\prime}}\|\nabla(r_T\Psi_T)\|_{L^2(T)}\\ &\lesssim h_T^{-1}\|\mathcal{P}_z^\ell(f)\|_{S_z^{\ell\prime}}\|r_T\|_{L^2(T)}.\end{aligned} \tag{2.119}$$

Here, we used the equivalence of norms on $\mathbb{P}_{\ell-1}(T)$, $\|\Psi_T\|_{L^\infty(T)} \lesssim |T|^{-1}$ and an inverse estimate.

For the face contributions, $F \in \mathcal{F}_z^i$, we have with the equivalence of norms on $\mathbb{P}_{\ell-1}(F)$ and Poincaré's inequality that

$$\begin{aligned}\|r_F\|_{L^2(F)}^2 &\lesssim |F| \int_F r_F E_F(r_F)\Psi_F = |F|\big(\langle \mathcal{P}_z^\ell(f), E_F(r_F)\Psi_F\rangle - \sum_{T\in\mathcal{T}_F}\int_T r_T E_F(r_F)\Psi_F\big)\\ &\lesssim \Big(\|\mathcal{P}_z^\ell(f)\|_{S_z^{\ell\prime}}^2 + \sum_{T\in\mathcal{T}_F} h_F^2\|r_T\|_{L^2(T)}^2\Big)^{\frac12}|F|\|\nabla(E_F(r_F)\Psi_F)\|_{L^2(\omega_F)}.\end{aligned} \tag{2.120}$$

Again, this technique is similar to [Ver13, section 1.4.5].

We continue with the estimate of the last factor

$$\begin{aligned}|F|\|\nabla(E_F(r_F)\Psi_F)\|_{L^2(\omega_F)} &\lesssim |F|h_F^{-1}\|E_F(r_F)\Psi_F\|_{L^2(\omega_F)} \lesssim h_F^{-1}\|E_F(r_F)\|_{L^2(\omega_F)}\\ &\lesssim h_F^{-1}\Big(\frac{|\omega_F|}{|F|}\Big)^{\frac12}\|r_F\|_{L^2(F)} \eqsim h_F^{-\frac12}\|r_F\|_{L^2(F)},\end{aligned} \tag{2.121}$$

where we used an inverse estimate and $\|\Psi_F\|_{L^\infty(\omega_F)} \lesssim |F|^{-1}$.

Inserting (2.121) into (2.120) and bounding the element contributions with (2.119) yields the estimate:

$$\|\mathcal{P}_z^\ell(f)\|_{H^{-1}(\omega_z)} \lesssim \big(\sum_{T\in\mathcal{T}_z} h_z^2\|r_T\|_{L^2(T)}^2\big)^{\frac12} + \big(\sum_{F\in\mathcal{F}_z^i} h_F\|r_F\|_{L^2(F)}^2\big)^{\frac12} \lesssim \|\mathcal{P}_z^\ell(f)\|_{S_z^{\ell\prime}}.$$

Hence, we have shown (2.21) and Proposition 2.20 ensures the existence of the stable projection $\mathcal{P}_z^\ell : S_z^{\ell\prime} \to D_z^\ell$. Since $S_z^\ell \subset H_0^1(\omega_z)$, we also have $\|\mathcal{P}_z^\ell(f)\|_{S_z^{\ell\prime}} \leqslant \|\mathcal{P}_z^\ell(f)\|_{H^{-1}(\omega_z)}$ and thus (2.116) is valid.

The hierarchical equivalence (2.117), follows by Lemma 2.26. Because the basis functions on the elements and faces are constructed from a reference situation, the constant depends in addition to d, $\sigma(\mathcal{T})$ and ℓ also on the chosen bases on $\mathbb{P}_{\ell-1}(\hat{T})$ and $\mathbb{P}_{\ell-1}(\hat{F})$. □

Proposition 2.75 settles the local situation. To obtain a global result, we replace $\mathcal{P}_z^1$ by $\mathcal{P}_z^\ell$ in Theorem 2.62 and lift the restriction $\ell = 1$. Then the modified Theorem provides global a-posteriori error equivalences for all considered, arbitrary order methods for the Laplace problem by using the projection $\mathcal{P}_z^\ell$. Within the Theorem, the parameter γ is again the stability parameter of the DG-methods or 0 in the case of the Lagrange and Crouzeix-Raviart methods.

Also for this estimator, we can bound the new oscillation $\|f - \mathcal{P}_z^\ell f\|_{H^{-1}(\omega_z)}$, $z \in \mathcal{V}$, by the classical one if $f \in L^2(\omega_z)$. More precisely, if we use $g \in S_{\ell-1}^0(\mathcal{T}_z)$ in the computation (2.82a) then we get the following extension of Corollary 2.63 (c.f. [KRVZ25, Theorem 4.19]):

Corollary 2.76. *Given any $f \in H^{-1}(\omega_z)$, the oscillation term of $\mathcal{P}_z^\ell$ defined via D_z^ℓ and $S_z^\ell \subset H_0^1(\omega_z)$ satisfies:*

$$\|f - \mathcal{P}_z^\ell f\|_{H^{-1}(\omega_z)} \lesssim \inf_{g\in S_{\ell-1}^0(\mathcal{T}_z)} \|f - g\|_{H^{-1}(\omega_z)}. \tag{2.122}$$

If $f \in L^2(\omega_z)$, then we have

$$\|f - \mathcal{P}_z^\ell f\|_{H^{-1}(\omega_z)} \lesssim h_z \inf_{g\in S_{\ell-1}^0(\mathcal{T}_z)} \|f - g\|_{L^2(\omega_z)}.$$

To evaluate the estimator, we have to treat both face and element contributions. In the lowest order case, we constructed an alternative projection which sometimes avoids or simplifies the evaluation of the face contributions. We would like to transfer this trick to the higher order case. In the lowest order case, the special structure of the lowest-order Crouzeix-Raviart element made this possible. For the higher order cases, we proceed similarly. We define an alternative test function space using higher order Crouzeix-Raviart functions and show that the induced projection exposes similar cancellation properties as $\mathcal{P}_z^{1+,\mathsf{cr}}$.

Remark 2.77. In the higher order case, one can already cancel some parts of $\mathcal{P}_z^\ell$ in some cases. If ℓ is large enough, some of the functions in $S_z^\ell \subset H_0^1(\omega_z)$ are contained in V_h. Thus, for such a function $w_h \in S_z^\ell \cap V_h$, we have $\langle \mathrm{res}_{\mathrm{h}}^{\mathsf{c}}(f, u_h), w_h\rangle \lesssim \mathbf{CE}_z(u_h)\|w_h\|_{\tilde{V}}$, where $\mathbf{CE}_z(u_h)$ is the local conformity error introduced in Definition 2.67. But in any case, on each side and element, some test functions cannot be bounded by the local conformity error. Therefore, we do not provide further details on this topic but construct a new projection which is similar to $\mathcal{P}_z^{1+,\mathsf{cr}}$.

Also in the higher order case, we would like to have bounds on the oscillation as in (2.87). But we cannot obtain these while using the space D_z^ℓ as the space of simple functionals. This was illustrated for the lowest order case with the inequality (2.85). Therefore, similar to the construction of $\mathcal{P}_z^{1+,\mathsf{cr}}$, we enhance the space D_z^ℓ, $\ell \geqslant 1$, with element contributions outside ω_z (c.f. (2.86) and [KRVZ25, (4.18)])

$$\begin{aligned} D_z^{+,\ell} := \Big\{ \chi \in H^{-1}(\Omega) \;\Big|\; \chi(\psi) = \sum_{T\in\mathcal{T}_z^+} \int_T r_T\psi + \sum_{F\in\mathcal{F}_z^i} \int_F r_F\psi \\ \text{for } \psi \in H_0^1(\Omega) \text{ with } r_T \in \mathbb{P}_{\ell-1}(T) \text{ and } r_F \in \mathbb{P}_{\ell-1}(F) \text{ for } T \in \mathcal{T}_z^+, F \in \mathcal{F}_z^i \Big\}. \end{aligned} \tag{2.123}$$

This space $D_z^{+,\ell}$ enables us to build a projection using a test function space S_z not contained in $H_0^1(\omega_z)$ and still have the bounds on the oscillation corresponding to (2.87). We comment on the resulting bounds in Remark 2.85 and turn to the construction of the test function space S_z.

Inspired by the construction of $\mathcal{P}_z^{1+,\mathsf{cr}}$ in the lowest order case, we use higher order Crouzeix-Raviart elements to build test functions. At this point, the Gauß-Legendre quadrature becomes important. Since we need it also for later purposes, we recall it for the reader's convenience.

The Gauß-Legendre quadrature employs the zeros of the Legendre polynomials. Their definition can be found in e.g. [EG04, Definition 1.14]. These polynomials are initially defined only on the interval $[-1,1]$. With an affine transformation, we can map these from the interval to any face $F \in \mathcal{F}$, and we call these mapped polynomials also Legendre polynomials. On that particular face F, the zeros of the affinely mapped Legendre polynomial of order ℓ are well-defined, and we denote them by $\xi_i^F; 1 \leqslant i \leqslant \ell$. These zeros are the nodes of the Gauß-Legendre quadrature formula.

Proposition 2.78 (Gauß-Legendre quadrature ([EG04, Definition 8.1]))**.** *Given some $F \in \mathcal{F}$, then there exist weights $\omega_i^F \in \mathbb{R}$; $1 \leqslant i \leqslant \ell$ such that for all $p \in \mathbb{P}_{2\ell-1}(F)$*

$$\int_F p = \sum_{i=1}^{\ell} \omega_i^F p(\xi_i^F). \tag{2.124}$$

The generalized Crouzeix-Raviart element of order $\ell \in \mathbb{N}$ is defined in (1.39). The integral restriction can be rephrased with the Gauß-Legendre quadrature as

$$0 = \int_F [\![v]\!]\, r = \sum_{i=1}^{\ell} \omega_i^F\, [\![v]\!]\,(\xi_i^F)\; r(\xi_i^F) \quad \forall v \in \mathrm{CR}_\ell \quad \text{and} \quad r \in \mathbb{P}_{\ell-1}(F).$$

Since we can prescribe the values of r at the points ξ_i^F, we see that $[\![v]\!]$ vanishes at all these points. Thus, $v \in \mathrm{CR}_\ell$ is continuous at $\xi_i^F \in F$, $i = 1, \dots, \ell$.

From now on, we restrict ourselves to $d = 2$ and ℓ odd, see Remark 2.86. We have the following set of degrees of freedom of CR_ℓ spaces:

Proposition 2.79 ([SB06, Lemma 1])**.** *For ℓ odd and $d = 2$, a set of degrees of freedom for CR_ℓ is given by*

1. *the point evaluations at the nodes ξ_{ijk}^T in the interior of the elements $T \in \mathcal{T}$ given by barycentric coordinates*

$$\lambda_1 = \frac{i}{\ell}, \quad \lambda_2 = \frac{j}{\ell}, \quad \lambda_3 = \frac{k}{\ell} \quad \text{with } i,j,k \in \{1,\dots,\ell-2\} \text{ and } i+j+k = \ell$$

2. *the point evaluations at the Legendre nodes ξ_j^F for all $F \in \mathcal{F}^i$ and $1 \leqslant j \leqslant \ell$.*

In the case ℓ odd, we introduce the basis functions of the Crouzeix-Raviart element for the degrees of freedom on the faces.

Definition 2.80. *We define $\Psi_{F,m}^{\mathsf{cr}} \in \mathrm{CR}_\ell$, ℓ odd, $1 \leqslant m \leqslant \ell$ and $F \in \mathcal{F}^i$ to be the unique function that satisfies*

$$\begin{aligned} \Psi_{F,m}^{\mathsf{cr}}(\xi_j^{F'}) &= \delta_{F,F'}\delta_{j,m} \text{ for } j = 1,\dots,\ell \text{ and } F' \in \mathcal{F}^i \text{and} \\ \Psi_{F,m}^{\mathsf{cr}}(\xi_{i,j,k}^T) &= 0 \text{ for all } i,j,k \in \{1,\dots,\ell-2\} \text{ with } i+j+k = \ell \text{ and } T \in \mathcal{T}. \end{aligned}$$

With these functions, we define the following space of test functions for ℓ odd and the space of simple functionals $D_z^{+,\ell}$ as

$$S_z^{\mathsf{cr},\ell+} = \operatorname{span}\big(\{\mathcal{E}_\ell(\Psi_{F,j}^{\mathsf{cr}}) \mid F \in \mathcal{F}_z^i, j = 1,\dots,\ell\} \cup \{p\Psi_T \mid p \in \mathbb{P}_{\ell-1}(T), T \in \mathcal{T}_z^+\}\big),$$

where $\mathcal{E}_\ell$ is the smoothing operator of Definition 1.35 and Ψ_T the bubble defined in (2.70). A disadvantage of this space is that $\operatorname{supp}(S_z^{\mathsf{cr},\ell+}) \subset \omega_z^+$ because $\mathcal{A}_\ell v_h$ might have larger support than its argument v_h. We added the element contributions in $\mathcal{T}_z^+ \backslash \mathcal{T}_z$ such that the new oscillation is again bounded by the classical one, c.f. page 67.

This new test function space $S_z^{\mathsf{cr},\ell+}$ is a possible choice to construct a projection $\mathcal{P}_z^{\ell,\mathsf{cr}}$ into the space $D_z^{+,\ell}$. Its advantage is that we can employ the local quasi-Galerkin orthogonality again.

Proposition 2.81. *Let $d = 2$ and $\ell \in \mathbb{N}$ be odd, then the pair $D_z^{+,\ell}$ and $S_z^{\mathsf{cr},\ell+}$ defines a well-posed and stable projection $\mathcal{P}_z^{\ell,\mathsf{cr}} : \big(S_z^{\mathsf{cr},\ell+}\big)' \to D_z^{+,\ell}$. For each $f \in H^{-1}(\omega_z^+)$, there are polynomials $r_T \in \mathbb{P}_{\ell-1}(T)$, $T \in \mathcal{T}_z$ and $r_F \in \mathbb{P}_{\ell-1}(F)$, $F \in \mathcal{F}_z^i$, such that*

$$\begin{aligned}\|\mathcal{P}_z^{\ell,\mathsf{cr}}(f)\|_{H^{-1}(\omega_z)} &\lesssim \Big(\sum_{T\in\mathcal{T}_z} h_z^2\|r_T\|_{L^2(T)}^2\Big)^{\frac12} + \Big(\sum_{F\in\mathcal{F}_z^i} h_F\|r_F\|_{L^2(F)}^2\Big)^{\frac12}\\ &\lesssim \|\mathcal{P}_z^{\ell,\mathsf{cr}}(f)\|_{\left(S_z^{\mathsf{cr},\ell+}\right)'} \leqslant \|\mathcal{P}_z^{\ell,\mathsf{cr}}(f)\|_{H^{-1}(\omega_z^+)}.\end{aligned} \tag{2.125}$$

On a reference element $\hat{T}$, we choose a basis $\hat{p}_i^T$ of $\mathbb{P}_{\ell-1}(\hat{T})$ with $1 \leqslant i \leqslant \binom{d+\ell-1}{\ell-1} =: d_{\hat{T}}$. This reference basis induces for each $T \in \mathcal{T}_z^+$ with the affine map $F_T : \hat{T} \to T$ the basis p_i^T, $1 \leqslant i \leqslant d_{\hat{T}}$ on $\mathbb{P}_{\ell-1}(T)$. This allows us to write

$$\begin{aligned}\|\mathcal{P}_z^{\ell,\mathsf{cr}}(f)\|_{H^{-1}(\omega_z)} &\lesssim \|\mathcal{P}_z^{\ell,\mathsf{cr}}(f)\|_{\left(S_z^{\mathsf{cr},\ell+}\right)'} \lesssim \sum_{T\in\mathcal{T}_z^+}\sum_{i=1}^{d_T}\frac{\langle f, p_i^T\Psi_T\rangle^2}{\|\nabla(p_i^T\Psi_T)\|_{L^2(\omega_z)}^2} + \sum_{F\in\mathcal{F}_z^i}\sum_{j=1}^{\ell-1}\frac{\langle f, \mathcal{E}_\ell\Psi_{F,j}^{\mathsf{cr}}\rangle^2}{\|\nabla(\mathcal{E}_\ell\Psi_{F,j}^{\mathsf{cr}})\|_{L^2(\omega_z^+)}^2}\\ &\lesssim \|\mathcal{P}_z^{\ell,\mathsf{cr}}(f)\|_{\left(S_z^{\mathsf{cr},\ell+}\right)'} \leqslant \|\mathcal{P}_z^{\ell,\mathsf{cr}}(f)\|_{H^{-1}(\omega_z^+)}.\end{aligned} \tag{2.126}$$

The hidden constants only depend on $\sigma(\mathcal{T})$, ℓ, and in the latter display also on the basis chosen on $\hat{T}$.

Proof. We apply the framework of section 2.1.3 and check Assumption 2.19. The Assumption 2.2 is fulfilled with $V_z = H_0^1(\omega_z)$, as before shown in Proposition 2.30, hence it remains to verify (2.21).

The dimensions of $D_z^{+,\ell}$ and $S_z^{\mathsf{cr},\ell+}$ coincide. To prove the second part of (2.21), we use the L^2-representation provided by Lemma 2.57. The element contributions are bounded as in (2.119).

Therefore, it remains to bound the face contributions. Fix $F \in \mathcal{F}_z^i$, $z \in \mathcal{V}$. We build the corresponding test polynomial in $\mathbb{P}_\ell(F)$ by specifying its values at the Gauß-Legendre points: $\Psi^F := \sum_{j=1}^\ell r_F(\xi_j^F)\Psi_{F,j}^{\mathsf{cr}} \in \mathrm{CR}_\ell$. Since the Gauß-Legendre quadrature of order ℓ is exact for polynomials of degree $2\ell-1$, see Proposition 2.78, we conclude with the momentum condition of $\mathcal{E}_\ell$ that

$$\begin{aligned}\|r_F\|_{L^2(F)}^2 &= \int_F r_F^2 = \sum_{i=1}^\ell r_F(\xi_i^F)^2\omega_i^F = \sum_{i=1}^\ell r_F(\xi_i^F)\Psi^F(\xi_i^F)\omega_i^F = \int_F r_F\Psi^F = \int_F r_F\mathcal{E}_\ell\Psi^F\\ &\lesssim \Big(\|\chi\|_{S_z^{\mathsf{cr},\ell+\prime}}^2 + \sum_{T\in\mathcal{T}_z^+} h_z^2\|r_T\|_{L^2(T)}^2\Big)^{\frac12}\|\nabla\mathcal{E}_\ell(\Psi^F)\|_{L^2(\omega_F^+)}.\end{aligned}$$

We have to add the element contributions outside ω_z since $\mathcal{E}_\ell(\Psi^F)$ can have a support larger than ω_z.

Now, the stability of $\mathcal{E}_\ell$ implies

$$\|\nabla\mathcal{E}_\ell(\Psi^F)\|_{L^2(\omega_F^+)} \lesssim \|\nabla_h\Psi^F\|_{L^2(\omega_F)} \leqslant \sum_{j=1}^\ell \big|r_F(\xi_j^F)\big|\|\nabla_h\Psi_{F,j}^{\mathsf{cr}}\|_{L^2(\omega_F)} \lesssim h_F^{-\frac12}\|r_F\|_{L^2(F)}. \tag{2.127}$$

We used that $\sum_{i=1}^\ell \big|r_F(\xi_i^F)\big|$ is a norm on the space $\mathbb{P}_{\ell-1}(F)$.

The alternative representation in (2.126) is obtained by Lemma 2.26 □

We have to show that the projection fulfills (2.29) to obtain a similar result as in Corollary 2.66.

Corollary 2.82. *Let $d = 2$ and $u \in H_0^1(\Omega)$ be the solution of (1.30) and $u_h \in V_h$ the discrete solution for $f \in H^{-1}(\Omega)$ of P-LAG, P-CRL, P-SIP, P-NSIP, P-RSIP or P-RNSIP for $\ell \in \mathbb{N}$ odd, then we have*

$$\begin{aligned}\|\operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f,u_h)\|_{H^{-1}(\omega_z)} &\leqslant \|\mathcal{P}_z^{\ell,\mathsf{cr}}\operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f,u_h)\|_{H^{-1}(\omega_z)} + \|\mathcal{P}_z^{\ell,\mathsf{cr}}f - f\|_{H^{-1}(\omega_z)}\\ &\leqslant \big(1 + 2\|\mathcal{P}_z^{\ell,\mathsf{cr}}\|_{\mathbb{L}\left(\left(S_z^{\mathsf{cr},\ell+}\right)',H^{-1}(\omega_z)\right)}\big)\|\operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f,u_h)\|_{H^{-1}(\omega_z^+)}\end{aligned} \tag{2.128}$$

and the latter factor is bounded by

$$\|\operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f,u_h)\|^2_{H^{-1}(\omega_z^+)} \leqslant C_{\mathsf{P}}^2 \sum_{y\in\mathcal{V}\cap\omega_z^+} \|\operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f,u_h)\|^2_{H^{-1}(\omega_y)},$$

where $C_{\mathsf{P}} > 0$ is the constant in Corollary 2.23.

Proof. Condition (2.29) is fulfilled by the construction of $D_z^{+,\ell}$, $\tilde{V}_z = H_0^1(\omega_z) + S_z^{\mathsf{cr},\ell+}$: For $u_h \in V_h$ and $w \in \tilde{V}_z \subset H_0^1(\omega_z^+)$, we have

$$\tilde{a}(u_h,w) = \int_{\omega_z^+} \nabla_h u_h \cdot \nabla w = -\sum_{T\in\mathcal{T}_z^+} \int_T \Delta u_h w + \sum_{F\in\mathcal{F}_z^{i,+}} \int_F [\![\nabla u_h \cdot \mathsf{n}_F]\!]\, w. \tag{2.129}$$

We define a $\chi \in D_z^{+,\ell}$ by setting $r_T := \Delta u_{h|T}$ and $r_F := [\![\nabla u_h \cdot \mathsf{n}_F]\!]_{|F}$ for all $T \in \mathcal{T}_z^+$ and $F \in \mathcal{F}_z^i$ and show $\langle \tilde{A}(u_h),\psi\rangle = \langle \chi,\psi\rangle$ for all $\psi \in \tilde{V}_z$. For $\psi \in H_0^1(\omega_z)$ this identity is clear due to (2.64). The remaining functions in $\tilde{V}_z$ are of two types, c.f. Corollary 2.66.

First, there are the functions of the form $p\Psi_T$, $T \in \mathcal{T}_z^+\backslash\mathcal{T}_z$ and $p \in \mathbb{P}_{\ell-1}(T)$. For a fixed $T \in \mathcal{T}_z^+\backslash\mathcal{T}_z$, we have similar to the proof of Corollary 2.66 that

$$\langle \tilde{A}(u_h), p\Psi_T\rangle = \int_T r_T p\Psi_T = \langle \chi, p\Psi_T\rangle.$$

Second, there are functions of the form $\mathcal{E}_\ell(w_h)$ for an appropriate $w_h \in \mathrm{CR}_\ell$ with $\operatorname{supp}(w_h) \subset \omega_z$. Then, we can write the contributions in (2.129) as

$$\begin{aligned}
&-\sum_{T\in\mathcal{T}_z^+} \int_T \Delta u_h \mathcal{E}_\ell(w_h) + \sum_{F\in\mathcal{F}_z^{i,+}} \int_F [\![\nabla u_h \cdot \mathsf{n}_F]\!]\, \mathcal{E}_\ell(w_h)\\
&\quad = -\sum_{T\in\mathcal{T}_z} \int_T \Delta u_h \mathcal{E}_\ell(w_h) - \sum_{T\in\mathcal{T}_z^+\backslash\mathcal{T}_z} \int_T \Delta u_h \mathcal{E}_\ell(w_h)\\
&\qquad + \sum_{F\in\mathcal{F}_z^i} \int_F [\![\nabla u_h \cdot \mathsf{n}_F]\!]\, \mathcal{E}_\ell(w_h) + \sum_{F\in\mathcal{F}_z^{i,+}\backslash\mathcal{F}_z^i} \int_F [\![\nabla u_h \cdot \mathsf{n}_F]\!]\, \mathcal{E}_\ell(w_h).
\end{aligned}$$

Because the smoother $\mathcal{E}_\ell$ fulfills the momentum conditions in Definition 1.33 and $\operatorname{supp}(w_h) \subset \omega_z$, we have

$$\begin{aligned}
-\sum_{T\in\mathcal{T}_z^+\backslash\mathcal{T}_z} \int_T \Delta u_h \mathcal{E}_\ell w_h = -\sum_{T\in\mathcal{T}_z^+\backslash\mathcal{T}_z} \int_T \Delta u_h w_h = 0 \quad \text{and}\\
\sum_{F\in\mathcal{F}_z^{i,+}\backslash\mathcal{F}_z^i} \int_F [\![\nabla u_h \cdot \mathsf{n}_F]\!]\, \mathcal{E}_\ell w_h = \sum_{F\in\mathcal{F}_z^{i,+}\backslash\mathcal{F}_z^i} \int_F [\![\nabla u_h \cdot \mathsf{n}_F]\!]\, w_h = 0.
\end{aligned}$$

Therefore, this setting with $D_z^{+,\ell}$ and $S_z^{\mathsf{cr},\ell+}$ satisfies condition (2.29).

The bounds (2.128) then follow by Proposition 2.22. The last bound is obtained by Proposition 2.35. □

In this setting, we can again eliminate the evaluation of the side contributions due to the usage of the test functions based upon the discrete space.

Proposition 2.83. *Let $d = 2$ and $\ell \in \mathbb{N}$ be an odd number and consider one of the methods P-CRL, P-SIP, P-NSIP, P-RSIP or P-RNSIP of degree ℓ. We define η_z, for a fixed $z \in \mathcal{V}$, as either*

$$\eta_z^2 := \sum_{T\in\mathcal{T}_z} h_z^2 \|r_T\|^2_{L^2(T)} \quad \text{or} \quad \eta_z^2 := \sum_{T\in\mathcal{T}_z} \sum_{i=1}^{d_T} \frac{\langle \mathcal{P}_z^{\ell,\mathsf{cr}}(\operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f,u_h)), p_i^T\Psi_T\rangle^2}{\|\nabla(p_i^T\Psi_T)\|^2_{L^2(\omega_z)}},$$

where for $T \in \mathcal{T}_z$ the functions p_i^T, $1 \leqslant i \leqslant \binom{d+\ell-1}{\ell-1} =: d_T$, are a basis for $\mathbb{P}_{\ell-1}(T)$ constructed via a reference element $\hat{T}$ as described in Proposition 2.81. Using the projection $\mathcal{P}_z^{\ell,\mathsf{cr}}$ induced by $D_z^{+,\ell}$ and $S_z^{\mathsf{cr},\ell+}$, we have the bounds

$$\eta_z \lesssim \|\mathcal{P}_z^{\ell,\mathsf{cr}}(\operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f,u_h))\|_{H^{-1}(\omega_z)} \lesssim \eta_z + \mathbf{CE}_z(u_h), \tag{2.130}$$

where $\mathbf{CE}_z(u_h)$ is the local conformity error from Definition 2.67. The hidden constants only depend on $\sigma(\mathcal{T})$ and ℓ and in the case for the second choice of η_z also on the chosen basis on $\hat{T}$.

Proof. The first bound in (2.130) follows as in (2.119).

We turn to the second bound, at first only for the first choice of η_z. In view of Proposition 2.81, we only have to bound the face contributions by the local conformity error. Choose a $F \in \mathcal{F}_z^i$. Recall the zeros of the Legendre polynomials ξ_j^F and the Crouzeix-Raviart basis functions as defined in Proposition 2.79. We test the residual with the same test function that we used in Proposition 2.81, which is $\Psi^F = \sum_{j=1}^{\ell} r_F(\xi_j^F)\Psi_{F,j}^{\mathsf{cr}}$.

Then, the definitions of $\mathcal{P}_z^{\ell,\mathsf{cr}}$ and $\mathbf{CE}_z(u_h)$ provide us with

$$\sum_{T\in\mathcal{T}_z^+}\int_T r_T\mathcal{E}_\ell\Psi^F + \sum_{F'\in\mathcal{F}_z^i}\int_{F'} r_{F'}\mathcal{E}_\ell\Psi^F = \langle \mathcal{P}_z^{\ell,\mathsf{cr}}\operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f,u_h), \mathcal{E}_\ell\Psi^F\rangle \lesssim \mathbf{CE}_z(u_h)\|\Psi^F\|_{\tilde{V}}, \tag{2.131}$$

with $r_T \in \mathbb{P}_{\ell-1}(T), T \in \mathcal{T}_z^+$ and $r_F \in \mathbb{P}_{\ell-1}(F), F \in \mathcal{F}_z^i$ the components of $\mathcal{P}_z^{\ell,\mathsf{cr}}(\operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f,u_h)) \in D_z^{+,\ell}$. Putting the element contributions on the right-hand side, we obtain

$$\|r_F\|_{L^2(F)}^2 = \sum_{F'\in\mathcal{F}_z^i}\int_{F'} r_{F'}\mathcal{E}_\ell\Psi^F \lesssim \mathbf{CE}_z(u_h)\|\Psi^F\|_{\tilde{V}} + \left|\sum_{T\in\mathcal{T}_z^+}\int_T r_T\mathcal{E}_\ell\Psi^F\right|. \tag{2.132}$$

We estimate the latter summand as

$$\sum_{T\in\mathcal{T}_z^+}\int_T r_T\mathcal{E}_\ell\Psi^F \lesssim \Big(\sum_{T\in\mathcal{T}_z^+} h_z^2\|r_T\|_{L^2(T)}^2\Big)^{\frac{1}{2}}\|\nabla\mathcal{E}_\ell\Psi^F\|_{L^2(\omega_F^+)}. \tag{2.133}$$

The last factor is bounded with the help of (2.127) and the equivalence of norms on finite-dimensional vector spaces as $\|\Psi^F\|_{\tilde{V}} \lesssim h_F^{-\frac{1}{2}}\|r_F\|_{L^2(F)}$. Putting everything together, we obtain

$$h_F\|r_F\|_{L^2(F)}^2 \lesssim \mathbf{CE}_z(u_h)^2 + \sum_{T\in\mathcal{T}_z^+} h_z^2\|r_T\|_{L^2(T)}^2. \tag{2.134}$$

Inserting (2.134) into (2.125) proves (2.130) for the first choice of η_z.

The estimate (2.130) for the second choice of η_z follows by bounding the face contributions in (2.126) with the local conformity error, defined in Definition 2.67. □

Similar to Theorem 2.69, we have the following result.

Theorem 2.84. *Let $d = 2$ and $\ell \in \mathbb{N}$ be an odd number and consider one of the methods P-CRL, P-SIP, P-NSIP, P-RSIP or P-RNSIP. Let $u \in H_0^1(\Omega)$ be the analytic solution of (1.30) and $u_h \in V_h$ the corresponding discrete solution for a fixed right-hand side $f \in H^{-1}(\Omega)$. Furthermore, let $r_T \in \mathbb{P}_{\ell-1}(T)$, $T \in \mathcal{T}_z^+$ be the solution of*

$$\int_T r_T q\Psi_T = \langle \operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f,u_h), q\Psi_T\rangle \quad q \in \mathbb{P}_{\ell-1}(T).$$

We set

$$\eta_z^2 := \sum_{T\in\mathcal{T}_z^+} h_z^2\|r_T\|_{L^2(T)}^2 \quad \text{or} \quad \eta_z^2 := \sum_{T\in\mathcal{T}_z^+}\sup_{q\in\mathbb{P}_{\ell-1}(T)}\frac{\langle \operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f,u_h), q\Psi_T\rangle^2}{\|\nabla(q\Psi_T)\|_{L^2(\omega_z)}^2}.$$

We have the local bound:

$$\eta_z^2 + \|\mathcal{P}_z^{\ell,\mathsf{cr}}f - f\|_{H^{-1}(\omega_z)}^2 \lesssim \|\operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f,u_h)\|_{H^{-1}(\omega_z^+)}^2 \lesssim \|\nabla_h(u-u_h)\|_{L^2(\omega_z^+)}^2. \tag{2.135}$$

Summing over all elements, we have for

$$\operatorname{est}^2 := \sum_{z\in\mathcal{V}}\eta_z^2 + \|\mathcal{P}_z^{\ell,\mathsf{cr}}f - f\|_{H^{-1}(\omega_z)}^2 + \|\Pi_V u_h - u_h\|_{\tilde{V}}^2$$

the global estimates:

$$\operatorname{est}^2 \lesssim \|u-u_h\|_{\tilde{V}}^2 \lesssim \operatorname{est}^2 + \sum_{z\in\mathcal{V}}\mathbf{CE}_z^2(u_h) \lesssim \operatorname{est}^2. \tag{2.136}$$

The hidden constants only depend on $\sigma(\mathcal{T})$ and ℓ, and in the case of the dG-methods also on γ.

Proof. The local bound (2.135) follows by using the first bound in (2.130) in combination with (2.30a):

$$\begin{aligned}\eta_z^2 + \|\mathcal{P}_z^{\ell,\mathsf{cr}} f - f\|_{H^{-1}(\omega_z)}^2 &\lesssim \|\mathcal{P}_z^{\ell,\mathsf{cr}}(\operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f,u_h))\|_{\left(S_z^{\mathsf{cr},\ell+}\right)'}^2 + \|\mathcal{P}_z^{\ell,\mathsf{cr}} f - f\|_{H^{-1}(\omega_z)}^2 \\ &\lesssim \|\operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f,u_h)\|_{H^{-1}(\omega_z^+)}^2 \lesssim \|\nabla_h(u-u_h)\|_{L^2(\omega_z^+)}^2.\end{aligned}$$

Summing over all elements, the lower bound in (2.136) follows directly with (1.22). The upper bound in (2.136) results from the splitting of the error into conformity and smoothed error as in Proposition 2.27, followed by the localization in Proposition 2.32. Then, we use the splitting (2.128) with the projection $\mathcal{P}_z^{\ell,\mathsf{cr}}$ and (2.130).

The last inequality in (2.136) is derived with the bound for the local conformity error (2.96) in Lemma 2.68. □

Remark 2.85. The new oscillation term $\|\mathcal{P}_z^{\ell,\mathsf{cr}} f - f\|_{H^{-1}(\omega_z)}$, $z \in \mathcal{V}$, can be bounded by the classical oscillation if $f \in L^2(\omega_z^+)$ with the arguments of (2.87)

$$\|\mathcal{P}_z^{\ell,\mathsf{cr}} f - f\|_{H^{-1}(\omega_z)} \lesssim h_z \inf_{g \in S_{\ell-1}^0(\mathcal{T}_z^+)} \|f-g\|_{L^2(\omega_z^+)}.$$

Remark 2.86. Theorem 2.84 cannot be true if ℓ is even. The canonical degrees of freedom of CR_ℓ, ℓ even, are linear dependent, see [FS83, BS07]. In particular, in the case of ℓ even, the space CR_ℓ is the space $S_\ell^1(\mathcal{T})$ with corresponding Fortin-Soulie bubbles added for each element, see [CS22a, Theorem 3.3]. Since these bubbles vanish at all Gauß-Legendre points of the faces, they cannot be used for detecting polynomials on faces in D_z^ℓ or $D_z^{+,\ell}$ due to the Gauß-Legendre quadrature, c.f. Proposition 2.78. Hence, we achieve a simplification only by testing with the Lagrange basis functions thus only the simplifications explained in Remark 2.77 can be used.

2.3.2 Error estimators for the biharmonic problem

In this section, we construct a projection to obtain an a-posteriori error-estimator for the Morley (B-MR) and the C^0-interior penalty discretizations (B-SCZ) of the biharmonic problem. We assume $d = 2$ throughout this section. To derive the minimal space D_z that fulfills (2.29), which is $(\tilde{A}V_h)_{|\tilde{V}_z} \subset (D_z)_{|\tilde{V}_z}$, we rewrite the biharmonic problem (1.66) with an integration by parts, as done in [KRVZ25, p. 38].

We recall the normal and tangential vectors of an edge as defined in Definitions 1.28 and 1.55. Furthermore, we generalize the notion of the tangent vector. Let $\omega \subset \Omega \subset \mathbb{R}^2$ be an open set with polyhedral boundary, and let $\mathsf{n}_\omega \in \mathbb{R}^2$ be the normal vector from Definition 1.28. We select the tangential vector $\mathsf{t}_\omega = \left(-(\mathsf{n}_\omega)_2, (\mathsf{n}_\omega)_1\right)^T$. We also select for each $F \in \mathcal{F}$ the orthogonal tangential vector $\mathsf{t}_F = \left(-(\mathsf{n}_F)_2, (\mathsf{n}_F)_1\right)^T \in \mathbb{R}^2$ orthogonal to n_F.

For $T \in \mathcal{T}_z$, $u_h \in S_2^0(\mathcal{T})$ and $v \in H_0^2(\omega_z)$, we have:

$$\int_T D^2 u_h : D^2 v = \int_T \sum_{i,j=1}^2 \partial_i\partial_j u_h \partial_i\partial_j v = \int_{\partial T} \sum_{j=1}^2 \partial_{\mathsf{n}_T}\partial_j u_h \partial_j v - \int_T \sum_{i,j=1}^2 \partial_i\partial_i\partial_j u_h \partial_j v. \tag{2.137}$$

The last integral vanishes because $u_h \in S_2^0(\mathcal{T})$. We continue by writing the derivatives in terms of the normal and tangential and obtain:

$$\sum_{j=1}^2 \partial_{\mathsf{n}_T}\partial_j u_h \partial_j v = (\partial_{\mathsf{n}_T})^2 u_h \mathsf{n}_T \cdot \partial_{\mathsf{n}_T} v \mathsf{n}_T + \partial_{\mathsf{n}_T}\partial_{\mathsf{t}_T} u_h \mathsf{t}_T \cdot \partial_{\mathsf{t}_T} v \mathsf{t}_T.$$

If we now sum (2.137) over all elements $T \in \mathcal{T}_z$, apply Lemma 1.30 and that $v \in H_0^2(\omega_z)$, we get

$$\begin{aligned}\int_{\omega_z} D_h^2 u_h : D_h^2 v &= \sum_{F \in \mathcal{F}_z^i} \int_F \left[\!\left[(\partial_{\mathsf{n}_F})^2 u_h \partial_{\mathsf{n}_F} v + \partial_{\mathsf{n}_F}\partial_{\mathsf{t}_F} u_h \partial_{\mathsf{t}_F} v \right]\!\right] \\ &= \sum_{F \in \mathcal{F}_z^i} \int_F \left[\!\left[(\partial_{\mathsf{n}_F})^2 u_h \right]\!\right] \partial_{\mathsf{n}_F} v + \int_F \left[\!\left[\partial_{\mathsf{n}_F}\partial_{\mathsf{t}_F} u_h \right]\!\right] \partial_{\mathsf{t}_F} v.\end{aligned}$$

We use an integration by parts for the last expression and obtain for $F \in \mathcal{F}_z^i$ with $F = [z, y_z^F]$, $y_z^F \in \mathcal{V} \cap F$:

$$\int_F \left[\!\left[\partial_{\mathsf{n}_F}\partial_{\mathsf{t}_F} u_h \right]\!\right] \partial_{\mathsf{t}_F} v = \frac{z - y_z^F}{|z - y_z^F|} \cdot \mathsf{t}_F \left[\!\left[\partial_{\mathsf{n}_F}\partial_{\mathsf{t}_F} u_h \right]\!\right](z)\, v(z) + \frac{y_z^F - z}{|y_z^F - z|} \cdot \mathsf{t}_F \left[\!\left[\partial_{\mathsf{n}_F}\partial_{\mathsf{t}_F} u_h \right]\!\right](y_z^F)\, v(y_z^F).$$

The point evaluations are well-defined since $\partial\omega_z$ is regular enough that $H^2(\omega_z)$ embeds into $C^0(\omega_z)$ for $d = 2$. The last summand vanishes since $\operatorname{supp}(v) \subset \omega_z$ and $y_z^F \in \partial\omega_z$.

This suggests choosing the space [KRVZ25, (5.13)]:

$$D_z^{\mathsf{bhm}} = \Big\{\chi \in H^{-2}(\Omega) \mid \chi(\psi) = \sum_{F\in\mathcal{F}_z^i} \int_F r_F \partial_n\psi + r_z\psi(z),\ r_F, r_z \in \mathbb{R};\ r_z = 0 \text{ if } z \in \partial\Omega,\ \psi \in H_0^2(\Omega)\Big\}, \tag{2.138}$$

whose restriction to $\tilde{V}_z$ will fulfill (2.29) after choosing a suitable test function space S_z. If $z \in \partial\Omega$ then r_z cannot be detected, hence it can be omitted from the definition of D_z^{bhm}. In the other cases, we have to test for constants on the edges by choosing an appropriate normal derivative of the test functions and the constant in the middle of the patch.

These arguments show:

Proposition 2.87. *In the case $\tilde{V}_z = H_0^2(\omega_z)$ (i.e. $S_z \subset H_0^2(\omega_z)$), $z \in \mathcal{V}$, the space D_z^{bhm} fulfills* (2.31) *for the Morley method (B-MR) and the C^0-interior penalty method (B-SCZ).*

Remark 2.88. In the case of the Poisson equation, we enlarged the space D_z^ℓ with piecewise polynomials beyond the contributions strictly necessary for (2.29) to obtain an oscillation which might have a higher convergence rate than the error, see Corollary 2.63. Here, we omit this for the following reason: If $f \in L^2(\Omega)$, we have for a yet to define operator $\mathcal{P}_z : H^{-2}(\omega_z^+) \to H^{-2}(\omega_z)$ the bound $\|f - \mathcal{P}_z f\|_{H^{-2}(\omega_z)} \lesssim \|f\|_{H^{-2}(\omega_z^+)} \lesssim h_z^2\|f\|_{L^2(\omega_z^+)}$, the oscillation converges in this case with second order. Because both methods only use second order polynomials, we can only expect first order convergence of the error. Thus, the oscillation term converges with a higher order than the error, hence we do not need to augment the oscillation free data. Still, it is possible to adapt the estimator to achieve potentially higher convergence rates for the oscillation or bound the new oscillation term by the oscillation used in e.g. [CN22]. For this, the user simply has to add functionals to D_z^{bhm} and construct appropriate H^2 test functions.

To construct a test function-space, we recall a set of degrees of freedom for the Morley element.

Proposition 2.89 ([Mor68])**.** *A Morley finite element has on each $T \in \mathcal{T}$ the following degrees of freedom*

(i) point evaluation at the vertices $z \in \mathcal{V} \cap T \cap \Omega$ and

(ii) for each $F \in \mathcal{F}_T^i$ the evaluation ∂_{n_F} at the barycenter of F denoted by m_F.

The last degree of freedom is the same as $\frac{1}{|F|}\int_F \partial_{\mathsf{n}_F}\cdot$.

These degrees of freedom are preserved by the $\mathcal{E}_{\mathsf{hct}}$ operator, see (1.72). We introduce as in [KRVZ25, (5.25)] the basis functions corresponding to the degrees of freedom of the Morley element:

$$\Psi_F^{\mathsf{mr}} \in \mathrm{MR}: \quad \partial_{\mathsf{n}_F}\Psi_F^{\mathsf{mr}}(m_{F'}) = \delta_{FF'} \quad \text{and} \quad \Psi_F^{\mathsf{mr}}(y) = 0, \tag{2.139a}$$

and

$$\Psi_z^{\mathsf{mr}} \in \mathrm{MR}: \quad \partial_{\mathsf{n}_F}\Psi_z^{\mathsf{mr}}(m_{F'}) = 0 \quad \text{and} \quad \Psi_z^{\mathsf{mr}}(y) = \delta_{zy}, \tag{2.139b}$$

for all $F, F' \in \mathcal{F}^i$ and $z, y \in \mathcal{V} \cap \Omega$.

With this nodal basis, we define the space of test functions for $z \in \mathcal{V}$:

$$S_z^{\mathsf{bhm}} = \operatorname{span}(\{\mathcal{E}_{\mathsf{hct}}(\Psi_F^{\mathsf{mr}}) \mid F \in \mathcal{F}_z^i\} \cup \{\mathcal{E}_{\mathsf{hct}}(\Psi_z^{\mathsf{mr}})\}).$$

Remark 2.90. The choice of the pair S_z^{bhm} and D_z^{bhm} forms a biorthogonal system. This allows us to explicitly write down the induced projection, similar to [KV21].

As the Morley element is not affine equivalent, proving scaling estimates is more involved, compare e.g. with [Cia78, section 6.2.3].

Proposition 2.91. *The* MR*-element satisfies for all $z \in \mathcal{V} \cap \Omega$ and $F \in \mathcal{F}^i$ the estimates*

$$\|D_h^2\Psi_F^{mr}\|_{L^2(\omega_F)} \leqslant C \quad \text{and} \quad \|D_h^2\Psi_z^{mr}\|_{L^2(\omega_z)} \leqslant \frac{C}{h_z}$$

with a constant $C > 0$ independent of the mesh size h_F.

Proof. We show the estimates elementwisely. Thus, fix a simplex $T \in \mathcal{T}$. On this simplex, we introduce another finite element which is similar to the Morley element. In contrast to the Morley element, the new element is affine equivalent. This alternative element, used in [LL75], has the following degrees of freedom on T:

1. point evaluations at the vertices $z \in \mathcal{V} \cap T \cap \Omega$,
2. evaluation of the $\partial_{m_T - m_F}$ at m_F, for all $F \in \mathcal{F}_T^i$.

Here m_T is the barycenter of the element T and m_F the midpoint of the face $F \in \mathcal{F}_T^i$. We write the associated nodal basis functions as $\tilde{\Psi}_F^{\mathsf{mr}}$, which corresponds to the derivative degree of freedom at $F \in \mathcal{F}_T^i$, and $\tilde{\Psi}_z^{\mathsf{mr}}$, corresponding to the point evaluation at $z \in \mathcal{V} \cap T \cap \Omega$.

Next, we express the non-affine basis functions in terms of the affine basis functions. For $F \in \mathcal{F}_T^i$ and any $\Psi \in \mathrm{MR}$, we have on T:

$$\partial_{m_T - m_F} \Psi = (m_T - m_F) \cdot \mathsf{n}_F \partial_{\mathsf{n}_F} \Psi + (m_T - m_F) \cdot \mathsf{t}_F \partial_{\mathsf{t}_F} \Psi. \tag{2.140}$$

Now, we express the non-affine Morley basis in terms of the affine Morley basis. We start with Ψ_z^{mr}, $z \in \mathcal{V} \cap T \cap \Omega$. For $F \in \mathcal{F}_T^i$ with $z \in F$, we have $\partial_{\mathsf{t}_F} \Psi_z^{\mathsf{mr}}(m_F) = \frac{(z - y_F) \cdot \mathsf{t}_F}{|z - y_F|^2}$ with $y_F \in \mathcal{V} \cap F$ and $y_F \neq z$. Because $\partial_{\mathsf{n}_F} \Psi_z^{\mathsf{mr}}(m_F) = 0$, this shows with (2.140) that $\partial_{m_T - m_F} \Psi_z^{\mathsf{mr}}(m_F) = (m_T - m_F) \cdot \mathsf{t}_F \frac{(z - y_F) \cdot \mathsf{t}_F}{|z - y_F|^2}$. Hence, we have

$$\Psi_z^{\mathsf{mr}} = \tilde{\Psi}_z^{\mathsf{mr}} + \sum_{\substack{F \in \mathcal{F}_T^i \\ z \in F}} (m_T - m_F) \cdot \mathsf{t}_F \frac{(z - y_F) \cdot \mathsf{t}_F}{|z - y_F|^2} \tilde{\Psi}_F^{\mathsf{mr}}. \tag{2.141}$$

For the other basis functions Ψ_F^{mr}, $F \in \mathcal{F}_T^i$, we have $\partial_{\mathsf{t}_{F'}} \Psi_F^{\mathsf{mr}}(m_{F'}) = 0$ and $\partial_{\mathsf{n}_{F'}} \Psi_F^{\mathsf{mr}}(m_{F'}) = \delta_{F,F'}$ for all $F' \in \mathcal{F}_T^i$, hence with (2.140):

$$\Psi_F^{\mathsf{mr}} = (m_T - m_F) \cdot \mathsf{n}_F \tilde{\Psi}_F^{\mathsf{mr}}. \tag{2.142}$$

By noting $|m_T - m_F| \eqsim h_T$, the estimates in the Proposition follow. □

The next Proposition shows that the projection induced by S_z^{bhm} and D_z^{bhm} is well-posed.

Proposition 2.92 ([KRVZ25, Lemma 5.14])**.** *For each $z \in \mathcal{V}$, the projection $\mathcal{P}_z^{bhm}$ induced by the spaces S_z^{bhm} and D_z^{bhm} via (2.22) is well-defined and stable and for $f \in H^{-2}(\omega_z^+)$, we have the equivalence*

$$\|\mathcal{P}_z^{bhm}(f)\|_{H^{-2}(\omega_z)} \lesssim \Big(\sum_{F \in \mathcal{F}_z^i} h_z \|r_F\|_{L^2(F)}^2 + h_z^2 |r_z|^2 \Big)^{\frac{1}{2}} \lesssim \|\mathcal{P}_z^{bhm}(f)\|_{(S_z^{bhm})'}$$

Here, $r_F := \frac{1}{|F|} \langle f, \mathcal{E}_{hct} \Psi_F^{mr} \rangle$, $F \in \mathcal{F}_z^i$, $r_z = \langle f, \mathcal{E}_{hct} \Psi_z^{mr} \rangle$ if $z \in \Omega$ and $r_z = 0$ if $z \in \partial\Omega$. We also have the alternative representation

$$\|\mathcal{P}_z^{bhm}(f)\|_{H^{-2}(\omega_z)}^2 \lesssim \sum_{F \in \mathcal{F}_z^i} \frac{\langle f, \mathcal{E}_{hct}(\Psi_F^{mr}) \rangle^2}{\|D^2 \mathcal{E}_{hct}(\Psi_F^{mr})\|_{L^2(\omega_z^+)}^2} + \frac{\langle f, \mathcal{E}_{hct}(\Psi_z^{mr}) \rangle^2}{\|D^2 \mathcal{E}_{hct}(\Psi_z^{mr})\|_{L^2(\omega_z^+)}^2} \lesssim \|\mathcal{P}_z^{bhm}(f)\|_{(S_z^{bhm})'}^2. \tag{2.143}$$

The latter summand is omitted if $z \in \partial\Omega$. The hidden constants only depend on $\sigma(\mathcal{T})$.

Proof. We show that Assumption 2.19 is verified by the pair D_z^{bhm} and S_z^{bhm}, $z \in \mathcal{V}$. Obviously, the dimension of the two spaces matches. Let $\chi \in D_z^{\mathsf{bhm}}$ be given with the representation

$$\langle \chi, v \rangle = \sum_{F \in \mathcal{F}_z^i} \int_F r_F \partial_{\mathsf{n}_F} v + r_z v(z)$$

for $v \in H_0^2(\omega_z)$ and $r_F, r_z \in \mathbb{R}$ for all $F \in \mathcal{F}_z^i$ and $r_z = 0$ if $z \in \partial\Omega$. We have with the trace theorem and Poincaré's inequality that

$$\begin{aligned} \langle \chi, v \rangle &\lesssim \sum_{F \in \mathcal{F}_z^i} \|r_F\|_{L^2(F)} \|\partial_{\mathsf{n}_F} v\|_{L^2(F)} + h_z |r_z| \|D^2 v\|_{L^2(\omega_z)} \\ &\lesssim \Big(\sum_{F \in \mathcal{F}_z^i} h_z \|r_F\|_{L^2(F)}^2 + h_z^2 |r_z|^2 \Big)^{\frac{1}{2}} \|D^2 v\|_{L^2(\omega_z)}. \end{aligned}$$

We continue with the estimate of the face contributions:

$$\|r_F\|_{L^2(F)}^2 = \int_F r_F^2 \partial_{\mathsf{n}_F} \mathcal{E}_{\mathsf{hct}}(\Psi_F^{\mathsf{mr}}) \leqslant \|\chi\|_{(S_z^{\mathsf{bhm}})'} \|r_F D^2 \mathcal{E}_{\mathsf{hct}}(\Psi_F^{\mathsf{mr}})\|_{L^2(\omega_z^+)} \lesssim \|\chi\|_{(S_z^{\mathsf{bhm}})'} h_F^{-\frac{1}{2}} \|r_F\|_{L^2(F)}.$$

In the last step, we used the stability of $\mathcal{E}_{\mathsf{hct}}$ and the scaling properties of Ψ_F^{mr} given in Proposition 2.91:

$$\|r_F D^2 \mathcal{E}_{\mathsf{hct}}(\Psi_F^{\mathsf{mr}})\|_{L^2(\omega_z^+)} \lesssim \|r_F D_h^2 \Psi_F^{\mathsf{mr}}\|_{L^2(\omega_z)} \lesssim h_F^{-\frac{1}{2}} \|r_F\|_{L^2(F)}.$$

For the central value $r_z \in \mathbb{R}$ if $z \in \Omega$, we compute:

$$|r_z| = |r_z| \mathcal{E}_{\mathsf{hct}} \Psi_z^{\mathsf{mr}}(z) = |\langle \chi, \mathcal{E}_{\mathsf{hct}} \Psi_z^{\mathsf{mr}} \rangle| \leqslant \|\chi\|_{S_z^{\mathsf{bhm}\prime}} \|D^2 \mathcal{E}_{\mathsf{hct}} \Psi_z^{\mathsf{mr}}\|_{L^2(\omega_z^+)} \lesssim h_z^{-1} \|\chi\|_{S_z^{\mathsf{bhm}\prime}},$$

where we applied the stability of $\mathcal{E}_{\mathsf{hct}}$ to conclude again with the scaling provided in Proposition 2.91:

$$\|D^2 \mathcal{E}_{\mathsf{hct}} \Psi_z^{\mathsf{mr}}\|_{L^2(\omega_z^+)} \lesssim \|D_h^2 \Psi_z^{\mathsf{mr}}\|_{L^2(\omega_z)} \lesssim h_z^{-1}. \qquad \square$$

The next Corollary shows that the projection induced by the pair $(D_z^{\mathsf{bhm}}, S_z^{\mathsf{bhm}})$ can be used for the Morley B-MR and C^0-interior penalty B-SCZ methods to obtain an error-estimator with error-dominated oscillation.

Corollary 2.93 ([KRVZ25, Theorem 5.15])**.** *Let $u \in H_0^2(\Omega)$ be the solution of (1.66) and $u_h \in V_h$ the discrete solution of B-MR or B-SCZ. The projection $\mathcal{P}_z^{bhm}$, $z \in \mathcal{V}$, fulfills (2.29), and we have the local estimates*

$$\begin{aligned} \|\operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f, u_h)\|_{H^{-2}(\omega_z)} &\leqslant \|\mathcal{P}_z^{bhm} \operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f, u_h)\|_{H^{-2}(\omega_z)} + \|\mathcal{P}_z^{bhm} f - f\|_{H^{-2}(\omega_z)} \\ &\leqslant \left(1 + 2\|\mathcal{P}_z^{bhm}\|_{\mathbb{L}\left((S_z^{bhm})', H^{-2}(\omega_z)\right)}\right) \|\operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f, u_h)\|_{H^{-2}(\omega_z^+)}. \end{aligned} \tag{2.144}$$

The latter factor is bounded by

$$\|\operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f, u_h)\|_{H^{-2}(\omega_z^+)}^2 \leqslant C_{\mathsf{P}}^2 \sum_{y \in \mathcal{V} \cap \omega_z^+} \|\operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f, u_h)\|_{H^{-2}(\omega_y)}^2,$$

where C_{P} is the constant in Proposition 2.56.

Proof. First, we show that condition (2.29) is valid in this setting. In Proposition 2.87, we have seen that

$$(\tilde{A} V_h)_{|H_0^2(\omega_z)} \subset (D_z^{\mathsf{bhm}})_{|H_0^2(\omega_z)}.$$

Thus, to show condition (2.29), we have to show the inclusion also for functions in S_z^{bhm}. Indeed, using the calculations at the beginning of this section, we have for each $v_h \in V_h$ and $s \in \mathrm{MR}$ with $\operatorname{supp}(s) \subset \omega_z$ and appropriate $r_F, r_{z'} \in \mathbb{R}$, $F \in \mathcal{F}_z^{i,+}$ and $z' \in \mathcal{V} \cap \mathring{\omega}_z^+$ that

$$\tilde{a}(v_h, \mathcal{E}_{\mathsf{hct}} s) = \sum_{F \in \mathcal{F}_z^{i,+}} \int_F r_F \partial_n \mathcal{E}_{\mathsf{hct}} s + \sum_{z' \in \mathcal{V} \cap \mathring{\omega}_z^+} r_{z'} \mathcal{E}_{\mathsf{hct}}(s(z')) = \sum_{F \in \mathcal{F}_z^i} \int_F r_F \partial_n \mathcal{E}_{\mathsf{hct}}(s) + \sum_{z' \in \mathcal{V} \cap \mathring{\omega}_z} r_{z'} \mathcal{E}_{\mathsf{hct}}(s(z'))$$

because of the momentum condition of $\mathcal{E}_{\mathsf{hct}}$ in (1.72) and that $\operatorname{supp}(s) \subset \omega_z$. Hence, condition (2.29) is fulfilled and Proposition 2.22 is applicable. Estimate (2.30c) of Proposition 2.22 therefore proves (2.144).

The last statement follows with Corollary 2.23 and Proposition 2.56. $\square$

In the case of the B-MR method, we can simplify the estimator again by using the local conformity error of Definition 2.67 . Because B-MR is overconsistent, the local conformity errors disappear.

Proposition 2.94 ([KRVZ25, Theorem 5.15])**.** *Let $u \in H_0^2(\Omega)$ be the analytic solution of the biharmonic problem (1.66) and $u_h \in \mathrm{MR}$ be the discrete solution of the B-MR method presented in section 1.4.4.1. Then the estimator induced by $\mathcal{P}_z^{bhm}$ can be written as*

$$\|D_h^2(u_h - u)\|_{L^2(\Omega)}^2 \eqsim \sum_{F \in \mathcal{F}} \int_F \frac{1}{h_F} |[\![\nabla_h u_h]\!]|^2 + \sum_{z \in \mathcal{V}} \|f - \mathcal{P}_z^{bhm} f\|_{H^{-2}(\omega_z)}^2. \tag{2.145}$$

We also have for each $z \in \mathcal{V}$ the lower bound

$$\sum_{F \in \mathcal{F}_z^i} \int_F \frac{1}{h_F} |[\![\nabla_h u_h]\!]|^2 + \|f - \mathcal{P}_z^{bhm} f\|_{H^{-2}(\omega_z)}^2 \lesssim \|D_h^2(u_h - u)\|_{L^2(\omega_z^+)}^2. \tag{2.146}$$

The hidden constants only depend on $\sigma(\mathcal{T})$.

Proof. We localize the error with Theorem 2.50. After this, the projection $\mathcal{P}_z^{\mathsf{bhm}}$ splits the localized residuals $\|\operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f,u_h)\|_{H^{-2}(\omega_z)}$ as presented in (2.144). The expression $\|\mathcal{P}_z^{\mathsf{bhm}}\operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f,u_h)\|_{H^{-2}(\omega_z)}$ is equivalent to the hierarchical representation given in (2.143) after replacing f with $\operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f,u_h)$. Since $S_z^{\mathsf{bhm}} \subset \mathcal{E}_{\mathsf{hct}}(\mathrm{MR})$ and the fact that $\delta_{V_h} = 0$ for B-MR ([VZ19a]), we obtain with Theorem 2.5 that the hierarchical representation vanishes. Consequently, the whole expression $\|\mathcal{P}_z^{\mathsf{bhm}}\operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f,u_h)\|_{H^{-2}(\omega_z)}$ vanishes and only the local oscillation terms remain. This shows one direction in (2.145).

The other direction in (2.145) follows by taking the sum over all $z \in \mathcal{V}$ in (2.146), which we prove next.

To show (2.146), we observe that (2.30a) is applicable because Corollary 2.93 shows that (2.29) is fulfilled. Then (2.30a) bounds the oscillation by

$$\|f - \mathcal{P}_z^{\mathsf{bhm}} f\|_{H^{-2}(\omega_z)} \lesssim \|(\tilde{A}u_h) - f\|_{H^{-2}(\omega_z^+)} = \sup_{s\in H^{-2}(\omega_z^+)} \frac{\tilde{a}(u_h-u,s)}{\|s\|_{\tilde{V}}} \leqslant \|D_h^2(u_h-u)\|_{L^2(\omega_z^+)}.$$

Combining this with (2.51) yields the local bound (2.146). □

Since the Morley functions are not contained in $\mathring{S}_2^1(\mathcal{T})$, the simplification with the quasi-Galerkin orthogonality in Theorem 2.5 is not available for the B-SCZ method. Alternatively, one could use the test function space

$$S_z^{C^0} := \operatorname{span}\{\overline{\Phi}_{\mathsf{n}_F} \mid F \in \mathcal{F}_z^i\} \oplus \begin{cases} \operatorname{span}\{\mathcal{E}_{\mathsf{hct}}(\Phi_z^1)\} & \text{if } z \in \mathcal{V}\cap\Omega, \\ \{0\} & \text{if } z \in \mathcal{V}\cap\partial\Omega. \end{cases}$$

The function $\overline{\Phi}_{\mathsf{n}_F}$ is defined in Corollary 1.59 and Φ_z^1 is the Lagrange basis function defined in (1.28). Then one can at least simplify the evaluation of Φ_z^1 to detect r_z. We present some details.

Similar to Proposition 2.92, the next Proposition shows that the pair $(D_z^{\mathsf{bhm}}, S_z^{C^0})$ defines a stable projection $\mathcal{P}_z^{\mathsf{bhm},2}$.

Proposition 2.95. *For each $z \in \mathcal{V}$, the projection $\mathcal{P}_z^{\mathsf{bhm},2}$ induced by the spaces $S_z^{C^0}$ and D_z^{bhm} via (2.22) is well-defined and stable. For each $f \in H^{-2}(\omega_z^+)$, we have the equivalence*

$$\|\mathcal{P}_z^{\mathsf{bhm},2}(f)\|_{H^{-2}(\omega_z)} \lesssim \Big(\sum_{F\in\mathcal{F}_z^i} h_z\|r_F\|_{L^2(F)}^2 + h_z^2|r_z|^2\Big)^{\frac12} \lesssim \|\mathcal{P}_z^{\mathsf{bhm},2}(f)\|_{(S_z^{C^0})'}$$

Here, $r_F := \langle f, \overline{\Phi}_{\mathsf{n}_F}\rangle$, $F \in \mathcal{F}_z^i$, $r_z := \langle f, \mathcal{E}_{\mathsf{hct}}(\Phi_z^1)\rangle$ if $z \in \Omega$ and $r_z := 0$ if $z \in \partial\Omega$. Additionally, we can write

$$\|\mathcal{P}_z^{\mathsf{bhm},2}(f)\|_{H^{-2}(\omega_z)}^2 \lesssim \sum_{F\in\mathcal{F}_z^i} \frac{\langle f,\overline{\Phi}_{\mathsf{n}_F}\rangle^2}{\|D^2\overline{\Phi}_{\mathsf{n}_F}\|_{L^2(\omega_z)}^2} + \frac{\langle f,\mathcal{E}_{\mathsf{hct}}(\Phi_z^1)\rangle^2}{\|D^2\mathcal{E}_{\mathsf{hct}}(\Phi_z^1)\|_{L^2(\omega_z^+)}^2} \lesssim \|\mathcal{P}_z^{\mathsf{bhm},2}(f)\|_{(S_z^{\mathsf{bhm}})'}^2. \tag{2.147}$$

The latter summand is omitted if $z \in \partial\Omega$. The hidden constants only depend on $\sigma(\mathcal{T})$.

With the new projection, we obtain error-estimators for the (B-MR) and (B-SCZ) methods with similar estimates as in Corollary 2.93. Of particular interest in this case is the omission of the test function that tests the point source at the center of the patch in the case of (B-SCZ). This is the topic of the following theorem.

Theorem 2.96. *Let $u \in H_0^2(\Omega)$ be the solution of (1.66). Let $u_h \in V_h = \mathring{S}_2^1(\mathcal{T})$ be the discrete solution of the method (B-SCZ). The projection $\mathcal{P}_z^{\mathsf{bhm},2}$, $z \in \mathcal{V}$, fulfills*

$$\eta_z^2 \lesssim \|\mathcal{P}_z^{\mathsf{bhm},2}(\operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f,u_h))\|_{(S_z^{C^0})'}^2 \lesssim \eta_z^2 + \mathbf{CE}_z^2(u_h), \tag{2.148}$$

where $\mathbf{CE}_z(u_h)$ is the local conformity error and

$$\eta_z^2 := \sum_{F\in\mathcal{F}_z^i} h_z^2\|r_F\|_{L^2(F)}^2 \quad \text{or} \quad \eta_z^2 := \sum_{F\in\mathcal{F}_z^i} \frac{\langle \mathcal{P}_z^{\mathsf{bhm},2}(\operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f,u_h)),\overline{\Phi}_{\mathsf{n}_F}\rangle^2}{\|D^2\overline{\Phi}_{\mathsf{n}_F}\|_{L^2(\omega_z)}^2},$$

with $r_F := \langle \operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f,u_h), \overline{\Phi}_{\mathsf{n}_F}\rangle$, $F \in \mathcal{F}_z^i$.

For each $z \in \mathcal{V}$, we have

$$\eta_z^2 + \|\mathcal{P}_z^{\mathsf{bhm},2}f - f\|_{H^{-2}(\omega_z)}^2 \lesssim \|D_h^2(u-u_h)\|_{L^2(\omega_z^+)}^2, \tag{2.149}$$

and

$$\mathbf{CE}_z(u_h) \lesssim \left(\sum_{F' \in \mathcal{F}_z} \int_{F'} \frac{1+\gamma}{h_{F'}} \left| \left[\!\left[\partial_{\mathsf{n}_{F'}} u_h \right]\!\right] \right|^2 \right)^{\frac{1}{2}}.$$

The hidden constants only depend on $\sigma(\mathcal{T})$.

Turning to the global perspective, we have with a constant, depending only on $\sigma(\mathcal{T})$, the bound

$$\sum_{z \in \mathcal{V}} \mathbf{CE}_z^2(u_h) \lesssim \frac{1+\gamma}{\gamma} \|\Pi_V u_h - u_h\|_{\tilde{V}}^2.$$

With the definition

$$\mathrm{est}^2 := \sum_{z \in \mathcal{V}} \eta_z^2 + \|\mathcal{P}_z^{\mathsf{bhm},2} f - f\|_{H^{-2}(\omega_z)}^2 + \|\Pi_V u_h - u_h\|_{\tilde{V}}^2,$$

we obtain the global bounds

$$\mathrm{est}^2 \lesssim \|u - u_h\|_{\tilde{V}}^2 \lesssim \mathrm{est}^2 + \sum_{z \in \mathcal{V}} \mathbf{CE}_z^2(u_h) \lesssim \mathrm{est}^2.$$

In the last display, the hidden constants only depend on γ and $\sigma(\mathcal{T})$.

Proof. The proof works similar to the one of Theorem 2.69. The estimate (2.148) follows by absorbing the term $|r_z|$ in the terms $\|r_F\|_{L^2(F)}$, $F \in \mathcal{F}_z^i$, or using the definition of the local conformity error directly. The local bound (2.149) is a consequence that the projection $\mathcal{P}_z^{\mathsf{bhm},2}$ fulfills (2.29) and a Cauchy-Schwarz inequality.

The derivation of the local bound for the local conformity error is similar to the approach in Lemma 2.68 except that we use Lemma 1.67 instead of Lemma 1.46.

The global estimates follow by summing over all $z \in \mathcal{V}$ and using the localization and globalization results for the method (B-SCZ) in section 2.2.3. □

Remark 2.97. An alternative projection with corresponding estimator is presented in [KRVZ25, section 5.2]. There, the authors use a test function space constructed from the HCT-basis functions Υ_z^0 and Υ_F:

$$S_z := \operatorname{span}\left\{ \Upsilon_F \mid F \in \mathcal{F}_z^i \right\} \oplus \begin{cases} \operatorname{span} \Upsilon_z^0 & \text{if } z \in \mathcal{V}^i, \\ \{0\} & \text{if } z \in \mathcal{V} \backslash \mathcal{V}^i. \end{cases}$$

3 Pressure-robust finite element methods for the Stokes equation

In this chapter, we transfer the lessons learned in the previous chapters to the Stokes problem, which is a simplification of the Navier-Stokes equations. The Navier-Stokes equations models fluids by describing the relations between the fields of fluid velocity and pressure under some external forces.The fluid's density and viscosity are parameters of the equation. The mathematics behind this equation is described in [Gal11, LuK16], while the latter presents the physical motivation as well.

The Navier-Stokes equation contains a non-linear term. If, for example, the viscosity of the fluid is large, the non-linear term can be neglected and the Navier-Stokes equation becomes the Stokes equation (c.f [Gal11, chapter 4], [BF13, section 6.2.4]). Therefore, one often uses the Stokes equation to model highly viscous fluids like honey or tar.

Both, the Stokes and Navier-Stokes equation are available in a time-dependent and stationary variant. As the framework of [VZ18a] and [KV21, KVZ] deal with stationary linear equations, we restrict ourselves also to the stationary, linear Stokes equation.

In the previous sections, the solutions could be viewed as a minimizer of a convex optimization problem. The Stokes equation, however, is equivalent to a saddle-point problem hence the a-posteriori framework is not directly applicable. Moreover, the solution of the Stokes equation consists of two variables: the velocity field and the pressure field.

A discretization of the Stokes equation usually involves both: a discretization of the velocity and the pressure. As already pointed out in the introduction, the discrete velocity and its estimators in classical settings (as in [BBF13]) might be influenced by the analytical pressure. This leads to undesirable situations, see [JLM$^+$17] and section 3.3 below. Methods and error estimators without this flaw are called *pressure-robust*.

Several pressure-robust methods are available, see for example the publications [Lin14, LMT16, LMNN18, LLMS17]. Also, a pressure-robust a-posteriori analysis for these methods was developed in [LMS19]. But these methods still do not have a quasi-optimal velocity approximation, see [KZ20, KVZ21, VZ19b]. In view of the advantages of quasi-optimal methods pointed out in [VZ18a] and the advances in understanding quasi-optimal methods, pressure-robust and quasi-optimal methods seem desirable and advantageous.

Actually, pressure-robust and quasi-optimal methods for the Stokes equation were developed in [KZ20, KVZ21, VZ19b]. But the a-posteriori analysis of [LMS19] needs $f \in \underline{H}(\mathrm{curl})$ and exhibits classical oscillation terms, e.g. $\mathrm{osc}_\ell^{L^2}(h\,\mathrm{curl}_h(f+\nu\Delta_h u_h))$ or $\mathrm{osc}_\ell^{L^2}(f-\nabla p)$ with $\mathrm{osc}_\ell^{L^2}(\cdot)$ defined in (2.17). The problems of such terms are pointed out in section 2.1.3. Since the construction of the quasi-optimal methods resolved similar issues in the a-priori analysis, it is imprudent to apply the estimator of [LMS19] to the new quasi-optimal and pressure-robust methods.

For this reason, we will construct new pressure-robust error estimators for the pressure-robust, quasi-optimal methods of [KZ20, KVZ21, VZ19b]. The estimators are inspired by the ones in chapter 2 and have an error-dominated oscillation, which does not have the problems of the classical oscillation.

The outline of this chapter is as follows: First, we review some additional analytical tools and the well-posedness of the analytic Stokes equation. After that, we discuss the essentials of the discretization for the Stokes equation and recall the pressure-robust, quasi-optimal methods of the above papers. In the next chapter, we construct the pressure robust error estimator with an error-dominated oscillation.

3.1 Analytic preliminaries

In this section, we adapt the analytic setting of section 1.1 to the Stokes equation. For this equation, we need some new differential operators, which we introduce in the following section. After this, we introduce the weak formulation of the Stokes problem and discuss the analytical well-posedness. Throughout this chapter and the next, we assume that we are given an open domain Ω that has a polyhedral, Lipschitz continuous boundary.

3.1.1 Additional differential operators

Beyond the already used gradient and Laplace operator, the divergence of a vector field appears in the Stokes equation. As before, we write the distributional partial derivatives into the direction of the standard basis vectors e_i as ∂_i, $1 \leqslant i \leqslant d$. Given a vector field $v : \omega \to \mathbb{R}^d$, $\omega \subset \Omega$, we write v_i for its ith component. The following standard definition can be found in [BBF13, pp. 49].

Definition 3.1 (div-operator)**.** *Let $\omega \subset \Omega$ be open. Given the distributional derivatives $\partial_i v_i$, $1 \leqslant i \leqslant d$, of a vector field $v : \omega \to \mathbb{R}^d$, we define the divergence operator on ω as*

$$\operatorname{div} v = \sum_{i=1}^{d} \partial_i v_i.$$

For any measurable $\omega \subset \Omega$, we define the normed space

$$\begin{aligned} H(\operatorname{div};\omega) &:= \{v \in (L^2(\mathring{\omega}))^d \mid \operatorname{div}(v_{|\mathring{\omega}}) \in L^2(\mathring{\omega})\},\\ \|v\|^2_{H(\operatorname{div};\omega)} &:= \|v\|^2_{L^2(\mathring{\omega})} + \|\operatorname{div}(v_{|\mathring{\omega}})\|^2_{L^2(\mathring{\omega})}, \quad v \in H(\operatorname{div};\omega). \end{aligned}$$

For simplicity, we set $H(\operatorname{div}) := H(\operatorname{div};\Omega)$ and $\|\cdot\|_{H(\operatorname{div})} := \|\cdot\|_{H(\operatorname{div};\Omega)}$.

The divergence operator is used in Gauß theorem, see for example [BBF13, Lemma 2.1.1].

Lemma 3.2 (Integration by parts for div)**.** *For every measurable $\omega \subset \Omega$ with Lipschitz boundary and $q \in H(\operatorname{div};\omega)$, $v \in H^1(\omega)$, we have*

$$\int_\omega \operatorname{div} q\, v = -\int_\omega q \cdot \nabla v + \int_{\partial\omega} q \cdot \mathsf{n}_\omega v dS. \tag{3.1}$$

Here n_ω denotes the outer normal vector of ω.

To construct vector fields with vanishing divergence, we often use the curl operator, see [BBF13, pp. 61].

Definition 3.3 (curl-operator)**.** *If $d = 3$, let $\omega \subset \Omega \subset \mathbb{R}^3$ be a measurable set and $v : \omega \to \mathbb{R}^3$ a vector field. Then, we define the* curl*-operator on $\mathring{\omega}$ as*

$$\operatorname{curl} v = \begin{pmatrix} \partial_2 v_3 - \partial_3 v_2 \\ \partial_3 v_1 - \partial_1 v_3 \\ \partial_1 v_2 - \partial_2 v_1 \end{pmatrix}.$$

We define the space

$$H(\operatorname{curl};\omega) := \{v \in (L^2(\mathring{\omega}))^3 \mid \operatorname{curl}(v_{|\mathring{\omega}}) \in (L^2(\mathring{\omega}))^3\}$$

with the norm

$$\|v\|^2_{H(\operatorname{curl};\omega)} = \|v\|^2_{L^2(\mathring{\omega})^3} + \|\operatorname{curl}(v_{|\mathring{\omega}})\|^2_{L^2(\mathring{\omega})}, \quad v \in H(\operatorname{curl};\omega).$$

If $d = 2$ and $\omega \subset \Omega \subset \mathbb{R}^2$ a measurable set, we distinguish two cases. Given a vector field $v : \omega \to \mathbb{R}^2$, we define the curl *on $\mathring{\omega}$ as*

$$\operatorname{curl} v = \partial_1 u_2 - \partial_2 u_1.$$

For a function $v : \omega \to \mathbb{R}$, we set

$$\operatorname{curl} v = \begin{pmatrix} \partial_2 v \\ -\partial_1 v \end{pmatrix}.$$

We define the spaces

$$\underline{H}(\operatorname{curl};\omega) := \{v \in (L^2(\mathring{\omega}))^2 \mid \operatorname{curl}(v_{|\mathring{\omega}}) \in (L^2(\mathring{\omega}))\},$$

and

$$H(\operatorname{curl};\omega) := \{v \in (L^2(\mathring{\omega})) \mid \operatorname{curl}(v_{|\mathring{\omega}}) \in (L^2(\mathring{\omega}))^2\}.$$

On these spaces, we define the norms

$$\begin{aligned} \|v\|^2_{\underline{H}(\operatorname{curl};\omega)} &= \|v\|^2_{L^2(\mathring{\omega})} + \|\operatorname{curl}(v_{|\mathring{\omega}})\|^2_{L^2(\mathring{\omega})}, \quad v \in \underline{H}(\operatorname{curl};\omega) \quad \text{and}\\ \|v\|^2_{H(\operatorname{curl};\omega)} &= \|v\|^2_{L^2(\mathring{\omega})} + \|\operatorname{curl}(v_{|\mathring{\omega}})\|^2_{L^2(\mathring{\omega})}, \quad v \in H(\operatorname{curl};\omega). \end{aligned}$$

Similarly to $H(\operatorname{div})$, for $\omega = \Omega$, we abbreviate $H(\operatorname{curl}) := H(\operatorname{curl},\Omega)$ with $\|\cdot\|_{H(\operatorname{curl})} := \|\cdot\|_{H(\operatorname{curl};\Omega)}$ and $\underline{H}(\operatorname{curl}) := \underline{H}(\operatorname{curl},\Omega)$ with $\|\cdot\|_{\underline{H}(\operatorname{curl})} := \|\cdot\|_{\underline{H}(\operatorname{curl};\Omega)}$.

The curl-operator is related to the cross product of two vectors $v, w \in \mathbb{R}^3$

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \times \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} v_2 w_3 - v_3 w_2 \\ v_3 w_1 - v_1 w_3 \\ v_1 w_2 - v_2 w_1 \end{pmatrix}.$$

With this, we have the following integration by parts formula:

Lemma 3.4 (Integration by parts for curl c.f. [GR79, Theorem 2.11])**.** *Let $\omega \subset \Omega \subset \mathbb{R}^3$ be a measurable set with Lipschitz continuous boundary and $v, w : \omega \to \mathbb{R}^3$, $v \in H(\operatorname{curl}, \omega)$ and $w \in H^1(\omega)^3$ then we have*

$$\int_\omega \operatorname{curl} v \cdot w = \int_\omega v \cdot \operatorname{curl} w + \int_{\partial\omega} v \times \mathsf{n}_\omega w, \tag{3.2}$$

where n_ω denotes the outer normal vector.

If $\omega \subset \Omega \subset \mathbb{R}^2$ is a set with Lipschitz continuous boundary and $v : \omega \to \mathbb{R}^2$, $v \in \underline{H}(\operatorname{curl}, \omega)$, $w : \omega \to \mathbb{R}, w \in H^1(\omega)$, we have

$$\int_\omega \operatorname{curl} vw = \int_\omega v \cdot \operatorname{curl} w + \int_{\partial\omega} v \cdot \mathsf{t}_\omega w, \tag{3.3}$$

where $\mathsf{t}_\omega = \big(-(\mathsf{n}_\omega)_2, (\mathsf{n}_\omega)_1\big)^T$ is the tangential vector.

3.1.2 The Stokes equation

Here, we introduce the weak formulation of the Stokes equation and derive conditions for its well-posedness. In contrast to the previous equations, the Stokes problem has two unknown functions instead of only one. One is a vector field, modelling the velocity of the fluid, and the other is a pressure field, modelling the pressure within the fluid. We seek the velocity field in the space $H_0^1(\Omega)^d$ that are vector fields whose components are in $H_0^1(\Omega)$. In contrast, the pressure field is only a scalar field, which does not require any differentiability. But for uniqueness, we seek it in the smaller space $L_0^2(\Omega)$: For a measurable set $\omega \subset \Omega$ and $1 \leqslant p \leqslant \infty$, we define

$$L_0^p(\omega) := \left\{ f \in L^p(\omega) \,\middle|\, \frac{1}{|\omega|} \int_\omega f = 0 \right\}. \tag{3.4}$$

The Stokes problem in the distributional sense reads as [EG04, section 4.1.1]: Given $f \in H^{-1}(\Omega)^d$, find a velocity field $u \in H_0^1(\Omega)^d$ and a pressure field $p \in L_0^2(\Omega)$ such that

$$\begin{aligned} -\nu \Delta u + \nabla p &= f \quad \text{in } H^{-1}(\Omega)^d, \\ \operatorname{div} u &= 0 \quad \text{in } L_0^2(\Omega). \end{aligned} \tag{3.5}$$

The vector Laplace operator Δ maps a d-dimensional vector field on a d-dimensional vector field and is defined as $(\Delta u)_j = \sum_{i=1}^d \partial_i \partial_i u_j$ for all $1 \leqslant j \leqslant d$. The scalar $\nu > 0$ is the viscosity of the modelled fluid. The constraint $\operatorname{div} u = 0$ ensures the *mass-conservation*, see [LuK16].

Remark 3.5. In some expositions, the Stokes equation is written with the operator $\operatorname{div} \varepsilon$ instead of Δ, see for example [BBF13]. (The operator ε is the symmetric gradient as defined in 1.4.3. In the case of the incompressible Stokes problem with homogeneous boundary conditions (this is the only setting we consider in this thesis), the two formulations of the continuous Stokes equation are equivalent. This follows from the formula in [BF13, Remark I.4.2]. However, their discretizations differ in general.

Moreover, the finite element methods we consider use the Laplace operator Δ, c.f. [KVZ21, KZ20, VZ19b]. Because this thesis deals with the a-posteriori analysis and not with the construction and a-priori analysis of numerical methods, we use the methods in the form provided by the literature.

We reformulate this equation in the weak sense to apply the framework of chapter 1, see [EG04, section 4.1.2]: We set $V = H_0^1(\Omega)^d$ and $V' = H^{-1}(\Omega)^d$. But we need two bilinear forms instead of one. The first bilinear form is given by

$$a := a_{\mathsf{S}} : V \times V \to \mathbb{R}, \qquad a(u, v) := a_{\mathsf{S}}(u, v) := \nu \int_\Omega \nabla u : \nabla v. \tag{3.6}$$

The ∇-operator acts on vector fields componentwisely, thus $\nabla v = (\partial_i v_j)_{1 \leqslant j \leqslant d, 1 \leqslant i \leqslant d} \in \mathbb{R}^{d \times d}$, $v \in H_0^1(\Omega)^d$. The notation of section 1.4.3 reads in this setting as $\nabla u : \nabla v = \sum_{i,j=1}^d \partial_i u_j \partial_i v_j$. Again, the bilinear form a induces the operator $A : H_0^1(\Omega)^d \to H^{-1}(\Omega)^d$ with $A(u) := a(u, \cdot)$.

The second bilinear form is defined on the spaces $Q := L^2_0(\Omega)$ and $V := H^1_0(\Omega)^d$ and reads as

$$b : V \times Q \to \mathbb{R} \qquad b(v,p) := -\int_\Omega p \operatorname{div} v. \tag{3.7}$$

Again, we can write the form with an operator $B : V \to Q'$ such that $b(v,p) = \langle Bv, p\rangle$, $v \in V, p \in Q$. By Lemma 3.2, we see that for $v \in H^1_0(\Omega)^d$, we have $\int_\Omega \operatorname{div} v = \int_{\partial\Omega} v \cdot \mathsf{n}_\Omega = 0$ thus $\operatorname{div} v \in L^2_0(\Omega)$. If $b(v,q) = 0$ for all $q \in L^2_0(\Omega)$, then we have $\operatorname{div} v = 0$ a.e. in Ω.

The two bilinear forms allow us to formulate the Stokes problem in the weak form ([EG04, (4.2)]):

Given $f \in H^{-1}(\Omega)^d$, find $u \in H^1_0(\Omega)^d$ and $p \in L^2_0(\Omega)$ such that

$$a(u,v) + b(v,p) = \nu \int_\Omega \nabla u : \nabla v - \int_\Omega p \operatorname{div} v = \langle f, v\rangle, \text{ for all } v \in H^1_0(\Omega)^d, \tag{3.8a}$$

$$b(u,q) = -\int_\Omega q \operatorname{div} u = 0, \text{ for all } q \in L^2_0(\Omega). \tag{3.8b}$$

In [EG04, chapter 4.1] and [BBF13, chapter 8], it is shown that (3.8) is well-posed because of the assumptions on the domain Ω. We only recall the main aspects. The problem is solved in two steps: At first, we obtain the velocity field $u \in H^1_0(\Omega)^d$ and after that we compute the pressure $p \in L^2_0(\Omega)$.

Because of (3.8b), the velocity field is sought in the space of (exactly) divergence-free functions

$$Z := Z(\Omega) := \{v \in H^1_0(\Omega)^d \mid \operatorname{div} v = 0\}.$$

By restricting ourselves to the space $Z(\Omega)$ as test- and trail-space in (3.8a), the problem (3.8) simplifies: The term $b(v,p)$ and the constraint $b(u,q) = 0$ disappear because $b(v,\cdot) = 0$ if $v \in Z(\Omega)$. This reduces the problem (3.8) to

$$\text{Given } f \in H^{-1}(\Omega)^d, \text{ find } u \in Z(\Omega) \text{ such that} \quad \nu \int_\Omega \nabla u : \nabla v = \langle f, v\rangle \text{ for all } v \in Z(\Omega). \tag{3.9}$$

We also define $Z' := Z(\Omega)'$ as the corresponding dual space of $Z(\Omega)$. This equation is well-posed and uniquely solvable by standard methods: The form a is continuous and coercive on $Z(\Omega)$ since the space $Z(\Omega)$ is a closed subset of $H^1_0(\Omega)^d$.

After determining the velocity field, it remains to prove the existence of a unique pressure field $p \in L^2_0(\Omega)$ such that

$$\langle \nabla p, v\rangle = -\int_\Omega p \operatorname{div} v = \langle f, v\rangle - \nu a(u,v) \text{ for all } v \in H^1_0(\Omega)^d, \tag{3.10}$$

where the first expression is understood in a distributional sense. Thus, this equation is only solvable if $\langle f, \cdot\rangle - \nu a(u,\cdot) \in \operatorname{im}(\nabla) \subset H^{-1}(\Omega)^d$. The space $\operatorname{im}(\nabla)$ can be characterized by Banach's closed range theorem. This theorem is proven in e.g. [Yos95, p.205] but see also the version [EG04, Theorem A. 34]. For the reader's convenience, we provide the closed range theorem in the form we will need.

Theorem 3.6 (Banach's closed range theorem, [EG04, Theorem A.34])**.** *Let $V := H^1_0(\Omega)^d$ and $Q := L^2_0(\Omega)$. For the operator $\operatorname{div} \in \mathbb{L}(V,Q)$ and its adjoint $\nabla \in \mathbb{L}(Q', V')$ the following statements are equivalent:*

(i) $\operatorname{im}(\operatorname{div})$ *is closed in Q;*

(ii) $\operatorname{im}(\nabla)$ *is closed in V';*

(iii) $\operatorname{im}(\operatorname{div}) = \{q \in Q \mid \langle q, q'\rangle = 0 \quad \forall q' \in \operatorname{kern}(\nabla)\}$ *and*

(iv) $\operatorname{im}(\nabla) = \{v' \in V' \mid \langle v', v\rangle = 0 \quad \forall v \in \operatorname{kern}(\operatorname{div})\}$.

Theorem 3.6 implies: If $\operatorname{im}(\operatorname{div}) = \{q \in Q \mid \langle q, q'\rangle = 0 \quad \forall q' \in \operatorname{kern}(\nabla)\}$ is fulfilled, then point (iv) in Theorem 3.6 is true. Consequently, as $\langle f, v\rangle - a(u,v) = 0$ for all $v \in \operatorname{kern}(\operatorname{div})$, we have $\langle f, \cdot\rangle - a(u,\cdot) \in \operatorname{im}(\nabla)$ and (3.10) is uniquely solvable because $p \in L^2_0(\Omega)$ and Ω connected.

This argument reduces the question of the well-posedness of (3.10) to the analysis of $\operatorname{kern}(\nabla)$. The kernel of ∇ consists only of the zero function because the domain Ω is connected and the functions in $L^2_0(\Omega)$ have mean value zero. This insight shows that the condition $\operatorname{im}(\operatorname{div}) = \{q \in Q \mid \langle q, q'\rangle = 0 \quad \forall q' \in \operatorname{kern}(\nabla)\}$, simplifies to $\operatorname{im}(\operatorname{div}) = Q = L^2_0(\Omega)$. Consequently, (3.10) is uniquely solvable if $\operatorname{div} : H^1_0(\Omega)^d \to L^2_0(\Omega)$ is surjective.

This is one reason why showing the surjectivity of the divergence operator is of major importance. In [Gal11, Section III.3], it is shown that the divergence operator is surjective if Ω is sufficiently regular. We recall one of the more general theorems in [Gal11] in a shortened form.

Theorem 3.7 (Analytic divergence inversion [Gal11, Theorem III.3.1])**.** *Given a sufficiently regular bounded set $\omega \subset \Omega$ (for example a simplex), then for each $g \in L^2_0(\omega)$ there exists $v \in H^1_0(\omega)^d$ with*

$$\operatorname{div} v = g \quad \text{and} \quad \|\nabla v\|_{L^2(\omega)} \lesssim \|g\|_{L^2(\omega)}.$$

The hidden constant depends on the geometry of ω.

Therefore, if the domain Ω is regular enough such that conditions of [Gal11, Theorem III.3.1] are fulfilled, the divergence operator $\operatorname{div} : H^1_0(\Omega)^d \to L^2_0(\Omega)$ is surjective and has a continuous right inverse $\operatorname{div}^{-1} : L^2_0(\Omega) \to H^1_0(\Omega)^d$. The operator div^{-1} is also relevant beyond the well-posedness of the Stokes equation. We will later use the operator to correct the divergence on single elements and complete patches.

An example of regular enough sets, on which a continuous right inverse of the divergence operator exists, are the *star-shaped* sets, see [Gal11, Section III.3].

Definition 3.8 ([BS08, (4.2.2)/(4.2.16)])**.** *A subset $\omega \subset \Omega$ is called* star-shaped *with respect to a ball $B \subset \omega$ if for all $x \in \omega$ we have* convexhull$(\{x\} \cup B) \subset \omega$.

The analysis of the divergence operator allows us to formulate the following Proposition about the Stokes equation (3.8).

Proposition 3.9 ([EG04, Theorem 4.3, Lemma B.69])**.** *The Stokes equation* (3.8) *is a well-posed problem, if the bounded domain Ω has a Lipschitz continuous boundary and is the union of finitely many star-shaped sets.*

Due to the assumed regularity of $\partial\Omega$, Riesz representation theorem ensures that the velocity-equation (3.9) is always well-posed in our setting. Therefore, in our setting, the well-posedness of the Stokes equation (3.8) is equivalent to the solvability of $\operatorname{div} u = g$, $u \in H^1_0(\Omega)^d$ and $g \in L^2_0(\Omega)$. The latter can be reformulated with [EG04, Lemma A.39] and $A = \nabla$ in the following way:

Lemma 3.10 ([EG04, Theorem 4.3, Corollary B.71])**.** *In our setting, the Stokes problem* (3.8) *is well-posed if and only if there exists an $\alpha > 0$ such that*

$$\inf_{p \in L^2_0(\Omega)} \sup_{v \in H^1_0(\Omega)^d} \frac{b(v,p)}{\|p\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)}} \geqslant \alpha. \tag{3.11}$$

A discrete counterpart of this condition is used to determine if a discretization of the Stokes problem is well-posed.

3.2 Discrete preliminaries

To discretize the Stokes equation, one often uses the so-called *mixed finite element methods* ([BBF13]): One selects a finite-dimensional space V_h, not necessarily contained in $V = H^1_0(\Omega)^d$, and a finite-dimensional space Q_h. For simplicity, we assume $Q_h \subset Q = L^2_0(\Omega)$. To stay close to the setting of section 1.2 and of [KZ20, KVZ21, VZ19b], we define $\tilde{V} = V + V_h$ and a norm $\|\cdot\|_{\tilde{V}}$ that extends $\|\nabla \cdot\|_{L^2(\Omega)}$. Both spaces are connected via discrete bilinear forms $a_h : V_h \times V_h \to \mathbb{R}$ and $b_h : V_h \times Q_h \to \mathbb{R}$, which are yet to define. These ingredients build the discrete Stokes problem: Seek $u_h \in V_h$ and $p_h \in Q_h$ such that

$$a_h(u_h, v_h) + b_h(v_h, p_h) = \langle f, v_h \rangle \quad \forall v_h \in V_h \tag{3.12a}$$

$$b_h(u_h, q_h) = 0 \qquad \forall q_h \in Q_h. \tag{3.12b}$$

In this thesis, we are mainly interested in the approximation of the velocity. The velocity field approximation u_h of the discrete Stokes equation is often not exactly divergence-free but only discretely divergence-free, which is a weaker concept: A function $v_h \in V_h$ is discretely divergence free if

$$b_h(v_h, q_h) = 0 \quad \forall q_h \in Q_h.$$

This gives rise to the definition of the discrete divergence $\underline{\operatorname{div}}_h : V_h \to Q_h$ (c.f. [KZ20, (2.7)]):

$$\int_\Omega \underline{\operatorname{div}}_h v_h q_h = -b_h(v_h, q_h) \quad \forall q_h \in Q_h.$$

Consequently, the solution $u_h \in V_h$ needs not to be exactly divergence-free because the divergence constraint might be weaker. We define $Z_h := \{v_h \in V_h \mid b_h(v_h, q_h) = 0 \quad \forall q_h \in Q_h\}$, which is the space of *discretely divergence-free functions*.

We learned in the previous section that the analytic Stokes problem (3.8) is well-posed if (3.11) is true. The discrete problem is only well-posed if and only if V_h and Q_h fulfill a discrete counterpart of (3.11), c.f. [EG04, Remark 4.4]. Therefore, the choice of the pairs of discrete spaces is restricted.

Lemma 3.11 ([EG04, Proposition 4.13]). *The discrete Stokes problem* (3.12) *with the pair of spaces* (V_h, Q_h) *is well-posed if and only if the problem*

$$\text{Given } f \in Z_h', \text{ find } u_h \in Z_h \text{ such that } a_h(u_h, v_h) = \langle f, v_h \rangle \quad \forall v_h \in Z_h$$

is well-posed and the discrete inf-sup-condition is fulfilled:

$$\inf_{q_h \in Q_h} \sup_{v_h \in V_h} \frac{b_h(v_h, q_h)}{\|q_h\|_{L^2} \|v_h\|_{\tilde{V}}} \geqslant \alpha_h > 0. \tag{3.13}$$

A pair of finite element spaces (V_h, Q_h) fulfilling (3.13) is called *inf-sup stable*. Further details are found in [BBF13, Chapter 5.1].

An important tool for proving the inf-sup stability (3.13) is the *Fortin* operator.

Definition 3.12 (Fortin operator ([BBF13, Proposition 8.4.1])). *We call a stable projection* $\Pi_F : \tilde{V} := V + V_h \to V_h$ *a* Fortin operator *if it satisfies the following condition:*

$$\begin{aligned} &b(v, q_h) = b_h(\Pi_F(v), q_h) \quad \forall q_h \in Q_h \quad \text{and} \quad \forall v \in V, \\ &\|\Pi_F v\|_{\tilde{V}} \lesssim \|v\|_{\tilde{V}} \quad \forall v \in \tilde{V}. \end{aligned}$$

The relation of Fortin operators and the inf-sup-condition (3.13) is addressed by the following Proposition. A proof is found in [BBF13, Chapter 5.1], see also [BBF13, Proposition 8.4.1].

Proposition 3.13 ([GR79, Chapter II, Lemma 1.1], [For77]). *If the analytic setting satisfies* (3.11)*, then there exists a Fortin operator with a stability constant independent of* h *if and only if the discrete setting fulfills* (3.13) *with* α_h *independent of* h*.*

We now introduce our main examples of finite element pairs (V_h, Q_h) for the Stokes equation and state the conditions when they fulfill (3.13).

3.2.1 The $\mathbb{P}^\ell - \mathbb{P}^{\ell-d}$ element

We start with the $\mathbb{P}^\ell - \mathbb{P}^{\ell-d}$ element and follow the exposition of [KZ20]. Given any polynomial degree $\ell \geqslant d$ and dimension $d \geqslant 2$, we choose $V_h := \mathring{S}^1_\ell(\mathcal{T})^d \subset H^1_0(\Omega)^d$ and $Q_h := \hat{S}^0_{\ell-d}(\mathcal{T}) := \{q_h \in S^0_{\ell-d}(\mathcal{T}) | \int_\Omega q_h = 0\}$. We select $a_h(v_h, w_h) = \nu \int_\Omega \nabla_h v_h : \nabla_h w_h,\ v_h, w_h \in V_h$ and $b_h(v_h, q_h) = -\int_\Omega q_h \operatorname{div} v_h,\ v_h \in V_h, q_h \in Q_h$. The inf-sup stability of this element is the topic of the next Proposition.

Proposition 3.14 (inf-sup stability of $(\mathring{S}^1_\ell(\mathcal{T})^d - \hat{S}^0_{\ell-d}(\mathcal{T}))$, [BBF13, section 8.4.3]). *For any mesh* $\mathcal{T}$ *in the sense of Definition 1.25, the pair* $(V_h, Q_h) = (\mathring{S}^1_\ell(\mathcal{T})^d, \hat{S}^0_{\ell-d}(\mathcal{T}))$ *fulfills the inf-sup condition* (3.13)*. Moreover, there is a Fortin operator for this pair whose stability constant depends only on* ℓ*,* d *and* $\sigma(\mathcal{T})$*.*

For completeness, we provide the proof of the Proposition.

Proof. This proof constructs a Fortin operator. Then the inf-sup condition is fulfilled by Proposition 3.13. The proof is based upon the ideas in [BBF13, section 8.4.3 and 8.5.5].

The construction of the Fortin operator consists of two steps. First, we construct a map that has the same in and out flow in each element as the original function. Second, we deal with the inner degrees of freedom of each element with the help of [BBF13, Proposition 8.5.8].

The first step is accomplished by the map $\Pi_{\mathtt{io}} : H^1_0(\Omega)^d \to \mathring{S}^1_\ell(\mathcal{T})^d$. We define this map similar to Definition 1.35 and [BBF13, section 8.4.1]. The operator $\Pi_{\mathtt{io}}$ is a combination of two operators: The first is the vectorized Scott-Zhang operator $\Pi^d_{\mathsf{SZ}} : H^1_0(\Omega)^d \to \mathring{S}^1_1(\mathcal{T})^d$, see Proposition 2.31. The second is the vectorized bubble operator $\mathcal{B}^d_{\mathcal{F},\ell} : H^1(\mathcal{T})^d \to \mathring{S}^1_{d+\ell-1}(\mathcal{T})^d$, similar to (1.43).

The operator $\mathcal{B}^d_{\mathcal{F},\ell}$, originally defined in [KVZ21, (3.11, 3.12)], is constructed similar to (1.43) but works with vector fields. First, we define an operator $Q^d_F : L^2(F)^d \to \mathbb{P}_{\ell-1}(F)^d$, $F \in \mathcal{F}^i$, as

$$\int_F (Q^d_F v) \cdot q \Psi_F = \int_F v \cdot q \quad \forall q \in \mathbb{P}_{\ell-1}(F)^d.$$

Then we define $\mathcal{B}^d_{\mathcal{F},\ell}$ as

$$\mathcal{B}^d_{\mathcal{F},\ell} v := \sum_{F \in \mathcal{F}^i} \sum_{z \in \mathcal{L}_{\ell-1}(F)} (Q^d_F \{\{v\}\})(z) \Phi^{\ell-1}_z \Psi_F, \quad v \in H^1(\mathcal{T})^d. \tag{3.14}$$

With these definitions, we set for $v \in H^1_0(\Omega)^d$:

$$\Pi_{\mathtt{io}}(v) := \Pi^d_{\mathsf{SZ}}(v) + \mathcal{B}^d_{\mathcal{F},\ell-d+1}\big(v - \Pi^d_{\mathsf{SZ}}(v)\big).$$

This operator preserves moments up to order $\ell - d$ on each $F \in \mathcal{F}^i$:

$$\int_F \Pi_{\mathtt{io}}(v) \cdot p_F = \int_F v \cdot p_F \qquad \forall F \in \mathcal{F}^i, p_F \in \mathbb{P}_{\ell-d}(F)^d, v \in H^1_0(\Omega)^d. \tag{3.15}$$

Arguing as in Corollary 1.41 and using the results in [SZ90, section 4], we have the approximability estimate for each $T \in \mathcal{T}$ and $v \in H^1_0(\Omega)^d$

$$\|\nabla\big(v - \Pi_{\mathtt{io}}(v)\big)\|^2_{L^2(T)} \lesssim \sum_{T' \in \mathcal{T}_T^+} \|\nabla v\|^2_{L^2(T')}, \tag{3.16a}$$

$$\|v - \Pi_{\mathtt{io}}(v)\|^2_{L^2(T)} \lesssim h_T^2 \sum_{T' \in \mathcal{T}_T^+} \|\nabla v\|^2_{L^2(T')}, \tag{3.16b}$$

$$\|\nabla\Pi_{\mathtt{io}}(v)\|^2_{L^2(T)} \lesssim \sum_{T' \in \mathcal{T}_T^+} \|\nabla v\|^2_{L^2(T')}. \tag{3.16c}$$

The constants depend only on ℓ, $\sigma(\mathcal{T})$ and d.

We turn to the second step, which is based upon the argument in [BBF13, Proposition 8.5.8]. We recall the element bubble Ψ_T, $T \in \mathcal{T}$, from (2.70), and we set as in [BBF13, (8.5.26)]

$$B(\Psi_T Q_h) := \{v_h \in \mathring{S}^1_\ell(\mathcal{T})^d \mid \text{there exists a } q_h \in Q_h \text{ such that } (v_h)_{|T} = (\Psi_T \nabla_h q_h)_{|T} \quad \forall T \in \mathcal{T}\}.$$

Since $B(\Psi_T Q_h) \subset \mathring{S}^1_\ell(\mathcal{T})^d$, we define the operator $\Pi_{\mathtt{int}} : H^1_0(\Omega)^d \to B(\Psi_T Q_h) \subset \mathring{S}^1_\ell(\mathcal{T})^d$ for $v \in H^1_0(\Omega)^d$ as (c.f. [BBF13, (8.5.27)])

$$\int_T \Pi_{\mathtt{int}}(v) \cdot \nabla q_h = \int_T (v - \Pi_{\mathtt{io}}(v)) \cdot \nabla q_h \quad \forall q_h \in Q_h \quad \text{and} \quad T \in \mathcal{T}. \tag{3.17}$$

We define the Fortin operator $\Pi_F : H^1_0(\Omega)^d \to \mathring{S}^1_\ell(\mathcal{T})^d$ as $\Pi_F(v) := \Pi_{\mathtt{int}}(v) + \Pi_{\mathtt{io}}(v)$ for $v \in H^1_0(\Omega)^d$. This operator fulfills for each $T \in \mathcal{T}$ and $q_h \in Q_h$:

$$\int_T \operatorname{div} \Pi_F(v) q_h = -\int_T \Pi_F(v) \cdot \nabla q_h + \int_{\partial T} \Pi_F(v) \cdot \mathtt{n}_T q_h. \tag{3.18}$$

The element contribution satisfies with (3.17)

$$\int_T \Pi_F(v) \cdot \nabla q_h = \int_T (\Pi_{\mathtt{int}}(v) + \Pi_{\mathtt{io}}(v)) \cdot \nabla q_h = \int_T (v - \Pi_{\mathtt{io}}(v) + \Pi_{\mathtt{io}}(v)) \cdot \nabla q_h = \int_T v \cdot \nabla q_h.$$

The boundary contribution in (3.18) fulfills with (3.15)

$$\int_{\partial T} \Pi_F(v) \cdot \mathtt{n}_T q_h = \sum_{F \in \mathcal{F}_T} \int_F \Pi_F(v) \cdot \mathtt{n}_T q_h = \sum_{F \in \mathcal{F}_T} \int_F v \cdot \mathtt{n}_T q_h = \int_{\partial T} v \cdot \mathtt{n}_T q_h.$$

Combining the last two displays yields the result with another integration by parts:

$$\int_T \operatorname{div} \Pi_F(v) q_h = \int_T \operatorname{div} v q_h \quad \forall q_h \in Q_h \quad \text{and} \quad \forall T \in \mathcal{T}.$$

Let $T \in \mathcal{T}$ and $v \in H^1_0(\Omega)^d$. The operator $\Pi_{\mathtt{int}}$ fulfills the estimate

$$\begin{aligned} \|\Pi_{\mathtt{int}}(v)\|^2_{L^2(T)} = \int_T |\nabla q \Psi_T|^2 &\lesssim \int_T |\nabla q|^2 \Psi_T = \int_T (v - \Pi_{\mathtt{io}}(v)) \cdot \nabla q_h \\ &\lesssim \|v - \Pi_{\mathtt{io}}(v)\|_{L^2(T)} \|\Pi_{\mathtt{int}}(v)\|_{L^2(T)}. \end{aligned} \tag{3.19}$$

Here, $\Pi_{\mathtt{int}}(v) = \nabla q \Psi_T$, for $q \in Q_h$. We applied an inverse estimate, then (3.17) and finally an inverse estimate again. Continuing (3.19) with (3.16b) yields

$$\|\nabla\Pi_{\mathtt{int}}(v)\|^2_{L^2(T)} \lesssim \sum_{T' \in \mathcal{T}_T^+} \|\nabla v\|^2_{L^2(T')}. \tag{3.20}$$

Combining (3.20) and (3.16c), we derive that Π_F fulfills for all $T \in \mathcal{T}$ and $v \in H^1_0(\Omega)^d$ the following stability result:

$$\|\nabla\Pi_F(v)\|^2_{L^2(T)} \lesssim \|\nabla\Pi_{\mathtt{int}}(v)\|^2_{L^2(T)} + \|\nabla\Pi_{\mathtt{io}}(v)\|^2_{L^2(T)} \lesssim \sum_{T' \in \mathcal{T}_T^+} \|\nabla v\|^2_{L^2(T')}.$$

Again, the constants only depend on ℓ, $\sigma(\mathcal{T})$ and d. □

3.2.2 Crouzeix-Raviart element

We have already introduced the generalized CR-element in (1.39), where we used it for the discretization of the Laplace equation. However, originally, the CR_1 element was constructed to obtain an inf-sup-stable element for the discretization of the Stokes equation, see [CR73], and thus it will be one of our main examples.

Traditionally, the discrete CR-formulation is obtained by using the forms

$$a_h(v_h, w_h) = \nu \int_\Omega \nabla_h v_h : \nabla_h w_h \quad \text{and} \quad b_h(v_h, q_h) = \int_\Omega q_h \operatorname{div}_h v_h, \quad v_h, w_h \in V_h, q_h \in Q_h$$

in (3.12). Here, ∇_h and div_h are the piecewise defined versions of the operators ∇ and div. Thus, we will use these forms for the Stokes examples using Crouzeix-Raviart elements as well. Showing (3.13), which is a necessary condition for a stable discrete method, is involved.

The implicit, arbitrary order definition (1.39) is already found in [CR73], but the article only provides a proof of (3.13) in the case $\ell = 1$. Showing (3.13) for the pairs $(\mathrm{CR}_\ell^d, \hat{S}_{\ell-1}^0(\mathcal{T}))$ with $\ell > 1$ arbitrary is more challenging and was completed just recently.

To obtain a first understanding of the inf-sup stability of the Crouzeix-Raviart element, the results of Scott and Vogelius [SV85] are useful. Their results show that the pair $(\mathring{S}_\ell^1(\mathcal{T}), \hat{S}_{\ell-1}^0(\mathcal{T}))$, also called Scott-Vogelius element, is inf-sup stable for $\ell \geqslant 4$ and special meshes. Because the CR_ℓ elements contain the Scott-Vogelius element $(\mathring{S}_\ell^1(\mathcal{T}), \hat{S}_{\ell-1}^0(\mathcal{T}))$, the stability of the Scott-Vogelius element implies the stability of $(\mathrm{CR}_\ell^d, \hat{S}_{\ell-1}^0(\mathcal{T}))$. But this insight is only useful in situations in which the Scott-Vogelius element is inf-sup stable. Lifting these restrictions took some time.

First attempts concentrated on specific polynomial degrees and $d = 2$. The paper [FS83] proved that the second order generalization $(\mathrm{CR}_2^2, \hat{S}_1^0(\mathcal{T}))$ is inf-sup stable despite some difficulties finding a basis of CR_2. Hereafter, a third order generalization was proposed in [CF89] and the inf-sup stability was established for some common meshes but not in general. In [SB06, BS07], Stoyan and Baran showed the inf-sup stability of CR_ℓ for all $\ell \geqslant 4$ with ℓ even. In particular, [BS07, Theorem 3.1] shows that $(\mathrm{CR}_\ell^2, \hat{S}_{\ell-1}^0(\mathcal{T}))$ for all $\ell \geqslant 4$, ℓ even, is inf-sup stable if the macroelement conditions in [BS07, Definition 3.2] are fulfilled.

For the remaining cases $\ell \geqslant 3$, ℓ odd, the inf-sup condition (3.13) has been shown in [CS22a, CS22b] just recently. We recall [CS22b, Theorem 1.1] combined with the assumptions on the triangulation provided in [CS22b, Definition 1].

Theorem 3.15 ([CS22b, Theorem 1.1])**.** *Fix a polynomial degree $\ell \in \mathbb{N}$ and a number $M \in \mathbb{N}$. Let $\mathcal{T}$ be a regular triangulation with at least one vertex in $\mathcal{V} \cap \Omega$. Each element $T \in \mathcal{T}$ shall fulfill either*

(i) T has at least one vertex in $\mathcal{V} \cap \Omega$, or

(ii) there exist finitely many elements $T_i \in \mathcal{T}$, with $1 \leqslant i \leqslant k$ for a $k \in \mathbb{N}$ with $k \leqslant M$, such that $\partial T_i \cap \partial T_{i+1} \in \mathcal{F}$, for all $0 \leqslant i \leqslant k-1$, $T = T_0$, and T_k has a vertex in $\mathcal{V} \cap \Omega$.

Under these conditions, the pair $(\mathrm{CR}_\ell^2, \hat{S}_{\ell-1}^0(\mathcal{T}))$ satisfies the discrete inf-sup condition (3.13) with a constant α_h depending only on ℓ, $\sigma(\mathcal{T})$ and M.

Turning to higher dimensional cases, we note that the situation in two dimensions is simplified because the implicit definition (1.39) with face integral constraints allows a direct reformulation in terms of zeros of the Legendre polynomials due to the corresponding quadrature formulas. In higher dimensions, the orthogonal polynomials still induce quadrature formulas, see [Coo97], but the theory of orthogonal polynomials is much more involved, see [DX14, Mac15]. This theory is used in e.g. [vDE21] to derive quadrature formulas. It seems natural that a generalization of the CR-element is related to such quadrature formulas and indeed, the article [CDS18] constructs such a generalization.

The inf-sup stability of 3d CR-elements is only partly understood. The stability is already established for $\ell = 1$ in [CR73] and for $\ell = 2$ a proof is found in [ST23]. For higher orders, [ST23] provides only necessary conditions. Due to these issues, we restrict ourselves to the two-dimensional case if $\ell > 1$ and will comment only on the three-dimensional case if $\ell = 1$.

3.2.3 The Taylor-Hood element

Another popular stable pair is the conforming Taylor-Hood element. This finite element pair distinguishes itself from the other examples in that it has a continuous pressure. This pair uses $V_h = (\mathring{S}_\ell^1(\mathcal{T}))^d$ as the velocity space and $Q_h = \hat{S}_{\ell-1}^1(\mathcal{T})$ as the pressure space for all polynomial degrees $\ell \geqslant 2$. The inf-sup stability is proven in [BBF13, section 8.8.2].

Theorem 3.16 ([BBF13, Theorem 8.8.1]). *In the case $d = 2$: Given a polyhedral domain Ω with a mesh $\mathcal{T}$ with $\#\mathcal{T} \geqslant 3$. Then the pair $(\mathring{S}^1_\ell(\mathcal{T})^2, \hat{S}^1_{\ell-1}(\mathcal{T}))$ fulfills the discrete inf-sup condition* (3.13) *with a constant α_h depending on the mesh only via $\sigma(\mathcal{T})$.*

One problem is that we always have to choose at least a second order velocity space, even if the regularity of the analytic problem only allows for first order convergence. In such a case, the computational effort of the additional degrees of freedom is not properly utilized.

3.3 Pressure-robust methods

All methods presented so far have disadvantages, especially when we are primarily interested in a good approximation of the velocity field.

In this section, we describe the problems with references to numerical examples in the existing literature. After that we will discuss ways to overcome these problems and improve the approximation of the velocity field. One of the approaches is the foundation of the framework used in [KZ20, VZ19b, KVZ21], for which we will build an a-posteriori analysis in the next chapter.

A main goal of numerical computations is to accurately approximate the real world. To achieve this, a mathematical model of fluids should uphold the fundamental physical conservation laws. One of these conservation laws is the mass conservation, see [Gal11], [BF13]. The stationary, analytic Stokes equation (3.5) enforces the mass conservation of incompressible fluids by the constraint $\operatorname{div} u = 0$, [BF13, (I.7)], [Lio13, (1.35)].

This constraint is discretized by (3.12b), which can also be formulated as $\underline{\operatorname{div}}_h u_h = 0$. But we have already seen on page 91 that the discrete divergence is in general a weaker concept than the analytical divergence. A discrete function u_h with $\underline{\operatorname{div}}_h u_h = 0$ not necessarily satisfies $\operatorname{div} u_h = 0$, if well-defined at all. Therefore, the mass is in general not conserved in the discrete setting, see [Lin09] and [JLM$^+$17]. In the CR_ℓ and $\mathbb{P}^\ell - \mathbb{P}^{\ell-d}$ cases, the mass is conserved at least elementwise. The discrete models are, nevertheless, more unrealistic than the original analytic ones.

Another related issue with many discretizations is that they do not mirror an invariance property of the analytic Stokes equation. Recall that given $f \in H^{-1}(\Omega)^d$ the analytical velocity field is determined by (3.9). We see that the velocity field is only determined by $f_{|Z(\Omega)}$. Consequently, if we modify the right-hand side in (3.9) by adding ∇F with $F \in L^2(\Omega)$ (understood in a distributional sense), the analytic velocity field does no change, since $(\nabla F)_{|Z(\Omega)} = 0$, see also [Lin14, section 2.3]. The term ∇F only perturbs the pressure not the velocity.

However, this invariance is not mirrored by most discrete methods. The discrete velocity might change if one adds ∇F to the right-hand side of (3.12), see e.g. [JLM$^+$17, Lin14]. The reason for this is: A discretely divergence-free function is in general not exactly divergence free. Testing the discrete equation only with discretely divergence-free functions still eliminates the discrete pressure and one obtains the equation for the discrete velocity:

$$a_h(u_h, z_h) = \langle f, z_h \rangle \quad \forall z_h \in Z_h. \tag{3.21}$$

But since in general $Z_h \not\subset Z$, the equation is determined by $f_{|Z_h}$ and not $f_{|Z}$. It might happen that $(\nabla F)_{|Z_h} \neq 0$ and thus $f + \nabla F$ yields another discrete solution than f alone, see also [Lin14, section 2.4]. We therefore say that the discrete velocity is not robust regarding pressure perturbations that the method is not pressure-robust.

An extreme situation is when the right-hand side in (3.5) is completely irrotational, meaning $f = \nabla F$ for suitable F. Then the analytic velocity is zero, but the example in [GLBB97] shows that the discrete velocity is not necessarily zero because the discrete velocity field is influenced by the analytic pressure. Hence, we see that a best-approximation result of the form

$$\|u - u_h\|_{\tilde{V}} \lesssim \inf_{z_h \in Z_h} \|u - z_h\|_{\tilde{V}}, \quad u \in H^1_0(\Omega)^d \text{ the exact solution, } u_h \in V_h \text{ the discrete solution,} \tag{3.22}$$

cannot hold true in general. The method cannot be *velocity quasi-optimal*.

Therefore, since the analytical pressure influences the discrete velocity, a-priori estimates of the velocity need to involve the pressure approximability. For this reason, we usually obtain for conforming methods $V_h \subset H^1_0(\Omega)^d$ only an a-priori estimate of the following form (see [KZ20, (2.4)] and [BBF13, Proposition 8.2.1]):

$$\|\nabla_h(u - u_h)\|_{L^2(\Omega)} \lesssim \inf_{v_h \in V_h} \|\nabla_h(u - v_h)\|_{L^2(\Omega)} + \frac{1}{\nu} \inf_{q_h \in Q_h} \|p - q_h\|_{L^2(\Omega)}. \tag{3.23}$$

Here, $u \in H^1_0(\Omega)^d$, $p \in L^2_0(\Omega)$ is the velocity resp. pressure of (3.8) and $u_h \in V_h$ is the discrete solution of (3.12). In the case of nonconforming V_h, some variational crimes have to be added in (3.23). In

general, this estimate is sharp due to the examples provided in [JLM+17, Example 1.1]. We observe in (3.23) that the convergence rate of the velocity may decrease if we only change the pressure and keep the analytical velocity the same. The velocity convergence rate depends on both, the velocity approximability and the pressure approximability. The estimate (3.23) ensures that the velocity error is small only if the velocity best-error and the pressure best-error are small. Even if the space V_h has a better approximation available, the finite element method might not compute it if the pressure cannot be approximated accurately enough. Moreover, we observe by (3.23) that the pressure approximability influences the velocity error even more if ν tends to zero. This problem is also observed in numerical experiments, see e.g. [Lin14, section 5.2].

There are two main reasons why the pressure approximability is often worse than the velocity approximability, compare also the review [JLM+17]. One reason lies in the construction of the finite element pair. For example, in the case of the $\mathbb{P}^\ell - \mathbb{P}^{\ell-d}$ element, the pressure space $\hat{S}^0_{\ell-d}(\mathcal{T})$ has intrinsically worse L^2-approximation properties than the velocity space $\mathring{S}^1_\ell(\mathcal{T})^d$ has H^1_0-approximation properties. The other reason lies in the concrete problem itself. The analytical pressure might not have enough regularity. This causes bad pressure-approximation, which might in turn cause bad velocity-approximation due to (3.23), even if the space V_h could approximate the analytical velocity quite well. Thus, the approximation power of V_h is often not fully exploited.

One might argue that these problems are of purely academic nature but actually the numerical computations of [Lin09] show that they are also relevant for practice.

Methods without these problems are called *pressure-robust*, see [JLM+17]. Some possible ways to obtain such a method are pairs of finite elements whose discrete divergence coincides with the analytic divergence. Examples of those are the Scott Vogelius [SV85, GS19] element and the more recent constructions by Guzmán and Neilan in [GN14, GN18]. The problems of the Scott-Vogelius element are that it needs at least $\ell \geqslant 4$ and special requirements on the mesh. The constructions in [GN14, GN18] employ rational functions which are difficult to implement. Another approach is the so called *grad-div*-stabilization, which modifies the formulation of the equations, see [FH88, OR04] and [JLM+17, section 5.1]. More details on these topics can be found in the review paper [JLM+17].

Another way to obtain a pressure-robust method without these drawbacks was proposed in [Lin14]. The author uses a divergence-correction operator $\Pi : V_h \to H(\mathrm{div})$ that modifies the right-hand side in (3.12). The operator maps discretely divergence-free functions to exactly divergence-free functions (see also the later extensions and improvements in [BLMS15, LMNN18, LMT16, LLMS17]). Employing such a divergence-correction operator, the discrete velocity is computed through a modified equation (3.21)

$$a_h(u_h, z_h) = \langle f, \Pi z_h \rangle \quad \forall z_h \in Z_h,$$

where $\operatorname{div} \Pi z_h = 0$. Thus, the discrete velocity is computed by only using $f_{|Z}$. This ensures that only those components of f which determine the analytical velocity are used for the computation of the discrete velocity. Consequently, the analytical pressure cannot influence the discrete velocity.

In the cited papers, at first, the authors used the Raviart-Thomas projection, in the later papers they employed the BDM-projection. The reader finds the definition of both projections in [BBF13, section 2.3].

The a-priori analysis of these modified methods allows for a velocity error estimate. For example in [LMNN18, Theorem 5.1] the authors derive:

$$\|\nabla_h(u - u_h)\|_{L^2(\Omega)} \lesssim \inf_{z_h \in Z_h} \|\nabla_h(u - z_h)\|_{L^2(\Omega)} + \frac{1}{\nu} R(\mathbb{P}(f)). \tag{3.24}$$

Here, $\mathbb{P}(f)$ is the Helmholtz projection (see [LMNN18, Definition 2.2]) that maps f onto the curl-part of the Helmholtz decomposition recalled in Lemma 4.17. The function $R(\cdot)$ is similar to the classical oscillation defined in (2.17). This is an improvement in comparison to (3.23). Because the authors of [LMNN18] assume that the oscillation-like terms are of higher order (which is not necessarily true, see section 2.1.3), they call their methods quasi-optimal.

But the oscillation-like terms in the a-priori estimate (3.24) nonetheless reveal some weaknesses of these pressure-robust methods: they still do not have a quasi-optimal velocity approximation in the strict sense of [VZ18a]. The a-priori analysis either exposes classical oscillation-like terms or makes assumptions on the regularity of u beyond $H^1_0(\Omega)^d$, see [Lin14, JLM+17, LMNN18]. The problems of classical oscillation terms are pointed out in section 2.1.3.

Another drawback is that both, the Raviart-Thomas and BDM-projection, map into the space $H(\mathrm{div})$, c.f. [BBF13, section 2.3.1]. A function $H(\mathrm{div})$ has less regularity than a function in H^1. Therefore, the methods require a right-hand side f with more regularity than the natural H^{-1}-regularity for the Stokes equation (3.8).

Both problems, the non quasi-optimal a-priori estimates and additional regularity assumptions on the data, are also common in the classical analysis of nonconforming methods as discussed in section 1.2. Knowing the works of Veeser and Zanotti [VZ18a, VZ19a, VZ18b], estimates like (3.24) do not seem to be satisfactory. Can the insights of [VZ18a, VZ19a, VZ18b] help us to build pressure-robust and velocity *quasi-optimal* methods?

To answer this question, we observe that the approach chosen by A. Linke has similarities to the framework by Veeser and Zanotti, recalled in chapter 1.2. A. Linke's approach modifies the right-hand side by an operator. This is similar to the concept of the smoothing operator introduced by Veeser and Zanotti. The idea now is to enhance these smoothers such that they map discretely divergence-free functions to exactly divergence-free functions (as the Raviart-Thomas or BDM-projection do) to obtain a pressure-robust method.

Actually, Kreuzer, Verfürth and Zanotti ([KZ20, VZ19b, KVZ21]) were able to enhance the smoothing operators of [VZ19a, VZ18b] with a divergence correction component and obtained smoothers that map discretely divergence-free finite element functions into exactly divergence-free $H^1_0(\Omega)^d$-functions. Methods with these smoothers are both, velocity quasi-optimal and pressure-robust.

There are two different types of quasi-optimal and pressure-robust methods (although the distinction is not sharp): ones with a canonical, classical un-augmented bilinear form a_h and the ones with an augmented a_h.

In the category of the non-augmented methods, we have the first order CR_1 method [VZ19b], the arbitrary order, discontinuous Galerkin $(S^0_\ell(\mathcal{T})^d, \hat{S}^0_{\ell-1}(\mathcal{T}))$, $\ell \geqslant 1$ methods in [KVZ21] and the arbitrary order CR methods in [Gla22]. In the category of the augmented methods, we have methods for the $\mathbb{P}^\ell - \mathbb{P}^{\ell-d}$ element with $\ell \geqslant d$ and $d = 2, 3$, and for $\ell \geqslant 1$ a variant of the CR_1 method. Moreover, we have a variant of the Taylor-Hood element $(\mathring{S}^1_\ell(\mathcal{T})^2, \hat{S}^0_{\ell-1}(\mathcal{T}))$, $\ell \geqslant 2$, with an augmented bilinear form. The augmented methods are introduced by [KZ20].

3.3.1 The abstract framework

The papers [KZ20, KVZ21, VZ19b] use the framework presented in [VZ18a], which we recalled in section 1.1, to construct an enhanced setting for quasi-optimal, pressure-robust methods for the Stokes equation. In this section, we review this extended framework.

The new setting for the Stokes equation is based upon the one introduced in chapter 1. First, we define the spaces, scalar products and associated norms. We set $V = H^1_0(\Omega)^d$ with $a(v, w) := \nu \int_\Omega \nabla v : \nabla w$ for $v, w \in H^1_0(\Omega)^d$ and $\nu > 0$ the viscosity. In contrast to chapter 1, we also have another space $Q = L^2_0(\Omega)$. The bilinear form $b : V \times Q \to \mathbb{R}$ shall be as in (3.7). Within this setting, we assume that the analytic Stokes problem (3.8) is well-posed, see Corollary 3.9.

Having fixed the analytic setup, we discretize this problem with a finite-dimensional velocity space V_h and pressure space Q_h. The analytic bilinear forms a and b are discretized by $a_h : V_h \times V_h \to \mathbb{R}$ and $b_h : V_h \times Q_h \to \mathbb{R}$. The scalar product a shall have an extension $\tilde{a} : \tilde{V} \times \tilde{V} := (V + V_h) \times (V + V_h) \to \mathbb{R}$. The operator $\tilde{A} : \tilde{V} \to \tilde{V}'$ is defined as $\tilde{A}(\tilde{v}) = \tilde{a}(\tilde{v}, \cdot)$, $\tilde{v} \in \tilde{V}$ and we set $\|\tilde{v}\|^2_{\tilde{V}} = \|\tilde{v}\|^2 = \tilde{a}(\tilde{v}, \tilde{v})$.

After having recalled the norms and spaces, we turn to the crucial ingredient which makes the methods velocity quasi-optimal and pressure-robust. Inspired by the elliptic setting, we introduce a stable smoothing operator $\mathrm{E_M} : V_h \to H^1_0(\Omega)^d$. Accordingly, the modified discrete problem reads as

$$\begin{aligned} \text{Given } f \in H^{-1}(\Omega)^d, \text{ find } u_h \in V_h \text{ and } p_h \in Q_h \text{ such that} \\ a_h(u_h, v_h) + b_h(v_h, p_h) &= \langle f, \mathrm{E_M}\, v_h \rangle \quad \forall v_h \in V_h \\ b_h(u_h, q_h) &= 0 \quad \forall q_h \in Q_h. \end{aligned} \tag{3.25}$$

We assume that the problem is well-posed by requiring that

$$\text{Given } f \in H^{-1}(\Omega)^d, \text{ find } u_h \in Z_h \text{ such that } a_h(u_h, v_h) = \langle f, \mathrm{E_M}\, v_h \rangle \quad \forall v_h \in Z_h \tag{3.26}$$

is well-posed and the discrete inf-sup condition (3.13) is true.

We say, following [KZ20, Definition 2.5] that a method of the form (3.25) is pressure-robust and quasi-optimal if it fulfills the estimates:

$$\|u - u_h\| \leqslant C_{\mathsf{qopt}} \inf_{z_h \in Z_h} \|u - z_h\| \leqslant C_{\mathsf{qopt}} C_{b_h} \inf_{v_h \in V_h} \|u - v_h\|. \tag{3.27}$$

Here, $u \in Z(\Omega)$ is the analytical velocity of (3.9) and $u_h \in Z_h$ is the discrete velocity approximation obtained by (3.26). The last estimate follows from [BBF13, Proposition 5.1.3]. The constant C_{qopt} is independent of the viscosity ν and the right-hand side $f \in H^{-1}(\Omega)^d$ in (3.25). The constant C_{b_h} depends on the bilinear form b_h and α_h in (3.13) but also not on the viscosity ν. To obtain a velocity

quasi-optimal and pressure-robust method, we follow [KZ20, Lemmas 2.6 and 2.7] and derive necessary conditions.

First, in a pressure-robust method, pressure contributions must not affect the discrete velocity. This is ensured by requiring that (3.26) computes the discrete velocity by using only $f_{|Z}$. Therefore, we need that $\mathrm{E_M}(Z_h) \subset Z$, see also [KZ20, 2.13 a].

If we assume $\mathrm{E_M}(Z_h) \subset Z$, then the setting for the velocity approximation (3.26) almost fits into the framework of chapter 1: Considering Z as the analytic space and Z_h as the discrete space, then we can view the discrete velocity problem (3.26) as a nonconforming discretization of (3.9). With this in mind, the abstract framework of chapter 1 can guide us to a framework of pressure-robust and quasi-optimal methods.

Therefore, with the insights of the framework of chapter 1, we should be able to construct methods with velocity a-priori estimates of the desired form (3.22). It remains to reformulate the conditions for quasi-optimality of chapter 1.

The condition for algebraic consistency (1.12) reads in this setting as

$$a_h(u, v_h) = \int_\Omega \nabla u : \nabla E v_h \text{ for all } u \in Z \cap Z_h \text{ and } v_h \in Z_h, \tag{3.28}$$

see [KZ20, (2.13 b)].

We see that these smoothing operators are of a different flavour than in the first part of the thesis. Beyond mapping nonconforming functions onto conforming ones, the smoothers in this chapter also have constraints on their divergence. The condition $\mathrm{E_M}(Z_h) \subset Z$ implies that they map discretely divergence-free functions into exactly divergence-free functions.

The differences suggest adapting the notion of a quasi-optimal smoother as well. Although the smoothers are defined on V_h, we apply the elliptic framework for the spaces Z and Z_h. Therefore, the notion of a quasi-optimal smoother in this setting would mean invariance on $Z \cap Z_h$ not $V \cap V_h$. Because the smoothers are also defined on V_h, we need a new terminology to distinguish this concept from the quasi-optimal smoothers invariant on $V \cap V_h$. Thus, we introduce the term *Z-quasi-optimal*: we say a smoother $E : V_h \to V$ is Z-quasi-optimal if for all $z_h \in Z_h$

$$\|z_h - E z_h\|_{\tilde{V}} \leqslant C_{\mathsf{Z-qopt}}(E) \inf_{z \in Z} \|z - z_h\|_{\tilde{V}} \tag{3.29}$$

for some constant $C_{\mathsf{Z-qopt}}(E)$. In cases of ambiguity, we call a smoother fulfilling Definition 1.17 *V-quasi-optimal*. The theory in section 1.3 transfers to the Z-quasi-optimal smoothers $E : Z_h \to Z$ with $\tilde{Z} := Z_h + Z$. In particular, (3.29) is equivalent to E being invariant on $Z \cap Z_h$.

In the examples, we will see that a smoother can be Z-quasi-optimal but not V-quasi-optimal. But a V-quasi-optimal smoother is always Z-quasi-optimal. In the a-posteriori analysis, we will see that a Z-quasi-optimal smoother serves our needs better than a V-quasi-optimal smoother.

To finish this section, we summarize properties of the methods in the abstract framework without proof as set forth in [KZ20, (2.12),(2.13)].

Lemma 3.17. *All quasi-optimal, pressure-robust methods within the framework of this section fulfill:*

$$\int_\Omega q_h(\underline{\mathrm{div}}_h v_h - \operatorname{div} E(v_h)) = 0 \text{ for all } v_h \in V_h \text{ and } q_h \in Q_h, \tag{3.30a}$$

$$E(Z_h) \subset Z \quad \text{and} \tag{3.30b}$$

$$a_h(u, v_h) = \int_\Omega \nabla u : \nabla E v_h \text{ for all } u \in Z \cap Z_h \text{ and } v_h \in V_h. \tag{3.30c}$$

3.3.2 Examples of quasi-optimal and pressure-robust methods

Based upon this framework, the papers [KZ20, KVZ21, VZ19b] develop velocity quasi-optimal and pressure-robust methods. As they are our main examples for our a-posteriori analysis, we review these methods.

3.3.2.1 Methods using $\mathbb{P}^\ell - \mathbb{P}^{\ell-d}$ element

We start with the augmented $\mathbb{P}^\ell - \mathbb{P}^{\ell-d}$ method from [KZ20] because it has the simplest smoothing operator. This new method differs from the standard formulation for the $\mathbb{P}^\ell - \mathbb{P}^{\ell-d}$ element pair in some regards. It is necessary to augment the form a_h as specified below, see [KZ20, p. 11] and below. Although the space $V_h = \mathring{S}^1_\ell(\mathcal{T})^d \subset H^1_0(\Omega)^d$ is conforming, a smoothing operator has to

correct the divergence because in this setting a discretely divergence-free function needs not to be exactly divergence-free. To achieve the correction, the smoother applies an element-wise divergence correction operator, see [KZ20, section 3.2]. This correction operator will also be used in the other pressure-robust, quasi-optimal methods.

We recall the construction of the divergence correction operator as presented in [KZ20, section 3.2], see also [FGN20]. The divergence correction operator is defined on a specific refinement of an element $T \in \mathcal{T}$. To obtain this refinement, each vertex $\mathcal{V} \cap T$ is connected with the barycenter m_T of T. This creates $d+1$ new simplices, which form the so-called *Alfeld refinement*, [FGN20]. One example of an Alfeld refinement is shown in figure 3.1. This subtriangulation is denoted by $\mathcal{M}_T$.

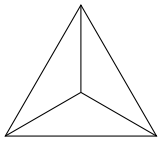


Figure 3.1: The Alfeld refinement on a sample triangle for $d = 2$, as shown in [KZ20, Figure 1].

On this Alfeld refinement, we define the divergence correction operator via a local Stokes problem. For this, we define the spaces $\mathring{S}^1_\ell(\mathcal{M}_T)$ and $\hat{S}^0_\ell(\mathcal{M}_T)$ analogously to their global counterparts in (1.26). Due to [GN18, Theorem 3.1], we know that the pairs $\left(\mathring{S}^1_\ell(\mathcal{M}_T)^d, \hat{S}^0_{\ell-1}(\mathcal{M}_T)\right)$ are inf-sup stable for all $T \in \mathcal{T}$ and $\ell \geqslant d$. Hence, the discrete Stokes problem based upon the pair of spaces $\mathring{S}^1_\ell(\mathcal{M}_T)^d$ and $\hat{S}^0_{\ell-1}(\mathcal{M}_T)$ is well-defined for each $T \in \mathcal{T}$. Moreover, the pair has a surjective discrete divergence. Thus, we can define a right inverse of the discrete divergence on each $T \in \mathcal{T}$ with the help of the following local Stokes problem as in [KZ20, section 3.2]:

$$\begin{aligned} &\text{Given } q \in L^2(\Omega) \text{ find } u_T \in \mathring{S}^1_\ell(\mathcal{M}_T)^d \text{ and } p_T \in \hat{S}^0_{\ell-1}(\mathcal{M}_T) \\ &\forall v_T \in \mathring{S}^1_\ell(\mathcal{M}_T)^d \qquad \int_T \nabla u_T : \nabla v_T - \int_T p_T \operatorname{div} v_T = 0, \\ &\forall q_T \in \hat{S}^0_{\ell-1}(\mathcal{M}_T) \qquad\qquad \int_T q_T \operatorname{div} u_T = \int_T q_T q. \end{aligned} \tag{3.31}$$

The right inverse is now defined as the linear operator $R^T_\ell : L^2(\Omega) \to H^1_0(\Omega)^d$ via

$$R^T_\ell q := u_T \text{ in } T \text{ and } R^T_\ell q := 0 \text{ in } \Omega \backslash T, \tag{3.32}$$

where $u_T \in \mathring{S}^1_\ell(\mathcal{M}_T)^d$ is the solution of (3.31) for $q \in L^2(\Omega)$. The properties of this operator are clarified by the following result (c.f. [KZ20, Proposition 3.2]).

Proposition 3.18 ([KZ20, Proposition 3.2])**.** *For any element $T \in \mathcal{T}$ and number $\ell \geqslant d$, the operator R^T_ℓ is well-defined. Furthermore, we have for each $q \in L^2(\Omega)$ the stability estimate*

$$\|\nabla R^T_\ell q\|_{L^2(T)} \lesssim \|q\|_{L^2(T)} \tag{3.33}$$

and

$$q_{|T} \in \hat{S}^0_{\ell-1}(\mathcal{M}_T) \Longrightarrow \operatorname{div} R^T_\ell q = q \text{ in } T. \tag{3.34}$$

With this new operator, we build the global divergence correction operator, (c.f. in [KZ20, section 3.3]). Although similar constructions are possible in any dimension $d > 2$, see [KZ20, p. 10], the cited paper only provides the construction for the 2-dimensional. We however also consider the 3-dimensional case for this method.

The global divergence correction operator reads as

$$R^{\mathrm{ub}}_h v_h := \sum_{T \in \mathcal{T}} R^T_\ell (\underline{\operatorname{div}}_h v_h - \operatorname{div} v_h). \tag{3.35}$$

The discrete divergence $\underline{\operatorname{div}}_h v_h$, $v_h \in \mathring{S}^1_\ell(\mathcal{T})^d$ is the L^2-projection of $\operatorname{div} v_h$ onto $\hat{S}^0_{\ell-d}(\mathcal{T})$. The authors of [KZ20] use the superscript "ub" to indicate that the method is unbalanced in the sense that the maximal velocity approximation power of $\mathring{S}^1_\ell(\mathcal{T})^d$ in the H^1-norm is strictly better than the maximal pressure approximation power of $\hat{S}^0_{\ell-d}(\mathcal{T})$ in the L^2-norm, c.f. [KZ20, Remark 3.1].

The operator R^{ub}_h allows us to define the smoothing operator $\mathcal{E}^{\mathsf{ub}} : \mathring{S}^1_\ell(\mathcal{T})^d \to H^1_0(\Omega)^d$ as in [KZ20, Proposition 3.4]

$$\mathcal{E}^{\mathsf{ub}} v_h := v_h + R^{\mathrm{ub}}_h v_h. \tag{3.36}$$

If $\operatorname{supp}(v_h) \subset \omega_z$, $v_h \in V_h$, $z \in \mathcal{V}$, then $\operatorname{supp}(\mathcal{E}^{\mathsf{ub}} v_h) \subset \omega_z$ because $\operatorname{supp}(R_h^{\mathrm{ub}}(v_h)) \subset \omega_z$. An estimate of the smoothing operator is provided in [KZ20, Proposition 3.4]. For all $v_h \in \mathring{S}^1_\ell(\mathcal{T})^d$, we have

$$\|\nabla(v_h - \mathcal{E}^{\mathsf{ub}} v_h)\|_{L^2(\Omega)} \eqsim \|\underline{\operatorname{div}}_h v_h - \operatorname{div} v_h\|_{L^2(\Omega)}. \tag{3.37}$$

This implies that $\mathcal{E}^{\mathsf{ub}}$ is invariant on $Z \cap Z_h$ and thus is a Z-quasi-optimal smoother. Moreover, from the proof of [KZ20, Proposition 3.4], we see that

$$\operatorname{div} \mathcal{E}^{\mathsf{ub}} v_h = \underline{\operatorname{div}}_h v_h \quad \forall v_h \in \mathring{S}^1_\ell(\mathcal{T})^d. \tag{3.38}$$

The next step to define a pressure-robust, quasi-optimal method is to construct a suitable bilinear form $a_h(\cdot,\cdot)$. As discussed in [KZ20, p. 2564], one cannot use the standard form. We recall their argument. With condition (3.30c) and [KZ20, (2.12b)] we see that the definition of $\mathcal{E}^{\mathsf{ub}}$ implies

$$a_h(u, w_h) = \int_\Omega \nabla u : \nabla w_h + \int_\Omega \nabla u : \nabla R_h^{\mathrm{ub}} w_h \tag{3.39}$$

for all $u \in Z \cap Z_h$ and $w_h \in \mathring{S}^1_\ell(\mathcal{T})^d$. In general, the second summand does not vanish. This can be seen by an integration by parts $\int_\Omega \nabla u : \nabla R_h^{\mathrm{ub}} w_h = -\sum_{T\in\mathcal{T}} \int_T \Delta u \cdot R_h^{\mathrm{ub}} w_h$. Thus, the form (3.39) is not symmetric and might be degenerated. In the setting of discontinuous Galerkin methods, one has similar problems, see [DPE12, section 4.2.1]. The authors of [KZ20] therefore suggest to use use a bilinear form inspired by the one used in P-SIP. The form a_h, as defined in [KZ20, (3.14) and (3.4)] reads:

$$\begin{aligned} a_h(v_h, w_h) = a_h^{\mathsf{ub}}(v_h, w_h) &:= \int_\Omega \nabla v_h : \nabla w_h + \int_\Omega \nabla v_h : \nabla R_h^{\mathrm{ub}} w_h \\ &+ \int_\Omega \nabla R_h^{\mathrm{ub}} v_h : w_h + \eta \int_\Omega \nabla R_h^{\mathrm{ub}} v_h : \nabla R_h^{\mathrm{ub}} w_h \text{ for some } \eta > 1 \quad \text{and} \quad v_h, w_h \in \mathring{S}^1_\ell(\mathcal{T})^d. \end{aligned} \tag{3.40}$$

This bilinear form is coercive on $\mathring{S}^1_\ell(\mathcal{T})^d$ as shown in [KZ20, Lemma 3.5].

The form $b_h(\cdot,\cdot)$ is defined as

$$b_h(v_h, q_h) = b_h^{\mathsf{S}}(v_h, q_h) = -\int_\Omega q_h \operatorname{div} v_h \,\mathrm{d}x \quad \text{for} \quad v_h \in \mathring{S}^1_\ell(\mathcal{T})^2, q_h \in \hat{S}^0_{\ell-d}(\mathcal{T}). \tag{3.41}$$

Obviously, in this setting $\operatorname{div} u = 0$ implies $b_h^{\mathsf{S}}(u, q_h) = 0$ for all $u \in H^1_0(\Omega)^2$ and $q_h \in \hat{S}^0_{\ell-d}(\mathcal{T})$ after canonically extending the form b_h. The common bilinear form $\tilde{a}$ is defined by $\tilde{a} = a_h$ with the canonical extension of a_h. The second form b_h is chosen as $b_h(v_h, q_h) = b_h^{\mathsf{S}}(v_h, q_h) = -\int_\Omega q_h \operatorname{div} v_h$, $v_h \in \mathring{S}^1_\ell(\mathcal{T})^d, q_h \in \hat{S}^0_{\ell-d}(\mathcal{T})$. Finally, [KZ20, Lemma 3.5, Theorem 3.12] show that problem (3.12) with the augmented form a_h, $\eta > 1$, and $\mathcal{E}^{\mathsf{ub}}$ is well-posed and the method is quasi-optimal and pressure-robust.

We summarize the method in the following display:

$$\begin{gathered} \tilde{V} = H^1_0(\Omega)^d + \mathring{S}^1_\ell(\mathcal{T})^d \subset H^1_0(\Omega)^d \quad \text{with} \quad V = H^1_0(\Omega)^d, \quad V_h = \mathring{S}^1_\ell(\mathcal{T})^d, \\ Q_h = \hat{S}^0_{\ell-d}(\mathcal{T}), \quad b_h = b_h^{\mathsf{S}} \\ \tilde{a} = a_{\mathsf{S}}, \quad \|\cdot\| = \|\nabla \cdot\|_{L^2(\Omega)}, \\ a_h = a_h^{\mathsf{ub}}; \quad \text{and} \quad E = \mathcal{E}^{\mathsf{ub}}. \\ \text{Given } f \in H^{-1}(\Omega)^d, \text{ seek } u_h \in \mathring{S}^1_\ell(\mathcal{T})^d \text{ such that} \\ a_h^{\mathsf{ub}}(u_h, v_h) = \langle f, \mathcal{E}^{\mathsf{ub}} v_h \rangle, \quad \forall v_h \in \mathring{S}^1_\ell(\mathcal{T})^d \quad \text{with} \quad \underline{\operatorname{div}}_h v_h = 0. \end{gathered} \tag{S-AP}$$

3.3.2.2 Methods using the Crouzeix-Raviart element

The classical Crouzeix-Raviart methods yield solutions which are exactly divergence-free on each element but not globally. Consequently, only a-priori estimates similar to (3.23) are valid. Thus, a quasi-optimal and pressure-robust method will be advantageous here as well.

Currently, two kinds of quasi-optimal and pressure-robust CR-methods are available, and one method which is between the two categories. The first type uses an augmented bilinear form as worked out in [KZ20, section 4.1]. The method is inspired by the augmented $\mathbb{P}^\ell - \mathbb{P}^{\ell-d}$ setting. We label this method by S-ACRL.

In the case $\ell = 1$, of S-ACRL, the augmentation is removable and we obtain the intermediate variant. As laid out in [KZ20, Remark 4.2], certain terms in the augmented bilinear form vanish if

$\ell = 1$. Without these terms, the stabilization parameter η can be set to 0. As a result, a quasi-optimal, pressure-robust CR_1 method becomes available using the standard bilinear form. This method is explored in [VZ19b] in more details.

The other category are CR_ℓ-methods ($\ell \geqslant 1$) with the standard bilinear form, but these methods have a more involved smoothing operator. These methods were introduced in [Gla22] and are based upon the discontinuous Galerkin methods introduced in [KVZ21], which we discuss in section 3.3.2.4. We comment on these Crouzeix-Raviart methods after that section.

In this section, we only recall the quasi-optimal, pressure-robust CR-methods which involve an augmented bilinear form as introduced in [KZ20]. These methods are labeled with S-ACRL. As in [KZ20], we set $V_h = (\mathrm{CR}_\ell)^2$ for $\ell \geqslant 1$ and $Q_h = \hat{S}^0_{\ell-1}(\mathcal{T})$. Further, we set

$$b_h(v_h, q_h) = b_h^{\mathsf{S}}(v_h, q_h) = -\int_\Omega q_h \operatorname{div}_h v_h, \quad v_h \in V_h \quad \text{and} \quad q_h \in Q_h, \tag{3.42}$$

where div_h is the piecewise div-operator. The scalar product

$$\tilde{a}(\tilde{v}, \tilde{w}) := \tilde{a}_{\mathsf{S}}^{\mathsf{CR}}(\tilde{v}, \tilde{w}) := \sum_{T\in\mathcal{T}} \int_T \nabla\tilde{v} : \nabla\tilde{w}, \quad \tilde{v}, \tilde{w} \in \tilde{V} = H^1_0(\Omega)^2 + (\mathrm{CR}_\ell)^2,$$

makes $\tilde{V}$ a Hilbert space.

Then, the authors of [KZ20] built the preliminary smoothing operator $M_h^{\mathsf{CR}} : (\mathrm{CR}_\ell)^2 \to \mathring{S}^1_\ell(\mathcal{T})^2$: (in the case $\ell = 1$, the operator M_h^{CR} maps into $\mathring{S}^1_2(\mathcal{T})^2$)

$$M_h^{\mathsf{CR}} v_h := \mathcal{A}_\ell v_h + \mathcal{B}_h^{\mathrm{CR}}\big(v_h - \mathcal{A}_\ell v_h\big)$$

where $\mathcal{A}_\ell$ is the vectorized averaging operator from Definition 1.34 and $\mathcal{B}_h^{\mathrm{CR}}$ is the vectorized bubble operator introduced in (1.37). According to [KZ20, Proposition 4.1], we have

$$\int_T \operatorname{div} M_h^{\mathsf{CR}} v_h = \int_T \operatorname{div} v_h \text{ for all } T \in \mathcal{T} \text{ and } v_h \in (\mathrm{CR}_\ell)^2 \quad \text{and} \tag{3.43}$$

$$\|\nabla_h(v_h - M_h^{\mathsf{CR}}(v_h))\|_{L^2(\Omega)} \lesssim \inf_{v\in H^1_0(\Omega)^2} \|\nabla_h(v - v_h)\|_{L^2(\Omega)} \leqslant \|\nabla_h v_h\|_{L^2(\Omega)} \text{ for all } v_h \in (\mathrm{CR}_\ell)^2. \tag{3.44}$$

The operator M_h^{CR} is a V-quasi-optimal smoother but does not necessarily fulfill the conditions $\mathrm{E}_{\mathrm{M}}(Z_h) \subset Z$ and (3.28) for pressure-robustness. This problem is solved by adding the divergence correction analyzed in Proposition 3.18. The final operator $\mathcal{E}^{\mathrm{CR}}_{\mathsf{aug}} : (\mathrm{CR}_\ell)^2 \to H^1_0(\Omega)^2$ reads (c.f. [KZ20, (4.4)])

$$\mathcal{E}^{\mathrm{CR}}_{\mathsf{aug}} v_h = M_h^{\mathsf{CR}} v_h + \sum_{T\in\mathcal{T}} R^T_\ell(\underline{\operatorname{div}}_h v_h - \operatorname{div} M_h^{\mathsf{CR}} v_h), \quad v_h \in (\mathrm{CR}_\ell)^2.$$

The operator R^T_ℓ is applicable because (3.43) implies

$$\int_T \underline{\operatorname{div}}_h v_h - \operatorname{div} M_h^{\mathsf{CR}} v_h = \int_T \underline{\operatorname{div}}_h v_h - \operatorname{div} v_h = 0$$

by the definition of the discrete divergence. In the case $\ell = 1$, we do not use R^T_1 but R^T_2, c.f. [KZ20, Remark 4.2] because the operator M_h^{CR} maps into $\mathring{S}^1_2(\mathcal{T})^2$ thus we need to use R^T_2 to correct the divergence. The operator $\mathcal{E}^{\mathrm{CR}}_{\mathsf{aug}}$ fulfills for all $v_h \in (\mathrm{CR}_\ell)^2$ ([KZ20, (4.5)]):

$$\|\nabla_h(v_h - \mathcal{E}^{\mathrm{CR}}_{\mathsf{aug}} v_h)\|_{L^2(\Omega)} \lesssim \inf_{v\in H^1_0(\Omega)^2} \|\nabla_h(v - v_h)\|_{L^2(\Omega)}.$$

Hence, the operator $\mathcal{E}^{\mathrm{CR}}_{\mathsf{aug}}$ is V-quasi-optimal in the sense of Definition 1.17 and also Z-quasi-optimal, c.f. (3.29). Additionally, the operator $\mathcal{E}^{\mathrm{CR}}_{\mathsf{aug}}$ fulfills (3.38), see [KZ20, (4.4)]:

$$\operatorname{div} \mathcal{E}^{\mathrm{CR}}_{\mathsf{aug}} v_h = \underline{\operatorname{div}}_h v_h \quad \forall v_h \in (\mathrm{CR}_\ell)^2. \tag{3.45}$$

The smoother is also used to define the discrete augmented bilinear form $a_h^{\mathrm{CR}}(\cdot,\cdot) : V_h \times V_h \to \mathbb{R}$ (cf. [KZ20, p. 2573]):

$$a_h^{\mathrm{CR}}(u_h, v_h) = \int_\Omega \nabla\mathcal{E}^{\mathrm{CR}}_{\mathsf{aug}} u_h : \nabla\mathcal{E}^{\mathrm{CR}}_{\mathsf{aug}} v_h + (\eta - 1)\int_\Omega \nabla_h R^{\mathrm{CR}}(u_h) : \nabla_h R^{\mathrm{CR}}(v_h) \quad u_h, v_h \in (\mathrm{CR}_\ell)^2. \tag{3.46}$$

Here, R^{CR} is defined as $R^{\mathrm{CR}} := (\mathcal{E}^{\mathrm{CR}}_{\mathsf{aug}} - \mathrm{id})$ and $\eta > 1$ is a penalty parameter. The result in [KZ20] shows that (3.25) is pressure-robust and quasi-optimal if we choose in the abstract problem (3.25) of the framework the forms $a_h := a_h^{\mathrm{CR}}$, $b_h(q_h, v_h) := \int_\Omega q_h \operatorname{div}_h v_h$ and $\mathcal{E}^{\mathrm{CR}}_{\mathsf{aug}}$ as the smoother.

Remark 3.19. In [KZ20, Remark 3.10], the relation between (3.46) and (3.40) is discussed. The definition of $R^{\mathrm{CR}} := (\mathcal{E}^{\mathrm{CR}}_{\mathsf{aug}} - \mathrm{id})$ is inspired by the fact that $R^{\mathrm{ub}}_h v_h = (\mathcal{E}^{\mathsf{ub}} - \mathrm{id})$ by (3.36). With this in mind, we can rewrite the definition (3.40) as

$$a_h(v_h, w_h) := \int_\Omega \nabla \mathcal{E}^{\mathsf{ub}}(v_h) : \nabla \mathcal{E}^{\mathsf{ub}}(w_h) + (\eta - 1) \int_\Omega \nabla R^{\mathrm{ub}}_h w_h : \nabla R^{\mathrm{ub}}_h w_h$$
$$\text{for some } \eta > 1 \quad \text{and} \quad v_h, w_h \in \mathring{S}^1_\ell(\mathcal{T})^2,$$

see [KZ20, (3.16)]. With this rewrite, the similarities between (3.40) and (3.46) become obvious.

We summarize the method for $d = 2$ in the following display:

$$\begin{gathered}
\tilde{V} = H^1_0(\Omega)^2 + \mathrm{CR}_\ell \quad \text{with} \quad V = H^1_0(\Omega)^2, \quad V_h = \mathrm{CR}^2_\ell, \\
Q_h = \hat{S}^0_{\ell-1}(\mathcal{T}), \quad b_h = b^{\mathsf{S}}_h \\
\tilde{a} = \tilde{a}^{\mathsf{CR}}_{\mathsf{S}}, \quad \|\cdot\| = \|\nabla_h \cdot\|_{L^2(\Omega)}, \\
a_h = a^{\mathrm{CR}}_h; \quad \text{and} \quad E = \mathcal{E}^{\mathrm{CR}}_{\mathsf{aug}}. \\
\text{Given } f \in H^{-1}(\Omega)^d, \text{ seek } u_h \in \mathrm{CR}_\ell \text{ such that} \\
a^{\mathrm{CR}}_h(u_h, v_h) = \langle f, \mathcal{E}^{\mathrm{CR}}_{\mathsf{aug}} v_h \rangle, \quad \forall v_h \in \mathrm{CR}_\ell \quad \text{and} \quad \underline{\mathrm{div}}_h v_h = 0.
\end{gathered} \tag{S-ACRL}$$

3.3.2.3 Methods using the Taylor-Hood element

Next, we recall pressure-robust methods for the Taylor-Hood element. The paper [LLMS17] already proposes a pressure-robust method with the Taylor-Hood element. But this method is not quasi-optimal in the sense that it does not fulfill (3.27). For this reason, Kreuzer and Zanotti adapted the ideas of [LLMS17] to design a two-dimensional, quasi-optimal and pressure-robust method using the Taylor-Hood element in [KZ20, section 4.2]. Similarly to the $\mathbb{P}^\ell - \mathbb{P}^{\ell-d}$ method, the quasi-optimal and pressure-robust Taylor-Hood method uses an augmented bilinear form. We recall this method as one of our examples.

As already discussed in section 3.2, the Taylor-Hood method in $d = 2$ uses $V_h = \mathring{S}^1_\ell(\mathcal{T})^2$ and $Q_h = \hat{S}^1_{\ell-1}(\mathcal{T})$ with $\ell \geqslant 2$. Since the velocity functions are already H^1-functions, the smoother only has to correct the divergence. But this is more difficult for the Taylor-Hood element because of the continuity of the pressure space Q_h.

The continuity of the space Q_h for the Taylor-Hood pair is the main difference to the previous methods. But at the same time it makes correcting the divergence difficult because in general the divergence cannot be corrected elementwisely, see [KZ20, p. 2574] . Therefore, [KZ20, p. 2575] builds a global operator $\mathcal{R}^{\mathsf{TH}} : \mathring{S}^1_\ell(\mathcal{T})^2 \to H^1_0(\Omega)^2$, which corrects for each patch ω_z, $z \in \mathcal{V}$, the divergence with a local operator. The local correction in each patch ω_z, $z \in \mathcal{V}$, is performed by a local operator $\mathcal{R}^z_\ell : L^2(\Omega) \to H^1_0(\Omega)^2$.

In [KZ20, pp. 2574], the operator $\mathcal{R}^z_\ell$ is defined on the patch ω_z, $z \in \mathcal{V}$, via a local Stokes problem. The system is defined on a subtriangulation of ω_z that is created by performing in each element of $\mathcal{T}_z$ an Alfeld split, see page 98. We write $\mathcal{M} = \{\mathcal{M}_T | T \in \mathcal{T}\}$ resp. $\mathcal{M}_z = \{\mathcal{M}_T | T \in \mathcal{T}_z\}$, $z \in \mathcal{V}$, for the sets that contain the new elements within Ω resp. ω_z.

These additional splittings induce new, local, finite element spaces, which we denote corresponding to their global counterparts. For each $z \in \mathcal{V}$, we write $\mathring{S}^1_\ell(\mathcal{M}_z)$ for the H^1-conforming space of piecewise, degree-ℓ polynomials with respect to the triangulation $\mathcal{M}_z$ with zero boundary values on $\partial\omega_z$. The space $\hat{S}^0_\ell(\mathcal{M}_z)$ contains the piecewise polynomials of degree ℓ with mean-value zero over ω_z.

Within the new spaces, we intend to solve local Stokes problems. In addition to the new spaces, we need two operators to define these problems. One of them is the simplified averaging operator $\mathcal{A}_\ell$, already defined in (1.34). The other one is a modified Lagrange interpolation operator ([KZ20, p. 2574]). For each $z \in \mathcal{V}$, we define the operator via:

$$\mathcal{I}^z_{\ell-1} : S^0_\ell(\mathcal{M}_z) \to S^0_{\ell-1}(\mathcal{M}_z) \qquad \mathcal{I}^z_{\ell-1} v = \sum_{T \in \mathcal{M}_z} \sum_{z' \in \mathcal{L}_{\ell-1}(T)} v_{|T}(z') \tilde{\Phi}^{z',T}_{\ell-1}, \quad v \in S^0_\ell(\mathcal{M}_z).$$

The function $\tilde{\Phi}^{z',T}_{\ell-1}$ is the Lagrange function $\Phi^{z'}_{\ell-1}$ on T and zero elsewhere.

The local Alfeld split together with the two operators $\mathcal{A}_{\ell-1}$ and $\mathcal{I}^z_{\ell-1}$ allow us to formulate the local Stokes problem which defines the local divergence correction operator R^z_ℓ. For a patch ω_z, $z \in \mathcal{V}$, the

problem reads as ([KZ20, (4.8)]): Given $q \in L^2(\Omega)$, find $u_z \in \mathring{S}^1_\ell(\mathcal{M}_z)^2$ and $p_z \in \hat{S}^0_{\ell-1}(\mathcal{M}_z)$ such that

$$\begin{aligned} \int_{\omega_z} \nabla u_z : \nabla v_z - \int_{\omega_z} p_z \operatorname{div} v_z &= 0 && \forall v_z \in \mathring{S}^1_\ell(\mathcal{M}_z)^2, \\ \int_{\omega_z} q_z \operatorname{div} u_z &= \int_{\omega_z} \big(\mathcal{A}_{\ell-1}(q_z \Phi^z_1) - \mathcal{I}^z_{\ell-1}(q_z \Phi^z_1)\big) q && \forall q_z \in \hat{S}^0_{\ell-1}(\mathcal{M}_z). \end{aligned} \tag{3.47}$$

This problem is well-posed as shown in [GN18, Corollary 6.2], see also [KZ20, p. 2575]. Its unique solution $u_z \in \mathring{S}^1_\ell(\mathcal{M}_z)^2 \subset H^1_0(\omega_z)^2$ for a given $q \in L^2(\Omega)$ defines the local divergence correction operator $\mathcal{R}^z_\ell : L^2(\Omega) \to \mathring{S}^1_\ell(\mathcal{M}_z)^2 \subset H^1_0(\Omega)^2$ as in [KZ20, p. 2575]:

$$\mathcal{R}^z_\ell q := u_z \text{ in } \omega_z, \qquad \mathcal{R}^z_\ell q := 0 \text{ elsewhere.} \tag{3.48}$$

Summing up the local operators $\mathcal{R}_\ell$ yields the global correction operator $\mathcal{R}^{\mathsf{TH}} : \mathring{S}^1_\ell(\mathcal{T})^2 \to H^1_0(\Omega)^2$, as in [KZ20, (4.9)]:

$$\mathcal{R}^{\mathsf{TH}} v_h := \sum_{z \in \mathcal{V}} \mathcal{R}^z_\ell(\operatorname{div} v_h), \qquad v_h \in \mathring{S}^1_\ell(\mathcal{T})^2. \tag{3.49}$$

This correction operator $\mathcal{R}^{\mathsf{TH}}$ allows us to define the smoothing operator $\mathcal{E}^{\mathsf{TH}} : \mathring{S}^1_\ell(\mathcal{T})^2 \to H^1_0(\Omega)^2$ as in [KZ20, Proposition 4.5]:

$$\mathcal{E}^{\mathsf{TH}}(v_h) := v_h + \mathcal{R}^{\mathsf{TH}}(v_h), \quad v_h \in \mathring{S}^1_\ell(\mathcal{T})^2.$$

The next Proposition states some properties of the operator:

Proposition 3.20 ([KZ20, Proposition 4.5])**.** *The operator $\mathcal{E}^{TH}$ fulfills* (3.30a), (3.30b) *and for all $v_h \in \mathring{S}^1_\ell(\mathcal{T})^2$*

$$\|\nabla(v_h - \mathcal{E}^{TH}(v_h))\|_{L^2(\Omega)} \lesssim \|\operatorname{div} v_h\|_{L^2(\Omega)}. \tag{3.50}$$

We directly see that $\mathcal{E}^{\mathsf{TH}}$ is a Z-quasi-optimal smoother.

The smoother $\mathcal{E}^{\mathsf{TH}}$ is the key to obtain the pressure-robust and quasi-optimal method based upon the Taylor Hood element in [KZ20, section 4.2], which we label as S-ATH. In addition to the smoother $\mathcal{E}^{\mathsf{TH}}$, the method S-ATH uses the spaces $V_h := \mathring{S}^1_\ell(\mathcal{T})^2$ and $Q_h := \hat{S}^1_{\ell-1}(\mathcal{T})$ and the augmented bilinear form as in (3.46):

$$a^{\mathsf{TH}}_h(v_h, w_h) := \int_\Omega \nabla \mathcal{E}^{\mathsf{TH}}(v_h) : \nabla \mathcal{E}^{\mathsf{TH}}(w_h) + (\eta - 1) \int_\Omega \nabla \mathcal{R}^{\mathsf{TH}}(v_h) : \nabla \mathcal{R}^{\mathsf{TH}}(w_h) \quad v_h, w_h \in \mathring{S}^1_\ell(\mathcal{T}),$$

with a penalty parameter $\eta > 1$. To discretize the divergence, we set $b_h(v_h, q_h) := \int_\Omega q_h \operatorname{div} v_h$, $q_h \in \hat{S}^1_{\ell-1}(\mathcal{T})$, $v_h \in \mathring{S}^1_\ell(\mathcal{T})$. Regarding the connection to the other augmented bilinear forms, see Remark 3.19.

Inserting the spaces and forms together with the smoother $\mathcal{E}^{\mathsf{TH}}$ into the abstract formulation (3.25) leads to a quasi-optimal, pressure-robust method using the Taylor-Hood element, c.f. [KZ20].

We summarize the method S-TH in the following display:

$$\begin{gathered} \tilde{V} = H^1_0(\Omega)^2 + \mathring{S}^1_\ell(\mathcal{T})^2 \subset H^1_0(\Omega)^2 \quad \text{with} \quad V = H^1_0(\Omega)^2, \quad V_h = \mathring{S}^1_\ell(\mathcal{T})^2, \\ Q_h = \hat{S}^1_{\ell-1}(\mathcal{T}), \quad b_h = b^{\mathsf{S}}_h \\ \tilde{a} = a_{\mathsf{S}}, \quad \|\cdot\| = \|\nabla \cdot\|_{L^2(\Omega)}, \\ a_h = a^{\mathsf{TH}}_h; \quad \text{and} \quad E = \mathcal{E}^{\mathsf{TH}}. \\ \text{Given } f \in H^{-1}(\Omega)^2, \text{ seek } u_h \in \mathring{S}^1_\ell(\mathcal{T})^2 \text{ such that} \\ a^{\mathsf{TH}}_h(u_h, v_h) = \langle f, \mathcal{E}^{\mathsf{TH}} v_h \rangle, \quad \forall v_h \in \mathring{S}^1_\ell(\mathcal{T})^2 \quad \text{and} \quad \underline{\operatorname{div}}_h v_h = 0 \end{gathered} \tag{S-TH}$$

3.3.2.4 A discontinuous Galerkin method

Up to now, we have discussed methods with an augmented bilinear form. In this section, we present a method from [KVZ21] which uses the classical non-augmented bilinear form as it was used e.g. in [HL02]. The drawback is that the smoothing operator is more complicated than in the augmented methods.

To define the method for $d = 2, 3$, we choose $V_h = S^0_\ell(\mathcal{T})^d$, $\tilde{V} = H^1_0(\Omega)^d + S^0_\ell(\mathcal{T})^d$ and $Q_h = \hat{S}^0_{\ell-1}(\mathcal{T})$, $\ell \in \mathbb{N}$, within the framework of section 3.3.1, see [KVZ21]. Furthermore, we set $\tilde{a} := \tilde{a}^{\mathsf{dG}} : \tilde{V} \times \tilde{V} \to \mathbb{R}$ defined as

$$\tilde{a}^{\mathsf{dG}}(\tilde{v}, \tilde{w}) := \sum_{T\in\mathcal{T}} \int_T \nabla\tilde{v} : \nabla\tilde{w} + \sum_{F\in\mathcal{F}} \frac{\gamma}{h_F} \int_F [\![\tilde{v}]\!] \cdot [\![\tilde{w}]\!], \quad \tilde{v}, \tilde{w} \in \tilde{V} = H^1_0(\Omega)^d + S^0_\ell(\mathcal{T})^d,$$

where $\gamma > 0$ is a suitable stabilization parameter. This is the vectorized version of the form used for the dG-method P-SIP, P-NSIP, P-RSIP and P-RNSIP. The form $\tilde{a}^{\mathsf{dG}}$ induces the norm

$$\|\cdot\|^2_{\tilde{V}} = \|\cdot\|^2 := \|\cdot\|^2_{\mathsf{dG}} := \|\nabla_h \cdot\|^2_{L^2(\Omega)} + \sum_{F\in\mathcal{F}} \int_F \frac{\gamma}{h_F} |[\![\cdot]\!]|^2 \quad \text{on } \tilde{V} = H^1_0(\Omega)^d + S^0_\ell(\mathcal{T})^d.$$

For the discrete bilinear form, we choose the vectorized version of the bilinear form $a_h^{\mathsf{dG}} : V_h \times V_h \to \mathbb{R}$, see (1.57) and [KVZ21, (9)],

$$\begin{aligned} a_h^{\mathsf{dG}}(v_h, w_h) := &\int_\Omega \nabla_h v_h : \nabla_h w_h + \sum_{F\in\mathcal{F}} \int_F \frac{\gamma}{h_F} [\![v_h]\!] \cdot [\![w_h]\!] \\ &+ \sum_{F\in\mathcal{F}} \left\{ \theta \int_F [\![v_h]\!] \cdot \{\{\nabla_h w_h \cdot \mathsf{n}_F\}\} - \int_F \{\{\nabla_h v_h \cdot \mathsf{n}_F\}\} \cdot [\![w_h]\!] \right\}, \quad v_h, w_h \in V_h = S^0_\ell(\mathcal{T})^d \end{aligned} \tag{3.51}$$

whereas we only consider the symmetric case with $\theta = -1$ in the Stokes setting. If the penalty parameter γ is chosen large enough (which we assume to be the case), the form a_h^{dG} is coercive and continuous as already mentioned on page 28, see also [KVZ21, section 2.4]. For the second bilinear form b_h, we choose as in [KVZ21, (10)]:

$$b_h : V_h \times Q_h \to \mathbb{R},$$
$$b_h(v_h, q_h) := \tilde{b}_h^{\mathsf{S}}(v_h, q_h) := -\int_\Omega \operatorname{div}_h v_h q_h + \int_\Sigma [\![v_h]\!] \cdot n \{\{q_h\}\}, \quad v_h \in V_h, q_h \in Q_h.$$

This bilinear form differs from the preceding choices of b_h. Nonetheless, if $v_h \in (\mathrm{CR}_\ell)^d$ then the form b_h reduces to the one in (3.42). Therefore, the new form b_h and the pair $(\mathring{S}^0_\ell(\mathcal{T})^d, \hat{S}^0_{\ell-1}(\mathcal{T}))$ fulfill (3.13), if the pair $(\mathrm{CR}^d_\ell, \hat{S}^0_{\ell-1}(\mathcal{T}))$ fulfills (3.13) because $(\mathrm{CR}_\ell)^d \subset S^0_\ell(\mathcal{T})^d$.

To apply the framework of section 3.3.1, a smoothing operator is required, which maps the discontinuous functions into the space $H^1_0(\Omega)^d$ and preserves the divergence in the sense of (3.30). Such a smoothing operator is built in [KVZ21] for the two-dimensional case and with comments for three dimensions.

The authors of [KVZ21] construct a suitable smoothing operator as the sum of four simpler operators. They enhance the smoothing operator $\mathcal{E}_\ell$ of Definition 1.35 with the element-wise divergence correction operator in (3.35). This has been done in such a way that the smoothing operator still satisfies the vectorized momentum conditions similar to (1.41) such that the resulting method is algebraic consistent. In fact, the divergence correction operator implicitly preserves some interior moments, the bubble operator $\mathcal{B}_\ell$ does not need to correct these.

The result [KVZ21, Lemma 13] characterizes the moments preserved by the divergence correction operator. The result shows that the polynomial space $\mathbb{P}_{\ell-2}(T)$ can be split into a direct sum. The divergence correction operator ensures that the momentum condition for the first part of the sum is fulfilled. Thus, the bubble operator only needs to ensure the momentum condition for the second part of the sum.

With this insight, the authors of [KVZ21] build the smoothing operator as the sum of four operators ([KVZ21, section 3.3]):

$$\mathcal{E}^{\mathrm{DG_{pr}}} : S^0_\ell(\mathcal{T})^d \to H^1_0(\Omega)^d, \qquad \mathcal{E}^{\mathrm{DG_{pr}}} := E_1 + E_2 + E_3 + E_4.$$

The first operator is a vectorized averaging operator $E_1 := \mathcal{A}^d_\ell$. Here $\mathcal{A}^d_\ell : S^0_\ell(\mathcal{T})^d \to \mathring{S}^1_\ell(\mathcal{T})^d$ can be defined in the same way as the scalar version in Definition 1.34. In contrast to our definition, [KVZ21] uses an average instead of choosing a fixed element. But this difference is not relevant for the final result.

The second operator is uses a vectorized bubble operator $\mathcal{B}^d_{\mathcal{F},\ell}$ similar to (1.43), see [KVZ21, (3.11, 3.12)]. This operator is already defined in (3.14). Then [KVZ21, (3.12)] defines $E_2(v_h) := \mathcal{B}^d_{\mathcal{F},\ell}(v_h - E_1(v_h))$.

The operator E_2 ensures the vectorized facial momentum conditions (similar to the scalar one in (1.41)):

$$\int_F E_2(v_h) \cdot p_F = \int_F \{\{v_h\}\} \cdot p_F \qquad \forall F \in \mathcal{F}^i, p_F \in \mathbb{P}_{\ell-1}(F)^d, v_h \in S^0_\ell(\mathcal{T})^d. \tag{3.52}$$

The third operator is the divergence correction operator based upon the Alfeld split, similar to the ones already discussed in (3.32), c.f. [KVZ21, (42),(43)].

The last operator then ensures the momentum preservation on each element for the remaining polynomials which are not preserved by the divergence correction operator.

Due to these involved issues, we do not recall the complete construction and refer the reader to the original literature [KVZ21]. We only state the stability results.

Proposition 3.21 ([KVZ21, Theorem 15, Proposition 18])**.** *The smoothing operator $\mathcal{E}^{\mathrm{DG}_{pr}}$ fulfills for all $T \in \mathcal{T}$ the estimate*

$$h_T \|\nabla(v_h - \mathcal{E}^{\mathrm{DG}_{pr}} v_h)\|_{L^2(T)} \lesssim \|v_h - \mathcal{E}^{\mathrm{DG}_{pr}} v_h\|_{L^2(T)} \lesssim \sum_{F \in \mathcal{F}_T^{i,+}} h_T^{\frac{1}{2}} \|[\![v_h]\!]\|_{L^2(F)}, \quad \forall v_h \in V_h.$$

As a consequence, the smoother $\mathcal{E}^{\mathrm{DG}_{pr}}$ is V-quasi-optimal and Z-quasi-optimal. Moreover, $\mathcal{E}^{\mathrm{DG}_{pr}}$ fulfills for all $v_h \in V_h$ the vectorized momentum conditions (see [KVZ21, Lemma 9])

$$\begin{aligned} \forall F \in \mathcal{F}^i, p_F \in \mathbb{P}_{\ell-1}(F)^d \quad & \int_F \mathcal{E}^{\mathrm{DG}_{pr}}(v_h) \cdot p_F = \int_F \{\{v_h\}\} \cdot p_F \quad \text{and} \\ \forall T \in \mathcal{T}, p_T \in \mathbb{P}_{\ell-2}(T)^d \quad & \int_T \mathcal{E}^{\mathrm{DG}_{pr}}(v_h) \cdot p_T = \int_T v_h \cdot p_T \end{aligned} \tag{3.53}$$

and preserves the divergence

$$\operatorname{div} \mathcal{E}^{\mathrm{DG}_{pr}}(v_h) = \underline{\operatorname{div}}_h v_h. \tag{3.54}$$

Using this smoother, [KVZ21, Theorem 20] shows that this setup builds a quasi-optimal and pressure-robust method. We label this method with S-SIP.

We summarize the method S-SIP for $d = 2$ and $\ell \geqslant 1$ arbitrary or $d = 3$ and $\ell = 1$:

$$\begin{gathered} \tilde{V} = H^1_0(\Omega)^d + S^0_\ell(\mathcal{T})^d \quad \text{with} \quad V = H^1_0(\Omega)^d, \quad V_h = S^0_\ell(\mathcal{T})^d, \\ Q_h = \hat{S}^0_{\ell-1}(\mathcal{T}), \quad b_h = \tilde{b}_h^{\mathsf{S}} \\ \tilde{a} = \tilde{a}^{\mathsf{dG}}, \quad \|\cdot\| = \|\cdot\|_{\mathsf{dG}}, \\ a_h = a_h^{\mathsf{dG}}; \quad \text{and} \quad E = \mathcal{E}^{\mathrm{DG}_{\mathsf{pr}}}. \\ \text{Given } f \in H^{-1}(\Omega)^d, \text{ seek } u_h \in S^0_\ell(\mathcal{T})^d \text{ such that} \\ a_h^{\mathsf{dG}}(u_h, v_h) = \langle f, \mathcal{E}^{\mathrm{DG}_{\mathsf{pr}}} v_h \rangle, \quad \forall v_h \in S^0_\ell(\mathcal{T})^d \quad \text{and} \quad \underline{\operatorname{div}}_h v_h = 0 \end{gathered} \tag{S-SIP}$$

Remark 3.22. In [Gla22], the S-SIP method serves as the basis to construct an arbitrary order, quasi-optimal and pressure-robust Crouzeix-Raviart method with the classical, non-augmented bilinear form of [CR73, SB06, BS07]. More precisely, the author uses the spaces $V_h = \mathrm{CR}^d_\ell$, $Q_h = \hat{S}^0_{\ell-1}(\mathcal{T})$, $\ell \in \mathbb{N}, \ell \geqslant 1$ and the bilinear forms

$$b_h(v_h, q_h) = -\int_\Omega q_h \operatorname{div}_h v_h, \quad v_h \in V_h \quad \text{and} \quad q_h \in Q_h,$$

to define the discrete divergence and

$$\tilde{a}(\tilde{v}, \tilde{w}) := \sum_{T \in \mathcal{T}} \int_T \nabla \tilde{v} : \nabla \tilde{w}, \quad \tilde{v}, \tilde{w} \in \tilde{V} = H^1_0(\Omega)^d + \mathrm{CR}^d_\ell,$$

together with $a_h(v_h, w_h) := \tilde{a}(v_h, w_h)$, $v_h, w_h \in \mathrm{CR}^d_\ell$. With the smoother $\mathcal{E}^{\mathrm{DG}_{\mathsf{pr}}}$, one gets a quasi-optimal, pressure-robust method using the CR^d_ℓ-element. We will refer to this method as S-CRL. We can summarize this method for $d = 2$ and $\ell \geqslant 1$ arbitrary or $d = 3$ and $\ell = 1$ as:

$$\begin{gathered} \tilde{V} = H^1_0(\Omega)^d + \mathrm{CR}^d_\ell \quad \text{with} \quad V = H^1_0(\Omega)^d, \quad V_h = \mathrm{CR}^d_\ell, \\ Q_h = \hat{S}^0_{\ell-1}(\mathcal{T}), \quad b_h = b_h^{\mathsf{S}} \\ \tilde{a} = \tilde{a}^{\mathsf{CR}}_{\mathsf{S}}, \quad \|\cdot\| = \|\nabla_h \cdot\|_{L^2(\Omega)}, \\ a_h = \tilde{a}^{\mathsf{CR}}_{\mathsf{S}}; \quad \text{and} \quad E = \mathcal{E}^{\mathrm{DG}_{\mathsf{pr}}}. \\ \text{Given } f \in H^{-1}(\Omega)^d, \text{ seek } u_h \in \mathrm{CR}^d_\ell \text{ such that} \\ \tilde{a}^{\mathsf{CR}}_{\mathsf{S}}(u_h, v_h) = \langle f, \mathcal{E}^{\mathrm{DG}_{\mathsf{pr}}} v_h \rangle, \quad \forall v_h \in \mathrm{CR}^d_\ell \quad \text{and} \quad \underline{\operatorname{div}}_h v_h = 0. \end{gathered} \tag{S-CRL}$$

4 Pressure-robust, a-posteriori velocity estimators for the Stokes equation

After we have recalled the a-priori analysis for pressure-robust, quasi-optimal methods, we turn to the a-posteriori analysis of the velocity field for these methods. In view of the benefits of pressure-robust methods, especially the pressure independence of the velocity, as laid out in section 3.3, it seems reasonable to demand that an error estimator for the velocity fields of these methods is also unaffected by pressure perturbations. If this property is satisfied by an error estimator for the velocity field of the Stokes equation, we call this estimator *pressure-robust*, which is in line with the a-priori terminology.

Nevertheless, many classical error estimators for the Stokes equation, such as the estimators in [Ver89, Ver91], estimate the velocity and pressure error at the same time. If such an estimator indicates a large error, we cannot distinguish whether the velocity, the pressure or both errors are large, therefore these estimators are not pressure-robust.

In contrast, the article [LMS19] presents a pressure-robust a-posteriori analysis in two dimensions. They achieve pressure-robust bounds by testing the residual with exactly divergence-free functions [LMS19, Theorem 5]. Similar approaches are presented in [HM25, LM22]. But the estimators pose additional regularity assumptions on the right-hand side f in (3.8) beyond $f \in H^{-1}(\Omega)^d$. In particular, they require that $f \in L^2(\Omega)^2$ and $f_{|T} \in \underline{H}(\mathrm{curl}, T)$ for each $T \in \mathcal{T}$ in (3.8). Besides, the estimators involve several pressure-robust but still classical oscillation terms as defined in (2.17). In [LMS19, Theorem 6], the authors derive the efficiency estimates and obtain several oscillation terms, such as $\mathrm{osc}_\ell^{L^2}(h\,\mathrm{curl}_h(f+\nu\Delta_h u_h))$, $\mathrm{osc}_\ell^{L^2}(f-\nabla p)$. The problems of these terms are laid out in [KV21, CDN12] and are recalled in Lemmas 2.17 and 2.18.

In particular, we use the ideas of the elliptic estimators, discussed in chapter 2, to build an error estimator with an error-dominated oscillation for the velocity error. As the velocity discretization of a pressure-robust method is independent of the pressure, our error-estimator shall also be independent of the pressure, it shall be pressure-robust.

For these reasons, we develop new pressure-robust a-posteriori estimators for the velocity field of the Stokes equations for the natural data regularity. The estimators work for the natural data regularity $H^{-1}(\Omega)^d$ and have an error-dominated oscillation term, which does not have the issues of the classical oscillations discussed in Lemmas 2.17 and 2.18. In particular, we use the ideas of the elliptic estimators, discussed in chapter 2 and combine it with the ideas of [LMS19], to build an error estimator with an error-dominated oscillation for the velocity error.

At first, we present an abstract framework for pressure-robust velocity estimators for the Stokes equation. The framework is similar to the elliptic one in section 2.1. The main difference is that the pressure-robust estimator only uses exactly divergence-free test functions to detect only the parts of the right-hand side $f \in H^{-1}(\Omega)^d$ in (3.8) which determine the velocity and not the parts determining the pressure. It cannot detect pressure components and thus it is pressure-robust.

The construction of the estimator contains again three steps, similar to the ones outlined in section 2.1.4. First, we split the error into the smoothed error and the conformity error. Then we localize the residual from the global space $Z(\Omega)'$ to the local spaces $Z(\omega_z)'$, $z \in \mathcal{V}$. Finally, we construct several projection operators, which only use exactly divergence-free test functions, to split off the oscillation term and make the estimator computable. Throughout this section, we assume the abstract setting of section 3.3.1.

We exemplify the abstract localization and the application of the projection operators for the quasi-optimal, pressure-robust methods in [VZ19b, KZ20, KVZ21] and recalled in section 3.3.2. If new quasi-optimal, pressure-robust methods within fitting into the abstract framework should appear, it should not be too complicated to adapt the results below to the new method, c.f. Remark

4.1 Pressure-robust localization for the Stokes equation

In this section, we construct an abstract, pressure-robust localization of the residual, based upon one of the localizations presented in chapter 2. In contrary to those localizations, the one in this chapter is explicitly tailored for the Stokes setting and its discretizations. Thus, even the abstract setting

works with concrete Sobolev spaces and a mesh. We start with the splitting of the error similar to the discussion in section 2.1.1.

4.1.1 Splitting of the error

We refine the concept of splitting the error into the conformity error and the conforming residual for the Stokes setting. Given the discrete solution $u_h \in Z_h \subset V_h$ of (3.25) and the analytic solution $u \in Z(\Omega) \subset H^1_0(\Omega)^d$ of (3.8), we split the error similar to (2.5) into

$$\|u - u_h\|^2_{\tilde{V}} = \|u - \Pi_Z u_h\|^2_{\tilde{V}} + \|\Pi_Z u_h - u_h\|^2_{\tilde{V}}, \tag{4.1}$$

where $\Pi_Z : Z_h \to Z$ is the $\tilde{a}$-orthogonal projection onto Z. Using this projection, the smoothed error can be quantified by testing f and u_h only with exactly divergence-free functions

$$\|u - \Pi_Z u_h\|_{\tilde{V}} = \sup_{z \in Z(\Omega)} \frac{\tilde{a}(\Pi_Z u_h, z) - \langle f, z\rangle}{\|z\|_V}. \tag{4.2}$$

Before further discussing the smoothed error and the resulting residual, we have a close look at the evaluation of the conformity error. With the help of a Z-quasi-optimal smoother $E : Z_h \to Z$, we have similar to (2.6):

$$\|\Pi_Z u_h - u_h\|_{\tilde{V}} = \inf_{z \in Z(\Omega)} \|z - u_h\|_{\tilde{V}} \leqslant \|Eu_h - u_h\|_{\tilde{V}} \leqslant C_{\mathsf{Z-qopt}}(E) \inf_{z \in Z(\Omega)} \|z - u_h\|_{\tilde{V}}.$$

Because all discussed quasi-optimal and pressure-robust examples expose a Z-quasi-optimal smoother and the method's smoother has to be implemented anyway, it is natural to use the method's smoother to assess the conformity error and the evaluation of $\|Eu_h - u_h\|_{\tilde{V}}$ is no obstacle. The applicant might apply one of the equivalences to jump terms or divergence expressions, as provided in (3.37) for S-AP, in (3.50) for S-TH, and in Proposition 3.21 for S-SIP and S-CRL.

With the quantification of the conformity error settled, we subsequently turn to the estimation of the smoothed error. The smoothed error in the form (4.2) defines the conforming residual for the Stokes equation, see also section 2.1:

$$\mathrm{res}^{\mathsf{c}}_{\mathrm{h}}(f, u_h) \in Z(\Omega)' \qquad \langle \mathrm{res}^{\mathsf{c}}_{\mathrm{h}}(f, u_h), v\rangle = \tilde{a}(u_h, v) - \langle f, v\rangle = \tilde{a}(\Pi_Z u_h, v) - \langle f, v\rangle, \quad v \in Z.$$

Here $u_h \in V_h$ is the discrete velocity solution of (3.25) with right-hand side $f \in H^{-1}(\Omega)^d$. This global residual is measured in the $\|\cdot\|_{Z(\Omega)'} = \sup_{v \in Z} \frac{\langle \cdot, v\rangle}{\|\nabla v\|_{L^2(\Omega)}}$ norm.

4.1.2 Local Fortin operators and other tools

Before turning to the localization, we introduce the notion of local, divergence-free spaces. Moreover, we present local Fortin operators and discuss their existence for our examples.

First, we introduce the local spaces and prove the globalization condition (2.7) as required by Assumption 2.2. Since our Stokes discretizations are based upon a mesh, we set $\mathcal{I} := \mathcal{V}$. For any non-empty, measurable subset $\omega \subset \Omega$ we write

$$Z(\omega) := \{z \in H^1_0(\omega)^d \mid \operatorname{div} z = 0\},$$

and $Z(\omega)'$ for the corresponding dual space. In particular, for all $z \in \mathcal{V}$, we define the local, divergence-free spaces

$$Z_z := Z(\omega_z) := \{z \in H^1_0(\omega_z)^d \mid \operatorname{div} z = 0\}; \quad Z_z^+ := Z(\omega_z^+) := \{z \in H^1_0(\omega_z^+)^d \mid \operatorname{div} z = 0\}.$$

We also define the corresponding dual spaces $Z_z' = Z(\omega_z)'$ and $Z_z^{+\prime} = Z(\omega_z^+)'$.

The norm $\|\cdot\|_V = \|\nabla \cdot\|_{L^2(\Omega)}$ is the same as in the Poisson case, hence condition (2.7) and in particular Assumption 2.2 is fulfilled with $V = Z(\Omega)$ and $V_z = Z(\omega_z)$, $z \in \mathcal{V}$ because of Proposition 2.30. The argument of Theorem 2.4 provides for all $R \in Z(\Omega)'$ the globalization result

$$\sum_{z \in \mathcal{V}} \|R\|^2_{Z(\omega_z)'} \lesssim \|R\|^2_{Z(\Omega)'}, \quad R \in Z(\Omega)'. \tag{4.3}$$

The counterpart of the globalization is the localization, for which we need the quasi-Galerkin orthogonality as in the elliptic case. For the localization, we need three ingredients. At first, we need a localization operator as on page 44. The localization operator $\Theta_z : H^1_0(\Omega) \to H^1_0(\omega_z)$ shall fulfill

$$\sum_{z \in \mathcal{V}} v\Theta_z = v, \qquad \operatorname{supp}(v\Theta_z) \subset \omega_z, \quad \forall z \in \mathcal{V}; v \in H^1_0(\Omega) \tag{4.4}$$

The second ingredient is the quasi-Galerkin orthogonality, which is similar to the one in Theorem 2.5, in the specialized setting for the Stokes equations. To derive such a result, we benefit from the quasi-optimality of the methods. The equation (3.26) for the velocity field in Z_h can be seen as a quasi-optimal discretization of the analytic velocity equation (3.9) in Z. Thus, the analysis of [VZ18a], which we summarized in section 1.2, is applicable. Accordingly, a quasi-Galerkin orthogonality, similar to the one in Theorem 2.5 is available. We abbreviate $\|w_h\|_{Z_h} := \|a_h(\cdot, w_h)\|_{Z_h'}$ and get the following Corollary.

Corollary 4.1. *Let $u \in Z$ be the analytic solution of (3.8) and $u_h \in Z_h$ the discrete solution of (3.25) for a quasi-optimal, pressure-robust method with smoother $\mathrm{E_M}$. Then for all $w_h \in Z_h$ we have*

$$\langle \mathrm{res}_{\mathrm{h}}^{\mathsf{c}}(f, u_h), \mathrm{E_M}\, w_h \rangle \leqslant \delta_{Z_h} \|u_h - \Pi_Z u_h\|_{\tilde{V}} \|w_h\|_{Z_h},$$

where δ_{Z_h} is the corresponding constant from (1.13), that is the smallest constant such that

$$\forall v_h, w_h \in Z_h \qquad a(\Pi_Z v_h, \mathrm{E_M}\, w_h) - a_h(v_h, w_h) \leqslant \delta_{Z_h} \|v_h - \Pi_Z v_h\|_{\tilde{V}} \|w_h\|_{Z_h}.$$

The last tool for the localization are local Fortin operators to correct the divergence. This is the main new tool in contrast to the localizations in section 2.1.2. In general, the application of the partition of unity Θ_z to a divergence-free function yields a non divergence-free function. Therefore, a locally operating Fortin operator has to correct the divergence in such a way that we test the residual only with exactly divergence-free functions.

The existence of a Fortin operator is ensured by the discrete inf-sup condition, see Proposition 3.13. But we need a Fortin operator which operates locally such that the localization procedure works. To construct such an operator, we use that the discrete inf-sup condition (3.13) is also valid on subsets of the mesh $\mathcal{T}$.

Ideally, we would like to have a Fortin operator which works only on the stars ω_z, $z \in \mathcal{V}$. To define such an operator, we restrict ourselves to ω_z and define the local discrete spaces $V_h(\omega_z)$ and $Q_h(\omega_z)$, which are defined in the same way as V_h and Q_h but use the triangulation $\mathcal{T}_z$. If the discrete inf-sup condition (3.13) is valid for $(V_h(\omega_z), Q_h(\omega_z))$, a suitable local Fortin operator can be defined.

Yet, in the case of the CR-element, the discrete inf-sup condition might not be valid on a patch ω_z with $z \in \mathcal{V} \cap \partial\Omega$. In such a case, the local Fortin operator has to be defined on a slightly larger patch but only for those $z \in \mathcal{V}$ which are on the boundary of Ω.

To hide these problems, we introduce the notation $\tilde{\omega}_z \subset \Omega$ to denote the patch of elements around z, on which the discrete inf-sup condition (3.13) is valid. The concrete definition of $\tilde{\omega}_z$ is then provided for each element separately. The triangulation associated with the patch $\tilde{\omega}_z$ is denoted by $\tilde{\mathcal{T}}_z$. We also write $V_h(\tilde{\omega}_z)$ and $Q_h(\tilde{\omega}_z)$ for the local spaces corresponding to V_h and Q_h. This new notation allows us to define the abstract local Fortin operator.

Definition 4.2 (Abstract local Fortin operator)**.** *A linear operator $\Pi_{Fl,z} : H_0^1(\tilde{\omega}_z)^d \to V_h$, $z \in \mathcal{V}$, is a* local Fortin operator *if it fulfills*

(i) $\mathrm{supp}(\Pi_{Fl,z}(w)) \subset \tilde{\omega}_z \quad \forall w \in H_0^1(\tilde{\omega}_z)^d$,

(ii) $b(w, q) = b_h(\Pi_{Fl,z} w, q) \quad \forall q \in Q_h$ *and* $\forall w \in H_0^1(\tilde{\omega}_z)^d$,

(iii) $\|\Pi_{Fl,z} w\|_{\tilde{V}} \lesssim \|w\|_{\tilde{V}} \quad \forall w \in H_0^1(\tilde{\omega}_z)^d$.

We now show that a local Fortin operator exists for all our examples and start with the S-CRL and S-SIP methods for $\ell = 1$. For these methods, the Fortin operator is the canonical CR-projection (see [CR73]) defined as

$$\Pi_{\mathsf{CR}}^z : H_0^1(\omega_z)^d \to \mathrm{CR}_1^d \qquad \Pi_{\mathsf{CR}}^z(v) = \sum_{F \in \mathcal{F}_z^i} \int_F v \cdot \mathsf{n}_F \;\; \Psi_F^{\mathsf{cr}} \mathsf{n}_F.$$

For this projection, the paper [AR08] provides the following stability Lemma, which we enhance a little.

Lemma 4.3 ([AR08, Lemma 2.4])**.** *The projection Π_{CR} fulfills for all $v \in H_0^1(\omega_z)^d$ and $T \in \mathcal{T}_z$:*

$$\|\Pi_{\mathsf{CR}} v\|_{L^2(T)} \lesssim \|v\|_{L^2(T)} + \sum_{F \in \mathcal{F}_T} h_T^{\frac{1}{2}} \|v\|_{L^2(F)}. \tag{4.5}$$

Furthermore, for each $v \in H_0^1(\omega_z)^d$

$$\|\nabla_h \Pi_{\mathsf{CR}} v\|_{L^2(\omega_z)} \lesssim \|\nabla v\|_{L^2(\omega_z)}.$$

Proof. The first estimate is shown in [AR08, Lemma 2.4]. The second one follows by an inverse estimate and (4.5):

$$h_z^2\|\nabla_h \Pi_{\mathsf{CR}} v\|^2_{L^2(\omega_z)} \lesssim \|\Pi_{\mathsf{CR}} v\|^2_{L^2(\omega_z)} \lesssim \sum_{T\in\mathcal{T}_z} \|\Pi_{\mathsf{CR}} v\|^2_{L^2(T)} \lesssim \sum_{T\in\mathcal{T}_z} \|v\|^2_{L^2(T)} + h_z \sum_{F\in\mathcal{F}_z^i} \|v\|^2_{L^2(F)}.$$

We apply the trace theorem to the edge contributions and get

$$h_z^2\|\nabla_h \Pi_{\mathsf{CR}} v\|^2_{L^2(\omega_z)} \lesssim \sum_{T\in\mathcal{T}_z} \|v\|^2_{L^2(T)} + h_z^2\|\nabla v\|^2_{L^2(T)} = \|v\|^2_{L^2(\omega_z)} + h_z^2\|\nabla v\|^2_{L^2(\omega_z)} \lesssim h_z^2\|\nabla v\|^2_{L^2(\omega_z)}.$$

In the last inequality, we applied Poincaré's inequality. □

This stability estimate helps us to show that the CR_1^d-projection is a local Fortin operator.

Lemma 4.4. *Assume the setting of S-CRL, S-ACRL and S-SIP $\ell = 1$. Then, the CR_1^d-projection Π_{CR} is a local Fortin operator $\Pi_{Fl,z} := \Pi_{\mathsf{CR}|H_0^1(\tilde{\omega}_z)^d} : H_0^1(\tilde{\omega}_z)^d \to \mathrm{CR}_1^d$ with $\tilde{\omega}_z = \omega_z$ for any mesh $\mathcal{T}$.*

Proof. Let $v \in H_0^1(\omega_z)^d$, $z \in \mathcal{V}$, be arbitrary. We observe that $\mathrm{supp}(\Pi_{\mathsf{CR}}(v)) \subset \omega_z$. Furthermore, we have for any $q_h \in Q_h = \hat{S}_0^0(\mathcal{T})$ and $T \in \mathcal{T}_z$ that

$$\int_T \mathrm{div}_h\, \Pi_{\mathsf{CR}}(v) q_h = \sum_{F\in\mathcal{F}_T} \int_F \Pi_{\mathsf{CR}}(v)\cdot \mathsf{n}_T q_{h|T} = \sum_{F\in\mathcal{F}_T} \int_F v\cdot \mathsf{n}_T q_{h|T} = \int_T \mathrm{div}\, v q_h.$$

The stability estimate follows by Lemma 4.3. □

In the case of S-CRL, S-ACRL or S-SIP and $\ell > 1$ the situation is more involved. Because the inf-sup condition is not completely settled in the case $d = 3$ (see [ST23]), we only consider $d = 2$ and for $d = 3$ only $\ell = 1$. With the results [CS22a, CS22b], recalled in Theorem 3.15, the local Fortin operator becomes available for $d = 2$.

Lemma 4.5. *Assume the setting of the methods S-CRL, S-ACRL or S-SIP for $\ell > 1$ and $d = 2$. For each $z \in \mathcal{V} \cap \Omega$, there exists a local Fortin operator $\Pi_{Fl,z} : H_0^1(\tilde{\omega}_z)^2 \to \mathrm{CR}_\ell^2$ with $\tilde{\omega}_z = \omega_z$ which fulfills the properties in Definition 4.2. For each $z \in \mathcal{V} \cap \partial\Omega$ with $\mathring{\omega}_z^+ \cap \mathcal{V} \neq \varnothing$, there exists a local Fortin $\Pi_{Fl,z} : H_0^1(\tilde{\omega}_z)^d \to \mathrm{CR}_\ell$ with $\tilde{\omega}_z = \omega_z^+$. The stability constant of $\Pi_{Fl,z}$ depends only on ℓ and $\sigma(\mathcal{T})$ and is uniformly bounded in $z \in \mathcal{V}$.*

Proof. Fix $z \in \mathcal{V}$ and assume the setting S-CRL or S-ACRL.

If $z \in \Omega$, set $\tilde{\omega}_z = \omega_z$ and let $v \in H_0^1(\omega_z)^2$. The pair $(V_h(\omega_z), Q_h(\omega_z))$ fulfills (3.13) due to Theorem 3.15. Thus, there exists a function $v_h \in V_h(\omega_z)$ with $b(v,q) = b_h(v_h,q) \quad \forall q \in Q_h(\omega_z)$ because $\int_{\omega_z} \mathrm{div}\, v = 0$. If $v \in V_h(\omega_z)$ then we choose $v_h = v$ with $\|\nabla_h v_h\|_{L^2(\omega_z)} = \|\nabla v\|_{L^2(\omega_z)}$. In the other case, we have the bound $\|\nabla_h v_h\|_{L^2(\omega_z)} \lesssim \|\,\mathrm{div}\, v\|_{L^2(\omega_z)} \lesssim \|\nabla v\|_{L^2(\omega_z)}$. The constant depends only on the value of α_h in the inf-sup condition (3.13) as shown in [BBF13, Remark 5.1.10]. Theorem 3.15 ensures that α_h only depends on ℓ and $\sigma(\mathcal{T})$ and not on $z \in \Omega \cap \mathcal{V}$.

The function $v_h \in V_h(\omega_z)$ can be extended by zero to the whole domain Ω. With a little abuse of notation, we name the extension $v_h \in V_h$ as well. The extension fulfills all properties required in Definition 4.2 with $\tilde{\omega}_z = \omega_z$. Therefore, we set $\Pi_z(v) = v_h$

If $z \in \partial\Omega$ with $\mathring{\omega}_z^+ \cap \mathcal{V} \neq \varnothing$, set $\tilde{\omega}_z = \omega_z^+$ and let $v \in H_0^1(\omega_z^+)^d$. The pair $(V_h(\omega_z^+), Q_h(\omega_z^+))$ fulfills the discrete inf-sup condition (3.13) due to Theorem 3.15. Property (1.22) ensures that $\mathcal{T}_z^+$ only contains finitely many triangles, and thus we can choose M in Theorem 3.15 in terms of $\sigma(\mathcal{T})$.

Arguing as before, we obtain a function $v_h \in V_h$ which satisfies all properties of Definition 4.2 with $\tilde{\omega}_z = \omega_z^+$. In this case, we set $\Pi_z(v) = v_h$. The operator again has a stability constant depending only on ℓ and $\sigma(\mathcal{T})$.

In the case of S-SIP, the same result is valid because $\mathrm{CR}_\ell \subset S_\ell^0(\mathcal{T})$ and the $\tilde{V}$-norms in the CR-setting and S-SIP setting are equivalent on the space CR_ℓ^2. □

Next, we deal with the local Fortin operator for the $\mathbb{P}_\ell - \mathbb{P}_{\ell-d}$ methods.

Lemma 4.6. *In the setting of S-AP, there exists a local Fortin operator $\Pi_{Fl,z} : H_0^1(\omega_z)^d \to \mathring{S}_\ell^1(\mathcal{T})^d$ for each $z \in \mathcal{V}$. The stability constants of the Fortin operators are uniformly bounded in $z \in \mathcal{V}$ by a constant that only depends on ℓ, d and $\sigma(\mathcal{T})$.*

Proof. Proposition 3.14 shows the existence of a Fortin operator without any restrictions on the mesh. Thus, it also provides a Fortin operator for the subdomain ω_z with the mesh $\mathcal{T}_z$ for any $z \in \mathcal{V}$. If we extend the resulting Fortin operator by zero to the whole domain Ω, the resulting operator satisfies the conditions in Definition 4.2 with $\tilde{\omega}_z = \omega_z$. The uniform boundedness of the Fortin operators for different patches follows from the fact that the stability constant of the Fortin operators in Proposition 3.14 only depends on ℓ, d and $\sigma(\mathcal{T})$. □

Finally, we discuss a local Fortin operator for the Taylor-Hood element.

Lemma 4.7. *Assume the setting of S-ATH, this means $d = 2$. For each $z \in \mathcal{V}$, there exists a local Fortin operator $\Pi_{Fl,z} : H^1_0(\tilde{\omega}_z)^2 \to \mathring{S}^1_\ell(\tilde{\mathcal{T}}_z)^2$, where $\tilde{\omega}_z$ is chosen in such a way that $\#\tilde{\mathcal{T}}_z \geqslant 3$. The stability constant of $\Pi_{Fl,z}$ is uniformly bounded in $z \in \mathcal{V}$ as it depends only on $\sigma(\mathcal{T})$.*

Proof. Fix $z \in \mathcal{V}$ and choose $\tilde{\mathcal{T}}_z$ such that $\#\tilde{\mathcal{T}}_z \geqslant 3$. Select $v \in H^1_0(\tilde{\omega}_z)^2$. Then Theorem 3.16 ensures the existence of a $v_h \in \mathring{S}^1_\ell(\tilde{\mathcal{T}}_z)^2$ with $\|\nabla v_h\|_{L^2(\tilde{\omega}_z)} \lesssim \|\nabla v\|_{L^2(\tilde{\omega}_z)}$, where the constant depends only on $\sigma(\mathcal{T})$, and

$$\int_{\tilde{\omega}_z} \operatorname{div} v_h \; q_h = \int_{\tilde{\omega}_z} \operatorname{div} v \; q_h, \quad \forall q_h \in \hat{S}^1_{\ell-1}(\tilde{\mathcal{T}}_z). \tag{4.6}$$

We observe that the identity is also valid for all $q_h \in S^1_{\ell-1}(\tilde{\mathcal{T}}_z)$.

The function v_h can be extended by zero to the whole domain Ω. The extension fulfills $\operatorname{div} v_h = 0$ on $\Omega \backslash \tilde{\omega}_z$ and $\|\nabla v_h\|_{L^2(\Omega)} \lesssim \|\nabla v\|_{L^2(\tilde{\omega}_z)}$. We define the local Fortin operator as $\Pi_{Fl,z}(v) := v_h$. It remains to prove property (ii) in Definition 4.2.

Let $q_h \in \hat{S}^1_{\ell-1}(\mathcal{T})$, we compute

$$\int_\Omega \operatorname{div} \Pi_{Fl,z}(v) \; q_h = \int_{\tilde{\omega}_z} \operatorname{div} v_h \; q_h + \underbrace{\int_{\Omega \backslash \tilde{\omega}_z} \operatorname{div} v_h \; q_h}_{=0} = \int_{\tilde{\omega}_z} \operatorname{div} v_h \; q_h.$$

Since each $q_h \in \hat{S}^1_{\ell-1}(\mathcal{T})$ satisfies $q_{h|\tilde{\omega}_z} \in S^1_{\ell-1}(\tilde{\mathcal{T}}_z)$, we can apply (4.6):

$$\int_\Omega \operatorname{div} \Pi_{Fl,z}(v) \; q_h = \int_{\tilde{\omega}_z} \operatorname{div} v \; q_h = \int_{\tilde{\omega}_z} \operatorname{div} v \; q_h + \underbrace{\int_{\Omega \backslash \tilde{\omega}_z} \operatorname{div} v \; q_h}_{=0} = \int_\Omega \operatorname{div} v \; q_h.$$

Here, we extended the function $v \in H^1_0(\tilde{\omega}_z)^2$ by zero to the whole domain Ω, thus $\operatorname{div} v = 0$ on $\Omega \backslash \tilde{\omega}_z$. This completes the proof of property (ii) in Definition 4.2. □

The previous Lemmas are summarized by the next result.

Lemma 4.8. *For the methods S-AP, S-ACRL, S-ATH, S-SIP and S-CRL and each $z \in \mathcal{V}$ there exists a local Fortin operator, if the mesh $\mathcal{T}$ satisfies the following restrictions:*

S-AP *no restriction,*

S-ACRL, S-SIP, S-CRL *$d = 2$ and for each $z \in \mathcal{V}$ the patch ω_z^+ contains an interior vertex, (if $\ell = 1$ no restriction).*

S-ATH *$d = 2$ and for each $z \in \mathcal{V}$ the patch ω_z^+ contains at least three elements.*

Proof. In the case of S-AP, the local Fortin operator is provided by Lemma 4.6. For the other methods, the local Fortin operator is provided by Lemma 4.5. In the first order case $\ell = 1$, Lemma 4.4 provides an improved local Fortin operator without restrictions on the mesh. The case of the Taylor-Hood method S-ATH is treated in Lemma 4.7. □

With these tools, we present an abstract localization result in the next subsection.

4.1.3 A pressure-robust localization

The localization, which we present, is based upon the localization given by Theorem 2.14. To stay in the abstract setting established by the Definition 4.2 of the local Fortin operator, we extended the notation.

Definition 4.9. *Let $\tilde{\omega}_z \subset \Omega$, $z \in \mathcal{V}$, be an abstract patch with the corresponding triangulation $\tilde{\mathcal{T}}_z$. Following the conventions introduced in Definition 1.26, we define*

$$\tilde{\mathcal{T}}_z^+ = \{T \in \mathcal{T} \mid T \cap \omega(\tilde{\mathcal{T}}_z) \neq \varnothing\} \quad \text{and} \quad \tilde{\omega}_z^+ = \omega(\tilde{\mathcal{T}}_z^+).$$

We now set forth the assumptions for the pressure-robust localization.

Assumption 4.10. *For each $w \in Z(\Omega)$, there is a localization operator $\Theta_z : H^1_0(\Omega)^d \to H^1_0(\omega_z)^d$ and for each $z \in \mathcal{V}$, there is a local Fortin operator $\Pi_{Fl,z}$ such that*

$$\Big\| \sum_{z\in\mathcal{V}} \Pi_{Fl,z}(w\Theta_z) \Big\|_{Z_h} \leqslant C_{\mathsf{I}} \|\nabla w\|_{L^2(\Omega)} \tag{4.7}$$

for a constant $C_{\mathsf{I}} > 0$. Let $\tilde{\omega}_z$ be the patch used in the definition of $\Pi_{Fl,z}$ and $\tilde{\omega}_z^+$ the corresponding patch introduced in Definition 4.9.

Let $\mathrm{E_M} : V_h \to V = H^1_0(\Omega)^d$ be smoothing operator of the method under consideration. We assume the existence of yet another linear and bounded operator $\hat{\mathrm{E}} : V \to V$ with $\hat{\mathrm{E}}_{|Z} = \mathrm{id}_{|Z}$. Finally, we assume for each $z \in \mathcal{V}$ and $w \in Z(\Omega)$ that

$$\hat{\mathrm{E}}(w\Theta_z) - \mathrm{E_M}(\Pi_{Fl,z}(w\Theta_z)) \in Z(\tilde{\omega}_z^+) \tag{4.8a}$$

$$\sum_{z\in\mathcal{V}} \big\| \hat{\mathrm{E}}(w\Theta_z) - \mathrm{E_M}(\Pi_{Fl,z}(w\Theta_z)) \big\|_V^2 \leqslant C_{\mathsf{apx}}^2 \|\nabla w\|^2_{L^2(\Omega)} \tag{4.8b}$$

with $C_{\mathsf{apx}} > 0$ independent of w.

Assumption 4.10 allows us to prove the following localization Theorem.

Theorem 4.11. *Let Assumption 4.10 be satisfied. Then the conforming residual localizes as*

$$\|\operatorname{res}^{\mathsf{c}}_{\mathrm{h}}(f,u_h)\|_{Z(\Omega)'} \leqslant C_{\mathsf{apx}} \Big(\sum_{z\in\mathcal{V}} \|\operatorname{res}^{\mathsf{c}}_{\mathrm{h}}(f,u_h)\|^2_{Z(\tilde{\omega}_z^+)'} \Big)^{\frac{1}{2}} + \delta_{Z_h} C_{\mathsf{I}} \|\Pi_Z u_h - u_h\|_{\tilde{V}} \tag{4.9}$$

where $C_{\mathsf{apx}}, C_{\mathsf{I}}$ are the constants from Assumption 4.10 and δ_{Z_h} is the constant from Corollary 4.1. For each $z \in \mathcal{V}$, the set $\omega_z \subset \tilde{\omega}_z^+ \subset \Omega$ is provided by the local Fortin operator $\Pi_{Fl,z}$ used in Assumption 4.10.

Proof. Given an arbitrary, exactly divergence-free function $w \in Z(\Omega)$ with $\|\nabla w\|_{L^2(\Omega)} = 1$. We insert the function $\mathrm{E_M}\big(\sum_{z\in\mathcal{V}} \Pi_{Fl,z}(w\Theta_z)\big)$ into the argument of the conforming residual:

$$\langle \operatorname{res}^{\mathsf{c}}_{\mathrm{h}}(f,u_h), w\rangle = \Big\langle \operatorname{res}^{\mathsf{c}}_{\mathrm{h}}(f,u_h), w - \mathrm{E_M}\Big(\sum_{z\in\mathcal{V}} \Pi_{Fl,z}(w\Theta_z)\Big)\Big\rangle + \Big\langle \operatorname{res}^{\mathsf{c}}_{\mathrm{h}}(f,u_h), \mathrm{E_M}\Big(\sum_{z\in\mathcal{V}} \Pi_{Fl,z}(w\Theta_z)\Big)\Big\rangle. \tag{4.10}$$

By property (4.4) of the localization operators Θ_z and item (ii) of Definition 4.2 for the local Fortin operator, we have for all $q_h \in Q_h$

$$b_h\Big(\sum_{z\in\mathcal{V}} \Pi_{Fl,z}(w\Theta_z), q_h\Big) = \sum_{z\in\mathcal{V}} b_h(\Pi_{Fl,z}(w\Theta_z), q_h) = \sum_{z\in\mathcal{V}} b(w\Theta_z, q_h) = b(w, q_h) = 0,$$

because $w \in Z(\Omega)$. Therefore, $\underline{\mathrm{div}}_h\Big(\sum_{z\in\mathcal{V}} \Pi_{Fl,z}(w\Theta_z)\Big) = 0$ and we can use $\sum_{z\in\mathcal{V}} \Pi_{Fl,z}(w\Theta_z)$ as an argument in Corollary 4.1.

Thus, we can estimate the last summand in (4.10) by Corollary 4.1 and (4.7):

$$\Big\langle \operatorname{res}^{\mathsf{c}}_{\mathrm{h}}(f,u_h), \mathrm{E_M}\Big(\sum_{z\in\mathcal{V}} \Pi_{Fl,z}(w\Theta_z)\Big)\Big\rangle \leqslant \delta_{Z_h} C_{\mathsf{I}} \|\Pi_Z u_h - u_h\|_{\tilde{V}}.$$

We turn to the first part of (4.10) and estimate with (4.8a):

$$\begin{aligned}
&\Big\langle \operatorname{res}^{\mathsf{c}}_{\mathrm{h}}(f,u_h), w - \sum_{z\in\mathcal{V}} \mathrm{E_M}(\Pi_{Fl,z}(w\Theta_z))\Big\rangle \\
&\quad= \Big\langle \operatorname{res}^{\mathsf{c}}_{\mathrm{h}}(f,u_h), \hat{\mathrm{E}}(w) - \sum_{z\in\mathcal{V}} \mathrm{E_M}(\Pi_{Fl,z}(w\Theta_z))\Big\rangle \\
&\quad= \sum_{z\in\mathcal{V}} \langle \operatorname{res}^{\mathsf{c}}_{\mathrm{h}}(f,u_h), \hat{\mathrm{E}}(w\Theta_z) - \mathrm{E_M}\big(\Pi_{Fl,z}(w\Theta_z)\big)\rangle \\
&\quad\leqslant \sum_{z\in\mathcal{V}} \|\operatorname{res}^{\mathsf{c}}_{\mathrm{h}}(f,u_h)\|_{Z(\tilde{\omega}_z^+)'} \big\| \hat{\mathrm{E}}(w\Theta_z) - \mathrm{E_M}\big(\Pi_{Fl,z}(w\Theta_z)\big)\big\|_V \\
&\quad\leqslant \Big(\sum_{z\in\mathcal{V}} \|\operatorname{res}^{\mathsf{c}}_{\mathrm{h}}(f,u_h)\|^2_{Z(\tilde{\omega}_z^+)'} \Big)^{\frac{1}{2}} \Big(\sum_{z\in\mathcal{V}} \big\| \hat{\mathrm{E}}(w\Theta_z) - \mathrm{E_M}(\Pi_{Fl,z}(w\Theta_z))\big\|^2_V \Big)^{\frac{1}{2}}.
\end{aligned}$$

Now, we apply estimate (4.8b) and obtain the result. □

Remark 4.12. The localization with Assumption 4.10 and Theorem 4.11 can be seen as the counterpart of the localization provided by Assumption 2.12 and Theorem 2.14 for the Stokes equation.

A generalization of the localization given by Assumption 2.6 and Theorem 2.11 is currently not available. To obtain such a generalization, we would need suitable localization operators $\Theta_z : Z(\Omega) \to Z(\omega_z)$, $z \in \mathcal{V}$. Currently, the author is not aware of any construction of such an operator that satisfies our needs.

The next Lemma helps checking condition (4.7) in the standard case.

Lemma 4.13. *Let $\Pi_{Fl,z} : H^1_0(\tilde{\omega}_z)^d \to V_h$, $z \in \mathcal{V}$, be one of the local Fortin operators for the S-AP, S-ACRL, S-ATH, S-SIP or S-CRL methods provided by Lemma 4.8. These operators fulfill* (4.7) *with $\Theta_z : H^1_0(\Omega)^d \to H^1_0(\omega_z)^d$ defined as $v\Theta_z := v\Phi_z$, $v \in H^1_0(\Omega)^d$.*

Proof. The statement is proved if we show (4.7) for the norm $\|\cdot\|_{Z_h} = \|\nabla_h \cdot\|_{L^2(\Omega)}$ because all forms a_h are continuous with respect to the corresponding $\tilde{V}$-norm. Also, S-SIP is covered, although its norm contains the stabilization term. In the case of S-SIP, the operators $\Pi_{Fl,z}$ map into the Crouzeix-Raviart space. On CR^d_ℓ, the $\tilde{V}$-norm of S-SIP and the one of the Crouzeix-Raviart methods are equivalent.

Thus, let us show (4.7) for the case $\|\cdot\|_{Z_h} = \|\nabla_h \cdot\|_{L^2(\Omega)}$. Fix $w \in Z(\Omega)$. We start by inserting $\sum_{z\in\mathcal{V}} c_z\Phi_z$ with $c_z = \int_\Omega w$ if $z \in \mathcal{V} \cap \Omega$ and $c_z = 0$ if $z \in \mathcal{V} \cap \partial\Omega$:

$$\Big\|\nabla_h\Big(\sum_{z\in\mathcal{V}} \Pi_{Fl,z}(w\Phi_z)\Big)\Big\|_{L^2(\Omega)} \leqslant \Big\|\nabla_h\Big(\sum_{z\in\mathcal{V}} c_z\Phi_z\Big)\Big\|_{L^2(\Omega)} + \Big\|\nabla_h\Big(\sum_{z\in\mathcal{V}} \Pi_{Fl,z}(w\Phi_z - c_z\Phi_z)\Big)\Big\|_{L^2(\Omega)}. \quad (4.11)$$

We used that all considered $\Pi_{Fl,z}$ are invariant under $c_z\Phi_z$. The first summand is bounded by $\|\nabla w\|_{L^2(\Omega)}$ as shown in (2.43) and (2.44). The latter term in (4.11) is estimated as

$$\begin{aligned}\Big\|\nabla_h\Big(\sum_{z\in\mathcal{V}} \Pi_{Fl,z}(w\Phi_z - c_z\Phi_z)\Big)\Big\|^2_{L^2(\Omega)} &= \int_\Omega \Big|\nabla_h\Big(\sum_{z\in\mathcal{V}} \Pi_{Fl,z}(w\Phi_z - c_z\Phi_z)\Big)\Big|^2 \\ &\lesssim \sum_{z\in\mathcal{V}} \int_{\tilde{\omega}_z} \Big|\nabla_h \Pi_{Fl,z}(w\Phi_z - c_z\Phi_z)\Big|^2 \lesssim \sum_{z\in\mathcal{V}} \|\nabla(w\Phi_z - c_z\Phi_z)\|^2_{L^2(\tilde{\omega}_z)},\end{aligned}$$

since at each point in Ω only finitely many summands are non-zero and all $\Pi_{Fl,z}$ are stable, see Definition 4.2. The resulting expression can again be bounded by $\|\nabla w\|_{L^2(\Omega)}$ as in (2.44). □

Theorem 4.11 is applicable to all our examples. The proof of Assumption 4.10 is the same for most methods, only in the case of the Taylor-Hood method A-TH, a different argument is needed. In the A-TH case, the main hindrance is that A-TH is the only method $1_T \notin Q_h + \mathbb{R}$ for $T \in \mathcal{T}$ as $Q_h \subset H^1_0(\Omega)^2$. Here, 1_T is the indicator function for the set T. Except the Taylor-Hood method A-TH, all other examples satisfy $1_T \in Q_h + \mathbb{R}$ for all $T \in \mathcal{T}$.

We start with the proof of Assumption 4.10 for all methods except A-TH. These are S-AP, S-ACRL, S-SIP and S-CRL. In these cases, we construct the auxiliary operator $\hat{\mathrm{E}} : V \to V$ with a special divergence operator. This new divergence operator $\widehat{\mathrm{div}} : V \to Q_h$ is defined as

$$\int_\Omega \widehat{\mathrm{div}} w\, q_h = \int_\Omega \mathrm{div}\, w\, q_h \quad \forall q_h \in Q_h. \quad (4.12)$$

Because we assume $Q_h \subset L^2_0(\Omega)$, (4.12) is also true for all test functions of the form $q_h + c$ with $q_h \in Q_h$ and $c \in \mathbb{R}$. Since $1_T \in Q_h$, we have $\int_T \widehat{\mathrm{div}} w - \mathrm{div}\, w = 0$ for all $w \in V$ and $T \in \mathcal{T}$. This insight allows us to define the operator $\hat{\mathrm{E}} : V \to V$ as

$$\hat{\mathrm{E}}(w) := w + \sum_{T\in\mathcal{T}} \mathrm{div}_T^{-1}\big(\widehat{\mathrm{div}} w - \mathrm{div}\, w\big), \quad w \in V.$$

Here, div_T^{-1} is a stable right inverse of div on $T \in \mathcal{T}$, whose existence is guaranteed by Theorem 3.7. The right inverse ensures that $\mathrm{div}\,\hat{\mathrm{E}}(w) = \widehat{\mathrm{div}} w$. As div_T^{-1} is bounded from $L^2_0(T) \to H^1_0(T)^d$, $\hat{\mathrm{E}}$ is also bounded.

The operator $\hat{\mathrm{E}}$ helps us to prove the next Theorem.

Theorem 4.14. *The methods S-AP, S-ACRL, S-SIP and S-CRL satisfy Assumption 4.10 if the mesh $\mathcal{T}$ fulfills the restrictions laid out in Lemma 4.8. Hence, under these restrictions, the localization provided in Theorem 4.11 is true.*

Proof. We show that Assumption 4.10 is true. We choose Lagrange hat functions, as a partition of unity $\Theta_z : H^1_0(\Omega)^d \to H^1_0(\omega_z)^d$ defined as $v\Theta_z := v\Phi_z$, $v \in H^1_0(\Omega)^d$, $z \in \mathcal{V}$.

The local Fortin operators $\Pi_{Fl,z}$, $z \in \mathcal{V}$, are available for the considered methods: In the case of S-AP, the local Fortin operator is provided by Lemma 4.6. For the other methods, the local Fortin operator is provided by Lemma 4.5. In the first order case $\ell = 1$, Lemma 4.4 provides an improved local Fortin operator, see also the overview in Lemma 4.8. From now on, we assume that $\Pi_{Fl,z} : H^1_0(\tilde{\omega}_z)^d \to V_h$, $z \in \mathcal{V}$, is a local Fortin operator provided by the above results for the method with $\tilde{\omega}_z$ chosen accordingly.

The bound (4.7) is shown in Lemma 4.13. We turn to the proofs (4.8). Fix a test function $w \in Z$, $w : \Omega \to \mathbb{R}^d$. Recall that the smoothers of all considered methods fulfill $\operatorname{div} \mathrm{E_M}\, w_h = \underline{\operatorname{div}}_h w_h$, $w_h \in V_h$, see (3.38) for S-AP, (3.45) for S-ACRL and (3.54) for S-SIP and S-CRL. Inserting $\mathrm{E_M}$ together with $\hat{\mathrm{E}}$ into (4.8a) we have

$$\operatorname{div}(\hat{\mathrm{E}}(w\Phi^1_z) - \mathrm{E_M}(\Pi_{Fl,z}(w\Phi^1_z))) = \widehat{\operatorname{div}} w\Phi^1_z - \underline{\operatorname{div}}_h \Pi_{Fl,z}(w\Phi^1_z) = 0,$$

owing to Definition 4.2 point (ii) and (4.12). Hence, $\hat{\mathrm{E}}(w\Phi^1_z) - \mathrm{E_M}(\Pi_{Fl,z}(w\Phi^1_z)) \in Z(\tilde{\omega}^+_z)$.

To show (4.8b), we insert the scalar Lagrange hat functions Φ^1_z, $z \in \mathcal{V}$, with a suitable coefficient $c_z \in \mathbb{R}^d$. We define this coefficient as $c_z := \frac{1}{|\omega_z|}\int_{\omega_z} w \in \mathbb{R}^d$ if $z \in \mathcal{V} \cap \Omega$ and $c_z := 0 \in \mathbb{R}^d$ otherwise. As $c_z\Phi^1_z \in H^1_0(\omega_z)^d \cap S^1_1(\mathcal{T}(\omega_z))^d$, we have $\operatorname{div} c_z\Phi^1_z \in \hat{S}^0_0(\mathcal{T}(\omega_z)) \subset Q_h$. Consequently, we have $\operatorname{div} c_z\Phi^1_z = \underline{\operatorname{div}}_h c_z\Phi^1_z = \widehat{\operatorname{div}} c_z\Phi^1_z$ because $\hat{S}^0_0 \subset Q_h$, thus the smoothing operators of all considered methods as well as $\hat{\mathrm{E}}$ are invariant under $c_z\Phi^1_z$. Also, the local Fortin operators are invariant on $c_z\Phi^1_z$ because $c_z\Phi^1_z \in V_h$ for all considered methods. Hence, we obtain

$$\begin{aligned}
&\sum_{z\in\mathcal{V}} \| \hat{\mathrm{E}}(w\Phi^1_z) - \mathrm{E_M}(\Pi_{Fl,z}(w\Phi^1_z))\|^2_V \\
&\quad= \sum_{z\in\mathcal{V}} \| \hat{\mathrm{E}}((w-c_z)\Phi^1_z) - \mathrm{E_M}\left(\Pi_{Fl,z}\big((w-c_z)\Phi^1_z\big)\right)\|^2_V \\
&\quad\lesssim \sum_{z\in\mathcal{V}} \|(w-c_z)\Phi^1_z\|^2_{\hat{V}} + \|\Pi_{Fl,z}\big((w-c_z)\Phi^1_z\big)\|^2_{\hat{V}} \\
&\quad\lesssim \sum_{z\in\mathcal{V}} \|(w-c_z)\Phi^1_z\|^2_{\hat{V}}.
\end{aligned} \tag{4.13}$$

The latter expression can be estimated as in the Poisson case, c.f. (2.44). In the case of S-SIP, the jump-term of $\|(w-c_z)\Phi^1_z\|_{\hat{V}}$ vanishes. Therefore, we have as in (2.44)

$$\|\nabla\big((w-c_z)\Phi^1_z\big)\|^2_{L^2(\omega_z)} \lesssim \|\nabla w\|^2_{L^2(\omega_z)} + \frac{1}{h_z}\|w-c_z\|^2_{L^2(\omega_z)} \lesssim \|\nabla w\|^2_{L^2(\omega_z)} \tag{4.14}$$

by Poincaré's inequality. Due to the finite overlap property, (1.22), the proof of Assumption 4.10 is complete. □

The last Theorem shows that the residual of S-AP, S-ACRL, S-SIP and S-CRL can be localized. The next Theorem shows that also the residual of S-ATH can be localized.

Theorem 4.15. *Assume that each ω^+_z, $z \in \mathcal{V}$, contains at least 3 elements. Then, the method S-ATH fulfills Assumption 4.10, thus the localization provided in Theorem 4.11 is true.*

Proof. This proof uses techniques that are inspired by [KZ20, section 4.2] and is similar to the one of Theorem 4.14. We show that Assumption 4.10 is true. The Assumption requires a partition of unity and a local Fortin operator. The partition of unity is again provided by the Lagrange hat functions Φ_z in this case. The local Fortin operators $\Pi_{Fl,z}$, $z \in \mathcal{V}$, in combination with the sets $\tilde{\omega}_z$, $z \in \mathcal{V}$ are provided by Lemma 4.7.

We define the operator $\hat{\mathrm{E}} : V \to V$ differently in this setting because we include an operator based upon $\mathcal{R}^{\mathsf{TH}}$. The latter is defined in [KZ20, (4.9)] and recalled in (3.49). The original operator $\mathcal{R}^{\mathsf{TH}}$ is only defined for functions in $\mathring{S}^1_\ell(\mathcal{T})^2$ as

$$\mathcal{R}^{\mathsf{TH}} v_h := \sum_{z\in\mathcal{V}} \mathcal{R}^z_\ell(\operatorname{div} v_h).$$

But the operators $\mathcal{R}^z_\ell$, $z \in \mathcal{V}$, are defined via the local Stokes problem (3.47), which is not only well-defined for $q \in \hat{S}^0_{\ell-1}(\mathcal{T})$ but for any $q \in L^2(\Omega)$, as already pointed out in [KZ20, (4.8)]. This insight allows us to immediately extend the operator $\mathcal{R}^{\mathsf{TH}}$ to $\mathcal{R}^{\mathsf{TH}} : H(\operatorname{div}) \to \mathring{S}^1_\ell(\mathcal{M})^2 \subset H^1_0(\Omega)^2$. Here, $\mathcal{M}$ is the mesh obtained by performing an Alfeld refinement in each element of $\mathcal{T}$, see page 102.

The extension of $\mathcal{R}^{\mathsf{TH}}$ enables us to define $\hat{\mathrm{E}} : H^1_0(\Omega)^2 \to H^1_0(\Omega)^2$ for any $w \in H^1_0(\Omega)^2$ as

$$\hat{\mathrm{E}}(w) := w + \sum_{T\in\mathcal{T}} \mathrm{div}_T^{-1}\left(\widehat{\mathrm{div}}w - \mathrm{div}\, w\right) + \mathcal{R}^{\mathsf{TH}}(w).$$

The operator $\widehat{\mathrm{div}} : H^1_0(\Omega)^2 \to \hat{S}^0_{\ell-1}(\mathcal{M})$ is defined for $w \in H^1_0(\Omega)^2$ by

$$\int_\Omega \widehat{\mathrm{div}}w\; q_h = \int_\Omega \mathrm{div}\, w\; q_h, \quad \forall q_h \in \hat{S}^0_{\ell-1}(\mathcal{M}),$$

similar to (4.12).

The stability of $\hat{\mathrm{E}}$ follows by the stability of the divergence inversion operator in Theorem 3.7, the stability of $\widehat{\mathrm{div}}$ and the stability of $\mathcal{R}^{\mathsf{TH}}$. The latter is shown on [KZ20, p. 2576].

This setting fulfills Assumption (4.10) with $\mathrm{E_M} = \mathcal{E}^{\mathsf{TH}}$. To prove this, we show (4.8a) first for $w \in H^1_0(\Omega)^2$. Let us evaluate the divergence of $\hat{\mathrm{E}}(w\Phi_z)$ and $\mathcal{E}^{\mathsf{TH}}(\Pi_{Fl,z}(w\Phi_z))$ for each $z \in \mathcal{V}$. The definitions of $\hat{\mathrm{E}}$ and $\mathcal{E}^{\mathsf{TH}}$ show

$$\begin{aligned} \mathrm{div}(\hat{\mathrm{E}}(w\Phi_z)) &= \widehat{\mathrm{div}}(w\Phi_z) + \mathrm{div}\,\mathcal{R}^{\mathsf{TH}}(w\Phi_z) \quad \text{and} \\ \mathrm{div}\left(\mathcal{E}^{\mathsf{TH}}(\Pi_{Fl,z}(w\Phi_z))\right) &= \mathrm{div}\left(\Pi_{Fl,z}(w\Phi_z)\right) + \mathrm{div}\,\mathcal{R}^{\mathsf{TH}}(\Pi_{Fl,z}(w\Phi_z)). \end{aligned} \tag{4.15}$$

We observe that $\mathrm{div}(\hat{\mathrm{E}}(w\Phi_z)) \in \hat{S}^0_{\ell-1}(\mathcal{M})$ and $\mathrm{div}\left(\mathcal{E}^{\mathsf{TH}}(\Pi_{Fl,z}(w\Phi_z))\right) \in \hat{S}^0_{\ell-1}(\mathcal{M})$. Thus, to show equality of the two expressions in (4.15), it suffices to test them with functions from $\hat{S}^0_{\ell-1}(\mathcal{M})$.

If we test $\mathrm{div}(\hat{\mathrm{E}}(w\Phi_z))$ with $q_h \in \hat{S}^0_{\ell-1}(\mathcal{M})$, we get

$$\begin{aligned} \int_\Omega \mathrm{div}(\hat{\mathrm{E}}(w\Phi_z))q_h &= \int_\Omega \widehat{\mathrm{div}}(w\Phi_z)q_h + \int_\Omega \mathrm{div}\,\mathcal{R}^{\mathsf{TH}}(w\Phi_z)q_h \\ &= \int_\Omega \mathrm{div}(w\Phi_z)q_h + \sum_{y\in\mathcal{V}}\int_{\omega_y} [\mathcal{A}_{\ell-1}(q_h\Phi_y) - \mathcal{I}^y_{\ell-1}(q_h\Phi_y)]\,\mathrm{div}(w\Phi_z). \end{aligned} \tag{4.16}$$

We note that $\sum_{y\in\mathcal{V}} \mathcal{A}_{\ell-1}(q_h\Phi_y) - \mathcal{I}^y_{\ell-1}(q_h\Phi_y) = \mathcal{A}_{\ell-1}(q_h) - q_h$ (c.f. [KZ20, Proposition 4.5]) and obtain with (4.16):

$$\int_\Omega \mathrm{div}(\hat{\mathrm{E}}(w\Phi_z))q_h = \int_\Omega \mathrm{div}(w\Phi_z)\mathcal{A}_{\ell-1}(q_h). \tag{4.17}$$

Next, we test $\mathrm{div}\left(\mathcal{E}^{\mathsf{TH}}(\Pi_{Fl,z}(w\Phi_z))\right)$ with $q_h \in \hat{S}^0_{\ell-1}(\mathcal{M})$. Inspired by [KZ20, Proposition 4.5], we get with (4.15)

$$\begin{aligned} \int_\Omega \mathrm{div}(\mathcal{E}^{\mathsf{TH}}(\Pi_{Fl,z}(w\Phi_z)))q_h &= \int_\Omega \mathrm{div}\left(\Pi_{Fl,z}(w\Phi_z)\right)q_h + \mathrm{div}\,\mathcal{R}^{\mathsf{TH}}(\Pi_{Fl,z}(w\Phi_z))q_h \\ &= \int_\Omega \mathrm{div}\left(\Pi_{Fl,z}(w\Phi_z)\right)q_h + \sum_{y\in\mathcal{V}}\int_{\omega_y} [\mathcal{A}_{\ell-1}(q_h\Phi_y) - \mathcal{I}^y_{\ell-1}(q_h\Phi_y)]\,\mathrm{div}(\Pi_{Fl,z}(w\Phi_z)) \\ &= \int_\Omega \mathrm{div}\left(\Pi_{Fl,z}(w\Phi_z)\right)\mathcal{A}_{\ell-1}(q_h) = \int_\Omega \mathrm{div}(w\Phi_z)\mathcal{A}_{\ell-1}(q_h). \end{aligned} \tag{4.18}$$

The last step follows because of $\mathcal{A}_{\ell-1}(q_h) \in Q_h$ and Definition 4.2. Combining (4.16) and (4.17) with (4.18), we deduce that

$$\int_\Omega \mathrm{div}(\hat{\mathrm{E}}(w\Phi_z))q_h = \int_\Omega \mathrm{div}(\mathcal{E}^{\mathsf{TH}}(\Pi_{Fl,z}(w\Phi_z)))q_h \quad \forall q_h \in \hat{S}^0_{\ell-1}(\mathcal{M}).$$

Hence, $\mathrm{div}(\hat{\mathrm{E}}(w\Phi_z)) = \mathrm{div}(\mathcal{E}^{\mathsf{TH}}(\Pi_{Fl,z}(w\Phi_z)))$, which shows (4.8a).

To show (4.8b), we use the arguments of (4.13), (4.14). These arguments are applicable if $\hat{\mathrm{E}}(c_z\Phi_z) = \mathcal{E}^{\mathsf{TH}}(\Pi_{Fl,z}(c_z\Phi_z))$ for all $z \in \mathcal{V}$ and $c_z \in \mathbb{R}^2$. Observing that $\widehat{\mathrm{div}}(c_z\Phi_z) = \mathrm{div}(c_z\Phi_z)$ for all $z \in \mathcal{V}$ and $c_z \in \mathbb{R}^2$ and $\Pi_{Fl,z}(c_z\Phi_z) = c_z\Phi_z$, we see

$$\hat{\mathrm{E}}(c_z\Phi_z) = c_z\Phi_z + \mathcal{R}^{\mathsf{TH}}(c_z\Phi_z) = \Pi_{Fl,z}(c_z\Phi_z) + \mathcal{R}^{\mathsf{TH}}\left(\Pi_{Fl,z}(c_z\Phi_z)\right) = \mathcal{E}^{\mathsf{TH}}\left(\Pi_{Fl,z}(c_z\Phi_z)\right).$$

Thus, (4.8b) is valid by the arguments of (4.13), (4.14).

□

4.1.4 A pressure-robust local localization

The localization in Theorem 4.11 has the disadvantage that it only localizes to $Z(\tilde{\omega}_z^+)'$. For the construction of the projection operator, which splits off the oscillation term (see the upcoming section 4.2), we would like to localize to $Z(\omega_z)'$ similar to the first part of this thesis. This problem is solved by a second localization similar to the one in Theorem 2.16. The application of the original Theorem 2.16 is, however, not possible because we have to test with divergence-free functions in the current setting. In this section, a new local localization Theorem, which tests the residual only with divergence-free functions, provides a suitable replacement for Theorem 2.16.

This new Theorem uses the Helmholtz decomposition and an H^2-partition of unity. We recall these two tools and start with the two-dimensional Helmholtz decomposition.

Theorem 4.16 (Helmholtz decomposition for $2d$, [GR79, Theorem 3.1])**.** *Let $d = 2$ and $\omega \subset \Omega$ a simply connected, bounded set with Lipschitz boundary. Fix $z \in Z(\omega)$. Then there exists a function $\hat{z} \in H^2(\omega)$ with*

$$\int_\omega \hat{z} = 0 \quad \textit{and} \quad \operatorname{curl}(\hat{z}) = z, \quad \textit{in } \omega$$

and the estimates

$$\|\nabla \hat{z}\|_{L^2(\omega)} = \|\operatorname{curl} \hat{z}\|_{L^2(\omega)} \quad \textit{and} \quad \|\hat{z}\|_{L^2(\omega)} \lesssim \operatorname{diam}(\omega) \|\operatorname{curl} \hat{z}\|_{L^2(\omega)},$$

where $\operatorname{diam}(\omega)$ *is the diameter of* ω*.*

Proof. In the case $d = 2$, [GR79, Theorem 3.1] provides a function $\hat{z} \in H^1(\omega)$ with $\operatorname{curl}(\hat{z}) = z$. Because $z \in H^1(\omega)^2$, we have $\hat{z} \in H^2(\omega)$. This function is not unique because we can add a constant. We choose this constant in such a way that $\int_\omega \hat{z} = 0$, c.f. [GR79, p. 39].

The first estimate follows by the observation that $\|\operatorname{curl} \hat{z}\|_{L^2(\omega)} = \|\nabla \hat{z}\|_{L^2(\omega)}$ because $|\operatorname{curl} \hat{z}| = |\nabla \hat{z}|$. The same observation in combination with Poincaré's inequality gives

$$\|\hat{z}\|_{L^2(\omega)} \lesssim \operatorname{diam}(\omega) \|\nabla \hat{z}\|_{L^2(\omega)} = \operatorname{diam}(\omega) \|\operatorname{curl} \hat{z}\|_{L^2(\omega)}. \tag{4.19}$$

□

Related to the previous Theorem, we have the following decomposition result:

Lemma 4.17 ([GR79, Theorem 3.2], [Gal11, section III.1])**.** *Every function $f \in L^2(\Omega)^2$ can be decomposed into*

$$f = \nabla f_1 + f_2$$

with $f_1, \in H^1(\Omega)$, $f_2 \in H(\operatorname{div}; \Omega)$ and $\operatorname{div} f_2 = 0$.

Next, we state the equivalent result for the three-dimensional case.

Theorem 4.18 (Helmholtz decomposition for $3d$ [BF13, Theorem IV.4.13])**.** *Assume $d = 3$. Let $\omega \subset \Omega$ be a simply connected, bounded set with Lipschitz boundary. Let $z \in Z(\omega)$ be given. Then there exists a function $\hat{z} \in H^2(\omega)^3$ with*

$$\operatorname{div}(\hat{z}) = 0 \quad \textit{and} \quad \operatorname{curl}(\hat{z}) = z \quad \textit{in } \omega.$$

The function $\hat{z}$ satisfies

$$\|\nabla \hat{z}\|_{L^2(\omega)} \lesssim \|\operatorname{curl} \hat{z}\|_{L^2(\omega)} \quad \textit{and} \quad \|\hat{z}\|_{L^2(\omega)} \lesssim \operatorname{diam}(\omega) \|\operatorname{curl} \hat{z}\|_{L^2(\omega)}.$$

Proof. As in the proof of [BF13, Theorem IV.4.13], we choose a ball $B \subset \mathbb{R}^3$ which contains ω and extend $z \in Z(\omega)$ to a function $z^* \in H^1(B)^3$ by zero. Besides $\operatorname{div}(z^*) = 0$ in B, the extension also fulfills $\|\nabla z^*\|_{L^2(B)} = \|\nabla z\|_{L^2(\omega)}$ and $\|z^*\|_{L^2(B)} = \|z\|_{L^2(\omega)}$. Then [BF13, Theorem IV.4.12] ensures the existence of a $\hat{z} \in H^2(B)^3$ with $\operatorname{div}(\hat{z}) = 0$, $\operatorname{curl}(\hat{z}) = z^*$ in B, $\partial_{\mathrm{n}_B} \hat{z} = 0$ on ∂B and $\|\nabla \hat{z}\|_{L^2(B)} \lesssim \|\operatorname{curl} \hat{z}\|_{L^2(B)}$. Moreover, [GR79, Lemma 3.6] shows that

$$\begin{aligned} \|\hat{z}\|_{L^2(\omega)} = \|\hat{z}\|_{L^2(B)} &\lesssim \operatorname{diam}(\omega) \|\operatorname{curl} \hat{z}\|_{L^2(B)} = \operatorname{diam}(\omega) \|z^*\|_{L^2(B)} \\ &= \operatorname{diam}(\omega) \|z\|_{L^2(\omega)} = \operatorname{diam}(\omega) \|\operatorname{curl} \hat{z}\|_{L^2(\omega)}. \end{aligned}$$

□

To establish the local localization Theorem, we also need a H^2-partition of unity. In the case $d = 2$, a suitable partition is provided in Lemma 2.46. In the case $d = 3$, we derive one in the next Lemma.

Lemma 4.19. *Let $d = 3$. For each $z \in \mathcal{V}$, there exists a function $\Upsilon_z^0 \in H^2(\Omega)$ with $\operatorname{supp}(\Upsilon_z^0) \subset \omega_z$ and $\sum_{z \in \mathcal{V}} \Upsilon_z^0 = 1$ and*

$$h_z^2 \|D^2 \Upsilon_z^0\|_{L^\infty(\Omega)} \eqsim h_z \|\nabla \Upsilon_z^0\|_{L^\infty(\Omega)} \eqsim \|\Upsilon_z^0\|_{L^\infty(\Omega)} \eqsim 1. \tag{4.20}$$

Proof. The two-dimensional HCT-element served as the basis for the H^2-conforming partition of unity, introduced in Lemma 2.46. The arbitrary dimensional generalization of the HCT-element developed in [WF87] allows us to construct an H^2-conforming three-dimensional partition of unity.

We write HCT^3 for the three-dimensional HCT-generalization of [WF87]. To describe the degrees of freedom of HCT^3 on a tetrahedron, we introduce some notation for its geometric entities. A tetrahedron $T \in \mathcal{T}$ has four faces denoted by $\mathcal{F}_T$. Each $F \in \mathcal{F}_T$ has three edges contained in $\mathcal{E}_F$.

In the two-dimensional case, the triangle is split by connecting each vertex of the triangle with the barycenter, c.f. page 35. In the three-dimensional case in [WF87] the tetrahedron is also split. For this splitting, an arbitrary *splitting point* m_F is chosen on each $F \in \mathcal{F}_T$, see [WF87, section 2]. This way, the tetrahedron is divided into 12 subtetrahedra, c.f. [WF87, p.101]. The shape functions of the HCT^3-element are the C^1 functions that are piecewise cubic polynomials on the mesh of the twelve subtetrahedra that was created by the splitting, see [WF87, Theorem 3.3].

In [WF87, (1.1)], the degrees of freedom of an HCT^3-element on a tetrahedron T are described. For our purposes, it suffices to know the simpler description of [WF87, p.102]: Taking the degrees of freedom of the $2d$ HCT element on each triangular face $F \in \mathcal{F}_T$ gives a possible set of degrees of freedom for the HCT^3 element, c.f. [WF87, Fig 3.1, Fig 3.2]. Thus, the values at the vertices of each T are contained in the degrees of freedom of HCT^3, and we write the corresponding basis functions as Υ^0_z, $z \in \mathcal{V}$.

These functions Υ^0_z, $z \in \mathcal{V}$, are a partition of unity $\sum_{z\in\mathcal{V}} \Upsilon^0_z = 1$ because the constant function 1 is contained in the space formed by the HCT^3 elements. It remains to show the estimates (4.20).

Similar to the $2d$-case, the degrees of freedom on HCT^3 just described are not affine-equivalent. Again, we overcome this problem by choosing alternative, affine-equivalent degrees of freedom on HCT^3. Inspired by the affine-equivalent degrees of freedom $\tilde{\mathcal{N}}_{\mathrm{HCT}}$ defined in (2.54), we define a set of affine-equivalent degrees of freedom for HCT^3: On each triangular face $F \in \mathcal{F}_T$, we choose the two-dimensional, affine-equivalent set of functionals $\tilde{\mathcal{N}}_{\mathrm{HCT}}$ (2.54) as degrees of freedom. Combining these degrees of freedom for all triangles $F \in \mathcal{F}_T$, we obtain a complete set of degrees of freedom for HCT^3 on T, c.f. [WF87, (1.1)]. Written explicitly, the affine-equivalent degrees of freedom read for each $T \in \mathcal{T}$ and function $v \in \mathrm{HCT}^3$ as

$$\begin{aligned}\tilde{\mathcal{N}}_{\mathrm{HCT}^3} := \{\tilde{N}^0_z(v) := v(z) \mid z \in \mathcal{V} \cap T\} \oplus \{\tilde{N}^y_z(v) := \nabla v(z)\cdot(y-z) \mid y, z \in \mathcal{V}\cap T,\ y \neq z\} \\ \oplus \bigoplus_{F\in\mathcal{F}_T} \{\tilde{N}_{F,E}(v) := \nabla v(m_E)\cdot(m_F - m_E) \mid E \in \mathcal{E}_F\}.\end{aligned} \tag{4.21}$$

Here, m_F is splitting point of F (see above and [WF87, section 2]) and m_E is the midpoint of the edge E.

We introduce the corresponding affine-equivalent basis functions $\tilde{\Upsilon}^0_z$ for $\tilde{N}^0_z$, $z \in \mathcal{V}\cap T$, and $\tilde{\Upsilon}_{F,E}$ for $\tilde{N}_{F,E}$, $F \in \mathcal{F}_T$ and $E \in \mathcal{E}_F$. The basis-functions for the gradients at the vertices are not needed in our discussion.

To derive the estimates (4.20), we express Υ^0_z in terms of the affine-equivalent basis-functions. Computing $\tilde{N}_{F,E}(\Upsilon^0_z)$, $E \in \mathcal{E}_F$, $F \in \mathcal{F}_T$, works as in (2.55) and (2.2.3.1) because HCT^3 restricted to $F \in \mathcal{F}_T$ is the two-dimensional HCT-element.

Hence, we can write Υ^0_z on a fixed $T \in \mathcal{T}$ as

$$\Upsilon^0_z = \tilde{\Upsilon}_z + \frac{3}{2} \sum_{F\in\mathcal{F}_T\cap\mathcal{F}_z} \sum_{\substack{E\in\mathcal{E}_F\\ z\in E}} \frac{z - y_E}{|z-y_E|^2}\cdot(m_F - m_E)\tilde{\Upsilon}_{F,E}. \tag{4.22}$$

Here, $z \in E$ means that E contains z and $y_E \in \mathcal{V}$ is the vertex on E, which is opposite of z. Again, m_F is the splitting point of F and m_E is the midpoint of the edge E. Finally, (4.22) shows that (4.20) is satisfied. □

The next Theorem is the announced local localization from $Z(\tilde{\omega}^+_z)'$ to $Z(\omega_z)'$.

Theorem 4.20. *Assume that the conditions in Lemma 4.8 are fulfilled for the considered method and $\tilde{\omega}_z$ is the patch provided by the corresponding local Fortin operator. Furthermore, we assume that $\tilde{\omega}^+_z$ is simply connected for all $z \in \mathcal{V}$. For each $z \in \mathcal{V}$ and functional $R \in Z(\tilde{\omega}^+_z)'$, we have*

$$\|R\|^2_{Z(\tilde{\omega}^+_z)'} \leqslant C^2_{\mathsf{P}} \sum_{y\in\mathcal{I}_z} \|R\|^2_{Z(\omega_y)'},$$

with $\mathcal{I}_z := \mathcal{V}\cap\tilde{\omega}^+_z$ and a constant $C_{\mathsf{P}} > 0$ independent of $z \in \mathcal{V}$.

Proof. Fix $z \in \mathcal{V}$. Let $v \in Z(\tilde{\omega}_z^+)$ be an exactly divergence-free function with $\operatorname{supp}(v) \subset \tilde{\omega}_z^+$. In the case of $d = 2$, let $\hat{v} \in H^2(\tilde{\omega}_z^+)$ be the function provided by Theorem 4.16 with $\operatorname{curl}(\hat{v}) = v$. In the case of $d = 3$, let $\hat{v} \in H^2(\tilde{\omega}_z^+)^3$ be the function provided by Theorem 4.18 with $\operatorname{curl}(\hat{v}) = v$. Let Υ_z^0, $z \in \mathcal{I}$, be the H^2-partition of unity which is provided for $2d$ in Lemma 2.46 and for $3d$ in Lemma 4.19. We compute:

$$\begin{aligned}\langle R, v\rangle = \langle R, \operatorname{curl}(\hat{v})\rangle &= \sum_{y\in\mathcal{I}_z} \langle R, \operatorname{curl}(\hat{v}\Upsilon_y^0)\rangle \leqslant \sum_{y\in\mathcal{I}_z} \|R\|_{Z(\omega_y)'} \|\nabla \operatorname{curl}(\hat{v}\Upsilon_y^0)\|_{L^2(\omega_y)} \\ &\leqslant \Big(\sum_{y\in\mathcal{I}_z} \|R\|^2_{Z(\omega_y)'}\Big)^{\frac{1}{2}} \Big(\sum_{y\in\mathcal{I}_z} \|\nabla \operatorname{curl}(\hat{v}\Upsilon_y^0)\|^2_{L^2(\omega_y)}\Big)^{\frac{1}{2}}.\end{aligned}$$

It remains to estimate the last factor.

We distinguish the cases $d = 2$ and $d = 3$ and start with $d = 2$. Fix $y \in \mathcal{I}_z$. The product rule provides

$$\operatorname{curl}(\hat{v}\Upsilon_y^0) = \operatorname{curl}(\hat{v})\Upsilon_y^0 + \hat{v}\operatorname{curl}\Upsilon_y^0. \tag{4.23}$$

Another application of the product rule on the first summand yields

$$\nabla(\operatorname{curl}(\hat{v})\Upsilon_y^0) = \nabla(v\Upsilon_y^0) = \nabla v \Upsilon_y^0 + v\nabla\Upsilon_y^0, \tag{4.24}$$

where $v\nabla\Upsilon_y^0$ is a multiplication of a column vector with a row vector and thus yields a 2×2 matrix. In combination with Lemma 2.46, we obtain the estimate

$$\|\nabla(\operatorname{curl}(\hat{v})\Upsilon_y^0)\|_{L^2(\omega_y)} \lesssim \|\nabla v\|_{L^2(\omega_y)} + \frac{1}{h_y}\|v\|_{L^2(\omega_y)} \leqslant \|\nabla v\|_{L^2(\tilde{\omega}_z^+)} + \frac{1}{h_y}\|v\|_{L^2(\tilde{\omega}_z^+)} \lesssim \|\nabla v\|_{L^2(\tilde{\omega}_z^+)}. \tag{4.25}$$

In the last bound, we used $h_y \eqsim h_z$ and Poincaré's inequality, which is applicable since $v = 0$ on $\partial\tilde{\omega}_z^+$ by assumption.

To estimate the second summand of (4.23), we compute

$$\nabla(\hat{v}\operatorname{curl}(\Upsilon_y^0)) = \operatorname{curl}(\Upsilon_y^0)\nabla\hat{v} + \hat{v}\,\nabla(\operatorname{curl}(\Upsilon_y^0)).$$

Here again, $\operatorname{curl}(\Upsilon_y^0)\nabla\hat{v}$ is the multiplication of a column vector with a row vector, yielding a 2×2 matrix. Therefore, we have the estimate

$$\begin{aligned}\|\nabla(\hat{v}\operatorname{curl}(\Upsilon_y^0))\|_{L^2(\omega_y)} &\leqslant \|\operatorname{curl}(\Upsilon_y^0)\nabla\hat{v}\|_{L^2(\omega_y)} + \|\hat{v}\nabla(\operatorname{curl}(\Upsilon_y^0))\|_{L^2(\omega_y)} \\ &\lesssim h_y^{-1}\|\nabla\hat{v}\|_{L^2(\omega_y)} + h_y^{-2}\|\hat{v}\|_{L^2(\omega_y)} \lesssim h_y^{-1}\|\nabla\hat{v}\|_{L^2(\tilde{\omega}_z^+)} + h_y^{-2}\|\hat{v}\|_{L^2(\tilde{\omega}_z^+)}.\end{aligned}$$

We continue the estimate with the bounds provided in Theorem 4.16:

$$\|\nabla(\hat{v}\operatorname{curl}(\Upsilon_y^0))\|_{L^2(\omega_y)} \lesssim h_y^{-1}\|\operatorname{curl}(\hat{v})\|_{L^2(\tilde{\omega}_z^+)} \lesssim \|\nabla v\|_{L^2(\tilde{\omega}_z^+)}.$$

In the last bound, we again employed Poincaré's inequality and $h_y \eqsim h_z$.

This finishes the case $d = 2$ and we turn to $d = 3$. The relation corresponding to (4.23) reads in the $3d$-setting as

$$\operatorname{curl}(\hat{v}\Upsilon_y^0) = \operatorname{curl}(\hat{v})\Upsilon_y^0 - \hat{v} \times (\nabla\Upsilon_y^0)^T.$$

Here, $(\cdot)^T$ denotes the transposed of a matrix, or in this case of a vector. This well-known formula can be verified by a simple calculation, see also [BF13, (A.5)]. We apply the ∇-operator to this expression. The resulting first summand $\nabla(\operatorname{curl}(\hat{v})\Upsilon_y^0)$ is bounded with (4.24) and (4.25) as in the $2d$-case.

For the second summand $\nabla(\hat{v} \times (\nabla\Upsilon_y^0)^T)$, we obtain with the product rule

$$\nabla(\hat{v} \times (\nabla\Upsilon_y^0)^T) = \nabla\hat{v} \times (\nabla\Upsilon_y^0)^T + \hat{v} \times D^2\Upsilon_y^0.$$

This follows from a direct computation. The cross-product of a matrix with a vector or vice versa is defined column-wise. We estimate the resulting terms with the help of the bounds in Lemma 4.19 and Theorem 4.18

$$\begin{aligned}\|\nabla(\hat{v} \times (\nabla\Upsilon_y^0)^T)\|_{L^2(\omega_y)} &\lesssim h_y^{-1}\|\nabla\hat{v}\|_{L^2(\omega_y)} + h_y^{-2}\|\hat{v}\|_{L^2(\omega_y)} \leqslant h_y^{-1}\|\nabla\hat{v}\|_{L^2(\tilde{\omega}_z^+)} + h_y^{-2}\|\hat{v}\|_{L^2(\tilde{\omega}_z^+)} \\ &\lesssim h_y^{-1}\|\operatorname{curl}\hat{v}\|_{L^2(\tilde{\omega}_z^+)} \lesssim \|\nabla v\|_{L^2(\tilde{\omega}_z^+)}.\end{aligned}$$

The last step follows again with an application of Poincaré's inequality. This completes the estimate in the $3d$-case.

In both cases, we obtain the estimate

$$\sum_{y\in\mathcal{I}_z} \|\nabla\operatorname{curl}(\hat{v}\Upsilon_y^0)\|^2_{L^2(\omega_y)} \lesssim \#\mathcal{I}_z \|\nabla v\|^2_{L^2(\tilde{\omega}_z^+)},$$

Due to the assumptions in Lemma 4.8 and the property (1.22), $\#\mathcal{I}_z$ is uniformly bounded in $z \in \mathcal{V}$ and thus C_{P} can be chosen independently of z. □

Theorem 4.20 finalizes the localization provided by Theorems 4.14 and 4.15. We obtain the equivalence of the global conforming residual in the $Z(\Omega)'$-norm to the local conforming residual in the $Z(\omega_z)'$-norm up to a conformity error. The next step is to build a projection which splits off the oscillation term and makes $\|\operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f,u_h)\|_{Z(\omega_z)'}$ computable.

4.2 Construction of a computable pressure-robust bound

We fix throughout this section a functional $f \in H^{-1}(\Omega)^d$, the corresponding analytic velocity solution $u \in H_0^1(\Omega)^d$ and the corresponding discrete solution $u_h \in Z_h$ of a quasi-optimal, pressure-robust method of degree $\ell \in \mathbb{N}$. Up to this point, we discussed the localization of the global residual $\|\operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f,u_h)\|_{Z(\Omega)'}^2$ to the local residua $\|\operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f,u_h)\|_{Z(\omega_z)'}^2$. The goal of this section is to make the local residua for a fixed $z \in \mathcal{V}$ computable.

To this end, we discuss the necessary modifications of the abstract framework in section 2.1.3 to obtain a computable pressure-robust bound on the localized residual $\|\operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f,u_h)\|_{Z(\omega_z)'}$. The major difference is that we only test with functions from $Z(\omega_z^+)$ to ensure the pressure-robustness. Therefore, we need to construct a projection operator $\mathcal{P}_z : S_z' \to D_z$, where $S_z \subset Z(\Omega)$ and D_z is a suitably chosen space of simple functionals.

We recall the recipe of section 2.1.3 to construct a stable projection operator $\mathcal{P}_z : S_z' \to D_z$: We need to specify the image of the projection $D_z \subset (Z(\omega_z)+S_z)' := \tilde{Z}_z'$ (the simple functionals) together with a space of (simple) test functions S_z. The test functions in S_z must be contained in $Z(\Omega)$ such that the velocity estimator is pressure-robust and is not affected by pressure contributions. The projection $\mathcal{P}_z$ defined via (2.22) induces a pressure-robust upper bound in the spirit of (2.28). In the pressure-robust setting of the Stokes equation, the bound reads as

$$\|\operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f,u_h)\|_{Z(\omega_z)'} \leqslant \underbrace{\|\mathcal{P}_z f - \tilde{A}u_h\|_{Z(\omega_z)'}}_{\text{estimator}} + \underbrace{\|\mathcal{P}_z f - f\|_{Z(\omega_z)'}}_{\text{oscillation}}. \tag{4.26}$$

We also get a lower bound similar to (2.30c) with an error-dominated oscillation, if the projection fulfills an adaption (2.29). This condition reads in the pressure-robust setting as

$$(\tilde{A}Z_h)_{|\tilde{Z}_z} \subset (D_z)_{|\tilde{Z}_z}. \tag{4.27}$$

Here, $\tilde{A}$ is the operator induced by the bilinear-form $\tilde{a}$. In contrast to the elliptic problems, the space $(\tilde{A}Z_h)_{|\tilde{Z}_z}$ is more involved. Therefore, we derive a preliminary version of its structure and refine it in the next subsections.

4.2.1 Construction of a space of simple functionals

To obtain a preliminary version of the space D_z for the Stokes equation, which contains $(\tilde{A}Z_h)_{|\tilde{Z}_z}$, we perform an integration by parts, similar to the Poisson case. At this point, we discuss the situation only for the case $S_z \subset Z(\omega_z)$ thus $\tilde{Z}_z = Z(\omega_z)$. In section 4.2.3, we construct an estimator, whose test functions S_z have a larger support than ω_z, we will comment on the special situation there.

For $w \in Z(\omega_z)$ and $u_h \in Z_h \subset S_\ell^0(\mathcal{T})^d$, $\ell \in \mathbb{N}$, we have that

$$\int_{\omega_z} \nabla_h u_h : \nabla w = -\sum_{T\in\mathcal{T}_z} \int_T \underbrace{\Delta u_h}_{\in\mathbb{P}_{\ell-2}(T)} \cdot w + \sum_{F\in\mathcal{F}_z^i} \int_F \underbrace{[\![\nabla_h u_h \mathsf{n}_F]\!]}_{\in\mathbb{P}_{\ell-1}(F)} \cdot w. \tag{4.28}$$

Although, this looks similar to the Poisson case, it is not the same situation. Instead of testing with $w \in H_0^1(\omega_z)^d$, we are testing with an exactly divergence-free function $w \in Z(\omega_z)$. This allows us to modify the functional in both, the element and the face contributions, without affecting the $Z(\omega_z)'$-norm. First, we discuss this effect on the elements and then on faces.

4.2.1.1 The element contributions

We start with the element contributions. The element contribution Δu_h in (4.28) for an fixed $T \in \mathcal{T}_z$ is a polynomial of degree $\ell-2$. But this polynomial cannot be completely detected by a divergence-free function with support on T. To analyze this effect, we decompose the polynomial. This decomposition is provided by the next two Lemmas, first for the $2d$ and then for the $3d$ case. Recall that we write $\mathbb{P}_{-1} = \{0\}$.

Lemma 4.21 (Polynomial decomposition in $2d$, [KVZ21, Lemma 14])**.** *Given $\ell \in \mathbb{N}_0$, $T \in \mathcal{T}$, we define for $x = (x_1, x_2)^T \in \mathbb{R}^2$ the space*

$$x^\perp \mathbb{P}_{\ell-1}(T) := \{(-x_2 r, x_1 r) \mid r \in \mathbb{P}_{\ell-1}(T)\}.$$

We have the decomposition $\mathbb{P}_\ell(T)^2 = \nabla \mathbb{P}_{\ell+1}(T) \oplus x^\perp \mathbb{P}_{\ell-1}(T)$. The operator curl $: x^\perp \mathbb{P}_{\ell-1}(T) \to \mathbb{P}_{\ell-1}(T)$ *is injective and thus surjective as the dimensions of the $x^\perp \mathbb{P}_{\ell-1}(T)$ and $\mathbb{P}_{\ell-1}(T)$ match.*

In $3d$, the Lemma reads a little different.

Lemma 4.22 (Polynomial decomposition in $3d$, [KVZ21, section 3.7])**.** *We define for $\ell \in \mathbb{N}_0$, $T \in \mathcal{T}$ and $x \in \mathbb{R}^3$ the space*

$$x \times \mathbb{P}_{\ell-1}(T)^3 := \{x \times r \mid r \in \mathbb{P}_{\ell-1}(T)^3\}.$$

Then, we have the decomposition

$$\mathbb{P}_\ell(T)^3 = \nabla \mathbb{P}_{\ell+1}(T) \oplus x \times \mathbb{P}_{\ell-1}(T)^3.$$

Moreover, the operator curl $: x \times \mathbb{P}_{\ell-1}(T)^3 \to \mathbb{P}_{\ell-1}(T)^3$ *is injective.*

As in [KVZ21, section 3.2], we fix the decomposition on a reference triangle $\hat{T}$. The decomposition on the remaining elements is then obtained through a suitable map. This well-known map and its companion, the Piola's transform, are discussed for example in [BBF13, section 2.1.3] and [Coh02, Appendix A]. We recall both maps.

For each $T \in \mathcal{T}$, let $\mathbf{F}_T : \hat{T} \to T$ be an affine map mapping the reference element onto T. Let $D\mathbf{F}_T$ be its Jacobian matrix. As in [KVZ21, (3.6)], we define for each $T \in \mathcal{T}$ and $\hat{v}, \hat{w} \in H^1(\hat{T})^d$ the maps

$$\mathcal{P}_T^{\mathrm{con}} \hat{v} := \frac{1}{|\det D\mathbf{F}_T|} D\mathbf{F}_T (\hat{v} \circ \mathbf{F}^{-1}) \in H^1(T)^d, \qquad \mathcal{P}_T^{\mathrm{cov}} \hat{w} := (D\mathbf{F}_T^{-1})^T (\hat{w} \circ \mathbf{F}^{-1}) \in H^1(T)^d.$$

Here, $(\cdot)^T$ is the transposed of a matrix. The map $\mathcal{P}_T^{\mathrm{con}}$ is also known as the Piola's transform, see [BBF13, (2.1.69)].

The two maps fulfill two identities, which are useful for our purposes. The first one stems from [KVZ21, (3.7)] and is a direct consequence of the definitions of the maps: For all $T \in \mathcal{T}$ and $\hat{v}, \hat{w} \in H^1(\hat{T})^d$, we have

$$\int_T \mathcal{P}_T^{\mathrm{con}} \hat{v} \cdot \mathcal{P}_T^{\mathrm{cov}} \hat{w} = \int_{\hat{T}} \hat{v} \cdot \hat{w}. \tag{4.29}$$

The other identity is the divergence preservation of the Piola's transform, see [KVZ21, (3.8)] and [BBF13, (2.1.71)]. For all $T \in \mathcal{T}$ and $\hat{v} \in H^1(\hat{T})^d$, we have on T that

$$\operatorname{div}(\mathcal{P}_T^{\mathrm{con}} \hat{v}) = \frac{1}{|\det D\mathbf{F}_T|} \operatorname{div}(\hat{v}) \circ \mathbf{F}^{-1}.$$

To minimize the notational differences between the $2d$ and $3d$ case and hide the connection to the reference element, we set for each $T \in \mathcal{T}$ and $\ell \in \mathbb{N}_0 \cup \{-1\}$

$$\mathbb{P}_\ell^{\mathrm{curl}}(T) := \begin{cases} \mathcal{P}_T^{\mathrm{cov}}\big(x^\perp \mathbb{P}_{\ell-1}(\hat{T})\big) & \text{if } d = 2, \\ \mathcal{P}_T^{\mathrm{cov}}\big(x \times \mathbb{P}_{\ell-1}(\hat{T})^3\big) & \text{if } d = 3. \end{cases}$$

Correspondingly, we set $\mathbb{P}_\ell^{\mathrm{curl}}(\hat{T}) := x^\perp \mathbb{P}_{\ell-1}(\hat{T})$ if $d = 2$ and $\mathbb{P}_\ell^{\mathrm{curl}}(\hat{T}) := x \times \mathbb{P}_{\ell-1}(\hat{T})^3$ if $d = 3$.

We use the two results to analyze the element contributions of (4.28). Fix $T \in \mathcal{T}$. Let $r_T \in \mathbb{P}_{\ell-2}(T)^d$, $T \in \mathcal{T}_z$ and $\hat{r}_T := \big(\mathcal{P}_T^{\mathrm{cov}}\big)^{-1}(r_T) \in \mathbb{P}_{\ell-2}(\hat{T})^d$. Choose an arbitrary $z \in Z(\omega_z)$ and set $\hat{z} := \big(\mathcal{P}_T^{\mathrm{con}}\big)^{-1}(z_{|T})$.

Using equation (4.29) and Lemma 4.21 resp. Lemma 4.22, we have

$$\int_T r_T \cdot z = \int_{\hat{T}} \hat{r}_T \cdot \hat{z} = \int_{\hat{T}} \Big(\nabla \hat{r}_T^{\nabla} \cdot \hat{z} + \hat{r}_T^{\mathrm{curl}} \cdot \hat{z} \Big), \tag{4.30}$$

with $\hat{r}_T^{\nabla} \in \mathbb{P}_{\ell-1}(\hat{T})$, $\hat{r}_T^{\mathrm{curl}} \in \mathbb{P}_{\ell-2}^{\mathrm{curl}}(\hat{T})$. The first summand can be transformed with an integration by parts into

$$\int_{\hat{T}} \nabla \hat{r}_T^{\nabla} \cdot \hat{z} = -\int_{\hat{T}} \hat{r}_T^{\nabla} \operatorname{div} \hat{z} + \sum_{F \in \mathcal{F}_{\hat{T}}} \int_F \hat{r}_T^{\nabla} \hat{z} \cdot \mathsf{n}_{\hat{T}} = \sum_{F \in \mathcal{F}_{\hat{T}}} \int_F \hat{r}_T^{\nabla} \hat{z} \cdot \mathsf{n}_{\hat{T}} = \sum_{F \in \mathcal{F}_T} \int_F \left(\hat{r}_T^{\nabla} \circ \mathbf{F}^{-1} \right) z \cdot \mathsf{n}_T. \tag{4.31}$$

The last equation is a consequence of [BBF13, (2.1.74)]. The set $\mathcal{F}_{\hat{T}}$ contains the edges of the reference element.

Viewed in the context of D_z, the last expression can be absorbed into the face contributions. With these insights, we formulate a preliminary space D_z as

$$D_z := \Big\{ \chi \in (H^{-1}(\Omega))^d \mid \chi(w) := \sum_{T\in\mathcal{T}_z} \int_T r_T^{\mathrm{curl}} \cdot w + \sum_{F\in\mathcal{F}_z^i} \int_F r_F \cdot w \\ \text{with } r_T^{\mathrm{curl}} \in \mathbb{P}^{\mathrm{curl}}_{\ell-2}(T) \text{ for } T\in\mathcal{T}_z,\ r_F \in \mathbb{P}_{\ell-1}(F)^d \text{ for } F\in\mathcal{F}_z^i \text{ and } w \in Z(\Omega) \Big\}. \tag{4.32}$$

Written in this way, we can detect for each $T \in \mathcal{T}_z$ the polynomials $r_T^{\mathrm{curl}} \in \mathbb{P}^{\mathrm{curl}}_{\ell-2}(T)$ with test functions from $Z(T)$. We construct these test functions later, first we discuss the face contributions.

4.2.1.2 The face contributions

The next step is to investigate the face contributions. In the two-dimensional case, the face contributions on each $F \in \mathcal{F}_z^i$, $z \in \mathcal{V}$ have two components, in the three-dimensional case three. It is convenient, to represent the components in the basis of the normal and tangential vectors of the face because the normal and tangential components of the face components exhibit different behaviour.

For the tangential and normal components, there is a profound difference what an $Z(\omega_z)$-function can detect and distinguish. While a $Z(\omega_z)$-function can detect the tangential components completely, this is not the case for the normal components. Some aspects of the normal components are indistinguishable for functions in $Z(\omega_z)$. If we add a constant to a normal component and compensate the modification on any other interior face within the patch, a $Z(\omega_z)$ test function cannot detect this change. A test function from $Z(\omega_z)$ can only detect certain sums of the normal components, not their particular values on particular faces.

We start to analyze this special effect of the normal components. The reasoning is simplified if we express the structure of the local mesh $\mathcal{T}_z$ with a *dual graph*. This is a classical concept in graph theory and described for example in [Die17, chapter 4.6]. The dual graph is a graph encoding the neighbourhood relations of the elements. To fix notation, we recall some aspects of graph theory.

Definition 4.23 (Graph [Die17, p. 2 and p. 28])**.** *Let* V *be a set of* nodes *and* A *a set of* arcs*. Each arc is a two element subset of* V*. The tuple* (V,A) *is called a* graph*.*

A graph $G' = (\mathsf{V}',\mathsf{A}')$ *is a subgraph of a graph* $G = (\mathsf{V},\mathsf{A})$ *if* $\mathsf{V}' \subset \mathsf{V}$ *and* $\mathsf{A}' \subset \mathsf{A}$*, in notation* $G' \subset G$*.*

Within the graph we consider paths, cycles and trees.

Definition 4.24 ([Die17, pp. 6-14])**.** *Given a graph* $G = (\mathsf{V},\mathsf{A})$*, we call a sequence of pairwise different graph nodes* $x_0,...,x_n \in \mathsf{V}$ *with* $n \in \mathbb{N}_0$ *and* $\{x_i, x_{i+1}\} \in \mathsf{A}$ *for all* $0 \leqslant i \leqslant n-1$ *a* path *of length* n*. A path* $x_0,...,x_n,x_0$ *is called a* cycle *of length* n*.*

A non-empty graph G *is* connected *if for any* $x_1, x_2 \in \mathsf{V}$ *there exists a path* $x_1,...,x_2$*. A connected graph without cycles is a* tree*. A* spanning tree T *of a graph* G *is a tree with* $T \subset G$ *with the property: for every other tree* $T' \subset G$ *with* $T \subset T'$*, we have* $T = T'$*.*

The following property of trees will become important in our investigation.

Proposition 4.25 ([Die17, Theorem 1.5.1])**.** *Given a tree* $T = (\mathsf{V},\mathsf{A})$*, there exists a unique path between two nodes. If we add* $\{x,y\}$ *to* A *for* $x,y \in \mathsf{V}$*,* $x \neq y$*, with* $\{x,y\} \notin \mathsf{A}$*, then the obtained graph contains a cycle.*

These definitions allow us to build a graph, which represents the neighbourhood properties of a finite element mesh. With this *dual graph*, we abstract the essential properties of the mesh from its concrete geometry.

Definition 4.26 (Dual graph, [Die17, chapter 4.6])**.** *Given a local triangulation* $\mathcal{T}_z$*,* $z \in \mathcal{V}$*, we define its dual graph* G_z *as the graph* $(\mathsf{V}_z,\mathsf{A}_z)$ *with* $\mathsf{V}_z := \mathcal{T}_z$ *and*

$$\mathsf{A}_z := \big\{ \{T_1,T_2\} \mid T_1,T_2 \in \mathcal{T}_z \quad \text{and} \quad \exists F \in \mathcal{F}_z \text{ with } T_1 \cap T_2 = F \big\}. \tag{4.33}$$

Given $v \in \mathsf{V}_z$*, we write* $T(v) \in \mathcal{T}_z$ *for the corresponding element. Given* $e \in \mathsf{A}_z$*, we write* $F(e) \in \mathcal{F}_z$ *for the corresponding face in the triangulation. Given a subset* $\tilde{\mathsf{A}} \subset \mathsf{A}_z$*, we define* $F(\tilde{\mathsf{A}}) := \{F(e) \mid e \in \tilde{\mathsf{A}}\}$*.*

As $\mathcal{T}_z$ contains at least one element, the dual graph is non-trivial.

Remark 4.27. We comment on the two-dimensional case. Within a patch ω_z, $z \in \mathcal{V}$, each element $T \in \mathcal{T}_z$ has at most two neighbours sharing a common face. Therefore, for each node $\mathsf{v} \in \mathsf{V}_z$, there exist at most two arcs containing the node. Because we assume that Ω has Lipschitz continuous boundary, the dual graph of $\mathcal{T}_z$ has to be connected. Hence, the dual graph is either a circle or all its vertices together form a path that is not a cycle. Accordingly, every spanning tree in the dual graph of $\mathcal{T}_z$ contains either all arcs (vertices form a path but no cycle) or all but one arc (vertices form a cycle).

Now, we fix a $z \in \mathcal{V}$. We consider the dual graph $G_z = (\mathsf{V}_z, \mathsf{A}_z)$ of the mesh $\mathcal{T}(\omega_z)$, $z \in \mathcal{V}$. Within this graph, we choose a spanning tree $S_z = (\mathsf{V}_z^S, \mathsf{A}_z^S)$, which might not be unique. This spanning tree allows us to characterize the constant parts of the normal components of the simple edge functionals in the space D_z, which was defined in (4.32).

Theorem 4.28. *Given $z \in \mathcal{V}$ and a local mesh $\mathcal{T}_z$ with the dual graph $G_z = (\mathsf{V}_z, \mathsf{A}_z)$ and a corresponding spanning tree $S_z = (\mathsf{V}_z^S, \mathsf{A}_z^S)$, we define $\chi \in Z(\omega_z)'$ as*

$$\langle \chi, w \rangle := \sum_{F \in \mathcal{F}_z^i} \int_F r_F \mathsf{n}_F \cdot w, \quad \forall w \in Z(\omega_z) \tag{4.34}$$

for $r_F \in \mathbb{R}$, $F \in \mathcal{F}_z^i$. There exists a function $\xi \in S_0^0(\mathcal{T}_z)$ such that $\langle \nabla \xi, w \rangle = 0$ in a distributional sense for all $w \in Z(\omega_z)$ and

$$\langle \chi, w \rangle = \langle \chi + \nabla \xi, w \rangle = \sum_{F \in (\mathcal{F}_z^i \backslash F(\mathsf{A}_z^S))} \int_F \tilde{r}_F \mathsf{n}_F \cdot w \quad \forall w \in Z(\omega_z). \tag{4.35}$$

for $\tilde{r}_F \in \mathbb{R}$, $F \in (\mathcal{F}_z^i \backslash F(\mathsf{A}_z^S))$.

Remark 4.29. In a two-dimensional situation, the setting simplifies as already pointed out in Remark 4.27. We distinguish for $z \in \mathcal{V}$ two cases. If $z \in \partial\Omega$ then $\mathcal{F}_z^i = F(\mathsf{A}_z^S)$. If $z \in \Omega$ then the set $F \in (\mathcal{F}_z^i \backslash F(\mathsf{A}_z^S))$ contains only one face.

Proof. Before constructing ξ, we set up some additional facts about graphs.

For the rest of the proof, fix a node $R \in \mathsf{V}_z^S$. (Traditionally, this node is called *root*.) By Proposition 4.25, any node $\mathsf{v} \in \mathsf{V}_z^S \backslash \{R\}$ induces a unique path $\mathsf{path}(\mathsf{v})$ in the spanning tree S_z which connects R with v. The length of $\mathsf{path}(\mathsf{v})$ defines the *level* $L(\mathsf{v})$ of v. We set $L(R) = 0$. Since the graph is finite, there is a maximal level, which we write as $L_{\max} \in \mathbb{N}_0$.

Thus, for each node $\mathsf{v} \in \mathsf{V}_z^S \backslash \{R\}$, we have a unique path $\mathsf{path}(\mathsf{v})$. As $L(\mathsf{v}) \geqslant 1$, there is another unique node $\tilde{\mathsf{v}} \in \mathsf{V}_z^S$, $\tilde{\mathsf{v}} \neq \mathsf{v}$, with $L(\tilde{\mathsf{v}}) = L(\mathsf{v}) - 1$ such that $\mathsf{e} = \{\mathsf{v}, \tilde{\mathsf{v}}\}$ is an arc in $\mathsf{path}(\mathsf{v})$. This assignment of the node v to an arc e defines the map $\mathsf{pe} : \mathsf{V}_z^S \backslash \{R\} \to \mathsf{A}_z^S$. We show that pe is bijective.

If there is an arc $\tilde{\mathsf{e}} = \{\mathsf{v}_1, \mathsf{v}_2\} \in \mathsf{A}_z^S$; $\mathsf{v}_1, \mathsf{v}_2 \in \mathsf{V}_z^S$; $\mathsf{v}_1 \neq \mathsf{v}_2$, which is not contained in $\mathsf{path}(\mathsf{v})$ for any $\mathsf{v} \in \tilde{\mathsf{e}}$, then this $\tilde{\mathsf{e}}$ together with $\mathsf{path}(\mathsf{v}_1)$ and $\mathsf{path}(\mathsf{v}_2)$ form a loop. This is a contradiction to S_z being a tree. Hence, pe is surjective.

If there are two $\mathsf{v}, \tilde{\mathsf{v}} \in \mathsf{V}_z^S \backslash \{R\}$ with $\mathsf{v} \neq \tilde{\mathsf{v}}$ and $\mathsf{pe}(\mathsf{v}) = \mathsf{pe}(\tilde{\mathsf{v}})$, then $\mathsf{v} \in \mathsf{pe}(\tilde{\mathsf{v}})$ and $\tilde{\mathsf{v}} \in \mathsf{pe}(\mathsf{v})$. Therefore, $L(\tilde{\mathsf{v}}) = L(\mathsf{v}) - 1$ and $L(\mathsf{v}) = L(\tilde{\mathsf{v}}) - 1$, a contradiction. Thus, pe is a bijective map.

For convenience, we say that $\mathsf{pe}(\mathsf{v})$ has the level $L(\mathsf{v})$. Since pe is bijective, each arc has a level, and we write $L(\mathsf{e})$ for the level of the arc $\mathsf{e} \in \mathsf{A}_z^S$.

We turn to the construction of the function ξ. The function ξ is built in such a way that adding $\nabla\xi$ eliminates the face contributions in (4.34) whose corresponding arcs are in the spanning tree.

We perform the elimination step-wise via a sequence of $L_{\max}$ intermediate functionals χ_i, $1 \leqslant i \leqslant L_{\max}$. In each step, we eliminate the undesired face contributions of a specific level. Hence, the functionals χ_i shall only contain contributions of a face if the face is outside the spanning tree or has a level larger than i.

Consequently, with the convention $\chi_0 := \chi$, the intermediate functional χ_i is expressible in the form

$$\langle \chi_i, v \rangle = \sum_{k=i+1}^{L_{\max}} \sum_{\mathsf{v} \in L^{-1}(k)} \int_{F(\mathsf{pe}(\mathsf{v}))} r^i_{F(\mathsf{pe}(\mathsf{v}))} \mathsf{n}_{F(\mathsf{pe}(\mathsf{v}))} \cdot v + \sum_{F \in (\mathcal{F}_z^i \backslash F(\mathsf{A}_z^S))} \int_F r_F^i \mathsf{n}_F \cdot v, \quad v \in Z(\omega_z), \tag{4.36}$$

$0 \leqslant i \leqslant L_{\max}$. The values $r_F^i \in \mathbb{R}$, $F \in \mathcal{F}_z^i$, might be different in each χ_i.

For each $1 \leqslant i \leqslant L_{\max}$, the functional χ_i is obtained from χ_{i-1} by adding the function $\nabla\xi_i$. More precisely, the functionals are obtained via the recursion $\chi_i = \chi_{i-1} + \nabla\xi_i$, $1 \leqslant i \leqslant L_{\max}$, with $\chi_0 := \chi$. Therefore, we construct for each $1 \leqslant i \leqslant L_{\max}$ a function ξ_i. In the recursion, the function ξ_i has the purpose to eliminate the facial contributions which correspond to arcs of level i in the spanning tree S_z.

In particular, we define the function $\xi_i : \omega_z \to \mathbb{R}^d$, $1 \leqslant i \leqslant L_{\max}$, as

$$\xi_i = -\sum_{\mathsf{v}\in L^{-1}(i)} 1_{T(\mathsf{v})} r^{i-1}_{F(\mathsf{pe}(\mathsf{v}))} \mathsf{n}_{F(\mathsf{pe}(\mathsf{v}))} \cdot \mathsf{n}_{T(\mathsf{v})},$$

where $1_{T(\mathsf{v})} : \overline{\Omega} \to \mathbb{R}^d$ is the indicator function of the set $T(\mathsf{v})$.

We show that the resulting $\chi_i = \chi_{i-1} + \nabla\xi_i$ is of the form (4.36). To this end, we compute the distributional derivative of the functional ξ_i:

$$\langle \nabla\xi_i, v\rangle = -\sum_{\mathsf{v}\in L^{-1}(i)} \sum_{F\in\mathcal{F}_{T(\mathsf{v})}} \int_F r^{i-1}_{F(\mathsf{pe}(\mathsf{v}))} \mathsf{n}_{F(\mathsf{pe}(\mathsf{v}))} \cdot \mathsf{n}_{T(\mathsf{v})} \mathsf{n}_{T(\mathsf{v})} \cdot v, \quad v \in Z(\omega_z).$$

For a fixed $\mathsf{v} \in L^{-1}(i)$, the set $\mathcal{F}_{T(\mathsf{v})}$ contains $F(\mathsf{pe}(\mathsf{v}))$. All faces in $(\mathcal{F}_{T(\mathsf{v})}\backslash\{F(\mathsf{pe}(\mathsf{v}))\}) \cap \mathcal{F}^i_z$ have a level larger than i. Therefore, adding $\nabla\xi_i$ to (4.36) eliminates all contributions of the faces $F(\mathsf{pe}(\mathsf{v}))$, $\mathsf{v} \in L^{-1}(i)$ and only modifies contributions of faces which have a higher level or are outside the spanning tree. But especially it does not affect arcs in the spanning tree of a level smaller than i. Accordingly, $\chi_{i-1} + \nabla\xi_i$ has the representation (4.36). Thus, we have with possible new $r^i_F \in \mathbb{R}$ for all $1 \leqslant i \leqslant L_{\max}$

$$\langle \chi + \sum_{j=1}^{i} \nabla\xi_i, v\rangle = \sum_{k=i+1}^{L_{\max}} \sum_{\mathsf{v}\in L^{-1}(k)} \int_{F(\mathsf{pe}(\mathsf{v}))} r^i_{F(\mathsf{pe}(\mathsf{v}))} \mathsf{n}_{F(\mathsf{pe}(\mathsf{v}))} \cdot v + \sum_{F\in(\mathcal{F}^i_z\backslash F(\mathsf{A}^S_z))} \int_F r^i_F \mathsf{n}_F \cdot v, \quad v \in Z(\omega_z).$$

□

Remark 4.30. The result in Theorem 4.28 is inspired and related to the theory of gain graphs as introduced in [Zas89]. Actually, one can view Theorem 4.28 as an adaption of [Zas89, Corollary 3.2] to our situation. Moreover, the construction is similar to the construction of a divergence-free basis for the Crouzeix-Raviart element, c.f. Remark 4.41 for $2d$ and Remark 4.47 for $3d$.

Theorem 4.28 shows that not all parts of the lowest order normal components on the faces can be distinguished in the $Z(\omega_z)'$-norm. In contrast, the higher order components of the normal components of the faces can be detected completely, as we will show later. To express this discrepancy in the space D_z, we introduce spaces of special polynomials for $\ell \in \mathbb{N}_0$ and $F \in \mathcal{F}^i_z$:

$$\hat{\mathbb{P}}_\ell(F) := \Big\{ p \in \mathbb{P}_\ell(F) \,\Big|\, \int_F p = 0 \Big\} \quad \text{and} \quad \hat{\mathbb{P}}^{\mathsf{n}}_\ell(F) := \Big\{ p \in (\mathbb{P}_\ell(F))^d \,\Big|\, \int_F p \cdot \mathsf{n}_F = 0 \Big\}.$$

We are now in the position to write down a suitable space D_z, which can be used in the recipe of Assumption 2.19 and Proposition 2.20.

Theorem 4.31. *Let $z \in \mathcal{V}$ and ω_z be the corresponding patch with $(\mathsf{V}^S_z, \mathsf{A}^S_z)$ a spanning tree of the dual graph of $\mathcal{T}_z$. The space*

$$D^{\mathcal{S}}_z := \Big\{ \chi \in Z(\Omega)' \,\Big|\, \langle\chi, w\rangle = \sum_{T\in\mathcal{T}_z} \int_T r_T \cdot w + \sum_{F\in\mathcal{F}^i_z} \int_F r_F \cdot w + \sum_{F\in(\mathcal{F}^i_z\backslash F(\mathsf{A}^S_z))} \int_F r^c_F \mathsf{n}_F \cdot w$$
$$\text{for } w \in Z(\Omega), \text{ for } r_T \in \mathbb{P}^{\mathrm{curl}}_{\ell-2}(T),\ T \in \mathcal{T}_z, \text{ for } r_F \in \hat{\mathbb{P}}^{\mathsf{n}}_{\ell-1}(F),\ F \in \mathcal{F}^i_z \text{ and } r^c_F \in \mathbb{R}, F \in \mathcal{F}^i_z\backslash F(\mathsf{A}^S_z) \Big\} \tag{4.37}$$

fulfills (4.27) *for all considered methods with $\tilde{V}_z = Z(\omega_z)$. Given a functional $\chi \in D^{\mathcal{S}}_z$ with the same variables as above, we have the upper L^2-bound*

$$\|\chi\|^2_{Z(\omega_z)'} \lesssim \sum_{T\in\mathcal{T}_z} h_z^2 \|r_T\|^2_{L^2(T)} + \sum_{F\in\mathcal{F}^i_z} h_z \|r_F\|^2_{L^2(F)} + \sum_{F\in\mathcal{F}^i_z\backslash F(\mathsf{A}^S_z)} h_z \|r^c_F\|^2_{L^2(F)}. \tag{4.38}$$

Proof. Given $u_h \in Z_h$ and $w \in Z(\omega_z)$, integration by parts yields the representation

$$\int_{\omega_z} \nabla_h u_h : \nabla w = -\sum_{T\in\mathcal{T}_z} \int_T \Delta_h u_h \cdot w + \sum_{F\in\mathcal{F}^i_z} \int_F [\![\nabla_h u_h \mathsf{n}_F]\!] \cdot w. \tag{4.39}$$

Up to this point, we did not exploit that w is exactly divergence-free. We change the functional in such a way that is not detectable in the $\|\cdot\|_{Z'(\omega_z)}$-norm.

First, we see that $\Delta_h u_h$ can be replaced by a suitable $r_T \in \mathbb{P}^{\mathrm{curl}}_{\ell-2}(T)$ owing to simplification that led to (4.32). Second, we split $[\![\nabla_h u_h \mathsf{n}_F]\!] \in \mathbb{P}_{\ell-1}(F)^d$ for each $F \in \mathcal{F}_z^i$ into polynomials $\hat{r}_F \in \hat{\mathbb{P}}^{\mathsf{n}}_{\ell-1}(F)$ and remainders $r_F^c \in \mathbb{R}$ in the following way

$$\sum_{F\in\mathcal{F}_z^i} \int_F [\![\nabla_h u_h \mathsf{n}_F]\!] \cdot w = \sum_{F\in\mathcal{F}_z^i} \int_F \hat{r}_F \cdot w + \sum_{F\in\mathcal{F}_z^i} \int_F r_F^c \mathsf{n}_F \cdot w.$$

The last summand can be rewritten with Theorem 4.28, and we obtain with $w \in Z(\omega_z)$

$$\sum_{F\in\mathcal{F}_z^i} \int_F r_F^c \mathsf{n}_F \cdot w = \sum_{F\in(\mathcal{F}_z^i \setminus F(\mathsf{A}_z^S))} \int_F \hat{r}_F^c \mathsf{n}_F \cdot w,$$

for new $\hat{r}_F^c \in \mathbb{R}$. Due to the construction, D_z^{S} fulfills (4.27) if $S_z \subset Z(\omega_z)$.

The L^2-estimate follows in the same way as in Lemma 2.57. □

In the case of the Stokes equation, it is customary to use the same classical oscillation defined in (2.17), see for example [Ver89, Ver91, CPR13, CKPS15] and [Joh16, section 4.4].

Also in the Stokes setting, we would like to have an oscillation which is bounded by the classical one as pointed out in Corollary 2.63 and Corollary 2.76. Let us say that we would like to enhance D_z^{S} in such a way that the resulting projection $\mathcal{P}_z$ is invariant under the functionals induced by $S_k^0(\mathcal{T}_z)$, $k \geqslant \ell - 1$. With (4.30) and (4.31), we observe that in this case the element contributions r_T, $T \in \mathcal{T}_z$ in D_z^{S} have to be in $\mathbb{P}_k^{\mathrm{curl}}(T)$ but also the face contributions r_F, $F \in \mathcal{F}_z^i$, have to be in $\mathbb{P}_{k+1}(F)$.

This entails that a corresponding test function space S_z not only needs additional test functions for each element (as in the elliptic examples but only if $k \geqslant 1$) but also for each face. Yet, adding test functions for the faces hinders us to completely hide these test functions with a local Galerkin orthogonality as shown in 4.2.3. It is still possible to hide some test functions but the construction and implementation are more involved hence we do not pursue this further. Therefore, we use the non-enhanced space of simple functionals D_z^{S} and accept the drawback that we cannot bound the new oscillation by the classical one.

Anyway, the final goal is not to bound the new oscillation term by the classical one but to ensure that the oscillation term converges with a higher rate than the error. For $\ell = 1$, this can also be ensured by the following argument.

Indeed, the regularity assumption $f \in H(\mathrm{curl})$ for the pressure-robust estimator in [LMS19] yields a second order convergence rate of the new oscillation. To see this, we need results similar to Theorems 4.16 and 4.18. However, as they differ in some regard, we provide the details.

Fix $z \in \mathcal{V}$ and assume that ω_z is simply connected. In two-dimensions, [GR79, Theorem 3.1] shows that for each $z \in Z(\omega_z)$ there exists a $\hat{z} \in H^1(\omega_z)$ with $\mathrm{curl}(\hat{z}) = z$ because ω_z is simply connected. The function $\hat{z}$ is further characterized on [GR79, p. 39]: We even have $\partial_{\mathsf{t}_{\omega_z}} \hat{z} = 0$ on $\partial\omega_z$. Here t_{ω_z} is the tangential vector as defined Lemma 3.4. Since $\hat{z}$ is only unique up to an additive constant, we can assume that $\hat{z} = 0$ on $\partial\omega_z$. Therefore, we have with Poincaré's inequality and $|\mathrm{curl}\,\hat{z}| = |\nabla\hat{z}|$ (similar to (4.19)) that

$$\|\hat{z}\|_{L^2(\omega_z)} \lesssim h_z \|\nabla\hat{z}\|_{L^2(\omega_z)} = h_z \|\mathrm{curl}\,\hat{z}\|_{L^2(\omega_z)} = h_z \|z\|_{L^2(\omega_z)}. \tag{4.40}$$

In three dimensions, [GR79, Theorem 3.6] shows that for each $z \in Z(\omega_z)$ there exists a $\hat{z} \in H(\mathrm{curl};\omega_z)$ with $\mathrm{curl}(\hat{z}) = z$ and $\hat{z} \times \mathsf{n}_{\omega_z} = 0$ on $\partial\omega_z$. The L^2-norm of $\mathrm{curl}(\hat{z})$ is bounded in [GR79, Lemma 3.4] as

$$\|\hat{z}\|_{L^2(\omega_z)} \lesssim h_z \|\mathrm{curl}(\hat{z})\|_{L^2(\omega_z)} = h_z \|z\|_{L^2(\omega_z)}. \tag{4.41}$$

Independent of the dimension, we conclude for $f \in H(\mathrm{curl};\Omega)$, $w \in Z(\omega_z)$ and its stream function $\hat{w}$ with $\mathrm{curl}(\hat{w}) = w$ with the integration by parts formulas (3.3) resp. (3.2) that

$$\int_{\omega_z} f \cdot w = \int_{\omega_z} f \cdot \mathrm{curl}(\hat{w}) = \int_{\omega_z} \mathrm{curl}(f) \cdot \hat{w} \leqslant \|\mathrm{curl}\, f\|_{L^2(\omega_z)} \|\hat{w}\|_{L^2(\omega_z)}.$$

The bounds (4.40) and (4.41) show with Poincaré's inequality that

$$\|\hat{w}\|_{L^2(\omega_z)} \lesssim h_z \|\mathrm{curl}(\hat{w})\|_{L^2(\omega_z)} = h_z \|w\|_{L^2(\omega_z)} \lesssim h_z^2 \|\nabla w\|_{L^2(\omega_z)}.$$

Consequently, we have for $f \in H(\mathrm{curl})$ that

$$\|f - \mathcal{P}_z f\|_{Z(\omega_z)'} \lesssim \|f\|_{Z(\omega_z)'} \lesssim h_z^2 \|\mathrm{curl}\, f\|_{L^2(\omega_z)}$$

with an abstract, stable projection $\mathcal{P}_z : Z(\omega_z)' \to Z(\omega_z)'$. If $S_z \not\subset Z(\omega_z)'$, a similar result applies. Hence, the oscillation already converges with second order, without additions to the space of oscillation-free data.

4.2.2 A pressure-robust velocity estimator using bubble test functions

To complement the upper bound (4.38) with a lower bound, we need appropriate test functions to construct a projection operator. The test functions need to be exactly divergence-free to obtain a pressure-robust error-estimator for the velocity field.

One method to construct an exactly divergence-free test function is to use the curl-operator. In two dimensions, the curl of a sufficiently regular function $v : \omega \to \mathbb{R}$, $\omega \subset \Omega$, is divergence-free because $\operatorname{div}\operatorname{curl} v = \partial_1\partial_2 v - \partial_2\partial_1 v = 0$. In three dimensions, a similar computation shows that $\operatorname{curl} v$ of a vector field $v : \omega \to \mathbb{R}^3$ is divergence-free.

Another approach to obtain a function in $Z(\omega_z^+)$ is to construct a function $w_h \in Z_h$ with $\operatorname{supp}(w_h) \subset \omega_z$ and then use a divergence correction operator (e.g. a smoother) to obtain an exactly divergence free function. Using these functions might simplify the estimator by exploiting a local quasi-Galerkin orthogonality, see Definition 2.67.

We exemplify both techniques. First, we construct a set of test functions using the curl-operator and afterwards we show how one obtains test functions for the faces with a smoothing operator.

We start with the curl-operator construction, at first only for $2d$ and discuss the three-dimensional case later. With the curl-operator, we construct appropriate functions which test the degrees of freedom of D_z^{S}. This construction is understandably more involved than the ones in chapter 2. We first construct the functions and then prove that they form a set of test functions.

First, we construct test functions to detect the element contributions for $d = 2$. Fix $T \in \mathcal{T}_z$. To obtain an exactly divergence-free $H_0^1(T)$-function with the curl-operator, we need a more regular function before we apply the curl-operator. Therefore, we define a $H_0^2(T)$ bubble $\Psi_{2,T} := \Psi_T^2$ as the square of the $H_0^1(T)$ bubble Ψ_T. Then $\operatorname{curl}(\Psi_{2,T}) \in H_0^1(T)^2 \cap Z(T)$, as needed. The bubble $\Psi_{2,T}$ allows us to build test functions for the element-interior degrees of freedom in D_z^{S}. For $T \in \mathcal{T}_z$, these test functions read as $\operatorname{curl}(\operatorname{curl}(q)\Psi_{2,T}) \in Z(T) \subset Z(\omega_z)$ with $q \in \mathbb{P}_{\ell-2}^{\operatorname{curl}}(T)$. These functions detect all degrees of freedom in the interior of T in D_z^{S}, which is shown in Theorem 4.33.

After that, we turn to the construction of the facial test functions in two dimensions. The facial polynomials $r_F \in \mathbb{P}_{\ell-1}(F)^2$, $F \in \mathcal{F}_z^i$, have two components. We express these vectors in the basis formed by the normal n_F and tangential vector t_F, see Definition 1.55, written as $(\mathsf{t}_F, \mathsf{n}_F)$. We assume that this basis is chosen in such a way that $(\mathsf{t}_F, \mathsf{n}_F)$ can be mapped onto (e_1, e_2) by an orthogonal, orientation preserving, affine map. Thus, for a vector field $v \in H^1(\omega_F)$, the operator $\operatorname{curl}(v)$ expressed in the coordinates $(\mathsf{t}_F, \mathsf{n}_F)$ reads as $(\partial_{\mathsf{n}_F} v, -\partial_{\mathsf{t}_F} v)^T$ in the coordinates $(\mathsf{t}_F, \mathsf{n}_F)$.

For both directions, $\mathsf{t}_F, \mathsf{n}_F$, $F \in \mathcal{F}_z^i$, we construct different test functions. They are both based upon the bubble $\Psi_{2,F}$, which was introduced in (1.77) as a part of the smoothing operator $\mathcal{E}_{\mathsf{hct}}$. The functions $E_F(q)\Psi_{2,F}$ with $q \in \mathbb{P}_{\ell-2}(F)$ detect the normal components on $F \in \mathcal{F}_z^i$ (except the lowest order components). The operator E_F is defined in (2.114).

The more elaborated bubbles $E_F(q)\overline{\Phi}_{\mathsf{n}_F}$ with $q \in \mathbb{P}_{\ell-1}(F)$ detect the tangential components of $F \in \mathcal{F}_z^i$. The function $\overline{\Phi}_{\mathsf{n}_F}$ is defined in Corollary 1.59. Together, these two test functions detect the components on the faces except the constants of the normal components, these need special treatment.

We already saw in the construction of D_z^{S} that the constants in the normal components of different faces are not linear independent within $Z(\omega_z)'$. This resembles in the test function space. Test Functions to detect the lowest order normal components are only needed for the faces in $(\mathcal{F}_z^i \backslash F(\mathsf{A}_z^S))$. Here, A_z^S are the arcs of a spanning tree S_z in the dual graph of $\mathcal{T}_z$.

We now construct a test function in two dimensions for one $F \in \mathcal{F}_z^i \backslash F(\mathsf{A}_z^S)$. Recall that in the two-dimensional case this set contains at most one face, see Remark 4.27. As before, we write λ_z^T, $z \in \mathcal{V}$, for the barycentric coordinate function of T, which is 1 in z, compare with its definition on page 19. We extend these functions to the whole $\mathbb{R}^d$. This allows us to define for each $F \in \mathcal{F}_z^i \backslash F(\mathsf{A}_z^S)$ a function $\Psi_{n,F} \in H_0^2(\omega_z)$

$$\Psi_{n,F}(x) = \begin{cases} \left(\prod_{T \in \mathcal{T}_z} \lambda_z^T(x)\right)^2 & \text{if } x \in \omega_z, \\ 0 & \text{else.} \end{cases} \tag{4.42}$$

After all test functions are available, we can define the space of simple test functions for $z \in \mathcal{V}$ and the spanning tree $S_z = (\mathsf{V}_z^S, \mathsf{A}_z^S)$

$$S_z^{\mathsf{SB}} :=$$
$$\operatorname{span}\Big(\{\operatorname{curl}(\operatorname{curl}(q_T)\Psi_{2,T}) \mid q_T \in \mathbb{P}_{\ell-2}^{\operatorname{curl}}(T);\ \ T \in \mathcal{T}_z\} \cup \{\operatorname{curl}(E_F(q_F^{\mathsf{n}})\Psi_{2,F}) \mid q_F^{\mathsf{n}} \in \mathbb{P}_{\ell-2}(F);\ \ F \in \mathcal{F}_z^i\}$$
$$\cup \{\operatorname{curl}(E_F(q_F^{\mathsf{t}})\overline{\Phi}_{\mathsf{n}_F}) \mid q_F^{\mathsf{t}} \in \mathbb{P}_{\ell-1}(F);\ \ F \in \mathcal{F}_z^i\} \cup \{\operatorname{curl}(\Psi_{n,F}) \mid F \in \mathcal{F}_z^i \backslash F(\mathsf{A}_z^S)\}\Big).$$

Here, E_F is the polynomial extension operator defined in (2.114).

The space S_z^{SB} induces in combination with D_z^{S} a suitable projection operator. Before proving the stability of the projection induced by the pair D_z^{S} and S_z^{SB}, we need the following auxiliary Lemma.

Lemma 4.32. *For $d = 2, 3$ and $\ell \in \mathbb{N}$ with $\ell \geqslant 2$, let $T \in \mathcal{T}$ and $r_T \in \mathbb{P}_{\ell-2}^{\mathrm{curl}}(T)$ be arbitrary. We have*

$$\|r_T\|_{L^2(T)}^2 \lesssim h_T^2 \|\operatorname{curl}(r_T)\|_{L^2(T)}^2.$$

The hidden constant depends only on d, ℓ and $\sigma(\mathcal{T})$.

Proof. At first, we proof the result for $d = 3$ and comment on $d = 2$ below. We use the estimates provided in [BBF13, Lemma 2.1.10] and obtain

$$\|r_T\|_{L^2(T)} \lesssim h_T^{\frac{d}{2}-1} \|\hat{r}_T\|_{L^2(\hat{T})},$$

with $r_T = \mathcal{P}_T^{\mathrm{cov}}(\hat{r}_T)$ by the definition of D_z^{S}. Because the operator curl is injective on $\mathbb{P}_{\ell-2}^{\mathrm{curl}}(\hat{T})$ (c.f. Lemmas 4.21 and 4.22), we apply an inverse estimate followed by [BBF13, (2.1.103)]:

$$\|r_T\|_{L^2(T)} \lesssim h_T^{\frac{d}{2}-1} \|\operatorname{curl}(\hat{r}_T)\|_{L^2(\hat{T})} \lesssim h_T \|\operatorname{curl}(r_T)\|_{L^2(T)}.$$

This finishes the proof for $d = 3$.

To transfer the result to $d = 2$, we use the following argument. We extend the triangles T and $\hat{T}$ to a three-dimensional prism with height 1. The functions r_T and $\hat{r}_T$ are extended to three-dimensional vector fields in such a way that they are constant in the third dimension. The third component of both vector fields is set to 0. We extend $\mathbf{F}_T : \hat{T} \to T$ in such a way, that $D\mathbf{F}_T$ is a matrix that acts only on the first two dimensions and leaves the third invariant. With these extensions, we directly have $r_T = \mathcal{P}_T^{\mathrm{cov}}(\hat{r}_T)$ in three dimensions. Then the result follows directly for $d = 2$. □

Theorem 4.33. *Let $d = 2$, $z \in \mathcal{V}$ and $\omega_z \subset \overline{\Omega}$ with a corresponding spanning tree $S_z = (\mathsf{V}_z^S, \mathsf{A}_z^S)$ for $\mathcal{T}_z$. The pair D_z^{S} and S_z^{SB} defines a stable projection $\mathcal{P}_z^{\ell,\mathsf{BS}}$ via (2.22). The projection can be written for $f \in Z(\omega_z)'$ as*

$$\|\mathcal{P}_z^{\ell,\mathsf{BS}}(f)\|_{Z(\omega_z)'}^2 \eqsim \sum_{T \in \mathcal{T}_z} h_z^2 \|r_T\|_{L^2(T)}^2 + \sum_{F \in \mathcal{F}_z^i} h_z \|r_F\|_{L^2(F)}^2 + \sum_{F \in \mathcal{F}_z^i \setminus F(\mathsf{A}_z^S)} h_z \|r_F^c\|_{L^2(F)}^2. \tag{4.43}$$

The hidden constants depend on ℓ and $\sigma(\mathcal{T})$.

Here, $r_T \in \mathbb{P}_{\ell-2}^{\mathrm{curl}}(T)$ is defined via

$$\int_T r_T \cdot \operatorname{curl}(\operatorname{curl}(q_T)\Psi_{2,T}) = \langle f, \operatorname{curl}(\operatorname{curl}(q_T)\Psi_{2,T})\rangle \quad \forall q_T \in \mathbb{P}_{\ell-2}^{\mathrm{curl}}(T). \tag{4.44}$$

The face contributions $r_F \in \hat{\mathbb{P}}_{\ell-1}^{\mathsf{n}}(F)$ for $F \in \mathcal{F}_z^i$ are determined by

$$\int_F r_F \cdot \operatorname{curl}(E_F(q_F^{\mathsf{t}})\overline{\Phi}_{\mathsf{n}_F}) = \langle f, \operatorname{curl}(E_F(q_F^{\mathsf{t}})\overline{\Phi}_{\mathsf{n}_F})\rangle - \sum_{T \in \mathcal{T}_F} \int_T r_T \cdot \operatorname{curl}(E_F(q_F^{\mathsf{t}})\overline{\Phi}_{\mathsf{n}_F}) \quad \forall q_F^{\mathsf{t}} \in \mathbb{P}_{\ell-1}(F)$$

and

$$\int_F r_F \cdot \operatorname{curl}(E_F(q_F^{\mathsf{n}})\Psi_{2,F}) = \langle f, \operatorname{curl}(E_F(q_F^{\mathsf{n}})\Psi_{2,F})\rangle - \sum_{T \in \mathcal{T}_F} \int_T r_T \cdot \operatorname{curl}(E_F(q_F^{\mathsf{n}})\Psi_{2,F}) \quad \forall q_F^{\mathsf{n}} \in \mathbb{P}_{\ell-2}(F).$$

The last contribution $r_F^c \in \mathbb{R}$ for $F \in \mathcal{F}_z^i \setminus F(\mathsf{A}_z^S)$ is computed as

$$\int_F r_F^c \mathsf{n}_F \cdot \operatorname{curl}(\Psi_{n,F}) = \langle f, \operatorname{curl}(\Psi_{n,F})\rangle - \sum_{T \in \mathcal{T}_z} \int_T r_T \cdot \operatorname{curl}(\Psi_{n,F}) - \sum_{F \in \mathcal{F}_z^i} \int_F r_F \cdot \operatorname{curl}(\Psi_{n,F}).$$

Proof. We apply Proposition 2.20 and therefore check Assumption 2.19. Assumption 2.2 has already been shown at the beginning of section 4.1. We show that the pair of spaces D_z^{S} and S_z^{SB} fulfills (2.21). The dimension condition is fulfilled by construction.

Let $\chi \in D_z^{\mathsf{S}}$ be arbitrary with

$$\langle \chi, w\rangle = \sum_{T \in \mathcal{T}_z} \int_T r_T \cdot w + \sum_{F \in \mathcal{F}_z^i} \int_F r_F \cdot w + \sum_{F \in \mathcal{F}_z^i \setminus F(\mathsf{A}_z^S)} \int_F r_F^c \mathsf{n}_F \cdot w,$$

$w \in Z(\omega_z)$ and $r_T \in \mathbb{P}^{\mathrm{curl}}_{\ell-2}(T)$, $T \in \mathcal{T}_z$, $r_F \in \hat{\mathbb{P}}^{\mathsf{n}}_{\ell-1}(F)$, $F \in \mathcal{F}_z^i$ and $r_F^c \in \mathbb{R}$, $F \in \mathcal{F}_z^i \backslash F(\mathsf{A}_z^S)$.

We show that the L^2-representation (4.38) is bounded by $\|\chi\|_{S_z^{\mathsf{SB}\prime}}$ and start with the element contributions. With the help of Lemma 4.32, we conclude

$$\begin{aligned}\|r_T\|^2_{L^2(T)} \lesssim h_T^2 \|\operatorname{curl}(r_T)\|^2_{L^2(T)} &\lesssim h_T^2 \int_T |\operatorname{curl}(r_T)|^2 \Psi_{2,T} = h_T^2 \int_T r_T \cdot \operatorname{curl}(\operatorname{curl}(r_T)\Psi_{2,T}) \\ &\lesssim h_T^2 \|\chi\|_{S_z^{\mathsf{SB}\prime}} \|\nabla\big(\operatorname{curl}(\operatorname{curl}(r_T)\Psi_{2,T})\big)\|_{L^2(T)}.\end{aligned} \tag{4.45}$$

An inverse estimate concludes

$$h_T^2 \|\nabla\big(\operatorname{curl}(\operatorname{curl}(r_T)\Psi_{2,T})\big)\|_{L^2(T)} \lesssim h_T^{-1} \|r_T\|_{L^2(T)}. \tag{4.46}$$

We turn to the faces. For $F \in \mathcal{F}_z^i$, we split the facial component r_F into its tangential $r_F^t := r_F \cdot \mathsf{t}_F \in \mathbb{P}_\ell(F)$ and normal component $r_F^n := r_F \cdot \mathsf{n}_F \in \hat{\mathbb{P}}_\ell(F)$ with

$$\|r_F\|^2_{L^2(F)} = \|r_F^n\|^2_{L^2(F)} + \|r_F^t\|^2_{L^2(F)}.$$

We start to estimate the tangential components. For each $F \in \mathcal{F}_z^i$, we have

$$\partial_{\mathsf{t}_F}(E_F(r_F^t)\overline{\Phi}_{\mathsf{n}_F}) = \partial_{\mathsf{t}_F}(E_F(r_F^t))\overline{\Phi}_{\mathsf{n}_F} + (E_F(r_F^t))\partial_{\mathsf{t}_F}\overline{\Phi}_{\mathsf{n}_F} = 0 \quad \text{and} \quad \partial_{\mathsf{n}_F}(E_F(r_F^t)\overline{\Phi}_{\mathsf{n}_F}) = E_F(r_F^t)\Psi_{2,F}$$

on F because of Corollary 1.59 and Lemma 1.58. Therefore, an inverse estimate provides

$$\begin{aligned}\|r_F^t\|^2_{L^2(F)} &\lesssim \int_F r_F^t E_F(r_F^t)\Psi_{2,F} = \int_F r_F^t \partial_{\mathsf{n}_F}(E_F(r_F^t)\overline{\Phi}_{\mathsf{n}_F}) \\ &= \int_F r_F^t \partial_{\mathsf{n}_F}(E_F(r_F^t)\overline{\Phi}_{\mathsf{n}_F}) - r_F^n \partial_{\mathsf{t}_F}(E_F(r_F^t)\overline{\Phi}_{\mathsf{n}_F}) = \int_F r_F \cdot \operatorname{curl}(E_F(r_F^t)\overline{\Phi}_{\mathsf{n}_F}).\end{aligned}$$

The lowest order normal components cannot be detected by $\operatorname{curl}(E_F(r_F^t)\overline{\Phi}_{\mathsf{n}_F})$ because

$$\int_F r_F^c \partial_{\mathsf{t}_F}(E_F(r_F^t)\overline{\Phi}_{\mathsf{n}_F}) = 0$$

by an integration by parts.

Thus, we only need to add the element contributions of the functional χ and obtain:

$$\begin{aligned}\|r_F^t\|^2_{L^2(F)} &\lesssim \langle \chi, \operatorname{curl}(E_F(r_F^t)\overline{\Phi}_{\mathsf{n}_F})\rangle - \sum_{T\in\mathcal{T}_F} \int_T r_T \cdot \operatorname{curl}(E_F(r_F^t)\overline{\Phi}_{\mathsf{n}_F}) \\ &\lesssim \Big(\|\chi\|^2_{S_z^{\mathsf{SB}\prime}} + h_z^2 \sum_{T\in\mathcal{T}_F} \|r_T\|^2_{L^2(T)}\Big)^{\frac{1}{2}} \|\nabla \operatorname{curl}(E_F(r_F^t)\overline{\Phi}_{\mathsf{n}_F})\|_{L^2(\omega_F)}.\end{aligned} \tag{4.47}$$

The terms $\|r_T\|_{L^2(T)}$ are estimated as in (4.45) and (4.46). For the last factor in (4.47), we have with inverse estimates

$$\|\nabla \operatorname{curl}(E_F(r_F^t)\overline{\Phi}_{\mathsf{n}_F})\|_{L^2(\omega_F)} \lesssim h_F^{-2} \|E_F(r_F^t)\overline{\Phi}_{\mathsf{n}_F}\|_{L^2(\omega_F)} \lesssim h_F^{-\frac{1}{2}} \|r_F^t\|_{L^2(F)}$$

because the function η_F in $\overline{\Phi}_{\mathsf{n}_F}$ can be estimated as $\|\eta_F\|_{L^\infty(\omega_F)} \lesssim h_z$, see Lemma 1.60. This finishes the estimate of the tangential components r_F^t, $F \in \mathcal{F}_z^i$.

The normal component is now estimated with Poincaré's inequality and $\int_F r_F^n = 0$:

$$\begin{aligned}\|r_F^n\|^2_{L^2(F)} &\lesssim h_F^2 \|\partial_{\mathsf{t}_F} r_F^n\|^2_{L^2(F)} \lesssim h_F^2 \int_F (\partial_{\mathsf{t}_F} r_F^n)^2 \Psi_{2,F} = h_F^2 \int_F r_F^n \partial_{\mathsf{t}_F}(\partial_{\mathsf{t}_F}(r_F^n)\Psi_{2,F}) \\ &= h_F^2 \Big(\int_F r_F \cdot \operatorname{curl}(\partial_{\mathsf{t}_F}(-E_F r_F^n)\Psi_{2,F}) + \int_F r_F^t \mathsf{t}_F \cdot \operatorname{curl}(\partial_{\mathsf{t}_F}(E_F r_F^n)\Psi_{2,F})\Big) \\ &= h_F^2 \Big(\langle \chi, \operatorname{curl}(\partial_{\mathsf{t}_F}(-E_F r_F^n)\Psi_{2,F})\rangle + \int_F r_F^t \mathsf{t}_F \cdot \operatorname{curl}(\partial_{\mathsf{t}_F}(E_F r_F^n)\Psi_{2,F}) \\ &\qquad + \sum_{T\in\mathcal{T}_F} \int_T r_T \cdot \operatorname{curl}(\partial_{\mathsf{t}_F}(E_F r_F^n)\Psi_{2,F})\Big) \\ &\lesssim h_F^2 \big(\|\chi\|^2_{S_z^{\mathsf{SB}\prime}} + h_F \|r_F^t\|^2_{L^2(F)} + \sum_{T\in\mathcal{T}_F} h_T^2 \|r_T\|^2_{L^2(T)}\big)^{\frac{1}{2}} \|\nabla\big(\operatorname{curl}(\partial_{\mathsf{t}_F}(E_F r_F^n)\Psi_{2,F})\big)\|_{L^2(\omega_F)}.\end{aligned}$$

The contributions $h_F\|r_F^t\|^2_{L^2(F)}$ and $\sum_{T\in\mathcal{T}_F} h_T^2\|r_T\|^2_{L^2(T)}$ are estimated as before in (4.47) and (4.45). The last factor is estimated by an inverse estimate as

$$h_F^2\|\nabla\big(\operatorname{curl}(\partial_{\mathsf{t}_F}(E_F r_F^n)\Psi_{2,F})\big)\|_{L^2(\omega_F)} \lesssim h^{-\frac{1}{2}}\|r_F^n\|_{L^2(F)}.$$

It remains to estimate the component $\|r_F^c\|_{L^2(F)}$ for $F\in\mathcal{F}_z^i\backslash F(\mathsf{A}_z^S)$. Owing to Remark 4.27, the set $\mathcal{F}_z^i\backslash F(\mathsf{A}_z^S)$ contains at most one element or is empty. If the set is not empty, we take that particular $F\in\mathcal{F}_z^i\backslash F(\mathsf{A}_z^S)$ and use the function $\Psi_{n,F}$, defined in (4.42). We have $\Psi_{n,F}(z)=1$ and $\Psi_{n,F}(x)=0$ for all $x\in\partial\omega_z$. Thus, $\int_F \partial_{\mathsf{t}_F}\Psi_{n,F} = \pm 1 = \alpha\in\mathbb{R}$ and then:

$$\|r_F^c\|^2_{L^2(F)} = h_F(r_F^c)^2 = \alpha h_F\int_F (r_F^c)^2\partial_{\mathsf{t}_F}\Psi_{n,F} = -\alpha h_F\int_F (r_F^c)^2\mathsf{n}_F\cdot\operatorname{curl}(\Psi_{n,F}).$$

Adding the other components of χ, we obtain

$$\begin{aligned}\|r_F^c\|^2_{L^2(F)} &\lesssim h_F\big(\langle\chi,\operatorname{curl}(-\alpha r_F^c\Psi_{n,F})\rangle + \sum_{T\in\mathcal{T}_z}\int_T r_T\cdot\operatorname{curl}(\alpha r_F^c\Psi_{n,F}) + \sum_{F\in\mathcal{F}_z^i}\int_F r_F\cdot\operatorname{curl}(\alpha r_F^c\Psi_{n,F})\big)\\ &\lesssim \big(\|\chi\|^2_{S_z^{\mathsf{SB}\prime}} + \sum_{T\in\mathcal{T}_z} h_T^2\|r_T\|^2_{L^2(T)} + \sum_{F\in\mathcal{F}_z^i} h_F\|r_F\|^2_{L^2(F)}\big)^{\frac{1}{2}} h_F\|\nabla\operatorname{curl}(r_F^c\Psi_{n,F})\|_{L^2(\omega_z)}.\end{aligned}$$

Again, an inverse estimate provides

$$h_F\|\nabla\operatorname{curl}(r_F^c\Psi_{n,F})\|_{L^2(\omega_z)} \lesssim h_F^{-\frac{1}{2}}\|r_F^c\|_{L^2(F)}. \qquad \square$$

Remark 4.34. Note the similarity to the biharmonic setting. We have one degree of freedom which is not element or side specific but patch specific. In both cases, we also use a similar test function to test this special degree of freedom.

After we have built a stable projection, we write down the estimator.

Corollary 4.35. *Let $\operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f,u_h)\in Z(\omega_z)'$, $z\in\mathcal{V}$, be the residual of the solution $u_h\in V_h$ for any inf-sup-stable, pressure-robust, quasi-optimal method discussed in section 3.3.1. For two dimensions that are S-AP, S-ATH, S-ACRL, S-SIP and S-CRL. Then the spaces D_z^{S} and S_z^{SB} (after choosing a spanning tree of the dual graph of $\mathcal{T}_z$) define a projection $\mathcal{P}_z^{\ell,\mathsf{BS}}: S_z^{\mathsf{SB}\prime}\to D_z^{\mathsf{S}}$ which induces a pressure-robust error-estimator with an error-dominated oscillation term in the sense of Proposition 2.22. The corresponding splitting reads*

$$\|\operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f,u_h)\|_{Z(\omega_z)'} \le \underbrace{\|\mathcal{P}_z^{\ell,\mathsf{BS}}(f-\tilde{A}u_h)\|_{Z(\omega_z)'}}_{\text{estimator }\eta_z} + \underbrace{\|\mathcal{P}_z^{\ell,\mathsf{BS}}f-f\|_{Z(\omega_z)'}}_{\text{oscillation}} \lesssim \|\operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f,u_h)\|_{Z(\omega_z)'}. \tag{4.48}$$

The hidden constant depends $\|\mathcal{P}_z^{\ell,\mathsf{BS}}\|_{\mathbb{L}(S_z',V_z')}$, which only depends on ℓ and $\sigma(\mathcal{T})$, c.f. Theorem 4.33.

The estimator can be rewritten in two forms. The term $\eta_z := \|\mathcal{P}_z^{\ell,\mathsf{BS}}(f-\tilde{A}u_h)\|_{Z(\omega_z)'}$ has the L^2-representation given in (4.43) *with the variables accordingly determined. Moreover, the estimator can also be written in a hierarchical form with the help of Lemma 2.26.*

Proof. Theorem 4.33 shows that the projection is well-defined. The splitting is a result of Proposition 2.22 because the space D_z^{S} fulfills (4.27) because $\tilde{Z}_z\subset Z(\omega_z)$ and the discussion in section 4.2.1. $\square$

The next Corollary clarifies the relation to the global error.

Corollary 4.36. *For a fixed $f\in H^{-1}(\Omega)^2$, let $u\in Z(\Omega)$ be the analytic velocity field of the solution of* (3.8)*. Let $u_h\in Z_h$ be the corresponding discrete solution of one of the methods S-AP, S-ACRL, S-ATH, S-SIP, S-CRL. Moreover, we assume that $\mathcal{T}$ fulfills the conditions given in Lemma 4.8 and that the corresponding $\tilde{\omega}_z^+$ is simply connected for all $z\in\mathcal{V}$. We define*

$$\eta_z^2 := \|\mathcal{P}_z^{\ell,\mathsf{BS}}f-\tilde{A}u_h\|^2_{Z(\omega_z)'}$$

which can be equivalently expressed in the hierarchical form provided by Lemma 2.26 or the L^2-representation (4.43) *with the variables determined as in Theorem 4.33. Then, we have the global estimate*

$$\|u-u_h\|^2_{\tilde{V}} \eqsim \sum_{z\in\mathcal{V}}\Big(\eta_z^2 + \|\mathcal{P}_z^{\ell,\mathsf{BS}}f-f\|^2_{Z(\omega_z)'}\Big) + \|\Pi_Z u_h - u_h\|^2_{\tilde{V}}. \tag{4.49}$$

Locally, we have the upper bound for each $z\in\mathcal{V}$

$$\eta_z^2 + \|\mathcal{P}_z^{\ell,\mathsf{BS}}f-f\|^2_{Z(\omega_z)'} \lesssim \|\nabla_h(u-u_h)\|^2_{L^2(\omega_z)}.$$

The hidden constants depend on ℓ and $\sigma(\mathcal{T})$.

Proof. This result is a combination of the previous results. At first, we split the error $\|u-u_h\|_{\tilde{V}}^2$, as demonstrated in section 4.1.1, into the conforming residual and the conformity error. Since $\mathcal{T}$ fulfills the conditions in Lemma 4.8, the localization of the conforming residual is provided by Theorem 4.14 or 4.15. An additional application of Theorem 4.20 shows that

$$\|\operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f,u_h)\|_{Z(\Omega)'} \lesssim \big(\sum_{z\in\mathcal{V}} \|\operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f,u_h)\|_{Z(\omega_z)'}^2\big)^{\frac{1}{2}} + \|\Pi_Z u_h - u_h\|_{\tilde{V}}.$$

The expression $\|\operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f,u_h)\|_{Z(\omega_z)'}$ is equivalent to the estimator and oscillation term by Corollary 4.35. This shows one direction of (4.49).

To show the other direction of (4.49), we observe that $\sum_{z\in\mathcal{V}} \|\operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f,u_h)\|_{Z(\omega_z)'}^2 \lesssim \|\operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f,u_h)\|_{Z(\Omega)'}^2$ by (4.3). The estimates (4.2) and (4.1) complete the proof of (4.49).

The local bound is obtained by using (4.48) and the local stability of the forms $\tilde{a}(\cdot,\cdot)$. □

4.2.3 A pressure-robust velocity estimator using finite element based test functions

We learned in chapter 2 that it might be beneficial if some test functions are smoothed finite element functions, i.e. functions from the space $\mathrm{E_M}(V_h)$, where $\mathrm{E_M}$ is the smoother of the method. For example in Theorem 2.69, we see that the estimator can be quantified by element contributions and the local conformity error. Because of Lemma 2.68, the evaluation of the local conformity error is simpler than the evaluation of face contributions. As pointed out on page 71, this technique has been already described in [DDPV96]. Nevertheless, the earlier paper [DDP95] already developed this technique for the Stokes setting. Therefore, in this section, we describe pressure-robust estimators for the Stokes equation which allow for the same simplification trick.

To apply this trick to the Stokes equation, we need test functions from $\mathrm{E_M}(Z_h)$. Now, the challenge is to construct discretely divergence-free functions within the finite element space which are still suitable test functions *after* smoothing. It turns out that the Crouzeix-Raviart CR_ℓ functions with ℓ odd in two dimensions are suitable for this task. Therefore, in this section, we assume $d=2$ and fix an odd $\ell\in\mathbb{N}$ to build an alternative space of test functions $S_z^{\mathsf{S\text{-}CR}}$. The construction of the space $S_z^{\mathsf{S\text{-}CR}}$ is done in two steps. First, we construct *model test functions* within the space CR_ℓ^2. Then, these functions are turned into the final test functions for the face contributions by an appropriate smoothing operator. We will comment on the case for $d=3$ at the end of section 4.2.4.

This construction has one disadvantage, which we were not able to resolve. The CR_ℓ^d model test functions that we will construct are only discretely divergence-free with respect to the space $\hat{S}_0^0(\mathcal{T})$, which is not necessarily the space Q_h of the method. Thus, the smoothing operator $\mathrm{E_M}$ of the method can only be used to map the model test functions into Z if $Q_h=\hat{S}_0^0(\mathcal{T})$. If this is not the case, we can still build a test function space with another appropriate smoother $\mathcal{E}^{\mathsf{Tf}}$ to construct $S_z^{\mathsf{S\text{-}CR}}$. We still obtain a projection operator that induces an estimator and an error-dominated oscillation. But in contrast to using $\mathrm{E_M}$ as the smoother, we cannot hide the face contributions of the estimator.

To distinguish the discrete divergence of the method induced by Q_h and the one induced by $\hat{S}_0^0(\mathcal{T})$, we introduce a new notation. The space $\hat{S}_0^0(\mathcal{T})$ induces the discrete divergence operator $\underline{\operatorname{div}}_{h,0}:\mathrm{CR}_\ell\to\hat{S}_0^0(\mathcal{T})$ of lowest order via

$$\int_\Omega \underline{\operatorname{div}}_{h,0} v_h q_h = \int_\Omega \operatorname{div}_h v_h q_h \qquad \forall q_h\in\hat{S}_0^0(\mathcal{T}). \tag{4.50}$$

We construct the new space $S_z^{\mathsf{S\text{-}CR}}$ by using an abstract bounded operator $\mathcal{E}^{\mathsf{Tf}}:\mathrm{CR}_\ell\to H_0^1(\Omega)^d$ to transform the discrete model test functions into exactly divergence-free $H_0^1(\Omega)^d$-functions. The operator $\mathcal{E}^{\mathsf{Tf}}$ shall fulfill $\operatorname{div}\mathcal{E}^{\mathsf{Tf}}v_h=\underline{\operatorname{div}}_{h,0}v_h$ for all $v_h\in\mathrm{CR}_\ell$. Moreover, $\mathcal{E}^{\mathsf{Tf}}$ shall fulfill the vectorized momentum conditions in (3.53). Such an operator $\mathcal{E}^{\mathsf{Tf}}$ actually exists because we can choose $\mathcal{E}^{\mathrm{DG_{pr}}}$ for a method with the pressure space $\hat{S}_0^0(\mathcal{T})$.

But regardless which concrete smoother $\mathcal{E}^{\mathsf{Tf}}$ one chooses, one always gets a suitable test function space $S_z^{\mathsf{S\text{-}CR}}$, which induces together with D_z^{S} a projection via (2.22). This projection can then be used to obtain an a-posteriori error-estimator with an error-dominated oscillation. If one can additionally choose $\mathcal{E}^{\mathsf{Tf}}$ in such a way that $\mathcal{E}^{\mathsf{Tf}}v_h=\mathrm{E_M}\,v_h$ for all model test functions v_h, then the evaluations of the residual with $\mathrm{E_M}\,v_h$ can be omitted with the help of a local Galerkin orthogonality and bounded by the local conformity error of Definition 2.67 after adaption to the Stokes setting.

We now turn to the construction of the space $S_z^{\mathsf{S\text{-}CR}}$. For the element contributions, we still employ the test functions $\operatorname{curl}(\operatorname{curl}(q)\Psi_{2,T})$ with $q\in\mathbb{P}_{\ell-2}^{\mathrm{curl}}(T)$.

To construct test functions for the face contributions, we use the CR_ℓ-basis functions in Definition 2.80 to construct model test functions which are discretely divergence-free with respect $\hat{S}_0^0(\mathcal{T})$.

After applying the smoothing operator $\mathcal{E}^{\mathsf{Tf}}$, we obtain the actual test functions, which are contained in $S_z^{\mathsf{S\text{-}CR}}$.

For each $F \in \mathcal{F}_z^i$, $z \in \mathcal{V}$ fixed, we construct test functions for detecting the tangential components $\int_F q_F \cdot \mathsf{t}_F$, $q_F \in \mathbb{P}_{\ell-1}(F)$ and the normal components $\int_F q_F \cdot \mathsf{n}_F$, $q_F \in \hat{\mathbb{P}}_{\ell-1}(F)$. Finally, we construct a test function which detects the lowest order normal components in ω_z. We assume $d = 2$ in this section.

For testing the tangential components on $F \in \mathcal{F}_z^i$, we use the CR_ℓ-functions $\Psi_{F,m}^{\mathsf{cr}}\mathsf{t}_F$, $1 \leqslant m \leqslant \ell$, as model test functions. The functions $\Psi_{F,m}^{\mathsf{cr}}$ are defined in Definition 2.80. These functions are discretely divergence-free with respect to $\hat{S}_0^0(\mathcal{T})$ because for each $T \in \mathcal{T}_z$, we have

$$\int_T \operatorname{div} \Psi_{F,m}^{\mathsf{cr}}\mathsf{t}_F = \sum_{F' \in \mathcal{F}_T} \int_{F'} \Psi_{F,m}^{\mathsf{cr}}\mathsf{t}_F \cdot \mathsf{n}_T = 0$$

as $\int_{F'} \Psi_{F,m}^{\mathsf{cr}} = \delta_{F,F'}$ and $\mathsf{n}_T \cdot \mathsf{t}_F = 0$ on F.

Turning to the normal components, we need more elaborated test functions such that the discrete divergence with respect to $\hat{S}_0^0(\mathcal{T})$ still vanishes. The intuition behind this is that the same amount of fluid which flows out of an element has to flow back into the element again such that the mass is preserved. For simplicity, we ensure this condition locally on the same face.

To enforce this condition, we use test functions whose mean values vanish over all $F \in \mathcal{F}$. Therefore, we define for $\ell \geqslant 1$, ℓ odd, $F \in \mathcal{F}_z^i$, the space

$$\hat{\mathrm{CR}}_\ell(F) = \Big\{ v_h \in \mathrm{CR}_\ell \;\Big|\; \int_F v_h = 0 \quad \text{and} \quad \int_{F'} v_h p = 0 \quad \forall p \in \mathbb{P}_{\ell-1}(F'), F' \in \mathcal{F}\backslash\{F\} \Big\}.$$

A function $v_h \in \hat{\mathrm{CR}}_\ell(F)$ fulfills for all $T \in \mathcal{T}_F$

$$\int_T \operatorname{div}(v_h \mathsf{n}_F) = \int_F v_h \mathsf{n}_F \cdot \mathsf{n}_T + \sum_{F' \in \mathcal{F}_T \backslash \{F\}} \int_{F'} v_h \mathsf{n}_{F'} \cdot \mathsf{n}_T = 0. \tag{4.51}$$

The space $\hat{\mathrm{CR}}_\ell(F)$ of model test functions is sufficient to detect the normal components as we will see later.

We turn to the last test function, which detects the lowest order normal components. Let ω_z, for a fixed $z \in \mathcal{V}$, be given and $(\mathsf{V}_z^S, \mathsf{A}_z^S)$ a spanning tree of the corresponding dual graph of $\mathcal{T}_z$. Then the last test function has to detect

$$\sum_{F \in \mathcal{F}_z^i \backslash F(\mathsf{A}_z^S)} \int_F r_F^c \mathsf{n}_F \cdot w, \qquad w \in Z(\omega_z^+), \tag{4.52}$$

with $r_F^c \in \mathbb{R}$ for $F \in \mathcal{F}_z^i \backslash F(\mathsf{A}_z^S)$. If $\mathcal{F}_z^i \backslash F(\mathsf{A}_z^S) = \varnothing$, nothing remains to detect. Thus, assume $\mathcal{F}_z^i \backslash F(\mathsf{A}_z^S) \neq \varnothing$ and let $F \in \mathcal{F}_z^i \backslash F(\mathsf{A}_z^S)$. Choose $T_1, T_2 \in \mathcal{T}$ such that $T_1 \cap T_2 = F$. Defining $\tilde{\mathsf{A}}_z^S = \{T_1, T_2\} \cup \mathsf{A}_z^S$, we obtain a cycle $\mathsf{v}_1, ..., \mathsf{v}_n, \mathsf{v}_1$ in $(\mathsf{V}_z^S, \tilde{\mathsf{A}}_z^S)$, $n \in \mathbb{N}$, according to [Die17, Theorem 1.5.1]. The arcs corresponding to the cycle are $\{\mathsf{e}_1, ..., \mathsf{e}_n\} \subset \tilde{\mathsf{A}}_z^S$, where e_i, $1 \leqslant i \leqslant n$, connects v_i with v_{i+1}, understood modulo n. Let $F_i = F(\mathsf{e}_i)$ and $\mathsf{n}_i := \mathsf{n}_{F_i}$, both $1 \leqslant i \leqslant n$. Without loss of generality, we assume that n_i points from $T(\mathsf{v}_i)$ to $T(\mathsf{v}_{i+1})$, understood modulo n.

We select for each F_i a CR_ℓ function. As ℓ is odd, the barycenter m_F of each face $F \in \mathcal{F}_z^i$ is a degree of freedom of the CR_ℓ element. We denote the associated basis functions as $\Psi_F^{\mathsf{cr,m}} \in \mathrm{CR}_\ell$. With these functions, we define the model test function for $F \in \mathcal{F}_z^i \backslash F(\mathsf{A}_z^S)$ as

$$\Psi_F^{\mathsf{cr,n,lo}} = \sum_{i=1}^n \frac{1}{\omega_i} \Psi_{F_i}^{\mathsf{cr,m}} \mathsf{n}_i, \tag{4.53}$$

where ω_i denotes the weight at m_{F_i} of the Gauß-Legendre quadrature on F_i, $1 \leqslant i \leqslant n$, see Proposition 2.78. The function $\Psi_F^{\mathsf{cr,n,lo}}$ for $F \in \mathcal{F}_z^i \backslash F(\mathsf{A}_z^S)$ fulfills $\underline{\operatorname{div}}_{h,0} \Psi_F^{\mathsf{cr,n,lo}} = 0$. To prove this, we fix $1 \leqslant i \leqslant n$ and compute:

$$\int_{T(\mathsf{v}_i)} \operatorname{div}\Big(\frac{1}{\omega_{i-1}} \Psi_{F_{i-1}}^{\mathsf{cr,m}} \mathsf{n}_{i-1} + \frac{1}{\omega_i} \Psi_{F_i}^{\mathsf{cr,m}} \mathsf{n}_i \Big) = \int_{F(\mathsf{e}_{i-1})} \frac{1}{\omega_{i-1}} \Psi_{F_{i-1}}^{\mathsf{cr,m}} \mathsf{n}_{i-1} \cdot \mathsf{n}_T + \int_{F(\mathsf{e}_i)} \frac{1}{\omega_i} \Psi_{F_i}^{\mathsf{cr,m}} \mathsf{n}_i \cdot \mathsf{n}_T = 0.$$

Here, $\mathsf{n}_i, 1 \leqslant i \leqslant n$, are the normal vectors of F_i and all indices are understood modulo n. In the last step, we used the Gauß-Legendre quadrature and $\mathsf{n}_{i-1} \cdot \mathsf{n}_T = -\mathsf{n}_i \cdot \mathsf{n}_T$.

Combining the test functions, we obtain the test function space for $z \in \mathcal{V}$:

$$\begin{aligned} S_z^{\mathsf{S\text{-}CR}} := \operatorname{span}\big(&\{\operatorname{curl}(\operatorname{curl}(q_T)\Psi_{2,T}) \mid q_T \in \mathbb{P}_{\ell-2}^{\operatorname{curl}}(T); T \in \mathcal{T}_z\} \\ &\cup \{\mathcal{E}^{\mathsf{Tf}}(\Psi_{F,i}^{\mathsf{cr}}\mathsf{t}_F) \mid F \in \mathcal{F}_z^i, 1 \leqslant i \leqslant \ell\} \cup \{\mathcal{E}^{\mathsf{Tf}}(\hat{\mathrm{CR}}_\ell(F)\mathsf{n}_F) \mid F \in \mathcal{F}_z^i\} \\ &\cup \{\mathcal{E}^{\mathsf{Tf}}(\Psi_F^{\mathsf{cr,n,lo}}) \mid F \in \mathcal{F}_z^i \backslash F(\mathsf{A}_z^S)\}\big). \end{aligned}$$

The next Theorem shows that this space $S_z^{\mathsf{S\text{-}CR}}$ in combination with D_z^{S} induces a stable projection.

Theorem 4.37. *Let $d = 2$, $\ell \in \mathbb{N}$ be odd and $z \in \mathcal{V}$. Let $(\mathsf{V}_z^S, \mathsf{A}_z^S)$ be a spanning tree of the mesh $\mathcal{T}_z$. Then the spaces D_z^{S} and $S_z^{\mathsf{S\text{-}CR}}$ define via (2.22) a projection operator $\mathcal{P}_z^{\ell,\mathsf{CRS}} : S_z^{\mathsf{S\text{-}CR}'} \to D_z^{\mathsf{S}}$. For all $f \in \tilde{Z}_z' = \big(Z(\omega_z) + S_z^{\mathsf{S\text{-}CR}}\big)'$, the projection satisfies the L^2-bound*

$$\|\mathcal{P}_z^{\ell,\mathsf{CRS}} f\|_{Z(\omega_z)'}^2 \lesssim \sum_{T\in\mathcal{T}_z} h_z^2 \|r_T\|_{L^2(T)}^2 + \sum_{F\in\mathcal{F}_z^i} h_z \|r_F\|_{L^2(F)}^2 + \sum_{F\in\mathcal{F}_z^i\backslash F(\mathsf{A}_z^S)} h_z \|r_F^c\|_{L^2(F)}^2 \lesssim \|f\|_{S_z^{\mathsf{S\text{-}CR}\prime}}^2,$$

with $r_T \in \mathbb{P}_{\ell-2}^{\operatorname{curl}}(T)$, $r_F \in \hat{\mathbb{P}}_{\ell-1}^{\mathsf{n}}(F)$ and $r_F^c \in \mathbb{R}$ for $T \in \mathcal{T}_z$ and $F \in \mathcal{F}_z^i$ determined via equation (2.22). The hidden constants only depend on ℓ and $\sigma(\mathcal{T})$ and the choice of $\mathcal{E}^{\mathsf{Tf}}$.

Proof. We show the well-posedness of $\mathcal{P}_z^{\ell,\mathsf{CRS}}$ with Proposition 2.20 via Assumption 2.19. Given $z \in \mathcal{V}$, let $(\mathsf{V}_z^S, \mathsf{A}_z^S)$ be a spanning tree of the dual graph of $\mathcal{T}_z$. The dimensions of D_z^{S} and $S_z^{\mathsf{S\text{-}CR}}$ coincide. Let $\chi \in D_z^{\mathsf{S}}$. Then, we can bound $\|\chi\|_{Z(\omega_z)'}$ by the L^2-representation given in (4.38)

$$\|\chi\|_{Z(\omega_z)'}^2 \lesssim \sum_{T\in\mathcal{T}_z} h_z^2 \|r_T\|_{L^2(T)}^2 + \sum_{F\in\mathcal{F}_z^i} h_z \|r_F\|_{L^2(F)}^2 + \sum_{F\in\mathcal{F}_z^i\backslash F(\mathsf{A}_z^S)} h_z \|r_F^c\|_{L^2(F)}^2. \tag{4.54}$$

The variables $r_T \in \mathbb{P}_{\ell-2}^{\operatorname{curl}}(T)$ for all $T \in \mathcal{T}_z$, $r_F \in \hat{\mathbb{P}}_{\ell-1}^{\mathsf{n}}(F)$ for all $F \in \mathcal{F}_z^i$ and $r_F^c \in \mathbb{R}$ for $F \in \mathcal{F}_z^i\backslash F(\mathsf{A}_z^S)$, are determined by (2.22).

We derive an upper bound with the help of the test functions $S_z^{\mathsf{S\text{-}CR}}$. The element contributions are treated as in (4.45). Only the estimates for the faces differ. On each $F \in \mathcal{F}_z^i$, we define $r_F^n = r_F \cdot \mathsf{n}_F \in \mathbb{P}_{\ell-1}(F)$ and $r_F^t = r_F \cdot \mathsf{t}_F \in \mathbb{P}_{\ell-1}(F)$ to split the face-contribution into

$$\|r_F\|_{L^2(F)}^2 = \|r_F^n\|_{L^2(F)}^2 + \|r_F^t\|_{L^2(F)}^2. \tag{4.55}$$

We estimate the tangential and normal components individually.

First, we start with the tangential components on a fixed $F \in \mathcal{F}_z^i$. Let ξ_j^F, $1 \leqslant j \leqslant \ell$, be the evaluation points of the Gauß-Legendre quadrature, as defined on page 77. We define the model test function as $\Psi^F := \sum_{j=1}^{\ell} r_F^t(\xi_j^F)\Psi_{F,j}^{\mathsf{cr}}$. This function allows us to derive an upper bound on $\|r_F^t\|_{L^2(F)}^2$:

$$\begin{aligned} \|r_F^t\|_{L^2(F)}^2 &= \sum_{i=1}^{\ell} \omega_i^F (r_F^t)^2(\xi_i^F) = \sum_{i=1}^{\ell} \omega_i^F r_F^t(\xi_i^F) \sum_{j=1}^{\ell} r_F^t(\xi_j^F)\Psi_{F,j}^{\mathsf{cr}}(\xi_i^F) \\ &= \int_F r_F^t \Psi^F = \int_F r_F^t \mathsf{t}_F \cdot \mathcal{E}^{\mathsf{Tf}}(\Psi^F \mathsf{t}_F) = \langle \chi, \mathcal{E}^{\mathsf{Tf}}(\Psi^F \mathsf{t}_F)\rangle - \sum_{T\in\mathcal{T}_z} \int_T r_T \cdot \mathcal{E}^{\mathsf{Tf}}(\Psi^F \mathsf{t}_F) \\ &\lesssim \big(\|\chi\|_{S_z^{\mathsf{S\text{-}CR}\prime}}^2 + \sum_{T\in\mathcal{T}_z} h_z^2 \|r_T\|_{L^2(T)}^2\big)^{\frac{1}{2}} \|\nabla(\mathcal{E}^{\mathsf{Tf}}(\Psi^F \mathsf{t}_F))\|_{L^2(\omega_z^+)}. \end{aligned} \tag{4.56}$$

We exploited the momentum condition of $\mathcal{E}^{\mathsf{Tf}}$. The stability of $\mathcal{E}^{\mathsf{Tf}}$ in combination with an inverse estimate yields

$$\begin{aligned} \|\nabla(\mathcal{E}^{\mathsf{Tf}}(\Psi^F \mathsf{t}_F))\|_{L^2(\omega_z^+)} &\lesssim \Big\|\nabla \sum_{j=1}^{\ell} r_F^t(\xi_j^F)\Psi_{F,j}^{\mathsf{cr}} \mathsf{t}_F\Big\|_{L^2(\omega_F)} \\ &\lesssim \sum_{j=1}^{\ell} \big|r_F^t(\xi_j^F)\big| \|\nabla_h \Psi_{F,j}^{\mathsf{cr}} \mathsf{t}_F\|_{L^2(\omega_F)} \lesssim h_F^{-\frac{1}{2}} \|r_F^t\|_{L^2(F)}. \end{aligned} \tag{4.57}$$

The last estimate follows because $\sum_{j=1}^{\ell} \big|r_F^t(\xi_j^F)\big|$ is a norm on $\mathbb{P}_{\ell-1}(F)$. After applying (4.45) once again to the terms $\|r_T\|_{L^2(T)}$ in (4.56), we obtain the upper bound for the tangential components.

Second, we perform a similar estimate for the normal components on each $F \in \mathcal{F}_z^i$. As in the tangential case, we define the model test function $\Psi^F = \sum_{j=1}^{\ell} r_F^n(\xi_j^F)\Psi_{F,j}^{\mathsf{cr}}$. This time, however, the function has to be in $\hat{\mathrm{CR}}_\ell(F)$ to fulfill the divergence constraint. Indeed, the Gauß-Legendre quadrature together with $r_F^n \in \hat{\mathbb{P}}_{\ell-1}(F)$ shows:

$$\int_F \Psi^F = \sum_{j=1}^{\ell} \omega_j^F r_F^n(\xi_j^F) = \int_F r_F^n = 0, \tag{4.58}$$

thus $\Psi^F \in \hat{\mathrm{CR}}_\ell(F)$. Therefore, (4.51) shows that $\int_T \operatorname{div}(\Psi^F \mathsf{n}_F) = 0$, $T \in \mathcal{T}$. The rest follows as in (4.56):

$$\begin{aligned}\|r_F^n\|_{L^2(F)}^2 &= \sum_{i=1}^{\ell} \omega_i^F (r_F^n)^2(\xi_i^F) = \sum_{i=1}^{\ell} \omega_i^F r_F^n(\xi_i^F) \sum_{j=1}^{\ell} r_F^n(\xi_j^F) \Psi_{F,j}^{\mathsf{cr}}(\xi_i^F) \\ &= \int_F r_F^n \Psi^F = \int_F r_F^n \mathsf{n}_F \cdot \mathcal{E}^{\mathsf{Tf}}(\Psi^F \mathsf{n}_F) = \langle \chi, \mathcal{E}^{\mathsf{Tf}}(\Psi^F \mathsf{n}_F)\rangle - \sum_{T\in\mathcal{T}_z} \int_T r_T \cdot \mathcal{E}^{\mathsf{Tf}}(\Psi^F \mathsf{n}_F) \qquad (4.59) \\ &\lesssim \big(\|\chi\|_{S_z^{\mathsf{S\text{-}CR}'}}^2 + \sum_{T\in\mathcal{T}_z} h_z^2 \|r_T\|_{L^2(T)}^2\big)^{\frac{1}{2}} \|\nabla(\mathcal{E}^{\mathsf{Tf}}(\Psi^F \mathsf{n}_F))\|_{L^2(\omega_z^+)}.\end{aligned}$$

The function Ψ^F does not interact with any component of the form $r_F^c \mathsf{n}_F$, $r_F^c \in \mathbb{R}$, in χ because of (4.58). The last factor in (4.59) is estimated analogously to (4.57).

It remains to bound the lowest order normal components, which are bounded with the help of the functions $\mathcal{E}^{\mathsf{Tf}}(\Psi_F^{\mathsf{cr,n,lo}})$. Here, Remark 4.27 applies again: in two dimensions the set $\mathcal{F}_z^i \backslash F(\mathsf{A}_z^S)$ is either empty or contains one face.

Assume $\mathcal{F}_z^i \backslash F(\mathsf{A}_z^S)$ contains an element F. We want to apply the Gauß-Legendre quadrature of order ℓ, see Proposition 2.78. This quadrature has ℓ evaluation points and weights but due to the construction of $\Psi_F^{\mathsf{cr,n,lo}}$, we are only interested in the evaluation point in the barycenter of F and its associated weight. Thus, we write $\xi_M^F \in F$ for barycenter of F and $\omega_M^F \in \mathbb{R}$ for the associated weight of the Gauß-Legendre quadrature of degree ℓ. Combining the quadrature formula with the momentum condition of $\mathcal{E}^{\mathsf{Tf}}$, we obtain:

$$\begin{aligned}\|r_F^c\|_{L^2(F)}^2 &= \omega_M^F (r_F^c)^2 = r_F^c r_F^c \omega_M^F \int_F \Psi_F^{\mathsf{cr,n,lo}} \cdot \mathsf{n}_F = \int_F r_F^c \mathsf{n}_F \cdot \mathcal{E}^{\mathsf{Tf}}(r_F^c \omega_M^F \Psi_F^{\mathsf{cr,n,lo}}) \\ &= \langle \chi, \mathcal{E}^{\mathsf{Tf}}(r_F^c \omega_M^F \Psi_F^{\mathsf{cr,n,lo}})\rangle - \sum_{T\in\mathcal{T}_z} \int_T r_T \cdot \mathcal{E}^{\mathsf{Tf}}(r_F^c \omega_M^F \Psi_F^{\mathsf{cr,n,lo}}) - \sum_{F\in\mathcal{F}_z^i} \int_F r_F \cdot \mathcal{E}^{\mathsf{Tf}}(r_F^c \omega_M^F \Psi_F^{\mathsf{cr,n,lo}}) \\ &\lesssim \Big(\|\chi\|_{S_z^{\mathsf{S\text{-}CR}'}}^2 + h_z^2 \sum_{T\in\mathcal{T}_z} \|r_T\|_{L^2(T)}^2 + h_z \sum_{F\in\mathcal{F}_z^i} \|r_F\|_{L^2(F)}^2\Big)^{\frac{1}{2}} \|\nabla \mathcal{E}^{\mathsf{Tf}}(r_F^c \omega_M^F \Psi_F^{\mathsf{cr,n,lo}})\|_{L^2(\omega_z^+)}.\end{aligned} \qquad (4.60)$$

With the definition of $\Psi_F^{\mathsf{cr,n,lo}}$ and $|F| \eqsim \omega_M^F \eqsim \omega_M^{F'}$ for all $F' \in \mathcal{F}_z^i$, we have with an inverse estimate the bound:

$$\|\nabla \mathcal{E}^{\mathsf{Tf}}(r_F^c \omega_M^F \Psi_F^{\mathsf{cr,n,lo}})\|_{L^2(\omega_z^+)} \lesssim h_z^{-\frac{1}{2}} \|r_F^c\|_{L^2(F)}.$$

The remaining parts in (4.60) are bounded as discussed before. □

The stable projection allows for the following local representation of the estimator.

Theorem 4.38. *Let $d = 2$ and ℓ be odd. Fix $f \in H^{-1}(\Omega)^2$ and $z \in \mathcal{V}$ and let $(\mathsf{V}_z^S, \mathsf{A}_z^S)$ be a spanning tree of the dual graph of the mesh $\mathcal{T}_z$. Let $u \in Z(\Omega)$ be the corresponding analytical velocity field determined by (3.8) and $u_h \in Z_h$ the corresponding discrete velocity field determined by one of the methods S-AP, S-ACRL, S-ATH, S-SIP or S-CRL with a mesh fulfilling the corresponding condition in Lemma 4.8 and that the corresponding $\tilde{\omega}_z^+$ is simply connected for all $z \in \mathcal{V}$. The projection $\mathcal{P}_z^{\ell,\mathsf{CRS}}$, $z \in \mathcal{V}$, induces the following splitting of the local conforming residual into an estimator and an oscillation*

$$\begin{aligned}\|\operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f,u_h)\|_{Z(\omega_z)'} &\lesssim \|\mathcal{P}_z^{\ell,\mathsf{CRS}} \operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f,u_h)\|_{Z(\omega_z)'} + \|\mathcal{P}_z^{\ell,\mathsf{CRS}} f - f\|_{Z(\omega_z)'} \\ &\lesssim \|\operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f,u_h)\|_{Z(\omega_z^+)'} \lesssim \sum_{z'\in\mathcal{V}\cap\omega_z^+} \|\operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f,u_h)\|_{Z(\omega_{z'})'}.\end{aligned} \qquad (4.61)$$

The induced estimator can be quantified further as

$$\|\mathcal{P}_z^{\ell,\mathsf{CRS}}(\operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f,u_h))\|_{Z(\omega_z)'}^2 \lesssim \eta_z^2 \lesssim \|\mathcal{P}_z^{\ell,\mathsf{CRS}}(\operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f,u_h))\|_{Z(\omega_z^+)'}^2 \lesssim \sum_{z'\in\mathcal{V}\cap\omega_z^+} \|\mathcal{P}_z^{\ell,\mathsf{CRS}}(\operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f,u_h))\|_{Z(\omega_{z'})'}^2. \qquad (4.62)$$

Here, η_z^2 might be either

$$\eta_z^2 = \sum_{T\in\mathcal{T}_z} h_z^2 \|r_T\|_{L^2(T)}^2 + \sum_{F\in\mathcal{F}_z^i} h_z \|r_F\|_{L^2(F)}^2 + \sum_{F\in\mathcal{F}_z^i\backslash F(\mathsf{A}_z^S)} h_z \|r_F^c\|_{L^2(F)}^2,$$

with $r_T \in \mathbb{P}^{\mathrm{curl}}_{\ell-2}(T)$, $r_F \in \mathbb{P}^{\mathsf{n}}_{\ell-1}(F)$ *and* $r^c_F \in \mathbb{R}$ *as in* (4.37) *and determined via equation* (2.22)*, or*

$$\eta_z^2 = \sum_{T\in\mathcal{T}_z} \max_{q_T\in\mathbb{P}^{\mathrm{curl}}_{\ell-2}(T)} \frac{\langle \mathrm{res}^{\mathsf{c}}_{\mathrm{h}}(f,u_h), \mathrm{curl}(\mathrm{curl}(q_T)\Psi_{2,T})\rangle^2}{\|\nabla \mathrm{curl}(\mathrm{curl}(q_T)\Psi_{2,T})\|^2_{L^2(T)}} + \sum_{F\in\mathcal{F}_z^i} \max_{\hat{w}\in\hat{\mathrm{CR}}_\ell(F)} \frac{\langle \mathrm{res}^{\mathsf{c}}_{\mathrm{h}}(f,u_h), \mathcal{E}^{\mathsf{Tf}}(\hat{w}\mathsf{n}_F)\rangle^2}{\|\nabla\mathcal{E}^{\mathsf{Tf}}(\hat{w}\mathsf{n}_F)\|^2_{L^2(\omega_F^+)}}$$
$$+ \sum_{F\in\mathcal{F}_z^i}\sum_{i=1}^{\ell} \frac{\langle \mathrm{res}^{\mathsf{c}}_{\mathrm{h}}(f,u_h), \mathcal{E}^{\mathsf{Tf}}(\Psi^{cr}_{F,i}\mathsf{t}_F)\rangle^2}{\|\nabla\mathcal{E}^{\mathsf{Tf}}(\Psi^{cr}_{F,i}\mathsf{t}_F)\|^2_{L^2(\omega_F^+)}} + \sum_{F\in\mathcal{F}_z^i\setminus F(\mathsf{A}_z^S)} \frac{\langle \mathrm{res}^{\mathsf{c}}_{\mathrm{h}}(f,u_h), \mathcal{E}^{\mathsf{Tf}}(\Psi^{cr,n,lo}_{F})\rangle^2}{\|\nabla\mathcal{E}^{\mathsf{Tf}}(\Psi^{cr,n,lo}_{F})\|^2_{L^2(\omega_z^+)}}. \tag{4.63}$$

The hidden constants depend on ℓ *and* $\sigma(\mathcal{T})$ *and the choice of* $\mathcal{E}^{\mathsf{Tf}}$*.*

Proof. To prove (4.61), we show (4.27). Because the functions in $\tilde{Z}_z$ are supported in ω_z^+ and not only in ω_z as it was the case for the projection $\mathcal{P}_z^{\ell,\mathsf{BS}}$, analyzed in Theorem 4.33, we need additional arguments. We compute for $z_h \in Z_h$ and $s \in \tilde{Z}_z = Z(\omega_z) + S_z^{\mathsf{S\text{-}CR}}$

$$\tilde{a}(z_h,s) = \int_{\omega_z^+} \nabla_h z_h : \nabla s = -\sum_{T\in\mathcal{T}_z^+}\int_T \Delta z_h \cdot s + \sum_{F\in\mathcal{F}_z^{i,+}}\int_F [\![\nabla z_h\cdot \mathsf{n}_F]\!]\cdot s. \tag{4.64}$$

In (4.64), the latter expression can be simplified due to the construction of the space $S_z^{\mathsf{S\text{-}CR}}$. All functions in $S_z^{\mathsf{S\text{-}CR}}$ which are not supported within ω_z stem from the space $\mathcal{E}^{\mathsf{Tf}}(\mathrm{CR}^2_\ell)$. Hence, the momentum conditions of the smoothing operator $\mathcal{E}^{\mathsf{Tf}}$ yield the simplification of (4.64)

$$\tilde{a}(z_h,s) = -\sum_{T\in\mathcal{T}_z}\int_T \Delta z_h\cdot s + \sum_{F\in\mathcal{F}_z^i}\int_F [\![\nabla z_h\cdot\mathsf{n}_F]\!]\cdot s. \tag{4.65}$$

Now, we can apply the same transformations as in (4.30), (4.31) and Theorem 4.28 to the functional in (4.65) without changing the $\|\cdot\|_{\tilde{Z}'_z}$-norm because $\tilde{Z}'_z \subset Z(\Omega)$ and test functions from $\tilde{Z}'_z$ cannot detect polynomials up to order $\ell-1$ on the faces $\mathcal{F}_z^{i,+}\backslash\mathcal{F}_z^i$. Therefore, the functional $\tilde{a}(z_h,s) = \sum_{F\in\mathcal{F}_z^i}\int_F r_F\cdot s + \sum_{T\in\mathcal{T}_z}\int_T r_T\cdot s$, $s\in\tilde{Z}_z$, satisfies $\tilde{a}(z_h,\cdot)_{|\tilde{Z}_z} \in D^{\mathsf{S}}_{z|\tilde{Z}_z}$. Hence, condition (4.27) is satisfied. Proposition 2.22 in combination with the local localization result Theorem 4.20 shows (4.61).

The L^2-representation of (4.62) is obtained with Theorem 4.37. The alternative hierarchical representation results from Lemma 2.26. □

The estimator induced by the projection $\mathcal{P}_z^{\ell,\mathsf{CRS}}$ can be applied to all quasi-optimal, pressure-robust Stokes methods with $d=2$ which we consider. However, this estimator plays its strengths when $\mathcal{E}^{\mathsf{Tf}}$ coincides with the smoother of the method. In this case, the expressions for all $F\in\mathcal{F}_z^i$ of the estimator can be bounded by the local conformity error $\mathbf{CE}_z(u_h)$ as introduced in Definition 2.67 (adapted to the current situation). The local conformity error's evaluation is cheaper because it often can reuse results from the global conformity error, which has to be computed anyway. Moreover, the local conformity error only depends on $z\in\mathcal{V}$ and the discrete solution $u_h\in Z_h$ but not on the right-hand side f, which might be expensive to evaluate. In consequence, the whole estimator simplifies.

The adapted version of $\mathbf{CE}_z(u_h)$ reads in this setting as

$$\mathbf{CE}_z(u_h) = \sup_{\substack{z_h\in Z_h\\ \mathrm{supp}(z_h)\subset\omega_z}} \frac{\langle \mathrm{res}^{\mathsf{c}}_{\mathrm{h}}(f,u_h), \mathrm{E_M}\, z_h\rangle}{\|z_h\|_{\tilde{V}}}.$$

To estimate the quantities $\mathbf{CE}_z(u_h)$ in the case of the Stokes problem, we have the following Lemma. Its results are similar to the ones for the Poisson problem.

Lemma 4.39. *For the different methods, we have different bounds for the local conformity error* $\mathbf{CE}_z(u_h)$*,* $z\in\mathcal{V}$*, where* $u_h\in Z_h$ *is the solution of the corresponding quasi-optimal, pressure-robust method.*

(i) In the case of S-CRL and in the lowest order case $\ell=1$ *of S-ACRL a solution* $u_h\in\mathrm{CR}^d_\ell$ *satisfies* $\mathbf{CE}_z(u_h)=0$*.*

(ii) In the case of arbitrary order S-ACRL with a solution $u_h\in\mathrm{CR}^d_\ell$*, we have:*

$$\sup_{\substack{z_h\in Z_h\\ \mathrm{supp}(z_h)\subset\omega_z}} \frac{\langle \mathrm{res}^{\mathsf{c}}_{\mathrm{h}}(f,u_h), \mathrm{E_M}\, z_h\rangle}{\|z_h\|_{\tilde{V}}} \lesssim \left(\sum_{F\in F_z^+}\int_F \frac{1}{h_F}|[\![u_h]\!]|^2\right)^{\frac{1}{2}}.$$

(iii) In the case of S-SIP with a solution $u_h \in S^0_\ell(\mathcal{T})$, we have:

$$\sup_{\substack{z_h \in Z_h \\ \operatorname{supp}(z_h) \subset \omega_z}} \frac{\langle \operatorname{res}^{\mathsf{c}}_{\mathrm{h}}(f, u_h), \mathrm{E_M}\, z_h \rangle}{\|z_h\|_{\tilde{V}}} \lesssim \left(\sum_{F' \in \mathcal{F}_z^+} \int_{F'} \frac{1+\gamma}{h_{F'}} |[\![u_h]\!]|^2 \right)^{\frac{1}{2}}.$$

Here, $\gamma > 0$ is the penalization parameter of the method.

Proof. Fix throughout the proof the discrete solution $u_h \in Z_h$ and a discrete test function $w_h \in Z_h$ with $\operatorname{supp}(w_h) \subset \omega_z$ for a fixed $z \in \mathcal{V}$.

In the cases S-CRL and S-ACRL lowest order, the smoothing operators fulfill the momentum conditions. Therefore, Lemma 1.46 is applicable for S-CRL, S-ACRL with $\gamma = 0$ and shows

$$\tilde{a}(u_h, \mathrm{E_M}\, w_h) = \tilde{a}(u_h, w_h) + \tilde{a}(u_h, \mathrm{E_M}\, w_h - w_h) = \tilde{a}(u_h, w_h) - \sum_{F' \in \mathcal{F}_z^+} \int_{F'} \{\{\nabla_h u_h \cdot \mathsf{n}_{F'}\}\} [\![w_h]\!].$$

The last summand vanishes due to the definition of the Crouzeix-Raviart element. This shows (i).

We turn to the proof of (ii). For the higher order case of the S-ACRL method, some additional arguments are needed. We start again similar to Lemma 2.68

$$\begin{aligned}
\langle \operatorname{res}^{\mathsf{c}}_{\mathrm{h}}(f, u_h), \mathcal{E}^{\mathrm{CR}}_{\mathsf{aug}} w_h \rangle &= \tilde{a}(u - u_h, \mathcal{E}^{\mathrm{CR}}_{\mathsf{aug}} w_h) = a_h^{\mathrm{CR}}(u_h, w_h) - \tilde{a}(u_h, \mathcal{E}^{\mathrm{CR}}_{\mathsf{aug}} w_h) \\
&= \int_\Omega \nabla_h \mathcal{E}^{\mathrm{CR}}_{\mathsf{aug}} u_h : \nabla_h \mathcal{E}^{\mathrm{CR}}_{\mathsf{aug}} w_h - \int_\Omega \nabla_h u_h : \nabla_h \mathcal{E}^{\mathrm{CR}}_{\mathsf{aug}} w_h \\
&\quad + (\eta - 1) \int_\Omega \nabla_h R^{\mathrm{CR}}(u_h) : \nabla_h R^{\mathrm{CR}}(w_h).
\end{aligned}$$

We have the bound

$$\begin{aligned}
\int_\Omega \nabla_h \mathcal{E}^{\mathrm{CR}}_{\mathsf{aug}} u_h : \nabla_h \mathcal{E}^{\mathrm{CR}}_{\mathsf{aug}} w_h - \nabla_h u_h : \nabla_h \mathcal{E}^{\mathrm{CR}}_{\mathsf{aug}} w_h &= \int_{\omega_z^+} \nabla_h (\mathcal{E}^{\mathrm{CR}}_{\mathsf{aug}} u_h - u_h) : \nabla_h \mathcal{E}^{\mathrm{CR}}_{\mathsf{aug}} w_h \\
&\leqslant \|\nabla_h (\mathcal{E}^{\mathrm{CR}}_{\mathsf{aug}} u_h - u_h)\|_{L^2(\omega_z^+)} \|\nabla_h w_h\|_{L^2(\omega_z^+)}.
\end{aligned}$$

The definition of R^{CR} yields

$$\int_\Omega \nabla_h R^{\mathrm{CR}}(u_h) : \nabla_h R^{\mathrm{CR}}(w_h) \lesssim \|\nabla_h (\mathcal{E}^{\mathrm{CR}}_{\mathsf{aug}} u_h - u_h)\|_{L^2(\omega_z^+)} \|\nabla_h w_h\|_{L^2(\omega_z^+)}.$$

Next, we bound $\|\nabla_h (\mathcal{E}^{\mathrm{CR}}_{\mathsf{aug}} u_h - u_h)\|_{L^2(\omega_z^+)}$ as

$$\|\nabla_h (\mathcal{E}^{\mathrm{CR}}_{\mathsf{aug}} u_h - u_h)\|_{L^2(\omega_z^+)} \leqslant \|\nabla_h (M_h^{\mathsf{CR}} u_h - u_h)\|_{L^2(\omega_z^+)} + \sum_{T \in \mathcal{T}_z^+} \|\nabla_h \big(R_\ell^T (\operatorname{div}_h u_h - \operatorname{div} M_h^{\mathsf{CR}} u_h)\big)\|_{L^2(T)}.$$

For a fixed element $T \in \mathcal{T}_z^+$, the latter term can be estimated with Proposition 3.18 as

$$\|\nabla_h \big(R_\ell^T (\operatorname{div}_h u_h - \operatorname{div} M_h^{\mathsf{CR}} u_h)\big)\|_{L^2(T)} \lesssim \|\operatorname{div}_h u_h - \operatorname{div} M_h^{\mathsf{CR}} u_h\|_{L^2(T)} \lesssim \|\nabla_h (u_h - M_h^{\mathsf{CR}} u_h)\|_{L^2(T)}.$$

In the proof of [KZ20, Proposition 4.1], the estimate

$$\|\nabla_h (u_h - M_h^{\mathsf{CR}} u_h)\|^2_{L^2(T)} \lesssim \sum_{F \in \mathcal{F}_T^+} \int_F \frac{1}{h_F} [\![u_h]\!]^2$$

is derived, which completes the proof of (ii).

The case (iii) for the discontinuous Galerkin method is handled as in Lemma 2.68. □

With the insights of Lemma 4.39, we can improve Theorem 4.38.

Theorem 4.40. *Let $d = 2$ and $\ell = 1$. We assume that $\mathcal{T}$ fulfills the conditions given in Lemma 4.8 and that the corresponding $\tilde{\omega}_z^+$ is simply connected for all $z \in \mathcal{V}$. Fix a functional $f \in H^{-1}(\Omega)^2$ and $z \in \mathcal{V}$. Assume that $u \in Z(\Omega)$ is the analytic solution of the Stokes problem (3.8) and $u_h \in Z_h$ the corresponding discrete solution of either S-ACRL, S-SIP or S-CRL. Furthermore, we assume $\mathcal{E}^{\mathsf{Tf}} = \mathrm{E_M}$.*

Then we have the estimate:

$$\|\mathcal{P}_z^{\ell, \mathsf{CRS}} (\operatorname{res}^{\mathsf{c}}_{\mathrm{h}}(f, u_h))\|^2_{Z(\omega_z)'} \lesssim \mathbf{CE}_z^2(u_h). \tag{4.66}$$

Globally, we can write

$$\|u-u_h\|_{\tilde{V}}^2 \eqsim \sum_{z\in\mathcal{I}} \|\mathcal{P}_z^{\ell,\mathsf{CRS}} f - f\|_{Z(\omega_z)'}^2 + \|\Pi_Z u_h - u_h\|_{\tilde{V}}^2. \tag{4.67}$$

Locally, we have for each $z \in \mathcal{V}$ the bound

$$\|\mathcal{P}_z^{\ell,\mathsf{CRS}} f - f\|_{Z(\omega_z)'}^2 \lesssim \|u_h - u\|_{\tilde{V}(\omega_z^+)}^2. \tag{4.68}$$

The hidden constants only depend on $\sigma(\mathcal{T})$.

Proof. The simplification of the estimator works similar to the elliptic cases, see e.g. Theorem 2.69 because we assume $\ell = 1$ and thus $\underline{\mathrm{div}}_{h,0} v_h = \underline{\mathrm{div}}_h v_h$ for all $v_h \in V_h$ of all considered methods. In the case of the hierarchical representation, Lemma 4.39 directly provides the estimates (4.66). In the case of the L^2-representation, a similar reasoning as in (2.132) and (2.133) yields the results. More precisely, we have seen in Theorem 4.37 that for each r_F, $F \in \mathcal{F}_z^i$, a Crouzeix-Raviart function $\Psi^F \in \mathrm{CR}^2$ exists with support in ω_z such that

$$\begin{aligned}\|r_F\|_{L^2(F)}^2 &= \int_F r_F \cdot \mathcal{E}^{\mathsf{Tf}}(\Psi^F) = \langle \mathrm{res}_{\mathrm{h}}^{\mathsf{c}}(f,u_h), \mathcal{E}^{\mathsf{Tf}}(\Psi^F)\rangle - \sum_{T\in\mathcal{T}_z}\int_T r_T \cdot \mathcal{E}^{\mathsf{Tf}}(\Psi^F)\\ &\lesssim \Big(\mathbf{CE}_z(u_h)^2 + h_z^2\|r_T\|_{L^2(T)}^2\Big)^{\frac{1}{2}} \|\Psi^F\|_{\tilde{V}}.\end{aligned}$$

As before, the element contributions are bounded with (4.45) and an inverse estimate shows the bound $\|\Psi^F\|_{\tilde{V}} \lesssim h^{-\frac{1}{2}}\|r_F\|_{L^2(F)}$. The contributions $\|r_F^c\|_{L^2(F)}^2$ can be estimated in the same spirit. This yields the result (4.66) in the case of the L^2-representation.

The global bound follows similarly to the reasoning in the proof of Corollary 4.36: First, we split the error as in section 4.1.1 into the conformity error and the conforming residual. The conforming residual is localized by Theorem 4.14 because Assumption 4.10 is fulfilled. Theorem 4.38 provides the splitting into the estimator and the oscillation term for the projection $\mathcal{P}_z^{\ell,\mathsf{CRS}}$. The equivalence (4.66) shows that the face contributions do not need to be evaluated if we add the local conformity error in the upper bound. The sum over all local conformity errors $\sum_{z'\in\mathcal{V}} \mathbf{CE}_{z'}(u_h)$ can be absorbed in the global conformity error:

- In the case of S-CRL and S-ACRL lowest order ($\ell = 1$), all local conformity errors are zero.
- In the case of S-ACRL higher order ($\ell > 1$), we can absorb $\sum_{z'\in\mathcal{V}} \mathbf{CE}_{z'}(u_h)$ into the global conformity error as following: The jump terms are bounded with the help of (2.34) by $\|\nabla_h(\mathcal{E}_\ell u_h - u_h)\|_{L^2(\Omega)}^2$. Since $\mathcal{E}_\ell$ is a quasi-optimal smoother and $Z \subset V$, we have the bound

  $$\|\mathcal{E}_\ell u_h - u_h\|_{\tilde{V}}^2 \lesssim \|\Pi_V u_h - u_h\|_{\tilde{V}}^2 \lesssim \|\Pi_Z u_h - u_h\|_{\tilde{V}}^2, \tag{4.69}$$

  see (2.33).
- In the case of S-SIP, we use (2.35) to bound the sum of the local conformity errors by $\|\mathcal{E}_\ell u_h - u_h\|_{\tilde{V}}$. Another application of (4.69) yields the result.

The other direction of the equivalence (4.67) follows by applying (4.61) and the globalization result (4.3). The global conforming residual and the conformity error can be bounded by the nonconforming residual as depicted in section 4.1.1.

The local lower bound follows by combining the estimator and oscillation back into the local conforming residual with (4.61). By the continuity of $\tilde{a}$, the local estimate (4.68) follows. □

Remark 4.41. In the case of $\ell = 1$, the test functions space $S_z^{\mathsf{S\text{-}CR}}$, $z \in \mathcal{V}$, can be rewritten as

$$S_z^{\mathsf{S\text{-}CR}} = \mathcal{E}^{\mathsf{Tf}}(\mathrm{CR}_{1,z}^{\mathsf{divfr}}) \quad \text{with}$$
$$\mathrm{CR}_{1,z}^{\mathsf{divfr}} = \Big(\mathrm{span}\left(\{\Psi_{F,1}^{\mathsf{cr}}\mathsf{t}_F \mid F\in\mathcal{F}_z^i\} \cup \{\Psi_F^{\mathsf{cr,n,lo}} \mid F \in \mathcal{F}_z^i \backslash F(\mathsf{A}_z^S)\}\right)\Big).$$

As pointed out in [BF91, section IV.8] (see also [VZ19b, p. 11] and [Tho81, Appendix 3]) the space $\mathrm{CR}_{1,z}^{\mathsf{divfr}}$ is a local, divergence-free basis for the local velocity space.

This finishes our derivation of an error estimator whose face components do not need to be evaluated in certain circumstances. But recall that in our case the elimination of the face contributions only works if $Q_h = \hat{S}_0^0(\mathcal{T})$. The question is: Does the elimination of the face contributions is only possible in the case of the pressure space $\hat{S}_0^0(\mathcal{T})$?

To discuss this issue, we focus on the situation $V_h = \mathrm{CR}_\ell^2$ and the detection of contributions on a fixed $F \in \mathcal{F}_z^i$, $z \in \mathcal{V}$. The two elements in $\mathcal{T}_F$ are denoted by T_1 and T_2. Recalling some notation, we write $\underline{\mathrm{div}}_{h,0}$ for the discrete divergence with respect to $\hat{S}_0^0(\mathcal{T})$ and $\underline{\mathrm{div}}_h$ for the discrete divergence with respect to a larger space Q_h. Correspondingly, we introduce the spaces $Z_h = \{w_h \in \mathrm{CR}_\ell^2 \,|\underline{\mathrm{div}}_h w_h = 0\}$ and $Z_{h,0} = \{w_h \in \mathrm{CR}_\ell^2 \,|\underline{\mathrm{div}}_{h,0} w_h = 0\}$.

We write $\mathrm{CR}_\ell(\omega_F)$ for Crouzeix-Raviart functions, whose degrees of freedom are non-zero only in the interior of ω_F. (The Crouzeix-Raviart degrees of freedom were defined in Proposition 2.79.) Analogously, we introduce $\mathrm{CR}_\ell(T_1)$, $\mathrm{CR}_\ell(T_2)$ and $\mathrm{CR}_\ell(\mathcal{T}_z)$. We also write $\mathrm{CR}_\ell(\omega_F)^2$, $\mathrm{CR}_\ell(T_1)^2$, $\mathrm{CR}_\ell(T_2)^2$ and $\mathrm{CR}_\ell(\mathcal{T}_z)^2$ for the corresponding spaces of vector fields.

After the introduction of the notation, we return to the main question: Can we eliminate the face contributions if Q_h is larger than $\hat{S}_0^0(\mathcal{T})$? To answer this question, let us recall why we can eliminate the evaluation of the face contributions in the simple case $\ell = 1$. The elimination of the face contributions relies on the fact that the corresponding test functions stem from the space $\mathrm{E_M}(Z_h)$. In this case, the local quasi-Galerkin orthogonality, introduced in Definition 2.67, eliminates the face contributions.

In the case $Q_h = \hat{S}_0^0(\mathcal{T})$, we have shown how to construct suitable test functions in $\mathrm{E_M}(Z_{h,0})$ because constructing functions $w_h \in \mathrm{CR}_\ell(\omega_F)^2$ with $\underline{\mathrm{div}}_{h,0}(w_h) = 0$ is rather straightforward. What is the problem in the case of a larger pressure space? Can we construct suitable functions $w_h \in \mathrm{CR}_\ell(\omega_F)^2$ with $\underline{\mathrm{div}}_h(w_h) = 0$ to detect the face contributions?

The problem of the condition $\underline{\mathrm{div}}_h(w_h) = 0$ is that it imposes more restrictions on w_h. We do not only need to have the same in- and out-flows on each element, as in the case $Q_h = \hat{S}_0^0(\mathcal{T})$, but we also have additional conditions in the interior of each element. If one has found a suitable CR-function w_h with the correct in and out flow on $T \in \mathcal{T}$, we have $\underline{\mathrm{div}}_{h,0} w_h = 0$, but we do not necessarily have $\underline{\mathrm{div}}_h w_h = 0$. Therefore, constructing a model test function $w_h \in \mathrm{CR}_\ell^2$ which fulfills $\underline{\mathrm{div}}_h w_h = 0$ and is still able to detect face contributions is involved.

We could try a two steps approach. First, we construct a model test function $w_h \in \mathrm{CR}_\ell^2$ with $\underline{\mathrm{div}}_{h,0} w_h = 0$ and which detects the desired aspect on F. We have constructed such functions in this section, hence the first step is no problem. In the second step, we add a correction function $\tilde{w}_h \in \mathrm{CR}_\ell^2$, such that $\underline{\mathrm{div}}_h(w_h + \tilde{w}_h) = 0$ and $w_h + \tilde{w}_h$ still serves as a model test function.

Ideally, the correction function $\tilde{w}_h$ is zero on the faces such that $\tilde{w}_h$ does not disturb the detection properties of w_h. More precisely, $\tilde{w}_h$ shall be the sum of two functions $\tilde{w}_{h,1} \in \mathrm{CR}_\ell(T_1)$ and $\tilde{w}_{h,2} \in \mathrm{CR}_\ell(T_2)$. Yet, the work of Scott and Vogelius [SV85, GS19] shows that such an element-supported correction function only exists if $(\operatorname{div} w_h)_{|T_i}(z) = 0$ at all $z \in \mathcal{V} \cap T_i$, $i = 1, 2$. We cannot fulfill this condition while w_h still detects the face-components. Thus, this approach to construct discretely divergence free CR-functions which are capable of detecting the face-components of $F \in \mathcal{F}_z$ and have support on $\mathcal{T}_F$ seems to be not feasible.

Therefore, we lift the restriction on the support and consider the possibility of a suitable divergence correction function $\tilde{w}_h$ with support in ω_z. But if one allows divergence correction with support outside T, this might interfere with the values on the sides of T, which are important for the testing process. However, we neglect this issue now and analyze the feasibility of this approach.

To determine, if it is possible at all to construct on a patch ω_z a discretely divergence-free CR-function which can detect the degrees of freedoms on the edges, we count the different degrees of freedom. We take a patch with 5 elements as an example and determine the dimensions of the relevant spaces in terms of the polynomial degree ℓ.

The relevant spaces are $\mathrm{CR}_\ell(\mathcal{T}_z)^2$ and $\hat{S}_{\ell-1}^0(\mathcal{T}_z)$ (this space poses the constraints). Then we compute the dimension of the discretely divergence-free functions (i.e. all functions $w_h \in \mathrm{CR}_\ell(\mathcal{T}_z)^2$ with $\underline{\mathrm{div}}_h w_h = 0$), which is the space in which we seek the model test functions. We compare its dimension to the dimension of the subspace of the face contributions in the space D_z. If the dimension of the subspace of face contributions in D_z is larger than $Z_h(\omega_z)$, then it is not possible to hide all face contributions of the estimator. Otherwise, the possibility is at least not ruled out, but one still has to construct these functions.

The following table provides the degrees of freedom of these spaces for a patch ω_z, $z \in \mathcal{V}$, with $N := 5$ elements. We assume that $\omega_z \subset \Omega$ such that we do not have to consider the special cases at the boundary of the domain. The dimensional analysis reveals that a set of test functions for the faces of the form $\mathrm{E_M}(w_h)$ with $w_h \in \mathrm{CR}_\ell(\mathcal{T}_z)^2 \cap Z_h$ is not always possible. We observe that this works for $\ell = 1$, which is the case we explicitly constructed, but for $\ell = 3, 5$ it is not possible. For $\ell \geqslant 7$ the

ℓ	$\dim(\mathrm{CR}_\ell(\mathcal{T}_z)^2)$ $N(\ell^2-\ell+2)$	$\dim(\hat{S}^0_{\ell-1}(\mathcal{T}_z))$ $(N/2)(\ell^2+\ell)-1$	$\dim(\ker(\underline{\mathrm{div}}_h(\mathrm{CR}_\ell(\mathcal{T}_z)^2)))$ $2N\ell-N+1$	$\dim($ face contr. in $D_z^{\mathsf{S}})$
1	10	4	6	6
3	40	29	11	26
5	110	74	36	46
7	220	139	81	66
9	370	224	146	86
11	560	329	231	106
13	790	454	336	126
15	1060	599	461	146
17	1370	764	606	166

Table 4.1: The dimensions of the spaces to analyze possibilities.

construction is no longer ruled out by a dimensional analysis. But if such test functions actually exist is not settled and remains open.

4.2.4 A pressure-robust error-estimator for 3d-situations

The previous estimators relied either on the definition of the curl operator or the Crouzeix-Raviart element in two dimensions. In this section, we derive another set of test functions for D_z^{S}, which are suitable for three-dimensional settings.

Many but not all of our test functions for the three-dimensional case are similar to the two-dimensional bubble estimator, which was introduced in section 4.2.2. One test function which fundamentally differs from the two-dimensional setting is the function to detect the lowest order normal components. This function differs fundamentally from the two-dimensional case because we have to consider the case that the set $\mathcal{F}_z^i \backslash F(\mathsf{A}_z^S)$ contains more than one element. For each of the faces in this set, we have to construct an appropriate function. We construct these functions by interpreting them as a flow through the cycle obtained by adding the corresponding face to the spanning tree. Moreover, these functions shall be only supported in the elements which induce this cycle.

As these functions are the most difficult ones in this setting, we start with their construction before turning to the other test functions which are more similar to the previous ones. The next Lemma provides the construction for the test functions to detect the lowest order normal components in D_z^{S}.

Lemma 4.42. *Let $z \in \mathcal{V}$ be arbitrary and ω_z be the corresponding patch. Let $(\mathsf{V}_z^S, \mathsf{A}_z^S)$ be a corresponding spanning tree. Fix $F \in \mathcal{F}_z^i \backslash F(\mathsf{A}_z^S)$. Adding F to the spanning tree induces a cycle according to Proposition 4.25. Let $\mathsf{v}_1, \dots, \mathsf{v}_n \in \mathsf{V}_z^S$ be the nodes and $\mathsf{e}_1, \dots, \mathsf{e}_n \in \mathsf{A}_z^S$ be the arcs of the cycle, $n \in \mathbb{N}$. Assume e_i connects v_i with v_{i+1}, $1 \leqslant i \leqslant n$, understood modulo n. There exists a function $\Psi_{n,F}^{3d} \in Z(\omega_z)$ with $\mathrm{supp}(\Psi_{n,F}^{3d}) \subset \bigcup_{i=1}^n T(\mathsf{v}_i)$ and*

$$\int_F r_F^c \mathsf{n}_F \cdot \Psi_{n,F}^{3d} = r_F^c,$$

for $r_F^c \in \mathbb{R}$.

Proof. On each face $F(\mathsf{e}_i)$, $1 \leqslant i \leqslant n$, we define a new normal vector $\tilde{\mathsf{n}}_{F(\mathsf{e}_i)} \in \mathbb{R}^3$ which points from v_i to v_{i+1} understood modulo n. This normal allows us to define the test function

$$\tilde{\Psi}_{n,F}^{3d} := \sum_{i=1}^n \Psi_{F(\mathsf{e}_i)} \tilde{\mathsf{n}}_{F(\mathsf{e}_i)}.$$

Here, $\Psi_{F(\mathsf{e}_i)}$ is the function defined in (1.37). The new function $\tilde{\Psi}_{n,F}^{3d}$ fulfills

$$\mathrm{supp}(\tilde{\Psi}_{n,F}^{3d}) \subset \bigcup_{i=1}^n T(\mathsf{v}_i) \quad \text{and}$$

$$\int_{T(\mathsf{v}_i)} \mathrm{div}\, \tilde{\Psi}_{n,F}^{3d} = \int_{F(\mathsf{e}_i)} \tilde{\Psi}_{n,F}^{3d} \cdot \tilde{\mathsf{n}}_{F(\mathsf{e}_i)} - \int_{F(\mathsf{e}_{i-1})} \tilde{\Psi}_{n,F}^{3d} \cdot \tilde{\mathsf{n}}_{F(\mathsf{e}_{i-1})} = 0, \quad \forall 1 \leqslant i \leqslant n.$$

We correct the divergence elementwisely with the elementwise right inverse of the divergence operator:

$$\Psi_{n,F}^{3d} := \tilde{\Psi}_{n,F}^{3d} - \sum_{i=1}^n \mathrm{div}_{T(\mathsf{v}_i)}^{-1}(\mathrm{div}\, \tilde{\Psi}_{n,F}^{3d}).$$

Here, $\mathrm{div}^{-1}_{T(\mathsf{v}_i)}$, $1 \leqslant i \leqslant n$, is a right inverse of div on the element $T(\mathsf{v}_i)$. A computable variant of $\mathrm{div}^{-1}_{T(\mathsf{v}_i)}$ is e.g. provided by $R_3^{T(\mathsf{v}_i)}$, which was defined in (3.32). Thus, the new function $\Psi^{3d}_{n,F} \in Z(\omega_z)$ fulfills the essential detection property

$$\int_F r_F^c \mathsf{n}_F \cdot \Psi^{3d}_{n,F} = r_F^c$$

for any $r_F^c \in \mathbb{R}$. □

After Lemma 4.42 provides the test functions to detect the lowest order normal components, we turn to the test functions for the other components. These new test functions are quite similar to the ones in the two-dimensional setting. We first generalize the two dimension bubble $\overline{\Phi}_{\mathsf{n}_F}$, introduced in Corollary 1.59, to three dimensions. With this three-dimensional bubble, we can define the remaining test functions.

To simplify matters, we fix for each $F \in \mathcal{F}$ two orthogonal, tangential vectors $\mathsf{t}_{F,1}$ and $\mathsf{t}_{F,2}$. These two vectors shall be chosen in such a way that the basis $(\mathsf{t}_{F,1}, \mathsf{t}_{F,2}, \mathsf{n}_F)$ can be mapped onto the standard basis (e_1, e_2, e_3) by an orthogonal, orientation preserving, affine map.

The basis $(\mathsf{t}_{F,1}, \mathsf{t}_{F,2}, \mathsf{n}_F)$ helps us to reformulate Corollary 1.59.

Corollary 4.43. *For $F \in \mathcal{F}^i$ with normal vector n_F, there is a function $\overline{\Phi}_{\mathsf{n}_F} \in H^2_0(\omega_F)$ with the following properties:*

1. $\int_F \partial_{\mathsf{n}_F} \overline{\Phi}_{\mathsf{n}_F} = 1$ *on* F,
2. $\partial_{\mathsf{t}_{F,1}} \overline{\Phi}_{\mathsf{n}_F} = \partial_{\mathsf{t}_{F,2}} \overline{\Phi}_{\mathsf{n}_F} = 0$ *on* F,
3. $\overline{\Phi}_{\mathsf{n}_F} = 0$ *on* F.

Proof. The first step is to obtain a generalization of the bubble function $\Psi_{2,F}$, defined in (1.77). If we adapt the scaling factor such that the function is normalized and interpret all objects in the three-dimensional setting, the definition is literally the same. With a little abuse of notation, we denote this three-dimensional variant also as $\Psi_{2,F} \in H^2_0(\omega_F)$, $F \in \mathcal{F}^i$.

The second step is to transfer the ramp-like function η_F of Lemma 1.58 to the three-dimensional setting. Here again, we only need to interpret its definition in three dimensions: $\eta_F : \mathbb{R}^3 \to \mathbb{R}$ defined as $\eta_F(x) := (x - m_F) \cdot \mathsf{n}_F$, where $m_F \in F$ is the barycenter of F.

Using both functions, we define the three-dimensional function $\overline{\Phi}_{\mathsf{n}_F} \in H^2_0(\omega_F)$ as $\overline{\Phi}_{\mathsf{n}_F} := \Psi_{2,F}\eta_F$. The properties given in the Corollary follow in the same manner as in the two-dimensional case. □

Given a function $\varphi : \Omega \to \mathbb{R}$, we define the operator $\widetilde{\mathrm{curl}}_F$ as the two-dimensional curl-operator on a face $F \in \mathcal{F}^i_z$ in terms of the basis $(\mathsf{t}_{F,1}, \mathsf{t}_{F,2})$ as

$$\widetilde{\mathrm{curl}}_F(\varphi) = (\partial_{\mathsf{t}_{F,2}} \varphi_{|F}, -\partial_{\mathsf{t}_{F,1}} \varphi_{|F})^T. \tag{4.70}$$

Given a vector field $\varphi : \Omega \to \mathbb{R}^3$, we analogously define the companion operator $\widehat{\mathrm{curl}}_F$ on F as:

$$\widehat{\mathrm{curl}}_F(\varphi) = \partial_{\mathsf{t}_{F,1}} \varphi_{\mathsf{t}_{F,2}} - \partial_{\mathsf{t}_{F,2}} \varphi_{\mathsf{t}_{F,1}}. \tag{4.71}$$

On $F \in \mathcal{F}^i_z$, we define in the basis $(\mathsf{t}_{F,1}, \mathsf{t}_{F,2}, \mathsf{n}_F)$ the vector space:

$$\mathbb{P}^{\mathsf{n},3d}_{\ell-1}(F) = \left\{ \begin{pmatrix} E_F(\widetilde{\mathrm{curl}}(q^{\mathsf{n}}_F) \cdot \mathsf{t}_{F,1})\Psi_{2,F} \\ E_F(\widetilde{\mathrm{curl}}(q^{\mathsf{n}}_F) \cdot \mathsf{t}_{F,2})\Psi_{2,F} \\ 0 \end{pmatrix} \middle| \, q^{\mathsf{n}}_F \in \hat{\mathbb{P}}_{\ell-1}(F) \right\}.$$

The operator E_F is the polynomial extension operator defined in (2.114).

With this definition, the test function space reads as

$$\begin{aligned} S^{\mathsf{3dS}}_z := \mathrm{span} \Big(& \{\mathrm{curl}(\mathrm{curl}(q_T)\Psi_{2,T}) \mid q_T \in \mathbb{P}^{\mathrm{curl}}_{\ell-2}(T); T \in \mathcal{T}_z\} \cup \{\mathrm{curl}(q^{\mathsf{n},3d}_F) \mid q^{\mathsf{n},3d}_F \in \mathbb{P}^{\mathsf{n},3d}_{\ell-1}(F), F \in \mathcal{F}^i_z\} \\ & \cup \{\mathrm{curl}(E_F(q^{\mathsf{t},1}_F)\overline{\Phi}_{\mathsf{n}_F}\mathsf{t}_{F,1}) \mid q^{\mathsf{t},1}_F \in \mathbb{P}_{\ell-1}(F); F \in \mathcal{F}^i_z\} \\ & \cup \{\mathrm{curl}(E_F(q^{\mathsf{t},2}_F)\overline{\Phi}_{\mathsf{n}_F}\mathsf{t}_{F,2}) \mid q^{\mathsf{t},2}_F \in \mathbb{P}_{\ell-1}(F); F \in \mathcal{F}^i_z\} \cup \{\Psi^{3d}_{n,F} \mid F \in \mathcal{F}^i_z \backslash F(\mathsf{A}^S_z)\}\Big). \end{aligned}$$

Similar to the $2d$-case, we used $\Psi_{2,T} := \Psi^2_T \in H^2_0(T)$, where Ψ_T is defined in (2.70). These test functions can be combined with the three-dimensional space D^{S}_z to define a projection operator.

Theorem 4.44. *Let $d = 3$ and $\ell \in \mathbb{N}$ arbitrary. Fix $z \in \mathcal{V}$ and a spanning tree $(\mathsf{V}_z^S, \mathsf{A}_z^S)$ of the dual graph of the mesh $\mathcal{T}_z$. Then the spaces S_z^{3dS} and D_z^S define via the equation (2.22) a stable projection operator $\mathcal{P}_z^{\ell,3dBS}$.*

Proof. Let $\chi \in D_z^{\mathsf{S}}$ be arbitrary with the variables $r_T \in \mathbb{P}_{\ell-2}^{\mathrm{curl}}(T)$ for all $T \in \mathcal{T}_z$, $r_F \in \hat{\mathbb{P}}_{\ell-1}^{\mathsf{n}}(F)$ for all $F \in \mathcal{F}_z^i$ and $r_F^c \in \mathbb{R}$ for $\mathcal{F}_z^i \backslash F(\mathsf{A}_z^S)$ as in (4.37). As in (4.38), we have the following L^2-representation

$$\|\chi\|_{Z(\omega_z)'}^2 \lesssim \sum_{T\in\mathcal{T}_z} h_z^2 \|r_T\|_{L^2(T)}^2 + \sum_{F\in\mathcal{F}_z^i} h_z \|r_F\|_{L^2(F)}^2 + \sum_{F\in\mathcal{F}_z^i\backslash F(\mathsf{A}_z^S)} h_z \|r_F^c\|_{L^2(F)}^2. \tag{4.72}$$

The terms $\sum_{T\in\mathcal{T}_z} h_z^2 \|r_T\|_{L^2(T)}^2$ and $\sum_{F\in\mathcal{F}_z^i\backslash F(\mathsf{A}_z^S)} h_z \|r_F^c\|_{L^2(F)}^2$ are estimated as in Theorem 4.33 by using the corresponding test functions in $S_z^{3\mathsf{dS}}$.

We estimate the remaining facial components and start splitting them into the normal and tangential components

$$\|r_F\|_{L^2(F)}^2 = \|r_F^n\|_{L^2(F)}^2 + \|r_F^t\|_{L^2(F)}^2, \tag{4.73}$$

with $r_F^n \in \hat{\mathbb{P}}_{\ell-1}(F)$ and $r_F^t = (r_F^{t,1}, r_F^{t,2}, 0)^T$ in the basis $(\mathsf{t}_{F,1}, \mathsf{t}_{F,2}, \mathsf{n}_F)$ for $r_F^{t,1} \in \mathbb{P}_{\ell-1}(F)$ and $r_F^{t,2} \in \mathbb{P}_{\ell-1}(F)$.

To apply the test functions, we recall the definition of the curl operator. The definition of the curl-operator written in the basis $(\mathsf{t}_{F,1}, \mathsf{t}_{F,2}, \mathsf{n}_F)$ for $F \in \mathcal{F}_z^i$ reads for a function $\varphi \in H^1(\Omega)^3$ as

$$\operatorname{curl}(\varphi) = \begin{pmatrix} \partial_{\mathsf{t}_{F,2}} \varphi_{\mathsf{n}_F} - \partial_{\mathsf{n}_F} \varphi_{\mathsf{t}_{F,2}} \\ -\partial_{\mathsf{t}_{F,1}} \varphi_{\mathsf{n}_F} + \partial_{\mathsf{n}_F} \varphi_{\mathsf{t}_{F,1}} \\ \partial_{\mathsf{t}_{F,1}} \varphi_{\mathsf{t}_{F,2}} - \partial_{\mathsf{t}_{F,2}} \varphi_{\mathsf{t}_{F,1}} \end{pmatrix}, \tag{4.74}$$

because the basis $(\mathsf{t}_{F,1}, \mathsf{t}_{F,2}, \mathsf{n}_F)$ can be mapped by an affine, orthogonal and orientation preserving map onto the standard basis by assumption.

We estimate the tangential components. Recall that $\partial_{\mathsf{n}_F} \eta_F = 1$. Then, we have

$$\begin{aligned} \int_F \left|r_F^t\right|^2 &= \int_F \left|r_F^{t,1}\right|^2 + \left|r_F^{t,2}\right|^2 = \int_F r_F^{t,1} r_F^{t,1} \partial_{\mathsf{n}_F} \eta_F + r_F^{t,2} r_F^{t,2} \partial_{\mathsf{n}_F} \eta_F \\ &\lesssim \int_F r_F^{t,1} r_F^{t,1} \partial_{\mathsf{n}_F} \eta_F \Psi_{2,F} + r_F^{t,2} r_F^{t,2} \partial_{\mathsf{n}_F} \eta_F \Psi_{2,F}. \end{aligned}$$

In the last inequality, we used an inverse estimate and the bubble $\Psi_{2,F} \in H_0^2(\omega_F)$, see the proof of Corollary 4.43. Because $\eta_F = 0$ on F, we can write

$$\int_F \left|r_F^t\right|^2 \lesssim \int_F r_F^{t,1} \partial_{\mathsf{n}_F}\big(E_F(r_F^{t,1}) \eta_F \Psi_{2,F}\big) + r_F^{t,2} \partial_{\mathsf{n}_F}\big(E_F(r_F^{t,2}) \eta_F \Psi_{2,F}\big).$$

With the definition $\overline{\Phi}_{\mathsf{n}_F} := \Psi_{2,F} \eta_F$ from Corollary 4.43, we obtain:

$$\|r_F^t\|_{L^2(F)}^2 \lesssim \int_F \begin{pmatrix} r_F^{t,1} \\ r_F^{t,2} \end{pmatrix} \cdot \begin{pmatrix} \partial_{\mathsf{n}_F}(E_F(r_F^{t,1}) \overline{\Phi}_{\mathsf{n}_F}) \\ \partial_{\mathsf{n}_F}(E_F(r_F^{t,2}) \overline{\Phi}_{\mathsf{n}_F}) \end{pmatrix} = \int_F \begin{pmatrix} r_F^{t,1} \\ r_F^{t,2} \end{pmatrix} \cdot \begin{pmatrix} -\partial_{\mathsf{n}_F}(E_F(-r_F^{t,1}) \overline{\Phi}_{\mathsf{n}_F}) \\ \partial_{\mathsf{n}_F}(E_F(r_F^{t,2}) \overline{\Phi}_{\mathsf{n}_F}) \end{pmatrix}. \tag{4.75}$$

We define on ω_F the function

$$\varphi := (E_F(r_F^{t,2}) \overline{\Phi}_{\mathsf{n}_F}, -E_F(r_F^{t,1}) \overline{\Phi}_{\mathsf{n}_F}, 0).$$

Because $\overline{\Phi}_{\mathsf{n}_F}$ and the tangential derivatives of $\overline{\Phi}_{\mathsf{n}_F}$ are zero on F, the function

$$(-\partial_{\mathsf{n}_F}(E_F(-r_F^{t,1}) \overline{\Phi}_{\mathsf{n}_F}), \partial_{\mathsf{n}_F}(E_F(r_F^{t,2}) \overline{\Phi}_{\mathsf{n}_F}), 0)^T = \operatorname{curl}\big((E_F(r_F^{t,2}) \overline{\Phi}_{\mathsf{n}_F}, -E_F(r_F^{t,1}) \overline{\Phi}_{\mathsf{n}_F}, 0)^T\big) = \operatorname{curl}(\varphi),$$

written in the basis $(\mathsf{t}_{F,1}, \mathsf{t}_{F,2}, \mathsf{n}_F)$ of F, is contained in $S_z^{3\mathsf{dS}}$. Combining (4.75) with $\operatorname{curl}\varphi$, we conclude

$$\|r_F^t\|_{L^2(F)}^2 \lesssim \big(\|\chi\|_{S_z^{3\mathsf{dS}}}^2 + \sum_{T\in\mathcal{T}_z} h_z^2 \|r_T\|_{L^2(T)}^2\big)^{\frac{1}{2}} \|\nabla \operatorname{curl}\varphi\|_{L^2(\omega_F)}.$$

The last factor is bounded with an inverse estimate

$$\|\nabla \operatorname{curl}\varphi\|_{L^2(\omega_F)} \lesssim h_z^{-2} \|\varphi\|_{L^2(\omega_F)} \lesssim h_z^{-\frac{1}{2}} \|r_F^t\|_{L^2(F)}$$

because the function η_F in $\overline{\Phi}_{\mathsf{n}_F}$ fulfills $\eta_F \leqslant h_z$, see Lemma 1.60.

We turn to the estimate of the normal component r_F^n, $F \in \mathcal{F}_z^i$. This component fulfills $\int_F r_F^n = 0$ because $r_F^n \in \hat{\mathbb{P}}_{\ell-1}(F)$ by the definition of D_z^{S}. Let $\widetilde{\nabla}_F v = (\partial_{\mathsf{t}_{F,1}} v_{|F}, \partial_{\mathsf{t}_{F,2}} v_{|F})^T$ and $\widetilde{\operatorname{curl}}_F$, $\widehat{\operatorname{curl}}_F$ as in (4.70) and (4.71) for each $F \in \mathcal{F}_z^i$. With an application of Poincaré's inequality and swapping coordinates, we obtain with an inverse estimate for each $F \in \mathcal{F}_z^i$:

$$\begin{aligned}\|r_F^n\|_{L^2(F)}^2 &\lesssim h_F^2 \|\widetilde{\nabla}_F r_F^n\|_{L^2(F)}^2 = h_F^2 \|\widetilde{\operatorname{curl}}_F r_F^n\|_{L^2(F)}^2 \\ &\lesssim h_F^2 \int_F \left|\widetilde{\operatorname{curl}}_F r_F^n\right|^2 \Psi_{2,F} = h_F^2 \int_F r_F^n \widehat{\operatorname{curl}}_F\big(\widetilde{\operatorname{curl}}_F(r_F^n)\Psi_{2,F}\big).\end{aligned} \tag{4.76}$$

Let us define $\varphi := \big(E_F(\widetilde{\operatorname{curl}}_F(r_F^n)\cdot \mathsf{t}_{F,1})\Psi_{2,F}, E_F(\widetilde{\operatorname{curl}}_F(r_F^n)\cdot \mathsf{t}_{F,2})\Psi_{2,F}, 0\big)^T$ in the basis $(\mathsf{t}_{F,1}, \mathsf{t}_{F,2}, \mathsf{n}_F)$. The normal component of $\operatorname{curl}(\varphi)$ is on F:

$$\operatorname{curl}(\varphi)\cdot \mathsf{n}_F = \partial_{\mathsf{t}_{F,1}}(\widetilde{\operatorname{curl}}(r_F^n)\cdot \mathsf{t}_{F,2}\Psi_{2,F}) - \partial_{\mathsf{t}_{F,2}}(\widetilde{\operatorname{curl}}(r_F^n)\cdot \mathsf{t}_{F,1}\Psi_{2,F}),$$

which is the same as $\widehat{\operatorname{curl}}_F\big(\widetilde{\operatorname{curl}}_F(r_F^n)\Psi_{2,F}\big)$ in (4.76). Therefore, we have the bound

$$\begin{aligned}\|r_F^n\|_{L^2(F)}^2 &\lesssim h_F^2 \int_F r_F^n \mathsf{n}_F \cdot \operatorname{curl}(\varphi) \\ &\lesssim h_F^2 \int_F \Big(\|\chi\|_{S_z^{3\mathsf{dS}}}^2 + \sum_{T\in\mathcal{T}_F} \|r_T\|_{L^2(T)}^2 + \sum_{F\in\mathcal{F}_z^i} \|r_F^t\|_{L^2(F)}^2\Big)^{\frac{1}{2}} \|\nabla \operatorname{curl}(\varphi)\|_{L^2(\omega_F)}.\end{aligned}$$

Due to (3.2), there is no interaction between $\operatorname{curl}(\varphi)$ and the contribution of r_F^c in χ. The remaining element and facial contributions are estimated as before. The last factor is again bounded with an inverse estimate

$$h_F^2 \|\nabla \operatorname{curl}(\varphi)\|_{L^2(\omega_F)} \lesssim \|r_F^n\|_{L^2(F)}. \qquad \square$$

This stable projection $\mathcal{P}_z^{\ell,3\mathsf{dBS}}$ can be used to define a pressure-robust error-estimator with an error-dominated oscillation for the three-dimensional Stokes problem. The details are provided in the next Corollary, which is similar to Corollary 4.35.

Corollary 4.45. *Given $f \in H^{-1}(\Omega)^3$, let $u \in Z(\Omega)$ be the solution of the Stokes problem (3.8). Let $\mathcal{T}$ be a mesh such that the discrete inf-sup condition (3.13) is fulfilled and let $u_h \in V_h$ be the discrete solution of one of the considered methods S-AP, S-SIP or S-CRL ($\ell = 1$), which are also available in three dimensions. Let $\operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f, u_h) \in Z(\omega_z)'$ be the corresponding local conforming residual. For each $z \in \mathcal{V}$, the projection $\mathcal{P}_z$ induced by the pair D_z^{S} and $S_z^{3\mathsf{dS}}$ (after choosing a spanning tree of the dual graph of $\mathcal{T}_z$) provides a pressure-robust error-estimator with an error-dominated oscillation term via Proposition 2.22. More precisely, we have*

$$\|\operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f,u_h)\|_{Z(\omega_z)'} \leqslant \underbrace{\|\mathcal{P}_z^{\ell,3\mathsf{dBS}} f - \tilde{A}u_h\|_{Z(\omega_z)'}}_{\text{estimator}} + \underbrace{\|\mathcal{P}_z^{\ell,3\mathsf{dBS}} f - f\|_{Z(\omega_z)'}}_{\text{oscillation}} \lesssim \|\operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f,u_h)\|_{Z(\omega_z)'}. \tag{4.77}$$

The estimator $\|\mathcal{P}_z^{\ell,3\mathsf{dBS}} f - \tilde{A}u_h\|_{Z(\omega_z)'}$ can be evaluated in a hierarchical manner as in Lemma 2.26 or via the L^2-representation given in (4.72), with the variables determined via (2.22).

Proof. Theorem 4.44 provides the stability of $\mathcal{P}_z^{\ell,3\mathsf{dBS}}$. The triangle inequality provides the first inequality in (4.77). Due to the construction of D_z^{S}, condition (4.27), which corresponds to (2.29) in the pressure-robust Stokes setting, is satisfied. Therefore, Proposition (2.22) shows the second estimate in (4.77). $\square$

We finish with the global result which corresponds to Corollary 4.36.

Corollary 4.46. *Let $u \in Z(\Omega)$ be the analytic velocity field solution of the Stokes equation (3.8) with $f \in H^{-1}(\Omega)^3$. Let $u_h \in Z_h$ be the corresponding discrete solution of one of the methods S-AP, S-SIP or S-CRL ($\ell = 1$) with a mesh, such that discrete inf-sup condition (3.13), the conditions in Lemma 4.8 are fulfilled and the corresponding $\tilde{\omega}_z^+$ is simply connected for all $z \in \mathcal{V}$.*

We define

$$\eta_z^2 := \|\mathcal{P}_z^{\ell,3\mathsf{dBS}} f - \tilde{A}u_h\|_{Z(\omega_z)'}^2$$

which can be equivalently expressed in the hierarchical form provided by Lemma 2.26 or the L^2-representation given in (4.72). The variables in the L^2-representation are determined via (2.22). Then, the global bound

$$\|u - u_h\|_{\tilde{V}}^2 \eqsim \sum_{z\in\mathcal{V}} \eta_z^2 + \|\mathcal{P}_z^{\ell,3\mathsf{dBS}} f - f\|_{Z(\omega_z)'}^2 + \|\Pi_Z u_h - u_h\|_{\tilde{V}}^2 \tag{4.78}$$

holds. Locally, we have the upper bound for $z \in \mathcal{V}$

$$\eta_z^2 + \|\mathcal{P}_z^{\ell,\mathsf{3dBS}} f - f\|_{Z(\omega_z)'}^2 \lesssim \|\nabla_h(u-u_h)\|_{L^2(\omega_z)}^2. \tag{4.79}$$

Proof. First, let us apply (4.1) to split the error into the conformity error and the conforming residual:

$$\|u-u_h\|_{\tilde{V}}^2 = \|\operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f,u_h)\|_{Z'}^2 + \|\Pi_Z u_h - u_h\|_{\tilde{V}}^2. \tag{4.80}$$

Next, we localize the conforming residual $\|\operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f,u_h)\|_{Z(\Omega)'}$ as shown in section 4.1. Because we assume that the mesh $\mathcal{T}$ fulfills the requirements of Lemma 4.8, Theorem 4.14 is applicable. This Theorem shows that Assumption 4.10 is valid for the considered methods, hence we can apply the localization Theorem 4.11.

For each $z \in \mathcal{V}$, the localized, conforming residual $\|\operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f,u_h)\|_{Z(\omega_z)'}$ is now split into the estimator and oscillation with the help of the projection $\mathcal{P}_z^{\ell,\mathsf{3dBS}}$ by (4.77) in Corollary 4.45. This implies one direction of (4.78).

To show the other direction of (4.78), use (4.77) again to bound the estimator and oscillation term by the local conforming residual. The globalization (4.3) shows that the sum of the local conforming residua is bounded by the global conforming residual. Combining the global conforming residual with the conformity error to the global error by (4.80) completes the proof of (4.78).

We turn to the local estimate (4.79). At first, we combine the estimator and oscillation again with (4.77) in Corollary 4.45. Then the statements follow by the continuity of $\tilde{a}(\cdot,\cdot)$. □

Finally, we remark that for $\ell = 1$, we also can construct a test function space based upon the $3d$ Crouzeix-Raviart element. We omit the cases $\ell > 1$ due to the problems discussed on page 94. Therefore, we only have to deal with the face contributions of D_z^{S}. Also in the $3d$ setting, we can proceed as laid out in section 4.2.3, therefore we only sketch the construction and do not provide the full results.

Also in the $3d$ setting, the barycenters of each face $F \in \mathcal{F}$ is a degree of freedom of the CR_1 element. We write $\Psi_F^{\mathsf{cr,m}} \in \mathrm{CR}_\ell$ for the corresponding basis function. Using these functions, we define the model test functions to detect the tangential components as

$$\Psi_F^{\mathsf{cr,m}}\mathsf{t}_{F,1} \qquad \Psi_F^{\mathsf{cr,m}}\mathsf{t}_{F,2} \qquad \forall F \in \mathcal{F}_z^i.$$

It remains to define the functions which are able to detect (4.52). Fix $F \in \mathcal{F}_z^i \backslash F(\mathsf{A}_z^S)$. We recall the required setup similar to the one on page 129. We select $T_1, T_2 \in \mathcal{T}$ such that $T_1 \cap T_2 = F$. As before, we define $\tilde{\mathsf{A}}_z^S = \{T_1, T_2\} \cup \mathsf{A}_z^S$ and obtain a cycle $\mathsf{v}_1, ..., \mathsf{v}_n, \mathsf{v}_1$ in V_z^S, $n \in \mathbb{N}$, which is ensured by [Die17, Theorem 1.5.1]. We denote the arcs corresponding to the cycle as $\{\mathsf{e}_1, ..., \mathsf{e}_n\} \subset \tilde{\mathsf{A}}_z^S$, where e_i, $1 \leqslant i \leqslant n$ connects v_i with v_{i+1}, understood modulo n. Let $F_i = F(\mathsf{e}_i)$ and $\mathsf{n}_i := \mathsf{n}_{F_i}$, both $1 \leqslant i \leqslant n$. Again, without loss of generality, we assume that n_i points from $T(\mathsf{v}_i)$ to $T(\mathsf{v}_{i+1})$, understood modulo n. With these notations, we can define similar to (4.53) as

$$\Psi_F^{\mathsf{cr,n,lo,3d}} = \sum_{i=1}^{n} \frac{1}{\omega_i} \Psi_{F_i}^{\mathsf{cr,m}} \mathsf{n}_i,$$

where $\omega_i = |F_i|$ is the weight corresponding to the mid-point quadrature rule on F_i.

With these test functions, we define the space

$$\begin{aligned} S_z^{\mathsf{S\text{-}CR3}} &= \mathcal{E}^{\mathsf{Tf}}(\mathrm{CR}_{1,z}^{\mathsf{divfr3}}) \quad \text{with} \\ \mathrm{CR}_{1,z}^{\mathsf{divfr3}} &= \Big(\operatorname{span}\big(\{\Psi_F^{\mathsf{cr,m}}\mathsf{t}_{F,1} \mid F \in \mathcal{F}_z^i\} \cup \{\Psi_F^{\mathsf{cr,m}}\mathsf{t}_{F,2} \mid F \in \mathcal{F}_z^i\} \\ &\qquad \cup \{\Psi_F^{\mathsf{cr,n,lo,3d}} \mid F \in \mathcal{F}_z^i \backslash F(\mathsf{A}_z^S)\}\big)\Big). \end{aligned}$$

The space of simple functionals D_z^{S} and the space of simple test function space $S_z^{\mathsf{S\text{-}CR3}}$ define via (2.22) a well-defined stable projection. The proof is similar to Theorems 4.44 and 4.37. This projection can be used to derive an estimator with an error-dominated oscillation. Of special interest is that a result corresponding to Theorem 4.40 can be derived if $\mathrm{CR}_1^d \subset V_h$ and $\mathcal{E}^{\mathsf{Tf}} = \mathrm{E_M}$.

Remark 4.47. Finally, we point out that $S_z^{\mathsf{S\text{-}CR3}}$ also forms a local, discretely divergence-free basis of space CR_1^3 restricted to ω_z. This follows from the initial paper [Hec81], which is refined in e.g. [RCnDLSR18], [AP99]. Compare also with Remark 4.41.

Remark 4.48. If one wants to adapt the a-posteriori analysis to new quasi-optimal, pressure-robust methods within the framework of [VZ19b, KZ20, KVZ21] that are not treated in this thesis, this can be most likely done in two steps. At first, one has to check Assumption 4.10. There the main difficulty probably is to provide a local Fortin operator, see Definition 4.2. Second, one has to apply one of the projection operators of section 4.2 to the new method. The main condition is (4.27). If the new method uses a velocity space consisting of piecewise polynomials, then one can often fulfill (4.27) by only adapting the polynomial degrees in D_z and then select the corresponding S_z.

5 A-priori estimates in weaker norms

We assume the setting of the framework of the chapter 1. In this chapter, we derive estimates in weaker norms within that framework. This is commonly referred to as the *Aubin-Nitsche* trick. The Aubin-Nitsche trick for classical conforming methods is described e.g. in [EG04, section 2.3.4] and for nonconforming methods in [ABCM02, section 5]. First, we derive an abstract setting. After that we apply it to our examples of chapter 1. Finally, we will look more closely at one important condition of the Aubin-Nitsche trick.

5.1 Abstract setting

The Aubin-Nitsche trick uses the analytical dual problem to obtain an auxiliary function. In the setting of chapter 1, the analytical dual problem for a given $g \in V'$ reads as

$$\text{Find } z \in V \text{ such that } a(v,z) = \langle g, v\rangle \quad \forall v \in V. \tag{5.1}$$

The estimates in weaker norms need a stronger bound than (1.13), which we call *double winged estimate*. This condition has already been introduced in [CN22] in display ($\hat{\mathbf{H}}$).

Definition 5.1. *We say two smoothers $E_1 : V_h \to V$ and $E_2 : V_h \to V$ fulfill the double winged estimate if*

$$a(E_1 v_h, E_2 w_h) - a_h(v_h, w_h) \leqslant C_{\mathsf{dw}} \|E_1 v_h - v_h\|_{\tilde{V}} \|E_2 w_h - w_h\|_{\tilde{V}} \quad \forall v_h, w_h \in V_h, \tag{5.2}$$

holds for some $C_{\mathsf{dw}} \geqslant 0$.

If the double winged estimate is fulfilled the following Theorem allows estimates of the discrete solution in weaker norms.

Theorem 5.2. *Let $u \in V$ be the analytic solution and $u_h \in V_h$ be the discrete solution within the setting of chapter 1. Assume we have two quasi-optimal smoothers $E_1 : V_h \to V$ and $E_2 : V_h \to V$ which fulfill the double winged estimate* (5.2)*. We suppose that we are given $l \in V'$ and $z \in V$ fulfilling*

$$\langle l, v\rangle = a(v,z) \quad \forall v \in V.$$

Then there is a constant $C_{AN} > 0$ such that

$$\langle l, u - E_1 u_h\rangle \leqslant C_{AN} \|u - u_h\|_{\tilde{V}} \|z - z_h\|_{\tilde{V}}$$

for any $z_h \in V_h$ with $E_2 z_h = \mathrm{E_M}\, z_h$ where $\mathrm{E_M} : V_h \to V$ is the smoother of the method.

Proof. Let $u \in V$ be the analytic solution of (1.3) and $u_h \in V_h$ the discrete solution of (1.12). Select $z_h \in V_h$ with $E_2 z_h = \mathrm{E_M}\, z_h$ then the definition of $z \in V$ yields:

$$\langle l, u - E_1 u_h\rangle = a(u - E_1 u_h, z) = a(u - E_1 u_h, z - \mathrm{E_M}\, z_h) + a(u - E_1 u_h, \mathrm{E_M}\, z_h). \tag{5.3}$$

The last term can be estimated with (5.2) as

$$a(u - E_1 u_h, \mathrm{E_M}\, z_h) = a_h(u_h, z_h) - a(E_1 u_h, \mathrm{E_M}\, z_h) \leqslant C_{\mathsf{dw}} \|E_1 u_h - u_h\|_{\tilde{V}} \|E_2 z_h - z_h\|_{\tilde{V}}. \tag{5.4}$$

Inserting (5.4) into the (5.3), we get

$$\begin{aligned}
\langle l, u - E_1 u_h\rangle &= a(u - E_1 u_h, z - \mathrm{E_M}\, z_h) + C_{\mathsf{dw}} \|E_1 u_h - u_h\|_{\tilde{V}} \|E_2 z_h - z_h\|_{\tilde{V}} \\
&\leqslant \|u - E_1 u_h\|_V \|z - E_2 z_h\|_V + C_{\mathsf{dw}} \|E_1 u_h - u_h\|_{\tilde{V}} \|E_2 z_h - z_h\|_{\tilde{V}} \\
&\leqslant (1 + C_{\mathsf{dw}}) \|\tilde{E_1}\|_{\mathbb{L}(\tilde{V},V)} \|\tilde{E_2}\|_{\mathbb{L}(\tilde{V},V)} \|u - u_h\|_{\tilde{V}} \|z - z_h\|_{\tilde{V}}
\end{aligned}$$

where we used Corollary 1.24 and (1.17). The operators $\tilde{E_1}$ and $\tilde{E_2}$ are the extended versions of E_1 and E_2 as introduced in Lemma 1.18. Consequently, we have $C_{\mathrm{AN}} \leqslant (1 + C_{\mathsf{dw}}) \|\tilde{E_1}\|_{\mathbb{L}(\tilde{V},V)} \|\tilde{E_2}\|_{\mathbb{L}(\tilde{V},V)}$. □

5.2 Application to examples

In this section, we show how to apply Theorem 5.2 to our examples of chapter 1 and start with P-CR1, P-CRL.

Corollary 5.3. *The Crouzeix-Raviart method fulfills the double winged estimate in* (5.2) *with* $C_{\mathsf{dw}} = 1$ *and* $E_1 = E_2 = \mathcal{E}_\ell$*, where* $\mathcal{E}_\ell : \mathrm{CR}_\ell \to H^1_0(\Omega)$ *is the quasi-optimal smoother introduced in Definition 1.35.*

This is an immediate consequence of the overconsistency. Therefore, we have the following estimate in the weaker norm.

Corollary 5.4. *Let* $l \in H^{-1}(\Omega)$ *and* $z \in H^1_0(\Omega)$ *solve*

$$\langle l, v\rangle = a_{\mathsf{P}}(v, z) \quad \forall v \in H^1_0(\Omega). \tag{5.5}$$

Then for some $C > 0$

$$\langle l, u - \mathcal{E}_\ell u_h\rangle \leqslant C \inf_{v_h \in \mathrm{CR}_\ell} \|u - v_h\|_{\tilde{V}} \inf_{z_h \in \mathrm{CR}_\ell} \|z - z_h\|_{\tilde{V}},$$

where $u \in H^1_0(\Omega)$ *is the analytic solution of* (1.3) *and* $u_h \in \mathrm{CR}_\ell$ *the discrete solution of P-CR1 or P-CRL. The constant* C *depends on* $\|\tilde{\mathcal{E}}_\ell\|_{\mathbb{L}(\tilde{V},V)}$*, which in turn depends only on* d *and* ℓ*.*

The Corollary enables us to derive a bound for the error in the L^2-norm. For this, we need a regularity result and a new definition. Let $D^2 v = (\partial_i \partial_j v)_{1 \leqslant i,j \leqslant d}$ denote the Hessian matrix of a function $v \in H^2(\Omega)$.

Proposition 5.5 ([Gri11, Theorem 3.2.1.2])**.** *If* $l \in L^2(\Omega)$ *and* Ω *a convex, bounded and open domain, then* (5.5) *has a unique solution* $z \in H^2(\Omega)$ *such that the bound* $\|D^2 z\|_{L^2(\Omega)} \lesssim \|l\|_{L^2(\Omega)}$ *holds.*

This regularity result allows for the following bound.

Proposition 5.6. *Assume that* Ω *is an open, bounded, polyhedral and convex domain such that Proposition 5.5 is applicable. Let* $u \in H^1_0(\Omega)$ *be the solution of* (1.30) *and* $u_h \in \mathrm{CR}_\ell$ *be the corresponding solution of P-CR1 or P-CRL with* $\ell \in \mathbb{N}$ *chosen appropriately. Let* $h_{\max} > 0$ *be such that* $h_{\max} \geqslant h_T$ *for all* $T \in \mathcal{T}$*. Then we have the estimate*

$$\|u - \mathcal{E}_\ell u_h\|_{L^2(\Omega)} \leqslant C h_{\max} \inf_{v_h \in \mathrm{CR}_\ell} \|u - v_h\|_{\tilde{V}}$$

for a $C \geqslant 1$ *independent of* $h_{\max}$*.*

Proof. Select $l \in H^{-1}(\Omega)$ as $\langle l, v\rangle := \int_\Omega (\mathcal{E}_\ell u_h - u)v$. Let $z \in H^1_0(\Omega)$ solve

$$\langle l, v\rangle = a(v, z) \quad \forall v \in H^1_0(\Omega),$$

then Proposition 5.5 provides a constant $C_1 > 0$ such that $\|D^2 z\|_{L^2(\Omega)} \leqslant C_1 \|u - \mathcal{E}_\ell u_h\|_{L^2(\Omega)}$. Corollary 5.4 shows

$$\int_\Omega (u - \mathcal{E}_\ell u_h)^2 = \langle l, u - \mathcal{E}_\ell u_h\rangle \leqslant \inf_{v_h \in \mathrm{CR}_\ell} \|u - v_h\|_{\tilde{V}} \inf_{z_h \in \mathrm{CR}_\ell} \|z - z_h\|_{\tilde{V}}.$$

To obtain the result we employ the Scott-Zhang operator (see Proposition 2.31 and [SZ90]) $\Pi_{\mathsf{SZ}} : H^1_0(\Omega) \to \mathring{S}^1_1(\mathcal{T}) \subset \mathrm{CR}_\ell$. In particular, Π_{SZ} fulfills for a $C_2 > 0$

$$\|z - \Pi_{\mathsf{SZ}} z\|_{L^2(\Omega)} \leqslant C_2 h_{\max} \|D^2 z\|_{L^2(\Omega)}.$$

This bound shows

$$\inf_{z_h \in \mathrm{CR}_\ell} \|z - z_h\|_{\tilde{V}} \leqslant \|z - \Pi_{\mathsf{SZ}} z\|_{\tilde{V}} \leqslant C_2 h_{\max} \|D^2 z\|_{L^2(\Omega)} \leqslant C_1 C_2 h_{\max} \|u - \mathcal{E}_\ell u_h\|_{L^2(\Omega)}.$$

Absorbing the last factor yields the result. □

After the Crouzeix-Raviart element, we turn to the dG-methods. The P-SIP and P-RSIP methods fulfill the double winged estimate as well.

Lemma 5.7. *The methods P-SIP and P-RSIP with* $\theta = -1$ *for arbitrary polynomial degree* $\ell \geqslant 1$ *fulfill the double winged estimate* (5.2) *with* $E_1 = E_2 = \mathcal{E}_\ell$*.*

Proof. We compute for $v_h, w_h \in S^0_\ell(\mathcal{T})$

$$\begin{aligned}
&a_{\mathsf{P}}(\mathcal{E}_\ell v_h, \mathcal{E}_\ell w_h) - a_h^{\mathsf{dG}}(v_h, w_h)\\
&\quad= \tilde{a}^{\mathsf{dG}}(\mathcal{E}_\ell v_h, w_h) + \tilde{a}^{\mathsf{dG}}(\mathcal{E}_\ell v_h, \mathcal{E}_\ell w_h - w_h) - a_h^{\mathsf{dG}}(v_h, w_h)\\
&\quad= \tilde{a}^{\mathsf{dG}}(v_h, w_h) + \tilde{a}^{\mathsf{dG}}(\mathcal{E}_\ell v_h - v_h, w_h) + \tilde{a}^{\mathsf{dG}}(v_h, \mathcal{E}_\ell w_h - w_h)\\
&\qquad+ \tilde{a}^{\mathsf{dG}}(\mathcal{E}_\ell v_h - v_h, \mathcal{E}_\ell w_h - w_h) - a_h^{\mathsf{dG}}(v_h, w_h).
\end{aligned}$$

Lemma 1.46 shows

$$\begin{aligned}
a_{\mathsf{P}}(\mathcal{E}_\ell v_h, \mathcal{E}_\ell w_h) - a_h^{\mathsf{dG}}(v_h, w_h) &= \int_\Omega \nabla_h v_h \cdot \nabla_h w_h + \int_\Sigma \frac{\gamma}{h} [\![v_h]\!] [\![w_h]\!]\\
-\int_\Sigma [\![v_h]\!] \{\{\nabla_h w_h \cdot \mathsf{n}\}\} - \int_\Sigma \frac{\gamma}{h} [\![v_h]\!] [\![w_h]\!] - \int_\Sigma \{\{\nabla_h v_h \cdot \mathsf{n}\}\} [\![w_h]\!] - \int_\Sigma \frac{\gamma}{h} [\![v_h]\!] [\![w_h]\!] &\qquad (5.6)\\
+ \int_\Omega \nabla_h(\mathcal{E}_\ell v_h - v_h) \cdot \nabla_h(\mathcal{E}_\ell w_h - w_h) + \int_\Sigma \frac{\gamma}{h} [\![v_h]\!] [\![w_h]\!] - a_h^{\mathsf{dG}}(v_h, w_h).
\end{aligned}$$

In this expression, many terms cancel owing to the definition of a_h^{dG}, thus

$$\begin{aligned}
\left|a(\mathcal{E}_\ell v_h, \mathcal{E}_\ell w_h) - a_h^{\mathsf{dG}}(v_h, w_h)\right| &= \left|\int_\Omega \nabla_h(\mathcal{E}_\ell v_h - v_h) \cdot \nabla_h(\mathcal{E}_\ell w_h - w_h) - \int_\Sigma \frac{\gamma}{h} [\![v_h]\!] [\![w_h]\!]\right|\\
&\lesssim \|\mathcal{E}_\ell v_h - v_h\|_{\mathsf{dG}} \|\mathcal{E}_\ell w_h - w_h\|_{\mathsf{dG}}.
\end{aligned}$$

□

Remark 5.8. The double winged inequality (5.2) cannot hold for P-NSIP with $E_1 = E_2 = \mathcal{E}_\ell$ because it would imply algebraic-consistency and hence quasi-optimality for the dual P-NSIP method since $\mathcal{E}_\ell$ is a quasi-optimal smoother. But quasi-optimality is ruled out for the dual P-NSIP method because for appropriate $z \in H^1_0(\Omega) \cap S^0_\ell(\mathcal{T})$ and $v_h \in S^0_\ell(\mathcal{T})$, we have

$$a_h^{\mathsf{dG}}(v_h, z) - a_{\mathsf{P}}(\mathcal{E}_\ell v_h, z) = 2\int_\Sigma [\![v_h]\!] \{\{\nabla_h z \cdot \mathsf{n}\}\} \neq 0.$$

Similarly as in Corollary 5.4, we formulate the Aubin-Nitsche trick applied to P-SIP. Theorem 5.2 directly yields the following Corollary.

Corollary 5.9. *Given $l \in H^{-1}(\Omega)$ and $z \in H^1_0(\Omega)$ solving*

$$\langle l, v\rangle = a_{\mathsf{P}}(v, z) \quad \forall v \in H^1_0(\Omega).$$

Then

$$\langle l, u - \mathcal{E}_\ell u_h\rangle \leqslant C \inf_{v_h \in S^0_\ell(\mathcal{T})} \|u - v_h\|_{\tilde{V}} \inf_{z_h \in S^0_\ell(\mathcal{T})} \|z - z_h\|_{\tilde{V}},$$

where $u \in H^1_0(\Omega)$ the analytic solution of (1.3), $u_h \in S^0_\ell(\mathcal{T})$ the discrete solution of the quasi-optimal P-SIP method given by (1.10), $a_h^{\mathsf{dG}}(\cdot,\cdot)$ as defined in (1.57) with $\theta = -1$ and the smoother $\mathrm{E_M} = \mathcal{E}_\ell$.

Interestingly, the double winged inequality also holds for the smoother $\mathcal{E}_{\mathsf{red},\ell}$ as shown in Lemma 5.7. Exploiting the generality of Theorem 5.2, we also have the Aubin-Nitsche trick for P-RSIP as well.

Corollary 5.10. *Let $l \in H^{-1}(\Omega)$ and $z \in H^1_0(\Omega)$ such that*

$$\langle l, v\rangle = a(v, z) \quad \forall v \in H^1_0(\Omega).$$

Then

$$\langle l, u - \mathcal{E}_\ell u_h\rangle \leqslant C \inf_{v_h \in S^0_\ell(\mathcal{T})} \|u - v_h\|_{\tilde{V}} \inf_{z_h \in \mathring{S}^1_1(\mathcal{T})} \|z - z_h\|_{\tilde{V}},$$

where $u \in H^1_0(\Omega)$ is the analytic solution of (1.3) and $u_h \in S^0_\ell(\mathcal{T})$ is the discrete solution of the quasi-optimal P-RSIP method.

Proof. We apply Theorem 5.2 with $E_1 = E_2 = \mathcal{E}_\ell$ and use that $\mathcal{E}_\ell z_h = \mathcal{E}_{\mathsf{red},\ell} z_h$ for all $z_h \in \mathring{S}^1_1(\mathcal{T})$. □

By choosing $l \in H^{-1}(\Omega)$ as $l : v \mapsto \int_\Omega (u - \mathcal{E}_\ell u_h)v$, $v \in H^1_0(\Omega)$, where $u \in H^1_0(\Omega)$ and $u_h \in S^0_\ell(\mathcal{T})$ are the analytic resp. P-SIP-discrete or P-RSIP-discrete solutions as in the Corollary, and Ω convex, bounded and open, we obtain a result which is analogous to Proposition 5.6. In the case P-RSIP, it is important to use the first order Scott-Zhang operator. We do not dwell on those details because the proof straightforwardly adapts.

We now turn to the symmetric gradient setting of the method E-CR1, which also fulfills the double winged estimate.

Lemma 5.11. *In the setting of E-CR1, the bilinear form $\tilde{a}^{\mathsf{LE}}$ in (1.61) satisfies the double winged estimate in (5.2) with the vectorized smoother $\mathcal{E}_1^d : V_h \to V$. Especially, the constant C_{dw} does not depend on μ or λ.*

Proof. We compute for all $v_h, w_h \in V_h$ with the help of (1.62) and (1.63)

$$\begin{aligned}
&a_{\mathsf{LE}}(\mathcal{E}_1^d v_h, \mathcal{E}_1^d w_h) - a_h^{\mathsf{LE}}(v_h, w_h)\\
&= \tilde{a}^{\mathsf{LE}}(\mathcal{E}_1^d v_h, w_h) + \tilde{a}^{\mathsf{LE}}(\mathcal{E}_1^d v_h, \mathcal{E}_1^d w_h - w_h) - a_h^{\mathsf{LE}}(v_h, w_h)\\
&= \tilde{a}^{\mathsf{LE}}(v_h, w_h) + \tilde{a}^{\mathsf{LE}}(\mathcal{E}_1^d v_h - v_h, w_h) + \tilde{a}^{\mathsf{LE}}(v_h, \mathcal{E}_1^d w_h - w_h)\\
&\quad + \tilde{a}^{\mathsf{LE}}(\mathcal{E}_1^d v_h - v_h, \mathcal{E}_1^d w_h - w_h) - a_h^{\mathsf{LE}}(v_h, w_h)\\
&= \int_\Omega 2\mu\varepsilon_h(\mathcal{E}_1^d v_h - v_h) : \varepsilon_h(\mathcal{E}_1^d w_h - w_h) + \lambda \operatorname{div}_h(\mathcal{E}_1^d v_h - v_h) \operatorname{div}_h(\mathcal{E}_1^d w_h - w_h)\\
&\quad - \int_\Sigma \frac{\mu\gamma}{h} [\![v_h]\!] \cdot [\![w_h]\!] \lesssim \|\mathcal{E}_1^d v_h - v_h\|_{\tilde{V}} \|\mathcal{E}_1^d w_h - w_h\|_{\tilde{V}}.
\end{aligned}$$

□

Corollary 5.12. *Let $u \in H_0^1(\Omega)^2$ and $u_h \in (\mathrm{CR}_1)^2$ be the analytic and discrete solution of the method E-CR1. Then we have for a constant $C > 0$*

$$\langle l, u - \mathcal{E}_1^d u_h\rangle \leqslant C \inf_{v_h \in (\mathrm{CR}_1)^2} \|u - v_h\|_{\tilde{V}} \inf_{z_h \in (\mathrm{CR}_1)^2} \|z - z_h\|_{\tilde{V}}.$$

If Ω is a convex, polyhedral, bounded and open domain and let $h_{\max} > 0$ be such that $h_{\max} \geqslant h_T$ for all $T \in \mathcal{T}$ then

$$\|u - \mathcal{E}_1^d u_h\|_{L^2(\Omega)} \leqslant C h_{\max} \inf_{v_h \in \mathrm{CR}_1} \|u - v_h\|_{\tilde{V}}.$$

Proof. The proof works as in the examples before, the corresponding regularity result can be found in [BS92, Lemma 2.2] and [Val88]. □

We turn to the biharmonic examples. The authors of [CN22] and [CN21] already derived the estimates in the weaker norms but for consistency with our notation, we recall them.

Lemma 5.13. *The methods B-MR and B-SCZ fulfill the double winged estimate (5.2) with $E_1 = E_2 = \mathcal{E}_{\mathsf{hct}}$.*

Proof. In the case B-MR the double winged estimate follows by the overconsistency with $C_{\mathsf{dw}} = 0$.

In the case B-SCZ the estimate follows in the same way as in Lemma 5.7: Inserting the appropriate terms, using Lemma 1.67 with (1.71b) and that $\mathcal{E}_{\mathsf{hct}}$ maps into $H_0^2(\Omega)$. □

Corollary 5.14. *Let $u \in H_0^2(\Omega)$ be the analytic solution of (1.66) and $u_h \in S_2^0(\mathcal{T})$ the discrete solution of the either B-MR or B-SCZ. Furthermore, given $l \in H^{-2}(\Omega)$, let $z \in H_0^2(\Omega)$ be the solution of the dual problem*

$$\int_\Omega D^2 v : D^2 z = \langle l, v\rangle \quad \forall v \in H_0^2(\Omega).$$

Then we have for $C > 0$

- *in the case of B-MR*

$$\langle l, u - \mathcal{E}_{\mathsf{hct}} u_h\rangle \leqslant C \inf_{v_h \in \mathrm{MR}} \|u - v_h\|_{\tilde{V}} \inf_{z_h \in \mathrm{MR}} \|z - z_h\|_{\tilde{V}},$$

- *and in the case of B-SCZ*

$$\langle l, u - \mathcal{E}_{\mathsf{hct}} u_h\rangle \leqslant C \inf_{v_h \in \mathring{S}_2^1(\mathcal{T})} \|u - v_h\|_{\tilde{V}} \inf_{z_h \in \mathring{S}_2^1(\mathcal{T})} \|z - z_h\|_{\tilde{V}}.$$

With the following result, we can derive with the Aubin-Nitsche estimates in the H^1-seminorm.

Proposition 5.15 ([Gri11])**.** *Under suitable boundary regularity, we have for $s \in \{0,1,2\}$ and the weak solution $z \in H_0^2(\Omega)$ of the biharmonic problem*

$$\int_\Omega D^2 v : D^2 z = \langle l, v\rangle \quad \forall v \in H_0^2(\Omega) \tag{5.7}$$

with $l \in H^{-2+s}$ that

$$\|z\|_{H^{2+s}} \leqslant C_{\mathsf{reg}} \|l\|_{H^{-2+s}} \tag{5.8}$$

holds.

The result and conditions on the boundary such that this regularity result is applicable are found in [Gri11]. In particular, Proposition 5.15 provides the estimates to bound the error in the H^1-seminorm. This is the topic of the next Proposition, see also [CN22].

Proposition 5.16 (Aubin Nitsche Trick for B-MR, B-SCZ)**.** *Assume that the boundary $\partial\Omega$ is regular enough that Proposition 5.15 is applicable for $s = 1$. Let $h_{\max} > 0$ be such that $h_{\max} \geqslant h_T$ for all $T \in \mathcal{T}$. Then there exists a $C > 0$ independent of $h_{\max}$ such that for the analytic solution $u \in H_0^2(\Omega)$ and the discrete solution $u_h \in V_h$ we have*

$$\|\nabla(u - \mathcal{E}_{\mathsf{hct}} u_h)\|_{L^2(\Omega)} \leqslant C h_{\max} \inf_{v_h \in V_h} \|u - v_h\|_{\tilde{V}}.$$

Proof. Define $l \in H^{-2}(\Omega)$ as

$$\langle l, v\rangle = \int_\Omega \nabla(\mathcal{E}_{\mathsf{hct}} u_h - u) \cdot \nabla v$$

with $v \in H_0^2(\Omega)$. Let $z \in H_0^2(\Omega)$ be the solution of (5.7). Because Proposition 5.15 is applicable, we have a constant C_{reg} such that (5.8) holds for $s = 1$. Together with Corollary 5.14 we have

$$\|\nabla(u - \mathcal{E}_{\mathsf{hct}} u_h)\|^2_{L^2(\Omega)} = \langle l, u - \mathcal{E}_{\mathsf{hct}} u_h\rangle \leqslant C \inf_{v_h \in V_h} \|u - v_h\|_{\tilde{V}} \inf_{z_h \in V_h} \|z - z_h\|_{\tilde{V}}.$$

The last factor is now estimated with the help of the Morley or Lagrange interpolation operator.

1. In the case of B-MR, we use the Morley interpolation operator in combination with the estimates provided in Proposition 2.48. Those estimates are extended to this situation by the arguments in [BS08, Theorem 4.4.4].

2. In the case of B-SCZ, we use the Lagrange interpolation operator with similar estimates: We use the trace theorem to eliminate the jumps and then [BS08, Theorem 4.4.4].

In both cases, we obtain with (5.8) the estimate

$$\inf_{z_h \in V_h} \|z - z_h\|_{\tilde{V}} \lesssim h_{\max} \|z\|_{H^3} \lesssim h_{\max} \|l\|_{H^{-1}(\Omega)} = h_{\max} \|\nabla(u - \mathcal{E}_{\mathsf{hct}} u_h)\|_{L^2(\Omega)},$$

which finishes the proof. □

Remark 5.17. Since the methods B-MR and B-SCZ only use second order polynomials, we cannot obtain L^2-estimates. Moreover, the Aubin-Nitsche trick will most likely be not available for the non-symmetric counterpart of B-SCZ, called B-NCZ. This is supported by numerical experiments in [SM07], although the article considers non quasi-optimal methods and a slightly different formulation of the bilinear forms.

5.3 Further considerations

After we have seen that many of our examples fulfill the double winged estimate, but some do not, the question arises: What is the fundamental mechanism behind the estimate? What are the essential properties of a method such that (5.2) is true?

The P-NSIP method does not fulfill the double winged estimate with the smoothers $E_1 = E_2 = \mathcal{E}_\ell$. But if we choose a different smoother for E_1 then the double winged estimate is true and the Aubin-Nitsche trick is applicable to get a bound on $\|u - \mathcal{E}_\ell u_h\|_{L^2(\Omega)}$.

In particular, if we choose $E_1 = 2\Pi_V - \mathcal{E}_\ell$, where Π_V is the $\tilde{a}$-orthogonal projection, then we can show the double winged estimate. The new smoother E_1 is quasi-optimal since

$$\|v_h - E_1 v_h\|_{\tilde{V}} \leqslant \|v_h - \Pi_V v_h\|_{\tilde{V}} + \|\Pi_V v_h - \mathcal{E}_\ell v_h\|_{\tilde{V}} \lesssim (1 + \gamma_{V_h})\|v_h - \Pi_V v_h\|_{\tilde{V}},$$

where γ_{V_h} is the constant for $\mathcal{E}_\ell$ from Proposition 1.23. To check the double winged estimate, we proceed as in the symmetric case of P-SIP. We have for $v_h, w_h \in S^0_\ell(\mathcal{T})$

$$\begin{aligned} &a_{\mathsf{P}}(E_1 v_h, \mathcal{E}_\ell w_h) - a_h^{\mathsf{dG}}(v_h, w_h) \\ &\qquad = \tilde{a}^{\mathsf{dG}}(v_h, \mathcal{E}_\ell w_h) + \tilde{a}^{\mathsf{dG}}(E_1 v_h - v_h, \mathcal{E}_\ell w_h) - a_h^{\mathsf{dG}}(v_h, w_h). \end{aligned}$$

We rewrite the second summand as $\tilde{a}^{\mathsf{dG}}(E_1 v_h - v_h, \mathcal{E}_\ell w_h) = \tilde{a}^{\mathsf{dG}}(v_h - \mathcal{E}_\ell v_h, \mathcal{E}_\ell w_h)$ and continue

$$\begin{aligned} &a_{\mathsf{P}}(E_1 v_h, \mathcal{E}_\ell w_h) - a_h^{\mathsf{dG}}(v_h, w_h) \\ &\quad = \tilde{a}^{\mathsf{dG}}(v_h, \mathcal{E}_\ell w_h) + \tilde{a}^{\mathsf{dG}}(v_h - \mathcal{E}_\ell v_h, \mathcal{E}_\ell w_h) - a_h^{\mathsf{dG}}(v_h, w_h) \\ &\quad = \tilde{a}^{\mathsf{dG}}(v_h, w_h) + \tilde{a}^{\mathsf{dG}}(v_h, \mathcal{E}_\ell w_h - w_h) \\ &\qquad + \tilde{a}^{\mathsf{dG}}(v_h - \mathcal{E}_\ell v_h, w_h) + \tilde{a}^{\mathsf{dG}}(v_h - \mathcal{E}_\ell v_h, \mathcal{E}_\ell w_h - w_h) - a_h^{\mathsf{dG}}(v_h, w_h). \end{aligned}$$

Similar to (5.6), we have with Lemma 1.46

$$\begin{aligned} a_{\mathsf{P}}(E_1 v_h, \mathcal{E}_\ell w_h) - a_h^{\mathsf{dG}}(v_h, w_h) &= \int_\Omega \nabla_h v_h \cdot \nabla_h w_h + \int_\Sigma \frac{\gamma}{h} [\![v_h]\!] [\![w_h]\!] \\ &+ \int_\Sigma [\![v_h]\!] \{\{\nabla_h w_h \cdot \mathsf{n}\}\} + \int_\Sigma \frac{\gamma}{h} [\![v_h]\!] [\![w_h]\!] - \int_\Sigma \{\{\nabla_h v_h \cdot \mathsf{n}\}\} [\![w_h]\!] - \int_\Sigma \frac{\gamma}{h} [\![v_h]\!] [\![w_h]\!] \\ &+ \int_\Omega \nabla_h(\mathcal{E}_\ell v_h - v_h) \cdot \nabla_h(\mathcal{E}_\ell w_h - w_h) + \int_\Sigma \frac{\gamma}{h} [\![v_h]\!] [\![w_h]\!] - a_h^{\mathsf{dG}}(v_h, w_h). \end{aligned}$$

Again, many terms cancel, and we arrive at

$$\begin{aligned} a_{\mathsf{P}}(E_1 v_h, \mathcal{E}_\ell w_h) - a_h^{\mathsf{dG}}(v_h, w_h) &= \int_\Omega \nabla_h(\mathcal{E}_\ell v_h - v_h) \cdot \nabla_h(\mathcal{E}_\ell w_h - w_h) + \int_\Sigma \frac{\gamma}{h} [\![v_h]\!] [\![w_h]\!] \\ &\lesssim \|\mathcal{E}_\ell v_h - v_h\|_{\mathsf{dG}} \|\mathcal{E}_\ell w_h - w_h\|_{\mathsf{dG}} \lesssim \|E_1 v_h - v_h\|_{\mathsf{dG}} \|\mathcal{E}_\ell w_h - w_h\|_{\mathsf{dG}} \end{aligned}$$

where we used the quasi-optimality of $\mathcal{E}_\ell$. Thus, the double-winged inequality is fulfilled and we can apply Theorem 5.2. If the domain Ω is in addition convex, this result provides similar to Corollary 5.9 an estimate for $\|E_1 u_h - u\|_{L^2(\Omega)}$, where $u_h \in S^0_\ell(\mathcal{T})$ the discrete solution of P-NSIP and $u \in H^1_0(\Omega)$ the corresponding analytic solution. Therefore, we obtain an L^2-error estimate if we employ the smoothing operator E_1.

This result raises the question if such a modification of the smoother is always possible to obtain an estimate in a weaker norm. This would imply that the solution u_h, obtained with the smoother $\mathcal{E}_\ell$ and discrete bilinear form a_h, already encodes the decisive aspects of the solution u that make the estimate in a weaker norm possible. Actually, the methods P-SIP and P-NSIP use the same information to compute their solutions.

More precisely, if we have a method M_1, which uses a smoother E, discrete space V_h and a discrete bilinear form $a_{1,h}$, only uses the information $f(Ew_h)$, $w_h \in V_h$, to compute the solution $u_{1,h} \in V_h$. We assume that M_1 allows an estimate in a weaker norm, e.g. L^2-estimates.

Now, we consider another method M_2, which uses the same discrete space V_h, the smoother E but possibly different bilinear form $a_{2,h}$ to compute a discrete solution $u_{2,h} \in V_h$. We assume that M_2 does not allow the estimate in the weaker norm. But in principle both methods, M_1 and M_2 have access to the same information.

Both methods use the information encoded in $f(Ew_h)$, $w_h \in V_h$, to compute the solution. In principle, this means that, given $u_{2,h}$, we can reconstruct $f(Ew_h)$ for all $w_h \in V_h$ and use method M_1 to compute the solution $u_{1,h} \in V_h$ and obtain an approximation, which satisfies the desired estimate in the weaker norm. Therefore, the information for the estimate of the weaker norm is already encoded in $u_{h,2}$.

We remark that the smoother $E_1 = 2\Pi_V - \mathcal{E}_\ell$, although not computability feasible, induces a quasi-optimal dual method. It fulfills the algebraic-consistency condition: For $v_h \in S^0_\ell(\mathcal{T})$ and $w \in H^1_0(\Omega)$, we have with Lemma 1.46

$$\begin{aligned} a_h^{\mathsf{dG}}(v_h, w) = \tilde{a}^{\mathsf{dG}}(v_h, w) + \int_\Sigma [\![v_h]\!] \{\{\nabla w\}\} &= \tilde{a}^{\mathsf{dG}}(v_h, w) - \tilde{a}^{\mathsf{dG}}(\mathcal{E}_\ell v_h - v_h, w) \\ &= \tilde{a}^{\mathsf{dG}}(\Pi_V v_h, w) - \tilde{a}^{\mathsf{dG}}(\mathcal{E}_\ell v_h - \Pi_V v_h, w) = \tilde{a}^{\mathsf{dG}}(E_1 v_h, w). \end{aligned}$$

Consequently, the primal and dual method are quasi-optimal.

Now one could guess that the double winged estimate holds if the smoothers are chosen in such a way that the discrete dual problem is quasi-optimal as well. However, this is not the case as the following example shows. We construct a sequence of spaces V_n, $n \in \mathbb{N}$, which approximates the analytic space V as $n \to \infty$. In this example, both primal and dual problem are quasi-optimal for each $n \in \mathbb{N}$. Even the consistency measures, introduced in Proposition 1.16, stay bounded as $n \to \infty$, while the constant C_{dw} in (5.2) tends to infinity.

We consider the standard sequence spaces, see [Aub00, Example 1.1.2].

Definition 5.18. *Given a sequence v, we denote with v_i, $i \in \mathbb{N}$, the individual components. To denote the sequence v, we also write $(v_i)_{i\in\mathbb{N}}$ or (v_i). Define the vector space*

$$\ell_2 = \left\{ (v_i)_{i\in\mathbb{N}} \;\middle|\; v_i \in \mathbb{R} \quad \forall i \in \mathbb{N} \quad \textit{and} \quad \sum_{i=1}^{\infty} |v_i|^2 < \infty \right\} \tag{5.9}$$

with componentwise addition and scalar multiplication. We equip the space ℓ_2 with the scalar product and norm

$$a_2(v,w) := \sum_{i=1}^{\infty} v_i w_i, \qquad \|v\|_2^2 = \sum_{i=1}^{\infty} |v_i|^2, \quad v, w \in \ell_2.$$

It is well known that ℓ_2 is a Hilbert space, see [Aub00, Theorem 1.1.2].

Having recalled the classical sequence space, we introduce our setting based upon the framework of chapter 1. We define the analytic space as

$$V := \{(v_i)_{i\in\mathbb{N}} \in \ell_2 \mid v_i = 0 \text{ for all } i \in \mathbb{N} \text{ which are odd}\}.$$

The topology of this space shall be induced by the scalar product $a_2(\cdot,\cdot)$. Denoting by V' the dual of V, we define the analytical problem as

$$\text{Given } f \in V', \text{ find } u \in V \text{ such that } a_2(u,w) = \langle f, w\rangle \quad \forall w \in V. \tag{5.10}$$

The problem is well posed because V is closed in ℓ_2.

We define for each $j \in \mathbb{N}$ the sequence $e^j := (e_i^j)_{i\in\mathbb{N}} \in \ell^2$ with $e_i^j := \delta_{i,j}$. To discretize (5.10), we introduce for each $n \in \mathbb{N}$ a discrete space defined as

$$V^n := \operatorname{span}\left\{ e^{2j} + \frac{1}{n} e^{2j-1} \;\middle|\; 1 \leqslant j \leqslant n \right\} \subset \ell^2.$$

These vector spaces are equipped with the topology induced by $a_2(\cdot,\cdot)$. Since they are finite-dimensional, they are closed within ℓ_2. The vector spaces V^n, $n \in \mathbb{N}$, are not contained in V, and thus we have a nonconforming discretization. To build a quasi-optimal method using the spaces V^n, $n \in \mathbb{N}$, we apply the theory recalled in chapter 1. There, we learned that a quasi-optimal method needs to be fully stable and fully algebraic-consistent, see Proposition 1.12. That Proposition also reveals that a quasi-optimal method necessarily has a smoother fulfilling (1.12). For this reason, we define for each V^n, $n \in \mathbb{N}$, a smoothing operator.

Given $n \in \mathbb{N}$ and V^n, we define the smoothing operator E^n as

$$\begin{aligned} E^n : V^n \to V, \quad (v_j^n)_{j\in\mathbb{N}} \mapsto (w_j)_{j\in\mathbb{N}}, \qquad & \text{with } v^n \in V^n \quad \text{and} \quad w \in V, \\ \text{the latter defined via} \quad w_j := \begin{cases} 0 & \text{if } j \text{ is odd,} \\ v_{j-1}^n + v_j^n & \text{if } j \text{ is even,} \end{cases} & \quad \forall j \in \mathbb{N}. \end{aligned}$$

Given $n \in \mathbb{N}$ and a sequence $v^n = (v_j^n)_{j\in\mathbb{N}} \in V^n$, we compute the ℓ_2-norm of $E^n(v^n)$ as

$$\|E^n(v^n)\|_2^2 = \sum_{\substack{j\in\mathbb{N} \\ j \text{ even}}} \left(v_{j-1}^n + v_j^n\right)^2 \leqslant 2 \sum_{\substack{j\in\mathbb{N} \\ j \text{ even}}} (v_{j-1}^n)^2 + (v_j^n)^2 = 2\|v^n\|_2^2.$$

Hence, all operators E^n, $n \in \mathbb{N}$, are bounded with a constant independent of n. These operators allow us to define for each $n \in \mathbb{N}$ the discrete method:

$$\text{Given } f \in V', \text{ find } u^n \in V^n \text{ such that } a_2(u^n, v^n) = \langle f, E^n v^n\rangle \quad \forall v^n \in V^n \text{ holds.} \tag{5.11}$$

Since we formulate the method (5.11) with a bounded smoothing operator, the method is fully stable by Proposition 1.10. Moreover, we have $V \cap V^n = \{(0)_{j\in\mathbb{N}}\}$. Thus, condition (1.12) is satisfied, and the method (5.11) is quasi-optimal.

Due to the symmetry of a_2, the dual problem of (5.10) is the same:

$$\text{Given } g \in V', \text{ find } z \in V \text{ such that } a_2(v,z) = \langle f, v\rangle \quad \forall v \in V. \tag{5.12}$$

We discretize the analytical dual problem with the same spaces. Due to the symmetry, it reads for $n \in \mathbb{N}$ as

$$\text{Find } z^n \in V^n, \text{ such that } a_2(v^n, z^n) = a_2(E^n v^n, z) \quad \forall v^n \in V^n \text{ holds.} \tag{5.13}$$

The function $z \in V$ is the solution of the analytical dual problem, which is the same as the primal one because $a(\cdot,\cdot)$ is symmetric.

We investigate the double winged estimate (5.2) in this setting. We show that the consistency measures are uniformly bounded with respect to $n \in \mathbb{N}$ but the constant C_{dw} in the double winged estimate tends to infinity as $n \to \infty$. First, we show the latter.

Fix an $n \in \mathbb{N}$. To quantify the constant C_{dw} in (5.2), we compute the conformity error $\|E^n v^n - v^n\|_2$ for $v^n = e^{2j} + \frac{1}{n}e^{2j-1} \in V^n, 1 \leqslant j \leqslant n$:

$$\|E^n v^n - v^n\|_2^2 = \Big(1 + \frac{1}{n} - 1\Big)^2 + \frac{1}{n^2} = \frac{2}{n^2}.$$

On the contrary, the value of $a_2(E^n v^n, Ev^n) - a_2(v^n, v^n)$ is

$$a_2(Ev^n, Ev^n) - a_2(v^n, v^n) = \Big(1 + \frac{1}{n}\Big)^2 - \Big(1^2 + \frac{1}{n^2}\Big) = \frac{2}{n}.$$

With this, we see that the inequality $|a_2(E^n v^n, E^n v^n) - a_2(v^n, v^n)| \leqslant C\|E^n v^n - v^n\|_2 \|E^n v^n - v^n\|_2$ cannot hold with a constant independent of n.

For a fixed $n \in \mathbb{N}$, we compute the consistency measures δ_V, δ_{V^n} for the primal method (5.11), as in Definition 1.15 or [VZ18a, Proposition 3.6 and Proposition 3.9], where we adapt the notation to the current setting. The corresponding measures for dual discretizations are the same because of the symmetry of (5.11).

For the reader's convenience, we recall the definition of δ_V: Let $\tilde{V}^n := V + V^n$ be the joint vector space equipped with the norm $\|\cdot\|_2$. Let $\Pi_{V^n} : \tilde{V}^n \to V^n$ be the orthogonal projection onto V^n with respect to the scalar product a_2. Then $\delta_V \geqslant 0$ is the smallest constant such that

$$\forall v \in V \qquad \|\Pi_{V^n} v - P^n v\|_2 \leqslant \delta_V \|v - \Pi_{V^n} v\|_2 \tag{5.14}$$

holds. Here, $P^n : V \to V^n$ is the map, which takes the analytic solution $u \in V$ to the discrete solution $u^n \in V^n$ of the method given by (5.11).

To quantify the constant δ_V for a fixed $n \in \mathbb{N}$, we compute $\Pi_{V^n} e^i$ and $P^n e^i$, for $i \in \mathbb{N}$ and i even. The orthogonal projection $\Pi_{V^n} e^i \in V^n$ is determined by the equation

$$a_2(\Pi_{V^n} e^i, w) = a_2(e^i, w) \quad \forall w \in V^n. \tag{5.15}$$

We express the sequence $\Pi_{V^n} e^i$ in terms of the functions which span the space V^n with coefficients $b_k \in \mathbb{R}$, $1 \leqslant k \leqslant n$:

$$\Pi_{V^n} e^i = \sum_{k=1}^{n} b_k \Big(e^{2k} + \frac{1}{n} e^{2k-1}\Big).$$

In equation (5.15), we set the function w to $e^{2j} + \frac{1}{n}e^{2j-1} \in V^n$ for a $1 \leqslant j \leqslant n$. Then we have on the left-hand side of (5.15) for all $1 \leqslant j \leqslant n$:

$$a_2\Big(\Pi_{V^n} e^i, e^{2j} + \frac{1}{n} e^{2j-1}\Big) = \sum_{k=1}^{n} b_k a_2\Big(e^{2k} + \frac{1}{n} e^{2k-1}, e^{2j} + \frac{1}{n} e^{2j-1}\Big) = \sum_{k=1}^{n} \delta_{k,j} b_k \Big(1 + \frac{1}{n^2}\Big) = b_j \Big(1 + \frac{1}{n^2}\Big). \tag{5.16}$$

On the right-hand side of (5.15), we have $a_2(e^i, e^{2j} + \frac{1}{n}e^{2j-1}) = 0$ for all $1 \leqslant j \leqslant n$ with $2j \neq i$ because i is even. Therefore, $b_j = 0$ for all $1 \leqslant j \leqslant n$ with $2j \neq i$. If $1 \leqslant j \leqslant n$ and $2j = i$, then we have $a_2(e^i, e^{2j} + \frac{1}{n}e^{2j-1}) = 1$.

Hence, if $i \leqslant 2n$, we have $b_{\frac{i}{2}} = \frac{n^2}{n^2+1}$ and

$$\Pi_{V^n}(e^i) = \frac{n^2}{n^2+1}\Big(e^i + \frac{1}{n} e^{i-1}\Big). \tag{5.17}$$

If $i > 2n$, we have $\Pi_{V^n}(e^i) = 0$.

It remains to compute $P(e^i)$ for a fixed $i \in \mathbb{N}$, i even. We again represent the sequence $P(e^i)$ in terms of the functions spanning V^n with coefficients $d_k \in \mathbb{R}$, $1 \leqslant k \leqslant n$:

$$P(e^i) = \sum_{k=1}^{n} d_k \Big(e^{2k} + \frac{1}{n} e^{2k-1} \Big).$$

Equation (5.11) determines $P(e^i)$, which reads in this situation

$$a_2(P(e^i), w) = a_2(e^i, E^n w) \quad \forall w \in V^n. \tag{5.18}$$

We replace w in (5.18) with the functions spanning V^n, which are $e^{2j} + \frac{1}{n} e^{2j-1}$, $1 \leqslant j \leqslant n$.

With a fixed $j \in \mathbb{N}$, $1 \leqslant j \leqslant n$, the left-hand side of (5.18) becomes as in (5.16):

$$\begin{aligned} a_2\Big(P(e^i), e^{2j} + \frac{1}{n} e^{2j-1} \Big) &= \sum_{k=1}^{n} d_k a_2\Big(e^{2k} + \frac{1}{n} e^{2k-1}, e^{2j} + \frac{1}{n} e^{2j-1} \Big) \\ &= \sum_{k=1}^{n} \delta_{j,k} d_k \Big(1 + \frac{1}{n^2} \Big) = d_j \Big(1 + \frac{1}{n^2} \Big). \end{aligned} \tag{5.19}$$

The right-hand side of (5.18) becomes

$$a_2\Big(e^i, E^n \Big(e^{2j} + \frac{1}{n} e^{2j-1} \Big) \Big) = a_2\Big(e^i, \Big(1 + \frac{1}{n} \Big) e^{2j} \Big) = \frac{1+n}{n} \delta_{i,2j}. \tag{5.20}$$

Consequently, the combination of (5.19), (5.20) yields together with (5.18) for all $j \in \mathbb{N}$, $1 \leqslant j \leqslant n$:

$$d_j = \frac{(n+1)n}{n^2+1} \delta_{i,2j}.$$

Concluding, we have $P(e^i) = \frac{(n+1)n}{n^2+1}(e^i + \frac{1}{n} e^{i-1})$ if $i \leqslant 2n$ and $P(e^i) = 0$ if $i > 2n$.

Now, we compute the norms in (5.14). For the functions $e^i \in V$, $1 \leqslant i \leqslant 2n$, i even, we have

$$P(e^i) - \Pi_{V^n}(e^i) = \Big(\frac{n^2+n}{n^2+1} - \frac{n^2}{n^2+1} \Big) \Big(e^i + \frac{1}{n} e^{i-1} \Big) = \frac{n}{n^2+1} \Big(e^i + \frac{1}{n} e^{i-1} \Big)$$

For a function e^i with $i > 2n$ we have $P(e^i) - \Pi_{V^n}(e^i) = 0$. We select an arbitrary function $v \in V$ which is expressed in terms of the e^i as

$$v = \sum_{\substack{i=1 \\ i \text{ even}}}^{\infty} v_i e^i, \tag{5.21}$$

for appropriate $v_i \in \mathbb{R}$, $i \in \mathbb{N}$ and i even. With this, the left-hand side of (5.14) becomes

$$\begin{aligned} \|P(v) - \Pi_{V^n}(v)\|^2 &= \Big\| \sum_{\substack{i=1 \\ i \text{ even}}}^{2n} v_i (P(e^i) - \Pi_{V^n}(e^i)) \Big\|^2 = \\ &\Big\| \sum_{\substack{i=1 \\ i \text{ even}}}^{2n} v_i \Big(\frac{n}{n^2+1} (e^i + \frac{1}{n} e^{i-1}) \Big) \Big\|^2 = \sum_{\substack{i=1 \\ i \text{ even}}}^{2n} v_i^2 \Big\| \Big(\frac{n}{n^2+1} (e^i + \frac{1}{n} e^{i-1}) \Big) \Big\|^2 = \\ &\sum_{\substack{i=1 \\ i \text{ even}}}^{2n} v_i^2 \Big(\frac{n^2}{(n^2+1)^2} + \frac{1}{(n^2+1)^2} \Big). \end{aligned}$$

To evaluate the right-hand side of (5.14), we notice with (5.17) for all $i \leqslant 2n$:

$$e^i - \Pi_{V^n}(e^i) = e^i - \frac{n^2}{n^2+1} \big(e^i + \frac{1}{n} e^{i-1} \big) = \frac{1}{n^2+1} e^i - \frac{n}{n^2+1} e^{i-1}.$$

For the $v \in V$ given in (5.21), we have

$$\begin{aligned} \|v - \Pi_{V^n}(v)\|^2 &= \Big\| \sum_{\substack{i>2n \\ i \text{ even}}} v_i e^i \Big\|^2 + \Big\| \sum_{\substack{i \leqslant 2n \\ i \text{ even}}} v_i (e^i - \Pi_{V^n}(e^i)) \Big\|^2 \\ &= \sum_{\substack{i>2n \\ i \text{ even}}} v_i^2 + \Big\| \sum_{\substack{i \leqslant 2n \\ i \text{ even}}} v_i \Big(\frac{1}{n^2+1} e^i - \frac{n}{n^2+1} e^{i-1} \Big) \Big\|^2 \\ &= \sum_{\substack{i>2n \\ i \text{ even}}} v_i^2 + \sum_{\substack{i \leqslant 2n \\ i \text{ even}}} v_i^2 \Big(\frac{1}{(n^2+1)^2} + \frac{n^2}{(n^2+1)^2} \Big). \end{aligned}$$

This shows that $\|P(v) - \Pi_{V^n}(v)\| \leqslant \|v - \Pi_{V^n}(v)\|$. Hence, we have $\delta_V = \delta_W = 1$, where δ_W is the corresponding constant for the dual discretization. Therefore, Proposition 3.6 in [VZ18a] tells us that $C_{\mathsf{qopt}} = \sqrt{2}$. Moreover, Proposition 3.9 in the same paper shows that $\delta_{V_h} = \delta_{W_h} \leqslant \sqrt{2}$, where δ_{W_h} is the corresponding constant for the dual discretization. We see that the constants do not go to infinity, so the uniform boundedness of $\delta_V = \delta_W$ and $\delta_{V_h} = \delta_{W_h}$ is not a sufficient condition for the double winged inequality with a uniformly bounded constant.

6 Estimating the local error-difference between two methods

The new quasi-optimal methods of [VZ18a, VZ19a, VZ18b] raise the question: How much better is the performance of the quasi-optimal nonconforming methods compared to the classical ones? This question was already investigated in [Zan17, chapter 5] through numerical experiments.

In this section, we choose a more theoretical approach. We derive an indicator that estimates *locally* how much the classical solution deviates from the quasi-optimal one. This extends and refines ideas of [Zan17, Lemma 5.3.1]. To compute our indicator, we only need the right-hand side f of the problem (1.3) and the smoother(s). In particular, the discrete solutions are not needed.

We point out possible (future) applications of the indicators.

- The indicators are one way to apply the estimators of chapter 2 to the classical nonconforming methods. We derive this in Theorem 6.8: We obtain an additional term, which corresponds to the difference between the quasi-optimal method and the classical one. This term can be bounded again by the error. Thus, instabilities, as e.g. created by the classical oscillation, c.f. section 2.1.3, are avoided.

- The indicators allow us to determine regions in the domain where the classical solution and quasi-optimal solution deviate only by a small amount and other regions where a quasi-optimal method might provide a better approximation. These regions can be determined a-priori without knowledge of the discrete solutions. In regions, where the difference of the two solutions is negligible, we might switch off the quasi-optimality. Solutions obtained in such a way, in combination with the indicators, may provide further understanding of quasi-optimal methods and general insights into how finite element methods work.

We construct a framework for these indicators at first and then apply the theory to the nonconforming examples for the Poisson problem.

Remark 6.1. There are already some results in the literature that compare different methods, c.f. [CPS12, CKPS15] and also [Bra09]. Those papers, however, compare methods using different elements, whereas we compare classical and quasi-optimal methods using the same elements.

Further developments might derive similar indicators for the quasi-optimal and pressure-robust Stokes methods of chapter 3. On the one hand, we could possibly apply the pressure-robust estimators to other, non quasi-optimal and non pressure-robust methods. On the other hand, also this indicator could provide insights into the differences between quasi-optimal, pressure-robust methods and classical approaches. Especially, new insights into of pressure-robust methods could be obtained. However, the necessary divergence correction hinders us to apply the framework of the elliptic setting to the Stokes equation.

This chapter consists of two parts. First we build a framework upon the setting in chapter 1 and then present applications to the Poisson discretization we have discussed.

6.1 Abstract setting

In this section, we show how one can estimate the difference between the discrete solutions of two different finite element methods M_1 and M_2. We assume that the analytic setting is the same for both methods. Hence, we have a fixed Hilbert space V with norm $\|\cdot\|_V$, as in chapter 1.

For the discretization, we have a finite-dimensional space V_h and a discrete bilinear form a_h. The extended bilinear form $\tilde{a} : \tilde{V} \times \tilde{V} \to \mathbb{R}$ is defined on $\tilde{V} := V + V_h$. We assume that there are two smoothers $E_1 : V_h \to V$ and $E_2 : V_h \to \hat{V}$, $\hat{V}$ a Hilbert space with $V \subset \hat{V}$. These two smoothers induce two methods M_1 resp. M_2. The discrete equations for $u_{h,1} \in V_h$ and $u_{h,2} \in V_h$ shall be well-posed and read as

$$a_h(u_{h,1}, w_{h,1}) = \langle f, E_1 w_{h,1} \rangle \quad \forall w_{h,1} \in V_h, \tag{6.1a}$$
$$a_h(u_{h,2}, w_{h,2}) = \langle f, E_2 w_{h,2} \rangle \quad \forall w_{h,2} \in V_h, \tag{6.1b}$$

for a given $f \in \hat{V}'$. Finally, we assume that the method M_1 is quasi-optimal, hence the quasi-Galerkin orthogonality result (Theorem 2.5) is applicable with a constant $\delta_{V_h,1}$. We do not assume that the method M_2 is quasi-optimal. Therefore, after splitting the error $\|u - u_{h,2}\|_{\tilde{V}}^2$ as in Lemma 2.1

$$\|u - u_{h,2}\|_{\tilde{V}}^2 \eqsim \|\operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f, u_{h,2})\|_{V'}^2 + \|u_{h,2} - \Pi_V(u_{h,2})\|_{\tilde{V}}^2,$$

we cannot continue with the localization as in section 2.1.2 because the quasi-Galerkin orthogonality Theorem 2.5 is not applicable. The following Lemma provides a compensation.

Lemma 6.2. *Within the setting of chapter 1, suppose that two methods M_1 and M_2 are given via (6.1), where M_1 is quasi-optimal. Furthermore, $u_{h,1}, u_{h,2} \in V_h$ shall be the respective discrete solutions. If the method M_1 fulfills Assumption 2.6, then the error of method M_2 can be equivalently expressed as*

$$\|u - u_{h,2}\|_{\tilde{V}}^2 \eqsim \sum_{z \in \mathcal{I}} \|\operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f, u_{h,2})\|_{V_z'}^2 + \|(E_1' - E_2')f\|_{V_h'}^2 + \|u_{h,2} - \Pi_V(u_{h,2})\|_{\tilde{V}}^2, \tag{6.2}$$

where the residual $\operatorname{res}_{\mathrm{h}}^{\mathsf{c}}$ is defined in (2.2a). The operators E_1' and E_2' are the adjoint operators (see Definition 1.4) of E_1 resp. E_2. The map $\Pi_V : \tilde{V} \to V$ is the $\tilde{a}(\cdot,\cdot)$-orthogonal projection onto V and $\|\cdot\|_{\tilde{V}}^2 := \tilde{a}(\cdot,\cdot)$.

Proof. Similar to the a-posteriori analysis in chapter 2, we split the error into the smoothed error and conformity error:

$$\|u - u_{h,2}\|_{\tilde{V}}^2 = \|u - \Pi_V(u_{h,2})\|_V^2 + \|u_{h,2} - \Pi_V(u_{h,2})\|_{\tilde{V}}^2 = \|\operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f, u_{h,2})\|_{V'}^2 + \|u_{h,2} - \Pi_V(u_{h,2})\|_{\tilde{V}}^2. \tag{6.3}$$

At this point, we would like to apply Theorem 2.11, but this is not possible due to the lacking quasi-Galerkin orthogonality of M_2. To circumvent this problem, we modify (2.12). We insert $E_1 w_h$, $w_h \in V_h$, with $\|w_h\|_{V_h} \leqslant C_{\mathsf{I}}$ (C_{I} as Assumption 2.6) into the conforming residual and estimate the additional term separately: Let $w \in V$ be arbitrary with $\|w\|_V = 1$ then

$$\langle \operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f, u_{h,2}), w \rangle = \langle \operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f, u_{h,2}), w - E_1 w_h \rangle + \langle \operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f, u_{h,2}), E_1 w_h \rangle. \tag{6.4}$$

The rest of the proof of Theorem 2.11 is also applicable to the method M_2, and we obtain

$$\|u - u_{h,2}\|_{\tilde{V}}^2 \lesssim \sum_{z \in \mathcal{I}} \|\operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f, u_{h,2})\|_{V_z'}^2 + \sup_{\substack{w_h \in V_h \\ \|w_h\|_{V_h} \leqslant C_{\mathsf{I}}}} \langle \operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f, u_{h,2}), E_1 w_h \rangle + \|u_{h,2} - \Pi_V(u_{h,2})\|_{\tilde{V}}^2. \tag{6.5}$$

The additional term can be expressed in terms of f and the smoothers. Writing $P_1 : V \to V_h$ for the projection operator of the quasi-optimal method M_1 (see (1.6)), we get for $w_h \in V_h$:

$$\begin{aligned} \langle \operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f, u_{h,2}), E_1 w_h \rangle &= \langle f, E_1 w_h \rangle - a(\Pi_V u_{h,2}, E_1 w_h) = a_h(u_{h,1}, w_h) - a(\Pi_V u_{h,2}, E_1 w_h) \\ &= a_h(u_{h,1}, w_h) - a_h(P_1(\Pi_V u_{h,2}), w_h) \\ &= a_h(u_{h,1} - u_{h,2}, w_h) - a_h(P_1(\Pi_V u_{h,2}) - u_{h,2}, w_h). \end{aligned} \tag{6.6}$$

The map P_1 can be extended to the bounded operator $\tilde{P}_1 : \tilde{V} \to V_h$, owing to the quasi-optimality of M_1, see Lemma 1.13. Therefore, the last term in (6.6) is bounded by the conformity error (similar to the proof of [VZ18a, Theorem 4.19]):

$$\begin{aligned} a_h(P_1(\Pi_V u_{h,2}) - u_{h,2}, w_h) = a_h(\tilde{P}_1(\Pi_V u_{h,2} - u_{h,2}), w_h) &\leqslant \|\tilde{P}_1(\Pi_V u_{h,2} - u_{h,2})\|_{\tilde{V}} \|w_h\|_{V_h} \\ &\lesssim \|\Pi_V u_{h,2} - u_{h,2}\|_{\tilde{V}} \|w_h\|_{V_h}. \end{aligned} \tag{6.7}$$

Inserting (6.7) into (6.6) yields

$$|\langle \operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f, u_{h,2}), E_1 w_h \rangle| \lesssim a_h(u_{h,1} - u_{h,2}, w_h) + \|\Pi_V u_{h,2} - u_{h,2}\|_{\tilde{V}} \|w_h\|_{V_h}. \tag{6.8}$$

With the equations of the methods (6.1), we have

$$\sup_{w_h \in V_h} \frac{a_h(u_{h,1} - u_{h,2}, w_h)}{\|w_h\|_{V_h}} = \sup_{w_h \in V_h} \frac{\langle f, E_1 w_h - E_2 w_h \rangle}{\|w_h\|_{V_h}} = \|(E_1' - E_2')f\|_{V_h'}. \tag{6.9}$$

This shows one direction of the equivalence (6.2).

To show the other direction, observe that

$$\sum_{z \in \mathcal{I}} \|\operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f, u_{h,2})\|_{V_z'}^2 \lesssim \|\operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f, u_{h,2})\|_{V'}^2$$

follows from Theorem 2.4 because Assumption 2.2 is true as it is required by Assumption 2.6.

By rearranging the terms in (6.6), we get

$$a_h(u_{h,1} - u_{h,2}, w_h) = a_h(P_1(\Pi_V u_{h,2}) - u_{h,2}, w_h) + \langle \operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f, u_{h,2}), E_1 w_h \rangle. \tag{6.10}$$

Applying (6.7) to (6.10), we obtain the bound

$$\begin{aligned} a_h(u_{h,1} - u_{h,2}, w_h) &\lesssim \langle \operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f, u_{h,2}), E_1 w_h \rangle + \|\Pi_V u_{h,2} - u_{h,2}\|_{\tilde{V}} \|w_h\|_{V_h} \\ &\lesssim \Big(\|\operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f, u_{h,2})\|_{V'} + \|\Pi_V u_{h,2} - u_{h,2}\|_{\tilde{V}} \Big) \|w_h\|_{V_h}. \end{aligned}$$

Another application of (6.3) completes the proof. □

Corollary 6.3. *Let $u_{h,1}, u_{h,2} \in V_h$ be the solutions of (6.1) and E_1 and E_2 the corresponding smoothers. Using (1.8), we can express (6.9) also as*

$$\|u_{h,1} - u_{h,2}\|_{\tilde{V}} = \sup_{w_h \in V_h} \frac{a_h(u_{h,1} - u_{h,2}, w_h)}{\|w_h\|_{V_h}} = \sup_{w_h \in V_h} \frac{\langle f, E_1 w_h - E_2 w_h \rangle}{\|w_h\|_{V_h}} = \|(E_1' - E_2')f\|_{V_h'}.$$

The Corollary shows that the global difference between the two discrete solutions in the $\tilde{V}$-norm can be computed a-priori without actually computing the solutions. Knowing the deviation of the non-quasi-optimal solution from the quasi-optimal, one might gain further understanding of quasi-optimal methods.

We show now, how we can estimate the expression $\|(E_1' - E_2')f\|_{V_h'}$ in terms of local quantities of the data. For this purpose, we define discrete local spaces and a corresponding seminorm. Assume that we have subspaces $V_{h,z}$, $z \in \mathcal{I}$, of V_h for a finite index set $\mathcal{I}$, which satisfy $V_h = \sum_{z \in \mathcal{I}} V_{h,z}$. We investigate the relationship between the following two quantities:

$$\|(E_1' - E_2')f\|_{V_h'} := \sup_{w_h \in V_h} \frac{\langle f, E_1 w_h - E_2 w_h \rangle}{\|w_h\|_{V_h}} \quad \text{and} \quad \|(E_1' - E_2')f\|_{V_{h,z}'} := \sup_{w_h \in V_{h,z}} \frac{\langle f, E_1 w_h - E_2 w_h \rangle}{\|w_h\|_{V_h}}. \tag{6.11}$$

Here, the operators E_1' and E_2' are the adjoint operators of E_1 resp. E_2 as defined in Definition 1.4.

Remark 6.4. The first term in (6.11) has some similarity with the left-hand side of [Gud10, (2.7)]. Nevertheless, that paper uses classical a-posteriori techniques to derive a-priori estimates, whereas we use the techniques of chapter 2 to compare quasi-optimal and classical methods and to apply the new a-posteriori error estimators to non-quasi-optimal methods.

Assumption 6.5. *Assume that there exist vector spaces $\mathbb{P}_z$, $z \in \mathcal{I}$, and linear localization operators $\Theta_z : V_h + \mathbb{P}_z \to V_{h,z}$ such that for all $w_h \in V_h + \mathbb{P}_z$*

$$\sum_{z \in \mathcal{I}} w_h \Theta_z = w_h. \tag{6.12}$$

Furthermore, for each $w_h \in V_h$ and $z \in \mathcal{I}$, there shall be a $p_z \in \mathbb{P}_z$ such that

$$\sum_{z \in \mathcal{I}} p_z \Theta_z \in V_h, \quad E_1\Big(\sum_{z \in \mathcal{I}} p_z \Theta_z \Big) = E_2\Big(\sum_{z \in \mathcal{I}} p_z \Theta_z \Big) \quad \textit{and}$$

$$\sum_{z \in \mathcal{I}} \|(w_h - p_z)\Theta_z\|_{V_h}^2 \leqslant C_{\mathsf{apx}}^2 \|w_h\|_{V_h}^2 \tag{6.13}$$

for a $C_{\mathsf{apx}} > 0$.

With Assumptions 6.5, we obtain the following Theorem.

Theorem 6.6. *Let $u_{h,1} \in V_h$ and $u_{h,2} \in V_h$ be the solutions of (6.1) with $f \in \hat{V}'$. Under the Assumption 6.5, the following estimate holds*

$$\|(E_1' - E_2')f\|_{V_h'}^2 = \sup_{w_h \in V_h} \frac{a_h(u_{h,1} - u_{h,2}, w_h)^2}{\|w_h\|_{V_h}^2} \lesssim \sum_{z \in \mathcal{I}} \|(E_1' - E_2')f\|_{V_{h,z}'}^2. \tag{6.14}$$

Proof. The left equality in (6.14) follows by Corollary 6.3

$$a_h(u_{h,2}-u_{h,1},w_h)=\langle f,E_2w_h-E_1w_h\rangle.$$

Next, we apply the localization operator and insert $E_1(\sum_{z\in\mathcal{I}}p_z\Theta_z)-E_2(\sum_{z\in\mathcal{I}}p_z\Theta_z)=0$:

$$\begin{aligned}\langle f,E_1w_h-E_2w_h\rangle&=\sum_{z\in\mathcal{I}}\langle f,E_1(w_h\Theta_z)-E_2(w_h\Theta_z)\rangle\\&=\sum_{z\in\mathcal{I}}\langle f,E_1(w_h\Theta_z-p_z\Theta_z)-E_2(w_h\Theta_z-p_z\Theta_z)\rangle\\&\lesssim\sum_{z\in\mathcal{I}}\sup_{\tilde w_h\in V_{h,z}}\frac{\langle f,E_1\tilde w_h-E_2\tilde w_h\rangle}{\|\tilde w_h\|_{V_h}}\|(w_h-p_z)\Theta_z\|_{V_h}\\&\lesssim\sum_{z\in\mathcal{I}}\|(E_1'-E_2')f\|_{V_{h,z}'}\|(w_h-p_z)\Theta_z\|_{V_h}\\&\lesssim\Big(\sum_{z\in\mathcal{I}}\|(E_1'-E_2')f\|_{V_{h,z}'}^2\Big)^{\frac12}\Big(\sum_{z\in\mathcal{I}}\|(w_h-p_z)\Theta_z\|_{V_h}^2\Big)^{\frac12}.\end{aligned}\tag{6.15}$$

The last factor is now estimated by the assumption (6.13).

$$\sum_{z\in\mathcal{I}}\|(w_h-p_z)\Theta_z\|_{V_h}^2\lesssim\|w_h\|_{V_h}^2.$$

This proves the Theorem. □

Next, we show that also the local quantities in (6.11) are bounded by the global quantities.

Theorem 6.7. *Assume that $u_{h,1}\in V_h$ and $u_{h,2}\in V_h$ are the solutions of (6.1). Moreover, assume that the norm for V_h fulfills for all choices of $w_{h,z}\in V_{h,z}$, $z\in\mathcal{I}$,*

$$\Big\|\sum_{z\in\mathcal{I}}w_{h,z}\Big\|_{V_h}^2\leqslant C_{\mathsf{loc}}\sum_{z\in\mathcal{I}}\|w_{h,z}\|_{V_h}^2.\tag{6.16}$$

Then we have the globalization result:

$$\sum_{z\in\mathcal{I}}\|(E_1'-E_2')f\|_{V_{h,z}'}^2\lesssim\sup_{w_h\in V_h}\frac{a_h(u_{h,1}-u_{h,2},w_h)^2}{\|w_h\|_{V_h}^2}=\|(E_1'-E_2')f\|_{V_h'}^2.$$

Proof. We argue as in Theorem 2.4 and [KV21, Lemma 4]. We select $w_{h,z}\in V_{h,z}$ with $\|w_{h,z}\|_{V_h}=1$ for all $z\in\mathcal{I}$. With these functions, we define the global functions $w=\sum_{z\in\mathcal{I}}\langle f,E_1w_{h,z}-E_2w_{h,z}\rangle(E_1w_{h,z}-E_2w_{h,z})\in V$ and $w_h=\sum_{z\in\mathcal{I}}\langle f,E_1w_{h,z}-E_2w_{h,z}\rangle w_{h,z}\in V_h$. Testing f with the local functions yields

$$\begin{aligned}\sum_{z\in\mathcal{I}}\langle f,E_1w_{h,z}-E_2w_{h,z}\rangle^2&=\Big\langle f,\sum_{z\in\mathcal{I}}\langle f,E_1w_{h,z}-E_2w_{h,z}\rangle(E_1w_{h,z}-E_2w_{h,z})\Big\rangle\\&=\langle f,w\rangle\leqslant\|(E_1'-E_2')f\|_{V_h'}\|w_h\|_{V_h}.\end{aligned}\tag{6.17}$$

We proceed to estimate $\|w_h\|_{V_h}$. With (6.16), we obtain

$$\|w_h\|_{V_h}^2\lesssim\sum_{z\in\mathcal{I}}\langle f,E_1w_{h,z}-E_2w_{h,z}\rangle^2\lesssim\|(E_1'-E_2')f\|_{V_h'}\|w_h\|_{V_h}\tag{6.18}$$

Inserting (6.18) into (6.17) and taking the supremum over all $w_{h,z}\in V_{h,z}$ with $z\in\mathcal{I}$ and $\|w_{h,z}\|_{V_h}=1$, we obtain

$$\sum_{z\in\mathcal{I}}\|(E_1'-E_2')f\|_{V_{h,z}'}^2\lesssim\|(E_1'-E_2')f\|_{V_h'}^2=\sup_{w_h\in V_h}\frac{\langle f,E_1w_h-E_2w_h\rangle^2}{\|w_h\|_{V_h}^2}.$$ □

The two previous results allow us to reformulate Lemma 6.2.

Theorem 6.8. *In addition to the assumptions of Lemma 6.2, suppose that Assumption 6.5 and condition (6.16) are true. Then, we have the following equivalence:*

$$\|u-u_{h,2}\|_{\tilde V}^2\eqsim\sum_{z\in\mathcal{I}}\big(\|\operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f,u_{h,2})\|_{V_z'}^2+\|(E_1'-E_2')f\|_{V_{h,z}'}^2\big)+\|u_{h,2}-\Pi_V(u_{h,2})\|_{\tilde V}^2.$$

Proof. The result follows by Lemma 6.2 and using the equivalence

$$\|(E_1' - E_2')f\|_{V_h'}^2 \eqsim \sum_{z\in\mathcal{I}} \|(E_1' - E_2')f\|_{V_{h,z}'}^2, \tag{6.19}$$

provided by Theorems 6.6 and 6.7. □

Remark 6.9. Theorem 6.8 rewrites the error $\|u - u_{h,2}\|_{\tilde{V}}$ in local terms depending on the data f, the discrete solution $u_{h,2}$ and the smoothers E_1 and E_2. Therefore, Theorem 6.8 provides an a-posteriori estimator for the error. Nevertheless, the quantity $\|\operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f, u_{h,2})\|_{V_z'}^2$ is not computable in general, c.f. section 2.1.3 and [KV21, Lemma 2]. But we can apply the techniques of section 2.1.3 here again: We employ a local projection operator to split $\|\operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f, u_{h,2})\|_{V_z'}^2$ into an estimator and an oscillation, as shown in (2.28). This splitting works even if $u_{h,2}$ is the solution of a non-quasi-optimal method.

The reverse direction shown in Proposition 2.22 relies on the inclusion (2.29). This condition depends on the spaces V_h, D_z and $\tilde{V}_z$ but not on the solutions or the method itself. In particular, (2.29) does not depend on the quasi-optimality of the considered method. Therefore, we can apply Proposition 2.22 also in this setting.

Consequently, the techniques discussed in section 2.1.3 can be reused here to make $\|\operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f, u_{h,2})\|_{V_z'}^2$ in (6.19) computable. Compare also with [KRVZ25, Remark 4.14].

The next Lemma deals with the evaluation of the expression $\|(E_1' - E_2')f\|_{V_{h,z}'}$. Similar to Lemma 2.26, it expresses the norm as a finite sum.

Lemma 6.10. *Assume the setting laid out at the beginning of the section and $V_{h,z}$ is a finite-dimensional vector space for each $z \in \mathcal{I}$. Then we have for a basis $w_{h,1}, \dots, w_{h,n}$ with $n = \dim(V_{h,z})$ that*

$$\|(E_1' - E_2')f\|_{V_{h,z}'}^2 \eqsim \sum_{i=1}^{n} \frac{\langle f, (E_1 - E_2)w_{h,i}\rangle^2}{\|w_{h,i}\|_{V_h}^2}.$$

Proof. Without loss of generality we assume that $\|w_{h,i}\|_{V_h} = 1$ for all $1 \leqslant i \leqslant n$. Let $w_h \in V_{h,z}$ be arbitrary with $w_h = \sum_{i=1}^n \lambda_i w_{h,i}$, $\lambda_i \in \mathbb{R}$. We compute

$$\begin{aligned}\langle f, (E_1 - E_2)w_h\rangle = \sum_{i=1}^{n} \lambda_i \langle f, (E_1 - E_2)w_{h,i}\rangle &\leqslant \left(\sum_{i=1}^{n} \lambda_i^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^{n} \langle f, (E_1 - E_2)w_{h,i}\rangle^2\right)^{\frac{1}{2}} \\ &\lesssim \|w_h\|_{V_h} \left(\sum_{i=1}^{n} \langle f, (E_1 - E_2)w_{h,i}\rangle^2\right)^{\frac{1}{2}}.\end{aligned}$$

The last bound is due to the equivalence of norms on finite-dimensional vector spaces. This shows one direction. As the other direction follows directly from the definitions, the proof is completed. □

6.2 Applications

We discuss the application to the Poisson problem for Crouzeix-Raviart method P-CR1, P-CRL and the dG-methods P-SIP, P-NSIP, P-RSIP, P-RNSIP. The two cases distinguish themselves by different interpolation operators, but the final result is the same. As all methods are based upon a mesh, we recall the notation introduced in section 1.4.1.

We start with the interpolation operator for the Crouzeix-Raviart method P-CRL.

Definition 6.11 ([AR08, Lemma 2.4])**.** *For any odd $\ell \geqslant 1$ and $d = 2$ if $\ell > 1$, we define the CR_ℓ-projection $\Pi_{\mathsf{CR}} : H^1(\mathcal{T}) \to \mathrm{CR}_\ell$ via the relations*

$$\begin{aligned}\int_T (v - \Pi_{\mathsf{CR}} v)q &= 0, \quad \forall q \in \mathbb{P}_{\ell-3}(T), \quad \forall T \in \mathcal{T}, \\ \int_F (v - \Pi_{\mathsf{CR}} v)q &= 0, \quad \forall q \in \mathbb{P}_{\ell-1}(F), \quad \forall F \in \mathcal{F}^i \quad \textit{and} \\ \int_F \Pi_{\mathsf{CR}} v q &= 0, \quad \forall q \in \mathbb{P}_{\ell-1}(F), \quad \forall F \in \mathcal{F}\backslash\mathcal{F}^i,\end{aligned}$$

for all $v \in H^1(\mathcal{T})$.

The next Lemma is a slight modification of [AR08, Lemma 2.4].

Lemma 6.12 ([AR08, Lemma 2.4]). *The projection* Π_{CR} *fulfills*

$$\|\Pi_{\mathsf{CR}} v\|_{L^2(T)} \lesssim \|v\|_{L^2(T)} + \sum_{F\in\mathcal{F}_T\cap\mathcal{F}^i} h_T^{\frac{1}{2}}\|v\|_{L^2(F)}$$

for all $v \in H^1(\mathcal{T})$ *and* $T \in \mathcal{T}$.

To discuss the discontinuous Galerkin methods, we need other approximation projection operators For its Definition, recall the notations in (1.23) and (1.25).

Definition 6.13. *For any* $\ell \in \mathbb{N}$ *with* $\ell \geqslant 1$, *we define the arbitrary order* DG*-projection as*

$$\Pi_{\mathsf{DG}} : C^0(\mathcal{T}) \to S^0_\ell(\mathcal{T}), \quad \Pi_{\mathsf{DG}}(v) := \sum_{T\in\mathcal{T}} \sum_{z\in\mathcal{L}_\ell(T)} v_{|T}(z)\Psi^\ell_z.$$

Here $C^0(\mathcal{T})$ *are the piecewise continuous functions.*

This operator satisfies the following stability estimate.

Lemma 6.14 ([BS08, Lemma 4.4.1]). *The operator* $\Pi_{\mathsf{DG}} : C^0(\mathcal{T}) \to S^0_\ell(\mathcal{T})$ *is bounded with*

$$h_T^{-\frac{d}{2}}\|\Pi_{\mathsf{DG}} v\|_{L^2(T)} + h_T^{1-\frac{d}{2}}\|\nabla\Pi_{\mathsf{DG}} v\|_{L^2(T)} \lesssim \|v\|_{C^0(\overline{T})}, \tag{6.20}$$

for all $v \in C^0(\mathcal{T})$ *and* $T \in \mathcal{T}$ *where* $\|v\|_{C^0(\overline{T})} := \sup_{x\in\overline{T}} |v(x)|$.

With these operators, we establish Assumption 6.5 for several nonconforming discretizations of the Poisson problem.

Theorem 6.15. *The methods for the Poisson problem P-CR1, P-CRL if* ℓ *is odd,* $\ell \in \mathbb{N}$, *(*$d = 2$ *if* $\ell > 1$*) and P-SIP, P-NSIP, P-RSIP and P-RNSIP for arbitrary* $\ell \in \mathbb{N}$, $\ell \geqslant 1$, *fulfill Assumption 6.5. More precisely, we set* $\mathcal{I} := \mathcal{V}$ *and define the discrete, local spaces as*

$$V_{h,z} := \{w_h \in V_h \mid \operatorname{supp}(w_h) \subset \omega_z\}, \quad z \in \mathcal{V}.$$

We set $\mathbb{P}_z := \mathbb{P}_0(\Omega)$ *for all* $z \in \mathcal{V}$. *The localization operators* Θ_z, $z \in \mathcal{V}$, *are induced by the Lagrange hat functions* Φ^1_z*:*

$$\Theta_z : V_h + \mathbb{P}_0(\Omega) \to V_{h,z}, \quad w_h\Theta_z := \Pi(w_h\Phi^1_z) \in V_{h,z}, \quad \text{for all } z \in \mathcal{V} \text{ and } w_h \in V_h + \mathbb{P}_0(\Omega), \tag{6.21}$$

where Π *is* Π_{CR} *for the Crouzeix-Raviart methods and* Π_{DG} *for the dG-methods.*

The operator $E_1 : V_h \to V$ *shall be the smoother of the quasi-optimal method and* $E_2 : V_h \to \hat{V} = L^2(\Omega)$ *shall be the identity. Then, Theorem 6.6 and also the globalization result in Theorem 6.7 are true.*

Remark 6.16. The technique to apply a partition of unity to localize a discrete function and then map it back into the discrete space is common in the literature about multilevel methods, see e.g. [Xu97].

Proof. To prove Theorem 6.15, we show that Assumption 6.5 is valid for the methods. The localization operator to these spaces is built with the Lagrange hat functions. In addition, we need the projection operators just defined.

We first prove the result for a method using Crouzeix-Raviart elements of order $\ell \in \mathbb{N}$, ℓ odd ($d = 2$ if $\ell > 1$), which are P-CR1 or P-CRL. Therefore, we choose the operator $\Pi_{\mathsf{CR}} : H^1(\mathcal{T}) \to \mathrm{CR}_\ell$ and write $\mathrm{CR}_\ell(\omega_z) := \{w_h \in \mathrm{CR}_\ell \mid \operatorname{supp}(w_h) \subset \omega_z\}$. For each $z \in \mathcal{V}$, the localization operator is defined as

$$\Theta_z : \mathrm{CR}_\ell + \mathbb{P}_0(\Omega) \to \mathrm{CR}_\ell(\omega_z), \qquad w_h\Theta_z \mapsto \Pi_{\mathsf{CR}}(w_h\Phi^1_z) \in V_{h,z} = \mathrm{CR}_\ell(\omega_z) \quad \forall w_h \in \mathrm{CR}_\ell + \mathbb{P}_0(\Omega).$$

Note that the operator Π_{CR} does not enlarge the support of its argument.

Fix $w_h \in \mathrm{CR}_\ell$. For each $z \in \mathcal{V} \cap \Omega$, we select $p_z := \frac{1}{|\omega_z|}\int_{\omega_z} w_h \in \mathbb{P}_0(\Omega)$ and $p_z := 0$ if $z \in \mathcal{V} \cap \partial\Omega$. An application of the localization operator Θ_z provides a discrete function $p_z\Theta_z \in \mathring{S}^1_1(\mathcal{T}_z) \subset \mathrm{CR}_\ell(\omega_z)$, which is also invariant under the smoother $\mathcal{E}_\ell$.

Therefore, only (6.13) of Assumption 6.5 needs to be verified. The stability of $\mathcal{E}_\ell$, which follows from [VZ19a, Proposition 3.3] or Corollary 1.41, yields

$$\sum_{z\in\mathcal{I}} \|\mathcal{E}_\ell\big((w_h - p_z)\Theta_z\big)\|^2_V \lesssim \sum_{z\in\mathcal{I}} \|(w_h - p_z)\Theta_z)\|^2_{\tilde{V}}. \tag{6.22}$$

By the stability of Π_{CR}, shown in Lemma 6.12 and an inverse estimate, we obtain

$$\begin{aligned}\big\|\big((w_h-p_z)\Theta_z\big)\big\|_{\hat V}^2 &\lesssim \Big\|\nabla_h\Big(\Pi_{\mathsf{CR}}\big((w_h-p_z)\Phi_z^1\big)\Big)\Big\|_{L^2(\omega_z)}^2 \lesssim h_z^{-2}\big\|\Pi_{\mathsf{CR}}\big((w_h-p_z)\Phi_z^1\big)\big\|_{L^2(\omega_z)}^2\\ &\lesssim h_z^{-2}\sum_{T\in\mathcal{T}_z}\|w_h-p_z\|_{L^2(T)}^2+\sum_{F\in\mathcal{F}_T\cap\mathcal{F}_z^i}h_z^{-1}\|w_h-p_z\|_{L^2(F)}^2\\ &\lesssim \sum_{T\in\mathcal{T}_z}h_z^{-2}\|w_h-p_z\|_{L^2(T)}^2+\|\nabla w_h\|_{L^2(T)}^2\lesssim\|\nabla_h w_h\|_{L^2(\omega_z)}^2.\end{aligned}\tag{6.23}$$

In the last step, we used Poincaré's inequality for piecewise H^1-functions, provided in [Bre03, Theorem 5.1], and that $\int_F[\![w_h]\!]=0$, $F\in\mathcal{F}_z$, for $w_h\in\mathrm{CR}_\ell$. Now, we have verified Assumption 6.5, thus the localization of Theorem 6.6 is true for P-CR1 and P-CRL.

The inverse bound, the globalization in Theorem 6.7 is true because (6.16) can be shown by a similar argument as in Proposition 2.30. To be more precise, we have by Jensen's inequality and the finite overlap property (1.22) for each choice of $w_z\in V_{h,z}=\mathrm{CR}_\ell(\omega_z)$, $z\in\mathcal{V}$:

$$\begin{aligned}\Big\|\sum_{z\in\mathcal{V}}w_z\Big\|_{V_h}^2 &= \sup_{\substack{v_h\in V_h\\ \|v_h\|_{\hat V}=1}}a_h\big(v_h,\sum_{z\in\mathcal{V}}w_z\big)^2\lesssim\sup_{\substack{v_h\in V_h\\ \|v_h\|_{\hat V}=1}}\sum_{z\in\mathcal{V}}\big(a_h(v_h,w_z)\big)^2\\ &\leqslant\sum_{z\in\mathcal{V}}\sup_{\substack{v_h\in V_h\\ \|v_h\|_{\hat V}=1}}\big(a_h(v_h,w_z)\big)^2=\sum_{z\in\mathcal{V}}\|w_z\|_{V_h}^2.\end{aligned}\tag{6.24}$$

We turn to the discontinuous Galerkin methods. If we consider one of the methods P-SIP, P-NSIP, P-RSIP and P-RNSIP with $\ell\geqslant 1$, we employ the operator $\Pi_{\mathsf{DG}}:C_0(\mathcal{T})\to S_\ell^0(\mathcal{T})=V_h$ and define the localization operator as

$$\Theta_z:V_h+\mathbb{P}_0(\Omega)\to V_{h,z},\quad z\in\mathcal{V},\qquad w_h\Theta_z\mapsto\Pi_{\mathsf{DG}}(w_h\Phi_z^1)\in V_{h,z},\quad\forall w_h\in V_h+\mathbb{P}_0(\Omega).$$

Again, we select for each $z\in\mathcal{V}\cap\Omega$ the constant polynomial $p_z:=\frac{1}{|\omega_z|}\int_{\omega_z}w_h\in\mathbb{P}_0(\Omega)$ and $p_z:=0$ if $z\in\mathcal{V}\cap\partial\Omega$ and we have $p_z\Theta_z\in S_\ell^0(\mathcal{T}_z)$. This function is invariant under $\mathcal{E}_\ell$ as well as $\mathcal{E}_{\mathsf{red},\ell}$.

Due to Corollary 1.41, we have a similar bound as in (6.22)

$$\sum_{z\in\mathcal{I}}\|\mathcal{E}_\ell\big((w_h-p_z)\Theta_z\big)\|_V^2\lesssim\sum_{z\in\mathcal{I}}\gamma^{-1}\|(w_h-p_z)\Theta_z)\|_{\hat V}^2.\tag{6.25}$$

Then a similar reasoning as in (6.23) yields with Lemma 6.14

$$\begin{aligned}\|\big((w_h-p_z)\Theta_z\big)\|_{\hat V}^2 &\lesssim\Big\|\nabla_h\Big(\Pi_{\mathsf{DG}}\big((w_h-p_z)\Phi_z^1\big)\Big)\Big\|_{L^2(\omega_z)}^2+\sum_{F\in\mathcal{F}_z^i}\frac{\gamma}{h_F}\big\|\,[\![\Pi_{\mathsf{DG}}\big((w_h-p_z)\Phi_z^1\big)]\!]\,\big\|_{L^2(F)}^2\\ &\lesssim\sum_{T\in\mathcal{T}_z}\frac{\gamma}{h_T^2}\Big\|\Pi_{\mathsf{DG}}\big((w_h-p_z)\Phi_z^1\big)\Big\|_{L^2(T)}^2+(1+\gamma)\Big\|\nabla_h\Big(\Pi_{\mathsf{DG}}\big((w_h-p_z)\Phi_z^1\big)\Big)\Big\|_{L^2(T)}^2\\ &\lesssim(1+2\gamma)h_z^{d-2}\|w_h-p_z\|_{C^0(\mathcal{T}_z)}^2\lesssim(1+2\gamma)h_z^{-2}\|w_h-p_z\|_{L^2(\omega_z)}^2\\ &\lesssim(1+2\gamma)\Big(\|\nabla_h w_h\|_{L^2(\omega_z)}^2+\sum_{F\in\mathcal{F}_z}\int_F\frac{1}{h_F}[\![w]\!]^2\Big)\end{aligned}\tag{6.26}$$

In the last step we exploited the piecewise H^1-Poincaré inequality from [Bre03, Theorem 5.1].

The condition (6.16) follows as in (6.24), thus Theorem 6.7 is also applicable to the dG-setting. □

The next Theorem is an application of Theorem 6.8 and Lemma 6.10.

Theorem 6.17. *Assume the setting of P-CR1, P-CRL, P-SIP or P-NSIP for a polynomial degree $\ell\in\mathbb{N}$. In the case of P-CRL, ℓ shall be odd and $d=2$. Fix $f\in L^2(\Omega)$ and consider the equations (6.1) with $\hat V=L^2(\Omega)$. Let $u_{h,1}\in V_h$ be the solution obtained by the quasi-optimal method (6.1a) with smoother $E_1:=\mathcal{E}_\ell$, whereas $u_{h,2}\in V_h$ shall be the solution obtained by the non-quasi-optimal method (6.1b) that uses $E_2:=\mathrm{id}_{V_h}$. Then, the error of the solution $u_{h,2}$ verifies the equivalence:*

$$\|u-u_{h,2}\|_{\hat V}^2\ \eqsim\ \sum_{z\in\mathcal{I}}\big(\|\operatorname{res}_{\mathrm{h}}^{\mathsf{c}}(f,u_{h,2})\|_{H^{-1}(\omega_z)}^2+\|(\mathcal{E}_\ell'-\mathrm{id}_{V_h})f\|_{V_{h,z}'}^2\big)+\|u_{h,2}-\Pi_V(u_{h,2})\|_{\hat V}^2.\tag{6.27}$$

In the case of the Crouzeix-Raviart methods P-CR1, P-CRL, the indicator $\|(\mathcal{E}_\ell' - \mathrm{id}_{V_h})f\|^2_{\mathrm{CR}_\ell(\omega_z)'}$ *fulfills for each* $z \in \mathcal{V}$ *the relations*

$$\sum_{i=1}^{n} \frac{\langle f, (\mathcal{E}_\ell - \mathrm{id}_{V_h})w_{h,i}\rangle^2}{\|w_{h,i}\|^2_{V_h}} \eqsim \|(\mathcal{E}_\ell' - \mathrm{id}_{V_h})f\|^2_{\mathrm{CR}_\ell(\omega_z)'} \lesssim \|\nabla_h(u_{h,1} - u_{h,2})\|^2_{L^2(\omega_z)} \tag{6.28}$$

after choosing a basis $w_{h,i} \in \mathrm{CR}_\ell(\omega_z)$, $1 \leqslant i \leqslant \dim\big(\mathrm{CR}_\ell(\omega_z)\big) = n$.

Correspondingly, in the case of the discontinuous Galerkin methods P-SIP or P-NSIP, the indicator $\|(\mathcal{E}_\ell' - \mathrm{id}_{V_h})f\|^2_{S^0_\ell(\mathcal{T}_z)'}$ *fulfills for each* $z \in \mathcal{V}$ *the relations*

$$\sum_{i=1}^{n} \frac{\langle f, (\mathcal{E}_\ell - \mathrm{id}_{V_h})w_{h,i}\rangle^2}{\|w_{h,i}\|^2_{V_h}} \eqsim \|(\mathcal{E}_\ell' - \mathrm{id}_{V_h})f\|^2_{S^0_\ell(\mathcal{T}_z)'} \lesssim \|u_{h,1} - u_{h,2}\|^2_{\tilde{V}} \tag{6.29}$$

where we have chosen a basis $w_{h,i} \in S^0_\ell(\mathcal{T}_z)$, $1 \leqslant i \leqslant \dim\big(S^0_\ell(\mathcal{T}_z)\big) = n$. *The hidden constants in* (6.27) *and* (6.29) *depend on* γ.

Proof. The equivalence (6.27) follows by Theorem 6.8 because Assumption 6.5 and condition (6.16) needed for Theorem 6.6 and 6.7 are shown in Theorem 6.15 for all considered methods.

In (6.28), (6.29), the representations in terms of a basis follow by Lemma 6.10. The lower bounds follow with equation (6.1), the continuity of a_h, and the equivalence of norms on finite-dimensional vector spaces:

$$\begin{aligned}\sup_{w_h \in V_{h,z}} \frac{\langle f, \mathcal{E}_\ell w_h - w_h\rangle^2}{\|w_h\|^2_{V_h}} &= \sup_{w_h \in V_{h,z}} \frac{a_h(u_{h,1} - u_{h,2}, w_h)^2}{\|w_h\|^2_{V_h}} \\ &\leqslant C(d, \ell, \sigma(\mathcal{T}), \gamma)\Big(\|\nabla_h(u_{h,1} - u_{h,2})\|^2_{L^2(\omega_z)} + \sum_{F \in \mathcal{F}_z} \int_F \frac{\gamma}{h_F} [\![u_{h,1} - u_{h,2}]\!]^2\Big).\end{aligned} \tag{6.30}$$

Here, $\gamma > 0$ is either the stability parameter of P-SIP or P-NSIP, or $\gamma = 0$ for the Crouzeix-Raviart methods. Moreover, in the case of P-SIP or P-NSIP, the hidden constant depends on γ because the stability of a_h depends on γ, c.f. [VZ18b, Lemma 3.5 and p. 2884]. □

Remark 6.18. A modified version Theorem 6.17 also applies to the methods P-RSIP and P-RNSIP if we substitute the smoother $\mathcal{E}_\ell$ by $\mathcal{E}_{\mathsf{red},\ell}$.

In view of Remark 6.9, we can apply the projections constructed in section 2.3.1 to split the terms $\|\operatorname{res}^{\mathsf{c}}_{\mathrm{h}}(f, u_{h,2})\|_{H^{-1}(\omega_z)}$ in (6.27) into an estimator and an oscillation term.

In this regard, the projections $\mathcal{P}^{1+,\mathsf{cr}}_z$, $\mathcal{P}^{\ell,\mathsf{cr}}_z$ were of special interest in the setting of the nonconforming, quasi-optimal methods, as they allowed us to replace the face contributions by a local conformity error (see Theorem 2.69 and Proposition 2.83). The local conformity error could then be easily evaluated by means of Lemma 2.68.

However, Lemma 2.68 relies on the structure of the quasi-optimal methods that we considered there. Therefore, the Lemma is not applicable to the current situation, at least not directly. The next Lemma thus serves as a replacement and shows that the local conformity error can also be bounded in the current situation.

Lemma 6.19. *Under the assumptions posed by Theorem 6.17, the local conformity error* $\mathbf{CE}_z(u_{h,2})$ *can be bounded for each* $z \in \mathcal{V}$ *as follows.*

1. *In the cases of the methods P-CR1 (*$\ell = 1$ *and* $d = 2, 3$*) and P-CRL (*$\ell \in \mathbb{N}$*, odd,* $d = 2$*), we have the bound*
$$\mathbf{CE}_z(u_{h,2}) \lesssim \|(\mathcal{E}_\ell' - \mathrm{id}'_{V_h})f\|_{V'_{h,z}}. \tag{6.31}$$
2. *In the cases of the P-SIP, P-NSIP, P-RSIP, P-RNSIP methods (*$\ell \in \mathbb{N}$ *and* $d = 2, 3$*), we have the bound*
$$\mathbf{CE}_z(u_{h,2}) \lesssim \|(\mathcal{E}_\ell' - \mathrm{id}'_{V_h})f\|_{V'_{h,z}} + \left(\sum_{F' \in \mathcal{F}_z} \int_{F'} \frac{1+\gamma}{h_{F'}} |[\![u_h]\!]|^2\right)^{\frac{1}{2}}. \tag{6.32}$$

Proof. We start with Crouzeix-Raviart methods. For a fixed $w_h \in \mathrm{CR}_\ell(\omega_z)$, we have with a straightforward adoption of Lemma 1.46

$$\begin{aligned}\tilde{a}^{\mathsf{CR}}(u_{h,2}, \mathcal{E}_\ell w_h) &= \tilde{a}^{\mathsf{CR}}(u_{h,2}, w_h) + \tilde{a}^{\mathsf{CR}}(u_{h,2}, \mathcal{E}_\ell w_h - w_h) \\ &= \tilde{a}^{\mathsf{CR}}(u_{h,2}, w_h) - \sum_{F' \in \mathcal{F}_z} \int_{F'} \{\{\nabla_h u_{h,2} \cdot \mathsf{n}_{F'}\}\} [\![w_h]\!] = \tilde{a}^{\mathsf{CR}}(u_{h,2}, w_h),\end{aligned}$$

owing to the definition of the Crouzeix-Raviart element, c.f. (1.33), (1.39). By this insight, we have

$$\tilde{a}^{\mathsf{CR}}(u-u_{h,2},\mathcal{E}_\ell w_h)=\langle f,\mathcal{E}_\ell w_h\rangle-\tilde{a}^{\mathsf{CR}}(u_{h,2},\mathcal{E}_\ell w_h)=\langle f,\mathcal{E}_\ell w_h\rangle-\tilde{a}^{\mathsf{CR}}(u_{h,2},w_h)=\langle f,\mathcal{E}_\ell w_h-w_h\rangle.$$

The equivalence of norm on finite-dimensional vector spaces yields (6.31).

Turning to the dG-methods, we again fix $w_h\in S^0_\ell(\mathcal{T}_z)$. Using the computation in (2.97), we have

$$\begin{aligned}\tilde{a}^{\mathsf{dG}}(u-u_{h,2},\mathcal{E}_\ell w_h)&=\langle f,\mathcal{E}_\ell w_h\rangle-\tilde{a}^{\mathsf{dG}}(u_{h,2},\mathcal{E}_\ell w_h)\\&=\langle f,\mathcal{E}_\ell w_h\rangle-a_h^{\mathsf{dG}}(u_{h,2},w_h)+\sum_{F'\in\mathcal{F}_z}\theta\int_{F'}[\![u_h]\!]\,\{\{\nabla_h w_h\cdot\mathsf{n}_{F'}\}\}+\sum_{F'\in\mathcal{F}_z}\int_{F'}\frac{\gamma}{h_{F'}}[\![u_h]\!]\,[\![w_h]\!]\\&\lesssim\langle f,\mathcal{E}_\ell w_h-w_h\rangle+\left(\sum_{F'\in\mathcal{F}_z}\int_{F'}\frac{1+\gamma}{h_{F'}}|[\![u_h]\!]|^2\right)^{\frac12}\|w_h\|_{V_h}.\end{aligned}$$

This shows (6.32) similar to Lemma 2.68. □

Lemma 6.19 has the consequence that, even in the setting of Theorem 6.17, we can bound the face contributions of $\mathcal{P}_z^{1+,\mathsf{cr}}$, $\mathcal{P}_z^{\ell,\mathsf{cr}}$ by quantities which are either easy to evaluate (e.g. the jump contributions in (6.32)) or can be absorbed in other quantities of (6.27) (e.g. the $\|(\mathcal{E}'_\ell-\mathrm{id}_{V_h})f\|_{V'_{h,z}}$ indicators).

Finally, we discuss an alternative bound of the local expressions $\|(\mathcal{E}'_\ell-\mathrm{id}'_{V_h})f\|_{V'_{h,z}}$ which is easier to bound. The idea of the following Corollary is inspired by the analysis in [LMNN18].

Corollary 6.20. *Assume the conditions required by Theorem 6.17 and the setting of one of the methods P-CRL, P-SIP or P-NSIP with $\ell\geqslant 2$ (ℓ odd in the case of P-CRL). Then, we have for each $z\in\mathcal{V}$ the bound*

$$\|(\mathcal{E}'_\ell-\mathrm{id}'_{V_h})f\|_{V'_{h,z}}\lesssim h_z\inf_{\overline{f}\in S^0_{\ell-2}(\mathcal{T}_z^+)}\|f-\overline{f}\|_{L^2(\omega_z^+)}.\tag{6.33}$$

Proof. The considered methods use the smoothing operator $\mathcal{E}_\ell:V_h\to V$ from Definition 1.35. This smoother satisfies the momentum conditions introduced in Definition 1.33. The interior momentum conditions enable for a fixed $\overline{f}\in S^0_{\ell-2}(\mathcal{T}_z^+)$ the estimate

$$\begin{aligned}\|(\mathcal{E}'_\ell-\mathrm{id}'_{V_h})f\|_{V'_{h,z}}&=\sup_{w_h\in V_h}\frac{\langle f,\mathcal{E}_\ell w_h-w_h\rangle}{\|w_h\|_{V_h}}\\&=\sup_{w_h\in V_h}\frac{\langle f-\overline{f},\mathcal{E}_\ell w_h-w_h\rangle}{\|w_h\|_{V_h}}\leqslant\sup_{w_h\in V_h}\frac{\|\mathcal{E}_\ell w_h-w_h\|_{L^2(\omega_z^+)}}{\|w_h\|_{V_h}}\|f-\overline{f}\|_{L^2(\omega_z^+)}.\end{aligned}$$

The last factor is bounded owing to the L^2-stability of $\mathcal{E}_\ell$, derived in Corollary 1.41, and the equivalence $\|w_h\|_{V_h}\eqsim\|w_h\|_{\tilde{V}}$ for all $w_h\in V_{h,z}$. This proves the Corollary. □

Remark 6.21. The applicability of the Corollary is limited. As pointed out in [BCNV24, Remark 4.13], the maximal convergence rate of one of the mentioned methods with polynomial degree ℓ is $O(h^\ell)$. For the Corollary to be useful, the expression $h_z\inf_{\overline{f}\in S^0_{\ell-2}(\mathcal{T}_z^+)}\|f-\overline{f}\|_{L^2(\omega_z^+)}$ should have at least this rate as well. This can be achieved, if f is sufficiently regular.

However, in such a situation, it is known that a quasi-optimal method does not perform much better than a classical one. For example, in [Gud10], Gudi shows for the classical Crouzeix-Raviart and a discontinuous Galerkin method the following a-priori estimate, see [Gud10, (2.9)]:

$$\|u-u_h\|\lesssim\inf_{v_h\in V_h}\|u-v_h\|+\sum_{T\in\mathcal{T}}h_T\inf_{\overline{f}\in\mathbb{P}_{\ell-1}(T)}\|f-\overline{f}\|_{L^2(T)}.\tag{6.34}$$

Here, $u\in H^1_0(\Omega)$ is the exact solution, $u_h\in V_h$ the discrete solution and $\|\cdot\|$ is a suitable norm for the method with discrete space V_h under consideration. Because for each $z\in\mathcal{V}$

$$\sum_{T\in\mathcal{T}_z^+}h_T^2\inf_{\overline{f}\in\mathbb{P}_{\ell-1}(T)}\|f-\overline{f}\|^2_{L^2(T)}\lesssim h_z^2\inf_{\overline{f}\in S^0_{\ell-2}(\mathcal{T}_z^+)}\|f-\overline{f}\|^2_{L^2(\omega_z^+)},\tag{6.35}$$

also (6.34) provides acceptable estimates if Corollary 6.20 provides a good estimate.

As already discussed in section 2.1.3, the oscillation term in (6.34) might be arbitrary larger than the of error of the classical method. In such a case, a quasi-optimal a-priori estimate is more advantageous than (6.34) and a quasi-optimal method is favorable. Due to (6.35), in such a case the upper bound in the Corollary is also arbitrary larger than the error of the classical method, thus diminishing usefulness of the result.

Remark 6.22. We want to discuss an improvement of the Corollary. Usually, one has a classical, non quasi-optimal method M_2 fixed, i.e. in the setting of the Corollary either a Crouzeix-Raviart or discontinuous Galerkin method. Then one chooses a corresponding quasi-optimal method. If one chooses a modified version of P-CRL, P-SIP or P-NSIP, one gets a better estimate in the Corollary.

One can modify the smoothing operator $\mathcal{E}_\ell$ such that it preserves interior moments up to order $\ell - 1$ instead of only up to order $\ell - 2$ (c.f. (1.41)). Because the algebraic consistency of the resulting method is not disturbed by this enhancement, one still has a quasi-optimal method.

With this modified quasi-optimal method, one can improve the estimate (6.33) to

$$\|(\mathcal{E}'_\ell - \mathrm{id}'_{V_h})f\|_{V'_{h,z}} \lesssim h_z \inf_{\overline{f} \in S^0_{\ell-1}(\mathcal{T}_z^+)} \|f - \overline{f}\|_{L^2(\omega_z^+)}.$$

Bibliography

[ABCM02] D. N. Arnold, F. Brezzi, B. Cockburn, and L. D. Marini. Unified analysis of discontinuous Galerkin methods for elliptic problems. *SIAM J. Numer. Anal.*, 39(5):1749–1779, 2001/02.

[ABCM02] D. N. Arnold, F. Brezzi, B. Cockburn, and L. D. Marini. Unified Analysis of Discontinuous Galerkin Methods for Elliptic Problems. *SIAM Journal on Numerical Analysis*, 39(5):1749–1779, January 2002.

[AFS68] J. H. Argyris, I. Fried, and D. W. Scharpf. The tuba family of plate elements for the matrix displacement method. *The Aeronautical Journal*, 72(692):701–709, 1968.

[AO00] M. Ainsworth and J. T. Oden. *A posteriori error estimation in finite element analysis*. Pure and Applied Mathematics (New York). Wiley-Interscience [John Wiley & Sons], New York, 2000.

[AP99] P. Alotto and I. Perugia. Mixed finite element methods and tree-cotree implicit condensation. *Calcolo*, 36(4):233–248, 1999.

[AR08] M. Ainsworth and R. Rankin. Fully computable bounds for the error in nonconforming finite element approximations of arbitrary order on triangular elements. *SIAM J. Numer. Anal.*, 46(6):3207–3232, 2008.

[Arn82] D. N. Arnold. An Interior Penalty Finite Element Method with Discontinuous Elements. *SIAM Journal on Numerical Analysis*, 19(4):742–760, August 1982.

[Arn93] D. N. Arnold. On nonconforming linear-constant elements for some variants of the Stokes equations. *Istit. Lombardo Accad. Sci. Lett. Rend. A*, 127(1):83–93, 1993.

[Aub00] J.-P. Aubin. *Applied functional analysis*. Pure and Applied Mathematics (New York). Wiley-Interscience, New York, second edition, 2000. With exercises by Bernard Cornet and Jean-Michel Lasry, Translated from the French by Carole Labrousse.

[BBF13] D. Boffi, F. Brezzi, and M. Fortin. *Mixed finite element methods and applications.*, volume 44. Berlin: Springer, 2013.

[BCNV24] A. Bonito, C. Canuto, R. H. Nochetto, and A. Veeser. Adaptive finite element methods, 2024.

[BF91] F. Brezzi and M. Fortin. *Mixed and hybrid finite element methods*, volume 15 of *Springer Ser. Comput. Math.* New York etc.: Springer-Verlag, 1991.

[BF13] F. Boyer and P. Fabrie. *Mathematical tools for the study of the incompressible Navier-Stokes equations and related models*, volume 183 of *Applied Mathematical Sciences*. Springer, New York, 2013.

[BLMS15] C. Brennecke, A. Linke, C. Merdon, and J. Schöberl. Optimal and pressure-independent L^2 velocity error estimates for a modified Crouzeix-Raviart Stokes element with BDM reconstructions. *J. Comput. Math.*, 33(2):191–208, 2015.

[Bra07] D. Braess. *Finite elements*. Cambridge University Press, Cambridge, third edition, 2007. Theory, fast solvers, and applications in elasticity theory, Translated from the German by Larry L. Schumaker.

[Bra09] D. Braess. An a posteriori error estimate and a comparison theorem for the nonconforming P_1 element. *Calcolo*, 46(2):149–155, 2009.

[Bre96] S. C. Brenner. Two-level additive Schwarz preconditioners for nonconforming finite element methods. *Math. Comp.*, 65(215):897–921, 1996.

[Bre03] S. C. Brenner. Poincaré-Friedrichs inequalities for piecewise H^1 functions. *SIAM J. Numer. Anal.*, 41(1):306–324, 2003.

[Bre04] S. C. Brenner. Korn's inequalities for piecewise H^1 vector fields. *Math. Comp.*, 73(247):1067–1087, 2004.

[Bre15] S. C. Brenner. Forty years of the Crouzeix-Raviart element. *Numer. Methods Partial Differential Equations*, 31(2):367–396, 2015.

[BS92] S. C. Brenner and L.-Y. Sung. Linear finite element methods for planar linear elasticity. *Math. Comp.*, 59(200):321–338, 1992.

[BS05] S. C. Brenner and L.-Y. Sung. C^0 interior penalty methods for fourth order elliptic boundary value problems on polygonal domains. *J. Sci. Comput.*, 22-23:83–118, 2005.

[BS07] A. Baran and G. Stoyan. Gauss-Legendre elements: a stable, higher order non-conforming finite element family. *Computing*, 79(1):1–21, February 2007.

[BS08] S. C. Brenner and L. R. Scott. *The mathematical theory of finite element methods*, volume 15 of *Texts in Applied Mathematics*. Springer, New York, third edition, 2008.

[Buc99] D. Buckholtz. Hilbert space idempotents and involutions. *Proceedings of the American Mathematical Society*, 128(5):1415–1418, October 1999.

[Car05] C. Carstensen. A unifying theory of a posteriori finite element error control. *Numer. Math.*, 100(4):617–637, 2005.

[CDN12] A. Cohen, R. DeVore, and R. H. Nochetto. Convergence rates of AFEM with H^{-1} data. *Found. Comput. Math.*, 12(5):671–718, 2012.

[CDS18] P. Ciarlet, C. F. Dunkl, and S. A. Sauter. A family of Crouzeix-Raviart finite elements in 3D. *Anal. Appl. (Singap.)*, 16(5):649–691, 2018.

[CF89] M. Crouzeix and R. S. Falk. Nonconforming finite elements for the Stokes problem. *Math. Comp.*, 52(186):437–456, 1989.

[CGH13] C. Carstensen, D. Gallistl, and J. Hu. A posteriori error estimates for nonconforming finite element methods for fourth-order problems on rectangles. *Numer. Math.*, 124(2):309–335, 2013.

[CGJ09] C. Carstensen, T. Gudi, and M. Jensen. A unifying theory of a posteriori error control for discontinuous Galerkin FEM. *Numerische Mathematik*, 112(3):363–379, May 2009.

[CH07] C. Carstensen and J. Hu. A unifying theory of a posteriori error control for non-conforming finite element methods. *Numerische Mathematik*, 107(3):473–502, August 2007.

[CHO07] C. Carstensen, J. Hu, and A. Orlando. Framework for the a posteriori error analysis of nonconforming finite elements. *SIAM J. Numer. Anal.*, 45(1):68–82, 2007.

[Chr08] O. Christensen. *Frames and bases: an introductory course*. Applied and numerical harmonic analysis. Birkhäuser, Boston, 2008.

[Cia78] P. G. Ciarlet. *The finite element method for elliptic problems*, volume Vol. 4 of *Studies in Mathematics and its Applications*. North-Holland Publishing Co., Amsterdam-New York-Oxford, 1978.

[Cia13] P. G. Ciarlet. *Linear and nonlinear functional analysis with applications*. Society for Industrial and Applied Mathematics, Philadelphia, PA, 2013.

[Cia22a] P. G. Ciarlet. *Mathematical elasticity. Volume I. Three-dimensional elasticity*, volume 84 of *Classics in Applied Mathematics*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2022. Reprint of the 1988 edition.

[Cia22b] P. G. Ciarlet. *Mathematical elasticity. Volume II. Theory of plates*, volume 85 of *Classics in Applied Mathematics*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2022. Reprint of the 1997 edition.

[Cia22c] P. G. Ciarlet. *Mathematical elasticity. Volume III. Theory of shells*, volume 86 of *Classics in Applied Mathematics*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2022. Reprint of the 2000 edition.

[CKPS15] C. Carstensen, K. Köhler, D. Peterseim, and M. Schedensack. Comparison results for the Stokes equations. *Appl. Numer. Math.*, 95:118–129, 2015.

[CKS00] B. Cockburn, G. E. Karniadakis, and C.-W. Shu. The development of discontinuous Galerkin methods. In *Discontinuous Galerkin methods (Newport, RI, 1999)*, volume 11 of *Lect. Notes Comput. Sci. Eng.*, pages 3–50. Springer, Berlin, 2000.

[CN21] C. Carstensen and N. Nataraj. A priori and a posteriori error analysis of the Crouzeix-Raviart and Morley FEM with original and modified right-hand sides. *Comput. Methods Appl. Math.*, 21(2):289–315, 2021.

[CN22] C. Carstensen and N. Nataraj. Lowest-order equivalent nonstandard finite element methods for biharmonic plates. *ESAIM Math. Model. Numer. Anal.*, 56(1):41–78, 2022.

[Coh02] G. C. Cohen. *Higher-order numerical methods for transient wave equations*. Sci. Comput. Berlin: Springer, 2002.

[Coo97] R. Cools. Constructing cubature formulae: the science behind the art. In *Acta numerica, 1997*, volume 6 of *Acta Numer.*, pages 1–54. Cambridge Univ. Press, Cambridge, 1997.

[CPR13] C. Carstensen, D. Peterseim, and H. Rabus. Optimal adaptive nonconforming FEM for the Stokes problem. *Numerische Mathematik*, 123(2):291–308, February 2013.

[CPS12] C. Carstensen, D. Peterseim, and M. Schedensack. Comparison Results of Finite Element Methods for the Poisson Model Problem. *SIAM Journal on Numerical Analysis*, 50(6):2803–2823, January 2012.

[CR73] M. Crouzeix and P.-A. Raviart. Conforming and nonconforming finite element methods for solving the stationary Stokes equations I. *Revue française d'automatique informatique recherche opérationnelle. Mathématique*, 7(R3):33–75, 1973.

[CS22a] C. Carstensen and S. Sauter. Critical functions and inf-sup stability of Crouzeix-Raviart elements. *Comput. Math. Appl.*, 108:12–23, 2022.

[CS22b] C. Carstensen and S. A. Sauter. Crouzeix-Raviart triangular elements are inf-sup stable. *Math. Comp.*, 91(337):2041–2057, 2022.

[CT65] R. W. Clough and J. L. Tocher. Finite element stiffness matrices and analysis of plate in bending. In *Proceedings of the Conference on Matrix Methods in Structural Mechanics*, pages 515–545. Wright Patterson AFB, 1965.

[dBD83] C. de Boor and R. DeVore. Approximation by smooth multivariate splines. *Trans. Amer. Math. Soc.*, 276(2):775–788, 1983.

[DDP95] E. Dari, R. Durán, and C. Padra. Error estimators for nonconforming finite element approximations of the Stokes problem. *Math. Comp.*, 64(211):1017–1033, 1995.

[DDPV96] E. Dari, R. Duran, C. Padra, and V. Vampa. A posteriori error estimators for nonconforming finite element methods. *ESAIM: Mathematical Modelling and Numerical Analysis*, 30(4):385–400, 1996.

[Die17] R. Diestel. *Graph theory*, volume 173 of *Graduate Texts in Mathematics*. Springer, Berlin, fifth edition, 2017.

[DPE12] D. A. Di Pietro and A. Ern. *Mathematical aspects of discontinuous Galerkin methods*, volume 69 of *Mathématiques & Applications (Berlin) [Mathematics & Applications]*. Springer, Heidelberg, 2012.

[DS96] P. Destuynder and M. Salaun. *Mathematical analysis of thin plate models*, volume 24 of *Math. Appl. (Berl.)*. Paris: Springer, 1996.

[DX14] C. F. Dunkl and Y. Xu. *Orthogonal polynomials of several variables*, volume 155 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, second edition, 2014.

[EG04] A. Ern and J.-L. Guermond. *Theory and practice of finite elements*, volume 159 of *Applied mathematical sciences ; 159*. Springer, New York, NY, 2004.

[Eva10] L. C. Evans. *Partial differential equations*, volume 19 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, second edition, 2010.

[FGN20] G. Fu, J. Guzmán, and M. Neilan. Exact smooth piecewise polynomial sequences on Alfeld splits. *Math. Comp.*, 89(323):1059–1091, 2020.

[FH88] L. P. Franca and T. J. R. Hughes. Two classes of mixed finite element methods. *Comput. Methods Appl. Mech. Engrg.*, 69(1):89–129, 1988.

[For77] M. Fortin. An analysis of the convergence of mixed finite element methods. *RAIRO Anal. Numér.*, 11(4):341–354, iii, 1977.

[FS83] M. Fortin and M. Soulie. A non-conforming piecewise quadratic finite element on triangles. *International Journal for Numerical Methods in Engineering*, 19(4):505–520, April 1983.

[Gal11] G. P. Galdi. *An introduction to the mathematical theory of the Navier-Stokes equations*. Springer Monographs in Mathematics. Springer, New York, second edition, 2011. Steady-state problems.

[Gal15] D. Gallistl. Morley finite element method for the eigenvalues of the biharmonic operator. *IMA J. Numer. Anal.*, 35(4):1779–1811, 2015.

[GH09] E. H. Georgoulis and P. Houston. Discontinuous Galerkin methods for the biharmonic problem. *IMA J. Numer. Anal.*, 29(3):573–594, 2009.

[Gla22] J. Glatzel. Quasioptimale und druckrobuste Crouzeix-Raviart-Methoden höherer Ordnung für das Stokes-Problem – Theorie und Implementierung. Master's thesis, TU Dortmund, 2022.

[GLBB97] J.-F. Gerbeau, C. Le Bris, and M. Bercovier. Spurious velocities in the steady flow of an incompressible fluid subjected to external forces. *Internat. J. Numer. Methods Fluids*, 25(6):679–695, 1997.

[GN14] J. Guzmán and M. Neilan. Conforming and divergence-free Stokes elements on general triangular meshes. *Math. Comp.*, 83(285):15–36, 2014.

[GN18] J. Guzmán and M. Neilan. inf-sup stable finite elements on barycentric refinements producing divergence-free approximations in arbitrary dimensions. *SIAM J. Numer. Anal.*, 56(5):2826–2844, 2018.

[GR79] V. Girault and P.-A. Raviart. *Finite element approximation of the Navier-Stokes equations*, volume 749 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin-New York, 1979.

[Gri11] P. Grisvard. *Elliptic problems in nonsmooth domains*, volume 69 of *Classics in Applied Mathematics*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2011. Reprint of the 1985 original [MR0775683], With a foreword by Susanne C. Brenner.

[GS19] J. Guzmán and L. R. Scott. The Scott-Vogelius finite elements revisited. *Math. Comp.*, 88(316):515–529, 2019.

[Gud10] T. Gudi. A new error analysis for discontinuous finite element methods for linear elliptic problems. *Math. Comp.*, 79(272):2169–2189, 2010.

[Hec81] F. Hecht. Construction d'une base de fonctions P_1 non conforme á divergence nulle dans $\mathbf{R_3}$. *RAIRO Anal. Numér.*, 15(2):119–150, 1981.

[HL02] P. Hansbo and M. G. Larson. Discontinuous Galerkin methods for incompressible and nearly incompressible elasticity by Nitsche's method. *Comput. Methods Appl. Mech. Engrg.*, 191(17-18):1895–1908, 2002.

[HM25] X. Hu and L. Mu. A pressure-robust a posteriori error estimator of Stokes problems for a low order nonconforming divergence-free method. *J. Sci. Comput.*, 104(3):27, 2025. Id/No 101.

[HS09] J. Hu and Z. Shi. A new a posteriori error estimate for the Morley element. *Numer. Math.*, 112(1):25–40, 2009.

[HSX12] J. Hu, Z. Shi, and J. Xu. Convergence and optimality of the adaptive Morley element method. *Numer. Math.*, 121(4):731–752, 2012.

[JLM+17] V. John, A. Linke, C. Merdon, M. Neilan, and L. G. Rebholz. On the Divergence Constraint in Mixed Finite Element Methods for Incompressible Flows. *SIAM Review*, 59(3):492–544, January 2017.

[Joh16] V. John. *Finite Element Methods for Incompressible Flow Problems*, volume 51 of *Springer Series in Computational Mathematics*. Springer International Publishing, Cham, 2016.

[KRVZ25] C. Kreuzer, M. Rott, A. Veeser, and P. Zanotti. Strictly equivalent a posteriori error estimators for quasi-optimal nonconforming methods, February 2025. arXiv:2502.01799.

[KV21] C. Kreuzer and A. Veeser. Oscillation in a posteriori error estimation. *Numer. Math.*, 148(1):43–78, 2021.

[KVZ] C. Kreuzer, A. Veeser, and P. Zanotti. Accurate error bounds for finite element methods. In preparation.

[KVZ21] C. Kreuzer, R. Verfürth, and P. Zanotti. Quasi-optimal and pressure robust discretizations of the Stokes equations by moment- and divergence-preserving operators. *Comput. Methods Appl. Math.*, 21(2):423–443, 2021.

[KZ20] C. Kreuzer and P. Zanotti. Quasi-optimal and pressure-robust discretizations of the Stokes equations by new augmented Lagrangian formulations. *IMA J. Numer. Anal.*, 40(4):2553–2583, 2020.

[Lax02] P. D. Lax. *Functional analysis*. Pure Appl. Math., Wiley-Intersci. Ser. Texts Monogr. Tracts. Chichester: Wiley, 2002.

[Lin09] A. Linke. Collision in a cross-shaped domain—a steady 2d Navier-Stokes example demonstrating the importance of mass conservation in CFD. *Comput. Methods Appl. Mech. Engrg.*, 198(41-44):3278–3286, 2009.

[Lin12] A. Linke. A divergence-free velocity reconstruction for incompressible flows. *C. R. Math. Acad. Sci. Paris*, 350(17-18):837–840, 2012.

[Lin14] A. Linke. On the role of the Helmholtz decomposition in mixed methods for incompressible flows and a new variational crime. *Comput. Methods Appl. Mech. Engrg.*, 268:782–800, 2014.

[Lio13] P.-L. Lions. *Mathematical topics in fluid mechanics. Volume 1: Incompressible models*. Oxford: Oxford University Press, reprint of the 1996 hardback ed. edition, 2013.

[LL75] P. Lascaux and P. Lesaint. Some nonconforming finite elements for the plate bending problem. *Rev. Française Automat. Informat. Recherche Opérationnelle Sér. Rouge Anal. Numér.*, 9:9–53, 1975.

[LLMS17] P. L. Lederer, A. Linke, C. Merdon, and J. Schöberl. Divergence-free Reconstruction Operators for Pressure-Robust Stokes Discretizations with Continuous Pressure Finite Elements. *SIAM Journal on Numerical Analysis*, 55(3):1291–1314, January 2017.

[LM22] P. L. Lederer and C. Merdon. Guaranteed upper bounds for the velocity error of pressure-robust Stokes discretisations. *J. Numer. Math.*, 30(4):267–294, 2022.

[LMNN18] A. Linke, C. Merdon, M. Neilan, and F. Neumann. Quasi-optimality of a pressure-robust nonconforming finite element method for the Stokes-Problem. *Mathematics of Computation*, 87(312):1543–1566, February 2018.

[LMS19] P. L. Lederer, C. Merdon, and J. Schöberl. Refined a posteriori error estimation for classical and pressure-robust Stokes finite element methods. *Numer. Math.*, 142(3):713–748, 2019.

[LMT16] A. Linke, G. Matthies, and L. Tobiska. Robust arbitrary order mixed finite element methods for the incompressible Stokes equations with pressure independent velocity errors. *ESAIM. Mathematical Modelling and Numerical Analysis*, 50(1):289–309, 2016.

[LuK16] G. Ł ukaszewicz and P. Kalita. *Navier-Stokes equations*, volume 34 of *Advances in Mechanics and Mathematics*. Springer, Cham, 2016. An introduction with applications.

[Mac15] I. G. Macdonald. *Symmetric functions and Hall polynomials*. Oxford Classic Texts in the Physical Sciences. The Clarendon Press, Oxford University Press, New York, second edition, 2015.

[Mor68] L. S. D. Morley. The triangular equilibrium element in the solution of plate bending problems. *The Aeronautical Quarterly*, 19:149–169, 1968.

[NV12] R. H. Nochetto and A. Veeser. Primer of adaptive finite element methods. In *Multiscale and adaptivity: modeling, numerics and applications*, volume 2040 of *Lecture Notes in Math.*, pages 125–225. Springer, Heidelberg, 2012.

[OR04] M. A. Olshanskii and A. Reusken. Grad-div stabilization for Stokes equations. *Math. Comp.*, 73(248):1699–1718, 2004.

[RCnDLSR18] A. A. Rodríguez, J. Camaño, E. De Los Santos, and F. Rapetti. A graph approach for the construction of high order divergence-free Raviart-Thomas finite elements. *Calcolo*, 55(4):Paper No. 42, 28, 2018.

[RWG99] B. Rivière, M. F. Wheeler, and V. Girault. Improved energy estimates for interior penalty, constrained and discontinuous Galerkin methods for elliptic problems. Part I. *Computational Geosciences*, 3(3):337–360, December 1999.

[RWG01] B. Rivière, M. F. Wheeler, and V. Girault. A Priori Error Estimates for Finite Element Methods Based on Discontinuous Approximation Spaces for Elliptic Problems. *SIAM Journal on Numerical Analysis*, 39(3):902–931, January 2001.

[SB06] G. Stoyan and A. Baran. Crouzeix-Velte decompositions for higher-order finite elements. *Comput. Math. Appl.*, 51(6-7):967–986, 2006.

[SM07] E. Süli and I. Mozolevski. *hp*-version interior penalty DGFEMs for the biharmonic equation. *Comput. Methods Appl. Mech. Engrg.*, 196(13-16):1851–1863, 2007.

[ST23] S. Sauter and C. Torres. On the inf-sup stability of Crouzeix-Raviart Stokes elements in 3D. *Math. Comp.*, 92(341):1033–1059, 2023.

[Str72] G. Strang. Variational crimes in the finite element method. Math. Found. finite Elem. Method Appl. part. differ. Equations, Sympos. Univ. Maryland, Baltimore 1972, 689-710 (1972)., 1972.

[SV85] L. R. Scott and M. Vogelius. Norm estimates for a maximal right inverse of the divergence operator in spaces of piecewise polynomials. *ESAIM: Mathematical Modelling and Numerical Analysis*, 19(1):111–143, 1985.

[SZ90] L. R. Scott and S. Zhang. Finite element interpolation of nonsmooth functions satisfying boundary conditions. *Mathematics of Computation*, 54(190):483–493, 1990.

[Tho81] F. Thomasset. *Implementation of finite element methods for Navier-Stokes equations*. Springer Series in Computational Physics. Springer-Verlag, New York-Berlin, 1981.

[TV16] F. Tantardini and A. Veeser. The L^2-Projection and Quasi-Optimality of Galerkin Methods for Parabolic Equations. *SIAM Journal on Numerical Analysis*, 54(1):317–340, January 2016. Publisher: Society for Industrial and Applied Mathematics.

[Val88] T. Valent. *Boundary value problems of finite elasticity*, volume 31 of *Springer Tracts in Natural Philosophy*. Springer-Verlag, New York, 1988. Local theorems on existence, uniqueness, and analytic dependence on data.

[vDE21] J. F. van Diejen and E. Emsiz. Cubature rules from Hall-Littlewood polynomials. *IMA J. Numer. Anal.*, 41(2):998–1030, 2021.

[Vee16] A. Veeser. Approximating gradients with continuous piecewise polynomial functions. *Found. Comput. Math.*, 16(3):723–750, 2016.

[Ver89] R. Verfürth. A posteriori error estimators for the Stokes equations. *Numer. Math.*, 55(3):309–325, 1989.

[Ver91] R. Verfürth. A posteriori error estimators for the Stokes equations. II. Nonconforming discretizations. *Numer. Math.*, 60(2):235–249, 1991.

[Ver94] R. Verfürth. A posteriori error estimation and adaptive mesh-refinement techniques. In *Proceedings of the Fifth International Congress on Computational and Applied Mathematics (Leuven, 1992)*, volume 50, pages 67–83, 1994.

[Ver13] R. Verfürth. *A posteriori error estimation techniques for finite element methods*. Numer. Math. Sci. Comput. Oxford: Oxford University Press, 2013.

[VV09] A. Veeser and R. Verfürth. Explicit upper bounds for dual norms of residuals. *SIAM J. Numer. Anal.*, 47(3):2387–2405, 2009.

[VZ18a] A. Veeser and P. Zanotti. Quasi-optimal nonconforming methods for symmetric elliptic problems. I: Abstract theory. *SIAM J. Numer. Anal.*, 56(3):1621–1642, 2018.

[VZ18b] A. Veeser and P. Zanotti. Quasi-optimal nonconforming methods for symmetric elliptic problems. III—Discontinuous Galerkin and other interior penalty methods. *SIAM Journal on Numerical Analysis*, 56(5):2871–2894, 2018.

[VZ19a] A. Veeser and P. Zanotti. Quasi-optimal nonconforming methods for symmetric elliptic problems. II—Overconsistency and classical nonconforming elements. *SIAM Journal on Numerical Analysis*, 57(1):266–292, 2019.

[VZ19b] R. Verfürth and P. Zanotti. A quasi-optimal Crouzeix-Raviart discretization of the Stokes equations. *SIAM J. Numer. Anal.*, 57(3):1082–1099, 2019.

[WF87] A. J. Worsey and G. Farin. An n-dimensional Clough-Tocher interpolant. *Constr. Approx.*, 3(2):99–110, 1987.

[Whe78] M. F. Wheeler. An Elliptic Collocation-Finite Element Method with Interior Penalties. *SIAM Journal on Numerical Analysis*, 15(1):152–161, February 1978.

[Xu97] J. Xu. An introduction to multilevel methods. In *Wavelets, multilevel methods and elliptic PDEs (Leicester, 1996)*, Numer. Math. Sci. Comput., pages 213–302. Oxford Univ. Press, New York, 1997.

[Yos95] K. Yosida. *Functional analysis*. Classics in Mathematics. Springer-Verlag, Berlin, 1995. Reprint of the sixth (1980) edition.

[Zan17] P. Zanotti. *Quasi-Optimal Nonconforming Methods For Symmetric Elliptic Problems*. PhD thesis, Universita‘ Degli Studi Di Milano, 2016/2017.

[Zas89] T. Zaslavsky. Biased graphs. I: Bias, balance, and gains. *J. Comb. Theory, Ser. B*, 47(1):32–52, 1989.