Construction Principles of Tight Wavelet Frames in Connection with Linear System Theory

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Construction Principles of Tight Wavelet Frames in Connection with Linear System Theory

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1. Introduction

In this thesis, we want to explore the connection between two seemingly separate fields of mathematics: Wavelet analysis and linear system (or control) theory, the former being our starting point.

The use of the term "wavelet" in its contemporary sense is usually attributed to the geophysicist Jean Morlet, first in its French form *ondelette* in the early 1980s ([81]). By then, it had already been used in seismology for several decades with a different meaning (see, for example, [88]). The field now known as wavelet analysis emerged in the early 1980s and developed rapidly during that decade. However, its origins can be traced back even further to Alfréd Haar's work on the theory of systems of orthogonal functions in the early 20th century (1910, [52]). His contributions predate both wavelet analysis and the use of the term wavelet in this context. However, the function later called *Haar wavelet* is still widely used, e.g., in teaching, since it is the simplest possible wavelet.

Similarly, the *Haar system* derived from this function, initially given by Haar as an example of an orthonormal system in $L^2([0,1])$, is the simplest possible example of a *wavelet basis*. Although the discontinuity of the Haar wavelet is a disadvantage in many scenarios, there are some applications. For example, the *Haar transform* based on this function is used in a modified form in the recent JPEG XL image compression standard ([74]). However, due in part to the loss of support in almost all major web browsers, the widespread adoption of this standard seems questionable at the time of writing.

Alfréd Haar is not the only mathematician whose ideas have resurfaced in wavelet theory. See, for example, Paul Lévy's work on Brownian motion, Littlewood and Paley's work on localizing the energy of a function, Antoni Zygmund's extension of their ideas to higher dimensions, and Alberto Calderón's work on harmonic analysis.

Although his methods are different from wavelet theory, Dennis Gabor is another name that is often mentioned in the history of wavelets because his *Gabor atoms* (1946, [42]) share their motivation with wavelets (and the whole field of time-frequency analysis): To overcome some limitations of the Fourier transform.

Fourier analysis ([38],[39]) has, of course, proven to be immensely useful in many areas of mathematics and physics, not only in signal processing but also, for example, in partial differential equations and quantum mechanics. However, when analyzing a signal, the classical Fourier transform provides no time localization because it is a decomposition into standing sine and/or cosine waves. This is often a significant drawback since real-world signals typically change over their duration. Furthermore, it implicitly assumes that the analyzed signal is infinite or periodic in time, which is another drawback since real-world signals are of finite duration. There are variations of the Fourier transform designed to overcome these limitations, such as the short-time Fourier transform (STFT), of which the *Gabor transform* is both a special case and the first example.

Dennis Gabor's work and wavelet analysis are based on different but similar ideas: Instead of using standing sine and cosine waves as a basis, Gabor (drawing on earlier work by John von Neumann in quantum mechanics, [82]) defined a basis of $L^2(\mathbb{R})$ by translating and modulating a single, in his original work Gaussian, function $g \in L^2(\mathbb{R})$. The resulting Gabor atoms then serve as window functions, dividing a signal into shorter, time-localized segments before applying the Fourier transform.

Wavelet bases and the continuous wavelet transform also have a single function $\psi \in L^2(\mathbb{R})$ (which has to satisfy some admissibility condition) as their starting point, the eponymous wavelet. Roughly speaking, wavelets are wavelike oscillations that decay quickly or, preferably, have a compact support. Then ψ, also called the mother wavelet, is shifted and *dilated* (instead of modulated), i.e., transformed into functions

$$\psi_{a,b}(x) := |a|^{-\frac{1}{2}}\, \psi\left(\frac{x-b}{a}\right)$$

with $a \in \mathbb{R} \setminus \{0\}$ and $b \in \mathbb{R}$. The idea of the *continuous wavelet transform (CWT)* is now to look at the integral transform $f \mapsto \langle f, \psi_{a,b} \rangle_{L^2(\mathbb{R})}$ over the whole parameter range, the *discrete wavelet transform (DWT)* is the expansion of $f \in L^2(\mathbb{R})$ in terms of the discrete set $\left\{\psi_{a,b} \,:\, (a,b) = (2^{-j}, k2^{-j}),\, j,k \in \mathbb{Z}\right\}$, a so-called *wavelet basis*. The general idea of using $\psi_{a,b}$ for an integral transform is attributed to Morlet. Together with the physicist Alex Grossmann and Pierre Goupillaud, a geophysicist like Morlet, he published what are often considered to be the first papers on what is now called the continuous wavelet transform ([46],[47],[49]). However, George Zweig discovered a version of the continuous wavelet transform called the cochlear transform several years earlier in 1975 ([110]) in the very different context of the ear's response to sound. Moreover, both can be seen as rediscoveries (with a different interpretation) of Calderón's work from 1964 ([15]).

The wavelet transform provides localization in both frequency and time. However, like other techniques in time-frequency analysis, it is subject to the *uncertainty principle of signal processing*, which means that it cannot provide arbitrarily good resolution in both time and frequency simultaneously. The use of dilation in the wavelet transform implies that the width in time of the analysis window decreases at higher frequencies, resulting in a better time resolution at the expense of lower frequency resolution. On the other hand, lower frequencies provide good frequency resolution at the expense of low time resolution.

The adaptive resolution this behavior provides is one of the main advantages over the short-time Fourier transform, where the analysis window has a fixed time width. This results in a uniform resolution across the frequency domain, which is a disadvantage when analyzing signals with fast transient characteristics. In addition, we have a wide range of options for choosing the mother wavelet and can thus fine-tune that choice for a given task, e.g., matching the shape of the wavelet to the shape of the characteristic we want to detect in the signal.

The field of wavelet analysis developed at a rapid pace during the 1980s. In 1982, Jan-Olov Strömberg discovered the first smooth orthonormal wavelet ([106]; however, his paper is actually on orthonormal bases for Hardy spaces and predates the term wavelet). Yves Meyer, unaware of Strömberg's construction at the time, also constructed an orthonormal wavelet basis in 1985 ([80]). In the following years, Battle ([9]) and Lemarié ([73]) independently developed orthonormal wavelet bases with exponential decay based on splines, and Ingrid Daubechies constructed orthogonal wavelet bases with compact support and some degree of smoothness ([25]).

Daubechies, Grossmann, and Meyer laid the foundations of the discrete wavelet transform in 1986 ([27]). Their expansions were based on *frames*, a generalization of bases first used by Richard Duffin and Albert Schaeffer in 1952 ([36]) and in the study of non-harmonic Fourier series. Frames are still generating systems of their respective Hilbert spaces (usually $L^2(\mathbb{R}^d)$ in the wavelet case) but are not required to be linearly independent. Thus, representations using frames have a certain degree of redundancy, which is often advantageous in signal processing. In contrast to bases, wavelet frames do not have one but several mother wavelets. Stéphane Mallat, coming from an image analysis background, noticed the similarities between orthogonal wavelets, *quadrature mirror filters* (invented for the digital telephone), and the pyramid algorithms (used in numerical image processing). The construction of wavelets with compact support by Daubechies is based on these ideas. Mallat also discovered (with some contributions from Meyer) that orthogonal wavelets can be constructed more systematically from a formal framework he called *multiresolution analysis* ([76],[78]). Based on all these ideas, he developed a fast wavelet transform (FWT) algorithm for the discrete wavelet transform (also called Mallat algorithm, 1989, [77]).

As an interesting aside, both Meyer and Mallat came into the field through chance encounters, as reported in [58]:

> The French mathematician Yves Meyer, a professor at the École Normale Supérieure in Paris, was waiting for his turn at a photocopier when a colleague showed him a paper on wavelets by Morlet and the theoretical physicist Alex Grossmann. Meyer was immediately fascinated and took the first available train to Marseille, where he began working with Grossman and Morlet, as well as the mathematician and physicist Ingrid Daubechies (now at Duke University). Meyer would go on to win the Abel prize for his work on wavelet theory.
>
> A few years later, a graduate student at Pennsylvania State University studying computer vision and image analysis named Stéphane Mallat bumped into an old friend at the beach. The friend, a graduate student with Meyer in Paris, told Mallat about their research in wavelets. Mallat understood the importance of Meyer's work for his own research right away, and quickly teamed up with Meyer.

Wavelets and wavelet transforms have applications in numerous areas. Many examples are in signal and image processing, the most prominent being the JPEG 2000 image compression standard ([79]). It has advantages over the older JPEG standard, including better compression performance. However, a lack of compatibility with the JPEG format, increased computational complexity, and extensions such as its metadata format not being in the public domain have prevented the widespread adoption of this standard (it is used to some extent in professional sectors, e.g., for medical imaging and digital cinema). Other examples include audio compression ([105]), noise reduction ([100]), and more specific applications such as the *Wavelet Scalar Quantization (WSQ) Gray-scale Fingerprint Image Compression Algorithm* developed by the FBI for digitizing fingerprints ([13]).

In addition to signal and image processing, there is a wide range of wavelet applications in other fields. To name a few, they are used at their origin in geophysics ([69]), in medicine (for example, the analysis of EEG, ECG, and EMG data, see [102]), and in solving partial differential equations ([24]). There are also several extensions of wavelets and techniques based on similar ideas designed to better handle specific tasks, such as shearlets ([50]), curvelets ([16]), and contourlets ([32]).

However, our focus here is not on the wavelet transform and its applications but on the construction of wavelet frames, particularly the construction of *tight* wavelet frames. While frames are a generalization of bases, tight frames generalize orthonormal bases. In particular, tight frames allow a simple computation of the coefficients when writing an element of the respective Hilbert space as a linear combination of the frame elements. Two well-known results in the construction of wavelet frames, which reduce the problem to a set of identities for trigonometric polynomials (or algebraic polynomials on the torus), are the *Unitary Extension Principle* by Amos Ron and Zuowei Shen (1997, [91]) and the *Oblique Extension Principle* by Charles Chui, Wenjie He, and Joachim Stöckler (2002, [23]) and Daubechies, Bin Han, Ron, and Shen (2003, [28]).

Numerous constructions are based on these extension principles. Ron and Shen themselves ([92]) constructed frames based on the Unitary Extension Principle based on both B-splines (bivariate) and box splines (d-variate; both types of functions are widespread examples throughout the theory of wavelets). Their wavelets have compact support and arbitrarily high smoothness. However, they only consider the case where the matrix $M = 2I$ is used for the dilation, and their number of mother wavelets increases with the desired degree of smoothness (even though the redundancy inherent in frames can be helpful, the number of mother wavelets should still be kept small to avoid computations becoming too costly). Karlheinz Gröchenig and Ron ([48]) combined these results and self-affine tilings to construct wavelet frames for general dilation matrices $M \in \mathbb{Z}^{d\times d}$ and arbitrary high smoothness. However, again, the number of mother wavelets increases with the desired degree of smoothness. Chui and He constructed frames based on B-splines ([21], Alexander Pethukov simultaneously proved a similar result, [85]) and on box splines ([23]) with a fixed number of mother wavelets. In [71], Ming-Jun Lai and Joachim Stöckler used a sum of squares decomposition of a specific nonnegative Laurent polynomial, which appears in the Unitary Extension Principle setting through the so-called sub-QMF condition, to reduce the number of required mother wavelets further.

Bin Han ([54]) used the Oblique Extension Principle to construct tight wavelet frames with arbitrary high smoothness and arbitrary high order of *vanishing moments* for any dimension and any dilation matrix $M \in \mathbb{Z}^{d\times d}$. To do this, he uses the Kronecker product and the univariate frames from [23] and [28] to obtain a frame for the Smith normal form of M, from which he constructs a frame for the original dilation matrix. The number of mother wavelets in this construction depends only on the dimension and the matrix M; in particular, it does not depend on the desired smoothness or the desired order of vanishing moments.

Of course, not all constructions of wavelet frames are based on the Unitary or the Oblique Extension Principle. Also, not all constructions focus on tight wavelet frames. For example, in [68], Aleksandr Krivosheim, Vladimir Protasov, and Maria Skopina use the *Matrix Extension Principle* and *polyphase components* to construct multivariate *dual frames* (and tight frames) with compact support. In [55], Bin Han constructed both tight and dual wavelet frames by integrating along parallel superplanes.

The field to which we want to connect wavelet analysis, particularly the construction of wavelets with the help of the Unitary and Oblique Extension principle, is linear system (or linear control) theory, which can be viewed from an engineering and a mathematical perspective.

The origins of linear system theory in engineering can be traced back to the early 20th century when the need arose to mathematically model and control real-world dynamic systems, such as electrical circuits, mechanical systems, and process control (for a brief historical overview, see [30]). Initially, frequency domain techniques were the predominant approach. These techniques were adequate for single-input-single-output (SISO) systems but not for multi-input-multi-output (MIMO) systems, which became increasingly important in the 1950s.

A significant advance was the state-space approach that emerged in the 1960s and was spearheaded by the seminal work of Rudolf E. Kálmán (beginning in [66] and continuing over several papers). The general idea is to use a system of coupled ordinary differential equations to model both the processes occurring within a real-world system (the eponymous "state") and the system's interactions with the external environment (i.e., its inputs and outputs).

State-space systems are used in contexts such as economics ([104]), neuroscience ([103]), and image processing ([90]).

From a mathematical point of view, control theory is a part of operator theory. In the early 1980s ([109],[40]), George Zames, Bruce Francis, and J. William Helton connected control theory, in particular the field of H^∞-control, to the Nevanlinna-Pick interpolation problem, which dates back to the early 20th century ([84],[87]). The Nevanlinna-Pick interpolation problem, in turn, has numerous connections to other aspects of operator theory ([97], [98]).

The goal of this thesis is as follows: In [18], Maria Charina, Mihai Putinar, Claus Scheiderer, and Joachim Stöckler linked the construction of wavelet frames using the Unitary Extension Principle in a constructive way to so-called *realizations* that appear in the study of state-space systems. We want to generalize this connection to multivariate wavelet frames based on the Oblique Extension Principle. In particular, this involves a step from an identity on the torus $\mathbb{T}^d$ (the Oblique Extension Principle) to the inside of the disc $\mathbb{D}^d$. We hope this connection will later allow insights from linear systems theory to help construct wavelet frames.

The thesis is structured in the following way: After some preliminaries and notations in Section 2, we will first use Section 3 to recall some basic results about frames (Section 3.1), with a special focus on (tight) wavelet frames (Section 3.2), i.e., a special kind of frame for the Hilbert space $L^2(\mathbb{R}^d)$ based on the wavelet concept. In particular, we will use Section 3.3 to present the Unitary and the Oblique Extension Principle since they are the actual starting point of our considerations. The Unitary and Oblique Extension Principle identities are formulated initially for trigonometric polynomials. However, in this thesis, we will mostly interpret the trigonometric polynomials involved as algebraic Laurent polynomials on the d-dimensional torus $\mathbb{T}^d$.

For our connection to linear system theory, we also have to rewrite the identities of the Oblique Extension Principle, which are scalar in their original form, into an identity for matrix (Laurent) polynomials. For this purpose, as well as for later results, we use Section 4 to recapitulate results on the factorization of trigonometric polynomials, i.e., the well-known Fejér-Riesz theorem (Section 4.1) and its generalizations, both to the the univariate matrix-valued case (Section 4.2) and to the scalar-valued multivariate case (Section 4.3), where we usually have only a sum-of-squares decomposition, not a single-square factorization $s = |\theta|^2$ as in the univariate case. However, we will recall a result by Geronimo and Woerdeman that gives an equivalent condition for the existence of single-square factorizations of bivariate trigonometric polynomials. Using these results, we formulate the desired matrix form of the Oblique Extension Principle in Section 4.4.

In Section 5, we will give a short introduction to system theory (Section 5.1) and reformulate some important equivalences in a way that fits our goal of linking the Oblique Extension Principle and linear system theory. Essentially, this result ensures that we have various "points" in linear system theory to which we can connect the Oblique Extension Principle.

Section 6 will then establish the desired connection in the univariate case. We can even formulate the results in a slightly more general way than is necessary for the Oblique Extension Principle. In Section 6.1, we will bring the problem back to the setting of the Unitary Extension Principle, where the desired link already exists (see Section 3 in [67]). This allows us to utilize similar methods to find the connection for the Oblique Extension Principle case. The proof of the main result is constructive. A similar equation to the matrix form of the Oblique Extension Principle also appears in the context of (a generalized version of) the Nevanlinna-Pick interpolation mentioned above. In Section 6.2, we will use these similarities to solve our problem by interpolation, both on the disc $\mathbb{D}$ and on the torus $\mathbb{T}$. We will also illustrate both construction methods with examples.

In Section 7, we will use the univariate results from the previous section to prove our first multivariate result. It is a well-known technique in wavelet analysis to construct multivariate wavelet frames (or bases) for diagonal scaling matrices from univariate wavelet frames (or bases) using the Kronecker product. These wavelets are called *separable* and are widely used because they are easy to construct, and their use is often less computationally expensive than using their non-separable counterparts. Now, using the results of Section 6, we can connect these separable wavelet frames to linear system theory, also with the help of the Kronecker product. Again, we will demonstrate this construction with an example.

In the last main section, Section 8, we will discuss the bivariate case. We have to restrict ourselves to the case where the so-called *vanishing moment recovery function*, a Laurent polynomial that appears in the Oblique Extension Principle, has a single square factorization. In Section 8.1, we then demonstrate that we can again adopt methods from the setting of the Unitary Extension Principle (a result from Anton Kummert, [70], an engineer, recently reproved in a more mathematical language by Greg Knese, [67]) to find the connection to system theory for the Oblique Extension Principle. The key to this method is to reduce the problem back to one dimension and utilize our univariate results. Again, we can formulate our result in a slightly more general way than is necessary for the Oblique Extension Principle setting. As in the univariate case, we can use interpolation for our constructions. We will explain how in Section 8.2 and demonstrate this method with an example.

The results in the examples are calculated using Matlab's Symbolic Math Toolbox (except for some parts of the example in Section 8.2, which had to be computed numerically). The code for these calculations can be found in the Appendix.

2. Preliminaries and Notations

Let us introduce some notations and definitions and recall some basic results.

2.1. Basic Notations

We use the standard symbols $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$ for the sets of natural numbers, integers, and real and complex numbers, respectively. Note that in this dissertation, 0 is not a natural number; we use the usual symbol $\mathbb{N}_0$ when we want to include it.

We denote by $\mathbb{T}^d$ the d-dimensional *torus*

$$\mathbb{T}^d := \{z \in \mathbb{C}^d \, : \, |z_1| = \ldots = |z_d| = 1\}$$

(also called *d-torus*) and by $\mathbb{D}^d$ the d-dimensional (open) unit *disc*

$$\mathbb{D}^d := \{z \in \mathbb{C}^d \, : \, |z_1|, \ldots, |z_d| < 1\}$$

(also called *d-disc*).

As usual, $\operatorname{Span} \Omega$ denotes the span, i.e., the set of all finite linear combinations of the elements of a set Ω. The *range* of a function (or operator) f is denoted by $\operatorname{ran}(f)$. In a topological space $\mathcal{T}$, $\operatorname{clos}_{\mathcal{T}}(U)$ is the *closure* of a subset $U \subset \mathcal{T}$ in $\mathcal{T}$.

2.2. Polynomials

We use the notation $\mathbb{C}[z] = \mathbb{C}[z_1, \ldots, z_d]$ for the ring of d-variate *polynomials* with complex coefficients and variables $z_1, \ldots, z_d$. Analogously we use $\mathbb{C}[z^{\pm 1}] = \mathbb{C}[z_1^{\pm 1}, \ldots, z_d^{\pm 1}]$ for the set of d-variate *Laurent polyomials* and $\mathbb{C}(z) = \mathbb{C}(z_1, \ldots, z_d)$ for the set of all d-variate rational functions with complex coefficients.

We further denote the set of all d-variate *matrix polynomials*, i.e., matrix-valued functions of the form

$$P(z) := \sum_{j=0}^{N} P_j z^j$$

with $P_j \in \mathbb{C}^{\nu_1 \times \nu_2}$, $0 \le j \le N$, with $\mathbb{C}[z]^{\nu_1 \times \nu_2} = \mathbb{C}[z_1, \ldots, z_d]^{\nu_1 \times \nu_2}$. As in the scalar-valued case, $\mathbb{C}[z^{\pm 1}]^{\nu_1 \times \nu_2} = \mathbb{C}[z_1^{\pm 1}, \ldots, z_d^{\pm 1}]^{\nu_1 \times \nu_2}$ and $\mathbb{C}(z)^{\nu_1 \times \nu_2} = \mathbb{C}(z_1, \ldots, z_d)^{\nu_1 \times \nu_2}$ are the sets of d-variate matrix Laurent polynomials and d-variate matrix-valued functions with rational entries respectively.

We use the analogous notation for polynomials (as well as Laurent polynomials and rational functions) with real coefficients, i.e., $\mathbb{R}[z] = \mathbb{R}[z_1, \ldots, z_d]$, $\mathbb{R}[z^{\pm 1}] = \mathbb{R}[z_1^{\pm 1}, \ldots, z_d^{\pm 1}]$, and $\mathbb{R}(z) = \mathbb{R}(z_1, \ldots, z_d)$, the notations for the matrix-valued counterparts are derived from this as above.

Using the substitution $z = e^{-2\pi i\omega}$ or $(z_1, \ldots, z_d) = (e^{-2\pi i\omega_1}, \ldots, e^{-2\pi i\omega_d})$ we can identify the set of *trigonometric polynomials*, i.e., functions of the form

$$p(\omega) := \sum_{k\in\mathbb{Z}^d} p_k e^{-2\pi i\langle k,\omega\rangle}$$

with finitely many non-zero coefficients $p_k \in \mathbb{C}$, with the set of algebraic Laurent polynomials on the torus $\mathbb{T}^d$.

2.3. Operators and Matrices

For Banach spaces $\mathcal{X}$ and $\mathcal{Y}$, let $\mathcal{B}(\mathcal{X}, \mathcal{Y})$ denote the set of *bounded, linear operators* $T : \mathcal{X} \to \mathcal{Y}$. As usual, we define the *norm* of an operator $T \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$ by

$$\|T\| := \sup_{\substack{x\in\mathcal{X},\\ \|x\|_{\mathcal{X}}=1}} \|Tx\|_{\mathcal{Y}}.$$

Now let $\mathcal{H}$ and $\mathcal{K}$ be two Hilbert spaces. An operator $T \in \mathcal{B}(\mathcal{H}, \mathcal{H}) =: \mathcal{B}(\mathcal{H})$ is called *positive semi-definite*, if

$$\langle Tx, x\rangle_{\mathcal{H}} \geq 0$$

holds for all $x \in \mathcal{H}$. We also use the short notation $T \geq 0$.

An operator $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ is *contractive*, if and only if

$$\|T\| \leq 1,$$

and *isometric*, if and only if

$$\|Tx\|_{\mathcal{H}} = \|x\|_{\mathcal{K}}$$

holds for all $x \in \mathcal{H}$. In particular, $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ is a contractive operator, if and only if

$$I - T^*T \geq 0,$$

(see [3], Proposition 1.12) and an isometric operator if and only if

$$I - T^*T = 0,$$

(see [3], Proposition 1.14), where T^* is the *adjoint operator* of T, i.e., the operator $T^* \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ with

$$\langle Tx, y\rangle_{\mathcal{K}} = \langle y, T^*y\rangle_{\mathcal{H}}$$

for all $x \in \mathcal{H}$ and $y \in \mathcal{K}$.

We use the same names and notations for matrices: As usual, N^* denotes the *conjugate transpose* of a matrix $N \in \mathbb{C}^{d\times d}$, i.e., $N^* := \overline{N}^T$. Similarly, if F is a matrix-valued function, we define

$$F(z)^* := \overline{(F(z))}^T.$$

If a matrix $N \in \mathbb{C}^{d\times d}$ is hermitian and positive semi-definite, i.e., $N^* = N$ and $z^*Nz \geq 0$ for all $z \in \mathbb{C}^d$, we write $N \geq 0$. Furthermore $N_1 \geq N_2 :\Leftrightarrow N_1 - N_2 \geq 0$. A matrix $N \in \mathbb{C}^{d\times d}$ is contractive, if $I - N^*N \geq 0$, and isometric, if $I - N^*N = 0$.

Let M be a *scaling matrix*, i.e., a matrix $M \in \mathbb{Z}^{d\times d}$ whose eigenvalues are all larger than 1 in modulus. We further define $q_M := |\det M|$. For this scaling matrix, Γ_M shall denote a complete set of representatives of $\left(M^T\right)^{-1}\mathbb{Z}^d/\mathbb{Z}^d$ with $0 \in \Gamma_M$. Note, that Γ_M contains exactly q_M elements.

The *dilation operator* $\mathcal{D}_N$ is defined by

$$\mathcal{D}_N f(x) := \sqrt{|\det N|}f(Nx)$$

for $f \in L^2(\mathbb{R}^d)$ and a regular matrix $N \in \mathbb{R}^{d\times d}$. The *translation operator* $\mathcal{T}_y$ is given by

$$\mathcal{T}_y f(x) := f(x - y)$$

for $f \in L^2(\mathbb{R}^d)$ and a fixed $y \in \mathbb{R}^d$. Both $\mathcal{D}_N$ and $\mathcal{T}_y$ are isometric operators on $L^2(\mathbb{R}^d)$.

2.4. Fourier Transformation and Series

The *Fourier transform* of a function $f \in L^1(\mathbb{R}^d)$ is given by

$$\mathcal{F}_1 f(x) := \hat{f}(x) := \int_{\mathbb{R}^d} f(y)e^{-2\pi i\langle x,y\rangle}\, dy.$$

$\mathcal{F}_1$ is then extended to an operator $\mathcal{F} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ in the usual way (see, for example, [5], Chapter 5.18).

A straightforward calculation yields

$$\begin{aligned} \mathcal{F}(\mathcal{D}_N f)(x) &= \frac{1}{\sqrt{|\det N|}}\hat{f}\left((N^T)^{-1}x\right) \\ \text{and} \quad \mathcal{F}(\mathcal{T}_y f)(x) &= e^{-2\pi i\langle x,y\rangle}\hat{f}(x) \end{aligned}$$

for $f \in L^2(\mathbb{R}^d)$.

The *convolution* of two functions $f, g : \mathbb{R}^d \to \mathbb{C}$ is given by

$$(f * g)(x) := \int_{\mathbb{R}^d} f(y)g(x - y)\, dy.$$

According to the *convolution theorem*, for any $f, g \in L^1(\mathbb{R}^d)$, the convolution $f * g$ is also in $L^1(\mathbb{R}^d)$ and satisfies

$$\mathcal{F}(f * g) = \mathcal{F}(f)\mathcal{F}(g)$$

(see [75], Theorem 2.2).

The *Fourier series* of a function $f \in L^2(\mathbb{R}^d)$ is given by

$$f(x) := \sum_{k \in \mathbb{Z}^d} c_k e^{-2\pi i \langle k, x \rangle}$$

with Fourier coefficients

$$c_k(f) := \langle f, e^{-2\pi i \langle k, \cdot \rangle} \rangle_{L^2(\mathbb{R}^d)} = \int_{\mathbb{R}^d} f(x) e^{2\pi i \langle k, x \rangle} \, dx.$$

3. Tight Wavelet Frames

In this section, we will recall some results about frames in Hilbert spaces, with a particular focus on wavelet frames, a special type of frames in the Hilbert space $L^2(\mathbb{R}^d)$.

Roughly speaking, frames in Hilbert spaces are a generalization of bases, where the elements of the frame still form a generating system, but their linear independence is no longer required. Consequently, although each element of the Hilbert space can still be written as a (potentially infinite) linear combination of the elements of the frame, the coefficients of this representation are generally not unique, unlike when using a basis. As mentioned in the introduction, this linear dependence has some advantages in signal analysis, as it provides redundancy of the data contained in a signal.

Frames were first used by Duffin and Schaeffer (1952, [36]) and (after a somewhat surprisingly long break) Young (1980, [108]) in the study of Fourier series and have been widely used in the context of wavelets (i.e., expansions of functions in $L^2(\mathbb{R}^d)$), beginning with Daubechies, Grossmann and Meyer (1985, [27]).

3.1. Frames in Hilbert Spaces

Let $\mathcal{H}$ be a Hilbert space. We start with the most prominent type of basis: The orthonormal basis, i.e., a (Schauder) basis $\{e_j\}_{j\in N}$ of $\mathcal{H}$ with $\langle e_k, e_j\rangle_{\mathcal{H}} = \delta_{j,k}$ for $j \neq k$. One of its main advantages is the easy calculation of the (unconditionally convergent) expansion

$$v = \sum_{j=1}^{\infty} \langle v, e_j\rangle_{\mathcal{H}}\, e_j \tag{3.1}$$

of each $v \in \mathcal{H}$. Another type of basis we will mention here because of its similarity to frames is the Riesz basis.

Definition 3.1
Let $\mathcal{H}$ be a Hilbert space with an orthonormal basis $\{e_j\}_{j\in N}$ and $U : \mathcal{H} \to \mathcal{H}$ be a bounded invertible operator. Then, the family $\{Ue_j\}_{j\in\mathbb{N}}$ is called *Riesz basis* for $\mathcal{H}$.

A Riesz basis is, in fact, a (Schauder) basis (see [19], Section 3.6). Because the trivial choice $U = \mathrm{id}_{\mathcal{H}}$ is always possible, every orthonormal basis is a Riesz basis. The converse is false since $\{Ue_j\}_{j\in\mathbb{N}}$ is an orthonormal basis only if the operator U is unitary.

The following theorem (see [19], Proposition 3.6.4) gives a property of Riesz bases that will clarify their connection to frames.

Theorem 3.2
Let $\mathcal{H}$ be a Hilbert space and $\{v_j\}_{j\in\mathbb{N}}$ be a Riesz basis for $\mathcal{H}$. Then there exist constants $0 < A \leq B < \infty$ such that

$$A\|v\|_{\mathcal{H}}^2 \leq \sum_{j=1}^{\infty} |\langle v, v_j\rangle|^2 \leq B\|v\|_{\mathcal{H}}^2. \tag{3.2}$$

Frames can now be defined as families $\{v_j\}_{j\in\mathbb{N}}$ that satisfy (3.2), but do not have to be a (Riesz) basis.

Definition 3.3
Let $\mathcal{H}$ be a separable Hilbert space. The family $\{v_j\}_{j\in\mathbb{N}}$ is a *frame* for $\mathcal{H}$ if there exist constants $0 < A \leq B < \infty$ such that

$$A\|v\|_{\mathcal{H}}^2 \leq \sum_{j=1}^{\infty} \left|\langle v, v_j\rangle_{\mathcal{H}}\right|^2 \leq B\|v\|_{\mathcal{H}}^2$$

holds for every $v \in \mathcal{H}$. The numbers A and B are called lower and upper *frame bound.* A frame is called a *tight frame* if $A = B$, and *normalized tight frame* (or *Parseval frame*) if $A = B = 1$.

Remark 3.4

a) If $\{v_j\}_{j\in\mathbb{N}}$ is a frame for $\mathcal{H}$, then

$$\operatorname{clos}_{\mathcal{H}}(\operatorname{Span}(\{v_j\}_{j\in\mathbb{N}})) = \mathcal{H}.$$

b) It is immediately clear from Theorem 3.2, that every Riesz basis for $\mathcal{H}$ is also a frame for $\mathcal{H}$. The reverse is not true. For example, if $\{e_j\}_{j\in\mathbb{N}}$ is an orthonormal (and thus also a Riesz) basis for $\mathcal{H}$, then the family $\{e_1, e_1, e_2, e_2, \ldots\}$ containing each element of the orthonormal basis exactly twice is a tight frame for $\mathcal{H}$ with frame bounds $A = B = 2$, but obviously no longer a basis.

c) A frame that is not a basis is called *redundant* or *overcomplete.* This name is justified, because if the family $\{v_j\}_{j\in\mathbb{N}}$ is a redundant frame there exists a sequence $\{c_j\}_{j\in\mathbb{N}} \in \ell^2(\mathbb{N}) \setminus \{0\}$ with

$$\sum_{j=1}^{\infty} c_j v_j = 0_{\mathcal{H}}.$$

In the following, we will not distinguish between the terms "tight frame" and "normalized tight frame" and will always assume that the frame bounds of a tight frame are equal to one. We can do this without loss of generality since it is only a matter of normalizing the frame elements v_j, $j \in \mathbb{N}$.

Tight frames are particularly interesting because, according to the following theorem (see [19], Theorem 5.1.6 and Lemma 5.7.1), they can be seen as a generalization of orthonormal bases; in particular, they "inherit" the essential and useful property of easily computable coefficients when writing an element of $\mathcal{H}$ as a linear combination of the frame elements.

Theorem 3.5
If $\{v_j\}_{j\in\mathbb{N}}$ is a tight frame for $\mathcal{H}$, the identities

$$\|v\|_{\mathcal{H}}^2 = \sum_{j=1}^{\infty} \left|\langle v, v_j\rangle_{\mathcal{H}}\right|^2$$

and

$$v = \sum_{j=1}^{\infty} \langle v, v_j\rangle_{\mathcal{H}} v_j, \tag{3.3}$$

hold for all $v \in \mathcal{H}$. The sum on the right side of (3.3) *is unconditionally convergent.*

The computation of coefficients for a representation with non-tight frames (or non-orthonormal Riesz bases) can be done using so-called *dual frames* (or *dual Riesz bases*). However, we will not look into this since we are concentrating on tight frames.

3.2. Multiresolution Analysis and Wavelet Frames

In this thesis, we are interested in frames for the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^d)$. To simplify the notation, we will use $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ for the scalar product and the norm in $L^2(\mathbb{R}^d)$ from now on. In particular, we will focus on a special type of frame for $L^2(\mathbb{R}^d)$ called *wavelet frame*, where the frame is generated from a finite set of functions by translation on the lattice $\mathbb{Z}^d$ and dilation by an integer scaling matrix $M \in \mathbb{Z}^{d\times d}$.

This approach has its roots in the structure of a *multiresolution analysis (MRA*, sometimes also called *multiresolution approximation)*, first introduced by Mallat (1989, [78]) as a tool for constructing wavelet bases.

Definition 3.6
A sequence $(V_j)_{j\in\mathbb{Z}}$ of closed subspaces $V_j \subseteq L^2(\mathbb{R}^d)$ is called *multiresolution analysis (MRA)* with scaling matrix M, if

i) $V_j \subset V_{j+1}$ for all $j \in \mathbb{Z}$ (monotonicity),

ii) $\operatorname{clos}_{L^2(\mathbb{R}^d)}\left(\bigcup_{j\in\mathbb{Z}} V_j\right) = L^2(\mathbb{R}^d)$ (density),

iii) $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$ (trivial intersection property),

iv) If $f \in V_j$, then $f(M^{-j}\cdot) \in V_0$ (dilation property),

v) There is a *refinable* (or *scaling*) *function* (sometimes also called *father wavelet*) $\phi \in V_0$, such that

$$\Phi := \left\{\mathcal{T}_k\phi \;:\; k \in \mathbb{Z}^d\right\} = \left\{\phi(\cdot - k) \;:\; k \in \mathbb{Z}^d\right\}$$

is a Riesz basis for V_0 (basis property).

We have some immediate remarks about this definition.

Remark 3.7

a) Some definitions require that the family Φ in v) is an orthonormal basis of V_0.

b) The basis property v) implies that the space V_0 is $\mathbb{Z}^d$-shift invariant, i.e., $\mathcal{T}_k V_0 \subseteq V_0$ for all $k \in \mathbb{Z}^d$.

c) We can use the basis property v) and the dilation property iv) to construct Riesz bases

$$\mathcal{D}_M^j(\Phi) := \left\{|\det M|^{j/2}\phi(M^j \cdot -k) : k \in \mathbb{Z}^d\right\}$$

for each space V_j, $j \in \mathbb{Z}$.

d) The trivial intersection property iii) is redundant since it follows from the other properties (see [65], Theorem 2.2).

One of the most important aspects of a multiresolution analysis is that the scaling function ϕ satisfies a so-called refinement equation, which we can consider in both the time and the frequency domain.

Remark 3.8
Since $\phi \in V_0 \subset V_1$ due to the nesting property, and $\mathcal{D}_M(\Phi)$ is a Riesz basis for V_1 according to Remark 3.7 c), the scaling function ϕ satisfies the *refinement* (or *scaling*) *equation*

$$\phi(x) = \sqrt{|\det M|} \sum_{k\in\mathbb{Z}^d} p_k \phi(Mx - k) \tag{3.4}$$

for a sequence $(p_k)_{k\in\mathbb{Z}^d} \in \ell^2(\mathbb{Z}^d)$. In the frequency domain, the refinement equation is given by

$$\hat{\phi}(\omega) = \tilde{p}\big((M^T)^{-1}\omega\big)\hat{\phi}\big((M^T)^{-1}\omega\big)$$

or equivalently

$$\hat{\phi}\big(M^T\omega\big) = \tilde{p}(\omega)\hat{\phi}(\omega), \tag{3.5}$$

with the $\mathbb{Z}^d$-periodic trigonometric function

$$\tilde{p}(\xi) := \frac{1}{\sqrt{|\det M|}} \sum_{k\in\mathbb{Z}^d} p_k e^{-2\pi i\langle k,\omega\rangle}. \tag{3.6}$$

The sequence $(p_k)_{k\in\mathbb{Z}^d}$ is also called the *refinement* (or *scaling*) *mask* of ϕ, the trigonometric function $\tilde{p}$ the corresponding *refinement* (or *scaling*) *symbol*.

Unlike the trivial intersection property, the density condition ii) in Definition 3.6 is not redundant. However, it can be replaced by the following equivalent condition (see [65], Theorem 2.1).

Theorem 3.9
Let $(V_j)_{j\in\mathbb{Z}}$ be a sequence of closed subspaces $V_j \subseteq L^2(\mathbb{R}^d)$ such that the properties i), iv), and v) in Definition 3.6 are satisfied. Then

$$\operatorname{clos}_{L^2(\mathbb{R}^d)}\Big(\bigcup_{j\in\mathbb{Z}} V_j\Big) = L^2(\mathbb{R}^d)$$

if and only if

$$\mathbb{R}^d \setminus \Big(\bigcup_{j\in\mathbb{Z}} M^j \operatorname{supp}\hat{\phi}\Big)$$

is a null set.

This theorem, in turn, allows us to derive an even simpler condition for ϕ (or rather $\hat{\phi}$) for the density condition ii) to hold.

Remark 3.10
If $\hat{\phi}$ is continuous at 0 and $\hat{\phi}(0) \neq 0$, then $\operatorname{supp}\hat{\phi}$ contains a neighborhood of 0. Since all eigenvalues of a scaling matrix M are larger than 1 in modulus, this implies that the equivalent condition from Theorem 3.9 holds, and thus also the density condition ii) from Definition 3.6.

As mentioned above, the structure of a multiresolution analysis was introduced in the univariate case as a tool for constructing bases of $L^2(\mathbb{R})$. Suppose we have a given multiresolution analysis $(V_j)_{j\in\mathbb{Z}}$ with refinement function ϕ for $d = 1$ and $M = 2$. Let W_j, $j \in \mathbb{Z}$, be the orthogonal complement of V_j in V_{j+1}. Then the properties of a multiresolution analysis imply (see, e.g., [19], Chapter 3.9)

$$L^2(\mathbb{R}) = \bigoplus_{j\in\mathbb{Z}} W_j, \tag{3.7}$$

so we can find a basis of $L^2(\mathbb{R})$ by finding and combining bases for all W_j, $j \in \mathbb{Z}$.

Furthermore, the spaces W_j, $j \in \mathbb{Z}$, satisfy the same dilation property as the spaces V_j, $j \in \mathbb{Z}$, i.e., if $f \in W_j$, then $f(2^{-j}\cdot) \in W_0$. So, to get a basis for W_j, we can first construct a basis of W_0 similar to v) in Definition 3.6, i.e., find $\psi \in W_0$ such that $\tilde{\Psi} := \{\psi(\cdot - k) \,:\, k \in \mathbb{Z}\}$ is a Riesz basis for W_0, and use this to obtain Riesz bases

$$\mathcal{D}_2^j(\tilde{\Psi}) := \left\{2^{j/2}\psi(2^j \cdot -k) \,:\, k \in \mathbb{Z}\right\}$$

for each W_j, $j \in \mathbb{Z}$, in the same way as in Remark 3.7 c). Combining this with (3.7) gives us a Riesz basis

$$X := \{2^{j/2}\psi(2^j \cdot -k) \,:\, j \in \mathbb{Z}; k \in \mathbb{Z}\}$$

for $L^2(\mathbb{R})$.

Remark 3.11

a) Usually, the function ψ is obtained using an analogon to the refinement equation (3.4), i.e., it is defined as

$$\psi(x) := \sqrt{2}\sum_{k\in\mathbb{Z}} q_k\phi(2x - k)$$

for some sequence $(q_k)_{k\in\mathbb{Z}}$, or equivalently in the frequency domain by

$$\hat{\psi}(2\omega) = \tilde{q}(\omega)\hat{\phi}(\omega).$$

b) Such a sequence can often be obtained quite easily. For example, if we have a multiresolution analysis with a refinable function $\phi(x) = \sqrt{2}\sum_{k\in\mathbb{Z}} p_k\phi(2x-k)$ such that Φ is an orthonormal basis of V_0, then

$$\psi(x) := \sqrt{2}\sum_{k\in\mathbb{Z}}(-1)^k\overline{p_{1-k}}\,\phi(2x-k)$$

generates an orthonormal basis X of $L^2(\mathbb{R})$ (see Mallat, [78], or Meyer, [80]). In the frequency domain, set

$$\tilde{q}(\omega) := \overline{p\left(\omega+\frac{1}{2}\right)}e^{-2\pi i\omega}.$$

Example 3.12
A well-known example of a multiresolution analysis is the *Haar multiresolution analysis* $(V_j)_{j\in\mathbb{Z}}$ with

$$V_j := \left\{f \in L^2(\mathbb{R}) \,:\, f \text{ is constant on } [2^{-j}k, 2^{-j}(k+1)) \text{ for each } k\in\mathbb{Z}\right\},$$

named after Alfréd Haar (1885–1933). It is easy to verify that this sequence satisfies all the properties in Definition 3.6 with the scaling function $\phi := \chi_{[0,1)}$. Obviously,

$$\phi = \chi_{[0,1)}(2\cdot) + \chi_{[0,1)}(2\cdot-1) = \phi(2\cdot)+\phi(2\cdot-1)$$

holds, so the refinement equation (3.4) is satisfied with the refinement mask $(p_k)_{k\in\mathbb{Z}}$,

$$p_k := \begin{cases} \frac{1}{\sqrt{2}}, & \text{if } k=0,1 \\ 0, & \text{otherwise} \end{cases}.$$

Thus, according to Remark 3.11 b), the function

$$\psi(x) := \phi(2x)-\phi(2x-1) = \begin{cases} 1, & \text{if } 0\le x<1/2 \\ -1, & \text{if } 1/2\le x<1 \\ 0, & \text{otherwise} \end{cases}$$

generates a orthonormal basis $X = \{2^{j/2}\psi(2^j\cdot -k) \,:\, j\in\mathbb{Z}; k\in\mathbb{Z}\}$ for $L^2(\mathbb{R})$.

The function ψ is known as the *Haar wavelet*, first introduced by Alfréd Haar in his dissertation (1909, [52]) to construct an orthonormal system in $L^2([0,1])$. Although much older, it is the simplest known example in the theory of wavelets. However, it is avoided in most applications because it is neither differentiable nor continuous. It can be seen as a special case of both the B-spline and the Daubechies wavelets, two other prominent classes of examples.

One can construct frames for $L^2(\mathbb{R})$ in a similar way to our construction of Riesz bases for $L^2(\mathbb{R})$ above. Analogous to Definition 3.6, we can define a *frame multiresolution analysis*, first introduced by Benedetto and Li (1992,[10], and 1994, [11]), the only difference from a "regular" multiresolution analysis being that the family $\Phi := \{\phi(\cdot - k) : k \in \mathbb{Z}\}$ in v) only needs to be a frame for V_0, not necessarily a Riesz basis.

For a given frame multiresolution analysis $(V_j)_{j\in\mathbb{Z}}$ we can then look again at the orthogonal complements W_j, $j \in \mathbb{Z}$, of V_j in V_{j+1} and find a frame $\tilde{\Psi} := \{\psi(\cdot - k) : k \in \mathbb{Z}\}$ with $\psi \in W_0$ for W_0. Then, similar to Remark 3.7 c) we get frames $\mathcal{D}_2^j(\tilde{\Psi})$ for each W_j, $j \in \mathbb{Z}$, which we can combine using (3.7) (which also holds for a frame multiresolution analysis) to construct a frame for $L^2(\mathbb{R})$.

However, we will not use this method to construct frames but will use a different approach that offers more freedom of choice (for limitations of the frame multiresolution analysis approach, see, e.g., [19], Section 17.6). On the one hand, we will not require our frames to be generated by a single function but will allow multiple functions to do so (although it is desirable to keep the number of these functions as small as possible to keep the computational effort low). This idea leads to the following definitions. On the other hand, we will construct a frame for $L^2(\mathbb{R}^d)$ directly instead of finding a frame for W_0 first. This approach will be the topic of the next section.

Definition 3.13
Let $r \in \mathbb{N}$ and $\Psi := \{\psi_1, \ldots, \psi_r\} \subset L^2(\mathbb{R}^d)$. The system

$$X(\Psi) := \{\psi_{\ell,j,k} : 1 \le \ell \le r; j \in \mathbb{Z}; k \in \mathbb{Z}^d\} \tag{3.8}$$

of functions

$$\psi_{\ell,j,k} := \mathcal{D}_M^j \mathcal{T}_k \psi_\ell = |\det M|^{j/2} \psi_\ell(M^j \cdot -k)$$

is called a *wavelet system*. The functions in Ψ are called *mother wavelets*.

Combining this with Definition 3.3 results in the following definition.

Definition 3.14
Let $X(\Psi) \subset L^2(\mathbb{R}^d)$ be a wavelet system. $X(\Psi)$ is called a *wavelet frame*, if it is a frame for $L^2(\mathbb{R}^d)$. Analogously, $X(\Psi)$ is a *tight wavelet frame*, if $X(\Psi)$ is a tight frame and a *(orthonormal) wavelet basis*, if $X(\Psi)$ is an (orthonormal) basis for $L^2(\mathbb{R}^d)$.

In particular, if $X(\Psi)$ is a tight wavelet frame (or an orthonormal wavelet basis), we can write any $f \in L^2(\mathbb{R}^d)$ as

$$f = \sum_{\ell=1}^{r} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^d} \langle f, \psi_{\ell,j,k} \rangle \psi_{\ell,j,k} \tag{3.9}$$

according to Theorem 3.5. The coefficients of the series on the right side of (3.9) are also called *wavelet coefficients*.

Remark 3.15
We can define wavelet systems and wavelet frames (and bases) more generally using translations on any arbitrary lattice Γ in $\mathbb{R}^d$, i.e., $\Gamma := A\mathbb{Z}^d$ for a regular matrix $A \in \mathbb{R}^{d\times d}$. In this case, the scaling matrix M is usually no longer in $\mathbb{Z}^{d\times d}$, but only in $\mathbb{R}^{d\times d}$, and must satisfy the property $M\Gamma \subseteq \Gamma$ in addition to having only eigenvalues larger than 1 in modulus.

However, since every lattice Γ is the image of $\mathbb{Z}^d$ under an invertible linear transformation, we can restrict ourselves to $\Gamma = \mathbb{Z}^d$ without loss of generality. In some applications, using a grid other than $\mathbb{Z}^d$ may still be advantageous. For example, functions on a different grid may have better symmetry properties.

3.3. Extension Principles

To simplify the construction of wavelet frames, the problem is often broken down into a set of criteria for trigonometric polynomials (or, more generally, periodic functions). For this, we no longer use a multiresolution analysis as our starting point, but only a scaling function $\phi \in L^2(\mathbb{R}^d)$, i.e., a function that satisfies the refinement equation (3.5), given in the frequency domain by

$$\hat{\phi}\big(M^T\omega\big) = \tilde{p}(\omega)\hat{\phi}(\omega)$$

with a trigonometric polynomial

$$\tilde{p}(\omega) := \frac{1}{\sqrt{|\det M|}} \sum_{k\in\mathbb{Z}^d} p_k e^{-2\pi i\langle k,\omega\rangle}$$

(note that $p_k \neq 0$ holds only for a finite number of indices $k \in \mathbb{Z}^d$). We also assume that

$$\lim_{\omega\to 0} \hat{\phi}(\omega) = 1 \tag{3.10}$$

holds ($\hat{\phi}(0) \neq 0$ is desirable anyway, see Remark 3.10).

Note that there are methods to construct scaling functions for any dilation matrix $M \in \mathbb{Z}^{d\times d}$ (see [12], Theorem 1) with compact support (see [107], Lemma 4.3 for the univariate case, the proof for the multivariate case is similar) and arbitrary smoothness (see [54], Theorem 2.2).

We use the scaling function ϕ to construct the mother wavelets $\psi_1, \ldots, \psi_r \in L^2(\mathbb{R}^d)$ of a tight wavelet frame $X(\Psi) := \{\psi_{\ell,j,k} \,:\, 1 \leq \ell \leq r; j \in \mathbb{Z}; k \in \mathbb{Z}^d\}$ in a manner similar to Remark 3.11 a), i.e., by

$$\hat{\psi}_j\big(M^T\omega\big) = \tilde{q}_j(\omega)\hat{\phi}(\omega), \qquad 1 \leq j \leq r, \tag{3.11}$$

with (trigonometric) polynomials $\tilde{q}_1, \ldots, \tilde{q}_r$,

$$\tilde{q}_j(\omega) := \frac{1}{\sqrt{|\det M|}} \sum_{k\in\mathbb{Z}^d} q_{j,k} e^{-2\pi i\langle k,\omega\rangle}, \qquad 1 \leq j \leq r,$$

called *wavelet symbols*. Their coefficient sequences $(q_{j,k})_{k\in\mathbb{Z}^d}$ are called *wavelet masks*.

Remark 3.16
If we construct a tight wavelet frame in the way described above and define the spaces V_j, $j \in \mathbb{Z}$, as

$$V_j := \operatorname{clos}_{L^2(\mathbb{R}^d)}\Big(\operatorname{Span}\Big\{\phi_{j,k} \,:\, k \in \mathbb{Z}^d\Big\}\Big)$$

with

$$\phi_{j,k} := \mathcal{D}_M^j \mathcal{T}_k \phi = |\det M|^{j/2}\phi(M^j \cdot - k), \qquad j \in \mathbb{Z}, k \in \mathbb{Z}^d,$$

the sequence $(V_j)_{j\in\mathbb{Z}}$ satisfies the conditions i)–iv) of a multiresolution analysis, but not necessarily condition v) (see [19], Lemma 18.2.5 and Lemma 17.3.1). Wavelet frames constructed in this way are still called *MRA-based*.

Note that the refinement symbol $\tilde{p}$ and the wavelet symbol $\tilde{q}_j$ do not necessarily have to be trigonometric polynomials. They can also be chosen more generally as $\mathbb{Z}^d$-periodic, measurable, and bounded functions (satisfying (3.5) and (3.11), respectively). However, it is more common to use trigonometric polynomials because they are easier to handle both in theory (e.g., due to their smoothness) and practice (faster computations and easier use of the FFT, among other things). Furthermore, their associated masks are finite - and finite masks appear in several places in signal processing (see Finite Impulse Response (FIR) masks). Therefore, we will mostly stick to trigonometric polynomials and use the more general form only in the following theorem.

We now turn to the criteria mentioned above. A well-known result for the construction of tight wavelet frames is the *Unitary Extension Principle* of Ron and Shen (1997, [91]), which places conditions on the wavelet symbols $\tilde{q}_1, \ldots, \tilde{q}_r$ such that $\psi_1, \ldots, \psi_r$ generate a tight wavelet frame.

Theorem 3.17 (Unitary Extension Principle (UEP))
Suppose that $\phi \in L^2(\mathbb{R}^d)$ satisfies (3.10) *as well as the refinement equation* (3.5) *for a $\mathbb{Z}^d$-periodic, measurable, and bounded refinement symbol $\tilde{p}$. Let $\tilde{q}_1, \ldots, \tilde{q}_r$ be $\mathbb{Z}^d$-periodic, measurable, and bounded functions, and let $\Psi = \{\psi_1, \ldots, \psi_r\}$ be constructed from $\tilde{q}_1, \ldots, \tilde{q}_r$ and ϕ by equation* (3.11)*.*

Then $X(\Psi)$ is a tight wavelet frame of $L^2(\mathbb{R}^d)$, if

$$\tilde{p}(\omega)\overline{\tilde{p}(\omega+\sigma)} + \sum_{j=1}^{r} \tilde{q}_j(\omega)\overline{\tilde{q}_j(\omega+\sigma)} = \delta(\sigma) \tag{3.12}$$

holds for almost all $\omega \in \mathbb{R}^d$ and all $\sigma \in \Gamma_M$.

Remark 3.18
The identities (3.12) imply that the so-called *sub-QMF condition* (quadrature mirror filter)

$$\sum_{\sigma \in \Gamma_M} |\tilde{p}(\omega+\sigma)|^2 \leq 1 \tag{3.13}$$

is a necessary condition on $\tilde{p}$ for the Unitary Extension Principle to hold.

Proof. Let $\sigma_1, \ldots, \sigma_{q_M}$ be the different elements of Γ_M with $\sigma_1 := 0$. Due to (3.12) and $\sigma_1 = 0$, the rows of the matrix

$$\begin{pmatrix} \tilde{p}(\omega+\sigma_1) & \tilde{q}_1(\omega+\sigma_1) & \cdots & \tilde{q}_r(\omega+\sigma_1) \\ \tilde{p}(\omega+\sigma_2) & \tilde{q}_1(\omega+\sigma_2) & \cdots & \tilde{q}_r(\omega+\sigma_2) \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{p}(\omega+\sigma_{q_M}) & \tilde{q}_1(\omega+\sigma_{q_M}) & \cdots & \tilde{q}_r(\omega+\sigma_{q_M}) \end{pmatrix}$$

are orthonormal for every $\omega \in \mathbb{R}^d$. If we extend this to a square unitary matrix, the first column of the extended matrix has the norm 1. This gives us the sub-QMF condition (3.13). □

A common example of a wavelet frame is one based on *B-splines* (short for basis splines). They can be constructed, for example, using the Unitary Extension Principle (see [91], Section 1.3).

Example 3.19
We consider the dyadic univariate case, i.e., the Hilbert space $L^2(\mathbb{R})$ with the scaling factor $M = 2$. Note, that $\Gamma_2 = \{0, 1/2\}$. The centered B-splines B_m of order $m \in \mathbb{N}$ are defined recursively by $B_1 := \chi_{[-1/2,1/2]}$ and

$$B_{m+1}(x) := B_m(x) * B_1(x) = \int_{\mathbb{R}} B_m(x-t)B_1(t)\,dt.$$

A simple calculation yields

$$\hat{B}_1(\omega) = \frac{\sin(\pi\omega)}{\pi\omega}.$$

Thus, the convolution theorem gives

$$\hat{B}_m(\omega) = \left(\frac{\sin(\pi\omega)}{\pi\omega}\right)^m.$$

We use the centered B-spline of order $2m$ as our scaling function, i.e., $\phi := B_{2m}$. It is refinable because the double-angle formula yields

$$\hat{\phi}(2\omega) = \left(\frac{\sin(2\pi\omega)}{2\pi\omega}\right)^{2m} = \left(\frac{2\sin(\pi\omega)\cos(\pi\omega)}{2\pi\omega}\right)^{2m} = \cos^{2m}(\pi\omega)\hat{\phi}(\omega).$$

In particular, the refinement symbol is given by

$$\tilde{p}(\omega) = \cos^{2m}(\pi\omega).$$

Obviously, the assumption (3.10) is also satisfied. We now define the $2m$ wavelet symbols $\tilde{q}_1, \ldots, \tilde{q}_{2m}$ by

$$\tilde{q}_j(\omega) := \sqrt{\binom{2m}{j}} \sin^j(\pi\omega)\cos^{2m-j}(\pi\omega).$$

For $\sigma = 0$ we have

$$\begin{aligned}
|\tilde{p}(\omega)|^2 + \sum_{j=1}^{2m} |\tilde{q}_j(\omega)|^2 &= \cos^{4m}(\pi\omega) + \sum_{j=1}^{2m} \binom{2m}{j} \sin^{2j}(\pi\omega)\cos^{2(2m-j)}(\pi\omega) \\
&= \sum_{j=0}^{2m} \binom{2m}{j} \sin^{2j}(\pi\omega)\cos^{2(2m-j)}(\pi\omega) \\
&= \left(\sin^2(\pi\omega) + \cos^2(\pi\omega)\right)^{2m} = 1
\end{aligned}$$

and for $\sigma = 1/2$, a simple shift of the sine and cosine yields

$$\begin{aligned}
&\tilde{p}(\omega)\overline{\tilde{p}(\omega+1/2)} + \sum_{j=1}^{r} \tilde{q}_j(\omega)\overline{\tilde{q}_j(\omega+1/2)} \\
&= \cos^{2m}(\pi\omega)\sin^{2m}(\pi\omega) + \sum_{j=1}^{2m} \binom{2m}{j} (-1)^j \sin^{2m}(\pi\omega)\cos^{2m}(\pi\omega) \\
&= \cos^{2m}(\pi\omega)\sin^{2m}(\pi\omega) \sum_{j=0}^{2m} \binom{2m}{j} (-1)^j = \cos^{2m}(\pi\omega)\sin^{2m}(\pi\omega)(1-1)^{2m} = 0.
\end{aligned}$$

Thus, according to the Unitary Extension Principle, the mother wavelets $\psi_1, \ldots, \psi_{2m}$ constructed from $\tilde{q}_1, \ldots, \tilde{q}_{2m}$ generate a tight wavelet frame.

Of course, depending on the application, specific properties of tight wavelet frames are desirable. The scaling function and the mother wavelets in Example 3.19 have compact support, meaning the associated masks are finite. Compact support of the scaling function ϕ is not strictly necessary but often helpful. Due to their construction, ϕ having compact support means that the supports of the mother wavelets are compact as well, as long as their associated masks are finite (which they usually are since we mostly use trigonometric polynomials as wavelet symbols). As mentioned above, compact supports/finite masks allow for faster computation and have other advantages (e.g., better localization properties, which can be important in signal and image analysis). Of course, the benefits of the compact support become clearer the shorter the length of the support is.

However, the number of $2m$ mother wavelets in Example 3.19 is quite large. In general, the number of mother wavelets should be kept as small as possible to keep the computational cost low (although using more mother wavelets means more redundancy, which may be desirable in some cases). Other useful characteristics of the mother wavelets may include symmetry or smoothness (both are useful in image processing, e.g., symmetry helps to avoid distortions, and smoother mother wavelets lead to smoother - and thus less visible - errors).

Another desirable property, which we will look at in some more detail, is a high number (or order) of vanishing moments.

Definition 3.20
Let $\nu \in \mathbb{N}_0$. The set $\Psi \subseteq L^2(\mathbb{R}^d)$ has ν *vanishing moments* or *vanishing moments of order* ν, if

$$\int_{\mathbb{R}^d} \psi(x)x^\beta dx = 0$$

with $x^\beta := x_1^{\beta_1} \ldots x_d^{\beta_d}$ holds for all $\psi \in \Psi$ and all $\beta \in \mathbb{N}_0^d$ with $|\beta| := \sum_{j=1}^{d} |\beta_j| < \nu$.

In other words, a set $\Psi \subseteq L^2(\mathbb{R}^d)$ has vanishing moments of order ν if all functions in Ψ are orthogonal to any polynomial of total degree smaller than ν. Alternatively, vanishing moments can also be defined in the frequency domain (see [75], Theorem 7.4).

Theorem 3.21
Let $\nu \in \mathbb{N}_0$ and $\Psi \subseteq L^2(\mathbb{R}^d)$ such that $(1+|\cdot|)^{\nu-1}\psi \in L^1(\mathbb{R}^d)$ for all $\psi \in \Psi$. Then Ψ has vanishing moments of order ν if and only if

$$\partial^\beta \hat{\psi}(0) = 0$$

for all $\psi \in \Psi$ and all $\beta \in \mathbb{N}_0^d$ with $|\beta| = \sum_{j=1}^d |\beta_j| < \nu$.

Proof. The assumption $(1+|\cdot|)^{\nu-1}\psi \in L^1(\mathbb{R}^d)$ for all $\psi \in \Psi$ is sufficient to guarantee that the derivatives $\partial^\beta \hat{\psi}$, $|\beta| < \nu$, exist and are continuous. The rules of the Fourier transform then imply

$$\partial^\beta \hat{\psi}(0) = (-2\pi i)^{|\beta|} \int_{\mathbb{R}^d} \psi(x) x^\beta \, dx$$

which immediately gives the equivalence from Theorem 3.21. □

The primary reason why a high order of vanishing moments is often advantageous is its connection to the approximation order of a tight wavelet frame. Roughly speaking, the approximation order measures how well functions from a Sobolev space are approximated when the sum over j in (3.9) is cut at some scaling level $J \in \mathbb{Z}$, i.e., how well

$$Q_J f := \sum_{\ell=1}^{r} \sum_{j<J} \sum_{k \in \mathbb{Z}^d} \langle f, \psi_{\ell,j,k} \rangle \psi_{\ell,j,k}.$$

approximates f (the operator $Q_J : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ is called *truncated operator*). Since one must work with finite sums when implementing methods based on wavelet frames, a high order of approximation is important in applications, and a higher order of vanishing moments allows a higher order of approximation (see [28]) since in that case, the wavelet coefficients are smaller at finer scales (i.e., large values of j). However, a higher order of vanishing moments leads to longer supports of the mother wavelets and, thus, longer masks. This increases the computational cost, so the right balance between these two factors has to be found. However, there is no trade-off between a high order of vanishing moments and a high degree of smoothness since the degree of smoothness of the mother wavelets increases with increasing order of vanishing moments.

Constructing tight wavelet frames with a high order of vanishing moments is difficult using the Unitary Extension Principle. For example, the B-spline-based frame in Example 3.19 has at least one mother wavelet with at most one vanishing moment.

A result that simplifies the construction of tight wavelet frames with a higher order of vanishing moments is the *Oblique Extension Principle* of Chui, He and Stöckler (2002, [23]) and (simultaneously, but independently) Daubechies, Han, Ron and Shen (2003, [28]).

Theorem 3.22 (Oblique Extension Principle (OEP))
Suppose $\phi \in L^2(\mathbb{R}^d)$ satisfies (3.10) as well as the refinement equation (3.5) for the refinement symbol $\tilde{p}$. Let $\tilde{q}_1, \ldots, \tilde{q}_r$ be trigonometric polynomials and $\tilde{s}$ be a $\mathbb{Z}^d$-periodic function that is essentially bounded on the torus $\mathbb{T}^d$ and satisfies $\lim_{\omega\to 0} \tilde{s}(\omega) = 1$ and $\tilde{s}(\omega) > 0$ for all $\omega \in \mathbb{R}^d$. Furthermore, let $\Psi = \{\psi_1, \ldots, \psi_r\}$ be constructed from $\tilde{q}_1, \ldots, \tilde{q}_r$ and ϕ by equation (3.11).

Then $X(\Psi)$ is a tight wavelet frame for $L^2(\mathbb{R}^d)$, if

$$\tilde{p}(\omega)\overline{\tilde{p}(\omega+\sigma)}\tilde{s}(M^T\omega) + \sum_{j=1}^{r} \tilde{q}_j(\omega)\overline{\tilde{q}_j(\omega+\sigma)} = \delta(\sigma)\tilde{s}(\omega) \tag{3.14}$$

holds for almost all $\omega \in \mathbb{R}^d$ and all $\sigma \in \Gamma_M$.

Obviously, the Unitary Extension Principle can be interpreted as the special case $\tilde{s} \equiv 1$ of the Oblique Extension Principle. On the other hand, the proof of the Oblique Extension Principle uses the Unitary Extension Principle.

Proof. Let $\phi, \tilde{p}, \tilde{q}_1, \ldots, \tilde{q}_r$ and $\tilde{s}$ satisfy the assumptions of the Oblique Extension Principle, in particular the identities (3.14). Furthermore, let $S := \sqrt{\tilde{s}}$ and φ be defined by $\hat{\varphi} := S\hat{\phi}$. Since $\tilde{s}$ and thus S is essentially bounded, φ is in $L^2(\mathbb{R}^d)$, and since the limit of both $\tilde{s}$ and $\hat{\phi}$ at 0 is 1, $\hat{\varphi}$ satisfies (3.10). We now set

$$P(\omega) := \frac{S(M^T\omega)\tilde{p}(\omega)}{S(\omega)} \qquad \text{and} \qquad Q_j(\omega) := \frac{\tilde{q}_j(\omega)}{S(\omega)},\ j = 1, \ldots, r.$$

Since $\tilde{s}$ is strictly positive, P and Q_j, $j = 1, \ldots, r$, are well-defined. The refinement equation for ϕ implies

$$P(\omega)\hat{\varphi}(\omega) = \frac{S(M^T\omega)\tilde{p(\omega)}}{S(\omega)} S(\omega)\hat{\phi}(\omega) = S(M^T\omega)\hat{\phi}(M^T\omega) = \hat{\varphi}(M^T\omega),$$

so φ is refinable with the refinement symbol P. Furthermore, because $\tilde{s}$ is $\mathbb{Z}^d$-periodic and $M^T\sigma \in \mathbb{Z}^d$ for $\sigma \in \Gamma_M$, the identities (3.14) yield

$$\begin{aligned} &P(\omega)\overline{P(\omega+\sigma)} + \sum_{j=1}^{r} Q_j(\omega)\overline{Q_j(\omega+\sigma)} \\ =&\frac{S(M^T\omega)\overline{S(M^T(\omega+\sigma))}\,\tilde{p}(\omega)\overline{\tilde{p}(\omega+\sigma)}}{S(\omega)\overline{S(\omega+\sigma)}} + \sum_{j=1}^{r} \frac{\tilde{q}_j(\omega)\overline{\tilde{q}_j(\omega+\sigma)}}{S(\omega)\overline{S(\omega+\sigma)}} \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{S(\omega)\overline{S(\omega+\sigma)}}\left(\tilde{p}(\omega)\overline{\tilde{p}(\omega+\sigma)}\left|S(M^T\omega)\right|^2 + \sum_{j=1}^{r} \tilde{q}_j(\omega)\overline{\tilde{q}_j(\omega+\sigma)}\right) \\
&= \frac{1}{S(\omega)\overline{S(\omega+\sigma)}}\left(\tilde{p}(\omega)\overline{\tilde{p}(\omega+\sigma)}\tilde{s}(M^T\omega) + \sum_{j=1}^{r} \tilde{q}_j(\omega)\overline{\tilde{q}_j(\omega+\sigma)}\right) \\
&= \frac{\delta(\sigma)\tilde{s}(\omega)}{\tilde{s}(\omega)} = \delta(\sigma)
\end{aligned}
$$

for all $\sigma \in \Gamma_M$ and all $\omega \in \mathbb{R}^d$. So, P and Q_j, $j = 1, \ldots, r$, satisfy the identities (3.12) from the Unitary Extension Principle and thus generate a tight wavelet frame. We have already shown that

$$\hat{\varphi}(M^T\omega) = P(\omega)\hat{\varphi}(\omega).$$

Furthermore,

$$Q_j(\omega)\hat{\varphi}(\omega) = \frac{\tilde{q}_j(\omega)}{S(\omega)} S(\omega)\hat{\phi}(\omega) = \tilde{q}_j(\omega)\hat{\phi}(\omega) = \hat{\psi}_j(\omega)$$

for $j = 1, \ldots, d$. Together, this means that the mother wavelets constructed from φ, P and Q_j, $j = 1, \ldots, r$, are the same as the mother wavelets ψ_j constructed from $\phi, \tilde{p}$ and $\tilde{q}_j$, $j = 1, \ldots, d$. So, $X(\Psi)$ is a tight wavelet frame for $L^2(\mathbb{R}^d)$. □

Since the Unitary Extension Principle can be seen as a corollary of the Oblique Extension Principle, and vice versa, any tight wavelet frame constructed using the Unitary Extension Principle can be constructed using the Oblique Extension Principle, and vice versa. Thus, the Oblique Extension Principle is not a generalization of the Unitary Extension Principle in the sense that it allows for the construction of "more" frames. Instead, the addition of the function $\tilde{s}$ gives us more flexibility to develop constructions that may be harder to find without it.

Note that even though the Oblique Extension Principle makes some constructions (e.g., with a higher order of vanishing moments) easier, constructions based on the Unitary Extension Principle may still have advantages in some situations. For example, frame decomposition and reconstruction algorithms are usually faster when using the Unitary Extension Principle. Roughly speaking, frame decomposition means computing the wavelet coefficients on the right side of a discrete version of (3.9) for a given signal on the left side. Frame reconstruction is the inverse operation - reconstructing the signal on the left side of (3.9) from given wavelet coefficients. When using frames based on the Oblique Extension Principle, the frame reconstruction requires an additional convolution and deconvolution with the mask of $\tilde{s}$, which is not necessary when using frames based on the Unitary Extension Principle (see [72], Section 3.4). Since these are the two most essential operations in the practical application of frames, a shorter computation time can be relevant.

Remark 3.23

a) In [23], the function $\tilde{s}$ is called *vanishing-moment recovery (VMR) function*, a name we will adopt here.

b) It is also possible to formulate and prove the Oblique Extension Principle for a non-negative vanishing-moment recovery function (see [28] for example). However, choosing a positive $\tilde{s}$ is common and makes many calculations and proofs easier.

c) The refinement symbol $\tilde{p}$ of a tight wavelet frame constructed with the Oblique Extension Principle generally does not satisfy the sub-QMF condition of Remark 3.18. However, for a rational vanishing-moment recovery function $\tilde{s}$, the Oblique Extension Principle identities (3.14) are equivalent to the so-called *oblique sub-QMF condition* (see [63], Theorem 1)

$$\frac{1}{\tilde{s}(M^T\omega)} - \sum_{\sigma\in\Gamma_M} \frac{|\tilde{p}(\omega)|^2}{\tilde{s}(\omega+\sigma)} \geq 0 \quad \text{for almost all } \omega \in \mathbb{R}^d. \tag{3.15}$$

As an example of a construction using the Oblique Extension Principle, we look at a slightly different form of the B-splines introduced in Example 3.19 (see [19], Example 18.6.3).

Example 3.24

Again, we consider the dyadic univariate case, i.e., $L^2(\mathbb{R})$ with scaling factor $M = 2$. Note, that $\Gamma_2 = \{0, 1/2\}$. A variant of the centered B-spline B_m from Example 3.19 is the shifted B-spline N_m, $m \in \mathbb{Z}$, where the centered B-spline is shifted so that the support is no longer centered around the origin, but is given by $[0, m]$. The translation rule of the Fourier transform yields

$$\hat{N}_m(\omega) = \left(\frac{1-e^{-2\pi i\omega}}{2\pi i\omega}\right)^m.$$

We use the shifted B-spline of order $2m$ as our scaling function, i.e., $\phi := N_{2m}$. It is refinable with the refinement symbol

$$\tilde{p}(\omega) := \left(\frac{1+e^{-2\pi i\omega}}{2}\right)^{2m}.$$

Obviously, the assumption (3.10) is also satisfied. Similar to Example 3.19 we can define the wavelet symbols $Q_1, \ldots, Q_{2m}$ by

$$Q_j(\omega) := i^j e^{-2\pi i m\omega} \sqrt{\binom{2m}{j}} \sin^j(\pi\omega) \cos^{2m-j}(\pi\omega).$$

(the factor i^j is optional) and use the Unitary Extension Principle to show that the corresponding mother wavelets generate a tight wavelet frame.

For example, in the case $m = 1$ (i.e., the piecewise linear B-spline), the two wavelet symbols are

$$Q_1(\omega) = \frac{\sqrt{2}}{4}(1 - e^{-4\pi i\omega}) \qquad \text{and} \qquad Q_2(\omega) = \frac{(1 - e^{-2\pi i\omega})^2}{4}.$$

Alternatively, we can use (as with other choices of m) the Oblique Extension Principle. To do this, let

$$\tilde{s}(\omega) := \frac{4 - \cos(2\pi\omega)}{3} = \frac{1}{3}\left(4 - \frac{e^{-2\pi i\omega}}{2} - \frac{e^{2\pi i\omega}}{2}\right).$$

It is easy to see that $\tilde{s}$ satisfies every requirement of the Oblique Extension Principle. With this $\tilde{s}$ and the wavelet symbols

$$\begin{aligned} \tilde{q}_1(\omega) &= Q_2(\omega) = \frac{(1 - e^{-2\pi i\omega})^2}{4} \\ \text{and} \quad \tilde{q}_2(\omega) &= \frac{\sqrt{6}}{24}(1 - e^{-2\pi i\omega})^2(1 + 4e^{-2\pi i\omega} + e^{-4\pi i\omega}), \end{aligned}$$

the identities (3.14) of the Oblique Extension Principle are satisfied. Thus, the associated mother wavelets generate a wavelet frame. Note that in this case, both mother wavelets have two vanishing moments. In contrast, as mentioned above, the construction using the Unitary Extension Principle has at least one mother wavelet with only one vanishing moment for any choice of m.

Since the refinement and the wavelet symbols involved are trigonometric functions, it is only natural to choose - as in Example 3.24 - a trigonometric $\tilde{s}$ as well.

Remark 3.25
As with the refinement and wavelet symbols, the vanishing moment recovery function $\tilde{s}$ is often chosen as a trigonometric polynomial, i.e.

$$\tilde{s}(\omega) = \sum_{k \in \mathbb{Z}^d} \tilde{s}_k e^{-2\pi i \langle k, \omega \rangle}$$

with finitely many non-zero coefficients $\tilde{s}_k \neq 0$. Since the dilation matrix M satisfies $M\mathbb{Z}^d \subset \mathbb{Z}^d$,

$$\tilde{s}(M^T\omega) = \sum_{k \in \mathbb{Z}^d} \tilde{s}_k e^{-2\pi i \langle k, M^T\omega \rangle} = \sum_{k \in \mathbb{Z}^d} \tilde{s}_k e^{-2\pi i \langle Mk, \omega \rangle} \tag{3.16}$$

is also a trigonometric polynomial.

We conclude this section by rewriting the identities (3.14) into another form that we will use for our further considerations. The first step is considering the symbols and the vanishing-moment recovery function as Laurent polynomials on the torus $\mathbb{T}^d$. To do this, we set $z = (z_1, \ldots, z_d) \in \mathbb{T}^d$ with $z_j := e^{-2\pi i \omega_j}$, $1 \leq j \leq d$, and

$$p(z) := \tilde{p}(\omega) \tag{3.17}$$

as well as

$$q_j(z) := \tilde{q}_j(\omega), \tag{3.18}$$

$1 \leq j \leq r$. In particular, this gives us

$$\hat{\phi}(M^T\omega) = p(z)\hat{\phi}(\omega) \quad \text{and} \quad \hat{\psi}_j(M^T\omega) = q_j(z)\hat{\phi}(\omega). \tag{3.19}$$

Moreover, set

$$p^\sigma(z) := p^\sigma(z_1, \ldots, z_d) = p(e^{-2\pi i \sigma_1} z_1, \ldots, e^{-2\pi i \sigma_d} z_d) = \tilde{p}(\omega + \sigma)$$

and analogously

$$q_j^\sigma(z) := q_j^\sigma(z_1, \ldots, z_d) = q_j(e^{-2\pi i \sigma_1} z_1, \ldots, e^{-2\pi i \sigma_d} z_d) = \tilde{q}_j(\omega + \sigma),$$

$1 \leq j \leq r$. Finally, let $d_M : \mathbb{T}^d \to \mathbb{T}^d$ be the coordinate transform that maps $z = (e^{-2\pi i \omega_1}, \ldots, e^{-2\pi i \omega_d}) \in \mathbb{T}^d$ to

$$d_M(z) := (e^{-2\pi i m_1 \omega}, \ldots, e^{-2\pi i m_d \omega}),$$

where m_j is the j-th row of M^T (and ω is written as a column vector). With this transformation,

$$\tilde{s}(M^T\omega) = (s \circ d_M)(z).$$

These notations now allow us to formulate the Oblique Extension Principle in terms of Laurent polynomials on the torus.

Corollary 3.26
Suppose that $\phi \in L^2(\mathbb{R}^d)$ satisfies (3.10) as well as the refinement equation in (3.19) for the refinement symbol p. Let $q_1, \ldots, q_r$ and s be Laurent polynomials on the torus $\mathbb{T}^d$ such that s is essentially bounded, $\lim_{z \to 1} s(z) = 1$ and $s(z) > 0$ for all $z \in \mathbb{T}^d$. Furthermore, let $\Psi = \{\psi_1, \ldots, \psi_r\}$ be constructed from $q_1, \ldots, q_r$ and ϕ by equation (3.19).

Then $X(\Psi)$ is a tight wavelet frame of $L^2(\mathbb{R}^d)$, if

$$p(z)\overline{p^\sigma(z)}(s \circ d_M)(z) + \sum_{j=1}^{r} q_j(z)\overline{q_j^\sigma(z)} = \delta(\sigma)s(z) \tag{3.20}$$

holds for all $\sigma \in \Gamma_M$ and $z \in \mathbb{T}^d$.

Remark 3.27

a) The function $s \circ d_M$ is a Laurent polynomial, see (3.16).

b) We still call p refinement symbol, $q_1, \dots, q_r$ wavelet symbols, and s vanishing-moment recovery function.

Example 3.28
Using this notation, the refinement symbol for the translated B-spline N_2 is

$$p(z) := \frac{(1+z)^2}{4}.$$

The vanishing-moment function known from Example 3.24 is

$$s(z) := \frac{1}{3}\left(4 - \frac{z}{2} - \frac{1}{2z}\right),$$

and the two wavelet symbols are

$$q_1(z) := \frac{(1-z)^2}{4} \qquad \text{and} \qquad q_2(z) := \frac{\sqrt{6}}{24}(1-z)^2(1+4z+z^2).$$

The second step in rewriting the identities (3.14) respectively (3.20) of the Oblique Extension Principle is to bring them into a matrix form. To do this, we first set $\Gamma_M = \{\sigma_1, \dots, \sigma_m\}$ and the matrix-valued functions $\mathcal{P}$, $\mathcal{S}$ and $\mathcal{Q}$ as

$$\mathcal{P}(z) := (p^{\sigma_1}(z), \dots, p^{\sigma_m}(z)), \qquad \mathcal{S}(z) := \operatorname{diag}(s^{\sigma_1}(z), \dots, s^{\sigma_m}(z))$$
$$\text{and} \qquad \mathcal{Q}(z) := \begin{pmatrix} q_1^{\sigma_1}(z) & \cdots & q_1^{\sigma_m}(z) \\ \vdots & \ddots & \vdots \\ q_r^{\sigma_1}(z) & \cdots & q_r^{\sigma_m}(z) \end{pmatrix},$$

where s^σ is defined in the same way as p^σ or q_j^σ. The identities (3.20) are now equivalent to

$$(s \circ d_M)(z)\mathcal{P}(z)^*\mathcal{P}(z) + \mathcal{Q}(z)^*\mathcal{Q}(z) = \mathcal{S}(z) \tag{3.21}$$

for all $z \in \mathbb{T}^d$.

In the next section, we will examine results on the factorization of trigonometric polynomials, allowing us to compress this form of the Oblique Extension Principle further.

4. Factorization of Trigonometric Polynomials

In order to further compress the matrix form (3.21) of the Oblique Extension Principle into its final form - which we will then use to explore the connection to linear system theory - it is necessary to find a decomposition of the vanishing moment recovery function s. Since we assume s to be a trigonometric polynomial or, alternatively, an algebraic Laurent polynomial on the torus, we will take a look at results concerning the factorization of such polynomials and, more generally, their representation as a sum of squares.

4.1. Univariate Polynomials

One of the most well-known results about trigonometric polynomials is the Fejér-Riesz theorem, named after Lipót Fejér (1916, [37]) and Friedrich Riesz (1916, [89], Lipót Fejér states that the original proof of the general case is due to Riesz).

Lemma 4.1 (Fejér-Riesz)
Let $p \in \mathbb{C}[z^{\pm 1}]$ *with* $\deg(p) = \nu$ *such that* $p(z) \geq 0$ *holds for all* $z \in \mathbb{T}$.

Then there exists $\theta \in \mathbb{C}[z]$ *with* $\deg(\theta) \leq \nu$ *and*

$$p(z) = |\theta(z)|^2$$

for all $z \in \mathbb{T}$.

We can prove this result relatively easily by using the fundamental theorem of algebra and observing that the roots of p always occur in pairs: If z_j is a root of p, then $(\overline{z}_j)^{-1}$ is a root as well. One of these roots is always inside the disc $\mathbb{D}$, the other outside. Now, selecting the roots outside of $\mathbb{D}$ to assemble θ from the corresponding linear factors (scaling p by a multiplicative constant may be necessary at the end) gives us part a) of the following remarks.

Remark 4.2

a) The polynomial θ can be chosen to have no roots in the disc $\mathbb{D}$. Such polynomials are also called *outer*.

b) In general, θ is not unique. However, if we choose θ to have no roots in $\mathbb{D}$, θ becomes unique up to a multiplicative constant of modulus one.

c) If p has only real coefficients, i.e., $p \in \mathbb{R}[z^{\pm 1}]$, we can choose θ with only real coefficients as well.

The Fejér-Riesz theorem and its generalizations have many applications, not only in wavelet theory but also, for example, in filter design in signal processing, control theory, electrical engineering, and quantum mechanics.

For this thesis, we need to examine two possible paths for generalizations: The Fejér-Riesz theorem for matrix-valued functions and the multivariate case.

4.2. Matrix Polynomials

The first generalization of the Fejér-Riesz theorem that we will need later is the matrix-valued Fejér-Riesz theorem, first proven by Marvin Rosenblum (1968, [94]).

Theorem 4.3
Let $P \in \mathbb{C}[z^{\pm 1}]^{n\times n}$ *be a matrix Laurent polynomial with* $\deg(P) = \nu$ *and* $P(z) \geq 0$ *for all* $z \in \mathbb{T}$.

Then, there exists $m \leq n$ *and a matrix polynomial* $A \in \mathbb{C}[z]^{m\times n}$ *with* $\deg(A) \leq \nu$ *and*

$$P(z) = A(z)^* A(z) \tag{4.1}$$

for all $z \in \mathbb{T}$.

Remark 4.4

a) It is possible to write the matrix A as

$$A = (A_0, 0_{m\times(m-n)})V,$$

where $A_0 \in \mathbb{C}[z]^{m\times m}$ and $V \in \mathbb{C}[z]^{n\times n}$ are two square matrix-valued polynomials with $\det A_0(z) \neq 0$ for all $z \in \mathbb{D}$ and V being invertible with $V^{-1} \in \mathbb{C}[z]^{n\times n}$ (see [67], Theorem 4.1)

b) The matrix A in Theorem 4.3 has a rational right inverse B which is analytic in $\mathbb{D}$.

c) If P has only real coefficients, i.e., $P \in \mathbb{R}[z^{\pm 1}]^{n\times n}$, then the matrix A also has only real coefficients, so (4.1) can be written as

$$P(z) = A(1/z)^T A(z)$$

for $z \in \mathbb{T}$ (see [56]).

The matrix-valued Fejér-Riesz theorem can also be seen as a finite-dimensional special case of the operator-valued Fejér-Riesz theorem (in its general form first proven by Marvin Rosenblum, 1968, [94], a special case was proven four years earlier by Israel Gohberg, [45]).

4.3. Multivariate Polynomials

Now, let us return to the scalar case but look at multivariate trigonometric polynomials. Fejér-Riesz type factorizations with a single polynomial do not always exist in the multivariate case. In some cases, it is still possible to write a function as the sum of the squares of several functions.

Definition 4.5
A Laurent polynomial $p \in \mathbb{C}[z_1^{\pm 1}, \dots z_d^{\pm 1}]$ has a sum of (hermitian) squares representation (sos representation) on a set $G \subset \mathbb{C}^d$ if there exist polynomials $\theta_1, \dots, \theta_N \in \mathbb{C}[z_1, \dots, z_d]$, such that

$$p(z) = \sum_{n=1}^{N} |\theta_n(z)|^2$$

holds for all $z \in G$. When this identity holds not for polynomials but for rational functions $\theta_1, \dots, \theta_N \in \mathbb{C}(z_1, \dots, z_d)$, it is called a sum of rational (hermitian) squares representation (sors representation).

A result by David Hilbert (1888, [59]) states that not every non-negative real polynomial on $\mathbb{R}^d$ has a sum of squares representation. Using this result, one can prove (see [14] and [96]) that the same holds in our case: Not every non-negative Laurent polynomial on the torus $\mathbb{T}^d$ has a sum of squares representation.

However, there are some special cases. In the bivariate case, i.e., for $p \in \mathbb{C}[z_1^{\pm 1}, z_2^{\pm 1}]$ with $p(z) \geq 0$ for all $z \in \mathbb{T}^2$, there is always a sum of squares representation on $\mathbb{T}^2$ (Scheiderer, 2006, [99]). For an arbitrary dimension $d > 1$, suppose we assume positivity instead of non-negativity, i.e., $p(z) > 0$ for all $z \in \mathbb{T}^d$. In that case, a sum of squares representation always exists for any Laurent polynomial in any number of variables (Dritschel, 2004, [33]). However, neither result contains any bound on the degrees of $\theta_1, \dots, \theta_N$.

The question of the existence of a sum of rational squares representation for non-negative real polynomials on $\mathbb{R}^d$ is Hilbert's seventeenth problem, solved in the affirmative by Hilbert himself for $d = 2$ (1893, [60]), and Emil Artin (1927, [4]) for the general case, a later result by Pfister (1967, [86]) gives the upper bound 2^d on the number of rational functions required.

One can use these results to prove that every non-negative Laurent polynomial on the torus $\mathbb{T}^d$ has a sum of rational squares representation (see [63], Section 2). In this thesis, however, we will not use sum of rational squares representations.

An important special case for us (see Section 8) are Laurent polynomials on $\mathbb{T}^2$, which can be written as the square of a *single* polynomial $\theta \in \mathbb{C}[z_1, z_2]$, analogous to the univariate Fejér-Riesz theorem. A result by Geronimo and Woerdemann (see [43], Theorem 1.1.3) gives us an equivalent condition for the existence of such a representation in the bivariate case. It is also crucial for us that this is possible with $\theta(z) \neq 0$ for all $z \in \overline{\mathbb{D}^2}$.

Theorem 4.6
Let $p \in \mathbb{C}[z_1^{\pm 1}, z_2^{\pm 1}]$ be a bivariate Laurent polynomial with

$$p(z_1, z_2) = \sum_{j=-n}^{n} \sum_{k=-m}^{m} p_{j,k} z_1^j z_2^k$$

and $p(z) > 0$ for $z = (z_1, z_2) \in \mathbb{T}^2$. We take the segment $c_{j,k}$, $-n \leq j \leq n$, $-m \leq k \leq m$, of the Fourier coefficients $c_{j,k}$, $j, k \in \mathbb{Z}$, of the reciprocal $1/p$ of p and define the block Toeplitz matrix Γ as

$$\Gamma := \begin{pmatrix} C_0 & \cdots & C_{-n} \\ \vdots & \ddots & \vdots \\ C_n & \cdots & C_0 \end{pmatrix}, \quad \textit{with} \quad C_\ell := \begin{pmatrix} c_{\ell,0} & \cdots & c_{\ell,-m} \\ \vdots & \ddots & \vdots \\ c_{\ell,m} & \cdots & c_{\ell,0} \end{pmatrix}.$$

Then, $\theta \in \mathbb{C}[z_1, z_2]$ with

$$p(z_1, z_2) = |\theta(z_1, z_2)|^2$$

and $\theta(z) \neq 0$ for $z = (z_1, z_2) \in \overline{\mathbb{D}^2}$ exists if and only if the $(n+1)m \times (m+1)n$ submatrix of Γ obtained by deleting the (scalar) rows $1 + \ell(m+1)$, $\ell = 0, \ldots, n$, and the (scalar) columns $1, 2, \ldots, m+1$ has rank nm.

We will demonstrate this result using two examples - one positive and one negative - which will also appear in later sections (where the question of the existence or non-existence of a representation using a single square will be important).

Example 4.7

a) Let us look at the Laurent polynomial

$$p(z_1, z_2) = \frac{z_1 z_2}{36} + \frac{z_1}{36 z_2} - \frac{2z_1}{9} - \frac{2z_2}{9} - \frac{2}{9z_2} + \frac{z_2}{36 z_1} + \frac{1}{36 z_1 z_2} - \frac{2}{9z_1} + \frac{16}{9}.$$

It is easy to see that $p(z_1, z_2) > 0$ holds for all $(z_1, z_2) \in \mathbb{T}^2$, since we can factorize p into

$$p(z_1, z_2) = \left(\frac{4}{3} - \frac{z_1}{6} - \frac{1}{6z_1}\right)\left(\frac{4}{3} - \frac{z_2}{6} - \frac{1}{6z_2}\right)$$

and both factors are obviously positive for $|z_1| = |z_2| = 1$. This form of p already tells us that p can be written as the square of the modulus of a single function $\theta \in \mathbb{C}[z_1^{\pm 1}, z_2^{\pm 1}]$ since we can use the classical Fejér-Riesz theorem for both factors and multiply the results.

In this case, $n = m = 1$, so the block Toeplitz matrix from Theorem 4.6 is given by

$$\Gamma := \begin{pmatrix} C_0 & C_{-1} \\ C_1 & C_0 \end{pmatrix} = \begin{pmatrix} c_{0,0} & c_{0,-1} & c_{-1,0} & c_{-1,-1} \\ c_{0,1} & c_{0,0} & c_{-1,1} & c_{-1,0} \\ c_{1,0} & c_{1,-1} & c_{0,0} & c_{0,-1} \\ c_{1,1} & c_{1,0} & c_{0,1} & c_{0,0} \end{pmatrix},$$

and the (2×2)-submatrix obtained by deleting the first and third rows and the first and second columns is

$$\begin{pmatrix} c_{-1,1} & c_{-1,0} \\ c_{0,1} & c_{0,0} \end{pmatrix}. \tag{4.2}$$

Now, e.g., using the residue theorem (for the first two integrands, only the pole $z = 4 - \sqrt{15}$ is in $\mathbb{D}$, for the third also $z = 0$), we get

$$\begin{aligned} \tilde{c}_{-1} &:= \int_0^1 \frac{e^{2\pi i t}}{\frac{4}{3} - \frac{e^{2\pi i t}}{6} - \frac{e^{-2\pi i t}}{6}}\, dt = \frac{1}{2\pi i}\oint_{|z|=1} \frac{6z}{8z - z^2 - 1}\, dz = 4\sqrt{\frac{3}{5}} - 3, \\ \tilde{c}_0 &:= \int_0^1 \frac{1}{\frac{4}{3} - \frac{e^{2\pi i t}}{6} - \frac{e^{-2\pi i t}}{6}}\, dt = \frac{1}{2\pi i}\oint_{|z|=1} \frac{6}{8z - z^2 - 1}\, dz = \sqrt{\frac{3}{5}}, \\ \text{and}\quad \tilde{c}_1 &:= \int_0^1 \frac{e^{-2\pi i t}}{\frac{4}{3} - \frac{e^{2\pi i t}}{6} - \frac{e^{-2\pi i t}}{6}}\, dt = \frac{1}{2\pi i}\oint_{|z|=1} \frac{6}{z(8z - z^2 - 1)}\, dz = 4\sqrt{\frac{3}{5}} - 3. \end{aligned}$$

with $z = e^{2\pi i t}$. Thus, utilizing the product form of p,

$$\begin{aligned} c_{0,0} &= \tilde{c}_0 \cdot \tilde{c}_0 = \frac{3}{5}, \qquad & c_{-1,0} &= \tilde{c}_{-1} \cdot \tilde{c}_0 = \frac{3}{5}\left(4 - \sqrt{15}\right) \\ c_{0,1} &= \tilde{c}_0 \cdot \tilde{c}_1 = \frac{3}{5}\left(4 - \sqrt{15}\right) \qquad \text{and} & c_{-1,1} &= \tilde{c}_{-1} \cdot \tilde{c}_1 = \frac{3}{5}\left(4 - \sqrt{15}\right)^2. \end{aligned}$$

If we insert these values into (4.2), it is easy to see that the resulting matrix has rank 1 since the rows are linearly dependent.

Thus, according to Theorem 4.6, there exists a polynomial $\theta \in \mathbb{C}[z_1, z_2]$ with $p(z_1, z_2) = |\theta(z_1, z_2)|^2$ and $\theta(z) \neq 0$ for all $z = (z_1, z_2) \in \overline{\mathbb{D}^2}$.

b) Let us look at another example motivated by the vanishing moment recovery function for a frame based on the piecewise linear box spline. Let

$$p(z_1, z_2) := \frac{3}{2} - \frac{1}{12}\left(z_1 + z_2 + z_1 z_2 + \frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_1 z_2}\right).$$

For $|z_1| = |z_2| = 1$ we have

$$\begin{aligned} p(z_1, z_2) &= \frac{3}{2} - \frac{1}{12}\left(z_1 + \overline{z_1} + z_2 + \overline{z_2} + z_1 z_2 + \overline{z_1 z_2}\right) \\ &= \frac{3}{2} - \frac{1}{12}\left(2\operatorname{Re}(z_1) + 2\operatorname{Re}(z_2) + 2\operatorname{Re}(z_1 z_2)\right) \\ &\geq \frac{3}{2} - \frac{1}{2} > 0, \end{aligned}$$

as $\operatorname{Re}(z) \leq 1$ for $|z| = 1$. Thus, according to the result by Scheiderer (or the result by Dritschel) mentioned above, there exists a sum of squares representation of p. Indeed,

$$p(z_1, z_2) = \sum_{n=1}^{4} |\theta_n(z_1, z_2)|^2$$

for the four polynomials

$$\theta_1(z_1,z_2) := 1, \quad \theta_2(z_1,z_2) := \frac{z_1-1}{2\sqrt{3}}, \quad \theta_3(z_1,z_2) := \frac{z_2-1}{2\sqrt{3}}$$
$$\text{and} \quad \theta_4(z_1,z_2) := \frac{z_1z_2-1}{2\sqrt{3}}.$$

For this example, we calculated the Fourier coefficients of the reciprocal numerically. The block Toeplitz matrix from Theorem 4.6 is given by

$$\Gamma = \begin{pmatrix} 0.6811 & 0.0434 & 0.0434 & 0.0434 \\ 0.0434 & 0.6811 & 0.0052 & 0.0434 \\ 0.0434 & 0.0052 & 0.6811 & 0.0434 \\ 0.0434 & 0.0434 & 0.0434 & 0.6811 \end{pmatrix},$$

and the (2×2)-submatrix obtained by deleting the first and third rows, and the first and second columns is

$$\begin{pmatrix} 0.0052 & 0.0434 \\ 0.0434 & 0.6811 \end{pmatrix},$$

which has rank 2 (its singular values are 0.6839 and 0.0024).

Thus, there is no $\theta \in \mathbb{C}[z_1,z_2]$ with $p(z_1,z_2) = |\theta(z_1,z_2)|^2$ and $\theta(z) \neq 0$ for all $z = (z_1,z_2) \in \overline{\mathbb{D}^2}$.

It is also possible to combine the paths of generalization from the last subsection (non-scalar polynomials) with the multivariate case: A recent result by Dritschel ([34]) states that every non-negative bivariate finite-dimensional operator-valued trigonometric polynomial P can be written as a sum of hermitian squares. The result gives bounds on both the number of polynomials needed and their degrees.

We can use this result to write any positive matrix-valued bivariate trigonometric polynomial P as the square of one (potentially quite large) matrix: If $P(z_1,z_2) \geq 0$ for all $(z_1,z_2) \in \mathbb{T}^2$, then

$$P(z_1,z_2) = \sum_{k=1}^{N} A_k(z_1,z_2)^* A_k(z_1,z_2) = A(z_1,z_2)^* A(z_1,z_2)$$

for all $(z_1,z_2) \in \mathbb{T}^2$ and

$$A(z_1,z_2) := \begin{pmatrix} A_1(z_1,z_2) \\ \vdots \\ A_N(z_1,z_2) \end{pmatrix}.$$

4.4. Application to the Oblique Extension Principle Identities

We can now use the results of this chapter to rewrite the Oblique Extension Principle identities (3.20) into a different matrix form. First, assume that the vanishing moment recovery function s is a Laurent polynomial with a sum-of-squares representation

$$s = \sum_{n=1}^{N} |\theta_n|^2$$

on $\mathbb{T}^d$. Then (3.21) is equivalent to

$$\sum_{n=1}^{N} ((\theta_n \circ d_M)\mathcal{P})(z)^* ((\theta_n \circ d_M)\mathcal{P})(z) + \mathcal{Q}(z)^* \mathcal{Q}(z) = S(z)^* S(z) \tag{4.3}$$

for all $z \in \mathbb{T}^d$ with

$$S(z) = \begin{pmatrix} \theta_1^{\sigma_1}(z) & \dots & \theta_N^{\sigma_1}(z) & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\ 0 & \dots & 0 & \theta_1^{\sigma_2}(z) & \dots & \theta_N^{\sigma_2}(z) & \dots & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & \dots & 0 & \dots & \theta_1^{\sigma_m}(z) & \dots & \theta_N^{\sigma_m}(z) \end{pmatrix}^T .$$

In particular, (3.21) and (4.3) are equivalent to

$$S(z)^* S(z) - R(z)^* R(z) = 0 \tag{4.4}$$

for all $z \in \mathbb{T}^d$ with

$$R(z) = \begin{bmatrix} \mathcal{Q}(z) \\ ((\theta_1 \circ d_M)\mathcal{P})(z) \\ \vdots \\ ((\theta_N \circ d_M)\mathcal{P})(z) \end{bmatrix} . \tag{4.5}$$

This equation is our final form of the Oblique Extension Principle identities and the form we will use for the remainder of this thesis.

Remark 4.8
In the case of the Unitary Extension Principle, i.e., $s \equiv 1$, (4.4) has the form

$$I - R(z)^* R(z) = 0 \tag{4.6}$$

with the identity matrix $I \in \mathbb{C}^{m \times m}$. The matrix R then becomes

$$R(z) = \begin{bmatrix} \mathcal{Q}(z) \\ \mathcal{P}(z) \end{bmatrix} .$$

5. Linear System Theory

We now turn our attention to the field to which we want to connect the construction of tight wavelet frames: Linear system theory. Linear system theory, particularly its subfield of linear control theory, is a topic at the intersection of functional analysis and engineering (there are also nonlinear versions of both, which are not of interest here).

5.1. A Brief Introduction to Linear System Theory

As mentioned in the introduction of this thesis, the engineering point of view in this field lies in the modeling of real-world dynamical systems. Of particular relevance to us is the state-space approach pioneered by Rudolf E. Kálmán in the 1960s ([66]). This approach is characterized by the systematic study of systems of coupled ordinary differential equations that model both the processes within a real-world system (the *state* of the system) and its interactions with the external environment (the inputs and outputs of the system).

In its linear, continuous, time-invariant form, a state-space model is given by the system

$$\begin{aligned}\dot{x}(t) &= Dx(t) + Cu(t); \quad x(0) = 0\\ y(t) &= Bx(t) + Au(t)\end{aligned}$$

of differential equations, where x, u, and y are continuous functions on $[0, \infty)$ representing a description of the state of the system, an input signal, and the output signal, respectively. The constant, complex matrices A (also called the feedthrough matrix), B (the output matrix), C (the input matrix), and D (the state matrix, which has to be square) describe the behavior of the system. Note that the designations A and D, respectively B and C, often appear the other way around. In the time-variant form of the system, the matrices also depend on t.

By applying the Laplace transform and utilizing the initial condition $x(0) = 0$, we obtain the system

$$\begin{aligned}\lambda X(\lambda) &= DX(\lambda) + CU(\lambda)\\ Y(\lambda) &= BX(\lambda) + AU(\lambda)\end{aligned}$$

(see [61], Example 2.3.18). Now, solving the first equation for X and substituting the result into the second equation gives us

$$Y(\lambda) = \big[A + B(\lambda I - D)^{-1}C\big]U(\lambda).$$

Thus, when represented by their respective Laplace transforms, the corresponding output to an input signal $U(\lambda)$ can be obtained by multiplication with the matrix-valued function

$$W(\lambda) := A + B(\lambda I - D)^{-1}C,$$

which is also called the transfer function of the system. Obtaining a transfer function becomes much more complicated when considering the time-variant case. However, the concept of a transfer function is not exclusive to the state-space approach; it was already in use before its emergence.

While time-continuous systems are widely used in contexts such as mechanical or electrical systems, there are instances where time-discrete systems may be more appropriate, including in economic models or digital control systems. The discrete, time-invariant form of a state-space model is given by

$$\begin{aligned} x(k+1) &= Dx(k) + Cu(k); \quad x(0) = 0 \\ y(k) &= Bx(k) + Au(k) \end{aligned}$$

with $k \in \mathbb{N}_0$. Its transfer function is identical to the transfer function of the continuous system above, just expressed in terms of (and calculated using) the Z-transform of the involved functions instead of the Laplace transform (see [61], Example 2.3.21). This fact makes it possible to transfer some results from one type of system to another. In this chapter, we will examine this type of system in a more general form, with multiple variables and - at least initially - Hilbert space operators A, B, C, and D. A good and more detailed overview of this topic can be found in [7].

Let $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{H}$ be three Hilbert spaces and $A \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$, $B \in \mathcal{B}(\mathcal{H}, \mathcal{Y})$, $C \in \mathcal{B}(\mathcal{X}, \mathcal{H})$, and $D \in \mathcal{B}(\mathcal{H}, \mathcal{H})$ be four bounded linear operators. Furthermore, we write $\mathcal{H}$ as a direct sum $\mathcal{H} = \mathcal{H}_1 \oplus \ldots \oplus \mathcal{H}_d$ of auxiliary Hilbert spaces. Then, our generalized linear state-space system of *Givone-Roesser type* ([44]) associated to A, B, C and D is given by

$$\begin{pmatrix} h_1(\alpha + e_1) \\ \vdots \\ h_d(\alpha + e_d) \end{pmatrix} = D \begin{pmatrix} h_1(\alpha) \\ \vdots \\ h_d(\alpha) \end{pmatrix} + Cu(\alpha), \tag{5.1}$$
$$y(\alpha) = Bh(\alpha) + Au(\alpha),$$

with some suitable initial conditions. Here, α is in $\mathbb{N}_0^d$, $\{e_k : 1 \leq k \leq d\}$ are the unit vectors in $\mathbb{C}^d$ and $u : \mathbb{N}_0^d \to \mathcal{X}$, $y : \mathbb{N}_0^d \to \mathcal{Y}$, and $h : \mathbb{N}_0^d \to \mathcal{H}$ are three functions, the latter partitioned according to the partitioning of $\mathcal{H}$, i.e.,

$$h(\alpha) = (h_1(\alpha), \ldots, h_d(\alpha))^T$$

with $h_j : \mathbb{N}_0^d \to \mathcal{H}_j$, $1 \leq j \leq d$. Analogous to the names above, $h(\alpha) \in \mathcal{X}$ is also called *state vector*, $u(\alpha) \in \mathcal{H}$ *input vector* and $y(\alpha) \in \mathcal{Y}$ *output vector*.

Similarly, the Hilbert space $\mathcal{H}$ is often called *state space*, $\mathcal{X}$ *input space*, $\mathcal{Y}$ *output space*, and the operators A, B, C, and D are called *feedthrough operator*, *output operator*, *input operator*, and *state operator*, respectively, just like their matrix counterparts above.

If we now define the operator-valued function $E : \mathbb{C}^d \to \mathcal{B}(\mathcal{H})$ by

$$E(z) := \begin{pmatrix} z_1 \operatorname{id}_{\mathcal{H}_1} & & \\ & \ddots & \\ & & z_d \operatorname{id}_{\mathcal{H}_d} \end{pmatrix}$$

for $z \in \mathbb{C}^d$, the associated *transfer function* W of the Givone-Roesser type system (5.1) is given by

$$W(z) := A + BE(z)(I - DE(z))^{-1}C, \tag{5.2}$$

defined for $z \in \mathbb{C}^d$ with sufficiently small value of $|z|$ and extended analytically as far as possible. The quadruple $\{A, B, C, D\}$ of operators is called a *realization* for W. If the state space $\mathcal{H}$ is finite-dimensional, D is a finite matrix, meaning that W is a (matrix-valued) rational function.

Since the properties of the system are encapsulated in the four operators, it is common to consider them in isolation, i.e., to look at the system

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} : \mathcal{X} \oplus \mathcal{H} \to \mathcal{Y} \oplus \mathcal{H}$$

of operators. This block matrix is also called *system matrix.*

As mentioned above, linear system theory has been studied from both an engineering and a functional analysis perspective. In the early 1980s, George Zames (see [109],[40], the latter in collaboration with Francis and Helton) established a link between control theory, in particular the then emerging field of H^∞-control, and the much older Nevanlinna-Pick interpolation problem ([84],[87]), which we will discuss in more detail in Section 6.2 and 8.2. The Nevanlinna-Pick interpolation problem, in turn, has numerous connections to other topics in complex analysis, in particular to operator theory and the theory of reproducing kernel Hilbert spaces. Since this connection has been established, these two fields have mutually benefited from each other. For more details, we again refer to Ball and ter Horst ([7]).

We now cite a first result that links linear system theory and the study of contractive analytic functions on $\mathbb{D}^d$ (see [2] and [8]).

Theorem 5.1
Let $\mathcal{X}$ and $\mathcal{Y}$ be two Hilbert spaces and $f : \mathbb{D}^d \to \mathcal{B}(\mathcal{X}, \mathcal{Y})$ be an analytic function. Then the following are equivalent:

(i) For each commutative tuple $T = (T_1, \dots T_d)$ of linear contractive operators on a Hilbert space $\mathcal{K}$,

$$\sup_{0<\varepsilon_k<1} \|f(\varepsilon_1 T_1, \dots, \varepsilon_d T_d)\| \leq 1$$

holds.

(ii) There are Hilbert spaces $\mathcal{H}_1, \dots, \mathcal{H}_d$ and functions $L_j : \mathbb{D}^d \to \mathcal{B}(\mathcal{X}, \mathcal{H}_j)$, $1 \leq j \leq d$, such that

$$I - f(w)^* f(z) = \sum_{j=1}^{d} (1 - \overline{w_j} z_j) L_j(w)^* L_j(z) \tag{5.3}$$

holds for all $z, w \in \mathbb{D}^d$.

(iii) There exists a Hilbert space $\mathcal{H} = \mathcal{H}_1 \oplus \dots \oplus \mathcal{H}_d$ and a unitary operator

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} : \mathcal{X} \oplus \mathcal{H} \to \mathcal{Y} \oplus \mathcal{H}$$

such that the identity

$$f(z) = A + BE(z)(I - DE(z))^{-1} C \tag{5.4}$$

holds for all $z \in \mathbb{D}^d$.

We will immediately add some remarks to this theorem.

Remark 5.2

a) The inequality in *(i)* is a variant of the *von Neumann inequality* ([83]), its general form being: If g is an analytic function defined in a neighborhood of the closed unit disc, and T is a contraction acting on a Hilbert space $\mathcal{K}$, then

$$\|f(T)\| \leq \|f\|_{\infty, \mathbb{D}}.$$

This still holds true in $\mathbb{D}^2$ (also called Andô's inequality, see [2], Theorem 10.27), but not in $\mathbb{D}^d$ for $d \geq 3$ (see [2], Section 10.7).

b) Let $\Omega \subset \mathbb{C}^d$ be an open set. The *Schur class* $\mathcal{S}(\Omega)$ is the unit ball of $H^\infty(\Omega)$, i.e., it consists of all holomorphic functions $\phi : \Omega \to \mathbb{C}$ with

$$\sup_{z\in\Omega} |\phi(z)| \leq 1.$$

The Schur class $\mathcal{S}(\mathbb{D}^d)$ is particularly interesting because it appears in control theory and is essential in the Nevanlinna-Pick interpolation mentioned above. Every function that satisfies the von Neumann inequality in *(i)* is in $\mathcal{S}(\mathbb{D}^d)$. The converse is true only for $d \leq 2$.

c) A constructive approach to finding a realization $\{A, B, C, D\}$ of a given function f can be found in Kummert (1989, [70], see also [67]).

d) The function g with $g(z) = (I - DE(z))^{-1}C$ satisfies

$$g(z) = C + DE(z)g(z),$$

and since the identity (5.4) in *(iii)* obviously implies $f(z) = A + BE(z)g(z)$, we have

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} I \\ E(z)g(z) \end{pmatrix} = \begin{pmatrix} f(z) \\ g(z) \end{pmatrix}. \tag{5.5}$$

e) The condition on the operator in *(iii)* is sometimes weakened, requiring it only to be contractive, not unitary. In this case, the identity (5.5) yields

$$\|f(z)\|^2 + \|g(z)\|^2 \leq 1 + \|E(z)g(z)\|^2 \leq 1 + \|g(z)\|^2$$

for $z \in \mathbb{D}^d$. The second inequality follows directly from the definition of E. In particular we get $\|f(z)\| \leq 1$ for all $z \in \mathbb{D}^d$.

5.2. Foundations of the Connection Between Linear System Theory and Tight Wavelet Frames

We now want to connect linear system theory to tight wavelet frames, in particular to the Unitary and the Oblique Extension Principle introduced in Section 3.3. The idea behind this connection is as follows: If we extend the identity (5.3) from $\mathbb{D}^d$ to the torus $\mathbb{T}^d$ and set $z = w \in \mathbb{T}^d$, the right side of the equation vanishes, i.e., we get

$$I - f(z)^* f(z) = 0.$$

We have already encountered an identity on $\mathbb{T}^d$ of this form in the matrix form (4.6) of the Unitary Extension Principle, which is given by $I - R(z)^* R(z) = 0$ with the matrix-valued function R defined in (4.5). For tight wavelet frames constructed using the Unitary Extension Principle in the special case where the coefficients of the refinement symbol p are non-negative, and p satisfies

$$p^\sigma\big((1,\dots,1)^T\big) = \delta_{0,\sigma},$$

Charina, Putinar, Scheiderer, and Stöckler (2015, [18], Theorem 4.4) gave a constructive result to find a realization, i.e., matrices A, B, C, and D such that

$$R(z) = A + BE(z)(I - DE(z))^{-1}C$$

for $z \in \mathbb{D}^d$. The objective of the following chapters will be to find similar results for tight wavelet frames based on the Oblique Extension Principle. The remainder of this section will be dedicated to laying the necessary groundwork.

As a first step, the following theorem serves as a connection between linear system theory and functions on $\mathbb{D}^d$, in a manner similar to Theorem 5.1, but with better suitability for the setting of the Oblique Extension Principle (it can be seen as a generalization of e.g. Theorem 2.1 in [67]; the equivalence between *(i)* and *(iii)* can be found in a similar form in [8], Theorem 5.2).

Theorem 5.3
Let $S : \mathbb{D}^d \to \mathbb{C}^{n_1 \times m}$ and $R : \mathbb{D}^d \to \mathbb{C}^{n_2 \times m}$ be two matrix-valued functions. Then the following are equivalent:

(i) There exist matrix-valued functions $G : \mathbb{D}^d \to \mathbb{C}^{r_0 \times m}$ and $F_j : \mathbb{D}^d \to \mathbb{C}^{r_j \times m}$, $1 \le j \le d$, such that the identity

$$S(w)^* S(z) - R(w)^* R(z) = G(w)^* G(z) + \sum_{j=1}^{d} (1 - \overline{w_j} z_j) F_j(w)^* F_j(z) \tag{5.6}$$

holds for all $z, w \in \mathbb{D}^d$.

(ii) There exist matrix-valued functions $F_j : \mathbb{D}^d \to \mathbb{C}^{r_j \times m}$, $1 \leq j \leq d$, *and a contractive complex matrix* T *such that the identity*

$$T \begin{pmatrix} S(z) \\ z_1 F_1(z) \\ \vdots \\ z_d F_d(z) \end{pmatrix} = \begin{pmatrix} R(z) \\ F_1(z) \\ \vdots \\ F_d(z) \end{pmatrix}$$

holds for all $z \in \mathbb{D}^d$.

(iii) There exists a contractive complex block matrix

$$T := \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

such that the identity

$$R(z) = \big[A + BE(z)(I - DE(z))^{-1}C\big]S(z) \tag{5.7}$$

holds for all $z \in \mathbb{D}^d$, *where*

$$E(z) := \begin{pmatrix} z_1 I_1 & & \\ & \ddots & \\ & & z_d I_d \end{pmatrix}$$

with identity matrices $I_1, \dots I_d$ *corresponding to the sizes of* $F_1, \dots, F_d$, *i.e.,* $I_j \in \mathbb{C}^{r_j \times r_j}$, $1 \leq j \leq d$.

In the proof of this theorem, we employ an argument that is frequently used in operator theory. It was named *lurking isometry argument* by Joseph Ball (Ball himself emphasizes that he only invented the name, and the argument itself is much older, see [101]). At its core is the following lemma (see [3], Lemma 2.18).

Lemma 5.4
Let $\mathcal{H}$ *and* $\mathcal{K}$ *be two Hilbert spaces,* Ω *a set, and* $f : \Omega \to \mathcal{H}$ *and* $g : \Omega \to \mathcal{K}$ *be two functions. Then the following are equivalent:*

(i) The identity

$$\langle f(\lambda), f(\mu)\rangle_{\mathcal{H}} = \langle g(\lambda), g(\mu)\rangle_{\mathcal{K}}$$

holds for all $\lambda, \mu \in \Omega$.

(ii) There is a linear isometry $V : \mathrm{clos}_{\mathcal{H}}(\mathrm{Span}(\mathrm{ran}(f))) \to \mathcal{K}$ *such that*

$$Vf(\lambda) = g(\lambda)$$

holds for all $\lambda \in \Omega$.

Remark 5.5

In most cases, Lemma 5.4 represents only the first step of a lurking isometry argument since it only yields an isometry on the subspace $\mathcal{U} := \operatorname{clos}_{\mathcal{H}}(\operatorname{Span}(\operatorname{ran}(f)))$ of $\mathcal{H}$. Therefore, it is usually followed by a so-called *extension step*, where V is extended to $\mathcal{U}^{\perp}$ in a manner suitable for the given situation to obtain an operator W on the whole space $\mathcal{H}$.

The trivial choice $W = V$ on $\mathcal{U}$ and $W = 0$ on $\mathcal{U}^{\perp}$ is always possible. With this choice, W is a partial isometry (partial isometries are continuous linear operators whose restriction to the orthogonal complement of their kernel is an isometry). Since every partial isometry is a contraction, this means that we can always extend V to a contraction W on the entire space $\mathcal{H}$.

Whether V can be extended to an isometry is a question of dimensions: If W is an extension of V to an isometry, then W isometrically maps $\mathcal{U}^{\perp}$ to $\operatorname{ran}(V)^{\perp}$. Thus, if (and only if)

$$\dim \mathcal{U}^{\perp} \leq \dim \operatorname{ran}(V)^{\perp},$$

V can indeed be extended to an isometry W. If (and only if)

$$\dim \mathcal{U}^{\perp} = \dim \operatorname{ran}(V)^{\perp},$$

V can even be extended to a unitary operator W (see [3], Remark 2.31).

Let us now prove Theorem 5.3. We will prove each implication of the equivalence of *(i)* and *(ii)*, respectively *(ii)* and *(iii)* separately; apart from the lurking isometry argument, most of the steps are relatively straightforward calculations.

Proof of Theorem 5.3.

(i) $\Rightarrow$ *(ii)* Let $F : \mathbb{D}^d \to \mathbb{C}^{r \times m}$ with $r := \sum_{j=1}^{d} r_j$ and $F(z) := (F_1(z), \dots, F_d(z))^T$. The identity in *(i)* implies

$$\begin{pmatrix} S(w) \\ E(w)F(w) \end{pmatrix}^* \begin{pmatrix} S(z) \\ E(z)F(z) \end{pmatrix} = \begin{pmatrix} R(w) \\ F(w) \\ G(w) \end{pmatrix}^* \begin{pmatrix} R(z) \\ F(z) \\ G(z) \end{pmatrix} \tag{5.8}$$

for all $z, w \in \mathbb{D}^d$.

We will now use the lurking isometry argument introduced above. To do this, define the set $\Omega := \mathbb{D}^d \times \{1, \dots, m\}$, the function $f : \Omega \to \mathbb{C}^{n_1 + r}$, which maps a tuple $(z, \ell) \in \Omega$ to the ℓ-th column of the matrix

$$\begin{pmatrix} S(z) \\ E(z)F(z) \end{pmatrix}, \tag{5.9}$$

and the function $g : \Omega \to \mathbb{C}^{n_2 + r + r_0}$, which maps a tuple $(z, \ell) \in \Omega$ to the ℓ-th column of the matrix

$$\begin{pmatrix} R(z) \\ F(z) \\ G(z) \end{pmatrix}. \tag{5.10}$$

Now, (5.8) implies that

$$\langle f((z,\ell)), f((w,k))\rangle_{\mathbb{C}^{n_1+r}} = \langle g((z,\ell)), g((w,k))\rangle_{\mathbb{C}^{n_2+r+r_0}}$$

holds for all $(z,\ell),(w,k) \in \Omega$. Thus, according to Lemma 5.4, there exists an isometry V on $\mathrm{clos}_{\mathbb{C}^{n_1+r}}(\mathrm{Span}(\mathrm{ran}(f)))$ such that

$$Vf((z,\ell)) = g((z,\ell)) \tag{5.11}$$

for all $(z,\ell) \in \Omega$. According to Remark 5.5 we can now extend V to a partial isometry $W : \mathbb{C}^{n_1+r} \to \mathbb{C}^{n_2+r+r_0}$. In particular, this W is a contraction. Since (5.11) implies that W maps each column of the matrix in (5.9) to the corresponding column of the matrix in (5.10), this means that (in a slight misuse of notation) we can identify W with a contractive matrix of the same name, satisfying

$$W \begin{pmatrix} S(z) \\ E(z)F(z) \end{pmatrix} = \begin{pmatrix} R(z) \\ F(z) \\ G(z) \end{pmatrix}$$

for all $z \in \mathbb{D}^d$. A "compression" (omitting the last r_0 rows of W) now gives us a contraction T satisfying

$$T \begin{pmatrix} S(z) \\ E(z)F(z) \end{pmatrix} = \begin{pmatrix} R(z) \\ F(z) \end{pmatrix}$$

which is the identity in *(ii)*.

(ii) $\Rightarrow$ *(i)* The identity in *(ii)* directly gives

$$\begin{pmatrix} S(w) \\ z_1F_1(w) \\ \vdots \\ z_dF_d(w) \end{pmatrix}^* T^*T \begin{pmatrix} S(z) \\ z_1F_1(z) \\ \vdots \\ z_dF_d(z) \end{pmatrix} = \begin{pmatrix} R(w) \\ F_1(w) \\ \vdots \\ F_d(w) \end{pmatrix}^* \begin{pmatrix} R(z) \\ F_1(z) \\ \vdots \\ F_d(z) \end{pmatrix} \tag{5.12}$$

or alternatively

$$\begin{pmatrix} S(w) \\ E(w)F(w) \end{pmatrix}^* T^*T \begin{pmatrix} S(z) \\ E(z)F(z) \end{pmatrix} = \begin{pmatrix} R(w) \\ F(w) \end{pmatrix}^* \begin{pmatrix} R(z) \\ F(z) \end{pmatrix}$$

for all $z, w \in \mathbb{D}^d$ with $E(z)$ defined as in *(iii)* and $F : \mathbb{D}^d \to \mathbb{C}^{r\times m}$ with $r := \sum_{j=1}^d r_j$ and $F(z) := (F_1(z),\ldots,F_d(z))^T$. Because T is a contraction, we have $I - T^*T \geq 0$, so the square root $H := \sqrt{I - T^*T}$ is well defined.

Defining G as

$$G(z) := H \begin{pmatrix} S(z) \\ E(z)F(z) \end{pmatrix}$$

gives us

$$G(w)^*G(z) = \begin{pmatrix} S(w) \\ E(w)F(w) \end{pmatrix}^* \begin{pmatrix} S(z) \\ E(z)F(z) \end{pmatrix} - \begin{pmatrix} S(w) \\ E(w)F(w) \end{pmatrix}^* T^*T \begin{pmatrix} S(z) \\ E(z)F(z) \end{pmatrix}$$

for all $z, w \in \mathbb{D}^d$, so according to (5.12)

$$\begin{pmatrix} S(w) \\ E(w)F(w) \end{pmatrix}^* \begin{pmatrix} S(z) \\ E(z)F(z) \end{pmatrix} = G(w)^*G(z) + \begin{pmatrix} R(w) \\ F(w) \end{pmatrix}^* \begin{pmatrix} R(z) \\ F(z) \end{pmatrix}$$

and thus

$$S(w)^*S(z) - R(w)^*R(z) = G(w)^*G(z) + (I - E(w)^*E(z))F(w)^*F(z)$$

for all $z, w \in \mathbb{D}^d$. This is exactly the identity in *(i)*.

(ii) $\Rightarrow$ *(iii)* Like in the first step of the proof, let $F : \mathbb{D}^d \to \mathbb{C}^{r\times m}$ with $r := \sum_{j=1}^d r_j$ and $F(z) := (F_1(z), \dots, F_d(z))^T$ and define $E(z)$ as in *(iii)*. We can write the identity in *(ii)* as

$$T \begin{pmatrix} S(z) \\ E(z)F(z) \end{pmatrix} = \begin{pmatrix} R(z) \\ F(z) \end{pmatrix}$$

for all $z \in \mathbb{D}^d$. We now divide T into

$$T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

with $A \in \mathbb{C}^{n_2\times n_1}$, $B \in \mathbb{C}^{n_2\times r}$, $C \in \mathbb{C}^{r\times n_1}$ and $D \in \mathbb{C}^{r\times r}$, which allows us to write the identity above as

$$\begin{aligned} AS(z) + BE(z)F(z) &= R(z) \\ CS(z) + DE(z)F(z) &= F(z) \end{aligned} \tag{5.13}$$

for all $z \in \mathbb{D}^d$. Rearranging the second equation yields

$$F(z) = (I - DE(z))^{-1}CS(z) \tag{5.14}$$

for all $z \in \mathbb{D}^d$, so by substituting this into the first equation, we get

$$\begin{aligned} R(z) &= AS(z) + BE(z)(I - DE(z))^{-1}CS(z) \\ &= \left[A + BE(z)(I - DE(z))^{-1}C\right]S(z) \end{aligned}$$

for all $z \in \mathbb{D}^d$.

(iii) $\Rightarrow$ *(ii)* Let

$$F(z) := (I - DE(z))^{-1}CS(z)$$

for $z \in \mathbb{D}^d$ and rearrange it to

$$CS(z) + DE(z)F(z) = F(z).$$

Now, (5.7) and the definition of $F(z)$ above give us

$$\begin{aligned} R(z) &= \left[A + BE(z)(I - DE(z))^{-1}C\right]S(z) \\ &= AS(z) + BE(z)(I - DE(z))^{-1}CS(z) \\ &= AS(z) + BE(z)F(z) \end{aligned}$$

for all $z \in \mathbb{D}^d$. Thus, the identities (5.13) are satisfied, which are equivalent to the identity from *(ii)* by the calculation from the previous steps of the proof.

□

We end this section with a few remarks about this theorem and its proof.

Remark 5.6

a) The calculations in the proof of the equivalence of *(ii)* and *(iii)* imply that the contractive matrix T can be chosen identically in *(ii)* and *(iii)*.

b) Equation (5.7) implies that the matrix function R is rational if S is rational and *(i)*–*(iii)* hold. The same is true for G according to the way it is defined in the step *(ii)* $\Rightarrow$ *(i)*, and for F (respectively $F_1, \ldots F_d$) according to (5.14) and the way it is defined in the step *(iii)* $\Rightarrow$ *(ii)*. Analogously, R, G and F (respectively $F_1, \ldots F_d$) are holomorphic functions if S is holomorphic and *(i)*–*(iii)* hold. The aforementioned result by Ball and Trent ([8]), which contains a statement similar to the equivalence of *(i)* and *(iii)*, deals only with the holomorphic case.

c) The identity in *(i)* yields

$$S(z)^*S(z) \geq R(z)^*R(z)$$

for all $z \in \overline{\mathbb{D}}^d$.

Remark 5.7

Theorem 5.3 remains largely valid when the matrix T in *(ii)* and *(iii)* is required to be isometric instead of contractive.

Looking at the proof of the equivalence of *(ii)* and *(iii)*, it is obvious that these statements are still equivalent for an isometric T without any changes in the proof. If the matrix T in *(ii)* is isometric, then $I - T^*T = 0$ and hence $H = 0$ in the proof of the implication *(ii)* $\Rightarrow$ *(i)*. This directly implies $G(z) = 0$ for all $z \in \mathbb{D}^d$.

The rest of the proof of this step implies that (5.6) still holds, i.e.,

$$S(w)^*S(z) - R(w)^*R(z) = \sum_{j=1}^{d}(1 - \overline{w_j}z_j)F_j(w)^*F_j(z)$$

for all $w, z \in \mathbb{D}^d$.

Conversely, if $G \equiv 0$, we get an isometry T in the implication *(i)* $\Rightarrow$ *(ii)* if V can be extended to an isometry W in the extension step of the lurking isometry argument. In particular, then $T = W$ (and thus T itself is an isometry), since the "compression step" is not needed. According to Remark 5.5, the extension of V to an isometry $W(= T)$ is possible if

$$\dim \operatorname{dom}(V)^{\perp} \leq \dim \operatorname{ran}(V)^{\perp}. \tag{5.15}$$

Since V is an isometry, $\dim \operatorname{dom}(V) = \dim \operatorname{ran}(V)$. Thus, (5.15) holds if $n_2 \geq n_1$, i.e., the extension to an isometry is possible if R has at least as many rows as S. We can add some rows of zeros to R to "enforce" this without changing the value of $S(w)^*S(z) - R(w)^*R(z)$. However, the matrix T in *(ii)* and *(iii)* can then only be chosen isometric for this new version of R with added rows of zeros. If we go back to the original R, we again only have a contractive T.

If T can be chosen isometrically (and accordingly $G \equiv 0$ holds in *(i)*), then it follows directly from *(i)* that

$$R(z)^*R(z) = S(z)^*S(z)$$

for all $z \in \mathbb{T}^d$.

In the following sections, we will focus on the aforementioned goal of finding realizations in the Oblique Extension Principle setting. First, in Section 6, we will focus on the univariate case.

6. Realizations Associated With Frames Based on the Oblique Extension Principle - The Univariate Case

Finally, we direct our attention to the primary objective of this dissertation: To find realizations associated with tight wavelet frames based on the Oblique Extension Principle. By this, we mean: Suppose we have a refinement symbol p, wavelet masks $q_1, \ldots, q_r$, and a vanishing moment recovery function s such that the Oblique Extension Principle is satisfied, i.e., the identity

$$S(z)^* S(z) - R(z)^* R(z) = 0$$

holds for all $z \in \mathbb{T}^d$ with the matrix-valued functions S and R constructed in Section 4.4. Can we then find an isometric or contractive complex block matrix

$$T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

such that the identity

$$R(z) = \big[A + BE(z)(I - DE(z))^{-1} C\big] S(z)$$

holds for all $z \in \mathbb{D}^d$? In this section, we will focus on the univariate case, so we will assume that $p, q_1, \ldots, q_r$ are univariate polynomials (in particular, we will no longer assume that they are Laurent polynomials; see Remark 6.1), s is a univariate Laurent polynomial and R and S are univariate matrix polynomials. This restriction to the univariate case allows for the application of the regular Fejér-Riesz Theorem 4.1.

In Section 6.1, we will employ the Fejér-Riesz Theorem and so-called positive semidefinite kernels to find such matrices. More precisely, we will construct matrix-valued functions F with

$$S(w)^* S(z) - R(w)^* R(z) = (1 - \overline{w} z) F(w)^* F(z),$$

which can be used to construct an isometric or contractive matrix T that satisfies the desired identity with the help of Theorem 5.3.

In Section 6.2, we conclude our discussion of the univariate case by presenting an alternative method for constructing the matrix T based on the Nevanlinna-Pick interpolation already briefly mentioned in Section 5.

6.1. Existence of Realizations Associated With Frames Based on the Univariate Oblique Extension Principle

We begin our considerations with a slight change in the assumptions for our symbols p and $q_1, \dots q_r$, which we have so far assumed to be Laurent polynomials (as we originally defined them as trigonometric polynomials, which we later identified with algebraic Laurent polynomials on the torus).

Remark 6.1
For the remainder of this thesis, we will assume that the refinement symbol p and the wavelet symbol $q_1, \dots q_r$ are polynomials, and thus that S and R are matrix polynomials. For this, the vanishing moment recovery function s does not necessarily have to be a polynomial, since the entries of S originate from its Fejér-Riesz factorization $s = |\theta|^2$ with a polynomial $\theta \in \mathbb{C}[z]$ (or, in the general multivariate case, from the sum-of-squares representation of s, which consists of polynomials $\theta_1, \dots, \theta_N \in \mathbb{C}[z_1, \dots, z_d]$).

This assumption is not a big loss of generality since we can always "enforce" polynomial symbols: Suppose that p and $q_1, \dots q_r$ are univariate Laurent polynomial symbols such that the Oblique Extension Principle holds for the vanishing moment recovery function s. Then, we can multiply the symbols by a suitable monomial z^ν (the same monomial for all symbols) to get polynomials. Plugging the new symbols into (3.20), we get

$$z^\nu p(z)\overline{p^\sigma(z)z^\nu}e^{2\pi i\sigma\nu}(s \circ d_M)(z) + \sum_{j=1}^{r} z^\nu q_j(z)\overline{q_j^\sigma(z)z^\nu}e^{2\pi i\sigma\nu} = \delta(\sigma)s(z),$$

and since $z^\nu\overline{z^\nu} = 1$ for $z \in \mathbb{T}$, this is equivalent to

$$e^{2\pi i\sigma\nu}\left(p(z)\overline{p^\sigma(z)}(s \circ d_M)(z) + \sum_{j=1}^{r} q_j(z)\overline{q_j^\sigma(z)}\right) = \delta(\sigma)s(z).$$

This equation holds for all $z \in \mathbb{T}$, since $p(z)\overline{p^\sigma(z)}(s\circ d_M)(z)+\sum_{j=1}^{r} q_j(z)\overline{q_j^\sigma(z)} = \delta(\sigma)s(z)$ and $e^{2\pi i\sigma\nu} = 1$ for $\sigma = 0$. Therefore, our new polynomial symbols still satisfy the Oblique Extension Principle identity (3.20).

In the multivariate case, we can similarly multiply the symbols with a suitable monomial $z_1^{\nu_1} \dots z_d^{\nu_d}$ to get polynomial symbols.

So, let p and $q_1, \dots q_r$ be univariate polynomials and s be a univariate Laurent polynomial on the torus $\mathbb{T}$ such that the Oblique Extension Principle holds. In particular, the scaling matrix M is now an integer with an absolute value greater than one, and the vanishing moment recovery function can be written as $s = |\theta|^2$ for some polynomial θ on $\mathbb{T}$ using the Fejér-Riesz Theorem 4.1. Furthermore, we define the matrix functions S and R as in Section 4.4.

Since the Oblique Extension Principle holds,

$$S(z)^*S(z) - R(z)^*R(z) = 0$$

for all $z \in \mathbb{T}$. Our goal now is to find a matrix function $F : \mathbb{D} \to \mathbb{C}^{n\times m}$ such that

$$S(w)^*S(z) - R(w)^*R(z) = (1 - \overline{w}z)F(w)^*F(z)$$

holds for all $z, w \in \mathbb{D}$, which means, in particular, that we go from an identity on the torus to an identity inside the disc. The version of this problem related to the Unitary Extension Principle, i.e., the case $s \equiv 1$, can be found in Section 3 of [67].

To achieve this objective, we have to find the answers to a few questions:

1. If and when such a decomposition exists, is it possible to choose F with polynomial entries, given $p, q_1, \ldots, q_r$, and θ and as a consequence S and R are also polynomials? In other words: Is the matrix-valued function $S(w)^*S(z) - R(w)^*R(z)$, whose entries are polynomials in $\overline{w}$ and z, divisible by $(1 - \overline{w}z)$?

2. To find such a decomposition,

$$K(w, z) := \frac{S(w)^*S(z) - R(w)^*R(z)}{1 - \overline{w}z}$$

   has to be positive semi-definite for all $z, w \in \mathbb{D}$. Can we guarantee this for every choice of S and R (that is, for every choice of $p, q_1, \ldots q_r$, and s)? Do the properties of R and especially S help us in any way?

3. If and when such a decomposition exists, is there a constructive way to find the matrix function F?

We can answer the first question in the affirmative: Yes, $S(w)^*S(z) - R(w)^*R(z)$ is divisible by $(1 - \overline{w}z)$. Note that on the torus $\mathbb{T}$,

$$S(z)^* = S(\overline{z}^{-1})^* \quad \text{and} \quad R(z)^* = R(\overline{z}^{-1})^*.$$

The latter forms have the advantage of being analytic on $\mathbb{C} \setminus \{0\}$. So, if we insert them into the Oblique Extension Principle identity (4.4), we get

$$S(\overline{z}^{-1})^*S(z) - R(\overline{z}^{-1})^*R(z) = 0$$

for all $z \in \mathbb{T}$; and because the entries of the matrix functions are analytic, we can employ the identity theorem to extend this identity to $\mathbb{C} \setminus \{0\}$.

Now, using this form of the Oblique Extension Principle identity, it is easy to see that for any fixed $w \in \mathbb{C} \setminus \{0\}$,

$$S(w)^*S(z) - R(w)^*R(z) = 0$$

holds for $z = 1/\overline{w}$. So, $z = 1/\overline{w}$ is a zero of $S(w)^*S(z) - R(w)^*R(z)$, which implies that

$$S(w)^*S(z) - R(w)^*R(z) = \left(\frac{1}{\overline{w}} - z\right)\tilde{K}_w(z) = (1 - \overline{w}z)\frac{\tilde{K}_w(z)}{\overline{w}}$$

for a matrix-valued function $\tilde{K}_w$ whose entries are polynomials in z. Similarly, for any fixed $z \in \mathbb{C} \setminus \{0\}$, there exists a matrix-valued function $\tilde{K}_z$ whose entries are polynomials in $\overline{w}$, with

$$S(w)^*S(z) - R(w)^*R(z) = \left(\frac{1}{z} - \overline{w}\right)\tilde{K}_z(w) = (1 - \overline{w}z)\frac{\tilde{K}_z(w)}{z},$$

since $\overline{w} = 1/z$ is a zero of $S(w)^*S(z) - R(w)^*R(z)$. Note that $z = 1/\overline{w}$ and $\overline{w} = 1/z$ are equivalent. Altogether, we have

$$\frac{S(w)^*S(z) - R(w)^*R(z)}{1 - \overline{w}z} = \frac{\tilde{K}_w(z)}{\overline{w}} = \frac{\tilde{K}_z(w)}{z} =: K(z, w)$$

for some K with polynomial entries in z and $\overline{w}$.

Part of the second question can be answered with relative ease as well: If we choose *any* matrix-valued function S (in other words, if we ignore the properties S has in the Oblique Extension Principle due to its construction), the Oblique Extension Principle identity $S(z)^*S(z) - R(z)^*R(z) = 0$ for $z \in \mathbb{T}$ is not sufficient to guarantee the desired positive semi-definiteness of $K(w, z)$.

If $K(w, z)$ were positive semi-definite on $\mathbb{D}$ for any S (and any R) under the assumption $S(z)^*S(z) - R(z)^*R(z) = 0$ for $z \in \mathbb{T}$ alone, this would imply that the matrix

$$-K(w, z) = \frac{R(w)^*R(z) - S(w)^*S(z)}{1 - \overline{w}z}$$

is also positive semi-definite on $\mathbb{D}$, since obviously

$$S(z)^*S(z) - R(z)^*R(z) = 0 \iff R(z)^*R(z) - S(z)^*S(z) = 0.$$

Now, both $K(w, z)$ and $-K(w, z)$ can be positive semi-definite on $\mathbb{D}$ only if $K(w, z) = 0$ and thus $S(w)^*S(z) - R(w)^*R(z) = 0$ for all $w, z \in \mathbb{D}$, which is clearly not true for any choice of S and R (see Example 6.9, for instance).

It is, therefore, necessary to take advantage of the properties of S. In our case, S is not any matrix-valued function; it is a diagonal matrix, and the diagonal entries come from the Fejér-Riesz factorization of the vanishing moment recovery s. Recall that the polynomial θ in the Fejér-Riesz factorization of s can be chosen zero-free on $\mathbb{D}$ according to Remark 4.2 a). Thus, the matrix inverse of S, given by

$$S^{-1}(z) = \operatorname{diag}\left(\frac{1}{\theta^{\sigma_1}(z)}, \dots, \frac{1}{\theta^{\sigma_m}(z)}\right) \tag{6.1}$$

is well defined on $\mathbb{D}$. If s is strictly positive on $\mathbb{T}$, S^{-1} is well defined on the torus as well; if we only require s to be non-negative, θ can, of course, have zeros on the torus, so S^{-1} may not be defined for some $z \in \mathbb{T}$.

Recall that in the version of the Oblique Extension Principle used here (Theorem 3.22 and Corollary 3.26, respectively), we assume that the vanishing moment recovery function is strictly positive. Thus, $S^{-1}(z)$ is well defined on $\overline{\mathbb{D}}$ under the assumptions used in this thesis. The following results would theoretically also hold for non-negative s.

The crucial point now is not the exact form of $S^{-1}(z)$, but the fact that S has a rational and analytic inverse on $\mathbb{D}$. We will therefore formulate the following results in terms of general (univariate) matrix polynomials $S \in \mathbb{C}[z]^{m\times m}$ and $R \in \mathbb{C}[z]^{n_0\times m}$ that satisfy $S(z)^*S(z) - R(z)^*R(z) = 0$ on $\mathbb{T}$, i.e., not necessarily those defined in Section 4.4, and will additionally require S to have a rational and analytic inverse S^{-1} on $\mathbb{D}$.

For such matrix polynomials S, we can reformulate the identity into a form that is relatively close to the identity found in the matrix form of the Unitary Extension Principle (i.e., the case $s \equiv 1$), albeit at the cost that the entries of the matrix functions are only rational and no longer polynomials. $S(z)^*S(z) - R(z)^*R(z) = 0$ for $z \in \mathbb{T}$ is equivalent to

$$I_m - \tilde{R}(w)^*\tilde{R}(z) = 0 \tag{6.2}$$

for every $z \in \mathbb{T}$ where

$$\tilde{R}(z) := R(z)S^{-1}(z)$$

is defined.

This new form of the Oblique Extension Principle identity allows us to use the following maximum principle (see [67], Proposition A.1):

Theorem 6.2
Let $R : \mathbb{D}^d \to \mathbb{C}^{n_0\times m}$ be a rational and analytic function with $I_m - R(z)^*R(z) \geq 0$ for all $z \in \mathbb{T}^d$ where R is defined.

Then $I_m - R(z)^*R(z) \geq 0$ holds for all $z \in \mathbb{D}^d$.

Since the function $\tilde{R}$ defined above is rational and analytic on $\mathbb{D}$ (since we assume S^{-1} to be rational and analytic and R to be polynomial), and by virtue of (6.2) also satisfies $I_m - \tilde{R}(z)^*\tilde{R}(z) \geq 0$ for $z \in \mathbb{T}$, we can apply this theorem to obtain

$$I_m - \tilde{R}(z)^*\tilde{R}(z) \geq 0 \tag{6.3}$$

for $z \in \mathbb{D}$, meaning we have made the step from $\mathbb{T}$ to $\mathbb{D}$.

To get from this statement to the desired positive semi-definiteness of $K(w,z)$, i.e., of $S(w)^*S(z) - R(w)^*R(z)$ divided by $(1 - \overline{w}z)$, we first introduce the concept of positive semi-definite kernels.

Definition 6.3
Let Ω be an arbitrary set and $\mathcal{H}$ be a complex Hilbert space. A function $K : \Omega\times\Omega \to \mathcal{B}(\mathcal{H})$ is called a *positive semi-definite kernel* if

$$\sum_{j,\ell=1}^{n} \langle K(\lambda_j, \lambda_\ell)h_\ell, h_j\rangle_{\mathcal{H}} \geq 0$$

holds for any choice of $\lambda_1, \ldots, \lambda_n \in \Omega$ and $h_1, \ldots, h_n \in \mathcal{H}$.

The following Lemma (see [67], Lemma 3.2) now states that dividing $I_m - \tilde{R}(w)^*\tilde{R}(z)$ by $1 - \overline{w}z$ yields a positive semi-definite kernel.

Lemma 6.4
Let $R : \mathbb{D} \to \mathbb{C}^{n_0\times m}$ be an analytic function with $I_m - R(z)^*R(z) \geq 0$.

Then, $\tilde{K}_R : \mathbb{D} \times \mathbb{D} \to \mathbb{C}^{m\times m}$ with

$$\tilde{K}_R(w,z) := \frac{I_m - R(w)^*R(z)}{1 - \overline{w}z}$$

is a positive semi-definite kernel.

According to (6.3), this can be applied to the function $\tilde{R}$. So,

$$\tilde{K}_{\tilde{R}}(w,z) := \frac{I_m - \tilde{R}(w)^*\tilde{R}(z)}{1 - \overline{w}z}$$

is a positive semi-definite kernel, i.e.,

$$\sum_{j,\ell=1}^{n} \left\langle \tilde{K}_{\tilde{R}}(z_j, z_\ell)v_\ell, v_j \right\rangle \geq 0 \tag{6.4}$$

holds for all possible choices of $z_1, \ldots, z_n \in \mathbb{D}$ and $v_1, \ldots, v_n \in \mathbb{C}^m$.

Of course, what we actually want is $K : \mathbb{D} \times \mathbb{D} \to \mathbb{C}^{m \times m}$ with

$$K(w,z) := \frac{S(w)^*S(z) - R(w)^*R(z)}{1 - \overline{w}z} = S(w)^*\tilde{K}_{\tilde{R}}(w,z)S(z)$$

to be positive semi-definite, but as

$$\begin{aligned}\sum_{j,\ell=1}^{n} \langle K(z_j, z_\ell)v_\ell, v_j\rangle &= \sum_{j,\ell=1}^{n} \left\langle S(z_j)^*\tilde{K}_{\tilde{R}}(z_j, z_\ell)S(z_\ell)v_\ell, v_j\right\rangle \\ &= \sum_{j,\ell=1}^{n} \left\langle \tilde{K}_{\tilde{R}}(z_j, z_\ell)S(z_\ell)v_\ell, S(z_j)v_j\right\rangle \\ &= \sum_{j,\ell=1}^{n} \left\langle \tilde{K}_{\tilde{R}}(z_j, z_\ell)\tilde{v}_\ell, \tilde{v}_j\right\rangle\end{aligned}$$

for $\tilde{v}_j = S(z_j)v_j$, $1 \leq j \leq n$, this follows directly from (6.4).

Before we address the third question, let us summarize our answers to the first two questions in the following Lemma (our answer to the first question is still valid in the more general setting we introduced for the answer to the second question).

Lemma 6.5
Let $S \in \mathbb{C}[z]^{m \times m}$ and $R \in \mathbb{C}[z]^{n_0 \times m}$ be two matrix polynomials such that

$$S(z)^*S(z) - R(z)^*R(z) = 0$$

for all $z \in \mathbb{T}$. Furthermore, let S have a rational and analytic inverse S^{-1} on $\mathbb{D}$.
Then, $K : \mathbb{D} \times \mathbb{D} \to \mathbb{C}^{m \times m}$ with

$$K(w,z) := \frac{S(w)^*S(z) - R(w)^*R(z)}{1 - \overline{w}z}$$

is

(i) polynomial in $\overline{w}$ and z and

(ii) a positive semi-definite kernel.

We now turn our attention to the third question: Find a matrix-valued function F such that

$$S(w)^*S(z) - R(w)^*R(z) = (1 - \overline{w}z)F(w)^*F(z)$$

holds for $z \in \mathbb{D}$. Fortunately, the construction of F from $K(w,z)$ in the context of Lemma 6.5 (i.e., again in our more general setting, which is somewhat detached from the Oblique Extension Principle) is relatively straightforward. Since the entries of K are polynomial, we can write it as

$$K(w,z) = \begin{pmatrix} I_m \\ wI_m \\ \vdots \\ w^{\nu-1}I_m \end{pmatrix}^* L \begin{pmatrix} I_m \\ zI_m \\ \vdots \\ z^{\nu-1}I_m \end{pmatrix} \tag{6.5}$$

with a uniquely defined constant matrix $L \in \mathbb{C}^{\nu m \times \nu m}$, where ν is the degree of $S(w)^*S(z) - R(w)^*R(z)$ in z, and $\overline{w}$ and m is the size of K (i.e., the number of columns of S and R). Thus, for any points $z_1, \dots, z_\nu \in \mathbb{D}$, the block matrix

$$\mathcal{K} := (K(z_j, z_\ell))_{1 \le j,\ell \le \nu} = \begin{pmatrix} K(z_1,z_1) & \cdots & K(z_1,z_\nu) \\ \vdots & \ddots & \vdots \\ K(z_\nu,z_1) & \cdots & K(z_\nu,z_\nu) \end{pmatrix} \tag{6.6}$$

can be written as $\mathcal{K} = V^*LV$ for

$$V := \begin{pmatrix} I_m & I_m & \cdots & I_m \\ z_1 I_m & z_2 I_m & \cdots & z_\nu I_m \\ \vdots & \vdots & \ddots & \vdots \\ z_1^{\nu-1} I_m & z_2^{\nu-1} I_m & \cdots & z_\nu^{\nu-1} I_m \end{pmatrix}.$$

Since V is the transposed version of a block Vandermonde matrix, it is nonsingular for distinct points $z_j \neq z_\ell$ for $j \neq \ell$. Since $\mathcal{K}$ is positive semi-definite, this implies that L is also positive semi-definite, so in particular

$$L = \tilde{F}^* \tilde{F}$$

for some $\tilde{F} \in \mathbb{C}^{n_1 \times m}$. Thus,

$$K(w,z) = F(w)^*F(z)$$

for $w, z \in \mathbb{D}$ and

$$F(z) := \tilde{F} \begin{pmatrix} I_m \\ zI_m \\ \vdots \\ z^{\nu-1}I_m \end{pmatrix}.$$

According to the definition of K, this means that

$$S(w)^*S(z) - R(w)^*R(z) = (1 - \overline{w}z)F(w)^*F(z)$$

holds for $z \in \mathbb{D}$, which is exactly the desired decomposition. The construction also immediately shows that F is a matrix polynomial.

In summary, we have obtained the following theorem:

Theorem 6.6
Let $S \in \mathbb{C}[z]^{m\times m}$ and $R \in \mathbb{C}[z]^{n_0\times m}$ be two matrix polynomials such that

$$S(z)^*S(z) - R(z)^*R(z) = 0$$

for all $z \in \mathbb{T}$. Furthermore, let S have a rational and analytic inverse S^{-1} on $\mathbb{D}$.

Then there exists a matrix polynomial $F : \mathbb{C} \to \mathbb{C}^{n_1\times m}$ such that

$$S(w)^*S(z) - R(w)^*R(z) = (1 - \overline{w}z)F(w)^*F(z)$$

for $w, z \in \mathbb{D}$.

Remark 6.7
The construction of F shows that if the matrix polynomials S and R have maximum degree ν, then F has the degree $\nu - 1$. The number of columns of F is pre-determined by the size of $S(z)^*S(z) - R(z)^*R(z)$, its number of rows, i.e., the number $n_1 \in \mathbb{N}$ in Theorem 6.6, is given by the number of rows of $\tilde{F}$ in the factorization $L = \tilde{F}^*\tilde{F}$. Thus, the smallest possible choice is $n_1 = \operatorname{rank} L$.

We can derive the following result for the Oblique Extension Principle, i.e., the matrix-valued polynomials S and R, from Section 4.4.

Corollary 6.8
Let $p \in \mathbb{C}[z]$ be a refinement symbol and $q_1, \ldots, q_r \in \mathbb{C}[z]$ be wavelet symbols such that the Oblique Extension Principle holds for the vanishing moment recovery function $s \in \mathbb{C}[z^{\pm 1}]$ with $s(z) > 0$ for $z \in \mathbb{T}$, i.e.,

$$S(z)^*S(z) - R(z)^*R(z) = 0$$

for $z \in \mathbb{T}$ with S and R defined as in Section 4.4 and extended to matrix functions $S : \mathbb{C} \to \mathbb{C}^{m\times m}$ and $R : \mathbb{C} \to \mathbb{C}^{(r+1)\times m}$. Furthermore, we assume that $s = |\theta|^2$ on $\mathbb{T}$ with $\theta \in \mathbb{C}[z]$ (and thus also S) zero-free on $\mathbb{D}$.

Then, there exists a matrix polynomial $F : \mathbb{C} \to \mathbb{C}^{n_1\times m}$ such that

$$S(w)^*S(z) - R(w)^*R(z) = (1 - \overline{w}z)F(w)^*F(z)$$

for $w, z \in \mathbb{D}$.

Proof. As mentioned in the answer to the second question (see (6.1)), S has the inverse

$$S^{-1}(z) = \operatorname{diag}\left(\frac{1}{\theta^{\sigma_1}(z)}, \ldots, \frac{1}{\theta^{\sigma_m}(z)}\right).$$

Because θ is zero-free on $\mathbb{D}$, S^{-1} is rational and analytic on the disc. Corollary 6.8 thus follows directly from Theorem 6.6. □

We will demonstrate the construction method derived in this section with an example.

Example 6.9
We revisit the piecewise linear (shifted) B-Spline N_2 from Examples 3.24 and 3.28, given by

$$N_2(x) = \begin{cases} x, & \text{if } 0 \leq x \leq 1 \\ 2-x, & \text{if } 1 < x \leq 2 \\ 0, & \text{otherwise} \end{cases}.$$

As we have already seen in the examples mentioned above, it is refinable with the scaling factor $M = 2$ and the refinement symbol (in terms of $z \in \mathbb{T}$)

$$p(z) = \frac{(1+z)^2}{4}.$$

Furthermore, using wavelet symbols

$$q_1(z) = \frac{(1-z)^2}{4} \quad \text{and} \quad q_2(z) = \frac{\sqrt{6}}{24}(1-z)^2(1+4z+z^2)$$

and the vanishing moment recovery function

$$s(z) = \frac{1}{3}\left(4 - \frac{z}{2} - \frac{1}{2z}\right),$$

the Oblique Extension Principle identities hold and thus the functions ψ_1, ψ_2 defined by $\hat{\psi}_j(2\omega) = \tilde{q}_j(z)\hat{N}_2(\omega)$, $j \in \{1,2\}$, are mother wavelets of a tight wavelet frame (with two vanishing moments).

For the matrix form of the Oblique Extension Principle identities, note that for $M = 2$ we have $\Gamma_2 = \{0, 1/2\}$, so $\{p^\sigma : \sigma \in \Gamma_2\}$ contains only p^0 with $p^0(z) = p(e^0 z) = p(z)$ and $p^{1/2}$ with $p^{1/2}(z) = p(e^{-\pi i}z) = p(-z)$ (the same is true for q_1 and q_2, of course). Furthermore, $s(z) = |\theta(z)|^2$ for $z \in \mathbb{T}$ and

$$\theta(z) = \frac{z - \left(4+\sqrt{15}\right)}{\sqrt{6\left(4+\sqrt{15}\right)}}.$$

The only zero of θ is $z_0 = 4 + \sqrt{15}$, which is not in $\mathbb{D}$, so all the assumptions of Theorem 6.8 are met.

The matrix polynomial S is given by

$$S(z) = \frac{1}{\sqrt{6\big(4+\sqrt{15}\big)}} \begin{pmatrix} z - \big(4+\sqrt{15}\big) & 0 \\ 0 & -z - \big(4+\sqrt{15}\big) \end{pmatrix}$$

and the matrix polynomial R is given by

$$R(z) = \begin{pmatrix} \frac{(1-z)^2}{4} & \frac{\sqrt{6}}{24}(1-z)^2(1+4z+z^2) & \frac{z^2-\big(4+\sqrt{15}\big)}{\sqrt{6\big(4+\sqrt{15}\big)}}\frac{(1+z)^2}{4} \\ \frac{(1+z)^2}{4} & \frac{\sqrt{6}}{24}(1+z)^2(1-4z+z^2) & \frac{z^2-\big(4+\sqrt{15}\big)}{\sqrt{6\big(4+\sqrt{15}\big)}}\frac{(1-z)^2}{4} \end{pmatrix}^T$$

(for $M = 2$ and $z = e^{-2\pi i\omega}$ we have $d_M(z) = e^{-2\pi i 2\omega} = \big(e^{-2\pi i\omega}\big)^2 = z^2$). Calculating the kernel $K(w, z)$ then shows that it is indeed polynomial in $\overline{w}$ and z and has degree 3 in both variables. It is given by

$$K(w,z) = \begin{pmatrix} I_2 \\ wI_2 \\ w^2 I_2 \\ w^3 I_2 \end{pmatrix}^* L \begin{pmatrix} I_2 \\ zI_2 \\ z^2 I_2 \\ z^3 I_2 \end{pmatrix}$$

with $L = \frac{1}{96}(L_1, L_2)$ for

$$L_1 = \begin{pmatrix} 15^{\frac{3}{2}}+53 & -\sqrt{15}-11 & -2\sqrt{15}-14 & 2\sqrt{15}-2 \\ -\sqrt{15}-11 & 15^{\frac{3}{2}}+53 & -2-2\sqrt{15} & 2\sqrt{15}+14 \\ -2\sqrt{15}-14 & 2-2\sqrt{15} & 73-5\sqrt{15} & 3\sqrt{15}+33 \\ 2\sqrt{15}-14 & 2\sqrt{15}+14 & 3\sqrt{15}+33 & 73-5\sqrt{15} \\ -\sqrt{15}-3 & -\sqrt{15}-3 & 4-4\sqrt{15} & -16 \\ -\sqrt{15}-3 & -\sqrt{15}-3 & 16 & 4\sqrt{15}-4 \\ 0 & 0 & -\sqrt{15}-3 & -\sqrt{15}-3 \\ 0 & 0 & -\sqrt{15}-3 & -\sqrt{15}-3 \end{pmatrix}$$

and

$$L_2 = \begin{pmatrix} -\sqrt{15}-3 & -\sqrt{15}-3 & 0 & 0 \\ -\sqrt{15}-3 & -\sqrt{15}-3 & 0 & 0 \\ 4-4\sqrt{15} & 16 & -\sqrt{15}-3 & -\sqrt{15}-3 \\ -16 & 4\sqrt{15}-4 & -\sqrt{15}-3 & -\sqrt{15}-3 \\ 25-5\sqrt{15} & 3\sqrt{15}-15 & 10-2\sqrt{15} & 10-2\sqrt{15} \\ 3\sqrt{15}-15 & 25-5\sqrt{15} & 2\sqrt{15}-10 & 2\sqrt{15}-10 \\ 10-2\sqrt{15} & 2\sqrt{15}-10 & 5-\sqrt{15} & 5-\sqrt{15} \\ 10-2\sqrt{15} & 2\sqrt{15}-10 & 5-\sqrt{15} & 5-\sqrt{15} \end{pmatrix}.$$

L is a positive semi-definite matrix (with rank 5) and can thus be written as $L = \tilde{F}^* \tilde{F}$ with $\tilde{F} = (\tilde{F}_1, \tilde{F}_2)$ for

$$\tilde{F}_1 = \begin{pmatrix} \sqrt{\frac{53+15\sqrt{15}}{96}} & -\frac{\sqrt{437\sqrt{15}-1129}}{4\sqrt{1698}} & -\frac{\sqrt{109\sqrt{15}-121}}{2\sqrt{1698}} & \frac{\sqrt{173\sqrt{15}-649}}{2\sqrt{1698}} \\ 0 & \frac{\sqrt{5943(17\sqrt{15}+72)}}{849} & -\frac{\sqrt{8772-2221\sqrt{15}}}{2\sqrt{5943}} & \frac{\sqrt{5(708-163\sqrt{15})}}{2\sqrt{1981}} \\ 0 & 0 & \frac{\sqrt{275+17\sqrt{15}}}{4\sqrt{42}} & \frac{\sqrt{13189255-416371\sqrt{15}}}{4\sqrt{1497090}} \\ 0 & 0 & 0 & \frac{2\sqrt{7(163\sqrt{15}-540)}}{\sqrt{106935}} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$\tilde{F}_2 = \begin{pmatrix} -\frac{\sqrt{7\sqrt{15}+13}}{4\sqrt{566}} & -\frac{\sqrt{7\sqrt{15}+13}}{4\sqrt{566}} & 0 & 0 \\ -\frac{\sqrt{17\sqrt{15}+72}}{2\sqrt{5943}} & -\frac{\sqrt{17\sqrt{15}+72}}{2\sqrt{5943}} & 0 & 0 \\ -\frac{\sqrt{522775-75091\sqrt{15}}}{2\sqrt{1497090}} & \frac{\sqrt{401815-38579\sqrt{15}}}{2\sqrt{1497090}} & -\frac{\sqrt{7(207\sqrt{15}+845)}}{4\sqrt{71290}} & -\frac{\sqrt{7(207\sqrt{15}+845)}}{4\sqrt{71290}} \\ -\frac{11\sqrt{163\sqrt{15}-540}}{2\sqrt{748545}} & -\frac{\sqrt{3(163\sqrt{15}-540)}}{2\sqrt{249515}} & -\frac{\sqrt{163\sqrt{15}-540}}{\sqrt{748545}} & -\frac{\sqrt{163\sqrt{15}-540}}{\sqrt{748545}} \\ \frac{\sqrt{15}-3}{3\sqrt{14}} & \frac{3-\sqrt{15}}{3\sqrt{14}} & \frac{\sqrt{15}-3}{6\sqrt{14}} & \frac{\sqrt{15}-3}{6\sqrt{14}} \end{pmatrix}.$$

We then get

$$S(w)^* S(z) - R(w)^* R(z) = (1 - \overline{w} z) F(w)^* F(z)$$

for $z \in \mathbb{D}$ and $F = (F_1, F_2)$ with

$$F_1(z) = \begin{pmatrix} -\frac{\sqrt{13+7\sqrt{15}}}{\sqrt{9056}} z^2 - \frac{\sqrt{109\sqrt{15}-121}}{2\sqrt{1698}} z + \sqrt{\frac{53+15\sqrt{15}}{96}} \\ -\frac{\sqrt{72+17\sqrt{15}}}{2\sqrt{5943}} z^2 - \frac{\sqrt{8772-2221\sqrt{15}}}{2\sqrt{5943}} z \\ -\frac{\sqrt{7(845+207\sqrt{15})}}{4\sqrt{71290}} z^3 - \frac{\sqrt{522775-75091\sqrt{15}}}{2\sqrt{1497090}} z^2 + \frac{\sqrt{275+17\sqrt{15}}}{4\sqrt{42}} z \\ \frac{\sqrt{163\sqrt{15}+540}(540\sqrt{7}-163\sqrt{105})}{1497090}\ z^2(2z+11) \\ \frac{(\sqrt{15}-3)}{6\sqrt{14}}\ z^2(z+2) \end{pmatrix}$$

and

$$F_2(z) = \begin{pmatrix} -\frac{\sqrt{13+7\sqrt{15}}}{\sqrt{9056}}z^2 + \frac{\sqrt{173\sqrt{15}-649}}{2\sqrt{1698}}z - \frac{\sqrt{437\sqrt{15}-1129}}{4\sqrt{1698}} \\ -\frac{\sqrt{72+17\sqrt{15}}}{2\sqrt{5943}}z^2 + \frac{\sqrt{5(708-163\sqrt{15})}}{2\sqrt{1981}}z + \frac{\sqrt{7(72+17\sqrt{15})}}{\sqrt{849}} \\ -\frac{\sqrt{275-17\sqrt{15}}}{120\sqrt{49903}}z\big(21(5+\sqrt{15})z^2 + (30-106\sqrt{15})z - 19\sqrt{15} - 1215\big) \\ -\frac{\sqrt{163\sqrt{15}-540}}{\sqrt{748545}}z^3 - \frac{\sqrt{3(163\sqrt{15}-540)}}{2\sqrt{249515}}z^2 + \frac{2\sqrt{7(163\sqrt{15}-540)}}{\sqrt{106935}}z \\ \frac{(\sqrt{15}-3)}{6\sqrt{14}}(z-2)z^2 \end{pmatrix}.$$

As we can see, F has degree 3, one smaller than the maximum degree of S and R.

Remark 6.10
The matrix-valued function F obtained by this procedure is not unique since the factorization $L = \tilde{F}^*\tilde{F}$ is not unique. In the example above, we have used the Cholesky factorization (or rather, the extension of the Cholesky factorization to positive semi-definite matrices). However, any other method of factorizing L would work as well. For example, using the singular value decomposition (and omitting the last rows of the resulting $\tilde{F}$, since they are numerically zero), we get $L = \tilde{F}^*\tilde{F}$ with

$$\tilde{F} = \begin{pmatrix} 0.72797 & 0.70746 & 0.35516 & 0.025504 & 0.0032691 \\ -0.72797 & 0.70746 & -0.35516 & 0.025504 & -0.0032691 \\ -0.40276 & -0.20493 & 0.59066 & 0.070434 & 0.025469 \\ -0.40276 & 0.20493 & 0.59066 & -0.070434 & 0.025469 \\ 0.089621 & -0.10247 & -0.18434 & 0.035217 & 0.070323 \\ -0.089621 & -0.10247 & 0.18434 & 0.035217 & -0.070323 \\ 0.04481 & 0 & -0.092171 & 0 & 0.035162 \\ 0.04481 & 0 & -0.092171 & 0 & 0.035162 \end{pmatrix}^T,$$

which is obviously a different matrix than $\tilde{F}$ in Example 6.9.

In the following section, we present an alternative method to obtain the function F based on a generalized version of the Nevanlinna-Pick interpolation.

6.2. Obtaining a Realization with the Nevanlinna-Pick Interpolation

To conclude this section, we present an alternative way to get from the Oblique Extension Principle identity (4.4) to one of the three equivalent statements in Theorem 5.3. To do this, we will use a generalized version of the *Nevanlinna-Pick interpolation* already briefly mentioned in Section 5.1. It is named after Georg Pick (1859–1942) and Rolf Nevanlinna (1895–1980).

The original Nevanlinna-Pick interpolation problem is to find a holomorphic function $f : \mathbb{D} \to \mathbb{D}$ such that for given points $w_1, \dots, w_n \in \mathbb{D}$ and $v_1, \dots, v_n \in \mathbb{D}$

$$f(w_j) = v_j$$

holds for all $1 \le j \le n$. This problem was first solved by Pick (1916, [87]) and independently by Nevanlinna (1919, [84]).

We cite the following generalized version of the Nevanlinna-Pick interpolation problem from [8]. We also generalize the Schur class $\mathcal{S}(\Omega)$ mentioned in Remark 5.2.

Definition 6.11
Let $\mathcal{X}$ and $\mathcal{Y}$ be two Hilbert spaces. The *generalized Schur class* $\mathcal{S}_d(\mathcal{X}, \mathcal{Y})$ consists of all functions $W : \mathbb{D}^d \to \mathcal{B}(\mathcal{X}, \mathcal{Y})$ which are analytic on $\mathbb{D}^d$ such that

$$\sup_{r<1} \|W(rT_1, \dots, rT_d)\| \le 1$$

for every commutative tuple $T = (T_1, \dots T_d)$ of linear contractive operators on a Hilbert space $\mathcal{K}$.

Note that according to Theorem 5.1, for any $W \in \mathcal{S}_d(\mathcal{X}, \mathcal{Y})$ there exist Hilbert spaces $\mathcal{H}_1, \dots, \mathcal{H}_d$ and functions $L_j : \mathbb{D}^d \to \mathcal{B}(\mathcal{X}, \mathcal{H}_j)$, $1 \le j \le d$, such that

$$I - W(w)^* W(z) = \sum_{j=1}^{d} (1 - \overline{w_j} z_j) L_j(w)^* L_j(z)$$

for $z, w \in \mathbb{D}^d$ as well as a unitary operator

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} : \mathcal{X} \oplus \mathcal{H} \to \mathcal{Y} \oplus \mathcal{H}$$

with $\mathcal{H} = \mathcal{H}_1 \oplus \ldots \oplus \mathcal{H}_d$ such that

$$W(z) = A + BE(z)(I - DE(z))^{-1} C$$

for all $z \in \mathbb{D}^d$. These are, of course, the versions of these statements from Section 5 that are more consistent with the Unitary Extension Principle; the following results will allow us to link to the respective Oblique Extension Principle versions instead.

Definition 6.12
Let $n \in \mathbb{N}$, $\mathcal{K}_1, \dots \mathcal{K}_n, \mathcal{X}$ and $\mathcal{Y}$ be Hilbert spaces and $U_j \in \mathcal{B}(\mathcal{K}_j, \mathcal{X})$, $V_j \in \mathcal{B}(\mathcal{K}_j, \mathcal{Y})$, $1 \leq j \leq n$, be bounded operators on these Hilbert spaces.

The associated *right tangential interpolation problem* is: Find a function $W \in \mathcal{S}_d(\mathcal{X}, \mathcal{Y})$ such that for given points $w_1, \dots, w_n \in \mathbb{D}^d$,

$$W(w_j)U_j = V_j \tag{6.7}$$

holds for all $1 \leq j \leq n$.

One can analogously define a left tangential interpolation problem (finding W such that $U_j W(w_j) = V_j$) and the bitangential interpolation problem (considering the right and the left tangential interpolation problem simultaneously).

The following theorem is part *(ii)* of Theorem 4.2 in [8] (based on earlier work from Agler in [1]) and gives a necessary and sufficient condition for the existence of a solution of the right tangential interpolation problem (the parts *(i)* and *(iii)* are similar results for the left tangential and bitangential interpolation problems, respectively).

Theorem 6.13
Let $n \in \mathbb{N}$, $\mathcal{K}_1, \dots \mathcal{K}_n, \mathcal{X}$ and $\mathcal{Y}$ be some Hilbert spaces, $U_j \in \mathcal{B}(\mathcal{K}_j, \mathcal{X})$, $V_j \in \mathcal{B}(\mathcal{K}_j, \mathcal{Y})$, $1 \leq j \leq n$, bounded operators on these Hilbert spaces, and $w_1, \dots, w_n \in \mathbb{D}^d$.

The associated right tangential interpolation problem has a solution $W \in \mathcal{S}_d(\mathcal{X}, \mathcal{Y})$ if and only if there exist positive semi-definite block matrices $\mathcal{N}_\ell = (\mathcal{N}_{\ell,jk})_{j,k=1,\dots,n}$, $1 \leq \ell \leq d$, with $(\mathcal{N}_{\ell,jk}) \in \mathcal{B}(\mathcal{K}_k, \mathcal{K}_j)$ and

$$U_j^* U_k - V_j^* V_k = \sum_{\ell=1}^{d} (1 - \overline{w_{j,\ell}} w_{k,\ell}) \mathcal{N}_{\ell,jk} \tag{6.8}$$

for all $1 \leq j, k \leq n$.

The equation (6.8) is quite similar to (5.6), and the proof of this theorem is in some ways similar to the proof of Theorem 5.3. To underline this, we will give a short sketch of the proof of the sufficiency of (6.8):

Since the block matrices $\mathcal{N}_\ell$ in (6.8) are positive semi-definite, we can write them as $\mathcal{N}_\ell = A_\ell^* A_\ell$. The A_ℓ are in $\mathcal{B}(\bigoplus_{m=1}^n \mathcal{K}_m, \tilde{\mathcal{C}}_\ell)$ for some auxiliary Hilbert spaces $\tilde{\mathcal{C}}_\ell$ and can be written in block form as well, i.e.,

$$A_\ell = \begin{pmatrix} A_{\ell,1}\,, & \dots, & A_{\ell,n} \end{pmatrix}$$

with $A_{\ell,m} \in \mathcal{B}(\mathcal{K}_m, \tilde{\mathcal{C}}_\ell)$. This gives us factorizations $\mathcal{N}_{\ell,jk} = A_{\ell,j}^* A_{\ell,k}$ of the respective blocks of $\mathcal{N}_\ell$.

Plugging this into (6.8) yields

$$\begin{aligned} U_j^* U_k - V_j^* V_k &= \sum_{\ell=1}^{d} (1 - \overline{w_{j,\ell}} w_{k,\ell}) A_{\ell,j}^* A_{\ell,k} \\ &= \sum_{\ell=1}^{d} A_{\ell,j}^* A_{\ell,k} - \sum_{\ell=1}^{d} (w_{j,\ell} A_{\ell,j})^* (w_{k,\ell} A_{\ell,k}) \end{aligned}$$

and thus

$$U_j^* U_k + \sum_{\ell=1}^{d} (w_{j,\ell} A_{\ell,j})^* (w_{k,\ell} A_{\ell,k}) = V_j^* V_k + \sum_{\ell=1}^{d} A_{\ell,j}^* A_{\ell,k} \tag{6.9}$$

for all $1 \leq j, k \leq n$. This equation now allows us to employ once again the lurking isometry argument we introduced in Lemma 5.4 and Remark 5.5. If we define the space $\mathcal{E}_*$ as the subspace of $\mathcal{X} \oplus (\bigoplus_{\ell=1}^{d} \tilde{\mathcal{C}}_\ell)$ given by the span of the elements

$$\left\{ \begin{pmatrix} U_j \\ w_{j,1} A_{1,j} \\ \vdots \\ w_{j,d} A_{d,j} \end{pmatrix} k_j \ : \ k_j \in \mathcal{K}_j, \ j = 1, \dots, n \right\},$$

and the space $\mathcal{E}$ as the subspace of $\mathcal{Y} \oplus \left(\bigoplus_{\ell=1}^{d} \tilde{\mathcal{C}}_\ell\right)$ given by the span of the elements

$$\left\{ \begin{pmatrix} V_j \\ A_{1,j} \\ \vdots \\ A_{d,j} \end{pmatrix} k_j \ : \ k_j \in \mathcal{K}_j, \ j = 1, \dots, n \right\},$$

then by (6.9) and Lemma 5.4 we can find a well-defined isometry $\tilde{T} : \mathcal{E}_* \to \mathcal{E}$, which maps

$$\begin{pmatrix} U_j \\ w_{j,1} A_{1,j} \\ \vdots \\ w_{j,d} A_{d,j} \end{pmatrix} k_j \qquad \text{to} \qquad \begin{pmatrix} V_j \\ A_{1,j} \\ \vdots \\ A_{d,j} \end{pmatrix} k_j$$

for every $k_j \in \mathcal{K}_j$ and every $1 \leq j \leq n$.

If the Hilbert spaces involved are infinite dimensional, choose any unitary extension

$$T : \mathcal{X} \oplus \left(\bigoplus_{\ell=1}^{d} \mathcal{C}_\ell \right) \to \mathcal{Y} \oplus \left(\bigoplus_{\ell=1}^{d} \mathcal{C}_\ell \right)$$

with $\tilde{\mathcal{C}}_\ell \subset \mathcal{C}_\ell$, if their dimension is finite, choose $T = \tilde{T}$, and write it as

$$T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

A simple calculation then shows that

$$W(w) = A + BE(w)(I - DE(w))^{-1}C \tag{6.10}$$

is the desired linear operator.

Remark 6.14
If we now return to the setting of Lemma 6.5, i.e., if we have two matrix polynomials $S \in \mathbb{C}[z]^{m \times m}$ and $R \in \mathbb{C}[z]^{n_0 \times m}$ such that

$$S(z)^* S(z) - R(z)^* R(z) = 0$$

for all $z \in \mathbb{T}$ and S has a rational and analytic inverse S^{-1} on $\mathbb{D}$, we know from Lemma 6.5 that

$$K(w, z) = \frac{S(w)^* S(z) - R(w)^* R(z)}{1 - \overline{w}z}$$

is positive semi-definite and polynomial in $\overline{w}$ and z. If we choose points $w_1, \ldots, w_n \in \mathbb{D}$ and define the operators U_j and V_j in Theorem 6.13 as the matrices $U_j := S(w_j)$ and $V_j := R(w_j)$, the blocks $N_{1,jk}$ are given by $K(w_j, w_k)$. In fact, the matrix N_1 is exactly the matrix $\mathcal{K}$ from (6.6). As we have already shown in the proof of Lemma 6.5, $\mathcal{K}$ and hence N_1 is positive semi-definite if we have distinct points $w_j \neq w_k$ for $j \neq k$, so in this case, the associated right tangential interpolation problem has a solution W by Theorem 6.13.

For this choice of U_j and V_j, the similarities between the sketch of the proof of the sufficiency in Theorem 6.13 and the proof of Theorem 5.3, in particular, the steps $(i) \Rightarrow (ii)$ and $(ii) \Rightarrow (iii)$, become even more apparent. Note, that in this scenario $(A_{1,j}, \ldots, A_{d,j})^T$ from the proof of Theorem 6.13 is F from Theorem 5.3 evaluated in w_j (and accordingly $A_{\ell,j}$ is $F_\ell(w_j)$). Note also that with the choice of W in (6.10), the condition (6.7) from the definition of right tangential interpolation problems is the same as the identity (5.7) in Theorem 5.3.

Since we can use T to construct not only W but also F by using (5.14), the Nevanlinna-Pick interpolation provides another way to obtain F, although applying the construction method introduced in Section 6.1 is arguably more straightforward than using the Nevanlinna-Pick interpolation. If the maximum degree of S and R is ν and thus $K(w, z)$ has degree $\nu - 1$ in $\overline{w}$ and z, we use ν points $w_1, \ldots, w_\nu \in \mathbb{D}$ for the interpolation.

We will demonstrate the construction of T using the Nevanlinna-Pick interpolation, once again using the B-spline wavelets as an example.

Example 6.15
We revisit the piecewise linear B-spline N_2. The refinement and wavelet symbols and the associated matrix functions R, S, and K can be found in Example 6.9. In this case, the Hilbert spaces $\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3, \mathcal{K}_4$ and $\mathcal{X}$ from Theorem 6.13 are given by $\mathbb{C}^2$ and the Hilbert space $\mathcal{Y}$ is given by $\mathbb{C}^3$. Since K has degree 3 in $\overline{w}$ and z, we pick four distinct points

$$w_1 = 0, \quad w_2 = \frac{1}{2}, \quad w_3 = -\frac{1}{2} \quad \text{and} \quad w_4 = \frac{\mathrm{i}}{2}$$

in $\mathbb{D}$ for the Nevanlinna-Pick interpolation. For this choice of points, the blocks of the matrix $\mathcal{N}_1$ in (6.8) are given by

$$\mathcal{N}_{1,11} = \frac{1}{96}\begin{pmatrix} 15\sqrt{15}+53 & -\sqrt{15}-11 \\ -\sqrt{15}-11 & 15\sqrt{15}+53 \end{pmatrix},$$

$$\mathcal{N}_{1,21} = \frac{1}{384}\begin{pmatrix} 55\sqrt{15}+181 & -9\sqrt{15}-43 \\ -\sqrt{15}-51 & 63\sqrt{15}+237 \end{pmatrix},$$

$$\mathcal{N}_{1,31} = \frac{1}{384}\begin{pmatrix} 63\sqrt{15}+237 & -\sqrt{15}-51 \\ -9\sqrt{15}-43 & 55\sqrt{15}+181 \end{pmatrix},$$

$$\mathcal{N}_{1,41} = \frac{1}{384}\begin{pmatrix} \sqrt{15}\,(61+4\mathrm{i})+215+28\mathrm{i} & \sqrt{15}\,(-3+4\mathrm{i})-41-4\mathrm{i} \\ \sqrt{15}\,(-3-4\mathrm{i})-41+4\mathrm{i} & \sqrt{15}\,(61-4\mathrm{i})+215-28\mathrm{i} \end{pmatrix},$$

$$\mathcal{N}_{1,22} = \frac{1}{6144}\begin{pmatrix} 619\sqrt{15}+3753 & -45\sqrt{15}-351 \\ -45\sqrt{15}-351 & 1019\sqrt{15}+5337 \end{pmatrix},$$

$$\mathcal{N}_{1,32} = \frac{1}{6144}\begin{pmatrix} 997\sqrt{15}+2247 & -3\sqrt{15}-1713 \\ -243\sqrt{15}-1025 & 997\sqrt{15}+2247 \end{pmatrix},$$

$$\mathcal{N}_{1,42} = \frac{1}{6144}\begin{pmatrix} \sqrt{15}\,(952+171\mathrm{i})+2792-727\mathrm{i} & \sqrt{15}\,(-8+19\mathrm{i})-600-735\mathrm{i} \\ \sqrt{15}\,(-144-117\mathrm{i})-688-311\mathrm{i} & \sqrt{15}\,(1008-13\mathrm{i})+3792-1599\mathrm{i} \end{pmatrix},$$

$$\mathcal{N}_{1,33} = \frac{1}{6144}\begin{pmatrix} 1019\sqrt{15}+5337 & -45\sqrt{15}-351 \\ -45\sqrt{15}-351 & 619\sqrt{15}+3753 \end{pmatrix},$$

$$\mathcal{N}_{1,43} = \frac{1}{6144}\begin{pmatrix} \sqrt{15}\,(1008+13\mathrm{i})+3792+1599\mathrm{i} & \sqrt{15}\,(-144+117\mathrm{i})-688+311\mathrm{i} \\ \sqrt{15}\,(-8-19\mathrm{i})-600+735\mathrm{i} & \sqrt{15}\,(952-171\mathrm{i})+2792+727\mathrm{i} \end{pmatrix},$$

$$\mathcal{N}_{1,44} = \frac{1}{6144}\begin{pmatrix} 899\sqrt{15}+4785 & \sqrt{15}\,(35+120\mathrm{i})-111+168\mathrm{i} \\ \sqrt{15}\,(35-120\mathrm{i})-111-168\mathrm{i} & 899\sqrt{15}+4785 \end{pmatrix},$$

and $\mathcal{N}_{1,jk} = \mathcal{N}_{1,kj}^*$ for all $1 \le j, k \le 4$.

As expected from our theoretical considerations in Remark 6.14, the complete block matrix $\mathcal{N}_1$ is a hermitian positive semi-definite $(8 \times 8)-$matrix. Its rank is 5, so we can write it as $\mathcal{N}_1 = A_1^* A_1$ with $A_1 = (A_{1,1}\,,\, A_{1,2}\,,\, A_{1,4}\,,\, A_{1,4})$, where the blocks of A_1 are $(5 \times 2)-$matrices (so the auxiliary Hilbert space $\tilde{\mathcal{C}_1}$ in the sketch of the proof of Theorem 6.13 is given by $\mathbb{C}^5$ in this case).

We can choose them, for example, as (for better readability, we only give the numerical values)

$$A_{1,1} = \begin{pmatrix} 1.0757 & -0.1440 \\ 0 & 1.0661 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad A_{1,2} = \begin{pmatrix} 0.9538 & -0.1328 \\ -0.0613 & 1.1570 \\ 0.2958 & 0.3792 \\ 0 & 0.1041 \\ 0 & 0 \end{pmatrix}$$

$$A_{1,3} = \begin{pmatrix} 1.1644 & -0.1885 \\ 0.0233 & 0.9370 \\ -0.3888 & -0.2798 \\ -0.0528 & -0.0855 \\ 0.0261 & -0.0435 \end{pmatrix} \text{ and } A_{1,4} = \begin{pmatrix} 1.0924 - 0.1053\mathrm{i} & -0.1274 + 0.0278\mathrm{i} \\ 0.0190 - 0.0423\mathrm{i} & 1.0851 + 0.1100\mathrm{i} \\ 0.0465 + 0.3664\mathrm{i} & -0.0497 + 0.3536\mathrm{i} \\ 0.0264 + 0.0353\mathrm{i} & -0.0093 + 0.1037\mathrm{i} \\ -0.0130 - 0.0217\mathrm{i} & 0.0217 + 0.0130\mathrm{i} \end{pmatrix}.$$

The space $\mathcal{E}_*$ mentioned in the sketch of the proof of Theorem 6.13 is now the subspace of $\mathcal{X} \oplus \tilde{\mathcal{C}}_1 = \mathbb{C}^2 \oplus \mathbb{C}^5 = \mathbb{C}^7$ given by the span of the elements

$$\begin{pmatrix} -1.1455 & 0 \\ 0 & -1.1455 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} h_1, \quad \begin{pmatrix} -1.0727 & 0 \\ 0 & -1.2182 \\ 0.4769 & -0.0664 \\ -0.0307 & 0.5785 \\ 0.1479 & 0.1896 \\ 0 & 0.0520 \\ 0 & 0 \end{pmatrix} h_2,$$

$$\begin{pmatrix} -1.2182 & 0 \\ 0 & -1.0727 \\ -0.5822 & 0.0942 \\ -0.0116 & -0.4685 \\ 0.1944 & 0.1399 \\ 0.0264 & 0.0427 \\ -0.0130 & 0.0217 \end{pmatrix} h_3 \quad \text{and} \quad \begin{pmatrix} -1.1455 + 0.0727\mathrm{i} & 0 \\ 0 & -1.1455 - 0.0727\mathrm{i} \\ 0.0526 + 0.5462\mathrm{i} & -0.0139 - 0.0637\mathrm{i} \\ 0.0211 + 0.0095\mathrm{i} & -0.0550 + 0.5426\mathrm{i} \\ -0.1832 + 0.0233\mathrm{i} & -0.1768 - 0.0249\mathrm{i} \\ -0.0177 + 0.0132\mathrm{i} & -0.0518 - 0.0046\mathrm{i} \\ 0.0109 - 0.0065\mathrm{i} & -0.0065 + 0.0109\mathrm{i} \end{pmatrix} h_4$$

with $h_1, h_2, h_3, h_4 \in \mathbb{C}^2$, and the space $\mathcal{E}$ is the subspace of $\mathcal{Y} \oplus \tilde{\mathcal{C}}_1 = \mathbb{C}^3 \oplus \mathbb{C}^5 = \mathbb{C}^8$ given by the span of the elements

$$\begin{pmatrix} 0.2500 & 0.2500 \\ 0.1021 & 0.1021 \\ -0.2864 & -0.2864 \\ 1.0757 & -0.1440 \\ 0 & 1.0661 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} h_1, \quad \begin{pmatrix} 0.0625 & 0.5625 \\ 0.0829 & -0.1722 \\ -0.6239 & -0.0693 \\ 0.9538 & -0.1328 \\ -0.0613 & 1.1570 \\ 0.2958 & 0.3792 \\ 0 & 0.1041 \\ 0 & 0 \end{pmatrix} h_2,$$

$$\begin{pmatrix} 0.5625 & 0.0625 \\ -0.1722 & 0.0829 \\ -0.0693 & -0.6239 \\ 1.1644 & -0.1885 \\ 0.0233 & 0.9370 \\ -0.3888 & -0.2798 \\ -0.0528 & -0.0855 \\ 0.0261 & -0.0435 \end{pmatrix} h_3 \quad \text{and} \quad \begin{pmatrix} 0.1875 - 0.2500\mathrm{i} & 0.1875 + 0.2500\mathrm{i} \\ 0.2615 + 0.0765\mathrm{i} & 0.2615 - 0.0765\mathrm{i} \\ -0.2216 - 0.2955\mathrm{i} & -0.2216 + 0.2955\mathrm{i} \\ 1.0924 - 0.1053\mathrm{i} & -0.1274 + 0.0278\mathrm{i} \\ 0.0190 - 0.0423\mathrm{i} & 1.0851 + 0.1100\mathrm{i} \\ 0.0465 + 0.3664\mathrm{i} & -0.0497 + 0.3536\mathrm{i} \\ 0.0264 + 0.0353\mathrm{i} & -0.0093 + 0.1037\mathrm{i} \\ -0.0130 - 0.0217\mathrm{i} & 0.0217 + 0.0130\mathrm{i} \end{pmatrix} h_4$$

with $h_1, h_2, h_3, h_4 \in \mathbb{C}^2$. A simple calculation shows that $\dim \mathcal{E}_* = \dim \mathcal{E} = 7$. The matrix

$$\tilde{T} = \begin{pmatrix} -0.2182 & -0.2182 & -0.4353 & 0.3804 & 0.3221 & -0.2582 & 0.6260 \\ -0.0891 & -0.0891 & 0.2018 & -0.1764 & -0.7729 & -0.3611 & 0.4260 \\ 0.2500 & 0.2500 & -0.5662 & 0.4949 & -0.4765 & 0.0325 & -0.2390 \\ -0.9391 & 0.1257 & -0.0687 & 0.0601 & -0.1284 & 0.1029 & -0.2495 \\ 0 & -0.9307 & -0.0786 & 0.0687 & -0.1469 & 0.1178 & -0.2855 \\ 0 & 0 & 0.6588 & 0.7298 & 0.0270 & -0.0831 & -0.0104 \\ 0 & 0 & 0.0573 & 0.1939 & -0.1447 & 0.4446 & 0.0557 \\ 0 & 0 & -0.0323 & 0.0283 & 0.1101 & -0.7568 & -0.4717 \end{pmatrix}$$

then describes the desired isometry $\tilde{T} : \mathcal{E}_* \to \mathcal{E}$. Since we are in the finite-dimensional case, we do not extend $\tilde{T}$ to a unitary matrix but choose $T := \tilde{T}$. We can write T as

$$T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

with

$$A = \begin{pmatrix} -0.2182 & -0.2182 \\ -0.0891 & -0.0891 \\ 0.2500 & 0.2500 \end{pmatrix},$$

$$B = \begin{pmatrix} -0.4353 & 0.3804 & 0.3221 & -0.2582 & 0.6260 \\ 0.2018 & -0.1764 & -0.7729 & -0.3611 & 0.4260 \\ -0.5662 & 0.4949 & -0.4765 & 0.0325 & -0.2390 \end{pmatrix},$$

$$C = \begin{pmatrix} -0.9391 & 0.1257 \\ 0 & -0.9307 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix},$$

$$\text{and} \quad D = \begin{pmatrix} -0.0687 & 0.0601 & -0.1284 & 0.1029 & -0.2495 \\ -0.0786 & 0.0687 & -0.1469 & 0.1178 & -0.2855 \\ 0.6588 & 0.7298 & 0.0270 & -0.0831 & -0.0104 \\ 0.0573 & 0.1939 & -0.1447 & 0.4446 & 0.0557 \\ -0.0323 & 0.0283 & 0.1101 & -0.7568 & -0.4717 \end{pmatrix}.$$

From these matrices we can obtain W and F by (6.10) and (5.14) respectively, i.e.,

$$W(w) = A + BE(w)(I - DE(w))^{-1}C$$

and

$$F(z) = (I - DE(z))^{-1}CS(z).$$

Remark 6.16
Note that similar to the results from Section 6.1, the functions W and F obtained through the Nevanlinna-Pick interpolation are not uniquely defined, not even for a fixed choice of points $w_1, \dots, w_n \in \mathbb{D}$, since the factorization $\mathcal{N}_1 = A_1^* A_1$ is not unique. A different choice of A_1 will result in a different matrix T and thus a different matrix polynomial F.

However, the block A is the same for any choice of A_1 and any choice of points $w_1, \dots, w_n$. The first equation in (5.13) yields

$$AS(0) = R(0)$$

for $z = 0$, since $E(0) = 0$ by definition. So, $A = R(0)(S(0))^{-1}$, meaning that A is uniquely defined and only depends on S and R. In our example,

$$\begin{aligned} R(0)(S(0))^{-1} &= \begin{pmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{\sqrt{6}}{24} & \frac{\sqrt{6}}{24} \\ -\frac{\sqrt{15}}{24} - \frac{1}{8} & -\frac{\sqrt{15}}{24} - \frac{1}{8} \end{pmatrix} \begin{pmatrix} 3 - \sqrt{15} & 0 \\ 0 & 3 - \sqrt{15} \end{pmatrix} \\ &= \begin{pmatrix} \frac{3}{4} - \frac{\sqrt{15}}{4} & \frac{3}{4} - \frac{\sqrt{15}}{4} \\ \frac{\sqrt{6}}{8} - \frac{\sqrt{10}}{8} & \frac{\sqrt{6}}{8} - \frac{\sqrt{10}}{8} \\ \frac{1}{4} & \frac{1}{4} \end{pmatrix} \approx \begin{pmatrix} -0.2182 & -0.2182 \\ -0.0891 & -0.0891 \\ 0.2500 & 0.2500 \end{pmatrix}, \end{aligned}$$

which is the same as above.

In their work, Ball and Trent only considered functions and operators on the polydisc, so they only chose points from $\mathbb{D}^d$ for the Nevanlinna-Pick interpolation. Our starting point here, however, is the Oblique Extension Principle and thus matrix-valued functions on the torus $\mathbb{T}^d$, which we later extended to the polydisc. Alternatively, we can perform the Nevanlinna-Pick interpolation in the "natural habitat" of R and S, i.e., on the torus $\mathbb{T}^d$.

In particular, in the univariate case considered here: Since S and R are defined on the torus $\mathbb{T}$, we can choose $w_1, \dots, w_n \in \mathbb{T}$, define U_j and V_j from Theorem 6.13 as $U_j := S(w_j)$ and $V_j := R(w_j)$ and then follow the steps in the sketch of the proof of Theorem 6.13 to solve the Nevanlinna-Pick interpolation problem for this choice of points.

On the torus, however, there is a problem that does not occur when interpolating in the disc: The blocks $N_{1,jk}$ are given by $K(w_j, w_k)$ from Lemma 6.5 only for $j \neq k$, since $K(w_j, w_k)$ has a singularity for $w_j = w_k \in \mathbb{T}$ (which it does not have for any $w_j, w_k \in \mathbb{D}$). This singularity is a consequence of the fact that, if we look at (6.8) for $d = 1$, i.e.,

$$U_j^* U_k - V_j^* V_k = (1 - \overline{w_j} w_k)\mathcal{N}_{1,jk},$$

the scalar factor on the right side of the equation is zero for $w_j = w_k \in \mathbb{T}$. While the above equation would be satisfied for any choice of $\mathcal{N}_{1,jj}$, $1 \leq j \leq n$, we cannot choose them completely freely since the complete matrix $\mathcal{N}_1$ must be hermitian positive semi-definite.

In practice, we have found that filling in the diagonal blocks using Lagrange interpolation is a good way to obtain a matrix $\mathcal{N}_1$ with the desired properties. So, to find $\mathcal{N}_{1,jj}$, we interpolate it from the non-diagonal blocks $\mathcal{N}_{1,jk}$ with $j \neq k$ in the same column (using the non-diagonal blocks in the respective row also works, of course).

However, this requires us to use an additional point for this first step. If the maximum degree of S and R is ν and therefore $K(w, z)$ has degree $\nu - 1$ in $\overline{w}$ and z, we would only need ν points for the Nevanlinna-Pick interpolation. However, we need ν non-diagonal blocks in each column (or row) to interpolate the respective diagonal block, so we need a total of $\nu + 1$ points. To avoid over-determination (and to simplify subsequent calculations), we discard one row and one column of blocks from $\mathcal{N}$ after filling in the diagonal blocks $\mathcal{N}_{1,jj}$ and use only the ν blocks needed for the Nevanlinna-Pick interpolation.

Example 6.17

Let us revisit Example 6.15, i.e., use the Nevanlinna-Pick interpolation to find the matrix T for the frame based on the piecewise linear B-spline N_2, but now with points on $\mathbb{T}$. Again, the matrix functions R, S, and K can be found in Example 6.9, the Hilbert spaces $\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3, \mathcal{K}_4$ and $\mathcal{X}$ from Theorem 6.13 are given by $\mathbb{C}^2$, and the Hilbert space $\mathcal{Y}$ is given by $\mathbb{C}^3$. As mentioned above, we pick five points instead of four. Let

$$w_1 = 1, \quad w_2 = -1, \quad w_3 = \mathrm{i}, \quad w_4 = -\mathrm{i} \quad \text{and} \quad w_5 = \frac{1}{\sqrt{2}}(1 + \mathrm{i})$$

on $\mathbb{T}$. The uniquely defined blocks of $\mathcal{N}_1$ are then given by

$$\mathcal{N}_{1,21} = \begin{pmatrix} 0.6455 & -0.8333 \\ -0.5000 & 0.6455 \end{pmatrix} = \mathcal{N}_{1,12}^*,$$

$$\mathcal{N}_{1,31} = \begin{pmatrix} 0.9682 - 0.1773\mathrm{i} & 0.0833 - 0.5833\mathrm{i} \\ -0.3227 - 0.3227\mathrm{i} & 1.2288 - 0.9167\mathrm{i} \end{pmatrix} = \mathcal{N}_{1,13}^*,$$

$$\mathcal{N}_{1,41} = \begin{pmatrix} 0.9682 + 0.1773\mathrm{i} & 0.0833 + 0.5833\mathrm{i} \\ -0.3227 + 0.3227\mathrm{i} & 1.2288 + 0.9167\mathrm{i} \end{pmatrix} = \mathcal{N}_{1,14}^*,$$

$$\mathcal{N}_{1,51} = \begin{pmatrix} 0.9845 + 0.0352\mathrm{i} & 0.2458 - 0.0446\mathrm{i} \\ -0.0582 - 0.2132\mathrm{i} & 1.9603 - 0.6137\mathrm{i} \end{pmatrix} = \mathcal{N}_{1,15}^*,$$

$$\mathcal{N}_{1,32} = \begin{pmatrix} 1.2288 + 0.9167\mathrm{i} & -0.3227 + 0.3227\mathrm{i} \\ 0.0833 + 0.5833\mathrm{i} & 0.9682 + 0.1773\mathrm{i} \end{pmatrix} = \mathcal{N}_{1,23}^*,$$

$$\mathcal{N}_{1,42} = \begin{pmatrix} 1.2288 - 0.9167\mathrm{i} & -0.3227 - 0.3227\mathrm{i} \\ 0.0833 - 0.5833\mathrm{i} & 0.9682 - 0.1773\mathrm{i} \end{pmatrix} = \mathcal{N}_{1,24}^*,$$

$$\mathcal{N}_{1,52} = \begin{pmatrix} 0.6640 + 0.4470\mathrm{i} & -0.5146 + 0.1404\mathrm{i} \\ -0.5791 + 0.5446\mathrm{i} & 0.7338 + 0.1830\mathrm{i} \end{pmatrix} = \mathcal{N}_{1,25}^*,$$

$$\mathcal{N}_{1,43} = \begin{pmatrix} 0.6455 - 0.1667\mathrm{i} & -0.6667 \\ -0.6667 & 0.6455 + 0.1667\mathrm{i} \end{pmatrix} = \mathcal{N}_{1,34}^*,$$

$$\mathcal{N}_{1,53} = \begin{pmatrix} 1.4135 + 0.5005\mathrm{i} & -0.1299 + 0.6780\mathrm{i} \\ 0.6039 + 0.2711\mathrm{i} & 1.8177 + 0.7693\mathrm{i} \end{pmatrix} = \mathcal{N}_{1,35}^*,$$

$$\text{and} \quad \mathcal{N}_{1,54} = \begin{pmatrix} 0.9263 - 0.0557\mathrm{i} & -0.3903 - 0.3181\mathrm{i} \\ -0.4171 - 0.4855\mathrm{i} & 0.7578 - 0.6930\mathrm{i} \end{pmatrix} = \mathcal{N}_{1,45}^*.$$

Using Lagrange interpolation to find the missing diagonal blocks yields

$$\mathcal{N}_{1,11} = \begin{pmatrix} 0.8545 & 0 \\ 0 & 2.1455 \end{pmatrix},$$

$$\mathcal{N}_{1,22} = \begin{pmatrix} 2.1455 & 0 \\ 0 & 0.8545 \end{pmatrix},$$

$$\mathcal{N}_{1,33} = \begin{pmatrix} 2.0727 & 0.5727 + 0.5000\mathrm{i} \\ 0.5727 - 0.5000\mathrm{i} & 2.0727 \end{pmatrix},$$

$$\mathcal{N}_{1,44} = \begin{pmatrix} 2.0727 & 0.5727 - 0.5000\mathrm{i} \\ 0.5727 + 0.5000\mathrm{i} & 2.0727 \end{pmatrix},$$

$$\text{and} \quad \mathcal{N}_{1,55} = \begin{pmatrix} 1.3299 & 0.2864 + 0.3536\mathrm{i} \\ 0.2864 - 0.3536\mathrm{i} & 2.2428 \end{pmatrix}.$$

The complete block matrix $\mathcal{N}_1$ is now a hermitian positive semi-definite $(10 \times 10)-$matrix of rank 5. By deleting the last row and column of blocks, we obtain a hermitian positive semi-definite $(8 \times 8)-$matrix of rank 5. We denote this matrix by $\tilde{\mathcal{N}}_1$ and write it as $\tilde{\mathcal{N}}_1 = A_1^* A_1$ with $A_1 = (A_{1,1}\,,\, A_{1,2}\,,\, A_{1,3}\,,\, A_{1,4})$ for

$$A_{1,1} = \begin{pmatrix} 0.9244 & 0 \\ 0 & 1.4648 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad A_{1,2} = \begin{pmatrix} 0.6983 & -0.5409 \\ -0.5689 & 0.4407 \\ 1.1551 & 0.5441 \\ 0 & 0.2678 \\ 0 & 0 \end{pmatrix},$$

$$A_{1,3} = \begin{pmatrix} 1.0474 + 0.1917\mathrm{i} & -0.3491 + 0.3491\mathrm{i} \\ 0.0569 + 0.3982\mathrm{i} & 0.8389 + 0.6258\mathrm{i} \\ 0.4587 - 0.7134\mathrm{i} & 0.6964 - 0.4078\mathrm{i} \\ -0.1150 - 0.0240\mathrm{i} & 0.1150 - 0.1579\mathrm{i} \\ 0.2097 & -0.1258 + 0.1678\mathrm{i} \end{pmatrix},$$

$$\text{and} \quad A_{1,4} = \begin{pmatrix} 1.0474 - 0.1917\mathrm{i} & -0.3491 - 0.3491\mathrm{i} \\ 0.0569 - 0.3982\mathrm{i} & 0.8389 - 0.6258\mathrm{i} \\ 0.4587 + 0.7134\mathrm{i} & 0.6964 + 0.4078\mathrm{i} \\ -0.1150 + 0.0240\mathrm{i} & 0.1150 + 0.1579\mathrm{i} \\ 0.1258 - 0.1678\mathrm{i} & -0.2097 \end{pmatrix}.$$

To keep this example from getting too long, we will not list the entire spanning sets of $\mathcal{E}_*$ and $\mathcal{E}$ again, but constructing them as in the sketch of the proof of Theorem 6.13 (as we did in Example 6.15) again allows us to find an isometry $\tilde{T} : \mathcal{E}_* \to \mathcal{E}$. It is given by the matrix $\tilde{T} = (\tilde{T}_1, \tilde{T}_2)$ with

$$\tilde{T}_1 = \begin{pmatrix} -0.2182 & -0.2182 & -0.2361 & 0.4904 \\ -0.0891 & -0.0891 & -0.0964 & -0.6360 \\ 0.2500 & 0.2500 & -0.8113 & 0.2203 \\ -0.8113 & 0.2704 & 0.1223 & 0.2384 \\ 0.0993 & -0.7820 & 0.1075 & 0.3107 \\ -0.4523 & -0.4227 & -0.4893 & -0.3726 \\ 0.0502 & -0.1086 & 0.0543 & -0.0958 \\ -0.0732 + 0.0366\mathrm{i} & 0.0732 - 0.0366\mathrm{i} & -0.0792 + 0.0396\mathrm{i} & 0.0645 - 0.0323\mathrm{i} \end{pmatrix}$$

and

$$\tilde{T}_2 = \begin{pmatrix} -0.2376 & 0.0138 & -0.6595 - 0.3298\mathrm{i} \\ 0.5515 & 0.0642 & -0.4552 - 0.2276\mathrm{i} \\ 0.3196 & 0.1500 & 0.1244 + 0.0622\mathrm{i} \\ 0.3457 & 0.162 & 0.1346 + 0.0673\mathrm{i} \\ 0.4696 & 0.0263 & 0.1965 + 0.0983\mathrm{i} \\ -0.3822 & -0.0517 & 0.2646 + 0.1323\mathrm{i} \\ -0.1361 & -0.0508 & -0.1247 - 0.0624\mathrm{i} \\ 0.1615 - 0.0808\mathrm{i} & -0.8676 + 0.4338\mathrm{i} & 0 \end{pmatrix}.$$

Since we are in the finite-dimensional case, we do not need to extend $\tilde{T}$ to a unitary matrix but choose $T := \tilde{T}$. Again, we can write T as a block matrix

$$T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

with $A \in \mathbb{C}^{3\times 2}$, $B \in \mathbb{C}^{3\times 5}$, $C \in \mathbb{C}^{5\times 2}$, and $D \in \mathbb{C}^{5\times 5}$. Note that although the blocks B, C, and D are different from their versions found in Example 6.15, the matrix A is the same, as expected from the comments in Remark 6.16.

This example concludes our discussion of the univariate case. In the next section, we will use the univariate results established here to prove a first multivariate result: By employing a Kronecker product approach, we can use the existence of realizations in the univariate case to construct realizations for the subset of multivariate cases where the scaling matrix M is diagonal.

7. Multivariate Realizations in the Case of Diagonal Scaling Matrices

Now, we come to our first multivariate result. A well-established and frequently utilized method in the theory and application of wavelets is the construction of multivariate wavelet frames for diagonal scaling matrices from univariate wavelet frames with the help of the Kronecker product. The exact method is briefly explained in the proof of Lemma 7.3. Although these so-called *separable* wavelet frames and their associated transforms are not optimal in terms of directional sensitivity, they are still widely used because of their simple construction and computational advantages. For example, the (unfortunately not widely adopted) JPEG 2000 image compression standard is based on separable wavelets.

Corollary 6.8 now allows us to use this method to transfer the existence of realizations from the univariate to the separable multivariate case. Just like the multivariate wavelet symbols, the multivariate matrix-valued functions $F_j : \mathbb{D}^d \to \mathbb{C}^{n_j \times m}$, $1 \le j \le d$, which we will construct, are composed of univariate matrix-valued functions using the Kronecker product. Note that these constructions work for any $d \ge 1$, while the multivariate results in Section 8 will only concern the case $d = 2$.

As in Section 6, we will first present a result for the more general setting of Lemma 6.5 and Theorem 6.6, which is slightly detached from the Oblique Extension Principle, and then consider the Oblique Extension Principle scenario as a corollary.

Theorem 7.1
Let $S_j \in \mathbb{C}[z]^{m_j \times m_j}$ and $R_j \in \mathbb{C}[z]^{n_{0,j} \times m_j}$, $1 \le j \le d$, be univariate matrix polynomials such that

$$S_j(z)^* S_j(z) - R_j(z)^* R_j(z) = 0$$

for every $z \in \mathbb{T}$ and every $1 \le j \le d$. Furthermore, let each S_j, $1 \le j \le d$, have a rational and analytic inverse S_j^{-1} on $\mathbb{D}$ and define the multivariate matrix polynomials $S \in \mathbb{C}[z_1, \ldots, z_d]^{m \times m}$ and $R \in \mathbb{C}[z_1, \ldots, z_d]^{n_0 \times m}$ with

$$m := \prod_{j=1}^{d} m_j \qquad \text{and} \qquad n_0 := \prod_{j=1}^{d} n_{0,j}$$

as $S(z) := S_1(z_1) \otimes \ldots \otimes S_d(z_d)$ and $R(z) := R_1(z_1) \otimes \ldots \otimes R_d(z_d)$.

Then there exist matrix polynomials $F_j \in \mathbb{C}[z_1, \ldots, z_d]^{n_j \times m}$, $1 \le j \le d$, such that

$$S(w)^* S(z) - R(w)^* R(z) = \sum_{j=1}^{d} (1 - \overline{w_j} z_j) F_j(w)^* F_j(z)$$

for $w, z \in \mathbb{D}^d$.

Proof. We use the definitions of R and S and the usual properties of the Kronecker product to rewrite the matrix polynomial $S(w)^*S(z) - R(w)^*R(z)$, constructing the desired realization in the process. To avoid notational confusion, we will refer to the matrix-valued functions from the univariate realizations with $\mathcal{F}_j$, $1 \le j \le d$, and their multivariate counterparts with F_j, $1 \le j \le d$, for the remainder of this proof.

We have

$$\begin{aligned}
&S(w)^*S(z) - R(w)^*R(z)\\
&= [S_1(w_1) \otimes \ldots \otimes S_d(w_d)]^*[S_1(z_1) \otimes \ldots \otimes S_d(z_d)]\\
&\qquad - [R_1(w_1) \otimes \ldots \otimes R_d(w_d)]^*[R_1(z_1) \otimes \ldots \otimes R_d(z_d)]\\
&= [S_1(w_1)^*S_1(z_1)] \otimes \ldots \otimes [S_d(w_d)^*S_d(z_d)] - [R_1(w_1)^*R_1(z_1)] \otimes \ldots \otimes [R_d(w_d)^*R_d(z_d)]\\
&= [S_1(w_1)^*S_1(z_1)] \otimes \ldots \otimes [S_d(w_d)^*S_d(z_d)] - [R_1(w_1)^*R_1(z_1)] \otimes \ldots \otimes [R_d(w_d)^*R_d(z_d)]\\
&\qquad - [R_1(w_1)^*R_1(z_1)] \otimes \ldots \otimes [R_{d-1}(w_{d-1})^*R_{d-1}(z_{d-1})] \otimes [S_d(w_d)^*S_d(z_d)]\\
&\qquad + [R_1(w_1)^*R_1(z_1)] \otimes \ldots \otimes [R_{d-1}(w_{d-1})^*R_{d-1}(z_{d-1})] \otimes [S_d(w_d)^*S_d(z_d)]\\
&= \Big([S_1(w_1)^*S_1(z_1)] \otimes \ldots \otimes [S_{d-1}(w_{d-1})^*S_{d-1}(z_{d-1})]\\
&\qquad\qquad - [R_1(w_1)^*R_1(z_1)] \otimes \ldots \otimes [R_{d-1}(w_{d-1})^*R_{d-1}(z_{d-1})]\Big) \otimes [S_d(w_d)^*S_d(z_d)]\\
&\qquad + [R_1(w_1)^*R_1(z_1)] \otimes \ldots \otimes [R_{d-1}(w_{d-1})^*R_{d-1}(z_{d-1})]\\
&\qquad\qquad \otimes [S_d(w_d)^*S_d(z_d) - R_d(w_d)^*R_d(z_d)].
\end{aligned}$$

Now, according to Theorem 6.6, there exists a matrix polynomial $\mathcal{F}_d \in \mathbb{C}[z_d]^{\tilde{n}_d \times m_d}$ such that

$$S_d(w_d)^*S_d(z_d) - R_d(w_d)^*R_d(z_d) = (1 - \overline{w_d}z_d)\mathcal{F}_d(w_d)^*\mathcal{F}_d(z_d)$$

for $w_d, z_d \in \mathbb{D}$. Plugging this into the equation above, we get

$$\begin{aligned}
&S(w)^*S(z) - R(w)^*R(z)\\
&= \Big([S_1(w_1)^*S_1(z_1)] \otimes \ldots \otimes [S_{d-1}(w_{d-1})^*S_{d-1}(z_{d-1})]\\
&\qquad\qquad - [R_1(w_1)^*R_1(z_1)] \otimes \ldots \otimes [R_{d-1}(w_{d-1})^*R_{d-1}(z_{d-1})]\Big) \otimes [S_d(w_d)^*S_d(z_d)]\\
&\qquad + [R_1(w_1)^*R_1(z_1)] \otimes \ldots \otimes [R_{d-1}(w_{d-1})^*R_{d-1}(z_{d-1})] \otimes [(1 - \overline{w_d}z_d)\mathcal{F}_d(w_d)^*\mathcal{F}_d(z_d)]\\
&= \Big([S_1(w_1)^*S_1(z_1)] \otimes \ldots \otimes [S_{d-1}(w_{d-1})^*S_{d-1}(z_{d-1})]\\
&\qquad\qquad - [R_1(w_1)^*R_1(z_1)] \otimes \ldots \otimes [R_{d-1}(w_{d-1})^*R_{d-1}(z_{d-1})]\Big) \otimes [S_d(w_d)^*S_d(z_d)]\\
&\quad + (1 - \overline{w_d}z_d)\Big([R_1(w_1)^*R_1(z_1)] \otimes \ldots \otimes [R_{d-1}(w_{d-1})^*R_{d-1}(z_{d-1})] \otimes [\mathcal{F}_d(w_d)^*\mathcal{F}_d(z_d)]\Big).
\end{aligned}$$

With the same calculation, we can again use Theorem 6.6 to get a matrix polynomial $\mathcal{F}_{d-1} \in \mathbb{C}[z_{d-1}]^{\tilde{n}_{d-1} \times m_{d-1}}$ such that

$$\begin{aligned}
&[S_1(w_1)^* S_1(z_1)] \otimes \ldots \otimes [S_{d-1}(w_{d-1})^* S_{d-1}(z_{d-1})] \\
&\qquad - [R_1(w_1)^* R_1(z_1)] \otimes \ldots \otimes [R_{d-1}(w_{d-1})^* R_{d-1}(z_{d-1})] \\
&= \Big([S_1(w_1)^* S_1(z_1)] \otimes \ldots \otimes [S_{d-2}(w_{d-2})^* S_{d-2}(z_{d-2})] \\
&\qquad - [R_1(w_1)^* R_1(z_1)] \otimes \ldots \otimes [R_{d-2}(w_{d-2})^* R_{d-2}(z_{d-2})] \Big) \otimes [S_{d-1}(w_{d-1})^* S_{d-1}(z_{d-1})] \\
&\quad + (1 - \overline{w_{d-1}} z_{d-1}) \Big([R_1(w_1)^* R_1(z_1)] \otimes \ldots \\
&\qquad \otimes [R_{d-2}(w_{d-2})^* R_{d-2}(z_{d-2})] \otimes [\mathcal{F}_{d-1}(w_{d-1})^* \mathcal{F}_{d-1}(z_{d-1})] \Big).
\end{aligned}$$

Repeating this, we get additional matrix polynomials $\mathcal{F}_j \in \mathbb{C}[z_j]^{\tilde{n}_j \times m_j}$, $1 \leq j \leq d-2$, such that

$$\begin{aligned}
&S(w)^* S(z) - R(w)^* R(z) \\
&= (1 - \overline{w_1} z_1) \Big([\mathcal{F}_1(w_1)^* \mathcal{F}_1(z_1)] \otimes [S_2(w_2)^* S_2(z_2)] \otimes \ldots \otimes [S_d(w_d)^* S_d(z_d)] \Big) \\
&\quad + (1 - \overline{w_2} z_2) \Big([R_1(w_1)^* R_1(z_1)] \otimes [\mathcal{F}_2(w_2)^* \mathcal{F}_2(z_2)] \\
&\qquad \otimes [S_3(w_3)^* S_3(z_3)] \otimes \ldots \otimes [S_d(w_d)^* S_d(z_d)] \Big) \\
&\quad \ldots \\
&\quad + (1 - \overline{w_d} z_d) \Big([R_1(w_1)^* R_1(z_1)] \otimes \ldots \\
&\qquad \otimes [R_{d-1}(w_{d-1})^* R_{d-1}(z_{d-1})] \otimes [\mathcal{F}_d(w_d)^* \mathcal{F}_d(z_d)] \Big).
\end{aligned}$$

So, overall, we have

$$S(w)^* S(z) - R(w)^* R(z) = \sum_{j=1}^{d} (1 - \overline{w_j} z_j) F_j(w)^* F_j(z)$$

for all $w, z \in \mathbb{D}^d$, with the matrix polynomials $F_1, \ldots, F_d$ given by

$$\begin{aligned}
F_1(z) &:= \mathcal{F}_1(z_1) \otimes S_2(z_2) \otimes \ldots \otimes S_d(z_d), \\
F_2(z) &:= R_1(z_1) \otimes \mathcal{F}_2(z_2) \otimes S_3(z_3) \otimes \ldots \otimes S_d(z_d), \\
&\vdots \\
\text{and} \quad F_d(z) &:= R_1(z_1) \otimes \ldots \otimes R_{d-1}(z_{d-1}) \otimes \mathcal{F}_d(z_d).
\end{aligned}$$

This proves the theorem. $\square$

As mentioned above, we can use this result to find realizations for so-called separable multivariate wavelet frames.

Definition 7.2
A wavelet frame $X(\Psi)$ with a set $\Psi := \{\psi_1, \dots, \psi_r\} \subset L^2(\mathbb{R}^d)$ of mother-wavelets is called *separable*, if we can write each $\psi_j \in \Psi$, $1 \leq j \leq r$, as

$$\psi_j(x) = \prod_{\ell=1}^{d} \psi_{j,\ell}(x_\ell)$$

with $\psi_{j,\ell} \in L^2(\mathbb{R})$, $1 \leq \ell \leq d$.

As mentioned above, it is widely known (see e.g., the proof of [54], Theorem 1.2) that for any diagonal dilation matrix, i.e., $M = \operatorname{diag}(m_1, \dots, m_d) \in \mathbb{Z}^{d \times d}$ with $m_j > 1$ for all $1 \leq j \leq d$, we can find a separable tight wavelet frame by choosing a univariate frame for each dilation factor m_j, $1 \leq j \leq d$, and combining them into a multivariate frame using the Kronecker product.

If these univariate frames are based on the univariate Oblique Extension Principle, this can be done so that the multivariate frame satisfies the multivariate Oblique Extension Principle. We give a brief sketch of this construction method as proof of the following lemma.

Lemma 7.3
For every dilation matrix $M = \operatorname{diag}(m_1, \dots, m_d) \in \mathbb{Z}^{d \times d}$ there exist a refinement symbol p, wavelet symbols $q_1, \dots, q_r$, and a vanishing moment recovery function s such that the Oblique Extension Principle identities hold.

Proof. For each diagonal entry $m_j, 1 \leq j \leq d$, of M, choose a univariate refinement symbol $p_j \in \mathbb{C}[z]$, wavelet symbols $q_{j,1}, \dots, q_{j,m_j} \in \mathbb{C}[z]$, and a vanishing moment recovery function $s_j \in \mathbb{C}[z^{\pm 1}]$, such that the (univariate) Oblique Extension Principle holds for the scaling factor m_j. In particular, we again assume that all symbols are polynomial (see Remark 6.1). For example, we can base these frames on B-splines. Since s_j is a univariate Laurent polynomial, the Fejér-Riesz theorem 4.1 guarantees the existence of a polynomial $\theta_j \in \mathbb{C}[z]$ that is zero-free on $\mathbb{D}$ and satisfies $s_j = \theta_j^* \theta_j$.

For the transition to a multivariate frame for the dilation matrix M, set the d-variate scaling symbol $p \in \mathbb{C}[z_1, \dots, z_d]$ as

$$p(z) = \prod_{j=1}^{d} p_j(z_j)$$

for $z = (z_1, \dots, z_d) \in \mathbb{T}^d$ and analogously the d-variate vanishing moment recovery function $s \in \mathbb{C}[z_1^{\pm 1}, \dots, z_d^{\pm 1}]$ as

$$s(z) = \prod_{j=1}^{d} s_j(z_j).$$

Then, $s = \theta^*\theta$ with $\theta \in \mathbb{C}[z_1, \dots, z_d]$ given by

$$\theta(z) = \prod_{j=1}^{d} \theta_j(z_j).$$

Note that θ is zero-free on $\mathbb{D}^d$ due to its construction. Similarly, we define the wavelet symbols $q_1, \dots, q_r \in \mathbb{C}[z_1, \dots, z_d]$ by

$$\{q_1, \dots, q_r\} := \left\{ \prod_{j=1}^{d} b_{j,k_j}(z_j) \ : \ k_j \in \{0, \dots, m_j\} \right\} \setminus \left\{ \prod_{j=1}^{d} b_{j,0}(z_j) \right\} \tag{7.1}$$

with $b_{j,0}(z_j) = \theta_j(z_j^{m_j})p_j(z_j)$ and $b_{j,\ell}(z_j) = q_{j,\ell}(z_j)$ for $1 \leq \ell \leq m_j$. Note, that

$$\Gamma_M = \left(M^T\right)^{-1}\mathbb{Z}^d/\mathbb{Z}^d = \Gamma_{m_1} \times \ldots \times \Gamma_{m_d}.$$

A simple calculation then shows that these symbols satisfy the Oblique Extension Principle for the dilation matrix M and the vanishing moment recovery function s. □

In the setting of the Oblique Extension Principle, the combination of this construction of multivariate tight wavelet frames and Theorem 7.1 yields the following result.

Corollary 7.4
For every separable tight wavelet frame constructed in the way described in the proof of Lemma 7.3, there exist matrix polynomials $F_j : \mathbb{D}^d \to \mathbb{C}^{n_j \times m}$, $1 \leq j \leq d$, such that

$$S(w)^*S(z) - R(w)^*R(z) = \sum_{j=1}^{d} (1 - \overline{w_j} z_j) F_j(w)^* F_j(z)$$

for all $w, z \in \mathbb{D}^d$.

Proof. Since the multivariate symbols and vanishing moment recovery function constructed in Lemma 7.3 satisfy the Oblique Extension Principle, we have

$$S(z)^*S(z) - R(z)^*R(z) = 0$$

for all $z \in \mathbb{T}^d$ with S and R as in (4.4).

In the process of constructing these symbols, we have chosen univariate symbols and univariate vanishing moment recovery functions for each diagonal entry m_j, $1 \leq j \leq d$, of M, such that the univariate Oblique Extension Principle holds. Now, take the respective matrix forms of the univariate Oblique Extension Principle identities, i.e.,

$$S_j(z_j)^* S_j(z_j) - R_j(z_j)^* R_j(z_j) = 0$$

for $z_j \in \mathbb{T}$ with matrix polynomials S_j and R_j as in (4.4).

If we arrange the multivariate masks $q_1, \ldots, q_r$ in the right order, we can write S and R as

$$S(z) = S_1(z_1) \otimes \ldots \otimes S_d(z_d)$$

and

$$R(z) = R_1(z_1) \otimes \ldots \otimes R_d(z_d).$$

The corollary now follows directly from Theorem 7.1. □

We will demonstrate this construction method in an example, again using the already familiar B-spline wavelets.

Example 7.5
We consider the bivariate case, i.e., $d = 2$, and the dilation matrix $M = 2I_2$. We choose the piecewise linear B-spline N_2 for both directions, so both p_1 and p_2 are given by the refinement symbol p from Example 6.9. Using these symbols, we have

$$p(z_1, z_2) := p_1(z_1)\, p_2(z_2) = \frac{(z_1+1)^2\,(z_2+1)^2}{16}.$$

Using the univariate wavelet symbols from Example 6.9, we use (7.1) to get the bivariate wavelet symbols

$$\begin{aligned}
q_1(z_1, z_2) &:= -\left(\frac{\sqrt{15}}{96} - \frac{1}{32}\right) (z_1+1)^2\,(z_2-1)^2 \left(-{z_1}^2 + \sqrt{15} + 4\right),\\
q_2(z_1, z_2) &:= (z_1+1)^2\,(z_2-1)^2 \left(\frac{\sqrt{6}}{192} - \frac{\sqrt{10}}{192}\right) \left({z_2}^2 + 4\,z_2 + 1\right) \left(-{z_1}^2 + \sqrt{15} + 4\right),\\
q_3(z_1, z_2) &:= -\left(\frac{\sqrt{15}}{96} - \frac{1}{32}\right) (z_1-1)^2\,(z_2+1)^2 \left(-{z_2}^2 + \sqrt{15} + 4\right),\\
q_4(z_1, z_2) &:= \frac{(z_1-1)^2\,(z_2-1)^2}{16},\\
q_5(z_1, z_2) &:= \frac{\sqrt{6}\,(z_1-1)^2\,(z_2-1)^2 \left({z_2}^2 + 4\,z_2 + 1\right)}{96},\\
q_6(z_1, z_2) &:= (z_1-1)^2\,(z_2+1)^2 \left(\frac{\sqrt{6}}{192} - \frac{\sqrt{10}}{192}\right) \left({z_1}^2 + 4\,z_1 + 1\right) \left(-{z_2}^2 + \sqrt{15} + 4\right),\\
q_7(z_1, z_2) &:= \frac{\sqrt{6}\,(z_1-1)^2\,(z_2-1)^2 \left({z_1}^2 + 4\,z_1 + 1\right)}{96},\\
q_8(z_1, z_2) &:= \frac{(z_1-1)^2\,(z_2-1)^2 \left({z_1}^2 + 4\,z_1 + 1\right) \left({z_2}^2 + 4\,z_2 + 1\right)}{96},
\end{aligned}$$

and using the univariate vanishing moment recovery function from Example 6.9, we get the bivariate vanishing moment recovery function

$$s(z_1, z_2) := s(z_1)\, s(z_2) = \left(\frac{z_1}{6} + \frac{1}{6\,z_1} - \frac{4}{3}\right) \left(\frac{z_2}{6} + \frac{1}{6\,z_2} - \frac{4}{3}\right)$$

with $s(z_1,z_2) = |\theta(z_1,z_2)|^2$ on $\mathbb{T}^2$ for

$$\theta(z_1,z_2) := -\left(\frac{\sqrt{15}}{6} - \frac{2}{3}\right)\left(z_1 - 4 - \sqrt{15}\right)\left(z_2 - 4 - \sqrt{15}\right).$$

The matrix function S is then given by

$$S(z_1,z_2) := \operatorname{diag}(\theta(z_1,z_2), \theta(z_1,-z_2), \theta(-z_1,z_2), \theta(-z_1,-z_2))$$

(recall that $\Gamma_M = \Gamma_2 \times \Gamma_2 = \{0,1/2\} \times \{0,1/2\}$), while the matrix function R has the first column

$$\frac{1}{192}\begin{pmatrix} -2\,(z_1+1)^2\,(z_2-1)^2\,\left(\sqrt{15}-(\sqrt{15}+3)\,{z_1}^2+3\right) \\ -(z_1+1)^2\,(z_2-1)^2\,\left({z_2}^2+4\,z_2+1\right)\left(\sqrt{6}+\sqrt{10}+(\sqrt{6}-\sqrt{10})\,{z_1}^2\right) \\ -2\,(z_1-1)^2\,(z_2+1)^2\,\left(\sqrt{15}+(3-\sqrt{15})\,{z_2}^2+3\right) \\ 12\,(z_1-1)^2\,(z_2-1)^2 \\ 2\sqrt{6}\,(z_1-1)^2\,(z_2-1)^2\,\left({z_2}^2+4\,z_2+1\right) \\ -(z_1-1)^2\,(z_2+1)^2\,\left({z_1}^2+4\,z_1+1\right)\left(\sqrt{6}+\sqrt{10}+(\sqrt{6}-\sqrt{10})\,{z_2}^2\right) \\ 2\sqrt{6}\,(z_1-1)^2\,(z_2-1)^2\,\left({z_1}^2+4\,z_1+1\right) \\ 2\,(z_1-1)^2\,(z_2-1)^2\,\left({z_1}^2+4\,z_1+1\right)\left({z_2}^2+4\,z_2+1\right) \\ 2\,(z_1+1)^2\,(z_2+1)^2\,\left(4\,{z_1}^2\,{z_2}^2+\sqrt{15}-{z_1}^2-{z_2}^2-\sqrt{15}\,{z_1}^2\,{z_2}^2+4\right) \end{pmatrix}$$

and the second, third and fourth columns can be obtained by substituting (z_1,z_2) with $(z_1,-z_2)$, $(-z_1,z_2)$ and $(-z_1,-z_2)$ respectively (note the way S and R are defined in Section 4.4).

Using the univariate results from Example 6.9, we can now construct the matrix polynomials F_1 and F_2 using the formulae from the proof of Theorem 7.1. For better readability, we give the coefficients in numerical form. The first column of F_1 is given by

$$\begin{pmatrix} -0.00968332(z_1^2+3.16399z_1-16.1633)(z_2-7.873) \\ 0 \\ -0.0110793(z_1^2+1.11089)(z_2-7.873) \\ 0 \\ -0.0146271(z_1^3+1.95772z_1^2-7.08425z_1)(z_2-7.873) \\ 0 \\ -0.00160684(z_1^3+5.5z_1^2)(z_2-7.873) \\ 0 \\ 0.0056578(z_1^3+2z_1^2)(z_2-7.873) \\ 0 \end{pmatrix}$$

and the second column can be obtained by substituting (z_1,z_2) in the first column with $(z_1,-z_2)$ and shifting the nonzero entries down by one row.

The third column is given by

$$\begin{pmatrix} -0.00968332(z_1^2 - 0.836023z_1 + 2.16402)(z_2 - 7.873) \\ 0 \\ -0.0110795(z_1^2 - 2.88911z_1 - 14.0004)(z_2 - 7.873) \\ 0 \\ -0.0146266(z_1^3 - 2.04223z_1^2 - 6.91556z_1)(z_2 - 7.873) \\ 0 \\ -0.0016069(z_1^3 + 1.5z_1^2 - 13.9995z_1)(z_2 - 7.873) \\ 0 \\ 0.0056578(z_1^3 - 2z_1^2)(z_2 - 7.873) \\ 0 \end{pmatrix},$$

and the fourth column can be obtained by substituting (z_1, z_2) in the third column with $(z_1, -z_2)$ and shifting the nonzero entries down by one row.

The first column of F_2 is given by

$$\begin{pmatrix} -0.016638(z_1 - 1)^2(z_2^2 + 3.16399z_2 - 16.1633) \\ -0.0190368(z_1 - 1)^2(z_2^2 + 1.11089z_2) \\ -0.0251325(z_1 - 1)^2(z_2^3 + 1.95772z_2^2 - 7.08425z_2) \\ -0.002761(z_1 - 1)^2(z_2^3 + 5.5z_2^2) \\ 0.0097214(z_1 - 1)^2(z_2^3 + 2z_2^2) \\ -0.0067923(z_1 - 1)^2(z_1^2 + 4z_1 + 1)(z_2^2 + 3.16399z_2 - 16.1633) \\ -0.00777191(z_1 - 1)^2(z_1^2 + 4z_1 + 1)(z_2^2 + 1.11089z_2) \\ -0.0102601(z_1 - 1)^2(z_1^2 + 4z_1 + 1)(z_2^3 + 1.95772z_2^2 - 7.08425z_2) \\ -0.00112716(z_1 - 1)^2(z_1^2 + z_1 + 1)(z_2^3 + 5.5z_2^2) \\ 0.0039688(z_1 - 1)^2(z_1^2 + 4z_1 + 1)(z_2^3 + 2z_2^2) \\ -0.00242076(z_1 + 1)^2(z_1^2 - 7.873)(z_2^2 + 3.16399z_2 - 16.1633) \\ -0.00276978(z_1 + 1)^2(z_1^2 - 7.873)(z_2^2 + 1.11089z_2) \\ -0.00365668(z_1 + 1)^2(z_1^2 - 7.873)(z_2^3 + 1.95772z_2^2 - 7.08425z_2) \\ -0.0004017(z_1 + 1)^2(z_1^2 - 7.873)(z_2^3 + 5.5z_2^2) \\ 0.0014144(z_1 + 1)^2(z_1^2 - 7.873)(z_2^3 + 2z_2^2) \end{pmatrix},$$

the second column is given by

$$\begin{pmatrix}
-0.016638(z_1-1)^2(z_2^2-0.836023z_2+2.16402)\\
-0.019037(z_1-1)^2(z_2^2-2.88911z_2-14.0004)\\
-0.0251319(z_1-1)^2(z_2^3-2.04223z_2^2-6.91556z_2)\\
-0.002761(z_1-1)^2(z_2^3+1.5z_2^2-13.9995z_2)\\
0.0097214(z_1-1)^2(z_2^3-2z_2^2)\\
-0.0067923(z_1-1)^2(z_1^2+4z_1+1)(z_2^2-0.836023z_2+2.16402)\\
-0.00777166(z_1-1)^2(z_1^2+4z_1+1)(z_2^2-2.88911z_2-14.0004)\\
-0.0102601(z_1-1)^2(z_1^2+4z_1+1)(z_2^3-2.04223z_2^2-6.91556z_2)\\
-0.00112715(z_1-1)^2(z_1^2+4z_1+1)(z_2^3+1.5z_2^2-13.9995z_2)\\
0.0039688(z_1-1)^2(z_1^2+4z_1+1)(z_2^3-2z_2^2)\\
-0.00242076(z_1+1)^2(z_1^2-7.873)(z_2^2-0.836023z_2+2.16402)\\
-0.00276981(z_1+1)^2(z_1^2-7.873)(z_2^2-2.88911z_2-14.0004)\\
-0.00365663(z_1+1)^2(z_1^2-7.873)(z_2^3-2.04223z_2^2-6.91556z_2)\\
-0.000401714(z_1+1)^2(z_1^2-7.873)(z_2^3+1.5z_2^2-13.9995z_2)\\
0.0014144(z_1+1)^2(z_1^2-7.873)(z_2^3-2z_2^2)
\end{pmatrix}$$

and the third and fourth columns can be obtained by substituting (z_1, z_2) with $(-z_1, z_2)$ in the first and second columns, respectively.

In the next section, we will return to tight wavelet frames for general, i.e., not necessarily diagonal, scaling matrices. However, we will restrict ourselves not only to the bivariate case but in particular to the bivariate case where the vanishing moment recovery function has a single-square representation $s(z_1, z_2) = |\theta(z_1, z_2)|^2$ according to Theorem 4.6.

8. Realizations Associated With Frames Based on the Oblique Extension Principle - The Bivariate Case

In our final main section, we will discuss the construction of realizations associated with tight wavelet frames based on the Oblique Extension Principle in the bivariate (not necessarily separable) case. One of the most significant differences to the univariate case is that the vanishing moment recovery function s now generally only has a sum-of-squares decomposition, i.e., there are polynomials $\theta_1, \dots, \theta_N \in \mathbb{C}[z_1, z_2]$ such that

$$s(z) = \sum_{n=1}^{N} |\theta_n(z)|^2$$

with $N > 1$ (see Definition 4.5 and the comments and citations below). So, in general, the matrix

$$S(z) = \begin{pmatrix} \theta_1^{\sigma_1}(z) & \dots & \theta_n^{\sigma_1}(z) & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\ 0 & \dots & 0 & \theta_1^{\sigma_2}(z) & \dots & \theta_n^{\sigma_2}(z) & \dots & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & \dots & 0 & \dots & \theta_1^{\sigma_m}(z) & \dots & \theta_n^{\sigma_m}(z) \end{pmatrix}^T$$

is not square as in the univariate case. However, this leads to some problems we will discuss in Section 9. We will therefore restrict ourselves to vanishing moment recovery functions $s \in \mathbb{C}[z_1^{\pm 1}, z_2^{\pm 1}]$ with a single-square decomposition $s = |\theta|^2$ for some polynomial $\theta \in \mathbb{C}[z_1, z_2]$, which means that

$$S(z) = \operatorname{diag}(\theta^{\sigma_1}(z), \dots, \theta^{\sigma_m}(z))$$

is - as in the univariate case - a square matrix. For a necessary and sufficient condition for the existence of such bivariate vanishing moment recovery functions, see Theorem 4.6.

In Section 8.1, we will bring the problem back down to a parameterized one-variable version to construct matrix polynomials F_1 and F_2 with

$$S(w)^* S(z) - R(w)^* R(z) = (1 - \overline{w_1} z_1) F_1(w)^* F_1(z) + (1 - \overline{w_2} z_2) F_2(w)^* F_2(z)$$

for $w, z \in \mathbb{D}^2$, which again also gives us an isometric matrix T with blocks A, B, C, and D satisfying

$$R(z) = \left[A + BE(z)(I - DE(z))^{-1} C\right] S(z)$$

for $z \in \mathbb{D}^2$ (cf. Theorem 5.3).

Similar to Section 6, we will conclude our consideration of the bivariate case in Section 8.2 by discussing the construction of the matrix-valued functions F_1 and F_2 with the help of interpolation.

8.1. Existence of Realizations Associated With Frames Based on the Bivariate Oblique Extension Principle

Similar to the univariate case, we will assume that the refinement symbol p and the wavelet symbols $q_1, \dots, q_r$ are polynomials in $\mathbb{C}[z_1, z_2]$ (cf. Remark 6.1), and s is a Laurent polynomial in $\mathbb{C}[z_1^{\pm 1}, z_2^{\pm 1}]$. As mentioned in the introduction of this section, we will also assume that there is a bivariate polynomial $\theta \in \mathbb{C}[z_1, z_2]$ such that

$$s(z_1, z_2) = |\theta(z_1, z_2)|^2$$

with $\theta(z) \neq 0$ for $z = (z_1, z_2) \in \overline{\mathbb{D}^2}$ (cf. Theorem 4.6).

Our objective here is essentially the same as in the univariate case: We again assume that the Oblique Extension Principle holds for $p, q_1, \dots q_r$ and s, i.e.,

$$S(z)^* S(z) - R(z)^* R(z) = 0$$

holds for all $z \in \mathbb{T}^2$ and the matrix polynomials S and R defined in Section 4.4; and we want to find matrix-valued functions $F_1 : \mathbb{C}^2 \to \mathbb{C}^{n_1 \times m}$ and $F_2 : \mathbb{C}^2 \to \mathbb{C}^{n_2 \times m}$ such that

$$S(w)^* S(z) - R(w)^* R(z) = (1 - \overline{w_1} z_1) F_1(w)^* F_1(z) + (1 - \overline{w_2} z_2) F_2(w)^* F_2(z)$$

holds for all $z, w \in \mathbb{D}^2$. As before, we want to choose these functions as matrix polynomials.

We directly formulate one of the main results of this chapter, which can be seen as a bivariate variant of Theorem 6.6. It is a generalization of a well-known result by Anton Kummert (s. [70], cf. also the version by Knese in [67]). Again, we will first formulate a more general result and later a corollary for the setting of the Oblique Extension Principle.

Theorem 8.1
Let $S \in \mathbb{C}[z_1, z_2]^{m \times m}$ and $R \in \mathbb{C}[z_1, z_2]^{n_0 \times m}$, $n_0 \geq m$, be two matrix polynomials such that

$$S(z)^* S(z) - R(z)^* R(z) = 0$$

holds for all $z \in \mathbb{T}^2$. Furthermore, let S have a rational and analytic inverse S^{-1} on $\mathbb{D}^2$.

Then there exist matrix-valued functions $F_1 : \mathbb{D}^2 \to \mathbb{C}^{n_1 \times m}$ and $F_2 : \mathbb{D}^2 \to \mathbb{C}^{n_2 \times m}$ with rational entries such that

$$S(w)^* S(z) - R(w)^* R(z) = \sum_{j=1}^{2} (1 - \overline{w_j} z_j) F_j(w)^* F_j(z) \tag{8.1}$$

holds for all $z, w \in \mathbb{D}^2$.

Proof. The method of this proof largely follows Knese's proof for the case $S = I_m$ in [67]. The first step of the proof brings the problem back to a single variable. To do this, we fix $z_2 = w_2 \in \mathbb{T}$. Analogous to our answer to the first question in Section 6.1, the matrix polynomial $S(w)^*S(z) - R(w)^*R(z)$ is then divisible by $(1 - \overline{w_1}z_1)$, so

$$K(w_1, z_1; z_2) = \frac{S(w)^*S(z) - R(w)^*R(z)}{1 - \overline{w_1}z_1}$$

is polynomial in $\overline{w_1}$ and z_1 (and Laurent polynomial in z_2). Similar to (6.5), this allows us to write K as

$$K(w_1, z_1; z_2) = \begin{pmatrix} I_m \\ w_1 I_m \\ \vdots \\ w_1^{\nu-1} I_m \end{pmatrix}^* L(z_2) \begin{pmatrix} I_m \\ z_1 I_m \\ \vdots \\ z_1^{\nu-1} I_m \end{pmatrix}.$$

By construction, the matrix L has the size $\nu m \times \nu m$, where ν is the maximum of the degrees of R and S in z_1, and its entries are Laurent polynomials in z_2. Analogous to our answer to the third question in Section 6.1, L is positive semi-definite, since by Lemma 6.5, $K(w_1, z_1; z_2)$ is a positive semi-definite kernel. We can thus use the matrix Fejér-Riesz theorem 4.3 to write it as

$$L(z_2) = A(z_2)^* A(z_2)$$

for some matrix polynomial $A \in \mathbb{C}[z_2]^{\tilde{n}_1 \times \nu m}$. So, by defining the univariate matrix polynomial $H \in \mathbb{C}[z_1]^{\tilde{n}_1 \times m}$ ($z_2 = w_2$ is still fixed) as

$$H(z_1; z_2) = A(z_2) \begin{pmatrix} I_m \\ z_1 I_m \\ \vdots \\ z_1^{\nu-1} I_m \end{pmatrix},$$

we have

$$K(w_1, z_1; z_2) = H(w_1; z_2)^* H(z_1; z_2)$$

for $w_1, z_1 \in \mathbb{D}$ and our fixed $z_2 \in \mathbb{T}$ or equivalently

$$S(w_1; z_2)^* S(z_1; z_2) - R(w_1; z_2)^* R(z_1; z_2) = (1 - \overline{w_1}z_1) H(w_1; z_2)^* H(z_1; z_2)$$

for $w_1, z_1 \in \mathbb{D}$ and our fixed $z_2 \in \mathbb{T}$. Thus, by Theorem 5.3 there exists an isometric matrix $\tilde{T}(z_2)$ such that

$$\tilde{T}(z_2) \begin{pmatrix} S(z_1; z_2) \\ z_1 H(z_1; z_2) \end{pmatrix} = \begin{pmatrix} R(z_1; z_2) \\ H(z_1; z_2) \end{pmatrix} \tag{8.2}$$

for $w_1, z_1 \in \mathbb{D}$ and our fixed $z_2 \in \mathbb{T}$. Note that $n_0 \geq m$, i.e., R has at least as many rows as S. This implies that choosing an isometric $\tilde{T}(z_2)$ is possible (see Remark 5.7).

In the second step of the proof, we return to the two-variable setting. To do this, we first extract the coefficient matrices of the powers of z_1, i.e., we write

$$R(z) = \sum_{j=0}^{\nu} R_j(z_2) z_1^j \qquad \text{and} \qquad S(z) = \sum_{j=0}^{\nu} S_j(z_2) z_1^j,$$

where ν is again given by the maximum of the degrees of R and S with respect to z_1, and $S_j \in \mathbb{C}[z_2]^{m\times m}$ and $R_j \in \mathbb{C}[z_2]^{n_0\times m}$, $1 \le j \le \nu$, are matrix polynomials with respect to z_2. Merging these matrices into $\tilde{S} \in \mathbb{C}[z_2]^{m\times(\nu+1)m}$ and $\tilde{R} \in \mathbb{C}[z_2]^{n_0\times(\nu+1)m}$ with

$$\tilde{S}(z_2) := (S_0(z_2), S_1(z_2), \ldots, S_\nu(z_2)) \quad \text{and} \quad \tilde{R}(z_2) := (R_0(z_2), R_1(z_2), \ldots, R_\nu(z_2))$$

allows us to write (8.2) as

$$\tilde{T}(z_2) \begin{pmatrix} \multicolumn{2}{c}{\tilde{S}(z_2)} \\ 0_{\tilde{n}_1\times m} & A(z_2) \end{pmatrix} = \begin{pmatrix} \multicolumn{2}{c}{\tilde{R}(z_2)} \\ A(z_2) & 0_{\tilde{n}_1\times m} \end{pmatrix} \tag{8.3}$$

for $z_2 \in \mathbb{T}$.

Now, according to our assumption, the matrix polynomial S has a rational and analytic inverse S^{-1} on $\mathbb{D}^2$. In particular, this implies that $S_0(z_2) = S(0, z_2)$ has a rational and analytic inverse for every $z_2 \in \mathbb{D}$. Furthermore, the matrix polynomial $A \in \mathbb{C}[z]^{\tilde{n}_1\times\nu m}$ has a rational and analytic right inverse $B \in \mathbb{C}(z)^{\nu m\times\tilde{n}_1}$ on $\mathbb{D}$ according to Remark 4.4. So, the $(m + \tilde{n}_1) \times (\nu + 1)m$ matrix on the left-hand side of (8.3) has the rational and analytic right inverse

$$\begin{pmatrix} S_0^{-1}(z_2) & -S_0^{-1}(z_2)(S_1(z_2), \ldots, S_\nu(z_2))B(z_2) \\ 0_{\nu m\times m} & B(z_2) \end{pmatrix}$$

of size $(\nu + 1)m \times (m + \tilde{n}_1)$ on $\mathbb{D}$. So,

$$\begin{pmatrix} \multicolumn{2}{c}{\tilde{S}(z_2)} \\ 0_{\tilde{n}_1\times m} & A(z_2) \end{pmatrix} \begin{pmatrix} S_0^{-1}(z_2) & -S_0^{-1}(z_2)(S_1(z_2), \ldots, S_\nu(z_2))B(z_2) \\ 0_{\nu m\times m} & B(z_2) \end{pmatrix} = I_{m+\tilde{n}_1}$$

for $z_2 \in \mathbb{D}$ and because of continuity also for $z_2 \in \mathbb{T}$, at least away from any singularities. Together with (8.3), this implies we can extend

$$\tilde{T}(z_2) = \begin{pmatrix} \multicolumn{2}{c}{\tilde{R}(z_2)} \\ A(z_2) & 0_{\tilde{n}_1\times m} \end{pmatrix} \begin{pmatrix} S_0^{-1}(z_2) & -S_0^{-1}(z_2)(S_1(z_2), \ldots, S_\nu(z_2))B(z_2) \\ 0_{\nu m\times m} & B(z_2) \end{pmatrix}$$

(note that even though the two matrices on the right are written as block matrices, the sizes of the respective blocks in the two matrices do not match) to a matrix-valued rational function in $\mathbb{C}(z)^{(n_0+\tilde{n}_1)\times(m+\tilde{n}_1)}$, that is analytic on $\mathbb{D}$ (which in turn implies we can extend (8.3) to $\mathbb{D}$).

Since $\tilde{T}$ is also isometric on $\mathbb{T}$, we have

$$I_{m+\tilde{n}_1} - \tilde{T}(z_2)^*\tilde{T}(z_2) = 0 \tag{8.4}$$

for $z_2 \in \mathbb{T}$. Now, since $\tilde{T}$ has no poles in $\mathbb{D}$, we can write it as $\tilde{T}(z_2) = (d(z_2))^{-1}\mathcal{R}(z_2)$ with a scalar polynomial d that is zero-free on $\mathbb{D}$ and a matrix polynomial $\mathcal{R}$. This allows us to rewrite (8.4) as

$$\mathcal{S}(z_2)^*\mathcal{S}(z_2) - \mathcal{R}(z_2)^*\mathcal{R}(z_2) = 0$$

for $z_2 \in \mathbb{T}$ with the matrix polynomial $\mathcal{S}(z_2) := d(z_2)I_{m+\tilde{n}_1}$. According to Theorem 6.6, there now exists a matrix polynomial $\tilde{F}$ of size $\tilde{n}_2 \times (m + \tilde{n}_1)$ such that

$$\mathcal{S}(w_2)^*\mathcal{S}(z_2) - \mathcal{R}(w_2)^*\mathcal{R}(z_2) = (1 - \overline{w_2}z_2)\tilde{F}(w_2)^*\tilde{F}(z_2)$$

for all $w_2, z_2 \in \mathbb{D}$. Dividing this by $\overline{d(w_2)}d(z_2)$ and again using Theorem 5.3 we get a matrix-valued function F of size $\tilde{n}_2 \times (m + \tilde{n}_1)$ and a constant isometric matrix $T \in \mathbb{C}^{(n_0+\tilde{n}_1+\tilde{n}_2)\times(m+\tilde{n}_1+\tilde{n}_2)}$ such that

$$T\begin{pmatrix} I_{m+\tilde{n}_1} \\ z_2F(z_2) \end{pmatrix} = \begin{pmatrix} \tilde{T}(z_2) \\ F(z_2) \end{pmatrix}$$

for $z_2 \in \mathbb{D}$ (choosing T isometric is again possible because $n_0 \geq m$). Multiplying this by

$$\begin{pmatrix} S(z) \\ z_1H(z_1; z_2) \end{pmatrix}$$

from the right yields

$$T\begin{pmatrix} S(z) \\ z_1H(z_1;z_2) \\ z_2F(z_2)\begin{pmatrix} S(z) \\ z_1H(z_1;z_2) \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \tilde{T}(z_2)\begin{pmatrix} S(z) \\ z_1H(z_1;z_2) \end{pmatrix} \\ F(z_2)\begin{pmatrix} S(z) \\ z_1H(z_1;z_2) \end{pmatrix} \end{pmatrix} \overset{(8.2)}{=} \begin{pmatrix} R(z) \\ H(z_1;z_2) \\ F(z_2)\begin{pmatrix} S(z) \\ z_1H(z_1;z_2) \end{pmatrix} \end{pmatrix}.$$

So, by defining the matrix-valued function F_2 of size $\tilde{n}_2 \times m$ as

$$F_2(z) := F(z_2)\begin{pmatrix} S(z) \\ z_1H(z_1;z_2) \end{pmatrix}$$

and renaming $F_1(z) := H(z_1; z_2)$ (as mentioned above, this matrix-valued function has size $\tilde{n}_1 \times m$, so in particular the sizes n_1 and n_2 mentioned in the theorem are given by $\tilde{n}_1$ and $\tilde{n}_2$, respectively), we finally get

$$T\begin{pmatrix} S(z) \\ z_1F_1(z) \\ z_2F_2(z) \end{pmatrix} = \begin{pmatrix} R(z) \\ F_1(z) \\ F_2(z) \end{pmatrix} \tag{8.5}$$

or equivalently (according to Theorem 5.3)

$$S(w)^*S(z) - R(w)^*R(z) = \sum_{j=1}^{2}(1 - \overline{w_j}z_j)F_j(w)^*F_j(z)$$

for $z, w \in \mathbb{D}^2$. □

Remark 8.2

a) The assumption $n_0 \geq m$ assures that we can choose $\tilde{T}(z_2)$ in (8.2) to be isometric. If S had more rows than R, we would only get a contractive $\tilde{T}(z_2)$. In the setting of the Unitary Extension Principle, we can always find an isometry since

$$I_m = R(z)^* R(z)$$

for $z \in \mathbb{T}^2$ implies that R has rank m on $\mathbb{T}^2$ (which is only possible if $n_0 \geq m$). In the Oblique Extension Principle setting, the same is true if S is invertible on $\overline{\mathbb{D}^2}$.

b) As long as $n_0 \geq m$ holds, we can also loosen the assumptions a little. If we look at the proof of Theorem 8.1, we only need a rational and analytic inverse of $S(0, z_2)$ for any $z_2 \in \mathbb{D}$. Alternatively, we could swap the roles of z_1 and z_2 in the proof and assume $S(z_1, 0)$ to have a rational and analytic inverse for any $z_1 \in \mathbb{D}$.

Even without taking the proof into account, we know that if there are matrix-valued functions F_1 and F_2 satisfying (8.1), they have to be at least rational according to Remark 5.6. If we do take the proof into account, however, we see that even though the statement of Theorem 8.1 is only about the existence of rational functions, the function F_1 is a matrix polynomial. So, the question is whether F_2 can also be chosen as a matrix polynomial.

The following theorem gives the answer: Not only can F_2 indeed be chosen polynomially, but for polynomial matrix functions R and S satisfying $S(z)^*S(z) - R(z)^*R(z) = 0$ on $\mathbb{T}^2$, the existence of a decomposition of the form (8.1) is in itself a sufficient condition for the functions F_1 and F_2 to be matrix polynomials. This result also holds in higher dimensions, so we formulate it for d-variate matrix polynomials.

Theorem 8.3

Let $S \in \mathbb{C}[z_1, \ldots, z_d]^{m \times m}$ and $R \in \mathbb{C}[z_1, \ldots, z_d]^{n_0 \times m}$ be two matrix polynomials such that

$$S(z)^*S(z) - R(z)^*R(z) = 0$$

holds for each $z \in \mathbb{T}^d$. Suppose further that

$$S(w)^*S(z) - R(w)^*R(z) = \sum_{j=1}^{d} (1 - \overline{w_j} z_j) F_j(w)^* F_j(z) \tag{8.6}$$

holds for all $z, w \in \mathbb{D}^d$ and some matrix functions $F_j : \mathbb{C}^d \to \mathbb{C}^{n_j \times m}$, $1 \leq j \leq d$, and let ν be the maximum of the total degrees of S and R.

Then, F_j is a matrix polynomial of total degree smaller than ν for each $1 \leq j \leq d$.

Proof. The method of this proof largely follows Knese's proof for the case $S = I_m$ in [67]. We already know from the proof of Theorem 5.3 (see also Remark 5.6) that $F_1, \dots, F_d$ are rational and holomorphic on $\mathbb{D}^d$. Again, the key to the proof is to break the problem down to one dimension. To do this, we fix an arbitrary direction $\tau \in \mathbb{T}^d$. Then, define $z, w \in \mathbb{D}^d$ as $z := \zeta\tau$ and $w := \eta\tau$ for some $\zeta, \eta \in \mathbb{D}$. Plugging this into (8.6) yields

$$\begin{aligned} S(\eta\tau)^* S(\zeta\tau) - R(\eta\tau)^* R(\zeta\tau) &= \sum_{j=1}^{d} (1 - \overline{\eta\tau_j} \cdot \zeta\tau_j) F_j(\eta\tau)^* F_j(\zeta\tau) \\ &= (1 - \overline{\eta}\zeta) \sum_{j=1}^{d} F_j(\eta\tau)^* F_j(\zeta\tau), \end{aligned}$$

since $\overline{\tau_j}\tau_j = |\tau_j|^2 = 1$ for each $1 \leq j \leq d$. Because R and S are matrix polynomials and τ is fixed, the left side of the equation is a matrix polynomial in $\overline{\eta}$ and ζ. Analogous to our answer to the first question in Section 6.1, the identity $S(z)^* S(z) - R(z)^* R(z) = 0$ now implies that $S(\eta\tau)^* S(\zeta\tau) - R(\eta\tau)^* R(\zeta\tau)$ is divisible by $(1 - \overline{\eta}\zeta)$. So,

$$\sum_{j=1}^{d} F_j(\eta\tau)^* F_j(\zeta\tau)$$

is a matrix polynomial in $\overline{\eta}$ and ζ, and its degree in each of those variables is smaller than ν. Now, let $n := \sum_{j=1}^{d} n_j$ and consider the homogeneous expansion of F, i.e., write

$$F(z) := \begin{pmatrix} F_1(z) \\ \vdots \\ F_d(z) \end{pmatrix} = \sum_{\ell=0}^{\infty} \tilde{F}_\ell(z)$$

where $\tilde{F}_\ell \in \mathbb{C}[z_1, \dots, z_d]^{n \times m}$, $\ell \in \mathbb{N}_0$, is the matrix polynomial containing all summands of $F \in \mathbb{C}[z_1, \dots, z_d]^{n \times m}$ that have total degree ℓ. Because

$$\sum_{j=1}^{d} F_j(\eta\tau)^* F_j(\zeta\tau) = F(\eta\tau)^* F(\zeta\tau) = \sum_{k,\ell=0}^{\infty} \tilde{F}_k(\eta\tau)^* \tilde{F}_\ell(\zeta\tau) = \sum_{k,\ell=0}^{\infty} \overline{\eta^k} \zeta^\ell \tilde{F}_k(\tau)^* \tilde{F}_\ell(\tau)$$

is polynomial in $\overline{\eta}$ and ζ, and its degree in each of those variables is smaller than ν, $\tilde{F}_k(\tau)^* \tilde{F}_\ell(\tau) = 0$ holds for $k \geq \nu$ or $\ell \geq \nu$. Since $\tau \in \mathbb{T}^d$ was arbitrary, this means that $\tilde{F}_k^* \tilde{F}_\ell \equiv 0$ holds on $\mathbb{T}^d$ for $k \geq \nu$ or $\ell \geq \nu$, so in particular, $\tilde{F}_\ell^* \tilde{F}_\ell \equiv 0$ holds on $\mathbb{T}^d$ for $\ell \geq \nu$. Because $\tilde{F}_\ell$ is a matrix polynomial, this implies $\tilde{F}_\ell \equiv 0$ for $\ell \geq \nu$. Therefore, F is a matrix polynomial of total degree smaller than ν, which, of course, implies that every F_j, $1 \leq j \leq d$, is a matrix polynomial of total degree smaller than ν. □

Combining Theorem 8.1 and Theorem 8.3, we get the following corollary.

Corollary 8.4
Let $S \in \mathbb{C}[z_1, z_2]^{m\times m}$ and $R \in \mathbb{C}[z_1, z_2]^{n_0\times m}$, $n_0 \geq m$, be two matrix polynomials such that

$$S(z)^*S(z) - R(z)^*R(z) = 0$$

holds for all $z \in \mathbb{T}^2$. Furthermore, define ν as the maximum of the total degrees of S and R and let S have a rational and analytic inverse S^{-1} on $\mathbb{D}^2$.

Then there exist matrix polynomials $F_1 : \mathbb{C}^2 \to \mathbb{C}^{n_1\times m}$ and $F_2 : \mathbb{C}^2 \to \mathbb{C}^{n_2\times m}$ with total degrees smaller than ν such that

$$S(w)^*S(z) - R(w)^*R(z) = \sum_{j=1}^{2}(1 - \overline{w_j}z_j)F_j(w)^*F_j(z)$$

holds for all $z, w \in \mathbb{D}^2$.

In particular, we have obtained the following result for the setting of the Oblique Extension Principle:

Corollary 8.5
Let $p \in \mathbb{C}[z_1, z_2]$ be a refinement symbol and $q_1, \ldots, q_r \in \mathbb{C}[z_1, z_2]$ be wavelet symbols such that the Oblique Extension Principle holds for the vanishing moment recovery function $s \in \mathbb{C}[z_1^{\pm 1}, z_2^{\pm 1}]$, i.e.,

$$S(z)^*S(z) - R(z)^*R(z) = 0$$

for $z \in \mathbb{T}^2$ with the matrix polynomials S and R defined as in Section 4.4 and extended to matrix functions $S : \mathbb{C}^2 \to \mathbb{C}^{m\times m}$ and $R : \mathbb{C}^2 \to \mathbb{C}^{(r+1)\times m}$. We further define ν as the maximum of the total degrees of S and R and assume that $s = |\theta|^2$ on $\mathbb{T}^2$ with $\theta \in \mathbb{C}[z_1, z_2]$ zero-free on $\overline{\mathbb{D}^2}$.

Then there exist matrix polynomials $F_1 : \mathbb{C}^2 \to \mathbb{C}^{n_1\times m}$ and $F_2 : \mathbb{C}^2 \to \mathbb{C}^{n_2\times m}$ with total degrees smaller than ν such that

$$S(w)^*S(z) - R(w)^*R(z) = \sum_{j=1}^{2}(1 - \overline{w_j}z_j)F_j(w)^*F_j(z)$$

for $z, w \in \mathbb{D}^2$.

Proof. Since $s = |\theta|^2$ on $\mathbb{T}^2$ with $\theta \in \mathbb{C}[z_1, z_2]$ zero-free on $\overline{\mathbb{D}^2}$, S is a diagonal matrix with each diagonal entry zero-free on $\overline{\mathbb{D}^2}$. This implies that S has a rational and analytic inverse S^{-1} on $\overline{\mathbb{D}^2}$ and that R has at least as many rows as S (see Remark 8.2). The result now follows directly from Corollary 8.4. □

In the following section, we present an alternative method to obtain the matrix polynomials F_1 and F_2. Like in Section 6.2, it will be based on interpolation.

8.2. Obtaining a Realization with the Nevanlinna-Pick Interpolation

If we want to construct the matrix polynomials F_1 and F_2 for given bivariate matrix polynomials S and R, Theorem 8.3, i.e., the fact that F_1 and F_2 are matrix polynomials, allows us to simplify the computations compared to the proof of Theorem 8.1. In particular, it makes it possible to skip the search for the isometries $\tilde{T}$ and T and to replace it with a numerical calculation, similar to the Nevanlinna-Pick interpolation known from Section 6.2.

We know that under the assumptions of Theorem 8.1, we can always find matrix-valued functions F_1 and F_2 such that (8.1) holds, i.e.,

$$S(w)^*S(z) - R(w)^*R(z) = \sum_{j=1}^{2}(1 - \overline{w_j}z_j)F_j(w)^*F_j(z),$$

for $z, w \in \mathbb{D}^2$. The proof also gives a method for constructing these functions. In this method, we obtain F_1 first and then use it to construct F_2. The first step in finding F_1, the factorization

$$\frac{S(w)^*S(z) - R(w)^*R(z)}{1 - \overline{w_1}z_1} = \begin{pmatrix} I_m \\ w_1 I_m \\ \vdots \\ w_1^{\nu-1} I_m \end{pmatrix}^* L(z_2) \begin{pmatrix} I_m \\ z_1 I_m \\ \vdots \\ z_1^{\nu-1} I_m \end{pmatrix}$$

is pretty straightforward. The more difficult part in finding F_1 is the factorization

$$L(z_2) = A(z_2)^*A(z_2).$$

Since L is positive semi-definite, we can, for example, use the Cholesky decomposition (or rather its extension to positive semi-definite polynomial matrices, to be precise); however, this may not be the optimal factorization in this case (see Section 9). Combining these two factorizations then immediately gives us the matrix polynomial F_1.

We can now take advantage of the fact that F_2 will also be a matrix polynomial according to Theorem 8.3 and compute it using polynomial interpolation, similar to the Nevanlinna-Pick interpolation from Section 6.2.

For this we consider the equation (6.8), i.e.,

$$U_j^*U_k - V_j^*V_k = \sum_{\ell=1}^{d}(1 - \overline{w_{j,\ell}}w_{k,\ell})\mathcal{N}_{\ell,jk}$$

for $d = 2$. Similar to our approach in the univariate case in Remark 6.14, we choose points $w_1, \dots, w_n \in \mathbb{D}^2$ and define $U_j := S(w_j)$ and $V_j := R(w_j)$. Since we have already calculated F_1, we know the values of the matrices

$$\mathcal{N}_{1,jk} = F_1(w_j)^*F_1(w_k), \qquad 1 \le j, k \le n.$$

We can thus solve (6.8) for the matrices $\mathcal{N}_{2,jk}$, $1 \leq j,k \leq n$. As in Theorem 6.13 and the sketch of its proof, we then construct the block matrix $\mathcal{N}_2 = (\mathcal{N}_{2,jk})_{j,k=1,\ldots,n}$ and write it as

$$\mathcal{N}_2 = A_2^* A_2.$$

If we now divide A_2 into suitable blocks, i.e., $A_2 = (A_{2,1}, \ldots, A_{2,n})$, we have obtained a factorization

$$\mathcal{N}_{2,jk} = A_{2,j}^* A_{2,k}$$

of each block $\mathcal{N}_{2,jk}$, $1 \leq j,k \leq n$. Because the block $\mathcal{N}_{2,jk}$ contains the values of $F_2(w_j)^* F_2(w_k)$, this means that $A_{2,k} = F_2(w_k)$. Since we know that F_2 is polynomial, we can now find it by interpolation, provided we have chosen enough points for it to be uniquely defined by its values in $w_1, \ldots, w_n$.

Remark 8.6
These calculations are simplest when done on a rectangular grid, i.e., when we pick

$$\{w_1, \ldots, w_n\} = \{z_0, \ldots, z_{\nu_1}\} \times \{\tilde{z}_0, \ldots, \tilde{z}_{\nu_2 - 1}\}$$

with $z_j \in \mathbb{D}$ for $1 \leq j \leq \nu_1$ and $\tilde{z}_k \in \mathbb{D}$ for $1 \leq k \leq \nu_2 - 1$, where ν_1 is the degree of z_1 (and $\overline{w_1}$) and ν_2 is the degree of z_2 (and $\overline{w_2}$) in the matrix polynomial $S(w)^* S(z) - R(w)^* R(z)$. Note that because of (8.1), we expect the degree of z_1 (and $\overline{w_1}$) in F_2 to be ν_1 and the degree of z_2 (and $\overline{w_2}$) in F_2 to be $\nu_2 - 1$.

On such a rectangular grid, finding the coefficients of F_2 now boils down to solving two systems of linear equations (see e.g. [62], Chapter 6.2): We rearrange the blocks of A_2 into a block matrix $\tilde{A}_2 = (\tilde{A}_{2,jk})_{0 \leq j \leq \nu_1, 0 \leq k \leq \nu_2 - 1}$ such that $\tilde{A}_{2,jk} = A_{2,\kappa}$ for some $1 \leq \kappa \leq n$ and

$$\tilde{A}_{2,jk} = F_2(z_j, \tilde{z}_k)$$

for all $0 \leq j \leq \nu_1$ and all $0 \leq k \leq \nu_2 - 1$. Those are of size $\rho \times m$, where $\rho := \operatorname{rank}(N_2)$ and m is the size of $S(w)^* S(z) - R(w)^* R(z)$. Now, define the block matrices Z_1 and Z_2 as

$$Z_1 := \begin{pmatrix} z_0^0 E_\rho & \ldots & z_0^{\nu_1} E_\rho \\ \vdots & \ddots & \vdots \\ z_{\nu_1}^0 E_\rho & \ldots & z_{\nu_1}^{\nu_1} E_\rho \end{pmatrix} \qquad \text{and} \qquad Z_2 := \begin{pmatrix} \tilde{z}_0^0 E_m & \ldots & \tilde{z}_{\nu_2-1}^0 E_m \\ \vdots & \ddots & \vdots \\ \tilde{z}_0^{\nu_2-1} E_m & \ldots & \tilde{z}_{\nu_2-1}^{\nu_2-1} E_m \end{pmatrix}$$

and solve the systems of linear equations

$$Z_1 \tilde{C} = \tilde{A}_2$$

and then

$$Z_2^T C^T = \tilde{C}^T.$$

The solution C of the second system is a block matrix $C = (C_{jk})_{0 \leq j \leq \nu_1, 0 \leq k \leq \nu_2 - 1}$, where the block C_{jk} is the matrix with the coefficients of $z_1^j z_2^k$ in F_2.

Example 8.7
We will use the method described above to construct the matrix polynomials F_1 and F_2 for the bivariate tight wavelet frame from Example 7.5. The refinement and wavelet symbol, the vanishing moment recovery function s, and the function θ with $s = |\theta|^2$ can be found there. In particular, note that the matrix polynomial $S(w)^*S(z) - R(w)^*R(z)$ has degree 4 in all its variables, i.e., $\nu_1 = \nu_2 = 4$.

We first follow the method in the proof of Theorem 8.1 and calculate

$$\frac{S(w)^*S(z) - R(w)^*R(z)}{1 - \overline{w_1} z_1} = \begin{pmatrix} I \\ w_1 I \\ \vdots \\ w_1^{\nu-1} I \end{pmatrix}^* L(z_2) \begin{pmatrix} I \\ z_1 I \\ \vdots \\ z_1^{\nu-1} I \end{pmatrix}.$$

The matrix $L(z_2)$ consists of 4×4 blocks of (4×4)-matrices, i.e., $L(z_2) = (L_{j,k}(z_2))_{0 \le j,k \le 3}$ where $L_{j,k}(z_2)$ is the coefficient matrix of the monomial $\overline{w_1}^j z_1^k$. The blocks $L_{j,k}$ can easily be calculated by differentiation. For example, the matrix with the constant coefficients, i.e., the block $L_{0,0}(z_2)$ of $L(z_2)$, is given by

$$L_{0,0}(z_2) = \begin{pmatrix} -\frac{15\sqrt{15}+53}{576}\ell(z_2) & 0 & \frac{\sqrt{15}+11}{576}\ell(z_2) & 0 \\ 0 & \frac{15\sqrt{15}+53}{576}\tilde{\ell}(z_2) & 0 & -\frac{\sqrt{15}+11}{576}\tilde{\ell}(z_2) \\ \frac{\sqrt{15}+11}{576}\ell(z_2) & 0 & -\frac{15\sqrt{15}+53}{576}\ell(z_2) & 0 \\ 0 & -\frac{\sqrt{15}+11}{576}\tilde{\ell}(z_2) & 0 & \frac{15\sqrt{15}+53}{576}\tilde{\ell}(z_2) \end{pmatrix}$$

with $\ell(z_2) := z_2 - 8 + z_2^{-1}$ and $\tilde{\ell}(z_2) := z_2 + 8 + z_2^{-1}$. The complete block matrix $L(z_2)$ is symmetrical and has rank 10. We can write it as

$$L(z_2) = A(z_2)^* A(z_2).$$

with a (10×16) matrix polynomial $A(z_2)$ of full rank. The matrix polynomial A now directly gives us

$$F_1(z_1, z_2) = A(z_2) \begin{pmatrix} I_4 \\ z_1 I_4 \\ z_1^2 I_4 \\ z_1^3 I_4 \end{pmatrix}.$$

F_1 is a matrix polynomial of size 10×4. Its first column can be obtained by multiplying

$$\begin{pmatrix} \sqrt{\frac{15\sqrt{15}-53}{54336}}\left(z_1^2+\left(\frac{4\sqrt{15}}{3}-2\right)z_1-\frac{4\sqrt{15}}{3}-11\right) \\ \frac{1}{6}\sqrt{\frac{33-4\sqrt{15}}{3962}}\left(z_1^2+(7\sqrt{15}-26)z_1\right) \\ \frac{1}{8}\sqrt{\frac{7(275-17\sqrt{15})}{106935}}\left(z_1^3+\left(\frac{86}{7}-\frac{8\sqrt{15}}{3}\right)z_1^2+\left(\frac{95}{7}-\frac{16\sqrt{15}}{3}\right)z_1\right) \\ -\frac{106}{3}\sqrt{\frac{2}{42071845+10857872\sqrt{15}}}\left(z_1^3+\left(\frac{11\sqrt{15}}{106}+\frac{947}{424}\right)z_1^2\right) \\ \frac{\sqrt{15}-4}{6}\sqrt{\frac{111+64\sqrt{15}}{32746}}\,z_1^3 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

with $(z_2-\sqrt{15}-4)$, and its second column can be obtained by multiplying

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -\frac{1}{8}\sqrt{\frac{73+5\sqrt{15}}{7431}}\left(z_1^3+\left(12-\frac{8\sqrt{15}}{3}\right)z_1^2+\left(49-\frac{44\sqrt{15}}{3}\right)z_1+\frac{4\sqrt{15}}{3}-2\right) \\ -\sqrt{\frac{257491}{6640856816\sqrt{15}}-\frac{177893}{19922570448}}\left(z_1^3+\left(\frac{167}{14}-\frac{5\sqrt{15}}{21}\right)z_1^2-\frac{127\sqrt{15}}{21}+\frac{655}{14}\right) \\ \frac{167399}{12\sqrt{7(307291202165+79273459418\sqrt{15})}}\left(z_1^3+\left(\frac{26838\sqrt{15}+489412}{167399}\right)z_1^2-\frac{384346\sqrt{15}+1455780}{167399}\right) \\ \frac{241}{4\sqrt{152962818\sqrt{15}+592573491}}\left(z_1^3+\frac{1610\sqrt{15}}{241}+\frac{6052}{241}\right) \\ \frac{1}{2\sqrt{25434+6576\sqrt{15}}}z_1^3 \end{pmatrix}$$

with $(z_2+\sqrt{15}+4)$.

Its third column can be obtained by multiplying

$$\begin{pmatrix} \sqrt{\frac{15\sqrt{15}-53}{54336}}\left(z_1^2+\left(\frac{4\sqrt{15}}{3}-6\right)z_1+\frac{4\sqrt{15}}{3}-3\right) \\ \frac{1}{6}\sqrt{\frac{33-4\sqrt{15}}{3962}}\left(z_1^2+\left(7\sqrt{15}-30\right)z_1-14\right) \\ \frac{1}{8}\sqrt{\frac{7\left(275-17\sqrt{15}\right)}{106935}}\left(z_1^3+\left(\frac{58}{7}-\frac{8\sqrt{15}}{3}\right)z_1^2+\left(\frac{16\sqrt{15}}{3}-\frac{193}{7}\right)z_1\right) \\ -\frac{106}{3}\sqrt{\frac{2}{42071845+10857872\sqrt{15}}}\left(z_1^3+\left(\frac{11\sqrt{15}}{106}-\frac{749}{424}\right)z_1^2-\left(\frac{22\sqrt{15}}{53}+\frac{99}{106}\right)z_1\right) \\ -\frac{1}{2\sqrt{25434+6576\sqrt{15}}}\left(z_1^3-4z_1^2+8z_1\right) \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

with $(z_2-\sqrt{15}-4)$, and its fourth column can be obtained by multiplying

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -\frac{1}{8}\sqrt{\frac{73+5\sqrt{15}}{7431}}\left(z_1^3+\left(8-\frac{8\sqrt{15}}{3}\right)z_1^2+\left(9-4\sqrt{15}\right)z_1-\frac{4\sqrt{15}}{3}+6\right) \\ -\sqrt{\frac{257491}{6640856816\sqrt{15}}-\frac{177893}{19922570448}}\,f_{1,4}(z_1) \\ \frac{167399}{12\sqrt{7\left(307291202165+79273459418\sqrt{15}\right)}}\,\tilde{f}_{1,4}(z_1) \\ \frac{241}{4\sqrt{592573491+152962818\sqrt{15}}}\left(z_1^3-4z_1^2+8z_1\right) \\ \frac{1}{2\sqrt{25434+6576\sqrt{15}}}\left(z_1^3-4z_1^2+8z_1\right) \end{pmatrix}$$

with $(z_2+\sqrt{15}+4)$.

The functions $f_{1,4}$ and $\tilde{f}_{1,4}$ in the fourth column are given by

$$f_{1,4}(z_1) = \left(z_1^3 + \left(\frac{333 - 10\sqrt{15}}{42} \right) z_1^2 + \left(\frac{20\sqrt{15} - 834}{21} \right) z_1 - \frac{1174\sqrt{15} + 1755}{42} \right)$$

$$\text{and} \quad \tilde{f}_{1,4}(z_1) = \left(z_1^3 + \left(\frac{26838\sqrt{15} - 180184}{167399} \right) z_1^2 - \left(\frac{107352\sqrt{15} + 618456}{167399} \right) z_1 \right).$$

As can be seen above, F_1 has a maximum degree of 3 in z_1 and 1 in z_2.

Since $S(w)^*S(z) - R(w)^*R(z)$ has degree 4 in all its variables, we now use

$$z_0 = 0, \quad z_1 = \frac{1}{2}, \quad z_2 = -\frac{1}{2}, \quad z_3 = \frac{1}{3} \quad \text{and} \quad z_4 = -\frac{1}{3}$$

to build the rectangular grid $\{w_1, \ldots, w_{20}\} = \{z_0, \ldots, z_4\} \times \{z_0, \ldots, z_3\}$ (cf. Remark 8.6). We calculate $U_j := S(w_j)$, $V_j := R(w_j)$ and $\mathcal{N}_{1,jk} = F_1(w_j)^*F_1(w_k)$, $1 \leq j, k \leq 20$, and solve

$$U_j^*U_k - V_j^*V_k = \sum_{\ell=1}^{2} (1 - \overline{w_{j,\ell}} w_{k,\ell}) \mathcal{N}_{\ell,jk}$$

for $\mathcal{N}_{2,jk}$, $1 \leq j, k \leq 20$, to get a (80×80)-matrix $\mathcal{N}_2 = (\mathcal{N}_{2,jk})_{j,k=1,\ldots,20}$ of rank 15. We can thus factor it into $\mathcal{N}_2 = A_2^*A_2$ with a (15×80)-matrix $A_2 = (A_{2,1}, \ldots, A_{2,20})$ with 20 blocks $A_{2,j}$ of size 15×4.

Due to the size of the matrices, we switched to numerical calculations at this point (up until now, even in the examples where we only gave numerical values, all underlying calculations were done symbolically). Using the interpolation method described in Remark 8.6 results in

$$F_2(z_1, z_2) = A_0(z_2) + z_1^1 A_1(z_2) + z_1^2 A_2(z_2) + z_1^3 A_3(z_2) + z_1^4 A_4(z_2),$$

where the first and third columns of $A_0(z_2)$ are given by

$$\begin{pmatrix}
-0.0028826\, z_2^3 - 0.010755\, z_2^2 + 0.047843\, z_2 - 0.27385 \\
-0.0033039\, z_2^3 - 0.044913\, z_2^2 - 0.058373\, z_2 + 0.30947 \\
-0.000048359\, z_2^3 - 0.00018042\, z_2^2 + 0.00080264\, z_2 - 0.0045941 \\
-0.000055428\, z_2^3 - 0.00075348\, z_2^2 - 0.00097928\, z_2 + 0.0051918 \\
0.038419\, z_2^3 + 0.073119\, z_2^2 - 0.27552\, z_2 + 0.0079415 \\
0.00064453\, z_2^3 + 0.0012267\, z_2^2 - 0.0046223\, z_2 + 0.00013323 \\
-0.00063834\, z_2^3 - 0.0023816\, z_2^2 + 0.010595\, z_2 - 0.060642 \\
-0.00073164\, z_2^3 - 0.0099459\, z_2^2 - 0.012926\, z_2 + 0.068531 \\
0.0085077\, z_2^3 + 0.016192\, z_2^2 - 0.061014\, z_2 + 0.0017586 \\
-0.00026257\, z_2^3 + 0.014546\, z_2^2 + 0.02957\, z_2 - 0.003385 \\
-0.000004405\, z_2^3 + 0.00024403\, z_2^2 + 0.00049608\, z_2 - 0.000056788 \\
0.015428\, z_2^3 + 0.031237\, z_2^2 - 0.0020947\, z_2 - 0.0047258 \\
0.00025883\, z_2^3 + 0.00052404\, z_2^2 - 0.000035141\, z_2 - 0.000079281 \\
-0.000058145\, z_2^3 + 0.0032212\, z_2^2 + 0.0065482\, z_2 - 0.0007496 \\
0.0034165\, z_2^3 + 0.0069173\, z_2^2 - 0.00046386\, z_2 - 0.0010465
\end{pmatrix}$$

and the second and fourth columns of $A_0(z_2)$ are given by

$$\begin{pmatrix} -0.0028826\, z_2^3 + 0.00077573\, z_2^2 + 0.067801\, z_2 + 0.34214 \\ -0.0033039\, z_2^3 - 0.031698\, z_2^2 + 0.094849\, z_2 + 0.22781 \\ -0.000048359\, z_2^3 + 0.000013014\, z_2^2 + 0.0011375\, z_2 + 0.0057398 \\ -0.000055428\, z_2^3 - 0.00053177\, z_2^2 + 0.0015912\, z_2 + 0.0038219 \\ 0.038419\, z_2^3 - 0.080556\, z_2^2 - 0.26065\, z_2 + 0.043343 \\ 0.00064453\, z_2^3 - 0.0013514\, z_2^2 - 0.0043728\, z_2 + 0.00072713 \\ -0.00063834\, z_2^3 + 0.00017178\, z_2^2 + 0.015014\, z_2 + 0.075765 \\ -0.00073164\, z_2^3 - 0.0070194\, z_2^2 + 0.021004\, z_2 + 0.050449 \\ 0.0085077\, z_2^3 - 0.017839\, z_2^2 - 0.057721\, z_2 + 0.0095981 \\ -0.00026257\, z_2^3 + 0.015596\, z_2^2 - 0.030714\, z_2 + 0.0014942 \\ -0.000004405\, z_2^3 + 0.00026165\, z_2^2 - 0.00051528\, z_2 + 0.000025067 \\ 0.015428\, z_2^3 - 0.030476\, z_2^2 - 0.0036165\, z_2 + 0.0048049 \\ 0.00025883\, z_2^3 - 0.00051127\, z_2^2 - 0.000060673\, z_2 + 0.00008061 \\ -0.000058145\, z_2^3 + 0.0034537\, z_2^2 - 0.0068016\, z_2 + 0.00033089 \\ 0.0034165\, z_2^3 - 0.0067488\, z_2^2 - 0.00080087\, z_2 + 0.001064 \end{pmatrix}.$$

The first column of $A_1(z_2)$ is given by

$$\begin{pmatrix} -0.0013533\, z_2^3 - 0.0050491\, z_2^2 + 0.022462\, z_2 - 0.12857 \\ -0.0015511\, z_2^3 - 0.021086\, z_2^2 - 0.027405\, z_2 + 0.14529 \\ 0.0057345\, z_2^3 + 0.021395\, z_2^2 - 0.095179\, z_2 + 0.54478 \\ 0.0065727\, z_2^3 + 0.08935\, z_2^2 + 0.11613\, z_2 - 0.61565 \\ 0.018037\, z_2^3 + 0.034328\, z_2^2 - 0.12936\, z_2 + 0.0037284 \\ -0.076429\, z_2^3 - 0.14546\, z_2^2 + 0.54812\, z_2 - 0.015799 \\ 0.0003999\, z_2^3 + 0.001492\, z_2^2 - 0.0066373\, z_2 + 0.03799 \\ 0.00045835\, z_2^3 + 0.0062308\, z_2^2 + 0.008098\, z_2 - 0.042932 \\ -0.0053298\, z_2^3 - 0.010144\, z_2^2 + 0.038223\, z_2 - 0.0011017 \\ -0.00012327\, z_2^3 + 0.0068291\, z_2^2 + 0.013883\, z_2 - 0.0015892 \\ 0.00052235\, z_2^3 - 0.028937\, z_2^2 - 0.058826\, z_2 + 0.006734 \\ 0.0072433\, z_2^3 + 0.014665\, z_2^2 - 0.00098342\, z_2 - 0.0022187 \\ -0.030692\, z_2^3 - 0.062142\, z_2^2 + 0.0041671\, z_2 + 0.0094013 \\ 0.000036426\, z_2^3 - 0.0020179\, z_2^2 - 0.0041022\, z_2 + 0.0004696 \\ -0.0021403\, z_2^3 - 0.0043334\, z_2^2 + 0.00029059\, z_2 + 0.0006556 \end{pmatrix},$$

the second column is given by

$$\begin{pmatrix} -0.0013533\, z_2^3 + 0.0003642\, z_2^2 + 0.031832\, z_2 + 0.16063 \\ -0.0015511\, z_2^3 - 0.014882\, z_2^2 + 0.04453\, z_2 + 0.10696 \\ 0.0057345\, z_2^3 - 0.0015432\, z_2^2 - 0.13488\, z_2 - 0.68064 \\ 0.0065727\, z_2^3 + 0.063059\, z_2^2 - 0.18869\, z_2 - 0.45321 \\ 0.018037\, z_2^3 - 0.03782\, z_2^2 - 0.12237\, z_2 + 0.020349 \\ -0.076429\, z_2^3 + 0.16026\, z_2^2 + 0.51853\, z_2 - 0.086225 \\ 0.0003999\, z_2^3 - 0.00010762\, z_2^2 - 0.009406\, z_2 - 0.047464 \\ 0.00045835\, z_2^3 + 0.0043974\, z_2^2 - 0.013158\, z_2 - 0.031604 \\ -0.0053298\, z_2^3 + 0.011175\, z_2^2 + 0.03616\, z_2 - 0.0060129 \\ -0.00012327\, z_2^3 + 0.0073222\, z_2^2 - 0.01442\, z_2 + 0.0007015 \\ 0.00052235\, z_2^3 - 0.031027\, z_2^2 + 0.061103\, z_2 - 0.0029725 \\ 0.0072433\, z_2^3 - 0.014308\, z_2^2 - 0.0016979\, z_2 + 0.0022559 \\ -0.030692\, z_2^3 + 0.060628\, z_2^2 + 0.0071947\, z_2 - 0.0095589 \\ 0.000036426\, z_2^3 - 0.0021636\, z_2^2 + 0.004261\, z_2 - 0.00020729 \\ -0.0021403\, z_2^3 + 0.0042279\, z_2^2 + 0.00050172\, z_2 - 0.00066658 \end{pmatrix},$$

and the third and fourth columns are the same as the first and second columns, respectively, apart from the sign. The first and third columns of $A_2(z_2)$ are given by

$$\begin{pmatrix} -0.002333\, z_2^3 - 0.0087041\, z_2^2 + 0.038721\, z_2 - 0.22163 \\ -0.002674\, z_2^3 - 0.03635\, z_2^2 - 0.047243\, z_2 + 0.25047 \\ -0.0019034\, z_2^3 - 0.0071013\, z_2^2 + 0.031591\, z_2 - 0.18082 \\ -0.0021816\, z_2^3 - 0.029657\, z_2^2 - 0.038544\, z_2 + 0.20435 \\ 0.031094\, z_2^3 + 0.059178\, z_2^2 - 0.22299\, z_2 + 0.0064273 \\ 0.025368\, z_2^3 + 0.048281\, z_2^2 - 0.18193\, z_2 + 0.0052438 \\ 0.0043673\, z_2^3 + 0.016294\, z_2^2 - 0.072487\, z_2 + 0.4149 \\ 0.0050057\, z_2^3 + 0.068047\, z_2^2 + 0.088439\, z_2 - 0.46887 \\ -0.058207\, z_2^3 - 0.11078\, z_2^2 + 0.41744\, z_2 - 0.012032 \\ -0.00021251\, z_2^3 + 0.011773\, z_2^2 + 0.023932\, z_2 - 0.0027396 \\ -0.00017338\, z_2^3 + 0.0096048\, z_2^2 + 0.019525\, z_2 - 0.0022351 \\ 0.012487\, z_2^3 + 0.025281\, z_2^2 - 0.0016953\, z_2 - 0.0038247 \\ 0.010187\, z_2^3 + 0.020626\, z_2^2 - 0.0013831\, z_2 - 0.0031205 \\ 0.00039781\, z_2^3 - 0.022038\, z_2^2 - 0.044801\, z_2 + 0.0051286 \\ -0.023375\, z_2^3 - 0.047326\, z_2^2 + 0.0031736\, z_2 + 0.0071599 \end{pmatrix}$$

and the second and fourth column are given by

$$\begin{pmatrix}
-0.002333\, z_2^3 + 0.00062783\, z_2^2 + 0.054874\, z_2 + 0.2769 \\
-0.002674\, z_2^3 - 0.025654\, z_2^2 + 0.076765\, z_2 + 0.18438 \\
-0.0019034\, z_2^3 + 0.00051222\, z_2^2 + 0.04477\, z_2 + 0.22591 \\
-0.0021816\, z_2^3 - 0.02093\, z_2^2 + 0.06263\, z_2 + 0.15043 \\
0.031094\, z_2^3 - 0.065197\, z_2^2 - 0.21095\, z_2 + 0.035079 \\
0.025368\, z_2^3 - 0.053192\, z_2^2 - 0.17211\, z_2 + 0.02862 \\
0.0043673\, z_2^3 - 0.0011753\, z_2^2 - 0.10272\, z_2 - 0.51836 \\
0.0050057\, z_2^3 + 0.048025\, z_2^2 - 0.1437\, z_2 - 0.34516 \\
-0.058207\, z_2^3 + 0.12205\, z_2^2 + 0.39491\, z_2 - 0.065668 \\
-0.00021251\, z_2^3 + 0.012623\, z_2^2 - 0.024858\, z_2 + 0.0012093 \\
-0.00017338\, z_2^3 + 0.010298\, z_2^2 - 0.020281\, z_2 + 0.00098663 \\
0.012487\, z_2^3 - 0.024665\, z_2^2 - 0.002927\, z_2 + 0.0038888 \\
0.010187\, z_2^3 - 0.020123\, z_2^2 - 0.002388\, z_2 + 0.0031727 \\
0.00039781\, z_2^3 - 0.02363\, z_2^2 + 0.046535\, z_2 - 0.0022638 \\
-0.023375\, z_2^3 + 0.046173\, z_2^2 + 0.0054794\, z_2 - 0.0072799
\end{pmatrix}.$$

The first column of $A_3(z_2)$ is given by

$$\begin{pmatrix}
0.00034597\, z_2^3 + 0.0012908\, z_2^2 - 0.0057422\, z_2 + 0.032867 \\
0.00039654\, z_2^3 + 0.0053905\, z_2^2 + 0.0070059\, z_2 - 0.037143 \\
0.00019161\, z_2^3 + 0.00071486\, z_2^2 - 0.0031802\, z_2 + 0.018203 \\
0.00021961\, z_2^3 + 0.0029854\, z_2^2 + 0.0038801\, z_2 - 0.020571 \\
-0.004611\, z_2^3 - 0.0087758\, z_2^2 + 0.033069\, z_2 - 0.00095315 \\
-0.0025537\, z_2^3 - 0.0048603\, z_2^2 + 0.018314\, z_2 - 0.00052788 \\
-0.0015768\, z_2^3 - 0.005883\, z_2^2 + 0.026171\, z_2 - 0.1498 \\
-0.0018073\, z_2^3 - 0.024568\, z_2^2 - 0.031931\, z_2 + 0.16929 \\
0.021016\, z_2^3 + 0.039998\, z_2^2 - 0.15072\, z_2 + 0.0043442 \\
0.000031514\, z_2^3 - 0.0017458\, z_2^2 - 0.003549\, z_2 + 0.00040627 \\
0.000017453\, z_2^3 - 0.00096688\, z_2^2 - 0.0019655\, z_2 + 0.000225 \\
-0.0018517\, z_2^3 - 0.0037491\, z_2^2 + 0.0002514\, z_2 + 0.00056719 \\
-0.0010255\, z_2^3 - 0.0020763\, z_2^2 + 0.00013923\, z_2 + 0.00031412 \\
-0.00014363\, z_2^3 + 0.0079569\, z_2^2 + 0.016175\, z_2 - 0.0018517 \\
0.0084395\, z_2^3 + 0.017087\, z_2^2 - 0.0011458\, z_2 - 0.0025851
\end{pmatrix},$$

the second column is given by

$$\begin{pmatrix} 0.00034597\,z_2^3 - 0.000093104\,z_2^2 - 0.0081375\,z_2 - 0.041063 \\ 0.00039654\,z_2^3 + 0.0038044\,z_2^2 - 0.011384\,z_2 - 0.027343 \\ 0.00019161\,z_2^3 - 0.000051564\,z_2^2 - 0.0045068\,z_2 - 0.022742 \\ 0.00021961\,z_2^3 + 0.002107\,z_2^2 - 0.0063047\,z_2 - 0.015143 \\ -0.004611\,z_2^3 + 0.0096684\,z_2^2 + 0.031284\,z_2 - 0.005202 \\ -0.0025537\,z_2^3 + 0.0053546\,z_2^2 + 0.017326\,z_2 - 0.002881 \\ -0.0015768\,z_2^3 + 0.00042434\,z_2^2 + 0.037089\,z_2 + 0.18715 \\ -0.0018073\,z_2^3 - 0.017339\,z_2^2 + 0.051884\,z_2 + 0.12462 \\ 0.021016\,z_2^3 - 0.044066\,z_2^2 - 0.14258\,z_2 + 0.023709 \\ 0.000031514\,z_2^3 - 0.0018719\,z_2^2 + 0.0036864\,z_2 - 0.00017933 \\ 0.000017453\,z_2^3 - 0.0010367\,z_2^2 + 0.0020416\,z_2 - 0.00009932 \\ -0.0018517\,z_2^3 + 0.0036577\,z_2^2 + 0.00043406\,z_2 - 0.00057669 \\ -0.0010255\,z_2^3 + 0.0020258\,z_2^2 + 0.00024039\,z_2 - 0.00031939 \\ -0.00014363\,z_2^3 + 0.0085314\,z_2^2 - 0.016801\,z_2 + 0.00081735 \\ 0.0084395\,z_2^3 - 0.016671\,z_2^2 - 0.0019783\,z_2 + 0.0026284 \end{pmatrix}$$

and the third and fourth columns are the same as the first and second columns, respectively, apart from the sign. Finally, the first and third columns of $A_4(z_2)$ are given by

$$\begin{pmatrix} 0.00017298\,z_2^3 + 0.00064539\,z_2^2 - 0.0028711\,z_2 + 0.016434 \\ 0.00019827\,z_2^3 + 0.0026953\,z_2^2 + 0.003503\,z_2 - 0.018571 \\ 0.000095804\,z_2^3 + 0.00035743\,z_2^2 - 0.0015901\,z_2 + 0.0091014 \\ 0.00010981\,z_2^3 + 0.0014927\,z_2^2 + 0.00194\,z_2 - 0.010285 \\ -0.0023055\,z_2^3 - 0.0043879\,z_2^2 + 0.016534\,z_2 - 0.00047657 \\ -0.0012769\,z_2^3 - 0.0024301\,z_2^2 + 0.0091572\,z_2 - 0.00026394 \\ -0.00078841\,z_2^3 - 0.0029415\,z_2^2 + 0.013086\,z_2 - 0.0749 \\ -0.00090365\,z_2^3 - 0.012284\,z_2^2 - 0.015966\,z_2 + 0.084643 \\ 0.010508\,z_2^3 + 0.019999\,z_2^2 - 0.075359\,z_2 + 0.0021721 \\ 0.000015757\,z_2^3 - 0.00087291\,z_2^2 - 0.0017745\,z_2 + 0.00020314 \\ 0.0000087267\,z_2^3 - 0.00048344\,z_2^2 - 0.00098277\,z_2 + 0.0001125 \\ -0.00092585\,z_2^3 - 0.0018745\,z_2^2 + 0.0001257\,z_2 + 0.00028359 \\ -0.00051276\,z_2^3 - 0.0010382\,z_2^2 + 0.000069617\,z_2 + 0.00015706 \\ -0.000071815\,z_2^3 + 0.0039785\,z_2^2 + 0.0080877\,z_2 - 0.00092583 \\ 0.0042198\,z_2^3 + 0.0085436\,z_2^2 - 0.00057291\,z_2 - 0.0012925 \end{pmatrix}$$

and the second and fourth columns are given by

$$\begin{pmatrix}
0.00017298\, z_2^3 - 0.000046552\, z_2^2 - 0.0040688\, z_2 - 0.020532 \\
0.00019827\, z_2^3 + 0.0019022\, z_2^2 - 0.005692\, z_2 - 0.013671 \\
0.000095804\, z_2^3 - 0.000025782\, z_2^2 - 0.0022534\, z_2 - 0.011371 \\
0.00010981\, z_2^3 + 0.0010535\, z_2^2 - 0.0031524\, z_2 - 0.0075715 \\
-0.0023055\, z_2^3 + 0.0048342\, z_2^2 + 0.015642\, z_2 - 0.002601 \\
-0.0012769\, z_2^3 + 0.0026773\, z_2^2 + 0.0086629\, z_2 - 0.0014405 \\
-0.00078841\, z_2^3 + 0.00021217\, z_2^2 + 0.018544\, z_2 + 0.093577 \\
-0.00090365\, z_2^3 - 0.0086696\, z_2^2 + 0.025942\, z_2 + 0.06231 \\
0.010508\, z_2^3 - 0.022033\, z_2^2 - 0.071291\, z_2 + 0.011855 \\
0.000015757\, z_2^3 - 0.00093594\, z_2^2 + 0.0018432\, z_2 - 0.000089667 \\
0.0000087267\, z_2^3 - 0.00051835\, z_2^2 + 0.0010208\, z_2 - 0.00004966 \\
-0.00092585\, z_2^3 + 0.0018289\, z_2^2 + 0.00021703\, z_2 - 0.00028835 \\
-0.00051276\, z_2^3 + 0.0010129\, z_2^2 + 0.0001202\, z_2 - 0.00015969 \\
-0.000071815\, z_2^3 + 0.0042657\, z_2^2 - 0.0084007\, z_2 + 0.00040868 \\
0.0042198\, z_2^3 - 0.0083354\, z_2^2 - 0.00098916\, z_2 + 0.0013142
\end{pmatrix}.$$

In particular, F_2 has a maximum degree of 4 in z_1 and 3 in z_2.

Note that although there are some similarities, both F_1 and F_2 are not the same as in Example 7.5, once again confirming that these functions are not uniquely defined. If we look at how F_1 is constructed in this example (see Section 9.3 in the Appendix), a different permutation P would yield an F_1 with a similar Kronecker product structure as in Example 7.5.

Remark 8.8

Theoretically, we could again interpolate on the bitorus $\mathbb{T}^2$ instead of $\mathbb{D}^2$, as we did in the univariate case in Example 6.17. The construction of the matrix polynomial F_1 is not affected by the choice of points. We would then use it to construct $\mathcal{N}_1$ for the chosen points on $\mathbb{T}^d$ and with its help $\mathcal{N}_2$ for the points where $\overline{w_{j,2}} w_{k,2} \neq 1$.

Then, as in the univariate case, we would fill in the missing blocks by interpolation, discard the now superfluous blocks, and use the rest to construct F_2. However, filling in the missing blocks by interpolation is more complex than in the univariate case (it is not an interpolation on a grid as in Remark 8.6).

9. Conclusion and Outlook

In this thesis, we have shown that it is indeed possible to link the construction of tight wavelet frames using the Oblique Extension Principle to linear system theory. This connection exists in the univariate case, in the bivariate case, and, to some extent, even in higher dimensions. Under some restrictions on the refinement symbol p, the wavelet symbols $q_1, \ldots, q_r$, and the vanishing moment recovery function s, we can always find an isometric block matrix

$$T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

such that

$$R(z) = \big[A + BE(z)(I - DE(z))^{-1}C\big]S(z)$$

holds for all $z \in \mathbb{D}^d$, $d \in \{1, 2\}$, and the matrix polynomials S and R defined in the matrix form of the Oblique Extension Principle. Equivalently, we can find a matrix polynomial F such that

$$S(w)^*S(z) - R(w)^*R(z) = (1 - \overline{w}z)F(w)^*F(z),$$

holds for $w, z \in \mathbb{D}$ in the univariate case, or matrix polynomials F_1 and F_2 such that

$$S(w)^*S(z) - R(w)^*R(z) = (1 - \overline{w_1}z_1)F_1(w)^*F_1(z) + (1 - \overline{w_2}z_2)F_2(w)^*F_2(z),$$

holds for $w, z \in \mathbb{D}^2$ in the bivariate case. For separable tight wavelet frames, we can even find realizations for any dimension $d \geq 1$. All of these results are constructive.

However, the degree of restriction we had to impose varied considerably: Our choice of refinement and wavelet symbols, and a vanishing moment recovery function is largely unrestricted in the univariate (and multivariate separable) case - we do have to choose polynomial masks, but this can be “enforced” (see Remark 6.1). In the bivariate case, we additionally had to focus on frames with a vanishing moment recovery function that allows a single square factorization $s(z_1, z_2) = |\theta(z_1, z_2)|^2$ on $\mathbb{T}^2$. The main reason for this restriction is to keep the matrix S square. If s had only a sum of squares representation

$$s(z_1, z_2) = \sum_{n=1}^{N} |\theta_n(z_1, z_2)|^2$$

for $(z_1, z_2) \in \mathbb{T}^2$ and $N > 1$, S would be of size $(\det M) \times (N \det M)$. Looking closely at the proof of Theorem 8.1, we can see the reason why this is a problem: The matrix $S_0(z_2)$ in (8.3) would also have the size $(\det M) \times (N \det M)$, so not only would it not have an inverse S_0^{-1}, it would not even have a right inverse, since $N \det M > \det M$. Since we need at least a right inverse of S_0 for the following steps, the method of this proof cannot be applied to this more general case.

Depending on the number N of polynomials θ_n in the sum of squares representation, the proof of Theorem 8.1 fails even earlier: If N is large enough that S has more rows than R, Theorem 5.3 yields only a *contractive* $\tilde{T}(z_2)$ in (8.2) according to Remark 5.7. Note that this very remark also implies that (8.5) will only be possible for a contractive matrix T.

Unlike the requirement that all symbols are polynomial, limiting ourselves to constructions featuring a vanishing moment recovery function s with a single-square factorization is a *real* restriction. For example, we have seen that the vanishing moment recovery function for a frame based on the piecewise linear box spline does *not* have a single-square factorization (see part b) of Example 4.7). It would, therefore, be desirable if this limitation could be removed in the future. For an as yet unpublished successor to [17] and [18], Maria Charina and Joachim Stöckler constructed a realization associated with this very frame, which suggests that it should be possible to generalize our results to rectangular matrix polynomials S. However, their construction involves a largely manual calculation, so there is no systematic approach (yet).

We hoped that the construction from Section 8.2 could also be applied when S is rectangular. However, this turned out to be too optimistic. For the same frame based on the piecewise linear box spline, when we followed the first steps of that construction, i.e., when we calculated F_1 as in the proof of Theorem 8.1 and used it to construct the block matrix $\mathcal{N}_1$, it was not positive semi-definite. This *might* - at least in some cases - be a matter of choosing the "right" matrix Fejér-Riesz factorization $L(z_2) = A(z_2)^* A(z_2)$ (see, for example, the different ways to choose a factorization in Theorem 4.6 in [53]).

Without knowing F_1 and thus $\mathcal{N}_1$ in advance, we face a different problem when trying to find the matrix polynomials F_1 and F_2 by interpolation. In this case, we know the left side of (6.8), i.e.,

$$U_j^* U_k - V_j^* V_k = \sum_{\ell=1}^{2} (1 - \overline{w_{j,\ell}} w_{k,\ell}) \mathcal{N}_{\ell,jk}$$

(since in our context, $U_j = S(z_j)$ and $V_j = R(z_j)$), but we have to decompose it blockwise into $\mathcal{N}_1$ and $\mathcal{N}_2$ before we can use these block matrices to find F_1 and F_2. So far, we have not found a systematic way to do this. In fact, this was the original reason why we performed the interpolation in Section 6.2 not only on $\mathbb{D}$ but also on the torus $\mathbb{T}$: We hoped that the additional freedom of being able to choose the diagonal blocks of $\mathcal{N}_1$ and $\mathcal{N}_2$ relatively freely (as long as the resulting matrices are hermitian positive semi-definite) would help us to make progress on this problem.

Alternatively, the solution may require an entirely different approach. For example, in [18], Charina, Putinar, Scheiderer, and Stöckler used a so-called adjunction formula to transform the Unitary Extension Principle identity from a matrix-valued to a scalar-valued equation. There is currently no generalization of this adjunction formula that does precisely the same for the Oblique Extension Principle case, but perhaps a corresponding approach could also help here.

Another possible route of generalization would be to extend the results to the d-variate non-separable setting for $d > 2$. The result by Anton Kummert ([70]), which we generalized in Theorem 8.1, deals only with the bivariate case. Whether a generalization of both Kummert's result and Theorem 8.1 to higher dimensions is possible, however, seems far from certain to us since many results in and around the theory of operators on $\mathbb{D}^d$ stop working when going from $d = 2$ to $d = 3$ (see part a) of Remark 5.2, for example; even the step from $d = 1$ to $d = 2$ is often far from trivial, see Chapter 11 of [2], for example). Since many applications of wavelets are one-dimensional (e.g., audio signals) or two-dimensional (e.g., image processing), the restriction to $d = 1$ and $d = 2$ would be acceptable.

Even if all these questions about possible generalizations have been answered (whether positively or negatively), remember that finding the connections between the construction of multivariate tight wavelet frames and linear system theory is only the first step. The next step would be determining if and how this connection can be used. Since we look at this topic from a wavelet perspective, we are mainly interested in whether and how results from linear system theory can be utilized in the construction of tight wavelet frames or at least whether the properties of the associated realization reveal something about the properties of a frame.

In summary, while this dissertation has addressed the connection between wavelet frames and linear system theory and hopefully contributed a small step forward, there is no lack of areas for further exploration.

Appendix: Matlab Code for the Examples

Example 6.9

The matrix-valued function F in the univariate Example 6.9 was constructed using the following Matlab script.

```
%Define symbolic variables z and w
syms z w

%Define the refinement symbol p, the wavelet symbols q1 and
  q2 and the vanishing moment recovery function s
p=(1+z)^2/4;
q1=(1-z)^2/4;
q2=sqrt(6)/24*(1-z)^2*(1+4*z+z^2);
s=1/3*(4+1/2*(-z-1/z));

%Define the polynomial theta from the Fejer-Riesz
  factorization s=|theta|^2
u=4+sqrt(sym(15));
theta=(z-u)/sqrt(6*u);

%Now build the matrix polynomials S and R
S=simplify([subs(theta,z,z) 0; 0 subs(theta,z,-z)]);
Q=simplify([q1 subs(q1,z,-z); q2 subs(q2,z,-z)]);
pvec=[subs(p,z,z),subs(p,z,-z)];
Rlower=simplify(subs(theta,z,z^2)*pvec);
R=[Q;Rlower];

%Build the matrix on the left-hand side of the desired
  identity, i.e., S(w)*S(z)-R(w)*R(z)
leftSide=simplify(subs(S',z,w)*S-subs(R',z,w)*R);

%Calculate the kernel K
K=simplify(expand(leftSide/(1-conj(w)*z)));

%Build the matrix L from (6.5)
L00=subs(K,[w z],[0 0]);
L10=subs(diff(K,w),[w z],[0 0]);
L20=1/2*subs(diff(K,w,2),[w z],[0 0]);
L30=1/6*subs(diff(K,w,3),[w z],[0 0]);
L01=subs(diff(K,z),[w z],[0 0]);
L11=subs(diff(diff(K,z),w),[w z],[0 0]);
L21=1/2*subs(diff(diff(K,z),w,2),[w z],[0 0]);
L31=1/6*subs(diff(diff(K,z),w,3),[w z],[0 0]);
```

```
L02=1/2*subs(diff(K,z,2),[w z],[0 0]);
L12=1/2*subs(diff(diff(K,z,2),w),[w z],[0 0]);
L22=1/2*1/2*subs(diff(diff(K,z,2),w,2),[w z],[0 0]);
L32=1/2*1/6*subs(diff(diff(K,z,2),w,3),[w z],[0 0]);
L03=1/6*subs(diff(K,z,3),[w z],[0 0]);
L13=1/6*subs(diff(diff(K,z,3),w),[w z],[0 0]);
L23=1/6*1/2*subs(diff(diff(K,z,3),w,2),[w z],[0 0]);
L33=1/6*1/6*subs(diff(diff(K,z,3),w,3),[w z],[0 0]);

L=[L00,L01,L02,L03;L10,L11,L12,L13;L20,L21,L22,L23;L30,L31,
  L32,L33];

%Factor L (the parameter 'nocheck' makes sure that chol tries
   to find tildeF even though L is not positive definite
tildeF=chol(L,'nocheck');

%Construct F
F=tildeF*Lambda;
```

Examples 6.15 and 6.17

The results in both Example 6.15 and Example 6.17 were calculated using the following Matlab script.

```
%Define symbolic variables z and w
syms z w

%Define the refinement symbol p, the wavelet symbols q1 and
  q2 and the vanishing moment recovery function s
p=(1+z)^2/4;
s=1/3*(4+1/2*(-z-1/z));
q1=(1-z)^2/4;
q2=sqrt(6)/24*(1-z)^2*(1+4*z+z^2);

%Define the polynomial theta from the Fejer-Riesz
  factorization s=|theta|^2
u=4+sqrt(sym(15));
theta=(z-u)/sqrt(6*u);

%Now build the matrix polynomials S and R
S=simplify([subs(theta,z,z) 0; 0 subs(theta,z,-z)]);
Q=simplify([q1 subs(q1,z,-z); q2 subs(q2,z,-z)]);
pvec=[subs(p,z,z),subs(p,z,-z)];
Rlower=simplify(subs(theta,z,z^2)*pvec);
R=[Q;Rlower];

%Choose four points in the disc or five points on the torus
  and evaluate S and R in these points
values={sym(1),sym(-1),sym(1i),sym(-1i),sym(1/sqrt(2)*(1+1i))
  };
nop=5;
points=cell(1,nop);
Scell=cell(1,nop);
Rcell=cell(1,nop);
for k=1:nop
   Scell{k}={simplify(subs(S,z,values{k})),k};
   Rcell{k}={simplify(subs(R,z,values{k})),k};
end

%Preallocate the necessary cell arrays and matrices
LS=cell(nop);
N=cell(nop);
Nmat=sym(zeros(nop));
test=sym(zeros(nop));
```

```
%The boolean variable lnec indicates if some of the points
  chosen for the interpolation are on the torus, i.e., if
  interpolation is necessary to fill some diagonal blocks.
  The variable countpoints saves the number of blocks that
  will have to be filled with interpolation, and the matrix
  lpoints saves the indices of those blocks.
lnec=false;
countpoints=0;
lpoints=zeros(nop^2,2);

for k=1:nop
   for j=1:nop
      %Calculate the left side of equation (6.8)
      LS{k,j}=simplify(Scell{k}{1}'*Scell{j}{1}-Rcell{k}{1}'*
        Rcell{j}{1});
      %Write the block in the matrix-form of the tableau
      LSmat((2*(k-1)+1):(2*k),(2*(j-1)+1):(2*j))=LS{k,j};

      %If there are blocks that are not uniquely defined, set
         lnec to true and adjust countpoints and lpoints
        accordingly
      if (abs(values{k})==1)&&(values{k}==values{j})
         lnec=true;
         countpoints=countpoints+1;
         lpoints(countpoints,:)=[k,j];
      %Else calculate the block N_{kj}, save it in the cell
        array N
      else
         N{k,j}=simplify(LS{k,j}/(1-values{k}'*values{j}));
         %...and write the block in the matrix form of the
           tableau.
         Nmat((2*(k-1)+1):(2*k),(2*(j-1)+1):(2*j))=N{k,j};
      end
   end
end

%Fill the remaining blocks using Lagrange interpolation and
  save the results both in N and in Nmat
if lnec
   lpoints=lpoints(1:countpoints,:);
   for nk=1:countpoints
      N{lpoints(nk,1),lpoints(nk,2)}=sym(zeros(2));
      pos=floor((lpoints(nk,2)-1)/nop)*nop;
      for mk=(pos+1):(pos+nop)
```

```
         if mk~=lpoints(nk,2)
            factor=1;
            for lk=(pos+1):(pos+nop)
               if lk~=mk && lk~=lpoints(nk,2)
                  factor=factor*((values{lpoints(nk,2)}-
                     values{lk})/(values{mk}-values{lk}));
               end
            end
            N{lpoints(nk,1),lpoints(nk,2)}=N{lpoints(nk,1),
               lpoints(nk,2)}+factor*N{lpoints(nk,1),mk};
         end
      end
   end

   for k=1:nop
      for j=1:nop
         if (abs(values{k})==1)&&(values{k}==values{j})
            Nmat((2*(k-1)+1):(2*k),(2*(j-1)+1):(2*j))=N{k,j};
         end
      end
   end
end

%If interpolation was necessary to fill some of the blocks,
  discard the last column and the last rows of blocks
if lnec
   NmatShort=Nmat(1:2*(nop-1),1:2*(nop-1));
else
   NmatShort=Nmat;
end

%Factor Nshort and discard the rows of Amat that contain only
   zeros
A=cell(1,nop-1);
Amat=chol(NmatShort,'nocheck');
AmatShort=Amat(1:5,:);

%Divide AmatShort into blocks
for j=1:(nop-1)
   A{j}=vpa(AmatShort(:,2*j-1:2*j),8);
end
```

```
%Preallocate cell arrays and matrices necessary for the
  construction of the isometry \tilde{T}
E=cell(1,nop-1);
Estar=cell(1,nop-1);
[rR,cR]=size(R);
[rS,cS]=size(S);
[rA,cA]=size(AmatShort);
Kmat=zeros(rR+rA,cA);
Kstarmat=zeros(rS+rA,cA);

%Build the vectors spanning the spaces E_* and E from the
  sketch of the proof of Theorem 6.13
for j=1:(nop-1)
    Estar{j}=[vpa(Scell{j}{1},8);vpa(values{j},8)*A{j}];
    E{j}=[vpa(Rcell{j}{1},8);A{j}];
    Kmat(:,2*j-1:2*j)=E{j};
    Kstarmat(:,2*j-1:2*j)=Estar{j};
end

%Calculate the isometry \tilde{T} (which is the T we are
  looking for)
Ttilde=mrdivide(vpa(Kmat,8),vpa(Kstarmat,8));

%Extract the blocks A,B,C,D
A=Ttilde(1:3,1:2);
B=Ttilde(1:3,3:7);
C=Ttilde(4:8,1:2);
D=Ttilde(4:8,3:7);

%Construct E and use it to construct W
E=w*sym(eye(5));
W=simplify(expand(A+B*E*inv(sym(eye(5))-D*E)*C));
```

Examples 8.7

The matrix polynomials F_1 and F_2 in Example 8.7 were calculated using the following Matlab script. The first part also gives us the symbols and vanishing moment recovery function from Example 7.5.

```
%Define symbolic variables z and w
syms z w

%Define the univariate refinement symbol p, the univariate
  wavelet symbols q1 and q2, and the univariate vanishing
  moment recovery function s
pu=(1+z)^2/4;
su=1/3*(4+1/2*(-z-1/z));
q1u=(1-z)^2/4;
q2u=sqrt(6)/24*(1-z)^2*(1+4*z+z^2);

%Define the polynomial theta from the Fejer-Riesz
  factorization su=|thetaU|^2
u=4+sqrt(sym(15));
thetaU=(z-u)/sqrt(6*u);

%Define symbolic variables for the bivariate case
syms z1 z2 w1 w2

%Build the bivariate refinement symbol, vanishing moment
  recovery function, and the polynomial theta for the
  factorization su=|thetaU|^2 (cf. the proof of Lemma 7.3)
p1=subs(pu,z,z1); p2=subs(pu,z,z2);
s1=subs(su,z,z1); s2=subs(su,z,z2);
theta1=subs(thetaU,z,z1); theta2=subs(thetaU,z,z2);

p=p1*p2;
s=s1*s2;
theta=theta1*theta2;

%Build the bivariate wavelet masks (cf. the proof of Lemma
  7.3)
b10=subs(theta1,z1,z1^2)*p1;
b11=subs(q1u,z,z1);
b12=subs(q2u,z,z1);
b20=subs(theta2,z2,z2^2)*p2;
b21=subs(q1u,z,z2);
b22=subs(q2u,z,z2);
```

```
q1=b10*b21;
q2=b10*b22;
q3=b11*b20;
q4=b11*b21;
q5=b11*b22;
q6=b12*b20;
q7=b12*b21;
q8=b12*b22;

%Now build the bivariate matrix polynomials S and R
S=simplify([subs(theta,[z1 z2],[z1 z2]) 0 0 0; 0 subs(theta,[
  z1 z2],[z1 -z2]) 0 0 ; 0 0 subs(theta,[z1 z2],[-z1 z2]) 0;
  0 0 0 subs(theta,[z1 z2],[-z1 -z2])]);
Q=simplify([subs(q1,[z1 z2],[z1 z2]) subs(q1,[z1 z2],[z1 -z2
  ]) subs(q1,[z1 z2],[-z1 z2]) subs(q1,[z1 z2],[-z1 -z2]);
subs(q2,[z1 z2],[z1 z2]) subs(q2,[z1 z2],[z1 -z2]) subs(q2,[
  z1 z2],[-z1 z2]) subs(q2,[z1 z2],[-z1 -z2]);
subs(q3,[z1 z2],[z1 z2]) subs(q3,[z1 z2],[z1 -z2]) subs(q3,[
  z1 z2],[-z1 z2]) subs(q3,[z1 z2],[-z1 -z2]);
subs(q4,[z1 z2],[z1 z2]) subs(q4,[z1 z2],[z1 -z2]) subs(q4,[
  z1 z2],[-z1 z2]) subs(q4,[z1 z2],[-z1 -z2]);
subs(q5,[z1 z2],[z1 z2]) subs(q5,[z1 z2],[z1 -z2]) subs(q5,[
  z1 z2],[-z1 z2]) subs(q5,[z1 z2],[-z1 -z2]);
subs(q6,[z1 z2],[z1 z2]) subs(q6,[z1 z2],[z1 -z2]) subs(q6,[
  z1 z2],[-z1 z2]) subs(q6,[z1 z2],[-z1 -z2]);
subs(q7,[z1 z2],[z1 z2]) subs(q7,[z1 z2],[z1 -z2]) subs(q7,[
  z1 z2],[-z1 z2]) subs(q7,[z1 z2],[-z1 -z2]);
subs(q8,[z1 z2],[z1 z2]) subs(q8,[z1 z2],[z1 -z2]) subs(q8,[
  z1 z2],[-z1 z2]) subs(q8,[z1 z2],[-z1 -z2])]);
pvec=[subs(p,[z1 z2],[z1 z2]),subs(p,[z1 z2],[z1 -z2]),subs(p
  ,[z1 z2],[-z1 z2]),subs(p,[z1 z2],[-z1 -z2])];
Rlower=simplify(subs(theta,[z1 z2],[z1^2 z2^2])*pvec);
R=[Q;Rlower];

%Build the matrix on the left-hand side of the desired
  identity, i.e., S(w)*S(z)-R(w)*R(z)
leftSide=simplify(subs(S',[z1 z2],[w1 w2])*S-subs(R',[z1 z2
  ],[w1 w2])*R);

%Fix z2=w2 on the bitorus
LS_fix_z2=simplify(subs(leftSide,w2,z2));
LS_fix_z2_torus=simplify(subs(LS_fix_z2,abs(z2),1));
```

```
%Calculate the kernel K
K=simplify(expand(LS_fix_z2_torus/(1-conj(w1)*z1)));

%To make sure that diff works as intended, substitute conj(w1
  ) with y
syms y
K_withY=simplify(subs(K,conj(w1),y));

Lambda=[eye(4);z1*eye(4);z1^2*eye(4);z1^3*eye(4)];

%Find matrix L(x2), s.t K=subs(Lambda,z1,w1)'*L*Lambda
L00=subs(K_withY,[z1 y],[0 0]);
L01=subs(diff(K_withY,y),[z1 y],[0 0]);
L02=1/2*subs(diff(K_withY,y,2),[z1 y],[0 0]);
L03=1/6*subs(diff(K_withY,y,3),[z1 y],[0 0]);
L10=subs(diff(K_withY,z1),[z1 y],[0 0]);
L11=subs(diff(diff(K_withY,z1),y),[z1 y],[0 0]);
L12=1/2*subs(diff(diff(K_withY,z1),y,2),[z1 y],[0 0]);
L13=1/6*subs(diff(diff(K_withY,z1),y,3),[z1 y],[0 0]);
L20=1/2*subs(diff(K_withY,z1,2),[z1 y],[0 0]);
L21=1/2*subs(diff(diff(K_withY,z1,2),y),[z1 y],[0 0]);
L22=1/4*subs(diff(diff(K_withY,z1,2),y,2),[z1 y],[0 0]);
L23=1/12*subs(diff(diff(K_withY,z1,2),y,3),[z1 y],[0 0]);
L30=1/6*subs(diff(K_withY,z1,3),[z1 y],[0 0]);
L31=1/6*subs(diff(diff(K_withY,z1,3),y),[z1 y],[0 0]);
L32=1/12*subs(diff(diff(K_withY,z1,3),y,2),[z1 y],[0 0]);
L33=1/36*subs(diff(diff(K_withY,z1,3),y,3),[z1 y],[0 0]);

L_withY=[L00,L10,L20,L30;L01,L11,L21,L31;L02,L12,L22,L32;L03,
  L13,L23,L33];

L=subs(L_withY,y,conj(w1));

%L has rank 10. To simplify subsequent calculations, we
  calculate Lnew=P'*L*P where everything except the 10x10
  block L0 in the upper left corner is zero
swap=[0 1;1 0];

P1=eye(16); P1(16,14)=-1;
P2=eye(16); P2(15,13)=-1;
P3=eye(16); P3(12,6)=-1/2;
P4=eye(16); P4(12,8)=1/2;
P5=eye(16); P5(12,10)=1;
P6=eye(16); P6(11,5)=-1/2;
```

```
P7=eye(16); P7(11,7)=1/2;
P8=eye(16); P8(11,9)=1;
P9=eye(16); P9(8,6)=-1;
P10=eye(16); P10(8,10)=4;
P11=eye(16); P11(8,14)=-8;
P12=eye(16); P12(7,5)=-1;
P13=eye(16); P13(7,9)=4;
P14=eye(16); P14(7,13)=-8;
P15=eye(16); P15([7,9],[7,9])=swap;
P16=eye(16); P16([8,10],[8,10])=swap;
P17=eye(16); P17([9,13],[9,13])=swap;
P18=eye(16); P18([10,14],[10,14])=swap;
P19=eye(16); P19([2,3],[2,3])=swap;
P20=eye(16); P20([3,5],[3,5])=swap;
P21=eye(16); P21([4,7],[4,7])=swap;
P22=eye(16); P22([5,9],[5,9])=swap;

P=(P22*P21*P20*P19*P18*P17*P16*P15*P14*P13*P12*P11*P10*P9*P8*
  P7*P6*P5*P4*P3*P2*P1)';

Lnew=simplify(P'*L*P);
L0=Lnew(1:10,1:10);

%L0 has a block structure where only the two 5x5 blocks on
  the diagonal are nonzero. The entries of those blocks each
  share the same Laurent polynomial factor. Using the scalar
  Fejer-Riesz theorem to factorize these two Laurent
  polynomials, we construct a diagonal matrix fr such that L0
  =fr'*L0numb*fr with a constant matrix L0numb
rootA=1/(sym(4)+sqrt(sym(15)));
rootB=-rootA;
frRA=1/(sqrt(1/rootA)*(1-rootA*z2))*eye(5);
frRB=1/(sqrt(-1/rootB)*(1-rootB*z2))*eye(5);
frR=[frRA zeros(5,5); zeros(5,5) frRB];
L0numb=simplify(subs(frR.',z2,1/z2)*L0*frR);

%Factorize L0numb=A0numb'*A0numb
A0numb=simplify(chol(L0numb,'nocheck'));

%Use A0numb and the scalar Fejer-Riesz factorization to build
   A0 such that L0=A0'*A0
frA=(sqrt(1/rootA)*(1-rootA*z2))*eye(5);
frB=(sqrt(-1/rootB)*(1-rootB*z2))*eye(5);
fr=[frA zeros(5,5); zeros(5,5) frB];
```

```
A0=simplify(A0numb*fr);

%Use A0 to build A such that L=A'*A
A=simplify([A0, zeros(10,6)]/P);

%Use A to build F1
F1=simplify(A*Lambda);

%Calculate F1'*F1 for the interpolation below
F1starF1=simplify(subs(F1,[z1 z2],[w1 w2])'*F1);

%Build the rectangular grid for the interpolation (see Remark
   8.5)
values=[0,sym(1/2),sym(-1/2),sym(1/3),sym(-1/3)];
nov=5;
nop=nov*(nov-1);
points=cell(1,nop);

for k=1:nov
   for j=1:(nov-1)
      points{(nov-1)*(k-1)+j}={[values(k), values(j)],k,j};
   end
end

%Precallocate the necessary matrices and cell arrays
N1=cell(nop,nop);
N2=cell(nop,nop);
N1mat=sym(zeros(4*nop,4*nop));
N2mat=sym(zeros(4*nop,4*nop));
LSmat=sym(zeros(4*nop,4*nop));
LS=cell(nop);
Rint=cell(nop);
Sint=cell(nop);

%For each point on the grid:
% 1. Calculate S(w1,w2)'*S(z1,z2)-R(w1,w2)'*R(z1,z2),
% 2. use the matrix-valued function F1 constructed above to
  calculate the respective block of N1 and
% 3. use the results from 1. and 2. to calculate the
  respective blocks of N2
for k=1:nop
   for j=1:nop
      LS{k,j}=simplify(subs(S,[z1 z2],points{k}{1})'*subs(S,[
        z1 z2],points{j}{1})-subs(R,[z1 z2],points{k}{1})'*
```

```
      subs(R,[z1 z2],points{j}{1}));
    LSmat((4*k-3):4*k,(4*j-3):4*j)=LS{k,j};
    N1{k,j}=simplify(subs(F1starF1,[w1,w2,z1,z2],[points{k
      }{1},points{j}{1}]));
    N1mat((4*k-3):4*k,(4*j-3):4*j)=N1{k,j};
    N2{k,j}=(LS{k,j}-(1-conj(points{k}{1}(1))*points{j
      }{1}(1))*N1{k,j})/(1-conj(points{k}{1}(2))*points{j
      }{1}(2));
    N2mat((4*k-3):4*k,(4*j-3):4*j)=N2{k,j};
  end
end

%To lower the computational effort, we change to numerical
  values
N2matNum=vpa(N2mat);

%We factorize N2matNum into N2matNum=A2matNum'*A2matNum,
  discarding some values that are numerically zero in the
  process.
[N2evcs, N2evls]=eig(N2matNum);

N2evlsClean=zeros(4*nop,4*nop);
r=rank(N2matNum);
for j=1:r
  N2evlsClean(j,j)=N2evls(j,j);
end

A2matNum=sqrt(N2evlsClean)*N2evcs';

%Divide A2matNum into blocks. Those blocks are F2 evaluated
  in the respective points of our grid
A2Num=cell(1,nop);
for j=1:nop
  A2Num{j}=A2matNum(1:r,4*(j-1)+1:4*j);
end

%We build the matrices tildeA2, Z1 and Z2 for the
  interpolation described in Remark 8.6
tildeA2=zeros(r*nov,4*(nov-1));
Z1mat=zeros(r*nov,r*nov);
Z2mat=zeros(4*(nov-1),4*(nov-1));

for k=1:nov
  for j=1:nov
```

```
      Z1mat((r*(k-1)+1):r*k,(r*(j-1)+1):r*j)=values(k)^(j-1)*
        eye(r);
   end
end

for k=1:(nov-1)
   for j=1:(nov-1)
      Z2mat((4*(k-1)+1):4*k,(4*(j-1)+1):4*j)=values(j)^(k-1)*
        eye(4);
   end
end

for j=1:nop
   z1p=points{j}{2};
   z2p=points{j}{3};
   tildeA2((r*(z1p-1)+1):r*z1p,(4*(z2p-1)+1):4*z2p)=A2Num{j};
end

%Solve the systems of linear equations described in Remark
  8.6
tildeCmat=Z1mat\tildeA2;
Cmat=(Z2mat'\tildeCmat')';

%Cmat contains the coefficient matrices of F2. Use it to
  build F2
F2=sym(zeros(r,4));

for j=1:nov
   for k=1:(nov-1)
      F2=F2+Cmat((r*(j-1)+1):r*j,(4*(k-1)+1):4*k)*z1^(j-1)*z2
        ^(k-1);
   end
end
```

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