Analyzing Paroli and Let-It-Ride Betting Systems: Expected Gain, Variance, and Risk Measures in Roulette

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Abstract

This study presents a comprehensive analysis of the Paroli (reverse Martingale) and Let-It-Ride betting systems in roulette, combining exact analytical formulas, recursive sequence methods, and Monte Carlo simulations to quantify expected gain, variance, and normalized risk measures. We derive closed-form expressions for net gain, total wager, and associated risk metrics, enabling systematic comparisons across sequences and bet sizes. A key theoretical insight is that Paroli constitutes a mathematical mirror of the classical Martingale system: their performance formulas are related by swapping win and loss probabilities. In the fair-game limit, Paroli and Martingale are statistically equivalent, with identical moments, differing only in whether bets increase after wins or losses. Extending this framework to Let-It-Ride variants, we analyze covariances and correlations between gain and total wager to characterize the risk structure of each system. Sequence-based tables illustrate outcome distributions, while simulations validate analytical results and highlight comparative volatility. Our findings confirm that, although Paroli and Let-It-Ride strategies can generate large individual gains, the underlying house edge remains unchanged.

Keywords: Paroli betting system; Martingale; Let-It-Ride; normalized variance; per-unit gain distribution; skewness; stochastic modeling.

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Contents

1 Introduction to the Paroli System

The **Paroli system** is a well-known betting strategy in roulette, often referred to as the *Anti-Martingale*. It is a **positive progression system**, meaning that the stake is increased after a win rather than after a loss. The central idea is to **capitalize on winning streaks** while limiting exposure during losing phases. A similar intuition about streak-driven risk-taking appears in literary fiction, in Lawrence Osborne's novel *The Ballad of a Small Player* (2014), which is set in the context of baccarat and portrays gambling-related psychological escalation. The term *Paroli* derives from the Italian *parola* ("word") and entered gambling terminology through the French card game *Pharaon* in the eighteenth century. A player calling "Paroli" reinvested both the original stake and accumulated winnings instead of collecting them. The term later evolved into the English word *parlay*, now used for accumulated bets in which winnings are rolled over into subsequent wagers.

1.1 Core Principles

- **Application:** The system is typically applied to *even-money bets* (Red/Black, Even/Odd, 1–18/19–36), which offer the highest single-spin winning probability (approximately 48.6% in European roulette).
- **Fixed Base Stake:** Each sequence begins with a fixed base unit.
- **Action after a Loss:** After a loss, the stake is reset to the base unit. The bet is never increased following a loss.
- **Action after a Win:** After a win, the stake is doubled for the next spin, allowing accumulated profits to "ride."
- **Termination Rule:** In the standard three-step version, the sequence ends after **three consecutive wins**. The accumulated profit is then secured, and a new sequence begins at the base stake.

1.2 Advantages and Disadvantages

Advantages

- **Limited Downside per Sequence:** The maximum loss within a sequence is restricted to one base unit.
- **Streak Exploitation:** Profits increase geometrically during short winning streaks without requiring a large initial bankroll.
- **Reduced Escalation Risk:** Unlike negative progression systems, the Paroli strategy avoids rapidly increasing stakes during losing streaks.

Disadvantages

- **Frequent Small Losses:** Substantial gains require consecutive wins (e.g., three in a row), which occur relatively infrequently.

- **Unchanged House Edge:** The progression does not alter the underlying probabilities; the casino's edge remains fully intact.

1.3 Key Strategy Takeaway

The Paroli system is often perceived as a comparatively **conservative progression strategy**. In the three-step version, a completed sequence yields a profit of 7 units while risking only one base unit at the start of each sequence.

However, the **expected value remains negative**: no progression system can eliminate the structural house advantage. The practical appeal of Paroli lies not in overcoming probability, but in **risk structuring**—limiting losses per sequence while allowing occasional amplified gains during short winning streaks.

2 A One-Night Paroli Strategy: What Can a Player Expect?

Defining idea

This section analyzes the short-run outcome distribution of a fixed one-night Paroli strategy using Monte Carlo simulation, focusing on terminal gains, intra-session risk (drawdowns), and total wagering exposure.

2.1 Simulation Design

We consider a player who spends one evening playing American roulette. The player performs exactly 85 spins, always wagering one unit on an even-money outcome (e.g., red), and applies a Paroli strategy with a maximum progression length of $m = 3$ consecutive wins.

To evaluate short-run variability, we simulate 100,000 independent sessions under this strategy. For each session, we record:

- the terminal gain W after 85 spins,
- the minimum session wealth level (pathwise drawdown),
- the total wagered amount S.

2.2 Simulation Results

Terminal gains. The distribution of terminal gains exhibits substantial dispersion around its center, as shown in Figure 1. While most sessions result in moderate losses, both positive outcomes and large negative outcomes occur with non-negligible probability.

Across 100,000 simulated sessions:

- the central 50% of outcomes lie approximately between -20 and $+4$,
- about 5% of sessions incur losses of 35 units or more,

- about 5% of sessions yield gains above 23 units.

The resulting distribution is strongly asymmetric, reflecting the impact of rare but amplified winning sequences under the Paroli progression.

Intra-session risk. The pathwise minimum wealth level provides a measure of intra-session risk (drawdown), as illustrated in Figure 2. Unlike the terminal gain, which is measured at the end of the 85-spin session, the minimum wealth level records the worst cumulative outcome attained at any point during the session. The median minimum wealth is approximately 16 units below the starting level, indicating that typical sessions experience meaningful temporary losses before recovery or termination.

In many unfavorable sessions, the minimum wealth level coincides with the terminal gain because the session ends near its worst point. However, in other cases substantial recovery occurs after an interim drawdown. In extreme cases, the drawdown exceeds 50 units, whereas in rare favorable trajectories the wealth remains positive throughout the entire session.

Total wagering exposure. Total wagered amounts remain relatively stable across sessions. The central 90% of observations lie approximately between 130 and 155 units, reflecting the bounded structure of the Paroli progression.

2.3 Interpretation

The simulation highlights that short-run outcomes of the Paroli system are dominated by variance rather than central tendency. Most sessions generate moderate losses, but the distribution exhibits both heavy left tails (drawdowns) and a non-negligible right tail (large gains).

From a risk perspective, a bankroll sized only to cover expected outcomes would underestimate exposure. In particular, intra-session drawdowns represent a key driver of required capital buffers.

2.4 Simulation Results Tables

Table 1: Summary statistics of total gains W after 85 spins (100,000 simulations)

Min	0.05-Q.	1st Qu.	Median	Mean	3rd Qu.	0.95-Q.	Max
-64	-35	-20	-8	-7.315	4	23	80

Table 2: Summary statistics of total wagered amount S during the session (100,000 simulations)

Min	0.05-Q.	1st Qu.	Median	Mean	3rd Qu.	0.95-Q.	Max
110	130	137	142	142.413	148	155	175

2.5 Figures

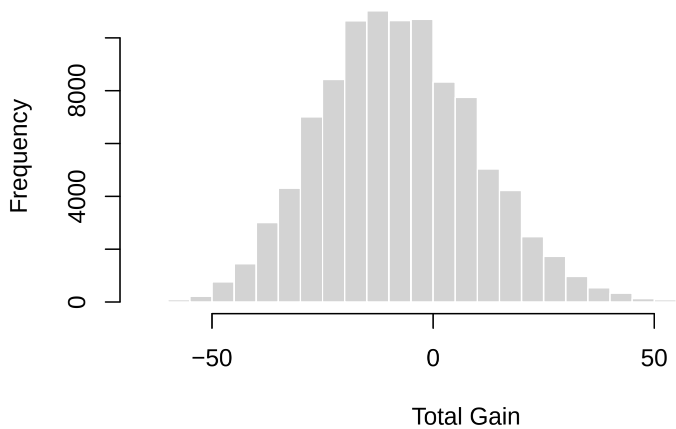

Figure 1: Histogram of total gains W from 100,000 simulated one-night Paroli sessions (85 spins, American roulette, $m = 3$). The distribution illustrates strong variability and asymmetry between losses and gains.

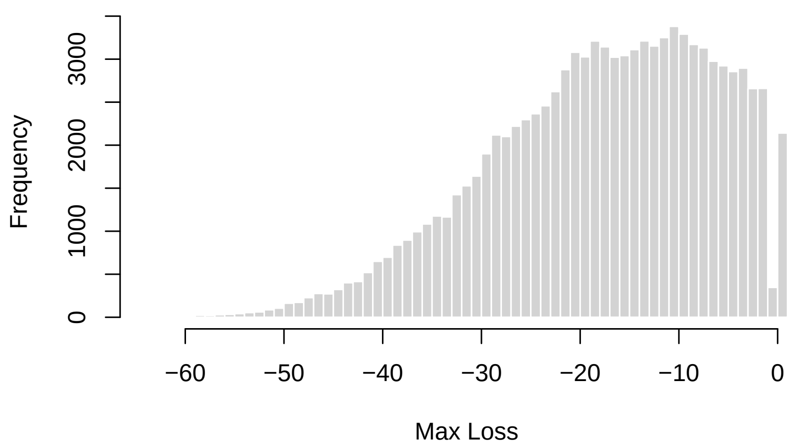

Figure 2: Histogram of minimum session wealth levels (path minima) during the 85-spin sessions. The figure illustrates intra-session drawdown risk under the Paroli progression.

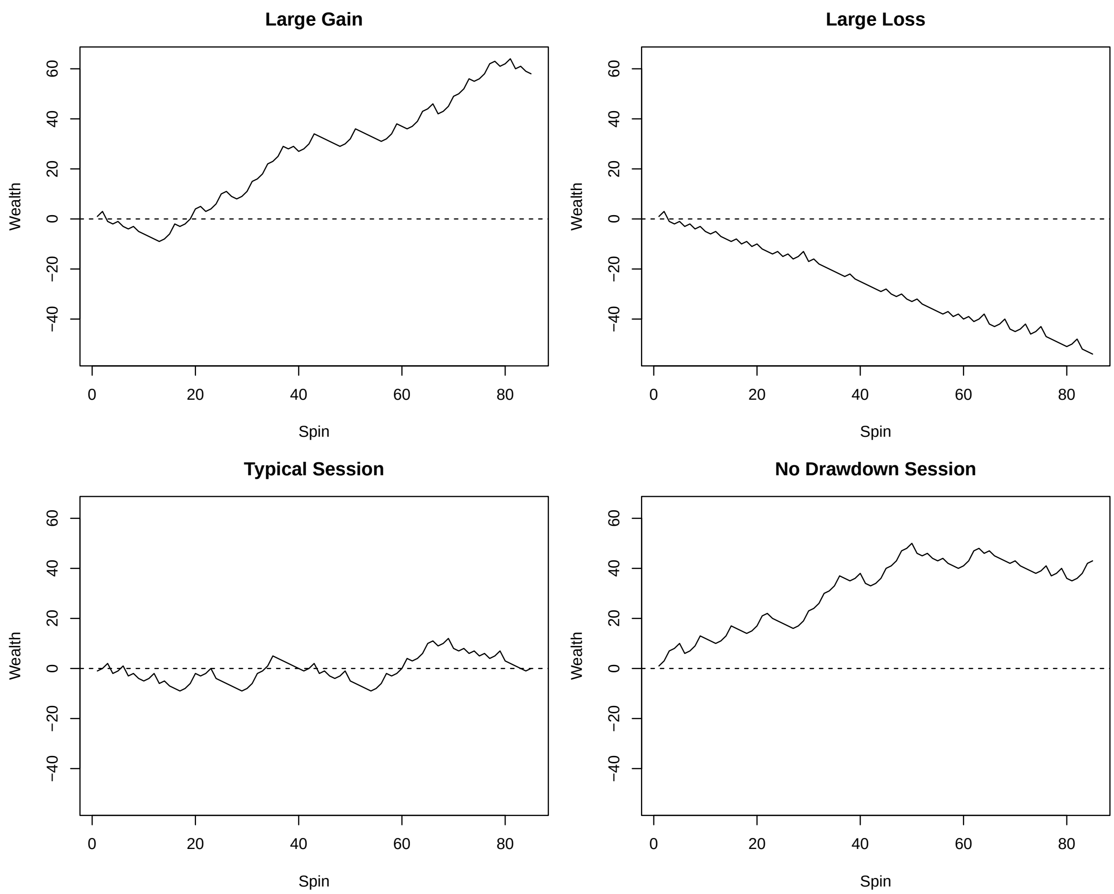

Figure 3: Four representative Paroli trajectories for American roulette (85 spins, maximum progression length $m = 3$). The trajectories illustrate a large gain outcome, a large loss outcome, a typical session, and a session without intra-period drawdown below zero.

Remark. The minimum wealth level corresponds to the lowest value reached during a session and is generally different from the terminal wealth. The two coincide only when the terminal outcome is attained at the point of the worst drawdown.

In a “no drawdown” trajectory, wealth remains non-negative throughout the session; in this case, the minimum wealth is typically either 0 or 1, depending on whether the process ever revisits zero after an initial gain (e.g., after an initial winning spin).

Illustrative outcomes: Large Gain (+58, min −9), Large Loss (−54, min −54), Typical Session (0, min −9), No Drawdown (+43, min +1).

Interpretation. The four paths highlight the strong asymmetry of short-run outcomes under the Paroli system: frequent moderate losses, occasional severe drawdowns, and rare but pronounced gains. Only a small fraction of simulated sessions remain non-negative throughout (see Figure 2).

3 Mathematical Analysis of the Paroli System on Even-Money Bets

3.1 Model Specification and Example

We consider a Paroli betting system applied to an even-money bet in American roulette with the following parameters:

- Let p denote the probability of losing a single bet ($p = 20/38$)
- Let $q = 1 - p = 18/38$ denote the probability of winning a single bet
- The system aims for m consecutive wins before stopping
- Initial bet: 1 unit
- After each win, the bet is doubled (standard Paroli progression)
- After a loss, the sequence ends and the bet resets to 1 unit

Table 3: Outcome sequences and probabilities for the Paroli system with $m = 3$ on an even-money bet. Probabilities are expressed in terms of p and q, with numeric values provided for clarity.

Sequence	Bets Placed	Total Wager S	Gain G	Probability	Numeric Probability
L	1	1	-1	p	0.5263
W,L	1, 2	3	-1	qp	0.2493
W,W,L	1, 2, 4	7	-1	q^2p	0.1181
W,W,W	1, 2, 4	7	7	q^3	0.1063

3.1.1 Moments

From Table 3, the expected gain, total wager, and their variances, as well as the expected sequence length, can be computed as follows:

$$E(G) = \sum_{i=1}^{4} G_i \cdot P_i \approx -0.150$$

$$Var(G) = \sum_{i=1}^{4} (G_i - E(G))^2 \cdot P_i \approx 6.079$$

$$E(S) = \sum_{i=1}^{4} S_i \cdot P_i \approx 2.845$$

$$Var(S) = \sum_{i=1}^{4} (S_i - E(S))^2 \cdot P_i \approx 5.671$$

$$E(L) = \sum_{i=1}^{4} L_i \cdot P_i \approx 1.698$$

The expected loss per unit wagered (house edge) is therefore

$$\text{House Edge} = -\frac{E(G)}{E(S)} \approx 0.053,$$

or approximately 5.3% of the total amount wagered.

Equivalently, the player's expected return per unit wagered is

$$\frac{E(G)}{E(S)} \approx -0.053.$$

This table-based approach provides a clear view of the Paroli sequence probabilities, potential gains, expected wagering volume, and expected sequence length. It also serves as a useful benchmark for verifying the results obtained from the closed-form formulas derived later.

3.1.2 Example Calculation of Correlation between Gain and Total Wager

From Table 3, we can also compute the expected product of gain and total wager:

$$E(GS) = \sum_{i=1}^{4} (G_i \cdot S_i) \cdot P_i \approx 3.107$$

The covariance and correlation between gain G and total wager S are then

$$\text{Cov}(G, S) = E(GS) - E(G)E(S) \approx 3.533,$$

$$\text{Corr}(G, S) = \frac{\text{Cov}(G, S)}{\sqrt{\text{Var}(G)\,\text{Var}(S)}} \approx 0.602.$$

The positive correlation indicates that large gains arise only in sequences with substantial cumulative wagers, consistent with the pattern observed in Table 3.

3.2 Random Variables

Table 3 also illustrates the distribution of the net gain G, which for $m = 3$ can take only a few discrete values depending on the number of consecutive wins. This discrete distribution corresponds to a two-point distribution, where the gain equals -1 with probability $1 - q^m$ and $2^m - 1$ with probability q^m. The expected gain and its variance can be computed directly from the table and agree with the closed-form formulas presented below.

Thus, the table provides an intuitive, sequence-based way to understand the gain distribution, while the closed-form formulas give a compact analytical expression for the same quantities.

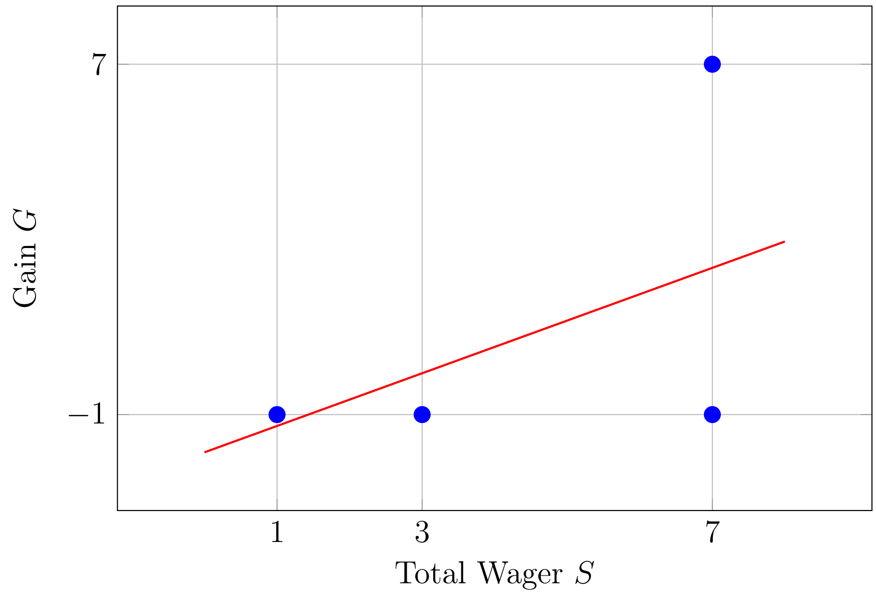

Figure 4: Scatter plot of Gain G versus Total Wager S for the Paroli system with $m = 3$. The regression line illustrates the positive correlation: larger gains occur in sequences with larger cumulative wagers.

3.2.1 Gain Distribution (G)

The net gain G in a Paroli sequence follows a **two-point distribution**:

$$G = \begin{cases} -1 & \text{with probability } 1 - q^m, \\ 2^m - 1 & \text{with probability } q^m. \end{cases}$$

3.2.2 Total Wager Distribution (S)

The total wager S in a sequence has the distribution:

$$S = \begin{cases} 2^{i+1} - 1 & \text{with probability } p \cdot q^i, \quad i = 0, 1, \dots, m-1, \\ 2^m - 1 & \text{with probability } q^m, \end{cases}$$

where i denotes the number of consecutive wins before a loss.

3.3 Moments and Expected Values

3.3.1 Expected Gain

$$E(G) = (-1) \cdot (1 - q^m) + (2^m - 1) \cdot q^m = -1 + 2^m q^m.$$

Recognizing that $2^m q^m = (2q)^m$, we obtain the compact form:

$$\boxed{E(G) = (2q)^m - 1}.$$

3.3.2 Expected Total Wager

$$E(S) = \sum_{i=0}^{m-1} (2^{i+1} - 1) \cdot p\, q^i + (2^m - 1) \cdot q^m$$

which simplifies to:

$$\boxed{E(S) = \frac{1 - (2q)^m}{1 - 2q}}, \quad 2q \neq 1.$$

3.3.3 Variance of Gain

$$\mathrm{Var}(G) = E(G^2) - (E(G))^2,$$

where

$$E(G^2) = 1 \cdot (1 - q^m) + (2^m - 1)^2 \cdot q^m.$$

Hence, the variance is:

$$\mathrm{Var}(G) = \big(1 - q^m + (2^m - 1)^2 q^m\big) - \big(-1 + 2^m q^m\big)^2.$$

This can be further expressed compactly as:

$$\boxed{\mathrm{Var}(G) = (4q)^m - (2q)^{2m}}.$$

3.3.4 Variance of Total Wager

$$\mathrm{Var}(S) = E(S^2) - (E(S))^2,$$

where

$$E(S^2) = \sum_{i=0}^{m-1} (2^{i+1} - 1)^2 \cdot p\, q^i + (2^m - 1)^2 \cdot q^m.$$

Using geometric series expansions, this simplifies to:

$$\boxed{E(S^2) = p \left[4\frac{1 - (4q)^m}{1 - 4q} - 4\frac{1 - (2q)^m}{1 - 2q} + \frac{1 - q^m}{1 - q} \right] + (2^m - 1)^2\, q^m},$$

where p is the loss probability, $q = 1 - p$ the win probability, and m the maximum number of consecutive wins.

3.3.5 Covariance of Total Gain and Total Wager in Paroli Systems

Consider a Paroli progression of length m (maximum consecutive wins before stopping) on an even-money bet. Let:

- $p =$ probability of **losing** a single bet,
- $q = 1 - p =$ probability of **winning** a single bet,
- Bet sizes follow a geometric progression: $1, 2, 4, \ldots, 2^{k-1}$ after $k-1$ consecutive wins.

The covariance between total gain G and total wager S for a Paroli round is:

$$\boxed{\mathrm{Cov}(G,S) = 2^m q^m \big(2^m - 1 - E(S)\big)},$$

where the expected total wager is

$$E(S) = \frac{1-(2q)^m}{1-2q}, \qquad 2q \neq 1.$$

For the special case $q = \frac{1}{2}$ (fair coin), take the limit:

$$\lim_{q \to 1/2} E(S) = 2m - 1.$$

Verification for $m = 3$, American Roulette. With $p = 20/38$, $q = 18/38$:

$$\begin{aligned}
2^m q^m &= 8 \cdot (18/38)^3 = 0.8503, \\
E(S) &= 2.8449, \\
2^m - 1 - E(S) &= 7 - 2.8449 = 4.1551, \\
\mathrm{Cov}(G,S) &= 0.8503 \cdot 4.1551 \approx 3.533,
\end{aligned}$$

matching direct computation from the distribution.

Interpretation. The covariance is **positive** for Paroli systems since $2^m - 1 > E(S)$ always (early stopping reduces the average total wager). This contrasts with Martingale systems, which exhibit **negative** covariance. The positive $\mathrm{Cov}(G,S)$ contributes to the discrepancy between the normalized variance $V(g) = \mathrm{Var}(G)/(E(S))^2$ and the true per-unit variance $\mathrm{Var}(G/S)$ in Paroli betting.

3.3.6 Correlation Coefficient of Total Gain and Total Wager in Paroli Systems

For a Paroli system with maximum m consecutive wins, win probability q $(0 < q < 1)$ and loss probability $p = 1 - q$, the correlation coefficient between G and S is

$$\boxed{\rho(G,S) = \frac{2^m q^m \big(2^m - 1 - E(S)\big)}{\sqrt{(4q)^m - (2q)^{2m}}\ \sqrt{\mathrm{Var}(S)}}}.$$

Here

$$E(S) = \frac{1-(2q)^m}{1-2q}, \qquad \mathrm{Var}(S) = E(S^2) - (E(S))^2,$$

$$E(S^2) = p\left[4\frac{1-(4q)^m}{1-4q} - 4\frac{1-(2q)^m}{1-2q} + \frac{1-q^m}{1-q}\right] + (2^m-1)^2 q^m.$$

For the special case $q = \frac{1}{2}$ (fair coin), we have:

$$\boxed{\rho(G,S) = \frac{2^m - m - 1}{\sqrt{(2^m-1)\big(3\cdot 2^m - m^2 - 2m - 3\big)}}.}$$

3.3.7 Expected Length of a Paroli Sequence

Let L denote the length of a Paroli sequence. The sequence continues as long as wins occur, or until a fixed maximum length m is reached. Its distribution is

$$\mathrm{P}(L=k) = \begin{cases} q^{k-1}p, & k = 1,\dots,m-1, \\ q^m, & k = m. \end{cases}$$

The expected length is

$$E(L) = \sum_{k=1}^{m-1} k\, q^{k-1}p + mq^m.$$

Using the geometric series identity

$$\sum_{k=0}^{m-1} q^k = \frac{1-q^m}{1-q},$$

this simplifies to

$$E(L) = \frac{1-q^m}{1-q} = \frac{1-(1-p)^m}{p}.$$

In the limit as $m \to \infty$, the expected length converges to

$$E(L) = \frac{1}{p},$$

the familiar mean waiting time until the first loss.

Numerical Example: For $p = 20/38 \approx 0.5263$ and $m = 3$:

$$E(L) = \frac{1-(1-p)^3}{p} = \frac{1-(18/38)^3}{20/38} \approx 1.698.$$

This shows that, on average, a Paroli sequence of length 3 lasts about 1.7 spins.

3.4 House Edge Verification

The house edge for an even-money bet in American roulette is

$$\text{House Edge} = p - q = \frac{1}{19} \approx 0.05263.$$

For the Paroli system, the expected gain per unit wagered is

$$\frac{E(G)}{E(S)} = q - p = -\frac{1}{19}.$$

This confirms that the Paroli progression does not change the fundamental house edge of the game.

Special Case: $m = 3$

For $m = 3$ consecutive wins:

$$E(G) = -1 + 8q^3,$$
$$E(S) = \frac{1 - 8q^3}{1 - 2q},$$
$$\text{Var}(G) = 64q^3(1 - q^3).$$

Numerical Example. For American roulette with $p = 20/38 \approx 0.5263$ and $q = 18/38 \approx 0.4737$:

$$E(G) \approx -0.1497, \quad E(S) \approx 2.845, \quad \text{Var}(G) \approx 6.079.$$

The expected gain per unit wagered is then

$$\frac{E(G)}{E(S)} \approx \frac{-0.1497}{2.845} \approx -0.0526,$$

confirming the house edge per unit wagered.

3.5 Key Takeaways

- Paroli reshapes the outcome distribution but does not overcome the house edge.
- $E(G)/E(S)$ measures the expected long-term loss per unit wagered.
- Longer rounds (larger m) increase variance, leading to more extreme gains or losses.
- Correlation between total gain and total wager highlights the risk of rare large outcomes.
- Approximate formulas provide near-accurate estimates, simplifying practical calculations.

4 Martingale and Paroli: Mirror Symmetry in Progression Systems

The Martingale and Paroli are two classic betting progressions that display a striking mathematical mirror symmetry. In a Martingale, the player doubles the bet after each *loss* and resets to the initial bet after a win; the goal is to recover all previous losses with a single win (see Pflaumer, 2019, for a detailed discussion). In a Paroli, the player doubles the bet after each *win* and resets after a loss; the aim is to capitalise on winning streaks while limiting losses. This section derives the statistical properties of a single round of each system, with emphasis on the correlation between total gain G and total wager S. All formulas are symmetric under swapping the loss probability p and the win probability $q = 1 - p$; for a fair coin they become exact opposites.

4.1 Distribution of Total Gain and Total Wager

Notation

- $p =$ probability of losing a single even-money bet.
- $q = 1 - p =$ probability of winning a single bet.
- Martingale round length n: stops at the first win, or after n consecutive losses if a loss limit is imposed.
- Paroli round length m: stops at the first loss, or after m consecutive wins.

All bets are assumed to be on an even-money outcome (e.g., red/black in roulette).

Martingale (stop at first win, max n losses)

$$\mathrm{G} = \begin{cases} -(2^n - 1), & \text{with probability } p^n \ (n \text{ losses}), \\ 1, & \text{with probability } 1 - p^n \ (\text{win occurs earlier}). \end{cases}$$

$$\mathrm{S} = \begin{cases} 2^k - 1, & \text{with probability } p^{k-1}(1-p),\ k = 1, \dots, n-1, \\ 2^n - 1, & \text{with probability } p^{n-1} \quad (n \text{ losses}). \end{cases}$$

Paroli (stop at first loss, max m wins)

$$\mathrm{G} = \begin{cases} -1, & \text{with probability } 1 - q^m \ (\text{loss occurs earlier}), \\ 2^m - 1, & \text{with probability } q^m \ (m \text{ wins}). \end{cases}$$

$$\mathrm{S} = \begin{cases} 2^{i+1} - 1, & \text{with probability } p\, q^i,\ i = 0, \dots, m-1, \\ 2^m - 1, & \text{with probability } q^m. \end{cases}$$

4.2 Moments and Expected Values

$$\text{Martingale:}\quad \mathrm{E(G)} = 1-(2p)^n,\quad \mathrm{Var(G)} = (4p)^n-(2p)^{2n},$$
$$\text{Paroli:}\quad \mathrm{E(G)} = (2q)^m-1,\quad \mathrm{Var(G)} = (4q)^m-(2q)^{2m}.$$

$$\mathrm{E(S)_{Mart}} = \frac{1-(2p)^n}{1-2p}\quad (p\neq 1/2),\qquad \lim_{p\to 1/2}\mathrm{E(S)_{Mart}} = n,$$
$$\mathrm{E(S)_{Par}} = \frac{1-(2q)^m}{1-2q}\quad (q\neq 1/2),\qquad \lim_{q\to 1/2}\mathrm{E(S)_{Par}} = m.$$

The variance of total wager is

$$\mathrm{Var_{Mart}(S)} = p^n\left(\frac{3\cdot 2^{2n}}{4p-1}+\frac{2^{n+1}}{1-2p}\right)+\frac{2p+1}{(2p-1)(4p-1)}-\left(\frac{1-(2p)^n}{1-2p}\right)^2,\quad p\neq 1/2,$$

$$\mathrm{Var_{Par}(S)} = \mathrm{E(S^2)_{Par}}-(\mathrm{E(S)_{Par}})^2,$$
$$\mathrm{E(S^2)_{Par}} = p\left(4\frac{1-(4q)^m}{1-4q}-4\frac{1-(2q)^m}{1-2q}+\frac{1-q^m}{1-q}\right)+(2^m-1)^2q^m.$$

For $p=q=1/2$ and $n=m$, both systems share the same variance:

$$\mathrm{Var(S)} = 3\cdot 2^n-n^2-2n-3.$$

4.3 Covariance Between Total Gain and Total Wager

Martingale (general $p\neq 1/2$)

$$\mathrm{Cov_{Mart}(G,S)} = \frac{2^{n+1}(1-p)p^n(1-(2p)^n)}{1-2p}-2^{2n}p^n(1-p^n),\qquad \lim_{p\to 1/2}\mathrm{Cov_{Mart}(G,S)} = n+1-2^n.$$

Paroli (general $q\neq 1/2$)

$$\mathrm{Cov_{Par}(G,S)} = 2^mq^m(2^m-1-\mathrm{E(S)_{Par}}),\qquad \lim_{q\to 1/2}\mathrm{Cov_{Par}(G,S)} = 2^m-1-m.$$

4.4 Correlation Coefficient $\rho(\mathrm{G,S})$

- **Martingale** $(p,\, n)$:

$$\boxed{\rho_{\mathrm{Mart}}(\mathrm{G,S}) = \frac{2^np^n(\mathrm{E(S)_{Mart}}-(2^n-1))}{\sqrt{(4p)^n-(2p)^{2n}}\,\sqrt{\mathrm{Var_{Mart}(S)}}}}.$$

- **Paroli** $(q,\, m)$:

$$\boxed{\rho_{\mathrm{Par}}(\mathrm{G,S}) = \frac{2^mq^m(2^m-1-\mathrm{E(S)_{Par}})}{\sqrt{(4q)^m-(2q)^{2m}}\,\sqrt{\mathrm{Var_{Par}(S)}}}}.$$

Swapping p and q (with $n=m$) transforms one formula into the other.

Fair-Coin Case ($p = q = 1/2$, $n = m$)

$$\rho_{\text{Mart}}(\text{G}, \text{S}) = -\rho_{\text{Par}}(\text{G}, \text{S}) = \frac{n + 1 - 2^n}{\sqrt{(2^n - 1)(3 \cdot 2^n - n^2 - 2n - 3)}}.$$

4.5 Numerical Illustration

Table 4 reports the correlation coefficients $\rho(\text{G}, \text{S})$ for American roulette ($p = 20/38$, $q = 18/38$) with progression lengths $n = m = 2$ to 10. The mirror relation $\rho_{\text{Par}}(m, q) = \rho_{\text{Mart}}(m, 1 - q)$ is clearly visible; as $p \to 1/2$, the coefficients become exact opposites.

Table 5 presents the correlation coefficients $\rho(\text{G}, \text{S})$ for Martingale rounds in American and European roulette, along with the approximate formula. The approximation provides nearly accurate values across all considered round lengths.

For $p = 20/38$, the mirror relation reproduces the numbers in Table 4, while in the fair-coin limit ($p = q = 1/2$) the two systems yield exact opposite correlations, as illustrated by the closed-form formula.

Table 4: Correlation coefficients $\rho(G, S)$ for Martingale and Paroli in American roulette ($p = 20/38$).

$n = m$	$\rho_{\text{Martingale}}$	ρ_{Paroli}
2	-0.5872	$+0.5669$
3	-0.6319	$+0.6017$
4	-0.6384	$+0.6001$
5	-0.6350	$+0.5901$
6	-0.6291	$+0.5791$
7	-0.6233	$+0.5696$
8	-0.6184	$+0.5622$
9	-0.6147	$+0.5568$
10	-0.6120	$+0.5530$

The Martingale and Paroli systems are mathematical mirrors: all moment formulas are related by swapping loss and win probabilities. Correlation between total gain and total wager is negative for Martingale (rare large losses) and positive for Paroli (rare large gains). For a fair coin they are exact opposites, illustrating the perfect symmetry of these progression systems.

Thus the "mirror" is algebraic: the same function evaluated at complementary probabilities. The table (with $p = 20/38$, $q = 18/38$) illustrates this near-symmetry – the values are not negatives because $p \neq q$, but they are mirror images with respect to the line $\rho = 0$.

Table 5: Correlation coefficients $\rho(G,S)$ for Martingale rounds in American and European Roulette, and the approximate formula ($n = 2$ to 14).

n	$\rho_{\text{Mart}}^{\text{Am}}$	$\rho_{\text{Mart}}^{\text{Eur}}$	$\rho_{\text{Mart}}^{\text{approx}}$
2	-0.5872	-0.5825	-0.5774
3	-0.6319	-0.6248	-0.6172
4	-0.6384	-0.6295	-0.6198
5	-0.6350	-0.6245	-0.6132
6	-0.6291	-0.6174	-0.6048
7	-0.6233	-0.6107	-0.5971
8	-0.6184	-0.6052	-0.5910
9	-0.6147	-0.6011	-0.5864
10	-0.6120	-0.5982	-0.5832
11	-0.6102	-0.5962	-0.5811
12	-0.6090	-0.5949	-0.5796
13	-0.6082	-0.5940	-0.5787
14	-0.6077	-0.5935	-0.5782

5 Skewness of Paroli and Martingale Rounds

For any round that results in only two possible net gains (a loss or a win), the skewness of the total gain G follows the universal two-outcome formula

$$\gamma_1 = \frac{2p_{\text{loss}} - 1}{\sqrt{p_{\text{loss}}(1 - p_{\text{loss}})}},$$

where p_{loss} is the probability of a *losing* round (see Pflaumer, 2026, pp. 203f.).

Applying this to the two progression systems:

- **Paroli** (length m, single-bet win probability q): The round loses (net gain -1) with probability $1 - q^m$ and wins (net gain $2^m - 1$) with probability q^m. Hence $p_{\text{loss}} = 1 - q^m$ and
$$\boxed{\gamma_1^{\text{Paroli}} = \frac{1 - 2q^m}{\sqrt{q^m(1 - q^m)}}}.$$
- **Martingale** (length n, single-bet loss probability p): The round wins (net gain $+1$) with probability $1 - p^n$ and loses (net gain $-(2^n - 1)$) with probability p^n. Hence $p_{\text{loss}} = p^n$ and
$$\boxed{\gamma_1^{\text{Martingale}} = \frac{2p^n - 1}{\sqrt{p^n(1 - p^n)}}}.$$

Remarks.

- For a fair coin ($p = q = \frac{1}{2}$ and $m = n$), $\gamma_1^{\text{Martingale}} = -\gamma_1^{\text{Paroli}}$, reflecting the mirror symmetry between the two systems.

- In typical casino roulette ($q < 1/2$), Paroli exhibits positive skew (rare large gains), while Martingale exhibits negative skew (rare large losses).
- These skewness measures depend only on the round win/loss probabilities, not on the specific bet sizes, analogous to the flat-bet case.
- **Use of skewness:** Skewness γ_1 provides a third parameter, alongside μ and σ^2, that could refine normal approximations of $\mathrm{P}(G < 0)$ by accounting for distributional asymmetry (see McCloskey, 2025). In this paper, the approximations are computed without this correction, but the remark illustrates why calculating skewness can be informative, particularly for progression systems.

Visual Illustration and Interpretation of Skewness

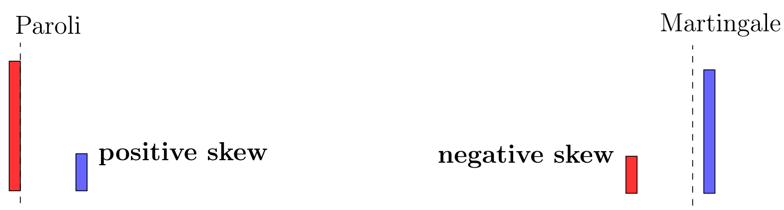

Figure 5: Illustration of skewness for Paroli and Martingale rounds. Red bars denote losses, blue bars denote wins. Bar heights indicate relative frequencies only; exact probabilities and payoffs are omitted for clarity.

- **Positive skew (Paroli):** The distribution has a long tail to the right. Most outcomes are small losses, but occasionally a large gain occurs.
- **Negative skew (Martingale):** The distribution has a long tail to the left. Most outcomes are small gains, but occasionally a large loss occurs.

The numerical values of skewness for Paroli and Martingale rounds across American Roulette, European Roulette, and a fair coin are summarized in Table 6, and the visual representation for American Roulette is shown in Figure 6.

6 Expected Return and Variance per Unit Bet

6.1 Per-Unit Gain Distribution for Paroli

Define the per-unit gain $R = G/S$. Its distribution is

$$\boxed{R = \begin{cases} -\dfrac{1}{2^{i+1}-1}, & \text{with probability } pq^i, \quad i = 0, 1, \ldots, m-1, \\ 1, & \text{with probability } q^m. \end{cases}}$$

Table 6: Skewness of Paroli and Martingale rounds. Columns correspond to American Roulette (double zero, single-bet win probability 18/38), European Roulette (single zero, 19/37), and a Fair Coin.

n/m	Mart. Am.	Par. Am.	Mart. Eur.	Par. Eur.	Mart. Fair	Par. Fair
1	0.1054	0.1054	0.0541	0.0541	0.0000	0.0000
2	-0.9966	1.3214	-1.0726	1.2391	-1.1547	1.1547
3	-2.0074	2.5549	-2.1311	2.4115	-2.2678	2.2678
4	-3.1804	4.1129	-3.3846	3.8617	-3.6148	3.6148
5	-4.6694	6.2416	-5.0042	5.8074	-5.3882	5.3882
6	-6.6383	9.2486	-7.1802	8.5117	-7.8113	7.8113
7	-9.2951	13.5607	-10.1594	12.3316	-11.1807	11.1807
8	-12.9167	19.7874	-14.2766	17.7691	-15.9061	15.9061
9	-17.8799	28.8082	-19.9938	25.5379	-22.5611	22.5611
10	-24.7004	41.8972	-27.9517	36.6574	-31.9531	31.9531

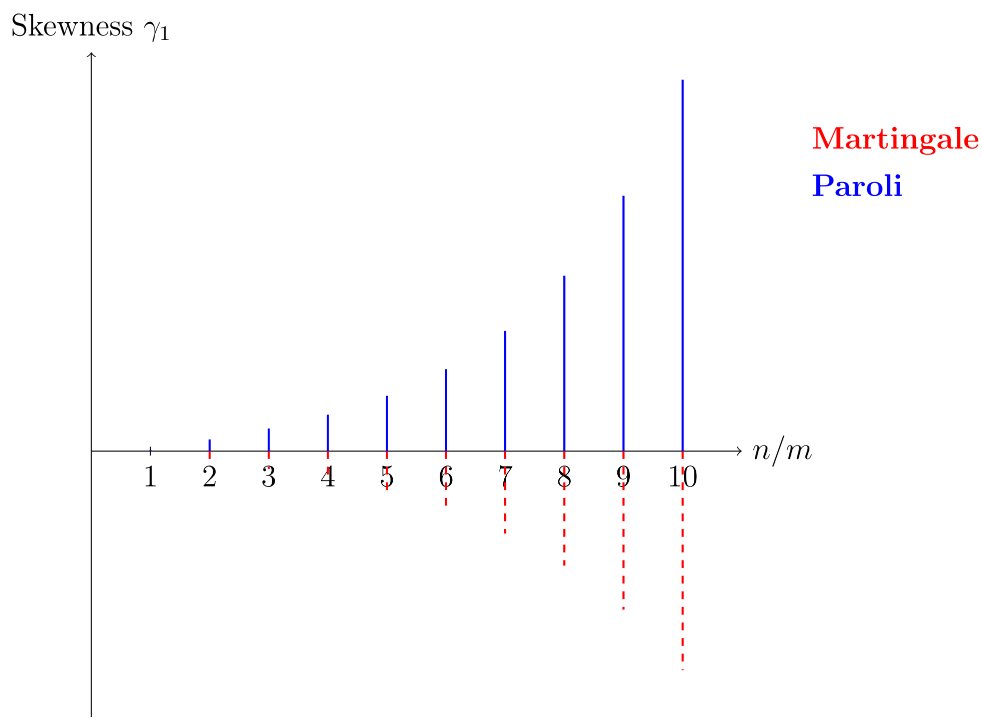

Figure 6: Skewness of Paroli and Martingale rounds for American Roulette (single-bet win probability 18/38). Red dashed: Martingale; Blue solid: Paroli.

Example: Paroli with $m = 3$ (American Roulette)

For a Paroli progression of length $m = 3$ in American roulette ($p = 20/38$, $q = 18/38$), the per-unit gain $R = G/S$ has the distribution

$$R = \begin{cases} -1 & \text{with probability } \dfrac{10}{19} \approx 0.5263, \\ -\dfrac{1}{3} & \text{with probability } \dfrac{90}{361} \approx 0.2493, \\ -\dfrac{1}{7} & \text{with probability } \dfrac{810}{6859} \approx 0.1181, \\ 1 & \text{with probability } \dfrac{729}{6859} \approx 0.1063. \end{cases}$$

The expected value equals

$$\mathrm{E}(R) = -\frac{24967}{48013} \approx -0.5200,$$

and the variance is

$$\mathrm{Var}(R) = \frac{222731}{336091} - \left(\frac{24967}{48013}\right)^2 \approx 0.392.$$

6.2 Per-Unit Gain Distribution for Martingale

Define again $R = G/S$. For a Martingale round of length n (stop at the first win, with at most n consecutive losses), the distribution is

$$\boxed{R = \begin{cases} \dfrac{1}{2^k - 1}, & \text{with probability } (1-p)p^{k-1}, \quad k = 1, \ldots, n-1, \\ -1, & \text{with probability } p^n. \end{cases}}$$

Here p denotes the probability of losing a single bet. For $k = 1$, we obtain $1/(2^1 - 1) = 1$, so $R = 1$ occurs with probability $(1 - p)$, corresponding to an immediate win.

6.3 Interpretation of $E(G/S)$ and $\mathrm{Var}(G/S)$

When evaluating a betting system, it is natural to ask: "How much do I win or lose per dollar wagered?" This is captured by the ratio

$$R = \frac{G}{S},$$

where G is the total gain and S is the total wager.

For a fixed betting amount (e.g., \$10 every spin), S is constant and the calculation is straightforward. However, in systems like Paroli or Martingale, the amount wagered varies from round to round — S is random — so we must compute R separately for each possible sequence of outcomes.

For deterministic betting schemes with fixed total wager S (non-random),

$$R = G/S, \qquad E(R) = E(G)/S, \qquad \mathrm{Var}(R) = \mathrm{Var}(G)/(S^2).$$

In this case, $E(R)$ represents the mean return per unit bet, whereas $\mathrm{Var}(R)$ quantifies the corresponding variability (risk).

For typical roulette parameters (without the Partager rule),

$$E(R) \approx -0.056 \quad \text{(American Roulette)}, \qquad E(R) \approx -0.027 \quad \text{(European Roulette)}.$$

When both G and S are random variables, as in Paroli or Martingale systems,

$$E(G/S) \neq E(G)/E(S), \qquad \mathrm{Var}(G/S) \neq \mathrm{Var}(G)/(E(S))^2.$$

This means you cannot simply take the average gain and divide by the average wager — the ratio of averages is not the same as the average of the ratio. Each possible sequence contributes its own G/S, and these must be weighted by their probabilities.

In Paroli sessions, $E(G/S) < 0$, indicating that losses dominate the average return. In contrast, Martingale sessions often yield $E(G/S) > 0$, since most rounds terminate with a win. Thus, $E(G/S)$ directly measures the profitability of a session. This difference is intuitive: in a typical Martingale round you win a small amount (e.g., 1 unit) after a short losing streak, while in a typical Paroli round you lose your initial bet before any winning streak can develop. Both systems are still subject to the house edge, but the distribution of returns feels very different to the player. From a behavioural perspective, this difference is important: in a typical Martingale round the player frequently experiences a small positive return, whereas in Paroli early losses are more common. This asymmetry influences the perceived success of the betting session, even though the underlying house edge remains unchanged.

Limiting Behavior. As the number of levels increases, both $E(G/S)$ and $\mathrm{Var}(G/S)$ approach finite limiting values:

- For Paroli, $E(G/S)$ converges to a small negative value, while $\mathrm{Var}(G/S)$ remains bounded.
- For Martingale, $E(G/S)$ converges to a small positive value, with $\mathrm{Var}(G/S)$ also remaining finite.

This demonstrates that per-unit proftability and risk remain stable even for long sessions, in contrast to naive approximations (such as delta-method expansions), which may diverge for large progression lengths.

6.4 Calculation of Expected Return and Variance per Unit Bet

For a Paroli betting system (stop after m consecutive wins), the return per unit bet is defined as

$$R = \frac{G}{S},$$

where G is the total gain of a sequence and S is the total amount wagered. We are interested in the expected return $E(R)$ and its variance $\mathrm{Var}(R)$.

Closed-form expressions for $E(R)$ and $\mathrm{Var}(R)$ are generally not available due to the path-dependent, nonlinear nature of G and S. In Paroli, gains accumulate geometrically on consecutive wins, while the probabilities of different sequences vary. A similar reasoning applies to Martingale strategies, where bets grow geometrically after consecutive losses.

In principle, $E(R)$ and $\mathrm{Var}(R)$ can be computed as follows:

Table 7: Expected return $E(G/S)$ and variance $\mathrm{Var}(G/S)$ for Paroli and Martingale strategies. Negative $E(G/S)$ for Paroli illustrates the anti-Martingale effect. Values are rounded to three digits.

m	Paroli		Martingale		m	Paroli		Martingale	
	$E(G/S)$	$\mathrm{Var}(G/S)$	$E(G/S)$	$\mathrm{Var}(G/S)$		$E(G/S)$	$\mathrm{Var}(G/S)$	$E(G/S)$	$\mathrm{Var}(G/S)$
1	-0.053	0.997	0.053	0.997	1	0.000	1.000	0.000	1.000
3	-0.520	0.392	0.430	0.465	3	-0.476	0.429	0.476	0.429
5	-0.607	0.212	0.541	0.252	5	-0.575	0.231	0.575	0.231
10	-0.631	0.160	0.580	0.170	10	-0.606	0.165	0.606	0.165
20	-0.631	0.158	0.582	0.166	20	-0.607	0.163	0.607	0.163

(a) American Roulette, $p = 20/38$ (b) Fair Roulette, $p = 0.5$

1. Enumerate all possible sequences up to the maximum number of levels m, or simulate them for large m.

2. For each sequence i, calculate the total gain G_i, total wager S_i, and return

$$R_i = \frac{G_i}{S_i}.$$

3. Assign the corresponding probability P_i to each sequence.

4. Compute

$$E(R) = \sum_i R_i\, P_i, \qquad \mathrm{Var}(R) = \sum_i R_i^2\, P_i - \Big(\sum_i R_i\, P_i\Big)^2.$$

Delta-method approximation. A common simplification is to approximate R by a first-order Taylor expansion around the expected total wager $E(S)$:

$$R = \frac{G}{S} \approx \frac{G}{E(S)} - \frac{G}{E(S)^2}(S - E(S)).$$

This approximation works well when S is close to its average, but fails when rare extreme sequences dominate — which is exactly the case for Martingale with large n.

Under this approximation,

$$E(R) \approx \frac{E(G)}{E(S)}, \qquad \mathrm{Var}(R) \approx \frac{\mathrm{Var}(G)}{E(S)^2} - 2\frac{E(G)}{E(S)^3}\mathrm{Cov}(G,S) + \frac{E(G)^2}{E(S)^4}\mathrm{Var}(S).$$

For Paroli, this approximation is reasonable for moderate m, but for Martingale strategies with increasing m, the first term $\mathrm{Var}(G)/E(S)^2$ can explode due to rare extreme sequences. Computing R pathwise, $R_i = G_i/S_i$, avoids this issue and provides bounded, accurate variance values.

Comparison with Martingale. While Paroli limits risk by stopping after m consecutive wins, Martingale strategies increase risk with each loss. As a result, the variance of the return per unit bet $\mathrm{Var}(R)$ remains moderate and bounded for Paroli, but for Martingale, extreme losing sequences can dominate the variance of total gain, making the delta-method approximation unreliable. Computing R pathwise ensures accurate, bounded variance values for both systems, highlighting the anti-Paroli nature of Martingale strategies.

Example

Let p be the probability of losing a single bet and $q = 1 - p$ the probability of winning.

6.4.1 Paroli ($m = 3$ consecutive wins)

$$E\left(\frac{G}{S}\right)_{\text{Paroli}} = -1 + \frac{2}{3}q + \frac{4}{21}q^2 + \frac{8}{7}q^3,$$
$$E\left(\left(\frac{G}{S}\right)^2\right)_{\text{Paroli}} = 1 - \frac{8}{9}q - \frac{40}{441}q^2 + \frac{48}{49}q^3,$$
$$\mathrm{Var}\left(\frac{G}{S}\right)_{\text{Paroli}} = E\left(\left(\frac{G}{S}\right)^2\right)_{\text{Paroli}} - \Big(E(G/S)_{\text{Paroli}}\Big)^2.$$

6.4.2 Martingale ($n = 3$ consecutive losses)

$$E\left(\frac{G}{S}\right)_{\text{Mart}} = 1 - \frac{2}{3}p - \frac{4}{21}p^2 - \frac{8}{7}p^3,$$
$$E\left(\left(\frac{G}{S}\right)^2\right)_{\text{Mart}} = 1 - \frac{8}{9}p - \frac{40}{441}p^2 + \frac{48}{49}p^3,$$
$$\mathrm{Var}\left(\frac{G}{S}\right)_{\text{Mart}} = E\left(\left(\frac{G}{S}\right)^2\right)_{\text{Mart}} - \Big(E(G/S)_{\text{Mart}}\Big)^2.$$

7 Standardized Variance for Paroli Betting System

To compare the risk of the Paroli strategy with other betting strategies, we standardize the variance to a 1-unit bet.

If the total wager S is deterministic (constant), the per-unit variance is simply

$$\mathrm{Var}(g) = \frac{\mathrm{Var}(G)}{(S)^2}.$$

For the general case where S is random, we standardize using the expected total wager:

$$\mathrm{Var}(g) = \frac{\mathrm{Var}(G)}{(E(S))^2}.$$

This formula allows a direct comparison of the relative risk of different strategies by normalizing the variance to the bet size, whether the stake is fixed or variable.

Note that $\mathrm{Var}(g)$ is not the variance of the per-unit return $R = G/S$; rather, it is a scaled version of the gain variance that enables a fair comparison between strategies with different average bet sizes. In other words, it answers the question: "If the average bet size were 1 unit, what would be the variance of the total gain?"

7.1 Example: Paroli Betting System

Consider the original Paroli betting system with $m = 3$. The expected gain $E(G)$, variance of gains $\mathrm{Var}(G)$, expected stake $E(S)$, and variance of stake $\mathrm{Var}(S)$ are as follows:

- $E(G) = -0.1497$
- $\mathrm{Var}(G) = 6.0792$
- $E(S) = 2.8449$
- $\mathrm{Var}(S) = 5.6712$
- $\mathrm{Var}(g) = 0.7511 \quad \big(\mathrm{Var}(G)/(E(S))^2\big)$

7.2 Interpretation of Standardized Variance

The standardized variance of the Paroli strategy, $\mathrm{Var}(g) = 0.7511$, allows for direct comparison with other betting systems, such as the James Bond strategy (see, e.g., Pflaumer (2026a), pp. 62 ff.). By standardizing the variance, we account for differences in bet sizes, making risk comparisons straightforward. This provides a consistent metric to evaluate the relative risk across various betting strategies, whether the stakes are fixed or variable.

This method of standardizing variance is also applicable to other betting strategies, including the Martingale system. Applying the same formula allows for a fair and consistent comparison of risk across different approaches. In both cases, the standardized variance offers a clear measure of the relative risk profile.

In conclusion, using $\mathrm{Var}(g) = 0.7511$ for Paroli demonstrates how standardized variance provides an effective tool for comparing betting systems based on their inherent risks.

8 Gain Distribution of an Even-Money Paroli

8.1 Gain of a Single Paroli Round

Let G denote the gain of one Paroli round consisting of m consecutive wins. Each round is independent and identically distributed with

$$E(G) = (2q)^m - 1, \qquad \mathrm{Var}(G) = (4q)^m - (2q)^{2m},$$

where q is the single-spin win probability.

For the most common roulette variants we have:

$$q = \begin{cases} \dfrac{18}{38}, & \text{American Roulette,} \\ \dfrac{18}{37}, & \text{European Roulette.} \end{cases}$$

These expressions fully characterize the gain distribution of one Paroli round and form the basis for aggregating multiple rounds and applying a normal approximation.

8.2 Aggregation over Multiple Paroli Rounds

Let

$$W = \sum_{i=1}^{M} G_i$$

denote the total gain after M independent Paroli rounds, where each G_i is identically distributed with

$$E(G) = (2q)^m - 1, \qquad \mathrm{Var}(G) = (4q)^m - (2q)^{2m}.$$

By linearity of expectation and independence we obtain

$$E(W) = M\big((2q)^m - 1\big),$$

and

$$\mathrm{Var}(W) = M\left[(4q)^m - (2q)^{2m}\right].$$

8.3 Normal Approximation

Each Paroli round G_i is a simple two-point distribution: either the player loses the base unit or achieves a positive gain after a winning sequence of length m. The total gain after M independent Paroli rounds is

$$W = \sum_{i=1}^{M} G_i,$$

which can be viewed as the sum of M independent two-point (binomial-type) random variables.

For sufficiently large M, the distribution of W can be approximated by a normal distribution using the Central Limit Theorem:

$$W \approx \mathcal{N}\Big(M\,E(G),\ M\ \mathrm{Var}(G)\Big) = \mathcal{N}\Big(M\big((2q)^m - 1\big),\ M\left[(4q)^m - (2q)^{2m}\right]\Big),$$

where q is the probability of winning a single bet. This approximation becomes increasingly accurate as M grows, allowing simple computation of probabilities, quantiles, and risk measures for the total gain.

While a single Paroli round has a highly skewed payoff distribution, the sum of many independent rounds converges to a smooth bell-shaped distribution, with mean and variance growing linearly in M.

Interpretation and Usefulness of the Normal Approximation Approximating the total gain W by a normal distribution is useful because each Paroli round is a simple two-point outcome, and summing many independent rounds produces a distribution that is roughly bell-shaped. This approach provides an intuitive understanding of the overall risk and reward without having to enumerate every possible sequence or run extensive simulations.

By using the normal approximation, we can quickly estimate probabilities of winning or losing, expected ranges of outcomes, and likely extreme results. It allows readers to grasp the key risk characteristics of the Paroli system in a clear and efficient way, complementing the empirical tables and figures shown earlier. This method becomes increasingly accurate as the number of Paroli rounds grows, making it a practical alternative to detailed simulation studies.

8.4 Probability Statements under the Normal Approximation

Using the normal approximation

$$W \approx \mathcal{N}(E(W), \mathrm{Var}(W)) \,,$$

we obtain for any threshold w

$$\mathrm{P}(W < w) \approx \Phi\left(\frac{w - E(W)}{\sqrt{\mathrm{Var}(W)}}\right),$$

where $\Phi(\cdot)$ denotes the standard normal distribution function.

In particular, the probability of an overall loss after M Paroli rounds is

$$\mathrm{P}(W < 0) \approx \Phi\left(\frac{-E(W)}{\sqrt{\mathrm{Var}(W)}}\right).$$

Interpretation. Although each Paroli round has a strictly negative expected value for realistic roulette parameters, the distribution of W is centered at $E(W)$ and spreads with rate $\sqrt{M}$, allowing short-term gains to occur with non-negligible probability.

8.5 Example: Total Gain Distribution after 50 Paroli Rounds

To illustrate the distribution of total gains under the Paroli system, we consider $M = 50$ independent Paroli rounds with a maximum progression of $m = 3$ consecutive wins. On average, each Paroli round lasts about 1.7 spins, so 50 rounds correspond to roughly 85 spins or plays.

We compare two common roulette variants: American Roulette and European Roulette.

Using the normal approximation for the sum of independent rounds, we compute the expected total gain E(W), variance, standard deviation, probabilities of net profit $P(W > 0)$, probabilities of a large gain $P(W > 25)$, and quantiles of the total gain distribution.

The summary statistics are presented in Table 9, while the corresponding quantiles for the 0.1 to 0.9 levels are shown in Table 8. From these results, it is evident that the house edge and the probability of achieving positive gains vary across the roulette variants. American roulette shows the largest negative expected value.

Overall, these tables demonstrate how the Paroli system produces a wide range of possible outcomes, with short-term gains possible even under a negative expected value, and how small changes in game rules can affect the distribution of total gains.

Table 8: Quantiles (0.1–0.9) of the total gain W after $M = 50$ Paroli rounds for different roulette variants.

Roulette	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
American	-29.83	-22.16	-16.63	-11.90	-7.49	-3.07	1.66	7.19	14.86
European	-27.09	-19.14	-13.41	-8.52	-3.95	0.63	5.52	11.25	19.19

The *quantile function* of the total gain W under the normal approximation is defined as the inverse of the cumulative distribution function:

$$w_\alpha = \Phi^{-1}(\alpha) \cdot \sqrt{\mathrm{Var}(W)} + E(W),$$

where w_α is the α-quantile, i.e., the gain level such that

$$\mathrm{P}(W < w_\alpha) = \alpha.$$

For example, the median total gain corresponds to $\alpha = 0.5$,

$$w_{0.5} = E(W),$$

while the probability of achieving a gain above w_α is

$$\mathrm{P}(W > w_\alpha) = 1 - \alpha.$$

Interpretation of Quantiles. Quantiles provide an intuitive way to understand the distribution of total gains. For example, consider the 0.6 quantile (60^{th} percentile) from Table 8:

- American Roulette: 0.6 quantile ≈ -3.07 (mean W ≈ -7.49, SD ≈ 17.43)
- European Roulette: 0.6 quantile ≈ 0.63 (mean W ≈ -3.95, SD ≈ 18.06)

Interpretation of the 0.6 Quantile. For American Roulette, the 0.6 quantile is approximately -3.07, meaning that in 60% of the simulations (or hypothetical sets of 50 Paroli rounds), the total loss exceeds 3 units.

For European Roulette, the 0.6 quantile is approximately 0.63. The mean total gain is negative ($E(W) \approx -3.95$), so most of these 60% of outcomes are still losses, but they are smaller than 0.63 units, with occasional outcomes corresponding to a very small gain.

Comparing the quantiles to the mean and standard deviation highlights the **skewed nature of the gain distribution**: although the mean is negative, a substantial fraction of outcomes still produces positive gains. The standard deviation shows the **spread of possible outcomes**, emphasizing that short-term results can differ significantly from the expected value.

Risk Perception. Although the Paroli system allows for the possibility of substantial short-term gains, the tables illustrate that outcomes are highly variable even over 50 rounds (approximately 85 spins). Players may experience frequent small losses interspersed with occasional winning streaks, creating the perception of success while the underlying expected value remains negative for standard roulette variants. The positive correlation between gain and total wager, as discussed earlier, further emphasizes that large gains only occur in long sequences, reinforcing the idea that risk is concentrated in a small fraction of plays.

Table 9: Summary statistics for total gain W after $M = 50$ Paroli rounds for different roulette variants.

Roulette	Mean (W)	Variance	Std. dev	P(W>0)	P(W>25)
American	-7.487	303.96	17.434	0.334	0.031
European	-3.945	326.02	18.056	0.414	0.055

Interpretation of the Normal Approximations. Figure 7 illustrates the approximate distribution of total gains W after 50 Paroli rounds ($m = 3$) for American, European, and European Partage Roulette. All three distributions are roughly bell-shaped, reflecting the central limit theorem applied to the sum of independent Paroli rounds.

From the plots, it is immediately apparent that:

- American Roulette has the lowest mean and a substantial probability of negative outcomes.
- European Roulette has a higher mean and a smaller chance of losing overall.
- European Roulette with Partage shows a shift of the mean to the right, reflecting the reduction of the house edge due to the Partage rule. This shift makes small losses less likely, while the variance remains similar to European Roulette without Partage.

Remark: The normal approximation provides an idealized, convenient representation of the gain distribution. In reality, the distribution of total gains remains *slightly skewed to the right* (see Fig. 1), particularly for a limited number of Paroli rounds. Only when the number of independent Paroli rounds becomes very large does the distribution approach symmetry. If precise probabilities or quantiles are required, these should be calculated directly using the underlying Binomial distribution of wins and losses in each Paroli round.

Remark on Partage or *En Prison* Rules. From an analytical perspective, the Partage or *En Prison* rules reduce the house edge of an even-money bet to approximately 0.0135 (European Roulette). This also slightly reduces the variance of a single round. As a result, the distribution of the total gain W after M Paroli rounds is shifted to a higher expected value while maintaining a similar variance compared to the standard European Roulette case without these rules.

Using the normal approximation introduced earlier, this means that the bell-shaped curve of W is centered slightly to the right, reflecting the smaller house edge, while its width remains largely unchanged.

For readers interested in specific numerical values, simulation studies provide concrete estimates. In our simulations with 100,000 sessions, the expected total gain increased by roughly 2 units compared to European Roulette without Partage, confirming the analytical insight that these rules improve the probability of positive outcomes without substantially altering the spread of possible gains.

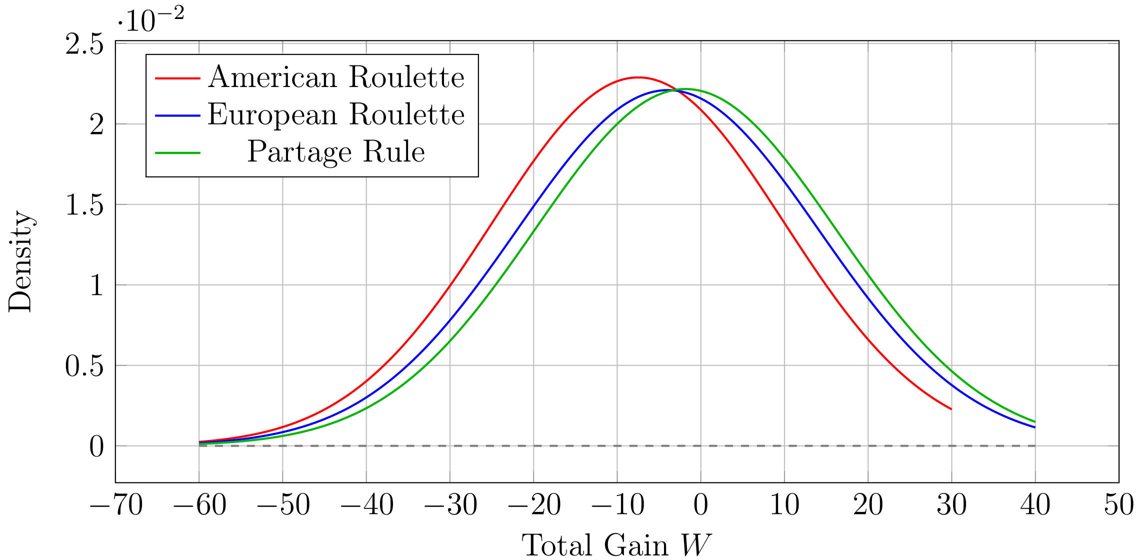

Figure 7: Normal approximations of total gain W after 50 Paroli rounds ($m = 3$) for American, European, and European Partage Roulette. Partage density uses the simulation-based mean of -1.835 from 100,000 runs.

9 Paroli System on Straight-Up Bets

9.1 Model Specification

Consider the Paroli system on a straight-up bet in American Roulette with:

- Probability of losing a single bet:

$$p = \frac{37}{38} \approx 0.9737$$

- Probability of winning a single bet:

$$q = 1 - p = \frac{1}{38} \approx 0.0263$$

- Payout multiplier: $\gamma = 35$ (win γ units per unit bet)
- Target: m consecutive wins before stopping
- Initial bet: 1 unit
- After each win: bet is doubled
- After any loss: the sequence ends

Table 10: Outcome sequences and probabilities for the Paroli system with $m = 3$ on a straight-up bet.

Sequence	Bets placed	Total wager S	Gain G	$R = G/S$	Probability
L	1	1	-1	-1.0000	0.973684
W,L	1,2	3	33	11.0000	0.025623
W,W,L	1,2,4	7	101	14.4286	0.000674
W,W,W	1,2,4	7	245	35.0000	0.000018

9.2 Decision Tree for $m = 3$

The possible sequences for $m = 3$ are summarized in Table 10.
As shown in Table 10, the probabilities of each sequence decay rapidly with the number of consecutive wins, while the potential gains grow quickly. From this table, the expected gain, total wager, and their variances can be computed directly:

$$E(G) = \sum_{i=1}^{4} (\text{Gain})_i \cdot (\text{Probability})_i \approx -0.0555$$

$$E(S) = \sum_{i=1}^{4} (\text{Total Wager})_i \cdot (\text{Probability})_i \approx 1.0554$$

$$E(R) = \sum_{i=1}^{4} (\text{Ratio } R_i = G_i/S_i) \cdot (\text{Probability})_i \approx -0.6815$$

$$\text{Var}(G) = \sum_{i=1}^{4} (\text{Gain}_i - E(G))^2 \cdot (\text{Probability})_i \approx 36.85$$

$$\text{Var}(S) = \sum_{i=1}^{4} (\text{Total Wager}_i - E(S))^2 \cdot (\text{Probability})_i \approx 0.12$$

$$\text{Var}(R) = \sum_{i=1}^{4} (R_i - E(R))^2 \cdot (\text{Probability})_i \approx 3.77$$

$$E(g) = \frac{E(G)}{E(S)} = -0.0526$$

This is the expected per-unit return, i.e., the negative value of the house edge.

9.3 General Distributions for Arbitrary m

9.3.1 Gain Distribution (G)

Let k denote the number of consecutive wins before the first loss.

For $k = 0, 1, \ldots, m-1$:

$$G_k = \gamma(2^k - 1) - 2^k = (\gamma - 1)2^k - \gamma,$$

with probability $q^k p$.

For $k = m$ (all m bets are won):

$$G_m = \gamma(2^m - 1),$$

with probability q^m.

Thus, G follows an $(m+1)$-point distribution.

Geometric Distribution (G)

Thus, the gain distribution is a transformation of the geometric distribution describing the number of wins before the first loss, with the exponential growth of bets and the payout multiplier γ modifying the distribution.

9.3.2 Total Wager Distribution (S)

For $k = 0, 1, \ldots, m-1$:

$$S_k = 2^{k+1} - 1,$$

with probability $q^k p$.

For $k = m$:

$$S_m = 2^m - 1,$$

with probability q^m.

9.3.3 Expected Values

In this section, we present only the expected values for the Paroli system: the expected gain $E(G)$, the expected total wager $E(S)$, and the per-unit expected return $E(g) = E(G)/E(S)$. These formulas are already rather complex and become even more cumbersome for second moments and variances. For practical applications, especially since the typical number of consecutive wins m is rather small, we recommend computing these moments using a working table of all possible sequences, as demonstrated in the introductory example. This approach is simpler and less error-prone than applying the fully general formulas.

9.3.4 Expected Gain

$$E(G) = \sum_{k=0}^{m-1} \left[(\gamma - 1)2^k - \gamma\right] q^k p + \gamma(2^m - 1)q^m$$

$$\text{Closed form:} \quad E(G) = (\gamma - 1)\frac{1 - (2q)^m}{1 - 2q} - \gamma\frac{1 - q^m}{1 - q} + \gamma(2^m - 1)q^m$$

9.3.5 Expected Total Wager

$$E(S) = \sum_{k=0}^{m-1} (2^{k+1} - 1)q^k p + (2^m - 1)q^m$$

$$\text{Closed form:} \quad E(S) = 2p\frac{1 - (2q)^m}{1 - 2q} - p\frac{1 - q^m}{1 - q} + (2^m - 1)q^m$$

9.3.6 Expected Per-Unit Gain

$$E(g) = \frac{E(G)}{E(S)}$$

This value represents the expected gain per unit wagered and corresponds to the negative value of the house edge for the game.

9.3.7 Comparison with Even-Money Bets

For even-money bets ($\gamma = 1$), the gain distribution collapses to a two-point distribution:

$$G = \begin{cases} -1 & \text{with probability } 1 - q^m \\ 2^m - 1 & \text{with probability } q^m \end{cases}$$

This simplification occurs because any loss results in exactly -1 unit gain, regardless of when it occurs:

$$G_k = (1-1)2^k - 1 = -1 \quad \text{for } k = 0, 1, 2, ...m-1.$$

9.4 Interpretation and Practical Implications

The Paroli system reshapes the distribution of outcomes while preserving the house edge, with the expected loss per unit wagered remaining constant at

$$-\frac{1}{19} \text{ (American Roulette) or } -\frac{1}{37} \text{ (European Roulette),}$$

regardless of the Paroli parameters. regardless of m. While higher values of m increase the potential size of gains, they also reduce their probability. The system's appeal lies in its positive skewness rather than in an improved expected value. Practically, it trades frequent small losses for rare large gains, with table limits imposing an upper bound on feasible values of m. Despite its psychological attractiveness, the Paroli system does not provide a mathematical advantage, as no betting progression can overcome the negative expectation of roulette.

10 Let-It-Ride System: Two-Point Distribution

10.1 Introduction: The Let-It-Ride System

The *Let-It-Ride* (LIR) system is a betting strategy in which, after a win, the player leaves both the **original bet** and **all winnings** on the table for the next spin. The player continues this process until either a loss occurs or a predetermined number of consecutive bets is reached.

Simple verbal definition:

> **"Let It Ride means: Win → Don't touch anything → Let everything grow → Keep going until you lose."**

Examples:

- **Straight-up number (zero):** Start with \$1 on zero → Win → Now \$36 on zero → Win → Now \$1,296 on zero → ... The bet grows very fast because of the high payout for a single number.
- **Even-money bet (red, Paroli system):** Start with \$1 on red → Win → Now \$2 on red → Win → Now \$4 on red → Win → Now \$8 on red → ... This is a special case of the Let-It-Ride system, where the payout is 1:1, and the bet doubles after each win.

Key phrase:

> **"Win? Don't touch it. Just point to the table and say: 'Let it all ride.'"**

10.2 Mathematical Formulation

Consider a let-it-ride betting system with the following parameters:

- m: Number of consecutive plays
- q: Probability of winning a single round
- p: Probability of losing a single round ($p = 1 - q$)
- γ: Payout factor (net odds received on a win)
- b: Initial bet size (typically 1 unit)

10.3 Total Gain Distribution

For m consecutive plays with the let-it-ride strategy, the total gain G follows a two-point distribution:

$$G = \begin{cases} G_w = b\left[(1+\gamma)^m - 1\right], & \text{with probability } q^m \\ G_l = -b, & \text{with probability } 1 - q^m \end{cases}$$

where:

- G_w is the total gain when winning all m rounds
- $G_l = -b$ is the loss when losing at least one round
- q^m is the probability of winning all m consecutive rounds

10.4 Expected Total Bet Function

The total amount wagered S is a random variable representing the cumulative bet over all rounds until either:

1. The player loses (stops betting after the first loss), or
2. The player completes all m rounds successfully

The bet progression follows:

$$S_i = \begin{cases} b, & \text{for } i = 1 \\ S_{i-1}(1+\gamma) = b(1+\gamma)^{i-1}, & \text{for } i \geq 2 \text{ if all previous rounds were won} \end{cases}$$

The probability of placing the i-th bet is q^{i-1} (probability of winning the first $i-1$ rounds). Therefore, the expected total amount wagered is:

$$E(S) = \sum_{i=1}^{m} E(\text{Bet on round } i) = \sum_{i=1}^{m} b(1+\gamma)^{i-1} q^{i-1}$$

This is a geometric series with ratio $r = q(1+\gamma)$:

$$E(S) = b \cdot \frac{1 - r^m}{1 - r}, \quad r \neq 1$$

Remark on the geometric series ratio r For roulette, the geometric series ratio $r = q(1+\gamma)$ is remarkably constant across all bet types:

- **European Roulette (37 numbers):** $r = 36/37 \approx 0.9730$ for straight-up, split, street, corner, six-line, dozen/column, and even-money bets.
- **American Roulette (38 numbers):** $r = 36/38 \approx 0.9474$ for all corresponding bets.

This happens because the payout factor γ is set such that the total return for a winning bet equals the number of numbers on the wheel minus one, making $q(1+\gamma)$ constant.

Normalized Gain and Variance For convenience, the *expected gain per unit of total bet* is defined as

$$E(g) = \frac{E(G)}{E(S)},$$

which corresponds to the effective *house edge* per unit wagered.

Similarly, the *standardized or normalized variance* is defined as

$$\text{Var}(g) = \frac{\text{Var}(G)}{E(S)^2}.$$

This normalization allows direct comparison of risk and expected return across different betting systems and numbers of consecutive plays.

The two-point distribution is extremely right-skewed.

10.5 Example: American Roulette Let-It-Ride Analysis

Table 11 summarizes the Let-It-Ride metrics for various bet types in American Roulette for $m = 1$ to 5 plays.

Columns: Plays, Final Gain, Win Probability, Expected Gain $E.G$, Total Wager S, Variance $Var.G$, Normalized Expected Gain $E(g)$, and Normalized Variance $Var(g)$.
Note that the normalized expected gain, defined as

$$E(g) = \frac{E(G)}{E(S)},$$

is identical across all bet types for American Roulette and equals -0.05263, corresponding to the negative house edge.
Note: For very small probabilities, the Win.Prob column rounds to 0. The exact probabilities of winning all m consecutive rounds for a straight-up bet on American Roulette are:

$$m = 4: \quad \text{Win.Prob} = \left(\frac{1}{38}\right)^4 = \frac{1}{2{,}085{,}136}, \quad G_w = 36^4 - 1 = 1{,}679{,}615,$$

$$m = 5: \quad \text{Win.Prob} = \left(\frac{1}{38}\right)^5 = \frac{1}{79{,}235{,}168}, \quad G_w = 36^5 - 1 = 60{,}466{,}175.$$

Comparison with German Lotto: For perspective, the probability of winning the top prize in the German Lotto (6 out of 49) is

$$\frac{1}{\binom{49}{6}} = \frac{1}{13{,}983{,}816} \approx 7.15 \cdot 10^{-8}.$$

By contrast, winning a straight-up bet on American Roulette five times in a row has probability

$$\left(\frac{1}{38}\right)^5 = \frac{1}{79{,}235{,}168} \approx 1.26 \cdot 10^{-8},$$

which is even smaller than the Lotto jackpot probability. This illustrates how extremely unlikely consecutive wins in a Let-It-Ride straight-up strategy are, even for just five plays.

Practical limit: In practice, the theoretical gains from a Let-It-Ride straight-up strategy are limited by casino table maximums. Even though the potential total gain after m consecutive wins is enormous (e.g., $G_w = 60{,}466{,}175$ units for $m = 5$), most casinos impose maximum bets far below this level. As a result, it is impossible to actually realize the full exponential growth of the bet in a real casino. This highlights that, while the mathematical analysis illustrates the extreme variance and potential gains, real-world constraints drastically reduce the achievable outcome.
Risk Consideration: The extremely high normalized variance $\mathrm{Var}(g)$ reported in Table 11 illustrates that the Let-It-Ride system is an exceptionally risky strategy. For example, even a single flat bet (one unit per spin) on American Roulette already carries $\mathrm{Var}(g) \approx 33.21$, whereas Let-It-Ride rapidly amplifies the variance with each consecutive win. This means that while the potential gains grow exponentially, so does the risk of losing the original stake, making the strategy highly volatile and impractical for most players.

Table 11: American Roulette Let-It-Ride analysis: key metrics for all bet types. Entries displayed as 0.0000 denote positive probabilities rounded to zero.

Bet Type	**Plays**	**Final Gain**	**Win.Prob**	**E.G**	**S**	**Var.G**	**Var.g**
Even-Money							
	1	1	0.4737	-0.0526	1.000	1.00	1.00
	2	3	0.2244	-0.1025	1.947	2.78	0.73
	3	7	0.1063	-0.1497	2.845	6.08	0.75
	4	15	0.0503	-0.1945	3.695	12.24	0.90
	5	31	0.0238	-0.2369	4.501	23.84	1.18
Dozen							
	1	2	0.3158	-0.0526	1.000	1.94	1.94
	2	8	0.0997	-0.1025	1.947	7.27	1.92
	3	26	0.0315	-0.1497	2.845	22.23	2.75
	4	80	0.0099	-0.1945	3.695	64.60	4.73
	5	242	0.0031	-0.2369	4.501	184.86	9.13
Quarter							
	1	8	0.1053	-0.0526	1.000	7.63	7.63
	2	80	0.0111	-0.1025	1.947	71.89	18.96
	3	728	0.0012	-0.1497	2.845	619.12	76.50
	4	6560	0.0001	-0.1945	3.695	5284.36	387.02
	5	59048	0.0000	-0.2369	4.501	45061.07	2224.58
Street							
	1	11	0.0789	-0.0526	1.000	10.47	10.47
	2	143	0.0062	-0.1025	1.947	128.44	33.87
	3	1727	0.0005	-0.1497	2.845	1468.54	181.45
	4	20735	0.0000	-0.1945	3.695	16702.59	1223.27
	5	248831	0.0000	-0.2369	4.501	189888.83	9374.46
Split							
	1	17	0.0526	-0.0526	1.000	16.16	16.16
	2	323	0.0028	-0.1025	1.947	289.99	76.47
	3	5831	0.0001	-0.1497	2.845	4958.05	612.61
	4	104975	0.0000	-0.1945	3.695	84559.48	6192.97
	5	1889567	0.0000	-0.2369	4.501	1441972.14	71187.50
Straight-Up							
	1	35	0.0263	-0.0526	1.000	33.21	33.21
	2	1295	0.0007	-0.1025	1.947	1162.36	306.51
	3	46655	0.0000	-0.1497	2.845	39669.46	4901.51
	4	1679615	0.0000	-0.1945	3.695	1352961.42	99088.31
	5	60466175	0.0000	-0.2369	4.501	46143126.67	2278000.87

10.6 Let-It-Ride as a Lottery

The Let-It-Ride system effectively transforms the original bet into a two-point lottery:

- With high probability $1 - q^m$: the player loses the initial bet b
- With low probability q^m: the player wins an exponentially large amount $b\left[(1+\gamma)^m - 1\right]$

Despite the strongly skewed distribution of outcomes, the fundamental relation

$$E(g) = \frac{E(G)}{E(S)} = -\frac{1}{37} \text{ (European Roulette)}, \quad -\frac{1}{19} \text{ (American Roulette)}$$

remains valid. This demonstrates that no Let-It-Ride strategy can overcome the house edge: it only alters the risk profile, increasing variance and potential gains, but the expected return per unit wagered remains negative.

Table 12: Let-It-Ride Gains by Bet Type, $m = 1$ to 5

Bet Type	k	$m=1$	$m=2$	$m=3$	$m=4$	$m=5$
Straight up	35	35	1,295	46,655	1,679,615	60,466,175
Split	17	17	323	5,831	104,975	1,889,567
Street	11	11	143	1,727	20,735	248,831
Corner	8	8	80	728	6,560	59,048
Six line	5	5	35	215	1,295	7,775
Dozen	2	2	8	26	80	242
Even money	1	1	3	7	15	31

Table 13: Let-It-Ride Gains by Bet Type, $m = 6$ to 8

Bet Type	$m=6$	$m=7$	$m=8$
Straight up	2,176,782,335	78,364,164,095	2,821,109,907,455
Split	34,012,223	612,220,031	11,019,960,575
Street	2,985,983	35,831,807	429,981,695
Corner	531,440	4,782,968	43,046,720
Six line	46,655	279,935	1,679,615
Dozen	728	2,186	6,560
Even money	63	127	255

Note: Gains are conditional on winning all m consecutive rounds; the corresponding probabilities q^m decrease exponentially with m.

Table 14: European Roulette Let-It-Ride analysis: key metrics for all bet types. Entries displayed as 0.0000 denote positive probabilities rounded to zero.

Bet Type	Plays	Final Gain	Win.Prob	E.G	S	Var.G	Var.g
Even-Money							
	1	1	0.4865	-0.0270	1.0000	0.9993	0.9993
	2	3	0.2367	-0.0533	1.9730	2.89	0.74
	3	7	0.1151	-0.0789	2.9196	6.52	0.76
	4	15	0.0560	-0.1038	3.8407	13.54	0.92
	5	31	0.0272	-0.1280	4.7369	27.14	1.21
Dozen							
	1	2	0.3243	-0.0270	1.0000	1.97	1.97
	2	8	0.1052	-0.0533	1.9730	7.62	1.96
	3	26	0.0341	-0.0789	2.9196	24.02	2.82
	4	80	0.0111	-0.1038	3.8407	71.79	4.87
	5	242	0.0036	-0.1280	4.7369	211.13	9.41
Quarter							
	1	8	0.1081	-0.0270	1.0000	7.81	7.81
	2	80	0.0117	-0.0533	1.9730	75.78	19.47
	3	728	0.0013	-0.0789	2.9196	670.63	78.67
	4	6560	0.0001	-0.1038	3.8407	5879.14	398.55
	5	59048	0.0000	-0.1280	4.7369	51488.47	2294.64
Street							
	1	11	0.0811	-0.0270	1.0000	10.73	10.73
	2	143	0.0066	-0.0533	1.9730	135.43	34.79
	3	1727	0.0005	-0.0789	2.9196	1590.80	186.62
	4	20735	0.0000	-0.1038	3.8407	18582.72	1259.74
	5	248831	0.0000	-0.1280	4.7369	216974.50	9669.71
Split							
	1	17	0.0541	-0.0270	1.0000	16.57	16.57
	2	323	0.0029	-0.0533	1.9730	305.83	78.57
	3	5831	0.0002	-0.0789	2.9196	5370.95	630.07
	4	104975	0.0000	-0.1038	3.8407	94078.29	6377.63
	5	1889567	0.0000	-0.1280	4.7369	1647655	73429.60
Straight-Up							
	1	35	0.0270	-0.0270	1.0000	34.08	34.08
	2	1295	0.0007	-0.0533	1.9730	1226.00	314.95
	3	46655	0.0000	-0.0789	2.9196	42973.55	5041.27
	4	1679615	0.0000	-0.1038	3.8407	1505265	102043.00
	5	60466175	0.0000	-0.1280	4.7369	52724975	2349748.00

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Appendix: R Code for Paroli and Martingale Betting Systems

Note: The R scripts below were generated with the assistance of ChatGPT (OpenAI).

Note on Parameters

The R scripts below allow readers to adjust key parameters for different betting scenarios:

- **p**: Probability of losing a single bet (e.g., $p = 20/38$ for American Roulette). To simulate European Roulette, set $p = 19/37$, and for a fair game, set $p = 0.5$.
- **gamma**: Payout multiplier for a winning bet. For even-money bets, use $\gamma = 1$; for straight-up bets, use $\gamma = 35$. This allows simulation of different bet types.
- **m_max**: Maximum number of consecutive wins (Paroli) or losses (Martingale) allowed before the sequence stops.

Readers can modify these values to simulate various betting strategies and outcomes.

Explanation of Metrics

The R scripts below compute the outcomes of Paroli and Martingale betting systems. For each value of m, the following metrics are calculated:

- **EG**: Expected total gain G over a betting sequence of up to m consecutive wins (Paroli) or losses (Martingale).
- **ES**: Expected total wager S over the sequence.
- **VarG**: Variance of the total gain G, measuring the risk associated with absolute outcomes.
- **EG/ES**: Average return per unit bet based on expected values, $E(G)/E(S)$.
- **VarG/ES^2**: Variance of return per unit bet based on expected values, $Var(G)/E(S)^2$.
- **E(L)**: Expected number of rounds played in the session.
- **Var(L)**: Variance of the number of rounds played, indicating variability of session length.
- **E(G/S)**: Expected gain per unit bet, computed pathwise for each sequence.
- **Var(G/S)**: Variance of gain per unit bet, computed pathwise for each sequence.
- **ER**: Expected realized return per sequence R, averaging G/S over all possible outcomes weighted by probability.

- **VarR**: Variance of the realized return per sequence R, measuring variability of per-sequence returns.
- **m**: The maximum number of consecutive wins (Paroli) or losses (Martingale) allowed before the sequence stops.

R Code: Paroli System

```
# --- Parameters ---
p <- 20/38          # Probability of losing (American roulette)
gamma <- 1          # Payout multiplier (adjustable: 1 for even money, 35 for straight-up)
m_max <- 10         # Maximum consecutive wins

# --- Recursive Paroli function ---
paroli_summary <- function(round=1, bet=1, gain=0, total_bets=0,
                           consecutive_wins=0, m) {
  if(consecutive_wins >= m) {
    R <- gain / total_bets
    return(data.frame(Gain=gain, TotalBets=total_bets,
                      R=R, Prob=1, Length=round))
  }

  # Loss branch
  loss_gain <- gain - bet
  loss_total <- total_bets + bet
  loss_R <- loss_gain / loss_total
  loss_df <- data.frame(Gain=loss_gain, TotalBets=loss_total,
                        R=loss_R, Prob=p, Length=round)

  # Win branch
  win_gain <- gain + gamma*bet
  win_total <- total_bets + bet
  next_bet <- 2*bet
  win_next <- paroli_summary(round+1, next_bet, win_gain,
                             win_total, consecutive_wins+1, m)
  win_next$Prob <- win_next$Prob*(1-p)
  win_next$R <- win_next$Gain / win_next$TotalBets

  rbind(loss_df, win_next)
}

# --- Metrics computation ---
compute_metrics <- function(m) {
  results <- paroli_summary(m=m)
  EG <- sum(results$Gain*results$Prob)
```

```
  ES <- sum(results$TotalBets*results$Prob)
  VarG <- sum(results$Gain^2*results$Prob) - EG^2
  ER <- sum(results$R*results$Prob)
  VarR <- sum(results$R^2*results$Prob) - ER^2
  data.frame(m=m, EG=EG, ES=ES, VarG=VarG, ER=ER, VarR=VarR)
}

# --- Compute metrics for all m ---
metrics_paroli <- do.call(rbind, lapply(1:m_max, compute_metrics))
print(metrics_paroli, digits=3)
```

R Code: Martingale System

```
# --- Parameters ---
p <- 20/38        # Probability of losing
gamma <- 1        # Payout multiplier (adjustable)
m_max <- 10       # Maximum consecutive losses

# --- Recursive Martingale function ---
martingale_summary <- function(round=1, bet=1, gain=0, total_bets=0,
                               consecutive_losses=0, m) {
  if(consecutive_losses >= m) {
    R <- gain / total_bets
    return(data.frame(Gain=gain, TotalBets=total_bets,
                      R=R, Prob=1, Length=round))
  }

  # Win branch (stops after win)
  win_gain <- gain + gamma*bet
  win_total <- total_bets + bet
  win_df <- data.frame(Gain=win_gain, TotalBets=win_total,
                       R=win_gain/win_total, Prob=1-p, Length=round)

  # Loss branch (double bet)
  loss_gain <- gain - bet
  loss_total <- total_bets + bet
  next_bet <- 2*bet
  loss_next <- martingale_summary(round+1, next_bet, loss_gain,
                                  loss_total, consecutive_losses+1, m)
  loss_next$Prob <- loss_next$Prob * p
  loss_next$R <- loss_next$Gain / loss_next$TotalBets

  rbind(win_df, loss_next)
}
```

```
# --- Metrics computation ---
compute_metrics <- function(m) {
  results <- martingale_summary(m=m)
  EG <- sum(results$Gain*results$Prob)
  ES <- sum(results$TotalBets*results$Prob)
  VarG <- sum(results$Gain^2*results$Prob) - EG^2
  ER <- sum(results$R*results$Prob)
  VarR <- sum(results$R^2*results$Prob) - ER^2
  data.frame(m=m, EG=EG, ES=ES, VarG=VarG, ER=ER, VarR=VarR)
}

# --- Compute metrics for all m ---
metrics_martingale <- do.call(rbind, lapply(1:m_max, compute_metrics))
print(metrics_martingale, digits=3)
```