Finite Sample Power of Cliff-Ord-Type-Tests for Spatial Disturbance Correlation in Linear Regression[1]

by

Walter Krämer
Fachbereich Statistik, Universität Dortmund
D-44221 Dortmund, Germany
walterk@statistik.uni-dortmund.de

Version August 2002

Abstract

The paper considers tests against for autocorrelation among the disturbances in linear regression models that can be expressed as ratios of quadratic forms. It shows that such tests are in general not unbiased and that power can even drop to zero for certain regressors and spatial weight matrices. Whether or not this can happen is however easily diagnosed for given regressors and for given spatial weights.

Keywords: spatial autocorrelation, unbiased tests, power.

---

[1] Research supported by Deutsche Forschungsgemeinschaft (DFG) under SFB 475. I am grateful to Badi Baltagi for useful discussion and comments and to Anurag Banerjee for providing the Gauss-Routines for the finite sample power computations (available via internet under http://www.american.deu/academic.depts/cas/econ/gaussres/Gausidx.htm).

1 The model and notation

This paper is concerned with the standard linear regression model

$$y = X\beta + u, \tag{1}$$

where y is $n \times 1$, X is $n \times k$ (nonstochastic of rank $k < n$), β is a $k \times 1$ vector of unknown regression coefficients, and u is an unobservable $n \times 1$ vector of normal zero mean disturbances.

Whenever the data in the model (1) refer to objects which are positioned in some "space", one has to allow for spatial correlation among the disturbances in (1). Such correlation may result from physical contiguity, interconnectedness in social networks, or exposure to some common risk, as in empirical models for capital asset prices and returns. This paper allows for correlation patterns which are induced when the regression disturbances follow the equation

$$u = \rho W u + \varepsilon, \tag{2}$$

where ε is an $n \times 1$ normal random vector with zero mean and scalar covariance matrix $\sigma_\varepsilon^2 I$, and W is some known $n \times n$ matrix of nonnegative weights with $\omega_{ii} = 0 (1 = 1, \ldots, n)$. The scalar ρ, which is unknown, determines the degree of correlation among the components of u. There is no correlation when $\rho = 0$.

This paper considers tests of $H_0 : \rho = 0$ against the one sided alternative $H_1 : 0 < \rho < 1/\lambda_{max}$, where λ_{max} is the Frobenius root of W (i.e. the unique positive real eigenvalue such that $\lambda_{max} \geq |\lambda_i|$ for arbitrary eigenvalues λ_i). From (1), we have under the alternative

$$u = (I - \rho W)^{-1} \varepsilon, \tag{3}$$

and $cov(u) = \sigma_\varepsilon^2 V$, where

$$V = [(I - \rho W')(I - \rho W)]^{-1}. \tag{4}$$

In many applications, W is scaled such that $\omega_{1i} + \ldots + \omega_{ni} = 1$ $(i = 1, \ldots, n)$, and therefore $\lambda_{max} = 1$. Since scaling has only minor effects on W in many applications (besides multiplication by a scalar), but destroys any existing and often very useful symmetry, we confine ourselves to symmetric weight matrices in unscaled form, where $V = (I - \rho\omega)^{-2}$.

2 Cliff-Ord-Type tests for spatial autocorrelation

We consider tests for spatial autocorrelation which reject for large values of some statistic

$$d = \frac{u'Q_1 u}{u'Q_2 u}, \tag{5}$$

where Q_1 and Q_2 are $n \times n$-matrices which in general depend on X and W. The prime example, and at the same time the test most often used in applications, is due to Cliff and Ord (1975) and has $Q_1 = (1/s)MWM$, $Q_2 = (1/n)M$, where $M = I - X(X'X)^{-1}X'$ and s is the sum of all elements in the spatial weights matrix. It is a ratio of quadratic forms in the OLS-residuals $\hat{u} = y - X\hat{\beta}$, where $\hat{\beta} = (X'X)^{-1}X'y$ is the OLS-estimator for β, and extends the Moran I-procedure to regression models. It is also equivalent fo the LM-test (Burridge 1980, Anselin 2001). Other tests along these lines include procedures where OLS-residuals are replaced by LUS or BLUS-residuals; see Bartels and Hardijk (1977) or Brandsma and Ketellapper (1978) for details.

The finite sample power of these tests has so far been investigated mostly by Monte Carlo; the consensus is that tests based on OLS-residuals are best and that power increases with increasing values of the autocorrelation coefficient ρ. Anselin and Rey (1991) and Anselin and Florax (1995) provide convenient surveys of this literature. As is shown in Theorem 1 below, however, power need not be increasing throughout the whole admissible range $(0, 1/\lambda_{max})$ of the spatial autocorrelation coefficient. Extending Krämer and Zeisel (1990), who investigated autocorrelation tests in the time series context, it is shown below that it can even drop to zero.

THEOREM 1: Let d_1 be the critical value corresponding to some significance level α, and let ω be the normalized eigenvector corresponding to λ_{max}. If λ_{max} is a simple eigenvalue of W and $\omega'(Q_1 - d_1 Q_2)\omega \neq 0$, then, depending upon X and W, the limiting power of all tests (5) as $\rho \to 1/\lambda_{max}$ is either zero or one. Given any matrix W of weights, and independently of sample size, there is always some regressor X such that for the Cliff-Ord-test the limiting power disappears.

PROOF: Rejection of H_0 occurs if

$$\begin{aligned} d &= u'Q_1u/u'Q_2u > d_1 \\ &\Leftrightarrow u'(Q_1 - d_1Q_2)u > 0 \\ &\Leftrightarrow \eta'V^{\frac{1}{2}}(Q_1 - d_1Q_2)V^{\frac{1}{2}}\eta > 0, \end{aligned} \tag{6}$$

where $\eta = V^{-\frac{1}{2}}u \sim N(0, I)$. Therefore,

$$P(\text{rejection}) = P\left(\sum_{i=1}^{u} \gamma_i \xi_i^2 > 0\right) = P\left((1-\rho\lambda_{max})^2 \Sigma \gamma_i \xi_i^2 > 0\right), \tag{7}$$

where the γ_i are the eigenvalues of $V^{\frac{1}{2}}(Q_1 - d_1Q_2)V^{\frac{1}{2}}$ and therefore also of $V(Q_1 - d_1Q_2)$, and the ξ_i are nid(0,1).

The limiting rejection probability as $\rho \to 1/\lambda_{max}$ depends upon the limiting behaviour of $(1-\rho\lambda_{max})^2V$. Let

$$W = \sum_{i=1}^{n} \lambda_i \omega_i \omega_i' \tag{8}$$

be the spectral decomposition of W, with the eigenvalues λ_i in increasing order (i.e. $\omega_n = \omega$). Then

$$V = \sum_{i=1}^{n} \frac{1}{(1-\rho\lambda_i)^2} \omega_i \omega_i' \tag{9}$$

is the spectral decomposition of V, and

$$lim_{\rho \rightarrow 1/\lambda_{max}} (1-\rho\lambda_{max})^2 V = \omega\omega', \tag{10}$$

a matrix of rank 1. Therefore, all limiting eigenvalues of $(1-\rho\lambda_{max})^2 V(Q_1 - d_1 Q_2)$ are zero except one, which is given by

$$tr(\omega\omega'(Q_1 - d_1Q_2) = \omega'(Q_1 - d_1Q_2)\omega. \tag{11}$$

If $\omega'(Q_1 - d_1Q_2)\omega$ is positive, the power tends to one; if $\omega'(Q_1 - d_2Q_2)\omega$ is negative, the power disappears.

For the Cliff-Ord-test, where $Q_1 = MWM$ and $Q_2 = M$, take $k = 1$ and $X = \omega$, except that the last component of X is set to zero. Then MW is zero except for the last component, so $W'MW > 0$. On the other hand, from $\omega_{uu} = 0$, $\omega' M\omega = 0$, so

$$\omega'(Q_1 - d_1Q_2)\omega = -d_1\omega' M\omega < 0, \tag{12}$$

since d_1 is strictly positive for all sample sizes and for all X and W. This shows that irrespective of W and sample size, one can always find examples such that the power disappears. •

Since $\omega'(Q_1 - d_1Q_2)\omega$ is known, the theorem also provides a means to avoid the zero-power trap: compute $\omega'(Q_1 - d_1Q_2)\omega$ and do not apply the test if $\omega'(Q_1 - d_1Q_2)\omega < 0$.

Theorem 1 does not apply when $\omega'(Q_1 - d_1Q_2)\omega = 0$. In the case of the Cliff-Ord-test, this occurs whenever ω is in the regression space of X. This seems to be a rather unusual situation, except when ω is a vector of constants and the regression has an intercept.

THEOREM 2: If $\omega'(Q_1 - d_1Q_2)\omega = 0$, then the limiting power of all tests (5) as $\rho \rightarrow 1/\lambda_{max}$ is in general strictly in between 0 and 1.

PROOF: Let $V(\rho) := (1-\rho\lambda_{max})^2 V$, and $\lim_{\rho\to 1/\lambda_{max}} V(\rho) = \omega\omega' =: \overline{V}$. Then

$$\overline{V}(Q_2 - d_1 Q_2) = 0 \quad \text{and} \tag{13}$$

$$\begin{aligned} & lim_{\rho\to 1/\lambda_{max}}(1-\rho\lambda_{max})^{-2} V(\rho)(Q_1 - d_1 Q_2) \\ & \quad = lim_{\rho\to 1/\lambda_{max}}(1-\rho\lambda_{max})^{-2}\left(V(\rho) - \overline{V}\right)(Q_1 - d_1 Q_2) \\ & \quad = V^*(Q_1 - d_1 Q_2), \quad \ldots \end{aligned} \tag{14}$$

where, by l'Hopital's rule,

$$V^* \; := \; lim_{\rho\to 1/\lambda_{max}}(1-\rho\lambda_{max})^{-2}\left(V(\rho) - \overline{V}\right) \tag{15}$$

$$= \sum_{i=1}^{n-1} \frac{\lambda_{max} - \lambda_i}{\lambda_{max}\left(1 - \frac{\lambda_{max}}{\lambda_i}\right)^3} \omega_i \omega_i' \tag{16}$$

is a matrix of rank $(n-1)$. Therefore

$$V^*(Q_2 - d_1 Q_2) \neq 0, \tag{17}$$

and the nonzero eigenvalues of this matrix uniquely determine the limiting power of the test. For most cases encountered in practice, some of these eigenvalues will be positive and some will be negative, so the limiting power will be strictly in between 0 and 1. •

Theorem 1 has also implications for efficient estimation. It is shown in Krämer and Donninger (1987, Theorem 2) that, for the X-matrix constructed in Theorem 1 to induce a limiting power of zero for the Cliff-Ord-test, the OLS-estimator has a limiting efficiency relative to the Generalized Least Squares estimator of zero as well, so in a sense, the autocorrelation test fares worst when it is needed most.

3 An empirical illustration

Here are some empirical examples. For ease of comparison with previous work, in particular Anselin and Rey (1991), the weight matrices are chosen for data which are arranged in regular square lattices of various sizes, with weights of 0 and 1 according to either the rook or the queen criterion: the rook criterion assigns a weight of 1 to all cells above, below, to the right and to the left of a given cell (except for cells along the border of the lattice), and the queen criterion assigns a weight of one to all cells immediately surrounding a given cell. A third class of weight matrices was obtained by a mechanism which with probability 1/3 assigns a value of 1 to all $\omega_{ij}(j > i), \omega_{ji} := \omega_{ij}$ and zero otherwise. Such weight matrices are much more densely packed than those obtained via the rook and queen criteria, with the proportion of nonzero weights remaining constant as sample size increases. Exact critical values d_1 and exact rejection probabilities, given X and W, were computed with the Imhof algorithm.

Figure 1 shows the exact rejection probabilities for a 4×4 lattice (i.e. $n = 16$), with W given by the queen criterion, and two X-matrices: a column of ones ($k = 1$) and the column of ones plus a column $(1, 2, \ldots, 16)'$. This is just for ease of replication. As is seen in the figure, the power of the test is initially higher when X is only a column of ones (this is not surprising, as the performance of the test depends upon the quality of the OLS-residuals, which suffers when additional regressors are added), but drops to zero as ρ approaches the boundary of the parameter space (which is scaled to the (0,1)-interval). With an additional regressor, however, power tends to one.

This dropping to zero of the power is not a fluke. Table 1 gives the number of cases (out of 1000 runs) where one obtains a limiting power of zero, with X ($n \times 2$) given by a column of ones plus a column of *nid* $(0, 25)$ variables.

Not surprisingly, this number of zero limits decreases as the significance level α of the test increases. It is very large for small samples and decreases as sample size increases. For weight matrices like rook or queen which are sparsely packed, a limiting power of zero for the Cliff-Ord-test does not seem to be an empirically relevant phenomenon for sample sizes beyond 25. This was also confirmed for other types of X-matrices (not shown in Table 1). However, if nonzero weights are dense, a disappearing power remains a distinct possibility also for larger samples.

References

Anselin, L. and Rey, S. (1991): "Properties of tests for spatial dependence in linear regression models." *Geographical Analysis* 23, 112 – 131.

Anselin, L. and Florex, R. (1995): "Small sample properties of tests for spatial dependence in regression models: Some further results." in Anselin, L. and Florax, R. (eds.): *New directions in spatial econometrics*, Berlin (Springer, 21 – 74.

Anselin, L. (2001): "Rao's score test in spatial econometrics." *Journal of statistical planning and inference* 97, 113 – 139.

Cliff, A.D. and Ord, J.K. (1975): *Spatial Autocorrelation*, London.

Burridge, P. (1980): "On the Cliff-Ord-Test for spatial autocorrelation." *Journal of the Royal Statistical Society* B 42, 107 – 108.

Krämer and Donninger, Chr. (1987): "Spatial autocorrelation among errors and the relative efficiency of OLS in the linear regression model." *Journal of the American Statistical Association* 82, 577 – 579.

Krämer, W. and Zeisel, H. (1990): "Finite sample power of linear regression autocorrelation tests." *Journal of Econometrics* 43, 363 – 372

Bartels, C.P.A. and Hordijk, L. (1977): "On the power of the generalized Moran Contiguity coefficient in testing for spatial autocorrelation among regression disturbances." *Regional Science and Urban Economics* 7, 83 – 101.

Brandsma, A. and Ketellapper, R. (1979): "Further evidence on alternative procedures for testing of spatial autocorrelation among regression disturbances." In Bartels, C.P.A. and Ketellaper, R. (eds.): *Exploratory and Explanatory Statistical Analysis of Spatial Data*, Boston.

Table 1: Number of zero power limits out of 1000 independent runs

	α		
n	1%	5%	10%
	a) rook		
16	999	956	236
25	8	2	0
36	1	0	0
	b) queen		
16	999	954	293
25	7	0	0
36	1	0	0
	c) random weights		
16	497	295	163
25	388	185	114
36	309	139	97

Figure 1: Power of Cliff-Ord test