Spectral estimation for psycho-physiological data: Estimating lower-dimensional representations in frequency space *

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Abstract

Two different estimation techniques for the spectrum of a nonstationary time series are compared empirically. Both of them are assuming a time-dependent autoregressive (AR-) model for the data.
The first estimation technique used is the Frequency State Dependent Model (FSDM-) technique (Schmitz and Urfer, 1997), a modification of the well known Kalman-filter approach. The FSD-Model is based on Priestleys SD-Models for the analysis of nonstationary time series (e.g.,Priestley, 1988) .
An alternative approach for estimating AR-parameters of nonstationary time series was proposed by Tsatsannis and Giannkis (1993). The basic idea is to directly decompose the time-dependent autoregressive parameters into their wavelet representation and to select suitable wavelet coefficients for reconstruction.
In either case, Kitagawa's (1983) "instantaneous spectrum" is calculated to obtain the actual spectral estimates. Applied to empirical data, both approaches lead to similar spectral estimates. However, simulations show how crucial the selection of wavelet coefficients is when applying the latter technique.

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1 Introduction

A helpful tool in the analysis of a continuous second order stationary process $X(t)$ is the so-called power spectrum. Following Parseval's theorem, the spectrum gives an equivalent representation of the process in the frequency domain. If the spectrum takes a high value at frequency ω_0, say, then $X(t)$ may be approximated by a complex exponential of infinite length of the same frequency. In fact, a superposition of several such exponentials tends to give an increasingly good approximation of $X(t)$.

Since complex exponentials are *of infinite length*, they are only of very limited use in the analysis of non-stationary time series. Therefore, some attempts were made to define *time-dependent* spectra giving insight in what frequencies play a predominant role in a particular *instance* of time. Particularly, the so-called instantaneous spectrum (Kitagawa, 1983) makes use of the AR-parameter estimates at each singular point of time. See section 2 for details. To apply this technique, it is important to find proper estimates of the time-varying sequences of the autoregressive coefficients governing the observed process. In the current paper, two different techniques are applied. The first method fits the so-called Frequency State Dependent Model (FSDM) to the data as it is decribed by Schmitz and Urfer (1997). This is a specially adapted form of Priestley's State Dependent Models (e.g. Priestley, 1988) and goes back to the well known Kalman filter. It will be described in section 3. Section 4 presents an alternative method based on a direct wavelet decomposition of the AR-parameters. This procedure was first described by Tsatsanis and Giannakis (1993). A comment on the limitations of wavelets in the analysis of nonstationary time series can be found in Priestley (1996). In his paper, Priestley compares wavelet methods and time-dependent spectral analysis in a mathematically more precise form.

A practical application of the Wavelet- and the FSDM-techniques to psychophysiological data is described in section 5. Both methods seem to give reasonable results. Yet, care has to be taken particularly when using wavelets, as shown in section 6. There, a small simulation study is described. The paper is ended by a summary of the findings and some concluding remarks.

2 The Instantaneous Spectrum

For second order stationary time series it is customary to use the power spectrum $f_X(\omega)$ as an equivalent representation of the underlying series. If the time series can be assumed to be an autoregressive (AR-) process of order p, it may be written as

$$x(t) = \sum_{i=1}^{p} x_{t-p} a_p + \epsilon_t \tag{1}$$

for $t \in \aleph$, where ϵ represents the error term with distribution N(0,σ_ϵ^2). The spectrum at frequency ω is then given by (e.g., Schlittgen and Streitberg, 1994)

$$f_X(\omega) = \frac{\sigma_\epsilon^2}{|1 - a_1 e^{i2\omega} - ... - a_p e^{i2\omega p}|^2} \tag{2}$$

Here, i denotes the imaginary unit. For nonstationary processes, this motivates the definition of the instantaneous spectrum (Kitagawa, 1983) for each instance t, given by

$$f_X(\omega, t) = \frac{\sigma_\epsilon^2(t)}{|1 - a_{t,1} e^{i2\omega} - ... - a_{t,p} e^{i2\omega p}|^2} \tag{3}$$

where now the $a_{t,j}$ are time dependent autoregressive parameters and the variance $\sigma_X^2(t)$ may also vary with time. For simplicity, constant variance over time is assumed in our case. An important condition for the instantaneous spectrum to exist is that all roots of the corresponding characteristic polynomial are larger than the unity in absolute value for all t.
The problem of finding a time-dependent spectral representation of a nonstationary time series may now be solved by estimating the p sequences $a_{t,j}$ for j=1,...,p and calculating the instantaneous spectrum for each t by applying equation (3). The two methods presented in the following sections will be used to do so.

3 The Frequency State Dependent Model

A well known approach for estimating nonstationary time series is given by the Kalman filter algorithm (Kalman, 1960). Schmitz and Urfer (1997) suggest a special form of this algorithm called Frequency State Dependent (FSD-) Model. In order to obtain an interpretable spectral estimate, they allow only for slowly varying AR-parameters. This is achieved by modelling the $a_{t,j}$ together with their gradients $\phi_{t,j,k}$ in a recursive state space model, leading to a dependence structure of the form

$$a_{t+1,j} = a_{t,j} + \sum_{k=1}^{p} \phi_{t,j,k} \left(a_{t,k} - a_{t-1,k} \right) \tag{4}$$

where $k \in 1, ..., p$. In detail, Schmitz et al. obtain the following filtering algorithm:

<u>System equation:</u>

$$z_{t+1} = A_{t+1,t} z_t + B_t \epsilon_{t+1} \tag{5}$$

where

$$A_{t+1,t} = \begin{bmatrix} I_p & I_p \otimes \triangle\alpha'_{t,p} \\ 0_{p^2 \times p} & I_{p^2} \end{bmatrix} \tag{6}$$

($\otimes$ denotes the Kronecker-product)

$$B_t = \begin{bmatrix} 0_p & 0_{p \times p^2} \\ 0_{p^2 \times p} & I_{p^2} \end{bmatrix} \tag{7}$$

$$\triangle\alpha_{t,p} = [\alpha_{t,1}, ..., \alpha_{t,p}]' \tag{8}$$

with $\alpha_{t,i} = a(t,i) - a(t-1,i), 1 \leq i \leq p$, and $\alpha_{t,p} = [a_{t,1}, ..., a_{t,p}]'$. Furthermore, $\epsilon_t \sim N(0, q^2 \cdot I)$ for all $t \in \aleph$ and

$$z_t = [a_{t,1}, ..., a_{t,p}, \phi_{t,1,1}, ..., \phi_{t,1,p}, ..., \phi_{t,p,1}, \phi_{t,p,p}]' \,. \tag{9}$$

<u>Observation equation:</u>

$$x_t = C_t z_t + \eta_t \tag{10}$$

or more explicitly,

$$x_t = [x_{t-1}, ..., x_{t-p}, 0, ..., 0] \begin{bmatrix} a_{t,1} \\ \vdots \\ a_{t,p} \\ \phi_{t,1,1} \\ \vdots \\ \phi_{t,p,p} \end{bmatrix} + \eta_t \tag{11}$$

with $\eta_t \sim N(0, r^2)$ for all $t \in \aleph$. The variances of the ϵ_t and η_t are assumed to be independent.

As it is usual for filters of the Kalman type, it is necessary to provide initial values for the vectors z_0, $\phi_{0,\cdot,\cdot}$ as well as the variances q^2 and r^2. A common procedure is to estimate z_0 from baseline data assumed to be stationary. The gradients are taken to be 0 at the beginning.
As shown by Kitagawa, it is not necessary to provide estimates for both the variances q^2 and r^2. It rather is sufficient to maximize a trade-off parameter $\mu^2 = q^2/r^2$ (Schmitz, 1995). Unfortunately, this has to be done numerically, leading to a considerable amount of computations.
The properties of the FSDM-filter are examined closely by Schmitz (1995). The author shows that important properties of the Kalman-filter estimates are preserved by the FSDM-approach. Particularly, if the initial starting values are normally distributed, the FSDM-filter is a linear and stable filter that gives optimal estimates in the conditional mean, mode and median sense.

4 The Wavelet Approach

An alternative way of determining autoregressive coefficients has been proposed by Tsatsanis und Giannakis (1993). They use wavelet techniques for time-dependent estimation of $p = 2$ AR-coefficients in a simple linear model setting.
The basic idea underlying the wavelet approach is to represent a (deterministic) function $a(t)$ using a (orthonormal) wavelet basis based on a mother wavelet $\psi(t)$, leading to the form

$$a(t) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} w_{m,n} \psi_{m,n}(t), \tag{12}$$

where $\psi_{m,n}(t)$ are the usual translated and dilated versions of $\psi(t)$ and

$$w_{m,n} = \int_{-\infty}^{\infty} a(t) \psi_{m,n}(t) dt \tag{13}$$

are the corresponding wavelet coefficients. The $w_{m,n}$ may be calculated from the finite number of equispaced observations $x(t), t = 1, ..., n$ by a lowpass-highpass filter algorithm (Mallat, 1989). The corresponding filters may be combined in an orthogonal matrix W, leading to the linear model representation

$$w = Wx \tag{14}$$

where w is the vector of the n wavelet coefficients $w_{m,n}$ and x contains the data.
In the stochastic autoregressive setting, the data $x(t)$ follow equation (1). If the p sequences of AR-coefficients are interpreted as deterministic functions of time, each of them can be transformed into its wavelet representation, i.e. into n coeffecients $w^j_{m,n}(t)$ for $j = 1, ..., p$. Obviously, not all of the $w^j_{m,n}(t)$ are then estimable from n data points.

For this reason, the following procedure is applied: Given the order p of the AR-Process, all p functions $a_j(t), j = 1, ..., p$ are treated equally by assigning not more than n/p wavelet coefficients to them. Assuming some smoothness in the $a_j(t)$ as it is done by the FSDM-approach, it seems reasonable to leave out complete sets of wavelet coefficients representing high frequencies. This amounts to a lowpass filtering of the $a_j(t)$.

Allthough the deletion of complete frequency bands gives reasonable results for the empirical data analysed here, it turns out that deleting complete frequency bands may as well lead to a complete breakdown of the spectral estimates in other situations. After describing the the application to empirical data in the following section, more details on this will be given in section 6.

5 Application to psychophysiological data

The two methods described above were applied to psychophysiological data of 5 hypertonic patients from a larger panel. In an experiment an attempt was made to simulate psycho-motorical stress by asking the participants to redraw lines graved into a metal plate. The task was not to touch the surrounding metal with the given pen (Comparable to study B in Schmitz, 1995). The heart rate data were measured in beat-to-beat-intervals. At the beginning, a baseline measurement of approximately 5 minutes was taken to evaluate the persons heart rate while being in calm state. Then, phases of stress and 2-minute breaks alternated over a period of about 15 minutes.

For statistical analysis, the beat-to-beat data were transformed to aequidistant "heart rate/min" at each second, using a Lagrange interpolation of order 5 (Mulder,1985). The linear trend was removed calculating the simple linear difference. Finally, the data were smoothed with a median filter of length 5. Earlier studies (Mulder, 1985) suggested that the predominant frequencies in heart rate data can be found in the frequency band between 0.07 Hz and 0.4 Hz. Particularly, frequencies around 0.1 Hz and around 0.3 Hz are to be expected. Therefore, the data were filtered by a bandpass filter of the form

$$g_u = \int_{-0.5}^{0.5} G(\lambda) exp(-i2\pi\lambda u) d\lambda \tag{15}$$

where

$$G(\lambda) = I_{[0.07,0.4]}(\lambda) \tag{16}$$

In order to obtain sequences of comparable length, each observed series was truncated to length n=1074. To calculate the time-dependent spectra, p=5 was chosen. This choice was due to the expected two peaks around 0.1 Hz and 0.3 Hz. Then, the FSD-analysis and the Wavelet analysis were applied as explained above. All calculations were done using S-Plus and the S+WAVELETS module (Mathsoft, 1995-1998). Some details about this programming language may be found in Scheffner and Krahnke (1998).

Figure 1 shows the result of the FSD-analysis for patient VP 23 as a typical example. The numerical optimization took several hours and resulted for this patient in a value of $\mu^2 = 20$ on a 100 MHz Pentium PC. The spectrum appears not very smooth. Two major peaks appear, varying around 0.1 Hz and 0.3 Hz as expected. Overall, the changes in the spectrum over time do not come out clearly.

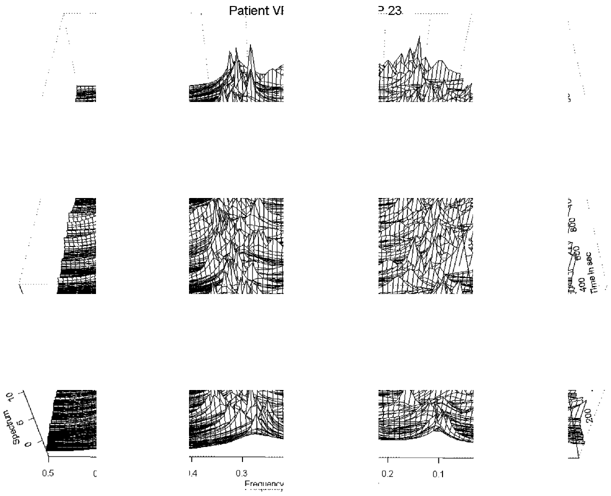

Figure 1: Spectrum obtained by FSDM-approach

The above figure may be directly compared to the corresponding graph obtained using the wavelet approach (Figure 2). Here, the Daubechies-wavelet of order 3 (Daubechies, 1988) was used. Due to the deleting of wavelet coefficients for higher frequencies it seems not surprising that a somewhat smoother spectrum is obtained. Much noise seems to be removed and the changes in frequency may be followed much easier over time. Yet, changes due to psychophysiological stress can not be seen clearly from the picture. It may be noted that calculations for the analysis took less than 5 minutes per patient since no numerical optimization was necessary.

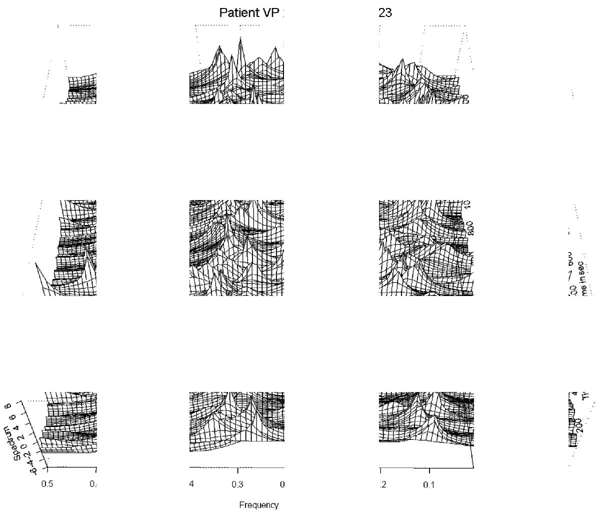

Figure 2: Spectrum obtained by Wavelet-technique

6 Simulation

To evaluate the performance of the FSDM-technique in comparison to the Wavelet-approach it seemed meaningful to perform a simulation with known predominant frequencies and AR-parameter sequences. The idea was to compare empirical and true parameters in both cases to decide which technique gives a spectral estimate closer to the true underlying spectrum. A first simulation was done by generating artificial data of the form

$$y_t = sin(2 * \pi * \lambda_1(t) * t) + sin(2 * \pi * \lambda_2(t) * t) + \epsilon(t) \tag{17}$$

where

$$\begin{aligned} t &= 1, ..., 512 & (18) \\ \lambda_1(t) &= 0.1 + t * \frac{3}{5120} & (19) \\ \lambda_2(t) &= 0.2 - 0.1 * \frac{t - 1^2}{511^2} & (20) \end{aligned}$$

and

$$\epsilon_t \sim N(0, 0.5^2) \tag{21}$$

The variation of frequencies between 0.1 and 0.4 Hz corresponds to findings in earlier studies of heart rate data (e.g., Mulder,1985). The two time-dependent sequences of frequencies are displayed in Figure 3.

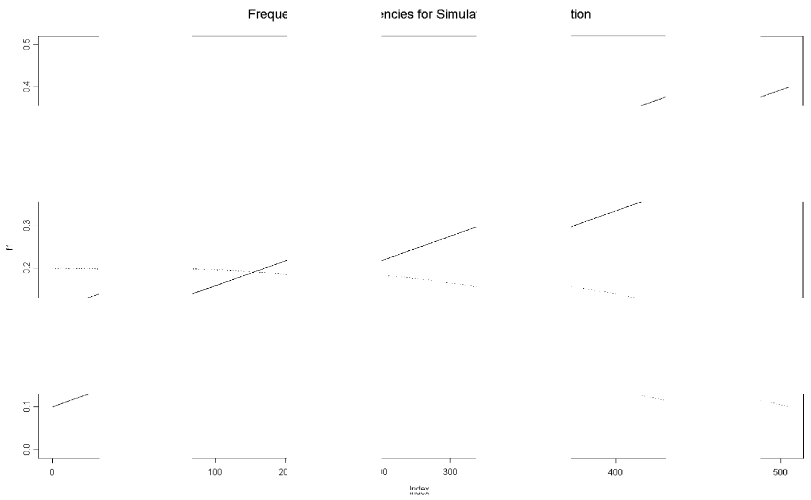

Figure 3: Dominating frequencies over time in simulation. Solid line represents $\lambda_1(t)$, dashed line represents $\lambda_2(t)$.

The "true" AR-parameters were estimated from these data by the running windows technique, taking 10 observations at once for each point of time. The "true" values are shown in Figure 4, whereas Figure 5 displays the AR-estimates obtained by the FSDM-technique ($\mu^2 = 1$). The corresponding spectracan be found in Figure 6 and 7.

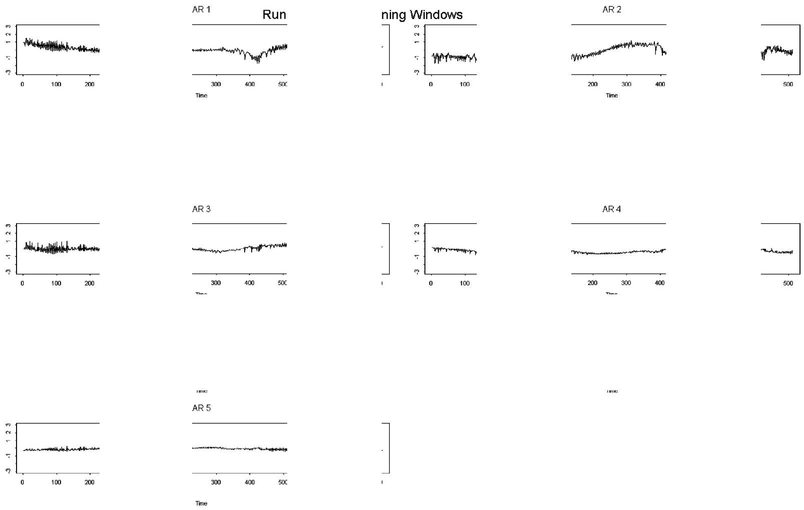

Figure 4: AR-estimates from Running Windows

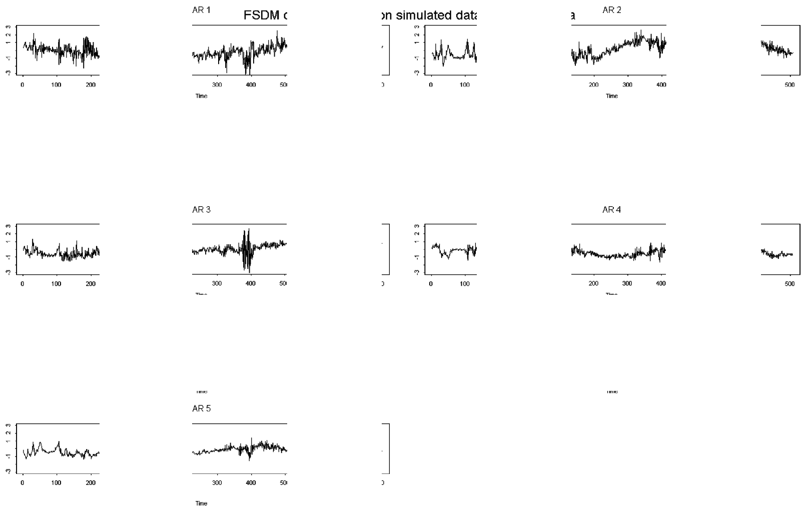

Figure 5: AR-estimates from FSDM-technique

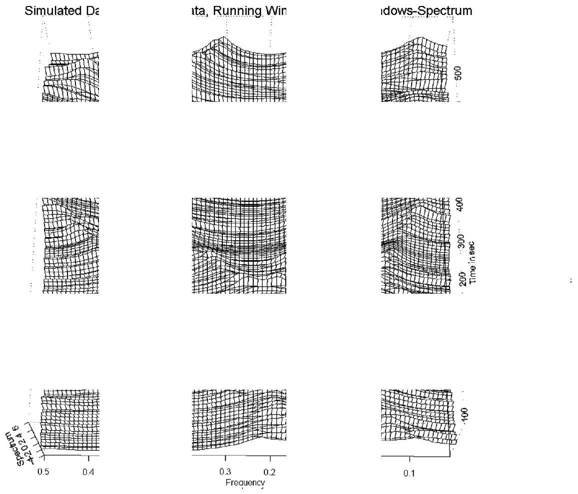

Figure 6: "True" spectrum (estimated from running windows)

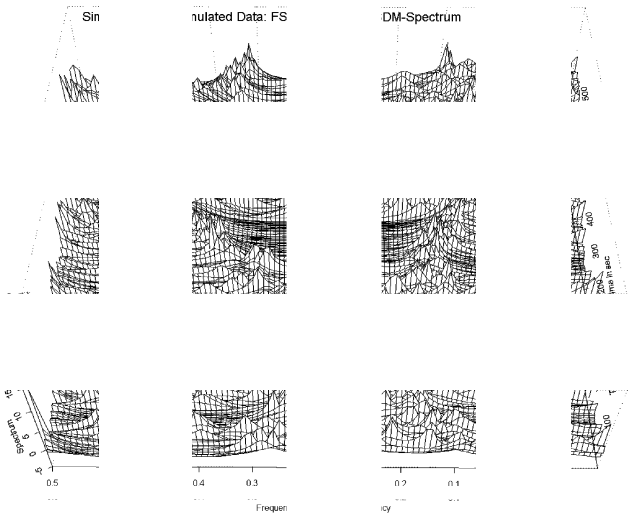

Figure 7: Spectral estimate following FSDM-technique

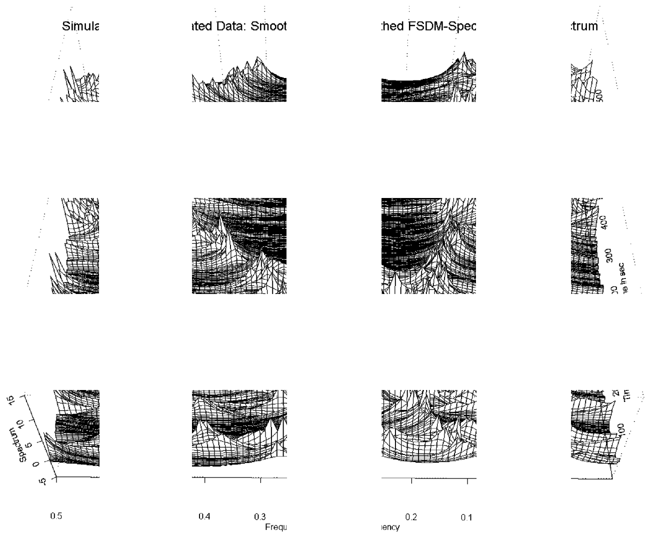

Figure 8: Spectral estimate, smoothed from FSDM

An improved estimate of the spectrum is obtained after smoothing the AR-coefficients by a robust smoothing algorithm [1] (Figure 8).
On the other hand, using the Wavelet-approach with only 64 coefficients corresponding to the lowest frequencies for each of the 5 AR-parameters is completely unsatisfactory. The AR-estimates don't even come close to the "true" values as obtained by the running windows approach (Figure 9). While the first four coefficients are essentially zero at all t, $\alpha(5,t)$ is constantly equal to one. Apparently, the wavelet technique as proposed above completely breaks down in this situation. A comparison between the two methods therfore seems not to be meaningful. A more careful choice of wavelet coefficients appears to be necessary, e.g. by selecting coefficients by a stepwise procedure. This is subject to current research. However, any deletion of a certain wavelet coefficient influences the estimate of an AR-coefficient over the whole time range. This fact by itself seems somewhat contradictory to the wavelet property of localizing a signal both in time and frequency(Mallat, 1989).

[1]The S-Plus 4.0-function smooth() was used here

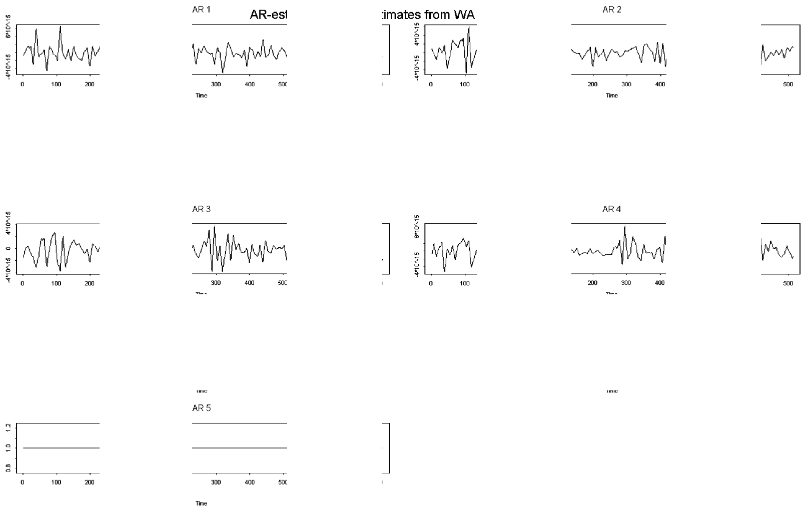

Figure 9: AR-coefficients from Wavelet-approach (64 coefficients each). Note the y-scale from $+6 * 10^{-15}$ to $-4 * 10^{-15}$ for the first 4 AR-coefficients!

7 Concluding Remarks

A comparison between the FSDM-technique and the Wavelet approach for estimating nonstationary time series is difficult to obtain, because the two techniques have very different backgrounds. Some promising empirical studies in favor of the WA-approach are overshadowed by the extremely poor performance in our simulation. It seems necessary to improve the wavelet technique described here by choosing the wavelet coefficients in a more refined way instead of restricting oneself to low-frequency bands of the AR-parameters. An alternative may be a stepwise selection of the coefficients, but this is a very time-consuming task. Besides that, selecting only a limited set of wavelet coefficients doesn't take into account the localizing properties of wavelets.
It should be noted, though, that the findings of Schmitz and Urfer (1997) concerning the good performance of the FSDM-technique was confirmed by the simple setting analyzed here. A visually even more appealing result was obtained after smoothing the AR-sequences. Including a smoothing procedure therefore seems to lead to an improvement of the results under certain

circumstances. The statistical properties of the corresponding estimates have to be investigated closely in future research, though.

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