A note on the behaviour of the process capability index C_{pmk} with asymmetric specification limits

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Abstract

The properties of C_{pmk} in the presence of asymmetric specification limits are discussed. It is shown that C_{pmk} tends to zero as the process variation increases and vice versa. Furthermore, if the process variation is small, C_{pmk} has its maximum near the target value but the maximum moves towards the specification midpoint as the variation increases. This is a desirable property as for large variation the percentage of items inside the specification limits is larger if the process mean is equal to the specification midpoint than if it is equal to the target value. Attention is drawn to the fact that for small process variations there is a shoulder in the graph of C_{pmk} when the process mean is equal to the specification midpoint.

1. Introduction

Process capability is extensively discussed in the recent past (Pignatiello, 1993, Gunter, 1989). Originally, the aim was to judge a process in terms of the ratio of the "allowable variation" to the "natural variation" of a (normally distributed) process as given by

$$C_p := \frac{USL - LSL}{6\sigma},$$

(Sullivan, 1984), where σ is the (positive) root of the process variation and LSL and USL are the lower and upper specification limits, respectively. Under the usual assumptions of process normality and process centering on the midpoint of the specification, the percentage of non-conforming items is $2\Phi\left(-3\, C_p\right) \cdot 100$, where Φ denotes the distribution function of the standard normal distribution (see e.g. Kotz & Johnson, 1993). But realising that this index does not take into account possible departure of the process mean μ from the specification midpoint $m := (USL + LSL)/2$, which was implicitly assumed to be also the target value T, the C_{pk} - and C_{pm} - Indices given by

$$C_{pk} := \frac{\min\{USL - \mu, \mu - LSL\}}{3\sigma} = \frac{d - |\mu - m|}{3\sigma}$$

and

$$C_{pm} := \frac{USL - LSL}{6\sqrt{\sigma^2 + (\mu - T)^2}}$$

were introduced. Here, $d := (USL - LSL)/2$ denotes the distance between the specification midpoint and the specification limits. C_{pk} originated in the Japanese process industries, C_{pm} was introduced independently by Chan, Cheng & Spiring (1988) and by Hsiang & Taguchi (1985). The former index is able to cope with departures from the specification midpoint, the latter is used to penalise deviations from the target value more severely than C_{pk}. Pearn, Kotz and Johnson (1992) combined these indices to their so-called third generation index:

$$C_{pmk} := \frac{\min\{USL - \mu, \mu - LSL\}}{3\sqrt{\sigma^2 + (\mu - T)^2}} = \frac{d - |\mu - m|}{3\sqrt{\sigma^2 + (\mu - T)^2}}$$

where d and m are defined as above. This index was originally introduced as C_{pn} by Choi and Owen (1990). Note that C_{pmk} and C_{pk} may take negative values if μ falls outside the specification. If the process is centered on target, i.e. for $\mu = T$, C_{pmk} and C_{pk} are identical, whereas $C_{pmk} < C_{pk}$ for $\mu \neq T$. For the special case of $\mu = T = m$, it is easily seen that $C_p = C_{pk} = C_{pm} = C_{pmk}$. For distributional properties of the estimator of C_{pmk} see Pearn et al. (1992) or Kotz & Johnson (1993).

There are a number of applications where asymmetric specifications are more desirable than symmetric ones. For example, in a process for the production of bolts it is possible to shorten the items whereas there is no way to bring them to the required length if they are too short. Thus, in the following we will discuss the behaviour of C_{pmk} for asymmetric specifications.

Boyles (1994) also considered C_{pmk} for asymmetric specifications. This note will somewhat extend his findings.

2. Properties of C_{pmk} as a function of the process variation

Let μ be fixed for the moment. Then the asymptotic behaviour of C_{pmk} in terms of σ^2 is immediately obvious from its definition. C_{pmk} approaches zero as σ^2 tends to infinity and approaches infinity as σ^2 tends to zero. The first derivative of C_{pmk} with respect to σ^2 is given by:

$$\frac{\partial C_{pmk}}{\partial \sigma^2} = - \frac{d - |\mu - m|}{3\left[\sigma^2 + (\mu - T)^2\right]^{3/2}},$$

which implies that C_{pmk} is decreasing in σ if $d > |\mu - m|$, i.e. if the process mean lies inside the specification. On the other hand, C_{pmk} is increasing in σ^2, if the process mean is situated outside the specification limits, i.e. although the process variation increases, the process seems to be more capable. At the first glance this result seems surprising, but in the case that the process mean falls outside the specification limits the percentage of non-conforming items is high if the variation of the process is small. Thus, a higher variation results in a higher percentage of conforming items, i.e. in larger process yield, and so this property of C_{pmk} is sensible.

3. Properties of C_{pmk} as a function of the process mean

Now let $\sigma^2 > 0$ be fixed. Then, the two cases $\mu < m$ and $\mu > m$ have to be discussed separately. The first derivative of C_{pmk} with respect to μ is given by

$$\frac{\partial C_{pmk}}{\partial \mu} = \begin{cases} \dfrac{(T-\mu)(T-(m-d))+\sigma^2}{3\left(\sigma^2+(\mu-T)^2\right)^{3/2}}, & \mu < m \\ \dfrac{(T-\mu)((m+d)-T)-\sigma^2}{3\left(\sigma^2+(\mu-T)^2\right)^{3/2}}, & \mu > m \end{cases} \tag{1}$$

Note that C_{pmk} is not differentiable in $\mu = m$. Furthermore, $m - d = LSL$ and $m + d = USL$.

From (1) it can be derived, that the maximum of C_{pmk} is located in one of the following points:

(a) $\mu_1^* := T + \frac{\sigma^2}{T - LSL}$, if $T + \frac{\sigma^2}{T - LSL} < m$.

(b) $\mu_2^* := T - \frac{\sigma^2}{USL - T}$, if $m < T - \frac{\sigma^2}{USL - T}$.

(c) $\mu_3^* := m$, if $T - \frac{\sigma^2}{USL - T} \le m \le T + \frac{\sigma^2}{T - LSL}$.

Thus, the position of the maximum depends on the size of σ^2. It is clear that only if σ^2 is sufficiently small, the maximum of the C_{pmk}-Index will lie close to the target value T, whereas for large σ^2 the maximum will be at the specification midpoint m. Furthermore it is interesting to note that even for small σ^2, the maximum will never actually reach the target value T.

Thus, C_{pmk} has its maximum value close to T, if σ^2 is sufficiently small. As σ^2 increases, the maximum moves from T to m. If we were only interested in process centering on target, this property is not really advantageous as one would expect the maximum of C_{pmk} at the target value. This is in fact the behaviour exhibited by the process capability index C_{pm}. But if we were primarily interested in process yield - which is the main interest in using C_{pk} -, this property would be desirable, because for increasing variation the percentage of non-conforming items decreases only if the process mean moves to the specification midpoint. Thus, considering C_{pmk} as a mixture of C_{pk} and C_{pm}, C_{pmk} behaves "more like C_{pm}" if σ^2 is small, whereas C_{pmk} behaves "more like C_{pk}" if σ^2 is large.

4. Behaviour of C_{pmk} if the process mean is near the specification midpoint

Let again $\sigma^2 > 0$ be fixed. The left and right limits of the derivative (1) for $\mu \to m$ are given by:

$$\lim_{\mu \uparrow m} \frac{\partial C_{pmk}^{-}}{\partial \mu} = \frac{(T-m)(T-LSL)+\sigma^2}{3\left(\sigma^2+(m-T)^2\right)^{3/2}} \text{ and } \lim_{\mu \downarrow m} \frac{\partial C_{pmk}^{+}}{\partial \mu} = \frac{(T-m)(USL-T)-\sigma^2}{3\left(\sigma^2+(m-T)^2\right)^{3/2}}$$

From this it can be seen that for any combination of $T \neq m$, m, μ and $\sigma^2 > 0$ the slope of the index is always larger for $\mu \uparrow m$ than for $\mu \downarrow m$. If, e.g., C_{pmk} has its maximum in μ_1^*, then $T + \frac{\sigma^2}{T-LSL} < m$, the slope of C_{pmk} is negative in m and C_{pmk} decreases more slowly for $\mu < m$. If the maximum is attained in μ_2^*, analogous considerations hold. Thus, in the case that $T \neq m$, there is always a "shoulder" in the graph of C_{pmk} at $\mu = m$. As the specification is explicitly assumed to be asymmetric the interest should be in the target value and not in the specification midpoint. Thus, it remains debatable, whether this property of C_{pmk} is desirable.

5. Example

A process with target value T = 50, lower and upper specification limits LSL = 45 and USL = 65 will be discussed (m = 55, d = 20). The behaviour of C_{pmk} for μ between 30 and 70 and σ^2 between 0.5 and 10 is shown in the Figure 1.

The limiting behaviour in terms of μ and σ^2 for C_{pmk} is easy to see. As was shown, C_{pmk} decreases if μ moves essentially away from the target value T, if σ^2 is small, or if μ moves away from the specification midpoint m, if σ^2 is large. For μ outside the specification limits and σ^2 increasing the increase in C_{pmk} is negligible.

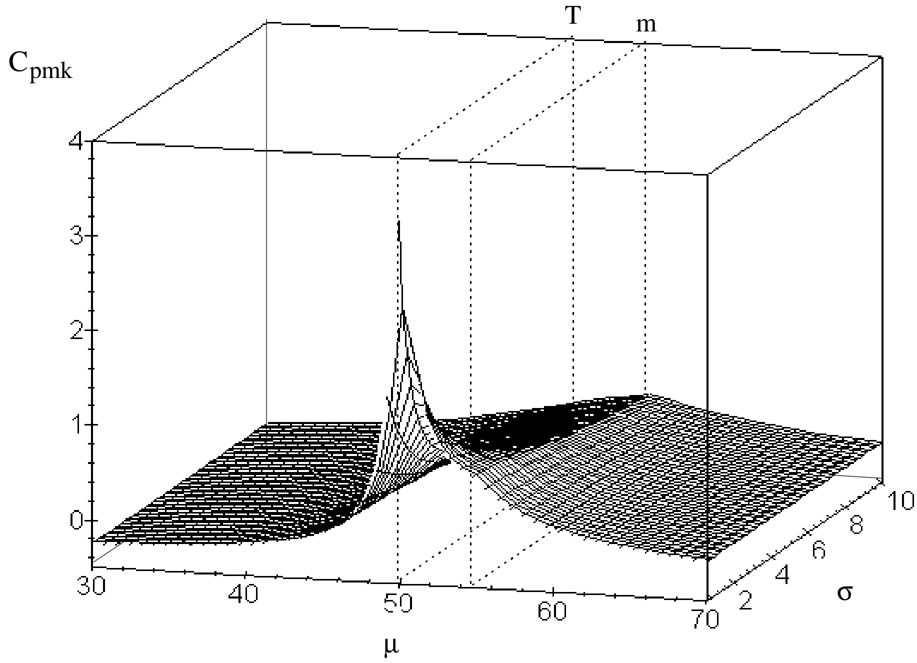

Figure 1: 3D-plot of C_{pmk} with respect to μ and σ

The maximum of C_{pmk} for σ^2 increasing moves from the target value T to the specification midpoint m. This is even better seen in Figure 2.

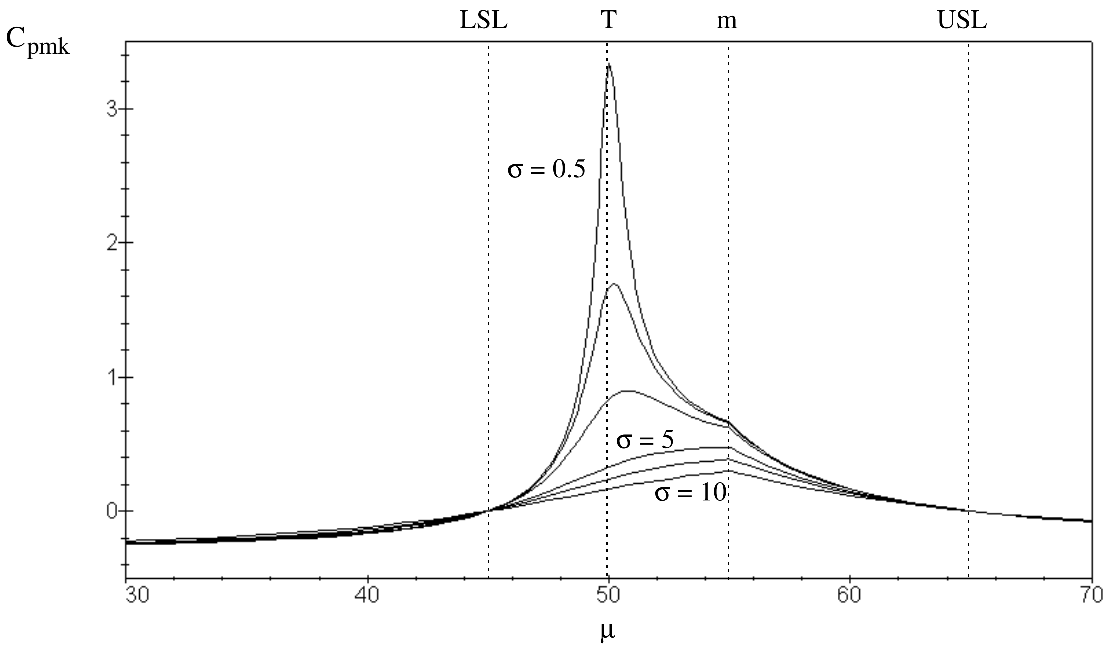

Figure 2 : Plot of C_{pmk} with respect to μ and σ (= 0.5, 1, 2, 5, 7, 10) fixed

From (c) we find that the maximum is located at m if σ is larger than 5. This may also be seen in Figure 2. Furthermore, the "shoulder" in the graph for $\sigma < 5$ and $\mu = m$ is obvious. Finally, Figure 2 also illustrates the fact that the maximum of C_{pmk} never equals the target value, since it is obviously to the right of the target value T.

Conclusion

The behaviour of C_{pmk} in the presence of asymmetric specifications has been discussed. The maximal value of C_{pmk} moves from near the target value to the specification midpoint if the variation of the process increases. C_{pmk} decreases as the variation increases if and only if the process mean varies inside the specification. It is argued that these properties constitute a sensible behaviour of a process capability index.

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Key Words Process Capability Index C_{pmk} , Statistical Process Control