# Proper Bounded Edge-Colorings

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**Abstract**

For fixed integers $k \geq 2$, and for $n$-element sets $X$ and colorings $\Delta: [X]^k \longrightarrow \{0, 1, \ldots\}$ where every color class is a matching and has cardinality at most $u$, we show that there exists a totally multicolored subset $Y \subseteq X$ with

$$|Y| \geq \max \left\{ c_1 \cdot \left( \frac{n^k}{u} \right)^{\frac{1}{2k-1}}, \ c_2 \cdot \left( \frac{n^k}{u} \right)^{\frac{1}{2k-1}} \cdot \left( \ln \left( \frac{u}{\sqrt{n}} \right) \right)^{\frac{1}{2k-1}} \right\}$$

where $c_1, c_2 > 0$ are constants. This lower bound is tight up to constant factors for $u = \Omega(n^{1/2+\epsilon})$ for every $\epsilon > 0$. For fixed values of $k$ we give a polynomial time algorithm for finding such a set $Y$ of guaranteed size.

## 1 Introduction

On each of $\binom{3n}{3}/n$ school days, in a school attended by $3n$ students, the students are asked to line up in $n$ rows, each containing three students. In 1851, Kirkman asked for the existence of such a schedule that would allow each triple of students to form a row on exactly one of the school days, cf. [Bi 81]. This classical problem was answered completely by Ray-Chaudhuri and Wilson [RW 71] who proved that such a schedule exists for each $n \equiv 1, 3 \bmod 6$. Here, we investigate a somewhat related combinatorial problem. Suppose that after such a schedule was prepared, the principle of the school wants (for unrevealed purposes) to select the largest group of, say, $m$ students with the property that no two triples of students form a row on the

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same day. For any schedule such an $m$ must satisfy

$$c_1 \cdot n^{2/5} \cdot (\ln n)^{1/5} \leq m \leq c_2 \cdot n^{2/3} \tag{1}$$

where $c_1, c_2 > 0$ are constants. While the upper bound is straightforward, the lower bound follows from [ALR 91]. There are schedules which, up to constant factors, match the lower bound. Here, we consider the general case in which one has $n$ students which are asked to line up in at most $u$ rows on a day, each containing $k$ people. Our results extend earlier work from [ALR 91] and [LRW 96] where the case $u = n/k$ respectively $k = 2$ was considered. We also give a polynomial time algorithm which finds a group of $m$ students satisfying the lower bound in (1).

It will be convenient to formulate our problem in terms of colorings.

**Definition 1** *Let $\Delta\colon [X]^k \longrightarrow \omega$ where $\omega = \{0, 1, \ldots\}$ be a coloring of the $k$-element subsets of a set $X$. The coloring $\Delta\colon [X]^k \longrightarrow \omega$ with color classes $C_0, C_1, \ldots$, i.e., $\Delta^{-1}(i) = C_i$ for $i \in \omega$, is called $u$-*bounded *if $|C_i| \leq u$ for $i = 0, 1, \ldots$. The coloring $\Delta\colon [X]^k \longrightarrow \omega$ is called* proper *if each color class $C_i$, $i = 0, 1, \ldots$, is a matching, i.e., sets of the same color are pairwise disjoint, thus, $\Delta(U) = \Delta(V)$ implies $U \cap V = \emptyset$ for all distinct sets $U, V \in [X]^k$. A subset $Y \subseteq X$ is called* totally multicolored *if the restriction of the coloring $\Delta$ to the set $[Y]^k$ of all $k$-element subsets of $Y$ is a one-to-one coloring.*

*For an $n$-element set $X$, define the parameter $f_u(n,k)$ by*

$$f_u(n,k) = min_\Delta max_{Y \subseteq X} \{|Y| \; ; \; Y \text{ is totally multicolored}\} \, ,$$

*where we minimize over all proper $u$-bounded colorings $\Delta\colon [X]^k \longrightarrow \omega$ with $|X| = n$.*

The first estimates on $f_u(n,k)$ were given by Babai, cf. [Ba 85], in connection with a Sidon-type problem. He showed for the case $u = n/2$ and $k = 2$ that

$$c_1 \cdot n^{1/3} \leq f_{n/2}(n, 2) \leq c_2 \cdot (n \cdot \ln n)^{1/3}$$

for constants $c_1, c_2 > 0$. In [ALR 91] the lower bound was improved by the factor $\Theta((\ln n)^{1/3})$, i.e., $f_{n/2}(n, 2) \geq c_3 \cdot (n \cdot \ln n)^{1/3}$ where $c_3 > 0$ is a constant. Moreover, for fixed integers $k \geq 2$

the results from [ALR 91] show that

$$f_{n/k}(n,k) = \Theta\left(n^{\frac{k-1}{2k-1}} \cdot (\ln n)^{\frac{1}{2k-1}}\right)$$

Here we will prove the following:

**Theorem 1** *Let $k \geq 2$ be a fixed integer. There exist constants $c_1, c_2, c_3 > 0$ such that for $2 \leq u \leq n/k$,*

$$\max\left\{c_1 \cdot \left(\frac{n^k}{u}\right)^{\frac{1}{2k-1}},\ c_2 \cdot \left(\frac{n^k}{u} \cdot \ln\left(\frac{u}{\sqrt{n}}\right)\right)^{\frac{1}{2k-1}}\right\} \leq f_u(n,k) \leq c_3 \cdot \left(\frac{n^k}{u} \cdot \ln n\right)^{\frac{1}{2k-1}}. \quad (2)$$

*Moreover, for every $n$-element set $X$ and every $u$-bounded proper coloring $\Delta\colon [X]^k \longrightarrow \omega$ one can find in time $O(u \cdot n^{2k-1})$ a totally multicolored subset $Y \subseteq X$ with*

$$|Y| \geq \max\left\{c_1 \cdot \left(\frac{n^k}{u}\right)^{\frac{1}{2k-1}},\ c_2 \cdot \left(\frac{n^k}{u}\right)^{\frac{1}{2k-1}} \cdot \left(\ln\left(\frac{u}{\sqrt{n}}\right)\right)^{\frac{1}{2k-1}}\right\}.$$

## 2 The Existence

In this section, we will prove the existence of a totally multicolored subset as guaranteed by Theorem 1. We will use the notion of of edge-colored hypergraphs. The vertices are the $n$ students, the edges correspond to the rows, and these edges are colored by the day.

Let $\mathcal{G} = (V, \mathcal{E})$ be a hypergraph with vertex set $V$ and edge set $\mathcal{E}$. For a vertex $v \in V$, let $d(v)$ denote the *degree* of $v$ in $\mathcal{G}$, i.e., the number of edges $E \in \mathcal{E}$ containing $v$. Let $d = \sum_{v\in V} d(v)/|V|$ denote the *average degree* of $\mathcal{G}$. If for some fixed $k$ we have $|E| = k$ for each edge $E \in \mathcal{E}$, then $\mathcal{G}$ is called *$k$-uniform.* A *2-cycle* in $\mathcal{G}$ is an (unordered) pair $E, E' \in \mathcal{E}$ of distinct edges which intersect in at least two vertices. The *independence number* $\alpha(\mathcal{G})$ is the largest size of a subset $I \subseteq V$ such that the induced hypergraph contains no edges, i.e., $E \not\subseteq I$ for every edge $E \in \mathcal{E}$.

### Lower Bounds

It turns out that the independence number is important in our considerations. Some of our arguments are based on a result of Ajtai, Komlós, Pintz, Spencer and Szemerédi, [AKPSS 82]. Here, we will use a modified version proved in [DLR 95].

**Theorem 2** *Let $\mathcal{G}$ be a $(k+1)$-uniform hypergraph on $n$ vertices. Assume that*

*(i) $\mathcal{G}$ contains no 2-cycles, and*

*(ii) the average degree satisfies $d \le t^k$ where $t \ge t_0(k)$,*

*then for some positive constant $c = c(k)$,*

$$\alpha(\mathcal{G}) \ge c \cdot \frac{n}{t} \cdot (\ln\ t)^{\frac{1}{k}} \ . \tag{3}$$

Now we are ready to prove the lower bounds given in Theorem 1.

**Proof:** We start by showing the two lower bounds in (2). Let $\Delta\colon [X]^k \to \omega$ be a $u$-bounded proper coloring where $|X| = n$. We construct a $2k$-uniform hypergraph $\mathcal{H} = (X, \mathcal{E})$ on $X$ where $E \in \mathcal{E} \subseteq [X]^{2k}$ if there exist two distinct $k$-element sets $S, T \in [X]^k$, $S, T \subseteq E$, so that $\Delta(S) = \Delta(T)$. As the number of $k$-element sets of the same color is at most $u$, the number of edges in $\mathcal{H}$ satisfies

$$|\mathcal{E}| = \sum_{i \in \omega} \binom{|\Delta^{-1}(i)|}{2} \le \frac{\binom{n}{k}}{u} \cdot \binom{u}{2} \ . \tag{4}$$

Observe that, if $I \subseteq X$ is an independent set of $\mathcal{H}$, then $I$ is totally multicolored with respect to the coloring $\Delta$. Concerning the first lower bound, it is enough to show that $\mathcal{H}$ contains an independent set of size $c_1 \cdot (n^k/u)^{1/(2k-1)}$. To see this, pick every vertex in $X$ at random independently of the other vertices with probability

$$p = (n^{k-1} \cdot u)^{-\frac{1}{2k-1}} \ . \tag{5}$$

By Chernoff's inequality, there exists a subset $Y \subseteq X$ of cardinality at least

$$(1 - o(1)) \cdot p \cdot n = (1 - o(1)) \cdot \left(n^k/u\right)^{\frac{1}{2k-1}} \ ,$$

and by Markov's inequality, the on $Y$ induced subhypergraph $\mathcal{H}_0 = (Y, \mathcal{E} \cap [Y]^{2k})$ of $\mathcal{H}$ contains at most

$$2 \cdot p^{2k} \cdot |\mathcal{E}| \le 2 \cdot p^{2k} \cdot \frac{\binom{n}{k}}{u} \cdot \binom{u}{2} \le \frac{1}{2} \cdot \left(\frac{n^k}{u}\right)^{\frac{1}{2k-1}}$$

edges since $k \ge 2$. By deleting one vertex from each edge in $[Y]^{2k} \cap \mathcal{E}$, we obtain a subset $Y' \subseteq Y$ with $|Y'| \ge |Y|/2 \ge (1/2 - o(1)) \cdot p \cdot n$. Clearly, $Y'$ is an independent set in $\mathcal{H}$, and hence $Y'$ is totally multicolored with respect to $\Delta$, i.e., $f_u(n,k) = \Omega((n^k/u)^{1/(2k-1)})$.

If $u = \sqrt{n} \cdot \omega(n)$, where $\omega(n) \longrightarrow \infty$ with $n \longrightarrow \infty$, we can improve the lower bound $f_u(n,k) \geq c_1 \cdot (n^k/u)^{1/(2k-1)}$ by a logarithmic factor. Let $\Delta: [X]^k \rightarrow \omega$ be a $u$-bounded proper coloring. Consider the $2k$-uniform hypergraph $\mathcal{H}$ with vertex set $X$ and with the set $\mathcal{E}$ of edges defined in the same way as above. Again, we want to show a large lower bound on the independence number of $\mathcal{H}$. Our strategy will be to find a random subset $Y \subseteq X$ such that the induced hypergraph has only a few 2-cycles. By deleting these 2-cycles the desired result will follow with Theorem 2.

Throughout this proof, let $c_1, c_2, \ldots$ denote positive constants. Recall that the number of edges of $\mathcal{H}$ satisfies inequality (4). For $j = 2, 3, \ldots, 2k-1$, let $\nu_j$ denote the number of $(2,j)$-*cycles* in $\mathcal{H}$, i.e., the number of pairs $\{E, E'\} \in [\mathcal{E}]^2$ of edges which intersect in exactly $j$ vertices. First, we estimate the total number $\nu_j$ of $(2,j)$-cycles in the hypergraph $\mathcal{H}$. We fix an edge $E \in \mathcal{E}$. The number of unordered pairs $\{U, V\}$ of distinct sets $U, V \in [X]^k$ with $\Delta(U) = \Delta(V)$ and $|(U \cup V) \cap E| = j$ and $1 \leq |U \cap E|, |V \cap E| \leq j-1$ is bounded from above by

$$\sum_{i=\lceil j/2 \rceil}^{j-1} \binom{2k}{i} \cdot \binom{n-2k}{k-i} \cdot \binom{2k-i}{j-i} \leq c_1 \cdot n^{k-\lceil j/2 \rceil} , \tag{6}$$

as either $|U \cap E| \geq \lceil j/2 \rceil$ or $|V \cap E| \geq \lceil j/2 \rceil$, and every color class is a matching.

If $U \cap E = \emptyset$ or $V \cap E = \emptyset$, but $|(U \cup V) \cap E| = j$, then the number of such pairs $\{U, V\}$ is at most

$$\binom{2k}{j} \cdot \binom{n-2k}{k-j} \cdot (u-1) \leq c_2 \cdot n^{k-j} \cdot u . \tag{7}$$

Now, (4), (6) and (7) imply that

$$\nu_j \quad \leq \quad |\mathcal{E}| \cdot \left(c_1 \cdot n^{k-\lceil j/2 \rceil} + c_2 \cdot n^{k-j} \cdot u\right) \leq c_3 \cdot u \cdot \left(n^{2k-\lceil j/2 \rceil} + n^{2k-j} \cdot u\right) .$$

As $u \leq n/k$ and $j \geq 2$, we have $n^{2k-\lceil j/2 \rceil} \geq n^{2k-j} \cdot u$, hence

$$\nu_j \leq c_4 \cdot u \cdot n^{2k-\lceil j/2 \rceil} . \tag{8}$$

With foresight we use a slightly larger value than in (5) for the probability $p$ of picking vertices, namely, we set

$$p = \left(\frac{1}{n^{k-1} \cdot u}\right)^{\frac{1}{2k-1}} \cdot \left(\frac{u}{\sqrt{n}}\right)^{\frac{1}{(k+1)(2k-1)}} .$$

Let $Y$ be a random subset of $X$ obtained by choosing vertices $v \in X$ with probability $p$ independently of the other vertices. The expected size $E(|Y|)$ of $Y$ is given by

$$E(|Y|) = p \cdot n = \left(\frac{n^k}{u}\right)^{\frac{1}{2k-1}} \cdot \left(\frac{u}{\sqrt{n}}\right)^{\frac{1}{(k+1)(2k-1)}} .$$

Let $\nu_j(Y)$, for $j = 2, 3, \ldots, 2k-1$, be random variables counting the number of $(2, j)$-cycles contained in $Y$. The random variable $\mu_2(Y) = \sum_{j=2}^{2k-1} \nu_j(Y)$ counts the total number of 2-cycles of the subhypergraph induced on $Y$. Let $E(\mu_2(Y))$ and $E(\nu_j(Y))$ denote the corresponding expected values.

We infer for $j = 2, 3, \ldots, 2k-1$ that

$$\begin{aligned} E(\nu_j(Y)) &\leq p^{4k-j} \cdot c_4 \cdot u \cdot n^{2k-\lceil j/2 \rceil} \\ &= pn \cdot c_4 \cdot u^{\frac{j-2k+\frac{1}{k+1}(4k-j-1)}{2k-1}} \cdot n^{\frac{k(j+1-2\lceil j/2 \rceil) - \lfloor j/2 \rfloor - \frac{1}{2(k+1)}(4k-j-1)}{2k-1}} , \end{aligned}$$

thus,

$$E(\nu_j(Y)) \leq \begin{cases} pn \cdot c_4 \cdot u^{\frac{j-2k+\frac{1}{k+1}(4k-j-1)}{2k-1}} \cdot n^{\frac{k-\frac{j}{2}-\frac{1}{2(k+1)}(4k-j-1)}{2k-1}} & \text{if } j \text{ is even} \\ pn \cdot c_4 \cdot u^{\frac{j-2k+\frac{1}{k+1}(4k-j-1)}{2k-1}} \cdot n^{\frac{-\frac{j-1}{2}-\frac{1}{2(k+1)}(4k-j-1)}{2k-1}} & \text{if } j \text{ is odd.} \end{cases}$$

Recall that $u = \sqrt{n} \cdot \omega(n) \leq n/k$ with $\omega(n) \longrightarrow \infty$ with $n \longrightarrow \infty$, hence, $\omega(n) = O(\sqrt{n})$. Then, for $j$ even,

$$\begin{aligned} E(\nu_j(Y)) &\leq pn \cdot c_4 \cdot u^{\frac{j-2k+\frac{1}{k+1}(4k-j-1)}{2k-1}} \cdot n^{\frac{k-\frac{j}{2}-\frac{1}{2(k+1)}(4k-j-1)}{2k-1}} \\ &= pn \cdot c_4 \cdot \omega(n)^{\frac{j-2k+\frac{1}{k+1}(4k-j-1)}{2k-1}} \\ &\leq pn \cdot c_4 \cdot \omega(n)^{\frac{-1}{(k+1)(2k-1)}} && \text{as } j \leq 2k-2 \\ &= o(pn) . && (9) \end{aligned}$$

For $j$ odd, we obtain

$$\begin{aligned} E(\nu_j(Y)) &\leq pn \cdot c_4 \cdot u^{\frac{j-2k+\frac{1}{k+1}(4k-j-1)}{2k-1}} \cdot n^{\frac{-\frac{j-1}{2}-\frac{1}{2(k+1)}(4k-j-1)}{2k-1}} \\ &= pn \cdot c_4 \cdot \omega(n)^{\frac{j-2k+\frac{1}{k+1}(4k-j-1)}{2k-1}} \cdot n^{-\frac{1}{2}} \\ &\leq pn \cdot c_4 \cdot \omega(n)^{\frac{k-1}{(k+1)(2k-1)}} \cdot n^{-\frac{1}{2}} && \text{as } j \leq 2k-1 \\ &= o(pn) && \text{as } \omega(n) = O(\sqrt{n}). \quad (10) \end{aligned}$$

Hence, by (9) and (10) we conclude $E(\mu_2(Y)) = \sum_{j=2}^{2k-1} E(\nu_j(Y)) = o(p \cdot n)$. Using Chernoff's and Markov's inequality, we infer that there exists a subset $Y \subseteq X$ with $|Y| = c_5 pn$, such that the induced hypergraph $\mathcal{H}_0 = (Y, \mathcal{E} \cap [Y]^{2k})$ contains at most $c_6 p^{2k} |\mathcal{E}|$ edges, and has only $o(pn)$ 2-cycles. We omit one vertex from each 2-cycle in $\mathcal{H}_0$. The resulting induced subhypergraph $\mathcal{H}_1$ has $(c_5 - o(1)) \cdot pn$ vertices, contains no 2-cycles anymore, and by (4) has average degree at most

$$d \leq t^{2k-1} = \frac{2k \cdot c_6 \cdot p^{2k} \cdot |\mathcal{E}|}{(c_5 - o(1)) \cdot pn} \leq c_7 \cdot p^{2k-1} \cdot n^{k-1} \cdot u \ ,$$

i.e., $t \leq c_8 \cdot p \cdot (n^{k-1} \cdot u)^{\frac{1}{2k-1}} = c_8 \cdot (u/\sqrt{n})^{\frac{1}{(k+1)(2k-1)}}$. As $u/\sqrt{n} \longrightarrow \infty$ with $n \longrightarrow \infty$ we can apply Theorem 2 to the subhypergraph $\mathcal{H}_1$ which yields

$$\begin{aligned} \alpha(\mathcal{H}) \geq \alpha(\mathcal{H}_1) \ & \geq \ c \cdot \frac{(c_5 - o(1)) \cdot p \cdot n}{c_8 \cdot p \cdot (n^{k-1} \cdot u)^{\frac{1}{2k-1}}} \cdot \left[ \ln \left( c_8 \cdot \left( \frac{u}{\sqrt{n}} \right)^{\frac{1}{(k+1)(2k-1)}} \right) \right]^{\frac{1}{2k-1}} \\ & \geq \ c' \cdot \left( \frac{n^k}{u} \right)^{\frac{1}{2k-1}} \cdot \left( \ln \left( \frac{u}{\sqrt{n}} \right) \right)^{\frac{1}{2k-1}} \ , \end{aligned}$$

i.e., $f_u(n,k) = \Omega((n^k/u)^{1/(2k-1)} \cdot (\ln n)^{1/(2k-1)})$. □

## Upper Bounds

Next, we will show the upper bound in (2) generalizing some arguments from [Ba 85].

**Proof:** Let $X$ be an $n$-element set where without loss of generality $n$ is divisible by $k$. Set $m = \lceil c \cdot n^k/u \rceil$, where $c > 0$ is a constant. Let $M_1, M_2, \ldots, M_m$ be random matchings, chosen uniformly and independently from the set of all matchings of size $u$ on $X$. We define a coloring $\Delta \colon [X]^k \to \omega$ in rounds as follows: in round $j = 1, 2, \ldots, m$, we color every $k$-element set in $M_j$ which has not been colored before, by color $j$. Let $\mathcal{C}_j$ be the set of all $k$-element subsets of $X$ which are colored in some round $i = 1, 2 \ldots, j-1$. In round $m+1$ we color the remaining $k$-elements sets in $[X]^k \setminus \mathcal{C}_{m+1}$ in an arbitrary way, such that each color class is a matching of size at most $u$. Let $Y \subseteq V$ be a fixed subset with $|Y| = x$, where $x = o(n/u^{1/k})$. We will prove that for $x \geq C \cdot (n^k/u \cdot \ln n)^{1/(2k-1)}$ with probability approaching to 1 the set $Y$ is not totally multicolored where $C > 0$ is a sufficiently large constant. This will give the desired result. We split the proof into several claims.

First, we give an upper bound on the probability that a certain number of $k$-element subsets of $Y$ is colored in round $j$.

**Claim 1** *For $j = 1, 2, \ldots, m$ and $t = 0, 1, \ldots,$*

$$\textit{Prob}\ \left[|M_j \cap [Y]^k| \geq t\right] \leq \left(\frac{u \cdot x^k}{n^k}\right)^t . \tag{11}$$

**Proof:** The left hand side of (11) does not depend on the particular choice of $Y$. Thus, assume that the matching $M_j$ is fixed. The set $Y$ can be chosen in $\binom{n}{x}$ ways. If $|M_j \cap [Y]^k| \geq t$, then from $M_j$ we can choose $t$ edges in $\binom{u}{t}$ ways, and the remaining elements of $Y$ can be chosen in at most $\binom{n-kt}{x-kt}$ ways, hence

$$\text{Prob}\ \left[|M_j \cap [Y]^k| \geq t\right] \leq \frac{\binom{u}{t} \cdot \binom{n-kt}{x-kt}}{\binom{n}{x}} \leq \left(\frac{u \cdot x^k}{n^k}\right)^t .$$

$\square$

Now, we estimate the probability that a certain number of $k$-element subsets of $Y$ is colored in some round $i \leq m$.

**Claim 2** *For $t = 0, 1, \ldots$ and for positive integers $n$,*

$$\textit{Prob}\ \left[|\mathcal{C}_{m+1} \cap [Y]^k| \geq t\right] \leq \left(\frac{e \cdot (t+m) \cdot u \cdot x^k}{t \cdot n^k}\right)^t . \tag{12}$$

**Proof:** For $j = 1, 2, \ldots, m$, consider the events $|M_j \cap [Y]^k| \geq t_j$. As the matchings are chosen independently of each other, these events are independent. By Claim 1 we have

$$\text{Prob}\ \left[|\ M_j \cap [Y]^k\ | \geq t_j\right] \leq \left(\frac{u \cdot x^k}{n^k}\right)^{t_j} .$$

Since $|\mathcal{C}_{m+1} \cap [Y]^k| \leq \sum_{j=1}^m |M_j \cap [Y]^k|$ we infer, using $\binom{n}{k} \leq (e \cdot n/k)^k$, that

$$\text{Prob}\ \left[|\mathcal{C}_{m+1} \cap [Y]^k| \geq t\right] \quad \leq \quad \text{Prob}\ \left[\sum_{j=1}^m |M_j \cap [Y]^k| \geq t\right]$$

$$
\begin{aligned}
&\leq \sum_{(t_j)_{j=1}^m, t_j \geq 0, \sum_{j=1}^m t_j = t} \prod_{j=1}^m \text{Prob } \left[ |M_j \cap [Y]^k| \geq t_j \right] \\
&\leq \sum_{(t_j)_{j=1}^m, t_j \geq 0, \sum_{j=1}^m t_j = t} \prod_{j=1}^m \left( \frac{u \cdot x^k}{n^k} \right)^{t_j} \\
&= \sum_{(t_j)_{j=1}^m, t_j \geq 0, \sum_{j=1}^m t_j = t} \left( \frac{u \cdot x^k}{n^k} \right)^{t} \\
&= \binom{t+m-1}{t} \cdot \left( \frac{u \cdot x^k}{n^k} \right)^{t_j} \\
&\leq \left( \frac{e \cdot (t+m)}{t} \right)^{t} \cdot \left( \frac{u \cdot x^k}{n^k} \right)^{t} \\
&= \left( \frac{e \cdot (t+m) \cdot u \cdot x^k}{t \cdot n^k} \right)^{t} .
\end{aligned}
$$

□

For $i = 1, 2, \ldots, m+1$, let $E_i$ denote the event $|\mathcal{C}_i \cap [Y]^k| \leq \lceil c_1 \cdot x^k \rceil$ where $c_1 > 0$ is a constant with $3ec \leq c_1 \leq 1/2 \cdot 1/k!$. Note that if $E_i$ does not hold for some $i$, then also $E_{m+1}$ does not hold.

It turns out that with high probability $E_{m+1}$ holds, i.e., only at most the constant fraction $c_1$ of all $k$-element subsets of $Y$ is colored before round $m+1$:

**Claim 3** *For large enough positive integers $n$,*

$$
\textit{Prob } [E_{m+1}] \geq 1 - 2^{-c_1 \cdot x^k} .
$$

**Proof:** Set $t = \lceil c_1 \cdot x^k \rceil$. Since $x = o\left(n/u^{1/k}\right)$ we have $t = o(n^k/u)$. For $n$ large enough, with $m = \lceil c \cdot n^k/u \rceil$, and as $e \cdot c/c_1 \leq 1/3$, the quotient $\frac{e \cdot (t+m) \cdot u \cdot x^k}{t \cdot n^k}$ is less than $1/2$, hence with (12) we have

$$
\text{Prob } [E_{m+1}] \;\; \geq \;\; 1 - \text{ Prob } \left[ |\mathcal{C}_{m+1} \cap [Y]^k| \geq t \right] \geq 1 - 2^{-t} \geq 1 - 2^{-c_1 \cdot x^k} .
$$

□

We define another random variable $Y_j$ by $Y_j = |[M_j]^2 \cap [[Y]^k \setminus \mathcal{C}_j]^2|$ for $j = 1, 2, \ldots, m$. Then $Y_j$ counts the number of pairs of distinct $k$-element subsets of $Y$ which are colored in round

$j$. For $j = 1, 2, \ldots, m$, we want to determine the probability Prob $[Y_j = 0]$. However, we do not know how many $k$-element sets of $Y$ were already colored in some round $i < j$. Therefore, we condition on the event that only at most the fraction $c_1$ of all $k$-element subsets of $Y$ has been colored before round $j$.

For a random variable $Z$ let $E(Z)$ denote the expected value of $Z$.

**Claim 4** *For some constant $c_2 > 0$, and sufficiently large positive integers $n$, and for $j = 1, 2, \ldots, m$,*

$$E(Y_j|E_j) > c_2 \cdot \frac{u^2 \cdot x^{2k}}{n^{2k}} .$$

**Proof:** As $E_j$ holds, we have for some constant $c_1' > 0$ that

$$|[Y]^k \setminus \mathcal{C}_j| \geq \binom{x}{k} - c_1 \cdot x^k \geq c_1' \cdot x^k .$$

For each set $S \in [Y]^k$ there are less than $k \cdot \binom{x-1}{k-1}$ $k$-element subsets of $Y$ which are not disjoint from $S$. Hence, for for some constant $c_2 > 0$ and $n$ large enough, the number of (unordered) pairs $\{S, T\} \in [[Y]^k \setminus \mathcal{C}_j]^2$ of sets with $S \cap T = \emptyset$ is at least

$$\frac{1}{2} \cdot c_1' \cdot x^k \cdot \left( c_1' \cdot x^k - k \cdot \binom{x-1}{k-1} \right) \geq c_2 \cdot x^{2k} . \tag{13}$$

For given disjoint $k$-element sets $S, T \in [X]^k$, the probability that both sets are in $M_j$ is given by

$$\text{Prob } [S, T \in M_j] \quad = \quad \frac{u \cdot (u-1)}{\binom{n}{k} \cdot \binom{n-k}{k}} \geq \frac{u^2}{n^{2k}} . \tag{14}$$

Hence, by (13) and (14) for the conditional expected value $E(Y_j|E_j)$ we have

$$E(Y_j|E_j) \geq c_2 \cdot \frac{u^2 \cdot x^{2k}}{n^{2k}} .$$

$\square$

**Claim 5** *For $j = 1, 2, \ldots, m$, and large positive integers $n$,*

$$\textit{Prob } [Y_j = 1 \mid E_j] \geq (c_2 - o(1)) \cdot \frac{u^2 \cdot x^{2k}}{n^{2k}} .$$

**Proof:** For $t = 1, 2, \ldots$, we claim that

$$\text{Prob}\,[Y_j \geq t \mid E_j] \leq \left(\frac{u \cdot x^k}{n^k}\right)^{\lceil \sqrt{2t+1}\,\rceil} . \tag{15}$$

Namely, $t$ pairwise distinct two-element sets span a set of cardinality at least $\lceil \sqrt{2t+1}\,\rceil$, i.e., $Y_j \geq t$ implies $|M_j \cap [Y]^k| \geq \lceil \sqrt{2t+1}\,\rceil$. By Claim 1 this shows inequality (15):

$$\text{Prob}\,[Y_j \geq t \mid E_j] \leq \text{Prob}\,\left[|M_j \cap [Y]^k| \geq \lceil \sqrt{2t+1}\,\rceil\right] \leq \left(\frac{u \cdot x^k}{n^k}\right)^{\lceil \sqrt{2t+1}\,\rceil} .$$

For $i = 0, 1, \ldots$, set $p_i = \text{Prob}\,[Y_j = i \mid E_j]$. Then we infer from (15), using $x = o\left(n/u^{1/k}\right)$, that

$$\begin{aligned} E(Y_j \mid E_j) = \sum_{i \geq 0} i \cdot p_i &\leq p_1 + \sum_{i \geq 2} i \cdot \left(\frac{u \cdot x^k}{n^k}\right)^{\lceil \sqrt{2i+1}\rceil} \\ &= p_1 + O\left(\left(\frac{u \cdot x^k}{n^k}\right)^3\right) \\ &= p_1 + o\left(\frac{u^2 \cdot x^{2k}}{n^{2k}}\right) . \end{aligned}$$

By Claim 4 we infer that $p_1 \geq (c_2 - o(1)) \cdot u^2 \cdot x^{2k}/n^{2k}$. □

Finally, for $j = 1, 2, \ldots, m$ let $A_j$ denote the event ($Y_j = 0$ and $E_{j+1}$).

**Claim 6** *For some constant $c_3 > 0$, and large enough positive integers $n$,*

$$\textit{Prob}\,[A_1 \wedge \ldots \wedge A_m] \leq \exp\left(-c_3 \cdot u \cdot \frac{x^{2k}}{n^k}\right) .$$

**Proof:** Notice that

$$\text{Prob}\,[A_1 \wedge \ldots \wedge A_m] = \text{Prob}\,[A_1] \cdot \prod_{i=2}^{m} \text{Prob}\,[A_i \mid A_1 \wedge \ldots \wedge A_{i-1}] . \tag{16}$$

By Claim 5 we have

$$\begin{aligned} \text{Prob}\,[A_1] &\leq \text{Prob}\,(Y_1 = 0\,|E_1) \leq \text{Prob}\,(Y_1 \neq 1\,|E_1) \leq \\ &\leq 1 - (c_2 - o(1)) \cdot \frac{u^2 \cdot x^{2k}}{n^{2k}} , \end{aligned} \tag{17}$$

while for $i \geq 2$ we infer

$$\begin{aligned}
\text{Prob}\,[A_i\,|A_1 \wedge \ldots \wedge A_{i-1}] &\leq \text{Prob}\,[Y_i = 0 \mid A_1 \wedge \ldots \wedge A_{i-1}] \\
&\leq \text{Prob}\,[Y_i = 0 \mid E_i] \\
&\leq \text{Prob}\,[Y_i \neq 1 \mid E_i] \\
&\leq 1 - (c_2 - o(1)) \cdot \frac{u^2 \cdot x^{2k}}{n^{2k}}\,. \qquad (18)
\end{aligned}$$

Using $(1-x)^m \leq \exp(-m \cdot x)$ where $m = \lceil c \cdot n^k/u \rceil$, inequalities (17), (18) together with (16) imply

$$\begin{aligned}
\text{Prob}\,[A_1 \wedge A_2 \wedge \ldots \wedge A_m] &\leq \left(1 - (c_2 - o(1)) \cdot \frac{u^2 \cdot x^{2k}}{n^{2k}}\right)^m \\
&\leq \exp\left(-(c_2 - o(1)) \cdot m \cdot \frac{u^2 \cdot x^{2k}}{n^{2k}}\right) \\
&\leq \exp\left(-c \cdot (c_2 - o(1)) \cdot \frac{u \cdot x^{2k}}{n^k}\right) \\
&\leq \exp\left(-c_3 \cdot \frac{u \cdot x^{2k}}{n^k}\right)\,.
\end{aligned}$$

$\square$

**Claim 7** *For large enough positive integers $n$, the probability that there exists a totally multicolored $x$-element subset is at most*

$$\binom{n}{x} \cdot \left(\exp\left(-c_3 \cdot u \cdot \frac{x^{2k}}{n^k}\right) + 2^{-c_1 \cdot x^k}\right)\,. \qquad (19)$$

**Proof:** If $Y$ is totally multicolored, then $Y_1 = Y_2 = \ldots = Y_m = 0$. Thus, either $A_1 \wedge A_2 \wedge \ldots \wedge A_m$ holds or some $E_i$, hence, $E_{m+1}$ fails. As there are exactly $\binom{n}{x}$ $x$-element sets $Y$, by combining the estimates from Claim 3 and Claim 6 we obtain (19). $\square$

We want to show that for $n \longrightarrow \infty$ expression (19) tends to 0 for $x \geq C \cdot \left(n^k/u\right)^{1/(2k-1)} \cdot (\ln n)^{1/(2k-1)}$ where $C > 0$ is a big enough constant. Namely,

$$\begin{aligned}
\binom{n}{x} \cdot 2^{-c_1 \cdot x^k} &\leq \left(\frac{e \cdot n}{x}\right)^x \cdot 2^{-c_1 \cdot x^k} \\
&\leq \exp\left(x \cdot \ln \frac{n}{x} - c_1 \cdot \ln 2 \cdot x^k\right) \\
&= o(1)
\end{aligned}$$

for $x \geq C \cdot (\ln n)^{1/(k-1)}$ where $C > 0$ is a large enough constant.

Moreover, we have

$$\begin{aligned}
\binom{n}{x} \cdot \exp\left(-c_3 \cdot u \cdot \frac{x^{2k}}{n^k}\right) &\leq \left(\frac{e \cdot n}{x}\right)^x \cdot \exp\left(-c_3 \cdot u \cdot \frac{x^{2k}}{n^k}\right) \\
&\leq \exp\left(2x \cdot \ln n - c_3 \cdot u \cdot \frac{x^{2k}}{n^k}\right) \\
&\leq \exp\left((2C - c_3 \cdot C^{2k}) \cdot \left(\frac{n^k}{u}\right)^{\frac{1}{2k-1}} \cdot (\ln n)^{\frac{2k}{2k-1}}\right) \\
&= o(1)
\end{aligned}$$

provided $C^{2k-1} > 2/c_3$ and $n$ is large enough. Thus, expression (19) tends to 0 with $n \longrightarrow \infty$. For $n \leq n_0$ one can obtain asymptotically the same upper bound by taking an appropriately large constant $C > 0$. □

# 3 An Algorithm

Here, we show that one can find in time $O(u \cdot n^{2k-1})$ a totally multicolored subset as guaranteed by Theorem 1. The algorithm follows the probabilistic arguments given before. It is based on recent results of Fundia [Fu 96] and from [BL 96].

**Proof:** Let $k \geq 2$ be a fixed integer and let $\Delta: [X]^k \rightarrow \omega$ with $|X| = n$ be a proper $u$-bounded coloring. First, we order the set $[X]^k$ of $k$-element subsets according to their color. This can be done in time $O(n^k \cdot \ln n)$. Then, by examining all $k$-element sets in $[X]^k$ we form a $2k$-uniform hypergraph $\mathcal{H} = (X, \mathcal{E})$, $\mathcal{E} \subseteq [X]^{2k}$, where $E \in \mathcal{E}$ if there exist two distinct $k$-element sets $S, T \in [X]^k$ with $S \cup T = E$ and $\Delta(S) = \Delta(T)$. By (4), we have $|\mathcal{E}| = O(n^k \cdot u)$, hence constructing the hypergraph $\mathcal{H}$ can be done in time $O(n^k \cdot u + n^k \cdot \ln n)$. We use the following algorithmic version of Turán's theorem, cf. [BL 96]. The existence result was given by Spencer [Sp 72].

**Lemma 1** *Let $\mathcal{G} = (V, \mathcal{E})$ be a $k$-uniform hypergraph on $n$ vertices with average degree $d^{k-1} \geq 1$. Then, one can find in time $O(|V| + |\mathcal{E}|)$ an independent set $I \subseteq V$ with*

$$|I| \geq \frac{k-1}{k} \cdot \frac{n}{d} \, .$$

**Proof:** We sketch the arguments. We use the method of conditional probabilities, cf. [AS 92]. Let $V = \{v_1, v_2, \ldots, v_n\}$. Every vertex $v_i$ will be assigned a probability $p_i \in [0,1]$, $i = 1, 2, \ldots, n$. Define a potential by

$$V(p_1, p_2, \ldots, p_n) = \sum_{i=1}^{n} p_i - \sum_{E \in \mathcal{E}} \prod_{v_i \in E} p_i \ .$$

The choice $p_1 = p_2 = \ldots = p_n = p = 1/d$ gives the initial value of the potential

$$V(p, \ldots, p) = p \cdot n - p^k \cdot \frac{n \cdot d^{k-1}}{k} = \frac{k-1}{k} \cdot \frac{n}{d} \ .$$

In each step $i$, $i = 1, 2, \ldots, n$, one after the other, we choose either $p_i = 0$ or $p_i = 1$ in order to maximize the current value of $V(p_1, p_2, \ldots, p_n)$. As $V(p_1, p_2, \ldots, p_n)$ is linear in each $p_i$, for $i = 1$, for example, either $V(p_1, \ldots, p_n) \leq V(1, p_2, \ldots, p_n)$ or $V(p_1, \ldots, p_n) \leq V(0, p_2, \ldots, p_n)$. If $V(p_1, \ldots, p_n) \leq V(1, p_2, \ldots, p_n)$, we set $p_1 = 1$, else let $p_1 = 0$. Iterating this, we obtain finally $p_1, p_2, \ldots, p_n \in \{0, 1\}$.

By our strategy, we infer $V(p_1, p_2, \ldots, p_n) \geq V(p, p, \ldots, p)$. For $V' = \{v_i \in V \mid p_i = 1\}$ we have

$$|V'| = \sum_{i=1}^{n} p_i = V(p_1, p_2, \ldots, p_n) + \sum_{E \in \mathcal{E}} \prod_{v_i \in E} p_i \ .$$

We can assume that $V'$ is independent as otherwise we omit one vertex from each edge contained in $V'$ and the value of $V(p_1, p_2, \ldots, p_n)$ will not decrease. Thus, $|V'| \geq V(p, p, \ldots, p) = \frac{k-1}{k} \cdot \frac{n}{d}$ and $V'$ is an independent set. The running time is $O(|V| + |\mathcal{E}|)$. □

By (4) the average degree $d$ of $\mathcal{H}$ satisfies $d^{2k-1} \leq 2k \cdot |\mathcal{E}|/|X| \leq c_1 \cdot n^{k-1} \cdot u$. By Lemma 1 we can find in time $O(|X| + |\mathcal{E}|) = O(n^k \cdot u)$ an independent set in $\mathcal{H} = (X, \mathcal{E})$ of size at least

$$\frac{k-1}{k} \cdot \frac{n}{d} \geq c' \cdot \frac{n}{(n^{k-1} \cdot u)^{\frac{1}{2k-1}}} = c' \cdot \left(\frac{n^k}{u}\right)^{\frac{1}{2k-1}}$$

where $c' > 0$ is a constant. With the sorting procedure in the beginning, this part of the algorithm can be done in time $O(n^k \cdot u + n^k \cdot \ln n)$.

Now, assume that $u = \sqrt{n} \cdot \omega(n)$ where $\omega(n) \longrightarrow \infty$ with $n \longrightarrow \infty$. Again we consider the hypergraph $\mathcal{H} = (X, \mathcal{E})$. First, we construct the sets $C_{2,j}$ of $(2,j)$-cycles in $\mathcal{H}$, $j = 2, 3, \ldots, 2k-1$. Using that the $k$-element sets are sorted according to their color, and that sets of the same color are pairwise disjoint, and using the considerations leading to (8), all 2-cycles in $\mathcal{H}$ can be constructed in time $O(|C_{2,j}|) = O(u \cdot n^{2k-\lceil j/2 \rceil})$.

We use the following lemma.

**Lemma 2** *Let $k \geq 3$ be an integer. Let $\mathcal{G} = (V, \mathcal{E})$ be a $k$-uniform hypergraph with $|V| = n$. Let $\mathcal{G}$ contain $\nu_j(\mathcal{G})$ many $(2,j)$-cycles which can be determined all in time $O(\nu_j(\mathcal{G}))$, $j = 2, 3, \ldots, k-1$. Then, for every real $p$ with $0 \leq p \leq 1$, one can find in time $O(|V| + |\mathcal{E}| + \sum_{j=2}^{k-1} \nu_j(\mathcal{G}))$ an induced subhypergraph $\mathcal{G}' = (V', \mathcal{E}')$ such that*

$$
\begin{aligned}
|V'| &\geq p/3 \cdot |V| \\
|\mathcal{E}'| &\leq 3 \cdot p^k \cdot |\mathcal{E}| \\
\nu_j(\mathcal{G}') &\leq 3k \cdot p^{2k-j} \cdot \nu_j(\mathcal{G})
\end{aligned}
$$

*for $j = 2, 3, \ldots, k-1$.*

**Proof:** As in the proof of Lemma 1, we use the method of conditional probabilities. Let $C_{2,j}$ be the set of all $(2,j)$-cycles in $\mathcal{G}$, $j = 2, 3, \ldots, k-1$.

Let $V = \{v_1, v_2, \ldots, v_n\}$. If $pn < 3.9$, any two-element subset $V' \subseteq V$ gives the desired subhypergraph, thus let $pn \geq 3.9$. Every vertex $v_i$ will be assigned a probability $p_i \in [0,1]$, $i = 1, 2, \ldots, n$. Define a potential $V(p_1, p_2, \ldots, p_n)$ by

$$
\begin{aligned}
V(p_1, p_2, \ldots, p_n) &= 3^{pn/3} \cdot \prod_{i=1}^{n} \left(1 - \frac{2}{3} \cdot p_i\right) + \\
&\quad + \frac{\sum_{E \in \mathcal{E}} \prod_{v_i \in E} p_i}{3 \cdot p^k \cdot |\mathcal{E}|} + \sum_{j=2}^{k-1} \frac{\sum_{C \in C_{2,j}} \prod_{v_i \in C} p_i}{3k \cdot p^{2k-j} \cdot |C_{2,j}|} \, .
\end{aligned}
$$

With $p_1 = p_2 = \ldots = p_n = p$ in the beginning, for $pn/3 \geq 1.3$ we have

$$
\begin{aligned}
V(p, \ldots, p) &= 3^{pn/3} \cdot \left(1 - \frac{2}{3} \cdot p\right)^n + \frac{p^k \cdot |\mathcal{E}|}{3 \cdot p^k \cdot |\mathcal{E}|} + \sum_{j=2}^{k-1} \frac{p^{2k-j} \cdot \nu_j(\mathcal{G})}{3k \cdot p^{2k-j} \cdot \nu_j(\mathcal{G})} \\
&\leq \left(\frac{3}{e^2}\right)^{pn/3} + \frac{2}{3} \\
&< 1 \, .
\end{aligned}
$$

Step by step, we decide which choice of $p_i \in \{0,1\}$ minimizes the current value of $V(p_1, p_2, \ldots, p_n)$. We set $p_1 = 1$, if $V(1, p_2, \ldots, p_n) \leq V(0, p_2, \ldots, p_n)$, else we set $p_1 = 0$. Iterating this for all vertices $v_1, v_2, \ldots, v_n$, we obtain finally $p_1, p_2, \ldots, p_n \in \{0,1\}$.

We have chosen the $p_i$'s to minimize the potential, thus, $V(p_1, p_2, \ldots, p_n) < 1$. The set $V' = \{v_i \in V \mid p_i = 1\}$ yields the desired induced subhypergraph as otherwise $V(p_1, p_2, \ldots, p_n) > 1$. The whole computation can be done in time $O(|V| + |\mathcal{E}| + \sum_{j=2}^{k-1} \nu_j(\mathcal{G}))$. □

We apply Lemma 2 to the hypergraph $\mathcal{H} = (X, \mathcal{E})$ with

$$p = \left(\frac{1}{n^{k-1} \cdot u}\right)^{\frac{1}{2k-1}} \left(\frac{u}{\sqrt{n}}\right)^{\frac{1}{(k+1)(2k-1)}} , \tag{20}$$

and we obtain in time $O(|X| + |\mathcal{E}| + \sum_{j=2}^{2k-1} \nu_j(\mathcal{H})) = O(u \cdot n^{2k-1})$ an induced subhypergraph $\mathcal{H}' = (X', \mathcal{E}')$ of $\mathcal{H}$ with $|X'| \geq pn/3$, and, $|\mathcal{E}'| \leq 3p^{2k} \cdot |\mathcal{E}|$ and, using the considerations (9), (10) the 2-cycles of $\mathcal{H}'$ satisfy $\sum_{j=2}^{2k-1} \nu_j(\mathcal{H}') \leq pn/6$ for $n$ large enough. Then, in time at most $O\left(u \cdot n^{2k-1}\right)$ we can determine all 2-cycles in $\mathcal{H}'$ and delete from $\mathcal{H}'$ one vertex from each 2-cycle. The resulting induced hypergraph $\mathcal{H}''$ on at least $pn/6$ vertices contains at most $c \cdot p^{2k} \cdot n^k \cdot u$ edges, thus, has average degree $d^{2k-1} \leq c' \cdot p^{2k-1} \cdot n^{k-1} \cdot u$. Then, we apply the following result from [BL 96] which gives an algorithmic version of the existence result from [DLR 95] and extends an algorithm of Fundia [Fu 96].

**Theorem 3** *Let $k \geq 3$ be a fixed integer. Let $\mathcal{G} = (V, \mathcal{E})$ be a $k$-uniform hypergraph on $n$ vertices with average degree at most $t^{k-1}$. If $\mathcal{G}$ does not contain any 2-cycles, then one can find for every fixed $\delta > 0$ in time $O(n \cdot t^{k-1} + n^3/t^{3-\delta})$ an independent set of size at least $c(k, \delta) \cdot n/t \cdot (\ln t)^{1/(k-1)}$.*

We apply Theorem 3 to $\mathcal{H}''$ and in time $O\left(p^{2k} \cdot n^k \cdot u + n^3 / \left(p \cdot n^{\frac{k-1}{2k-1}} \cdot u^{\frac{1}{2k-1}}\right)^{3-\delta}\right)$ $= o\left(n^{2k-1} \cdot u\right)$, where $\delta < 3$, we obtain an independent set in $\mathcal{H}''$ hence in $\mathcal{H}$ of size at least

$$c_2 \cdot \left(\frac{n^k}{u}\right)^{\frac{1}{2k-1}} \cdot \left(\ln\left(\frac{u}{\sqrt{n}}\right)\right)^{\frac{1}{2k-1}} .$$

The corresponding vertices form a totally multicolored set of size as desired. □

## 4 Concluding Remarks

The running time of the algorithm can be reduced slightly as follows. Similarly as in Lemma 2, we choose first a subhypergraph $\mathcal{H}' = (X', \mathcal{E}')$ of $\mathcal{H} = (X, \mathcal{E})$, where we do not control the 2-cycles, but where $|X'| = p_1 n/3$ and $|\mathcal{E}'| \leq 3p_1^{2k} \cdot |\mathcal{E}|$. Then, $\mathcal{H}'$ contains at most $O(u \cdot (p_1 \cdot n)^{2k-\lceil j/2 \rceil})$ many $(2,j)$-cycles. The value of $p_1 > 0$ should be chosen as small as possible such that for some constant $\gamma > 0$ and $j = 2, 3, \ldots, 2k-1$, cf. [DLR 95] or [BL 96]:

$$u \cdot (p_1 n)^{2k-\lceil j/2 \rceil - 1} = O\left( p_1 n \cdot \left( p_1 \cdot n^{\frac{k-1}{2k-1}} \cdot u^{\frac{1}{2k-1}} \right)^{4k-1-j-\gamma} \right) .$$

For this subhypergraph we apply Lemma 2 with a different parameter $p_2$ with $p \approx p_1 \cdot p_2$ and proceed as before where the value of $p$ is given by (20). Thus, we save some time by controlling the 2-cycles later. However, more interesting might be to find the real growth rate of $f_u(n,k)$ and a corresponding fast algorithm. It might be also of some interest to give explicitly a coloring which yields our, or possibly better upper bounds, on $f_u(n,k)$.

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