On the distribution of a test statistic for outlier identification in exponential samples

Jörg Pawlitschko*

Department of Statistics
University of Dortmund
D-44221 Dortmund

Abstract

In this paper we discuss the distribution of the ratio of the maximum and the appropriately standardized median of a (sub-) sample consisting of the m smallest observations in a sample of size N coming from a one-parameter exponential distribution. A statistics of this kind is useful when testing for the presence of outliers, especially when implemented within an inward or outward testing procedure. Besides giving a tractable expression for the survival function of this statistic we tabulate the critical values needed for corresponding outlier identification rules for samples of size up to $N = 50$.

1 Introduction

Let $\underline{x}_N = (x_1, \ldots, x_N)$ be a sample occuring in a lifetime experiment. A simple but nevertheless useful model for such lifetimes assumes that the x_i come i.i.d. from a one-parameter exponential distribution $Exp(\nu)$ with scale parameter $\nu > 0$ and distribution function

$$F_\nu(t) = 1 - \exp(-t/\nu), \qquad t > 0.$$

However, often one is concerned with the problem that an unknown number $k \leq k^* = \lfloor (N-1)/2 \rfloor$ of observations in $\underline{x}_N$ indeed do not come from $Exp(\nu)$ but are outliers with

*pawl@statistik.uni-dortmund.de

respect to this distribution. To give the notion of an outlier a more formal meaning we adopt the concept introduced in Davies and Gather (1993) and call any $x \in \mathbb{R}_+$ an α-outlier with respect to a distribution F if it is contained in the so-called α-outlier region of F. In case of $F = Exp(\nu)$ this α-outlier region is given by

$$out(\alpha, Exp(\nu)) \;=\; \{x > 0 : \; x > -\nu \ln(\alpha)\}.$$

Usually α is chosen depending on the sample size N as $\alpha = \alpha_N = 1 - (1 - \tilde{\alpha})^{1/N}$ for some given $\tilde{\alpha} \in (0, 1)$. The task of identifing all outliers in in $\underline{x}_N$ can then be formalized as the problem of deciding for each x_i whether it is located in $out(\alpha_N, Exp(\nu))$ or not.

Beginning with Cochran (1941) there is a vast literature on the topic of outlier identification in exponential samples, contributions have been made e.g. by Kale (1976), Kimber (1982), Sweeting (1983), Chikkagoudar and Kunchur (1987), Likeš (1967, 1987), Balasooriya (1989), Balasooriya and Gadag (1994), Tse and Balasooriya (1991), Jeevanand and Nair (1998), Schultze and Pawlitschko (2000a, 2000b). There are mainly three different types of identification rules: (i) one-step outlier identifiers, (ii) inward testing procedures, and (iii) outward testing procedures. A one-step outlier identifier is an empirical version, say OR, of the α_N-outlier region that is constructed from the given sample so that any observation located in OR is classified as outlier. The other two rules proceed in a stepwise manner. With an inward testing procedure, first the "most extreme" observation of the entire sample is checked with an appropriate discordancy test whether it is an outlier. If the test fails to reject the corresponding null hypothesis, no observation is declared as outlier and the procedure stops. Otherwise, the most extreme observation is identified as outlier and removed from the sample. Then in a second step the most extreme observation of the remaining subsample is tested. The procedure terminates if for the first time a discordancy test does not reject or if a given maximal number k^* of observations has been classified as outliers. The largest reasonable choice for k^* is $k^* = \lfloor (N-1)/2 \rfloor$ which is assumed furtheron. Outward testing procedures work in the reverse direction. In a first step, the k^* most extreme observations are removed from the sample. Then the least suspicious of these observations is rejoined with the remaining ones and checked by a discordancy test whether it sticks out as an outlier in this subsample of size $N - k^* + 1$. If the discordancy test rejects, all k^* removed observations are declared as outliers and the procedure terminates. Otherwise, the next of the removed observations is added to the subsample and tested with respect to its outlyingness. The procedure terminates if the first time a discordancy test rejects or if all observations that have been removed in the first step are eventually rejoined with the reduced sample.

In general the question which observations should be regarded as the k^* "most extreme" ones of a given sample has no unique solution. In the exponential case, however, clearly the k^* largest observations stand out as the most susceptible ones. Let $x_{(1)} \leq \cdots \leq x_{(N)}$ denote the ordered values in the sample and $\underline{x}_{N-i+1,N} \;=\; (x_{(1)}, \ldots, x_{(N-i+1)})$, $i = 1, \ldots, k^*$, the subsample considered in the i-th step of the inward testing procedure or

$(k^* - i + 1)$-th step of the outward testing procedure. Principally, for inward and outward testing procedures the same test statistics can be used for a certain subsample (of course some kind of standardization for the whole procedure has to be taken into account, see Section 3). However, many inward testing procedures suffer from their proneness to masking which generally means that an outlier is not discovered because it is hidden by further large outliers in the sample. This disadvantage has lead most authors to prefer outward testing procedures which are not susceptible to masking. However, as Davies and Gather (1993) already noted in the case of normal samples and Schultze and Pawlitschko (2000b) discuss in detail for the exponential case, the masking trap of inward testing procedures can be avoided if robust discordancy tests are applied.

Appealing are tests with test statistics of type

$$T^S_{N-i+1}(\underline{x}_{N-i+1,N}) \;=\; \frac{x_{(N-i+1)}}{S_{N-i+1}(\underline{x}_{N-i+1,N})}, \qquad i = 1, \ldots, k^*, \tag{1}$$

where S_n denotes an estimator of the scale parameter ν based on n observations. There are many possible choices for S_n that are also robust, see e.g. Gather and Schultze (1999). One possible choice is the standardized median (SM) which for a sample $\underline{x}_n = (x_1, \ldots, x_n)$ of size n is defined as

$$SM_n(\underline{x}_n) \;=\; \frac{1}{\ln 2}\,\mathsf{Med}(\underline{x}_n) \;=\; \frac{1}{\ln 2}\begin{cases} x_{((n+1)/2)}, & n \text{ odd},\\ \frac{1}{2}\left(x_{(n/2)} + x_{(n/2+1)}\right), & n \text{ even}.\end{cases}$$

The constant $1/\ln 2$ is needed to achieve Fisher-consistency. When used as component of a test statistic T^{SM}_{N-i+1} of type (1), multiplication with this constant is actually not necessary. However, we prefer to keep the constant since interpretation of the test statistic becomes easier and comparability with other possible test statistics of type (1) is guaranteed. Gather and Schultze (1999) prove that SM is a most B-robust scale estimator which has optimal explosion breakdown point 1/2. As Schultze and Pawlitschko (2000a, b) show, these good robustness properties also carry over to outlier identification rules that are based on SM.

When discordancy tests with test statistics of type (1) are applied within an inward or outward testing procedure, the corresponding critical values can usually be determined only via simulations since the finite sample distribution of these test statistics becomes intractable. However, it is possible to give explicit expressions for the survival function of T^{SM}_{N-i+1}, $i = 1, \ldots, k^*$, that are simple enough to allow the exact calculation of critical values if N is not too large. These expressions and their derivation are presented in Section 2. Section 3 contains some remarks concerning the choice of critical values and some tables for sample sizes up to $N = 50$.

2 Finite sample distribution of the test statistics

The following theorems give more general results than needed for the determination of critical values for inward and outward testing procedures based on the standardized median. We set

$$C(N,\, m,\, r,\, \ell) \;=\; \frac{(-1)^{m-\ell}}{(m-\ell)!\,(\ell-r-1)!\,(N-\ell+1)},$$

$1 \le r < \ell \le m \le N$, for short.

Theorem 1 Let $\underline{X}_N = (X_1, \ldots, X_N)$ be a random sample with elements coming i.i.d. from an $Exp(\nu)$-distribution and let $X_{(1)} \le \cdots \le X_{(N)}$ denote the corresponding order statistics. Then for $1 \le r < m \le N$ and $a > 0$

$$\begin{aligned} &P\Big(\frac{X_{(m)}}{a\, X_{(r)}} \;>\; t\Big) \\ &\quad=\; \frac{N!}{(N-m)!} \sum_{\ell=r+1}^{m} C(N,\, m,\, r,\, \ell) \prod_{i=1}^{r} \frac{1}{(N-\ell+1)\,(a\,t-1)+(N-i+1)} \end{aligned}$$

for $t > 1/a$.

Proof. From a well known result for order statistics from an exponential distribution we have that for $i = 1, \ldots, N$

$$X_{(k)} \;\stackrel{d}{=}\; \nu \sum_{i=1}^{k} \frac{U_i}{N-i+1}$$

where U_i, $i = 1, \ldots, N$, are independent $Exp(1)$-distributed random variables and $\stackrel{d}{=}$ denotes equality in distribution. Hence

$$\begin{aligned} &P\Big(\frac{X_{(m)}}{a\, X_{(r)}} \;>\; t\Big) \\ &\quad=\; P\Big(\sum_{i=r+1}^{m} \frac{U_i}{N-i+1} \;>\; (a\,t-1) \sum_{i=1}^{r} \frac{U_i}{n-i+1}\Big) \\ &\quad=\; \int \cdots \int_{\mathbb{R}_+^r} P\Big(\sum_{i=r+1}^{m} \frac{U_i}{N-i+1} \;>\; (a\,t-1) \sum_{i=1}^{r} \frac{u_i}{N-i+1}\Big) \times \cdots \\ &\qquad \cdots \times \exp\Big(-\sum_{i=1}^{r} u_i\Big)\, du_r \ldots du_1. \qquad (2) \end{aligned}$$

Now $\sum_{i=r+1}^{m} U_i/(N-i+1)$ has a so-called general gamma distribution with survival function

$$P\Big(\sum_{i=r+1}^{m} \frac{U_i}{N-i+1} > y\Big) = \sum_{i=r+1}^{m} \prod_{\substack{j=r+1\\ j\neq i}}^{m} \frac{N-j+1}{i-j} \exp\big(-(N-i+1)\,y\big) \tag{3}$$

for $y > 0$ (see e.g. Johnson et al., 1995). Inserting this result in (2), integrating out $u_1, \ldots, u_r$, and making use of

$$\begin{aligned}\prod_{\substack{j=r+1\\ j\neq i}}^{m} \frac{N-j+1}{j-i} &= \frac{(-1)^{m-i}}{(m-i)!\,(i-r-1)!} \prod_{\substack{j=r+1\\ j\neq i}}^{m} (N-j+1) \\ &= C(N,\,m,\,r,\,i) \prod_{j=r+1}^{m} (N-j+1) \end{aligned} \tag{4}$$

yields the representation of the survival function as stated in the theorem. $\square$

Theorem 2 Under the assumptions of Theorem 1, for $1 \le r < m-1$, $m \le N$, and $a > 0$

$$\begin{aligned} & P\Big(\frac{X_{(m)}}{a/2\,(X_{(r)} + X_{(r+1)})} > t\Big) \\ &= \frac{N!}{(N-m)!} \sum_{\ell=r+2}^{m} C(N,\,m,\,r,\,\ell)\, \frac{1}{(N-\ell+1)\,(a\,y/2-1)+(N-r)} \times \cdots \\ &\quad \cdots \times \prod_{i=1}^{r} \frac{1}{(N-\ell+1)\,(a\,t-1)+(N-i+1)} \end{aligned}$$

for $t \ge 2/a$ and

$$\begin{aligned} & P\Big(\frac{X_{(m)}}{a/2\,(X_{(r)} + X_{(r+1)})} > t\Big) \\ &= \frac{N!}{(N-m)!} \sum_{\ell=r+2}^{m} C(N,\,m,\,r,\,\ell)\, \frac{1}{(N-\ell+1)\,(a\,y/2-1)+(N-r)} \times \cdots \\ &\quad \cdots \times \Big(\prod_{i=1}^{r} \frac{1}{(N-\ell+1)\,(a\,t-1)+(N-i+1)} - \cdots \\ &\quad \cdots - \prod_{i=1}^{r} \frac{1}{(N-r)\,\dfrac{a\,t-1}{1-a\,t/2} + (N-i+1)}\Big) + \cdots \end{aligned}$$

$$\cdots + \prod_{i=1}^{r} \frac{N-i+1}{(N-r)\,\dfrac{a\,t-1}{1-a\,t/2} + (N-i+1)}$$

for $1/a < t < 2/a$.

Proof. With the same notations as in the proof of Theorem 1 we have

$$\begin{aligned} & P\Big(\frac{X_{(m)}}{a/2\,(X_{(r)} + X_{(r+1)})} \;>\; t\Big) \\ = \;\; & P\Big(\sum_{i=r+2}^{m} \frac{U_i}{N-i+1} \;>\; (a\,t-1)\,\sum_{i=1}^{r} \frac{U_i}{N-i+1} + (a\,t/2-1)\,\frac{U_{r+1}}{N-r}\Big). \qquad (5) \end{aligned}$$

Now we have to distinguish between two cases:

(i) $t \geq 2/a$: In this case the right hand side of the inequality that occurs in (5) is always nonnegative. Hence the proof can be carried through with similar arguments as the proof of Theorem 1.

(ii) $1/a < t < 2/a$: Now the right hand side of the above inequality may also take on negative values so that conditional on the realizations of $U_1, \ldots, U_{r+1}$ the corresponding event may occur with probability one. Thus (5) becomes

$$\begin{aligned} & \int\limits_{\mathbb{R}_+^r}\!\cdots\!\int \Big[\int_0^M P\Big(\sum_{i=r+1}^{m} \frac{U_i}{N-i+1} \;>\; (a\,t-1)\,\sum_{i=1}^{r} \frac{u_i}{N-i+1} + (a\,t/2-1)\,\frac{u_{r+1}}{N-r}\Big) \\ & \qquad\qquad \exp(-u_{r+1})\,du_{r+1}\Big]\;\exp\Big(-\sum_{i=1}^{r} u_i\Big)\,du_r \ldots du_1 + \cdots \\ & \qquad\qquad \cdots + \int\limits_{\mathbb{R}_+^r}\!\cdots\!\int \Big[\int_M^\infty \exp(-u_{r+1})\,du_{r+1}\Big]\;\exp\Big(-\sum_{i=1}^{r} u_i\Big)\,du_r \ldots du_1 \\ & \quad = \;\; (\mathrm{I}) + (\mathrm{II}), \end{aligned}$$

say, where

$$M \;=\; (N-r)\,\frac{a\,t-1}{1-a\,t/2}\,\sum_{i=1}^{r} \frac{u_i}{N-i+1}.$$

Integrating out (I) immediately gives

$$(\mathrm{I}) \;=\; \prod_{i=1}^{r} \frac{N-i+1}{(N-r)\,\dfrac{a\,t-1}{1-a\,t/2} + N-i+1}.$$

A lengthy but straightforward calculation using again the distributional result (3) and equation (4) now with $r+2$ instead of $r+1$ leads to

$$
\begin{aligned}
(\mathrm{II}) \;=\; & \frac{N!}{(N-m)!} \sum_{\ell=r+2}^{m} C(N,\, m,\, r,\, \ell)\, \frac{1}{(N-\ell+1)\,(a\,t/2-1)+(N-r)} \times \cdots \\
& \cdots \times \Bigg(\prod_{i=1}^{r} \frac{1}{(N-\ell+1)\,(a\,t-1)+(N-i+1)} - \cdots \\
& \cdots - \prod_{i=1}^{r} \frac{1}{(N-r)\,\dfrac{a\,t-1}{1-a\,t/2}+(N-i+1)} \Bigg).
\end{aligned}
$$

Combining these results gives the assertion of Theorem 2. □

3 Critical values

The results from the previous section are now used to find critical values for the discordancy tests based on SM which can be used within stepwise outlier identification rules. First we have to specify the test levels for each step. Usually, an outlier identification rule based on test statistics T_{N-i+1}, $i = 1, \ldots, k^*$, is standardized such that under the null model H_0 that $X_i \sim Exp(\nu)$, $i = 1, \ldots, N$, one has

$$P_{H_0}(\text{no observation is identified as } \alpha_N\text{-outlier}) \;\geq\; 1-\tilde{\alpha}. \tag{6}$$

For an inward testing procedure this requirement is already fulfilled if the critical value $t_N(\tilde{\alpha})$ for the discordancy test used in the first step is chosen such that this test keeps the level $\tilde{\alpha}$, that is

$$P_{H_0}\big(T_N(\underline{X}_N) \;>\; t_N(\tilde{\alpha})\big) \;\leq\; \tilde{\alpha}.$$

The critical values $t_{N-i+1}(\tilde{\alpha})$, $i = 2, \ldots, k^*$, for the following steps then can be chosen arbitrarily. Mostly they are also determined according to

$$P_{H_0}\big(T_{N-i+1}(\underline{X}_{N-i+1,N}) \;>\; t_{N-i+1}(\tilde{\alpha})\big) \;\leq\; \tilde{\alpha}. \tag{7}$$

For an outward testing procedure (6) is equivalent to the requirement that

$$P_{H_0}\Big(\bigcup_{i=1}^{k^*} \big\{T_{N-i+1}(\underline{X}_{N-i+1,N}) \;>\; t_{N-i+1}(\tilde{\alpha})\big\} \Big) \;\geq\; 1-\tilde{\alpha}.$$

For most choices of the test statistics their joint distribution is not tractable. However, a simple Bonferroni argument shows that (6) is fulfilled if the critical values are chosen according to

$$P_{H_0}\big(T_{N-i+1}(\underline{X}_{N-i+1,N}) \;>\; t_{N-i+1}(\tilde{\alpha})\big) \;=\; \tilde{\alpha}/k^*. \tag{8}$$

The following tables contain the critical values $t^{SM}_{N-i+1}(\tilde{\alpha})$, $i = 1, \ldots, k^*$, for the inward and outward testing procedures with test statistics (1) based on the standardized median for sample sizes $N = 10(10)50$ and $\tilde{\alpha} = 0.05, 0.1$. The null distribution of the test statistics is obtained from Theorems 1 and 2 by choosing $a = 1/\ln 2$ and $r = \lfloor (m-1)/2 \rfloor$ for $m = N - k^* + 1, \ldots, N$. The local levels of the tests are chosen according to (7) and (8), respectively.

Note that the critical values are not always monotone decreasing in i as might have possibly been expected. This is due to the fact that in case that $N - i + 1$ is even the median of the subsample $\underline{x}_{N-i+1,N}$ is defined as the mean of the two order statistics with greatest depth.

	Inward testing $\tilde{\alpha} =$		Outward testing $\tilde{\alpha} =$	
i	0.05	0.1	0.05	0.1
1	6.6208	5.3039	9.7130	8.0825
2	5.0377	4.0302	7.4780	6.1785
3	3.9756	3.2207	5.8028	4.8300
4	3.9184	3.0912	6.0392	4.8906

Table 1. Critical values for $N = 10$

	Inward testing $\tilde{\alpha} =$		Outward testing $\tilde{\alpha} =$	
i	0.05	0.1	0.05	0.1
1	7.0150	5.9053	10.9172	9.6113
2	5.1973	4.4351	7.8639	6.9717
3	4.3264	3.7251	6.4171	5.7193
4	3.9624	3.4023	5.9441	5.2771
5	3.5763	3.0852	5.3120	4.7280
6	3.4529	2.9576	5.2477	4.6369
7	3.2005	2.7518	4.8259	4.2727
8	3.1814	2.7060	4.9600	4.3457
9	2.9805	2.5448	4.6111	4.0479

Table 2. Critical values for $N = 20$

	Inward testing $\tilde{\alpha} =$		Outward testing $\tilde{\alpha} =$	
i	0.05	0.1	0.05	0.1
1	7.2223	6.2111	11.3471	10.2082
2	5.3631	4.6932	8.0338	7.3027
3	4.5275	3.9957	6.6249	6.0534
4	4.1027	3.6253	5.9960	5.4785
5	3.7444	3.3212	5.4185	4.9614
6	3.5520	3.1450	5.1795	4.7324
7	3.3312	2.9572	4.8256	4.4153
8	3.2300	2.8580	4.7380	4.3208
9	3.0709	2.7230	4.4805	4.0906
10	3.0195	2.6652	4.4801	4.0724
11	2.8934	2.5589	4.2722	3.8874
12	2.8755	2.5280	4.3376	3.9252
13	2.7683	2.4384	4.1560	3.7646
14	2.7773	2.4281	4.2837	3.8534

Table 3. Critical values for $N = 30$

	Inward testing $\tilde{\alpha} =$		Outward testing $\tilde{\alpha} =$	
i	0.05	0.1	0.05	0.1
1	7.3808	6.4265	11.6317	10.5878
2	5.5083	4.8888	8.1743	7.5289
3	4.6856	4.1937	6.7716	6.2702
4	4.2372	3.8024	6.0798	5.6368
5	3.8866	3.4999	5.5194	5.1276
6	3.6686	3.3035	5.2180	4.8450
7	3.4551	3.1182	4.8830	4.5395
8	3.3263	2.9985	4.7265	4.3882
9	3.1756	2.8675	4.4906	4.1729
10	3.0937	2.7882	4.4099	4.0902
11	2.9778	2.6876	4.2276	3.9241
12	2.9251	2.6334	4.1955	3.8851
13	2.8308	2.5518	4.0456	3.7488
14	2.7985	2.5148	4.0498	3.7419
15	2.7185	2.4458	3.9206	3.6249
16	2.7020	2.4218	3.9557	3.6449
17	2.6316	2.3615	3.8399	3.5403
18	2.6286	2.3481	3.9048	3.5854
19	2.5646	2.2937	3.7969	3.4886

Table 4. Critical values for $N = 40$

	Inward testing $\tilde{\alpha} =$		Outward testing $\tilde{\alpha} =$	
i	0.05	0.1	0.05	0.1
1	7.5130	6.5960	11.8595	10.8762
2	5.6345	5.0474	8.3005	7.7085
3	4.8181	4.3525	6.8941	6.4373
4	4.3575	3.9495	6.1681	5.7706
5	4.0097	3.6464	5.6144	5.2629
6	3.7784	3.4387	5.2821	4.9523
7	3.5673	3.2531	4.9549	4.6508
8	3.4249	3.1225	4.7664	4.4716
9	3.2765	2.9917	4.5387	4.2616
10	3.1803	2.9016	4.4234	4.1494
11	3.0673	2.8019	4.2502	3.9896
12	2.9994	2.7364	4.1794	3.9184
13	2.9086	2.6563	4.0399	3.7897
14	2.8597	2.6075	3.9998	3.7465
15	2.7840	2.5409	3.8827	3.6386
16	2.7491	2.5042	3.8660	3.6166
17	2.6840	2.4471	3.7643	3.5232
18	2.6601	2.4197	3.7671	3.5185
19	2.6028	2.3696	3.6765	3.4355
20	2.5879	2.3497	3.6966	3.4461
21	2.5364	2.3049	3.6140	3.3706
22	2.5294	2.2914	3.6510	3.3958
23	2.4822	2.2505	3.5739	3.3255
24	2.4826	2.2429	3.6282	3.3655

Table 5. Critical values for $N = 50$

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