Integer Optimization with Total Variation Regularization

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Integer Optimization with Total Variation Regularization

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Abstract

This thesis is concerned with the analysis and solution of integer optimization problems in function space with total variation regularization. We prove the existence of optimal solutions and provide first-order optimality conditions. A function space trust-region algorithm for the solution in the multi-dimensional case is proposed and its convergence to stationary points is analyzed.

In order to compute lower bounds for the global optimization of integer optimization problems with total variation regularization, we consider the relaxation that is obtained by relaxing the integrality condition to box constraints. In order to apply an outer-approximation algorithm in function space, we introduce a regularization of the relaxation which includes a regularized total variation and a Tikhonov regularization. We derive necessary and sufficient optimality conditions for an example from optimal control and use them for the construction of an instance with known optimal solution.

For the numerical solution of integer optimization problems with total variation regularization, we introduce a novel discretization to recover the total variation in the presence of integrality restrictions. Due to the restriction of the integer-valued discretized input functions to prescribed finite-element meshes, the known discretizations from literature generally fail to recover the total variation of limit functions correctly. Our novel discretization consists of two embedded meshes for the discretization of the input function and the discretization of the total variation term whose mesh sizes are superlinearly coupled. In order to conserve compactness, we add an additional constraint to the discretized problems that vanishes as the superlinearly coupled mesh sizes are driven to zero. This constraint contains a constant whose admissible range is determined and whose choice has a significant impact on the numerical results. We propose an outer-approximation algorithm to solve the discretized problems. Moreover, we transfer the discretization to the relaxation.

With our numerical experiments, we demonstrate the practicability of the proposed discretizations and algorithms. In particular, we illustrate that our novel discretization scheme is able to recover the interfaces of level sets of limit functions correctly while the anisotropic discretizations from literature lead to distortions.

Zusammenfassung

Diese Thesis befasst sich mit der Analyse und Lösung von ganzzahligen Optimierungsproblemen im Funktionenraum mit Totalvariationsregularisierung. Wir beweisen die Existenz von optimalen Lösungen und stellen Optimalitätsbedingungen erster Ordnung auf. Ein Trust-Region-Algorithmus zur Lösung im mehrdimensionalen Fall wird vorgeschlagen und dessen Konvergenz zu stationären Punkten wird analysiert.

Um untere Schranken für die globale Optimierung von ganzzahligen Optimierungsproblemen mit Totalvariationsregularisierung zu berechnen, betrachten wir die Relaxierung, die durch die Relaxierung der Ganzzahligkeitsbedingung zu Box-Beschränkungen erhalten wird. Um einen äußeren Approximationsalgorithmus im Funktionenraum anzuwenden, führen wir eine Regularisierung der Relaxierung ein, die eine regularisierte Totalvariation und eine Tikhonov-Regularisierung enthält. Wir leiten notwendige und hinreichende Optimalitätsbedingungen für ein Beispiel aus der Optimalsteuerung her und verwenden diese für die Konstruktion einer Instanz mit bekannter optimaler Lösung.

Für das numerische Lösen von ganzzahligen Optimierungsproblemen mit Totalvariationsregularisierung führen wir eine neuartige Diskretisierung ein, um die Totalvariation in der Anwesenheit von Ganzzahligkeitsrestriktionen approximieren zu können. Aufgrund der Restriktion der ganzzahligen diskretisierten Eingabefunktionen auf vorgegebene Finite-Elemente-Gitter schaffen es die bekannten Diskretisierungen aus der Literatur im Allgemeinen nicht, die Totalvariation von Grenzfunktionen korrekt zu approximieren. Unsere neuartige Diskretisierung besteht aus zwei eingebetteten Gittern für die Diskretisierung der Eingabefunktion und die Diskretisierung der Totalvariation, deren Gitterweiten superlinear gekoppelt sind. Um die Kompaktheitseigenschaft beizubehalten, fügen wir eine zusätzliche Nebenbedingung zu den diskretisierten Problemen hinzu, die verschwindet, wenn die superlinear gekoppelten Gitterweiten gegen Null getrieben werden. Diese Nebenbedingung beinhaltet eine Konstante, deren zulässiger Bereich bestimmt wird und dessen Wahl einen signifikanten Einfluss auf die numerischen Ergebnisse hat. Wir schlagen einen äußeren Approximationsalgorithmus zur Lösung der diskretisierten Probleme vor. Außerdem übertragen wir die Diskretisierung auf die Relaxierung.

Mit unseren numerischen Experimenten demonstrieren wir die praktische Umsetzbarkeit der vorgeschlagenen Diskretisierungen und Algorithmen. Insbesondere veranschaulichen wir, dass unser neues Diskretisierungsschema dazu in der Lage ist, die Schnittstellen der Niveaumengen von Grenzfunktionen korrekt wiederherzustellen, während die anisotropen Diskretisierungen aus der Literatur zu Verzerrungen führen.

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Contents

Chapter 1

Introduction

Integer optimization problems in function space with total variation regularization are of great interest, both from a mathematical and an application point of view. Many applications can be modeled by means of integer optimization problems. This includes for example material optimization, topology optimization, imaging, network planning, and many more.

While in finite dimensions the existence of optimal solutions to integer optimization problems is already guaranteed under very mild assumptions, this changes completely if the integer optimization problems are formulated in function space. The reason for this is that the feasible sets of integer optimization problems in function space are generally not sequentially weakly-$*$ closed in L^∞ such that the limits of weakly-$*$ convergent minimizing sequences in L^∞ might not be contained in it. The total variation regularization can remedy this issue by enforcing compactness on the minimizing sequence which yields the existence of convergent subsequences with regard to a stronger convergence notion with respect to which the feasible set is sequentially closed.

The total variation of an integer-valued function measures the perimeter of the level sets of its input function scaled by the jump heights. Consequently, the total variation regularization penalizes large perimeters as they for example result from high frequency chattering functions. This ensures that the resulting optimal solutions to integer optimization problems with total variation regularization can be implemented in practice, where the level of detail that can be realized is often limited.

Integer optimization problems with total variation regularization are therefore a powerful modeling tool, but they present some challenges in both theoretical analysis and numerical implementation that will be addressed and tackled in this thesis.

1.1 Problem formulation

We now formulate the superordinate optimization problem that is subject of this thesis. Throughout this thesis, we make the following assumptions, which are extended in the following chapters when required. The set $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}$, is assumed to be a bounded Lipschitz domain. The superordinate problem reads

$$\text{(P)} \qquad \begin{aligned} \min_{u \in L^2(\Omega)} \quad & J(u) \coloneqq F(u) + \alpha \mathrm{TV}(u) \\ \text{s.t.} \quad & u(x) \in U \text{ for almost all (a.a.) } x \in \Omega, \end{aligned}$$

where $U \coloneqq \{\nu_1, \dots, \nu_N\} \subset \mathbb{Z}$ is a non-empty and finite set of $N \in \mathbb{N}$ integers with $\underline{\nu} \coloneqq \min\{\nu : \nu \in U\}$ and $\overline{\nu} \coloneqq \max\{\nu : \nu \in U\}$ and $\alpha > 0$. For the function $F : L^2(\Omega) \to \mathbb{R}$, the following is assumed.

Assumption 1.1. The function $F : L^2(\Omega) \to \mathbb{R}$ is

(i) lower semicontinuous and

(ii) bounded from below, that is, there is some $B \in \mathbb{R}$ such that

$$F(u) \geq B \tag{1.1}$$

for all $u \in L^2(\Omega)$ with $\underline{\nu} \leq u(x) \leq \overline{\nu}$ for almost all $x \in \Omega$.

Moreover, $\mathrm{TV} : L^1(\Omega) \to [0, \infty]$ denotes the total variation and is for $u \in L^1(\Omega)$ defined by

$$\mathrm{TV}(u) = \sup \left\{ \int_\Omega u(x) \operatorname{div} \phi(x) \, \mathrm{d}x : \phi \in C_c^1(\Omega; \mathbb{R}^d), \|\phi\|_{L^\infty(\Omega;\mathbb{R}^d)} \leq 1 \right\} \tag{TV}$$

with $\|\phi\|_{L^\infty(\Omega;\mathbb{R}^d)} \coloneqq \operatorname{ess\,sup}_{x\in\Omega} \|\phi(x)\|_2$. We denote the space of functions of bounded variation by

$$\mathrm{BV}(\Omega) \coloneqq \left\{ u \in L^1(\Omega) : \mathrm{TV}(u) < \infty \right\}.$$

We denote the feasible set of (P) by

$$L^1_U(\Omega) \coloneqq \{u \in L^1(\Omega) : u(x) \in U \text{ for a.a. } x \in \Omega\}$$

and its subset of feasible solutions with finite objective value by

$$\mathrm{BV_U}(\Omega) \coloneqq \mathrm{BV}(\Omega) \cap L^1_U(\Omega).$$

1.2 Integer optimization with total variation regularization in literature

Mixed-integer optimization problems in function space are of great interest in current research. This is due to their wide range of real-life applications like optimal gear shifting [54], traffic light optimization in road networks [57], the control of transmission lines [56], the control of gas and fluid flow in networks [61], and material and topology optimization [10, 71, 99].

Solution methods for integer optimization problems in function space are for example trust-region algorithms [60], combinatorial integral approximation [64, 75, 92, 93], multibang regularization [37, 73], and outer-approximation algorithms [20].

Total variation regularization is a frequently used tool in the context of optimization. This includes for example image denoising and deblurring [28, 46, 87, 90] and image segmentation [47, 88]. Also in the field of optimal control, total variation regularization is subject to recent research, see for instance [24, 26, 36, 66].

The use of total variation regularization by means of constraints or penalty in the context of integer optimization is therefore a highly demanded and topical research field. Integer optimal control problems with total variation restriction in the constraints are for example considered in [94]. A parabolic optimal control problem with binary-valued controls whose total variation is restricted over the time horizon is investigated in [16] and an outer-approximation method [17] and a branch-and-bound algorithm [18] are proposed for its solution. Extended formulations for the closed convex hull of feasible controls in function space [15, 21] and after discretization [19] have been investigated. In [70], a trust-region algorithm for the solution of integer optimization problems with total variation regularization in the objective on one-dimensional domains is introduced. The solution of the resulting subproblems is addressed in [79, 97].

Despite the many possible applications, research on integer optimization problems with total variation regularization on multi-dimensional domains is still relatively scarce. This is due to the associated complex theoretical analysis as well as the non-trivial numerical implementation. We will fill this gap by developing holistic solution methods that include optimality conditions, algorithms in functions space, computations of lower bounds, appropriate finite-element discretizations, and algorithms for the solution after discretization.

1.3 Contribution

We will state first-order optimality conditions for (P) in terms of the application of local variations to the level sets of the input function. We propose a trust-region algorithm in function space to solve (P) in the multi-dimensional case $d \geq 2$. This trust-region algorithm is a development of the trust-region algorithm stated in [70] for the one-dimensional case. We will prove that the iterates of the algorithm converge strictly in $\mathrm{BV}(\Omega)$ to a stationary point that fulfills the introduced first-order optimality condition.

In order to solve (P) globally, the computation of lower bounds for the optimal value of (P) is crucial. To this end, we consider the relaxation of (P) that is obtained by relaxing the integrality constraint $u(x) \in U$ to box constraints $\underline{\nu} \leq u(x) \leq \overline{\nu}$. The optimal value of the relaxation yields a lower bound on the optimal value of (P). In order to solve the relaxation, we will state an outer-approximation algorithm that is applied to a regularization of the relaxation. This regularization contains a regularized total variation term and a Tikhonov regularization. In contrast to the total variation, the regularized total variation can be expressed by a unique maximizer of its dual formulation. This fact is used for the calculation of tight cutting planes for the outer-approximation algorithm. We will prove that the iterates of the outer-approximation algorithm converge in $L^2(\Omega)$ to an optimal solution to the regularized relaxation. Moreover, we will prove that the optimal solutions to the regularized relaxations converge weakly to an optimal solution of the relaxation if the regularization parameters are driven to zero. Additionally, we will state optimality conditions for the relaxation in order to construct an instance with known optimal solution for our numerical experiments.

In order to solve (P) numerically, we introduce a novel discretization scheme for (P). The common discretizations of the total variation known from literature generally fail to approximate the total variation term correctly in the presence of integrality restrictions. This is due to their anisotropic behavior caused by the restriction to prescribed finite-element meshes. We therefore introduce a discretization of the total variation and a corresponding discretization of (P) that involves the coupling of two embedded finite-element meshes, the finer one for the discretization of the input function and the coarser one for the discretization of the total variation. If the mesh sizes are coupled superlinearly, we are able to recover the total variation term in the presence of integrality constraints. The replacement of the total variation by the discretized total variation in (P) is accompanied with the loss of compactness when the superlinearly coupled mesh sizes are driven to zero. This makes it necessary to add an additional constraint that restores compactness and vanishes as the mesh sizes are driven to zero. This additional constraint involves a constant whose admissible range is determined and which has a significant influence on the numerical results as we demonstrate in our numerical examples. We prove convergence of the discretized problem to (P) in the sense of Γ-convergence if the superlinearly coupled mesh sizes are driven to zero. Moreover, we state and analyze an outer-approximation algorithm to solve the discretized optimization problems. We also introduce discretizations of the relaxation and the regularized relaxation and prove their convergence in the sense of Γ-convergence if the mesh sizes are driven to zero.

In our numerical examples we demonstrate the practicability of the established algorithms. The instances used are from imaging and optimal control and fulfill the assumptions made in this thesis. We also look at the practical implementation of the discretizations and algorithms including a reformulation of the occurring finite-dimensional subproblems as mixed-integer linear and quadratic programs that can be solved with well-known off-the-shelf solvers. The presented numerical results demonstrate the impact of our research findings. In particular, they demonstrate the ability of the novel discretization of (P) with superlinearly coupled meshes to recover the interfaces of level sets correctly and that the common discretizations from literature fail to do so.

Publications

The results from Chapter 3 are published in the article [77]. The results in Chapter 4 are self-contained but use concepts from the article [82], which is currently in preparation and contains further results. Sections 5.1 to 5.3 and Sections 6.1 and 6.2 have been submitted for publication in article [95], which has been accepted for publication and is in the production process. In addition to the results in this thesis, the article [96] has also been created and published during the PhD research of the author.

1.4 Structure of this work

Chapter 2 provides the groundwork for the findings of this thesis that are elaborated in the following chapters. In Chapter 3, we state the first-order optimality condition for (P) and the trust-region algorithm formulated in functions space to solve (P) in the multi-dimensional case. We introduce the relaxation of (P) and its regularization in Chapter 4 and state the function space outer-approximation algorithm to solve the regularized relaxation. Chapter 5 introduces the discretization of the total variation in presence of integrality restrictions and the corresponding discretization of (P) as well as an outer-approximation algorithm to solve the discretized problems. We present numerical examples in Chapter 6 and draw a conclusion in Chapter 7.

Chapter 2

Preliminaries and notation

This chapter provides the tools that are required for the analysis and the solution of optimization problem (P). In Section 2.1, we consider the space of functions of bounded variation and define a suitable notion of convergence. Section 2.2 introduces sets of finite perimeter and draws the connection to functions of bounded variation. With the preparations from Section 2.1, we will prove the existence of optimal solutions to (P) in Section 2.3. We introduce the space $H(\mathrm{div};\Omega)$ in Section 2.4 which will play an essential role in the formulation of the regularized total variation and the discretized total variation. The latter is discretized by means of Raviart–Thomas functions which are introduced in Section 2.5.

2.1 Functions of bounded variation

The distributional gradient of a function of bounded variation is a finite Radon measure. In order to have the necessary analytical tools at hand, we collect some essential definitions from measure theory as well as definitions and statements regarding a sensible notion of convergence in the space of functions of bounded variation from [4].

We denote measure spaces by $(X, \mathcal{E})$, where X is a nonempty set and $\mathcal{E} \subset \mathcal{P}(X)$ denotes the σ-algebra.

Definition 2.1 (Borel σ-algebra, [4, Def. 1.1])**.** Let $X \subset \mathbb{R}^n$. We define the Borel σ-algebra, denoted by $\mathcal{B}(X)$, as the smallest σ-algebra that contains all open subsets of X.

We define a measure and its total variation as follows.

Definition 2.2 (Measure and total variation, [4, Def. 1.4])**.** Let $(X, \mathcal{E})$ be a measure space. A measure on $(X, \mathcal{E})$ is a mapping $\mu : \mathcal{E} \to \mathbb{R}^m$ such that $\mu(\emptyset) = 0$ and for any sequence $\{E_k\}_{k\in\mathbb{N}}$ of pairwise disjoint elements of $\mathcal{E}$, there holds

$$\mu\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} \mu(E_k).$$

The total variation $|\mu| : \mathcal{E} \to [0,\infty]$ is for $E \in \mathcal{E}$ defined by

$$|\mu|(E) := \sup\left\{\sum_{k=1}^{\infty} \|\mu(E_k)\|_2 : E_k \in \mathcal{E} \text{ pairwise disjoint, } E = \bigcup_{k=1}^{\infty} E_k\right\}.$$

Definition 2.3 (Finite Radon measure, [4, Def. 1.40])**.** Let $X \subset \mathbb{R}^d$ be open with Borel σ-algebra $\mathcal{B}(X)$ and consider the measure space $(X, \mathcal{B}(X))$. If $\mu : \mathcal{B}(X) \to \mathbb{R}^m$ is a measure according to Definition 2.2, then μ is called a finite Radon measure. We denote by $\mathcal{M}(X;\mathbb{R}^m)$ the space of all finite $\mathbb{R}^m$-valued Radon measures on X.

The relation between functions of bounded variation and finite Radon measures is the following.

Proposition 2.4 ([4, Def. 3.1, Prop. 3.6])**.** *Let $u \in L^1(\Omega)$. The function u is a function of bounded variation in Ω, that is, $u \in \mathrm{BV}(\Omega)$, if and only if the distributional derivative of u can be represented by a finite Radon measure $Du = (D_1u, \dots, D_du)$ in Ω, that satisfies*

$$\int_\Omega u(x)\frac{\partial \phi}{\partial x_i}(x)\,\mathrm{d}x = -\int_\Omega \phi(x)\,\mathrm{d}D_iu(x) \quad \forall\, \phi \in C_c^\infty(\Omega)$$

for $i = 1, \dots, d$. Moreover, there holds

$$\mathrm{TV}(u) = |Du|(\Omega)$$

for $u \in \mathrm{BV}(\Omega)$.

The space $\mathrm{BV}(\Omega)$ equipped with the norm

$$\|u\|_{\mathrm{BV}(\Omega)} := \|u\|_{L^1(\Omega)} + \mathrm{TV}(u)$$

is a Banach space. As discussed in Remark 3.12 in [4], the dual space of $\mathrm{BV}(\Omega)$ is hard to characterize, which is the reason why we do not use weak convergence in $\mathrm{BV}(\Omega)$. Instead, we use that $\mathrm{BV}(\Omega)$ is the dual space of a separable space and that the usual weak-$*$ convergence corresponds to the following definition.

Definition 2.5 (Weak-$*$ convergence, [4, Def. 3.11])**.** Let $\{u_k\}_{k\in\mathbb{N}} \subset \mathrm{BV}(\Omega)$ be a sequence and $u \in \mathrm{BV}(\Omega)$. We say that $\{u_k\}_{k\in\mathbb{N}}$ converges weakly-$*$ in $\mathrm{BV}(\Omega)$ to u if

$u_k \to u$ in $L^1(\Omega)$ and $Du_k \overset{*}{\rightharpoonup} Du$ in $\mathcal{M}(\Omega;\mathbb{R}^d)$, that is,

$$\lim_{k\to\infty} \int_\Omega \phi(x)\,\mathrm{d}Du_k(x) = \int_\Omega \phi(x)\,\mathrm{d}Du(x)$$

for all $\phi \in C_0(\Omega)$.

In Proposition 3.13 in [4], a simpler criterion for weak-$*$ convergence in $\mathrm{BV}(\Omega)$ is stated.

Proposition 2.6 ([4, Prop. 3.13])**.** *Let $\{u_k\}_{k\in\mathbb{N}} \subset \mathrm{BV}(\Omega)$. Then $u_k \overset{*}{\rightharpoonup} u$ in $\mathrm{BV}(\Omega)$ as $k \to \infty$, if and only if $\{u_k\}_{k\in\mathbb{N}}$ is bounded in $\mathrm{BV}(\Omega)$ and $u_k \to u$ in $L^1(\Omega)$ as $k \to \infty$.*

Additionally, we define strict convergence in $\mathrm{BV}(\Omega)$.

Definition 2.7 (Strict convergence, [4, Def. 3.14])**.** Let $\{u_k\}_{k\in\mathbb{N}} \subset \mathrm{BV}(\Omega)$ be a sequence and $u \in \mathrm{BV}(\Omega)$. We say that $\{u_k\}_{k\in\mathbb{N}}$ converges strictly in $\mathrm{BV}(\Omega)$ to u if $u_k \to u$ in $L^1(\Omega)$ and $\mathrm{TV}(u_k) \to \mathrm{TV}(u)$ as $k \to \infty$.

The following compactness result will be essential to prove the existence of subsequences that converge weakly-$*$ in $\mathrm{BV}(\Omega)$.

Theorem 2.8 (Compactness in BV, [4, Thm. 3.23])**.** *If a sequence $\{u_k\}_{k\in\mathbb{N}}$ is bounded in $\mathrm{BV}(\Omega)$, then it admits a subsequence that converges weakly-$*$ in $\mathrm{BV}(\Omega)$ to some $u \in \mathrm{BV}(\Omega)$.*

Proposition 2.9. *The mapping $\mathrm{TV} : L^1(\Omega) \to [0,\infty]$ is convex and lower semicontinuous and therefore weakly lower semicontinuous.*

Proof. The lower semicontinuity of $\mathrm{TV} : L^1(\Omega) \to [0,\infty]$ is proven in [4, Prop. 3.6]. To prove the convexity, let $u, v \in L^1(\Omega)$ and $\lambda \in [0,1]$. There holds

$$\begin{aligned}
&\mathrm{TV}(\lambda u + (1-\lambda)v) \\
&\quad = \sup\left\{\int_\Omega (\lambda u(x) + (1-\lambda)v(x))\operatorname{div}\phi(x)\,\mathrm{d}x : \phi \in C_c^1(\Omega;\mathbb{R}^d), \|\phi\|_{L^\infty(\Omega;\mathbb{R}^d)} \le 1\right\} \\
&\quad \le \lambda \sup\left\{\int_\Omega u(x)\operatorname{div}\phi(x)\,\mathrm{d}x : \phi \in C_c^1(\Omega;\mathbb{R}^d), \|\phi\|_{L^\infty(\Omega;\mathbb{R}^d)} \le 1\right\} \\
&\qquad + (1-\lambda)\sup\left\{\int_\Omega v(x)\operatorname{div}\phi(x)\,\mathrm{d}x : \phi \in C_c^1(\Omega;\mathbb{R}^d), \|\phi\|_{L^\infty(\Omega;\mathbb{R}^d)} \le 1\right\} \\
&\quad = \lambda\mathrm{TV}(u) + (1-\lambda)\mathrm{TV}(v).
\end{aligned}$$

□

2.2 Sets of finite perimeter

We want to give a brief introduction to the field of sets of finite perimeter which were first introduced by Caccioppoli in [22]. The definitions and statements are mainly from [4]. We are making the necessary preparations for the following chapters, but we also want to go a little further in detail in order to gain sufficient understanding of the subject matter. However, for a deeper insight, we refer to [72] in addition to [4].

As in Section 1.1, Ω always denotes a bounded Lipschitz domain in $\mathbb{R}^d$. Whenever we need to define or state something on an open set that is not necessarily our Lipschitz domain Ω, we will denote the open set by $A \subset \mathbb{R}^d$. For a set $E \subset \mathbb{R}^d$, we define the $\{0,1\}$-valued indicator functional by $\chi_E : \mathbb{R}^d \to \{0,1\}$. For a Lebesgue-measurable set $E \subset \mathbb{R}^d$ and an open set $A \subset \mathbb{R}^d$, we define the perimeter $P(E,A)$ of E in A according to Definition 3.35 in [4] by

$$P(E,A) \coloneqq \sup\left\{ \int_E \operatorname{div}\phi(x)\,\mathrm{d}x : \phi \in C_c^1(A;\mathbb{R}^d), \|\phi\|_{L^\infty(A;\mathbb{R}^d)} \leq 1 \right\}.$$

If the set E is of finite perimeter $P(E,A')$ in A' for any open set $A' \subset A$ such that the set $\overline{A'}$ is compact and fulfills $\overline{A'} \subset A$, we say that E is a set of locally finite perimeter in A. A partition $\{E_i\}_{i\in I}$, $I \subset \mathbb{N}$, of A is called a Caccioppoli partition if $\sum_{i\in I} P(E_i,A) < \infty$.

We denote the topological boundary of a set E by the usual notation ∂E. The reduced boundary of a Lebesgue-measurable set $E \subset \mathbb{R}^d$ is denoted by $\partial^* E$ and was first introduced in [43].

Definition 2.10 (Reduced boundary, generalized outer normal, [4, Def. 3.54])**.** Let $E \subset \mathbb{R}^d$ be a Lebesgue-measurable set and $A \subset \mathbb{R}^d$ be the largest open set such that E is locally of finite perimeter in A. The reduced boundary $\partial^* E$ of E is the collection of all points $x \in \operatorname{supp}|D\chi_E| \cap A$ such that the limit

$$n_E(x) \coloneqq \lim_{\rho\searrow 0} \frac{-D\chi_E(B_\rho(x))}{|D\chi_E|(B_\rho(x))}$$

exists in $\mathbb{R}^d$ and satisfies $\|n_E(x)\|_2 = 1$. The function $n_E : \partial^* E \to \mathbb{S}^{d-1} = \{y \in \mathbb{R}^d : \|y\|_2 = 1\}$ is called the generalized outer normal to E.

Sets of finite perimeter and functions of bounded variation are linked in the following way.

Theorem 2.11 ([4, Thm. 3.36])**.** *Let $A \subset \mathbb{R}^d$ be an open set and $E \subset \mathbb{R}^d$ be a set of finite perimeter in A. Then the distributional derivative $D\chi_E$ of χ_E is an $\mathbb{R}^d$-valued finite Radon measure in A. There holds $P(E,A) = |D\chi_E|(A)$ and the generalized*

Gauss–Green formula

$$\int_E \operatorname{div} \phi(x) \, \mathrm{d}x = \int_A n_E(x) \cdot \phi(x) \, \mathrm{d}|D\chi_E|(x) \quad \forall \phi \in C_c^1(A; \mathbb{R}^d) \tag{2.1}$$

holds with $D\chi_E = -n_E|D\chi_E|$.

The theorem of De Giorgi, Theorem 3.59 in [4] and originally in [41–43], gives information about the structure of the reduced boundary. First, it states that the reduced boundary $\partial^* E$ of a Lebesgue-measurable set $E \subset \mathbb{R}^d$ is countably $(d-1)$-rectifiable, that is, there exist countably many Lipschitz functions $f_i : \mathbb{R}^{d-1} \to \mathbb{R}^d$, $i \in \mathbb{N}$, such that

$$\partial^* E \subset \bigcup_{i=1}^{\infty} f_i(\mathbb{R}^{d-1}),$$

see Definition 2.57 in [4], and there holds that

$$|D\chi_E| = \mathcal{H}^{d-1} \llcorner \partial^* E, \tag{2.2}$$

where $\mu \llcorner E_1$ for a measure μ on $(X, \mathcal{E})$ and a set $E_1 \in \mathcal{E}$ denotes the restriction measure defined by $\mu \llcorner E_1(E_2) = \mu(E_1 \cap E_2)$ for $E_2 \in \mathcal{E}$, see Definition 1.65 in [4]. Moreover, figuratively speaking, it states that the set E zoomed in close to a point $x_0 \in \partial^* E$ looks like the half space induced by the hyperplane that is orthogonal to the generalized outer normal $n_E(x_0)$ and the reduced boundary itself looks like the orthogonal vector $n_E^{\perp}(x_0)$ to $n_E(x_0)$. In particular, there holds

$$\lim_{\rho \searrow 0} \frac{\mathcal{H}^{d-1}(\partial^* E \cap B_\rho(x_0))}{\omega_{d-1} \rho^{d-1}} = 1, \tag{2.3}$$

where $B_\rho(x_0) = \{x \in \mathbb{R}^d : \|x - x_0\|_2 < \rho\}$ and ω_{d-1} is the Lebesgue measure of the unit ball in $\mathbb{R}^{d-1}$. The mathematically complete statements and their proofs are provided in Section 3.5 in [4] as well as Chapter 15 in [72].

The representation (2.2) yields that (2.1) in Theorem 2.11 can be rewritten as

$$\int_E \operatorname{div} \phi(x) \, \mathrm{d}x = \int_{\partial^* E} n_E(x) \cdot \phi(x) \, \mathrm{d}\mathcal{H}^{d-1}(x) \quad \forall \phi \in C_c^1(A; \mathbb{R}^d). \tag{2.4}$$

We denote the Lebesgue measure of a set $E \subset \mathbb{R}^d$ by $|E|$.

Definition 2.12 (Points of density t and essential boundary, [4, Def. 3.60])**.** Let $E \subset \mathbb{R}^d$ be a Lebesgue-measurable set and $t \in [0,1]$. We denote the set of points where E has density t by

$$E^t := \left\{ x \in \mathbb{R}^d : \lim_{\rho \searrow 0} \frac{|E \cap B_\rho(x)|}{|B_\rho(x)|} = t \right\}.$$

The essential boundary of E is defined by $\partial^e E := \mathbb{R}^d \setminus (E^0 \cup E^1)$.

We note that the notations for the reduced boundary and the essential boundary may differ in the literature. We use the notation from [72]. In [4], the essential boundary of a set E is denoted by $\partial^* E$ and the reduced boundary is denoted by $\mathcal{F}E$. The following relations between the reduced boundary $\partial^* E$, the set $E^{\frac{1}{2}}$ of points of density $\frac{1}{2}$, and the essential boundary $\partial^e E$ hold.

Theorem 2.13 (Federer, [4, Thm. 3.61])**.** *Let $A \subset \mathbb{R}^d$ be an open set and $E \subset \mathbb{R}^d$ be a set of finite perimeter in A. There holds*

$$\partial^* E \cap A \subset E^{\frac{1}{2}} \subset \partial^e E$$

and

$$\mathcal{H}^{d-1}\left(A \setminus \left(E^0 \cup \partial^* E \cup E^1\right)\right) = 0.$$

Theorem 2.14 ([4, Thm. 4.17])**.** *Let $A \subset \mathbb{R}^d$ be open and $\{E_i\}_{i\in I}$ be a Caccioppoli partition of A. Then*

$$\bigcup_{i\in I}(E_i)^1 \cup \bigcup_{\substack{i,j\in I\\ i\neq j}} (\partial^* E_i \cap \partial^* E_j)$$

contains $\mathcal{H}^{d-1}$-almost all of A.

Lemma 2.15 ([77, Lem. A.3])**.** *Let $E, F \subset \Omega$ be sets of finite perimeter. Then there holds*

$$\partial^*(E \cup F) \subset \partial^e E \cup \partial^e F \text{ and } \partial^*(E \setminus F) \subset \partial^e E \cup \partial^e F.$$

In particular, there holds

$$\mathcal{H}^{d-1}(\partial^*(E \cup F)) \leq \mathcal{H}^{d-1}(\partial^* E \cup \partial^* F) \text{ and } \mathcal{H}^{d-1}(\partial^*(E \setminus F)) \leq \mathcal{H}^{d-1}(\partial^* E \cup \partial^* F).$$

Proof. Let $x \in \partial^*(E \cup F)$. Then by Theorem 2.13, there holds that x has density $\frac{1}{2}$ regarding $E \cup F$, and therefore

$$\frac{1}{2} = \lim_{r\searrow 0} \frac{|(E \cup F) \cap B_r(x)|}{|B_r(x)|} \geq \limsup_{r\searrow 0} \frac{|E \cap B_r(x)|}{|B_r(x)|}$$

and

$$\frac{1}{2} = \lim_{r\searrow 0} \frac{|(E \cup F) \cap B_r(x)|}{|B_r(x)|} \geq \limsup_{r\searrow 0} \frac{|F \cap B_r(x)|}{|B_r(x)|}.$$

Moreover,

$$\frac{1}{2} = \lim_{r\searrow 0} \frac{|(E\cup F)\cap B_r(x)|}{|(B_r(x)|} \leq \liminf_{r\searrow 0} \frac{|E\cap B_r(x)|}{|B_r(x)|} + \frac{|F\cap B_r(x)|}{|B_r(x)|}$$

and thus

$$\frac{|E\cap B_r(x)|}{|B_r(x)|} \not\to 0 \quad \text{or} \quad \frac{|F\cap B_r(x)|}{|B_r(x)|} \not\to 0 \quad \text{as } r\searrow 0.$$

Therefore, $x\in\partial^e E\cup\partial^e F$. To prove $\partial^*(E\setminus F)\subset\partial^e E\cup\partial^e F$, we use that

$$\mathbb{R}^d\setminus(E\setminus F) = (\mathbb{R}^d\setminus E)\cup F$$

and obtain

$$\begin{aligned}\partial^*(E\setminus F) &= \partial^*(\mathbb{R}^d\setminus(E\setminus F))\\ &= \partial^*((\mathbb{R}^d\setminus E)\cup F) \subset \partial^e(\mathbb{R}^d\setminus E)\cup\partial^e F = \partial^e E\cup\partial^e F.\end{aligned}$$

The last claim follows from the monotonicity of $\mathcal{H}^{d-1}$ and Theorem 2.13. □

According to Section 3.9 in [4], the distributional derivative Du of a function $u\in\mathrm{BV}(\Omega)$ can be decomposed into an absolutely continuous part $D^a u$ with respect to the Lebesgue measure and a singular part $D^s u$ with respect to the Lebesgue measure. The singular part $D^s u$ can be decomposed further. To this end, we need to define approximate limits of u and approximate jump points of u.

Definition 2.16 (Approximate limit, [4, Def. 3.63])**.** The function $u\in L^1(\Omega)$ has an approximate limit at $x\in\Omega$ if there exists some $z\in\mathbb{R}$ such that

$$\lim_{\rho\searrow 0}\frac{1}{|B_\rho(x)|}\int_{B_\rho(x)}|u(y)-z|\,\mathrm{d}y = 0.$$

We then call z the approximate limit of u at x. The set S_u of points in which u has no approximation limit is called approximate discontinuity set.

Let us denote for $x,v\in\mathbb{R}^d$ the half balls

$$B_\rho^+(x,v) := \left\{y\in B_\rho(x) : (y-x)^T v > 0\right\}$$

and

$$B_\rho^-(x,v) := \left\{y\in B_\rho(x) : (y-x)^T v < 0\right\}.$$

Definition 2.17 (Approximate jump point, [4, Def. 3.67])**.** Let $u\in L^1(\Omega)$ and $x\in\Omega$. Then x is an approximate jump point of u if there exist $a,b\in\mathbb{R}$ and

$v \in \mathbb{S}^{d-1} = \{y \in \mathbb{R}^d : \|y\|_2 = 1\}$ such that $a \neq b$,

$$\lim_{\rho \searrow 0} \frac{1}{|B_\rho^+(x,v)|} \int_{B_\rho^+(x,v)} |u(y) - a| \,\mathrm{d}y = 0,$$

and

$$\lim_{\rho \searrow 0} \frac{1}{|B_\rho^-(x,v)|} \int_{B_\rho^-(x,v)} |u(y) - b| \,\mathrm{d}y = 0.$$

The set of approximate jump points is denoted by J_u.

For the characteristic function χ_E of a set $E \subset \Omega$, there holds $S_{\chi_E} = \partial^e E$ and $J_{\chi_E} \subset E^{\frac{1}{2}}$. If E is additionally of finite perimeter, then there holds $\partial^* E \subset J_{\chi_E}$, see Example 3.68 in [4].

As stated in (3.89) and Definition 3.91 in [4], the singular part $D^s u$ of the distributional derivative Du of a function $u \in \mathrm{BV}(\Omega)$ can be decomposed into two parts, the Cantor part $D^c u$, defined by

$$D^c u \coloneqq D^s u \llcorner (\Omega \setminus S_u),$$

and the jump part $D^j u$, defined by

$$D^j u \coloneqq D^s u \llcorner J_u.$$

In total, we may write

$$Du = D^a u + D^c u + D^j u.$$

The set of special functions with total variation $\mathrm{SBV}(\Omega)$ was first introduced in [44] and is a subset of $\mathrm{BV}(\Omega)$ that contains the functions $u \in \mathrm{BV}(\Omega)$ whose Cantor part $D^c u$ of Du is zero, that is,

$$Du = D^a u + D^j u.$$

We are interested in the space $\mathrm{SBV}(\Omega)$ because the feasible set $\mathrm{BV_U}(\Omega)$ of (P) is contained in it.

Lemma 2.18 ([77, Lem. 2.1])**.** (a) *Let $u \in \mathrm{BV_U}(\Omega)$. Then there exists a Caccioppoli partition $\{E_1, \ldots, E_N\}$ of Ω such that $u = \sum_{i=1}^N \nu_i \chi_{E_i}$.*

(b) *Let* $\sum_{i=1}^{N} \nu_i \chi_{E_i} = u \in \mathrm{BV}_{\mathrm{U}}(\Omega)$ *as in* (a). *Then it holds that*

$$\infty > \mathrm{TV}(u) = |Du|(\Omega) = \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} |\nu_i - \nu_j| \mathcal{H}^{d-1}(\partial^* E_i \cap \partial^* E_j) \text{ and} \tag{2.5}$$

$$\mathrm{TV}(u) \geq \frac{1}{2} \sum_{i=1}^{N} P(E_i, \Omega). \tag{2.6}$$

Proof. (a) By the coarea formula in BV, Theorem 3.40 in [4] and originally in [51], there holds for $u \in \mathrm{BV}_{\mathrm{U}}(\Omega)$ that the sets

$$F_i := \{x \in \Omega : u(x) > \nu_i - \varepsilon\} = \{x \in \Omega : u(x) \geq \nu_i\}$$

with $\varepsilon \in (0,1)$ and $i \in \{1, \ldots, N\}$ are of finite perimeter in Ω. Without loss of generality, we may assume $\nu_i > \nu_{i+1}$ for all $i \in \{1, \ldots, N-1\}$. We define $E_1 := F_1 = u^{-1}(\{\nu_1\})$ and $E_i := F_i \setminus F_{i-1} = u^{-1}(\{\nu_i\})$ for all $i \in \{2, \ldots, N\}$. This yields $P(E_1, \Omega) = P(F_1, \Omega) < \infty$, $P(E_i, \Omega) \leq P(F_i, \Omega) + P(F_{i-1}, \Omega) < \infty$ for all $i \in \{2, \ldots, N\}$, and $u = \sum_{i=1}^{N} \nu_i \chi_{E_i}$.

(b) Let $\sum_{i=1}^{N} \nu_i \chi_{E_i} = u \in \mathrm{BV}_{\mathrm{U}}(\Omega)$. By means of Theorem 2.11 and (2.2), we obtain for the distributional derivative Du of u

$$Du = \sum_{i=1}^{N} \nu_i D\chi_{E_i} = -\sum_{i=1}^{N} \nu_i n_{E_i} \mathcal{H}^{d-1} \llcorner (\partial^* E_i \cap \Omega), \tag{2.7}$$

where the functions χ_{E_i} are considered elements of $\mathrm{BV}(\Omega)$ and n_{E_i} denotes the generalized outer normal of E_i that is defined on $\partial^* E_i$. Next, we prove the identity

$$Du = \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} (\nu_j - \nu_i) n_{E_i} \mathcal{H}^{d-1} \llcorner (\partial^* E_i \cap \partial^* E_j \cap \Omega). \tag{2.8}$$

To this end, we observe that every $x \in \partial^* E_i \cap \Omega$ is a point of density $\frac{1}{2}$ for E_i by Theorem 2.13 and, consequently, cannot be a point of density 1 for any E_j, $j \in \{1, \ldots, N\}$. We apply Theorem 2.14 to the right hand side of (2.7) and obtain

$$Du = -\sum_{i=1}^{N} \nu_i n_{E_i} \mathcal{H}^{d-1} \llcorner \left(\bigcup_{\substack{j=1 \\ j \neq i}}^{N} \underbrace{\partial^* E_i \cap \partial^* E_j \cap \Omega}_{=: A_{\{i,j\}}} \right),$$

where we have used that $x \in \Omega$ can be a point of density $\frac{1}{2}$ for at most two of the disjoint sets E_k, $k \in \{1, \ldots, N\}$. Using this observation again, we obtain that the sets $A_{\{i,j\}}$ are pairwise disjoint which implies

$$Du = -\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}(\nu_i n_{E_i} + \nu_j n_{E_j})\mathcal{H}^{d-1} \llcorner (\partial^* E_i \cap \partial^* E_j \cap \Omega). \tag{2.9}$$

Then (2.8) follows because $n_{E_j} = -n_{E_i}$ on $\partial^* E_i \cap \partial^* E_j$. Because every $x \in \partial^* E_i \cap \partial^* E_j$ has density $\frac{1}{2}$ with respect to E_i and E_j and therefore density 1 with respect to Ω, there holds $x \notin \partial^*\Omega \subset \Omega^{\frac{1}{2}}$. Since Ω has Lipschitz boundary, there holds $\mathcal{H}^{d-1}(\partial\Omega \setminus \partial^*\Omega) = 0$ by Proposition 3.62 in [4]. In combination, we get

$$\mathcal{H}^{d-1}((\partial^* E_i \cap \partial^* E_j) \setminus \Omega) = 0.$$

Moreover, because the $A_{\{i,j\}}$ are pairwise disjoint, the measures $\mathcal{H}^{d-1} \llcorner A_{\{i,j\}}$ are pairwise singular. Thus we deduce from (2.8) with $\mathrm{TV}(u) = |Du|(\Omega)$ and $\|n_{E_i}(x)\|_2 = 1$ for $x \in \partial^* E_i$ that

$$\mathrm{TV}(u) = \sum_{i=1}^{N-1}\sum_{j=i+1}^{N} |\nu_i - \nu_j|\mathcal{H}^{d-1}(\partial^* E_i \cap \partial^* E_j),$$

which gives (2.5). Moreover,

$$\begin{aligned}
\infty > \mathrm{TV}(u) &= \sum_{i=1}^{N-1}\sum_{j=i+1}^{N} |\nu_i - \nu_j|\mathcal{H}^{d-1}(\partial^* E_i \cap \partial^* E_j) \\
&\geq \frac{1}{2}\sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N}\mathcal{H}^{d-1}(\partial^* E_i \cap \partial^* E_j) \\
&\geq \frac{1}{2}\sum_{i=1}^{N}\mathcal{H}^{d-1}(\partial^* E_i \cap \Omega) = \frac{1}{2}\sum_{i=1}^{N} P(E_i, \Omega),
\end{aligned}$$

which gives (2.6).

□

2.3 Existence of solutions to (P)

We will prove the existence of optimal solutions to our superordinate problem (P). For sequences of feasible functions $u \in \mathrm{BV}_{\mathrm{U}}(\Omega)$, we have boundedness in $L^\infty(\Omega)$, which shifts convergence in $L^1(\Omega)$ to convergence in $L^p(\Omega)$ for $1 \leq p < \infty$. Moreover, we prove that the limit is in $L^1_U(\Omega)$.

Lemma 2.19 (Variant of [70, Lem. 2.2] and [95, Lem. 2.1])**.** *Let the sequence* $\{u_k\}_{k\in\mathbb{N}} \subset L^1(\Omega)$ *converge to some* $u \in L^1(\Omega)$ *with* $\underline{\nu} \le u_k(x) \le \overline{\nu}$ *for almost all* $x \in \Omega$ *for all* $k \in \mathbb{N}$*. Then there holds* $\underline{\nu} \le u(x) \le \overline{\nu}$ *for almost all* $x \in \Omega$ *and* $u_k \to u$ *in* $L^p(\Omega)$ *for all* $1 \le p < \infty$*. If moreover* $\{u_k\}_{k\in\mathbb{N}} \subset L^1_U(\Omega)$*, there holds* $u \in L^1_U(\Omega)$*.*

Proof. There holds

$$\|u_k - u\|^p_{L^p(\Omega)} \le \|u_k - u\|_{L^1(\Omega)} \|(u_k - u)^{p-1}\|_{L^\infty(\Omega)} \to 0$$

because $\|(u_k - u)^{p-1}\|_{L^\infty(\Omega)} \le \max_{\nu_1,\nu_2\in U} |\nu_1 - \nu_2|^{p-1} < \infty$ for all $k \in \mathbb{N}$.

The convergence $u_k \to u$ in $L^1(\Omega)$ for some $u \in L^1(\Omega)$ implies the existence of a subsequence $\{u_{k_\ell}\}_{\ell\in\mathbb{N}}$ that converges pointwise almost everywhere in Ω to u according to Lemma 3.22 in [3]. Since $\underline{\nu} \le u_{k_\ell}(x) \le \overline{\nu}$ for almost all $x \in \Omega$ for all $\ell \in \mathbb{N}$, this yields $\underline{\nu} \le u(x) \le \overline{\nu}$ for almost all $x \in \Omega$.

Now let additionally $u_k(x) \in U$ for almost all $x \in \Omega$ for all $k \in \mathbb{N}$. Let $x \in \Omega$ such that $u_{k_\ell}(x) \to u(x)$ as $\ell \to \infty$. This yields the existence of some $L \in \mathbb{N}$ such that $|u_{k_\ell}(x) - u(x)| < \frac{1}{2}$ for all $\ell \ge L$ and since $U \subset \mathbb{Z}$, this implies the existence of some $\nu \in U$ such that $u_{k_\ell}(x) = \nu$ for all $\ell \ge L$, that is, the sequence $\{u_{k_\ell}(x)\}_{\ell\in\mathbb{N}}$ is constant for all $\ell \ge L$. Hence, there must hold $u(x) \in U$ which yields the claim. □

The feasible set $\mathrm{BV_U}(\Omega)$ of (P) is sequentially weakly-∗ and strictly closed in $\mathrm{BV}(\Omega)$ such that the limits of sequences of feasible functions that converge weakly-∗ or strictly in $\mathrm{BV}(\Omega)$ are feasible for (P).

Lemma 2.20 ([70, Lem. 2.2])**.** *The set* $\mathrm{BV_U}(\Omega)$ *is sequentially weakly-∗ and strictly closed in* $\mathrm{BV}(\Omega)$

Proof. Consider a sequence $\{u_k\}_{k\in\mathbb{N}} \subset \mathrm{BV_U}(\Omega)$ that converges weakly-∗ or strictly in $\mathrm{BV}(\Omega)$. This yields in particular $u_k \to u$ in $L^1(\Omega)$ for some $u \in \mathrm{BV}(\Omega)$. The claim then follows from Lemma 2.19. □

With these preparations, we can now prove the existence of minimizers for (P).

Theorem 2.21 ([70, Prop. 2.3])**.** *Let Assumption 1.1 hold. The optimization problem* (P) *admits an optimal solution.*

Proof. The feasible set of (P) is non-empty because U is non-empty. Consider a minimizing sequence $\{u_k\}_{k\in\mathbb{N}}$ for (P). Since F and TV are bounded from below, there holds that $\{\mathrm{TV}(u_k)\}_{k\in\mathbb{N}}$ is bounded. Moreover, the feasible set $\mathrm{BV_U}(\Omega)$ of (P) is bounded in $L^\infty(\Omega)$ such that the sequence $\{u_k\}_{k\in\mathbb{N}}$ is bounded in $L^\infty(\Omega)$ and consequently in $L^1(\Omega)$. Together we obtain that $\{u_k\}_{k\in\mathbb{N}}$ is bounded in $\mathrm{BV}(\Omega)$ such that Theorem 2.8 yields the existence of a subsequence that converges weakly-∗ in $\mathrm{BV}(\Omega)$ to some $u \in \mathrm{BV}(\Omega)$. We may assume that the subsequence is already the whole minimizing sequence. In particular, the sequence $\{u_k\}_{k\in\mathbb{N}}$ converges in $L^1(\Omega)$. By Lemma 2.19, it then converges in $L^p(\Omega)$ for all $1 \le p < \infty$ such that we obtain by Assumption 1.1 that $F(u) \le \liminf_{k\to\infty} F(u_k)$. By the lower semicontinuity of TV according to Proposition 2.9, there also holds $\mathrm{TV}(u) \le \liminf_{k\to\infty} \mathrm{TV}(u_k)$ such that $F(u)+\alpha\mathrm{TV}(u) \le \liminf_{k\to\infty} F(u_k)+\alpha\mathrm{TV}(u_k)$. Since the set $\mathrm{BV}_{\mathrm{U}}(\Omega)$ is sequentially weakly-$*$ closed in $\mathrm{BV}(\Omega)$, there holds $u \in \mathrm{BV}_{\mathrm{U}}(\Omega)$ such that u minimizes (P). $\square$

2.4 The space $H(\mathrm{div};\Omega)$

This section is dedicated to the space $H(\mathrm{div};\Omega)$. We base our definitions and notations on [14, 100, 103]. For a start, we define the operator $-\mathrm{div} : L^2(\Omega;\mathbb{R}^d) \to H^{-1}(\Omega)$, where H^{-1} denotes the dual space of $H_0^1(\Omega)$, as the adjoint operator to $\nabla : H_0^1(\Omega) \to L^2(\Omega;\mathbb{R}^d)$, that is, for given $\phi \in L^2(\Omega;\mathbb{R}^d)$, $\mathrm{div}\,\phi \in H^{-1}(\Omega)$ is defined by

$$\langle \mathrm{div}\,\phi, v\rangle_{H^{-1},H_0^1} \coloneqq -\int_\Omega \phi(x)\cdot\nabla v(x)\,\mathrm{d}x \quad \forall\, v \in H_0^1(\Omega).$$

The Hilbert space $H(\mathrm{div};\Omega)$ is defined by

$$H(\mathrm{div};\Omega) \coloneqq \{\phi \in L^2(\Omega;\mathbb{R}^d) : \mathrm{div}\,\phi \in L^2(\Omega)\}$$

and equipped with the scalar product

$$(\phi,\psi)_{H(\mathrm{div};\Omega)} = (\phi,\psi)_{L^2(\Omega;\mathbb{R}^d)} + (\mathrm{div}\,\phi, \mathrm{div}\,\psi)_{L^2(\Omega)},$$

for $\phi,\psi \in H(\mathrm{div};\Omega)$, which induces the norm

$$\|\phi\|_{H(\mathrm{div};\Omega)} = \sqrt{\|\phi\|^2_{L^2(\Omega;\mathbb{R}^d)} + \|\,\mathrm{div}\,\phi\|^2_{L^2(\Omega)}}$$

for $\phi \in H(\mathrm{div};\Omega)$. We define the space $H^{\frac{1}{2}}(\partial\Omega) \coloneqq \gamma_0(H^1(\Omega))$, where $\gamma_0 : H^1(\Omega) \to L^2(\partial\Omega)$ denotes the usual linear and continuous trace operator. Moreover, we define $H^{-\frac{1}{2}}(\partial\Omega)$ as the dual space of $H^{\frac{1}{2}}(\partial\Omega)$. Similarly to the trace operator γ_0, one can define the linear and continuous normal trace operator $\gamma_n : H(\mathrm{div};\Omega) \to H^{-\frac{1}{2}}(\partial\Omega)$ by Lemma 1.2.2 in [100], where n denotes the outer unit normal to Ω, such that

$$\gamma_n\phi = (\phi\cdot n)|_{\partial\Omega} \quad \forall\,\phi \in C^\infty(\bar{\Omega};\mathbb{R}^d).$$

By Lemma 1.2.3 in [100], Green's formula

$$\langle \gamma_n\phi, \gamma_0 v\rangle_{H^{-1/2},H^{1/2}} = \int_\Omega \phi(x)\cdot\nabla v(x)\,\mathrm{d}x + \int_\Omega v(x)\,\mathrm{div}\,\phi(x)\,\mathrm{d}x \tag{2.10}$$

holds for all $\phi \in H(\mathrm{div};\Omega)$ and all $v \in H^1(\Omega)$. We will often write

$$\gamma_0 v = v|_{\partial\Omega}$$

for $v \in H^1(\Omega)$,

$$\gamma_n \phi = (\phi \cdot n)|_{\partial\Omega}$$

for $\phi \in H(\mathrm{div};\Omega)$, and

$$\langle \gamma_n \phi, \gamma_0 v \rangle_{H^{-1/2},H^{1/2}} = \int_{\partial\Omega} v(x)\phi(x) \cdot n(x) \, \mathrm{d}\mathcal{H}^{d-1}(x)$$

for $v \in H^1(\Omega)$ and $\phi \in H(\mathrm{div};\Omega)$. The space $H_0(\mathrm{div};\Omega)$ is defined as the closure of the set $C_c^\infty(\Omega;\mathbb{R}^d)$ with respect to the $H(\mathrm{div};\Omega)$ norm. Since Ω is a bounded Lipschitz domain, there holds

$$H_0(\mathrm{div};\Omega) \coloneqq \{\phi \in H(\mathrm{div};\Omega) : (\phi \cdot n)|_{\partial\Omega} \equiv 0\},$$

see Theorem 1.3 in [103] or (1.16) in [14]

2.5 Raviart–Thomas finite elements

Within this section, we assume the bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$, $d \in \{1,2,3\}$, to be a finite union of axis-aligned intervals, squares, or cubes $Q \in \mathcal{Q}_h$ of height $h > 0$. We introduce the finite element spaces that we use for the discretization of the total variation and the optimization problem (P). Raviart–Thomas finite elements will play a central role for the discretization of the total variation. Raviart–Thomas finite elements were first introduced in [89]. We adhere to the definitions and results in [11] and refer to [11, 12, 14] for more details about the introduced finite-element spaces. For each element $Q \in \mathcal{Q}_h$, we define the space of polynomials of degree k_i in x_i on Q by

$$\begin{aligned}
P^h_{k_1}(Q) &\coloneqq \left\{ p : Q \to \mathbb{R} \,:\, p(x) = \sum_{i \le k_1} a_i x^i \right\} \text{ if } d = 1,\\
P^h_{k_1,k_2}(Q) &\coloneqq \left\{ p : Q \to \mathbb{R} \,:\, p(x_1,x_2) = \sum_{i \le k_1, j \le k_2} a_{ij} x_1^i x_2^j \right\} \text{ if } d = 2,\\
P^h_{k_1,k_2,k_3}(Q) &\coloneqq \left\{ p : Q \to \mathbb{R} \,:\, p(x_1,x_2,x_3) = \sum_{i \le k_1, j \le k_2, \ell \le k_3} a_{ij\ell} x_1^i x_2^j x_3^\ell \right\} \text{ if } d = 3.
\end{aligned}$$

For the discretization of the input function $u \in \mathrm{BV}(\Omega)$, we choose the space of piecewise constant functions $P0^h$ that is defined by

$$P0^h = \left\{ p \in L^\infty(\Omega) \,:\, p|_Q \in P0^h(Q) \text{ for all } Q \in \mathcal{Q}_h \right\},$$

where $P0^h(Q) = P_0^h(Q)$ if $d = 1$, $P0^h(Q) = P_{0,0}^h(Q)$ if $d = 2$, and $P0^h(Q) = P_{0,0,0}^h(Q)$ if $d = 3$. For the discretization of the test functions $\phi \in H_0(\mathrm{div};\Omega)$ for the dual formulation of the total variation, we choose the conforming finite-element space $RT0^h \subset H(\mathrm{div};\Omega)$, which is the lowest-order Raviart–Thomas space defined on the mesh $\mathcal{Q}_h$. The space $RT0^h$ contains piecewise linear functions ϕ whose normal components $\phi \cdot n_E$ are continuous and constant on each facet E of the elements $Q \in \mathcal{Q}_h$, where n_E denotes the respective outer unit normal vector to E. Note that Raviart–Thomas functions are continuous in the case $d = 1$ in accordance with $H(\mathrm{div};\Omega) = H^1(\Omega)$. Specifically, we define for each element $Q \in \mathcal{Q}_h$

$$\begin{aligned}
&RT0^h(Q) \coloneqq P_1^h(Q) \text{ for the case } d = 1,\\
&RT0^h(Q) \coloneqq P_{1,0}^h(Q) \times P_{0,1}^h(Q) \text{ for the case } d = 2, \text{and}\\
&RT0^h(Q) \coloneqq P_{1,0,0}^h(Q) \times P_{0,1,0}^h(Q) \times P_{0,0,1}^h(Q) \text{ for the case } d = 3.
\end{aligned}$$

The lowest-order Raviart–Thomas space is then defined by

$$RT0^h \coloneqq \left\{ \phi \in H(\mathrm{div};\Omega) \,:\, \phi|_Q \in RT0^h(Q) \;\text{ for all }\; Q \in \mathcal{Q}_h \right\}$$

and its restriction to functions whose normal trace is zero on the boundary of Ω by

$$RT0_0^h \coloneqq \left\{ \phi \in RT0^h \,:\, (\phi \cdot n)|_{\partial\Omega} \equiv 0 \right\} \subset H_0(\mathrm{div};\Omega),$$

where n denotes the outer normal of Ω. Note that the restriction to axis-aligned squares and cubes Q is mandatory for the above definition of Raviart–Thomas functions. In the following, we aim to define an interpolation operator that maps a given input function to $RT0^h$. To this end, we need to evaluate the normal trace of the input function ϕ on each facet F of a given element $Q \in \mathcal{Q}_h$ separately. In general, this is not possible for functions $\phi \in H(\mathrm{div};\Omega)$ as we explain in the following. For a facet F of an element $Q \in \mathcal{Q}_h$ with outer unit normal n_Q, there holds

$$\int_F \phi(x) \cdot n_Q(x) \,\mathrm{d}\mathcal{H}^{d-1}(x) = \int_{\partial Q} \phi(x) \cdot n_Q(x) \chi_F(x) \,\mathrm{d}\mathcal{H}^{d-1}(x), \tag{2.11}$$

which might be not well-defined because $\phi \cdot n_Q \in H^{-\frac{1}{2}}(\partial Q)$ but $\chi_F \notin H^{\frac{1}{2}}(\partial Q)$ since there exists no $v \in H^1(Q)$ such that $\chi_F = \gamma_0 v$, see Section 2.5.1 in [11]. According to (2.5.1) in [11], it is sufficient to require that ϕ is an element of the space

$$W(Q) = \{\phi \in L^s(Q;\mathbb{R}^d) : \operatorname{div}\phi \in L^2(\Omega)\}$$

with fixed $s > 2$ to make the integral (2.11) well-defined and finite. We do not want to go into further detail at this point because for our purposes, it will be sufficient to simply assume $\phi \in W^{1,\infty}(\Omega;\mathbb{R}^d)$ which makes ϕ a Lipschitz continuous function that immediately can be integrated over the facets of Q without any further considerations and refer to [11] for a more general definition of the interpolation operator.

We first define the local interpolation operator $I_Q : W^{1,\infty}(\Omega;\mathbb{R}^d) \to RT0^h(Q)$ following [11]. Let $\phi \in W^{1,\infty}(\Omega;\mathbb{R}^d)$, then $I_Q\phi$ is uniquely defined by

$$\int_{E_i} (\phi(x) - I_Q\phi(x)) \cdot n_i\, p(x)\, \mathrm{d}\mathcal{H}^{d-1}(x) = 0 \quad \forall\, p \in P0^h \tag{2.12}$$

for all $i = 1, \dots, 2d$, where E_i, $i \in \{1, \dots, 2d\}$, denote the $2d$ facets of Q and $n_i \in \mathbb{R}^d$ denote their corresponding outer normal vectors. Indeed, the above $2d$ equations determine a unique Raviart–Thomas function. To see this, we highlight that for a given function $\bar{\phi} \in RT0^h(Q)$ the normal trace $\bar{\phi} \cdot n_E$ along each facet E of the element Q is constant. This follows immediately from the definition of the facets of the axis-aligned squares and cubes and their outer normal vectors. For example, let $Q \coloneqq [0,1]^2$ and consider its facet $E = \{1\} \times [0,1]$ with outer normal $n_E = (1,0)^T$. Then $\bar{\phi}(x_1,x_2) = (a_1,a_2)^T + (c_1x_1, c_2x_2)^T$ is restricted to $(a_1,a_2)^T + (c_1, c_2x_2)^T$ along E with normal trace $(a_1 + c_1, a_2 + c_2x_2)^T \cdot n_E(x) \equiv a_1 + c_1$. Similar considerations yield that the normal trace is constant along arbitrary facets of the intervals or the axis-aligned squares or cubes. For an arbitrary $\phi \in W^{1,\infty}(\Omega;\mathbb{R}^d)$ on the interval, square, or cube $Q = [\alpha_1,\beta_1] \times \dots \times [\alpha_d,\beta_d]$ with $\alpha_i < \beta_i$ for $i \in \{1,\dots,d\}$, the above equations (2.12) can be rewritten as $I_Q\phi(x) = (a_1,\dots,a_d)^T + (c_1x_1,\dots,c_dx_d)^T$ with

$$\begin{aligned} a_i + \beta_i\ c_i &= \frac{1}{\mathcal{H}^{d-1}(E_{i_+})} \int_{E_{i_+}} \phi(x) \cdot n_{i_+}\, \mathrm{d}\mathcal{H}^{d-1}(x) \\ -a_i - \alpha_i c_i &= \frac{1}{\mathcal{H}^{d-1}(E_{i_-})} \int_{E_{i_-}} \phi(x) \cdot n_{i_-}\, \mathrm{d}\mathcal{H}^{d-1}(x), \end{aligned}$$

where $E_{i_-} = \{x \in Q \,|\, x_i = \alpha_i\}$, that is, $n_{i_-} = -e_i$, and $E_{i_+} = \{x \in Q \,|\, x_i = \beta_i\}$, that is, $n_{i_+} = e_i$, for $i = 1,\dots,d$. The above equation system admits a unique solution and therefore uniquely defines $I_Q\phi$. We define the global interpolation operator $I_{RT0^h} : W^{1,\infty}(\Omega;\mathbb{R}^d) \to RT0^h$ elementwise by the local interpolation operator, that is, for given $\phi \in W^{1,\infty}(\Omega;\mathbb{R}^d)$, we define $(I_{RT0^h}\phi)|_Q \coloneqq I_Q\phi$ for all $Q \in \mathcal{Q}_h$. We note that $I_{RT0^h}\phi$ is indeed an element of $RT0^h$ because the normal traces of $I_{RT0^h}\phi$ are continuous on the element boundaries. This is due to the fact that for two adjacent squares or cubes Q_1 and Q_2 there holds $n_{Q_1} = -n_{Q_2}$ along the intersecting face.

Lemma 2.22 (cf. [11, Prop. 2.1.2]). *Let $\mathcal{Q}$ be a mesh on Ω and let $\phi \in L^2(\Omega)$ such that $\phi|_Q \in C^1(\bar{Q};\mathbb{R}^d)$ for each $Q \in \mathcal{Q}$. Then there holds $\phi \in H(\operatorname{div};\Omega)$ if the normal traces $\phi \cdot n_Q$ are continuous for all $Q \in \mathcal{Q}$, where n_Q denotes the outer unit normal vector to $Q \in \mathcal{Q}$.*

Proof. The function ϕ is in $H(\operatorname{div};\Omega)$ if there is a function $f \in L^2(\Omega)$ such that

$$-\int_\Omega \phi(x) \cdot \nabla v(x) \,\mathrm{d}x = \int_\Omega f(x)v(x)\,\mathrm{d}x \quad \forall\, v \in H_0^1(\Omega), \tag{2.13}$$

that is, $\operatorname{div}\phi = f \in L^2(\Omega)$. Since $\phi_Q \coloneqq \phi|_Q \in C^1(\bar{Q};\mathbb{R}^d)$, we may apply the integration by parts formula on each $Q \in \mathcal{Q}$, which yields for $v \in H_0^1(\Omega)$ that

$$\begin{aligned}\int_\Omega \phi(x)\cdot\nabla v(x)\,\mathrm{d}x &= \sum_{Q\in\mathcal{Q}}\int_Q \phi(x)\cdot\nabla v(x)\,\mathrm{d}x\\ &= \sum_{Q\in\mathcal{Q}} -\int_Q \operatorname{div}\phi_Q(x)v(x)\,\mathrm{d}x + \int_{\partial Q} v(x)\phi_Q(x)\cdot n_Q(x)\,\mathrm{d}\mathcal{H}^{d-1}(x).\end{aligned}$$

If the normal components are continuous, we have that

$$\sum_{Q\in\mathcal{Q}}\int_{\partial Q} v(x)\phi_Q(x)\cdot n_Q(x)\,\mathrm{d}\mathcal{H}^{d-1}(x) = 0$$

because $n_{Q_1} = -n_{Q_2}$ on $\bar{Q}_1 \cap \bar{Q}_2$ with $Q_1 \neq Q_2$ and $\int_{\partial\Omega} v(x)\phi(x)\cdot n(x)\,\mathrm{d}\mathcal{H}^{d-1}(x) = 0$ due to $v \in H_0^1(\Omega)$ such that (2.13) is fulfilled with $f|_Q \coloneqq \operatorname{div}\phi_Q$ for all $Q \in \mathcal{Q}$. □

If $\phi \in W_0^{1,\infty}(\Omega;\mathbb{R}^d)$, that is, if the trace of ϕ is zero on the boundary of Ω we obtain that $I_{RT0^h}\phi \in RT0_0^h$ because $\phi|_{\partial\Omega} \equiv 0$ and therefore $(I_{RT0^h}\phi\cdot n)|_{\partial\Omega} \equiv 0$.

By Proposition 2.5.1 in [11], the interpolation operator $I_{RT0^h} : W^{1,\infty}(\Omega;\mathbb{R}^d) \to RT0^h$ has the following approximation property.

Lemma 2.23 (cf. [11, Prop. 2.5.1]). *Let $\phi \in W^{1,\infty}(\Omega;\mathbb{R}^d)$. There exists a constant $c > 0$ such that*

$$\|\phi - I_{RT0^h}\phi\|_{L^2(\Omega;\mathbb{R}^d)} \leq ch\|\nabla\phi\|_{L^2(\Omega;\mathbb{R}^{d\times d})}.$$

Functions $\phi \in RT0^h$ are piecewise linear on the elements $Q \in \mathcal{Q}_h$. This yields that their divergence is piecewise constant on the elements $Q \in \mathcal{Q}_h$. Moreover, the divergence of the interpolated function $I_{RT0^h}\phi$ coincides with the projection of the divergence of the input function ϕ onto the piecewise constant functions. We denote the projection operator onto $P0^h$ by $\Pi_{P0^h} : L^1(\Omega) \to P0^h$.

Lemma 2.24 ([11, Prop. 2.5.2], [23, Lem. 3.5]). *Let $\phi \in W^{1,\infty}(\Omega;\mathbb{R}^d)$. There holds $\operatorname{div} I_{RT0^h}\phi = \Pi_{P0^h}\operatorname{div}\phi$.*

Proof. For each $Q \in \mathcal{Q}_h$ there holds

$$
\begin{aligned}
(\Pi_{P0^h} \operatorname{div} \phi)\,|_Q &\equiv \frac{1}{|Q|} \int_Q \operatorname{div} \phi(x) \,\mathrm{d}x \\
&\overset{(2.10)}{=} \frac{1}{|Q|} \left(\int_{\partial Q} \phi(x) \cdot n_Q(x) \,\mathrm{d}\mathcal{H}^{d-1}(x) - \underbrace{\int_Q \phi(x) \cdot \nabla 1 \,\mathrm{d}x}_{=0} \right) \\
&= \frac{1}{|Q|} \int_{\partial Q} \phi(x) \cdot n_Q(x) \,\mathrm{d}\mathcal{H}^{d-1}(x) \\
&\overset{(2.12)}{=} \frac{1}{|Q|} \int_{\partial Q} I_Q \phi(x) \cdot n_Q(x) \,\mathrm{d}\mathcal{H}^{d-1}(x) \\
&= \frac{1}{|Q|} \left(\int_{\partial Q} I_Q \phi(x) \cdot n_Q(x) \,\mathrm{d}\mathcal{H}^{d-1}(x) - \underbrace{\int_Q I_Q \phi(x) \cdot \nabla 1 \,\mathrm{d}x}_{=0} \right) \\
&\overset{(2.10)}{=} \frac{1}{|Q|} \int_Q \underbrace{\operatorname{div} I_Q \phi(x)}_{\text{const.}} \,\mathrm{d}x \\
&\equiv \operatorname{div} I_Q \phi.
\end{aligned}
$$

□

This immediately gives that for $\phi \in W^{1,\infty}(\Omega; \mathbb{R}^d)$ there holds

$$
\| \operatorname{div}(\phi - I_{RT0^h} \phi) \|_{L^2(\Omega)} = \| \operatorname{div} \phi - \Pi_{P0^h} \operatorname{div} \phi \|_{L^2(\Omega)} \to 0 \text{ as } h \searrow 0. \tag{2.14}
$$

The following result was proved in [11] but is also a corollary of Lemma 2.24 and the Bramble–Hilbert lemma which is stated in Lemma 2.2.2 in [11].

Lemma 2.25 (cf. [11, Prop. 2.5.3])**.** *Let $\phi \in W^{1,\infty}(\Omega; \mathbb{R}^d)$ with $\operatorname{div} \phi \in H^1(\Omega)$, then there exists a constant $c > 0$ such that*

$$
\| \operatorname{div}(\phi - I_{RT0^h} \phi) \|_{L^2(\Omega)} \le ch \| \nabla \operatorname{div} \phi \|_{L^2(\Omega; \mathbb{R}^d)}.
$$

A similar statement as the following proposition was proven in Lemma 3.6 in [23] for the case $d = 2$. We adapt the proof for $d \in \{1, 2, 3\}$. Similar results are proven for tetrahedral meshes in Theorem 6.3 in [2].

Proposition 2.26 (cf. [23, Lem. 3.6])**.** *Let $\phi \in W^{1,\infty}(\Omega; \mathbb{R}^d)$. There exists a constant $C(\phi) > 0$ such that*

$$
\| \phi - I_{RT0^h} \phi \|_{L^\infty(\Omega; \mathbb{R}^d)} \le C(\phi) h.
$$

Proof. Since $\phi \in W^{1,\infty}(\Omega;\mathbb{R}^d)$, it is Lipschitz continuous by Theorem 4.5 in [50] with a constant $L(\phi) > 0$, that is,

$$\|\phi(x_1) - \phi(x_2)\|_2 \leq L(\phi)\|x_1 - x_2\|_2$$

for all $x_1, x_2 \in \Omega$. We denote $\phi^h := I_{RT0^h}\phi$. Consider an arbitrary but fixed $Q \in \mathcal{Q}_h$. Denote the extreme points of Q by $q_i \in Q$, $i \in \{1,\ldots,2^d\}$, and the center of Q by $c_Q \in Q$. The triangle inequality yields

$$\|\phi - \phi^h\|_{L^\infty(Q;\mathbb{R}^d)} \leq \|\phi - \phi(c_Q)\|_{L^\infty(Q;\mathbb{R}^d)} + \|\phi(c_Q) - \phi^h\|_{L^\infty(Q;\mathbb{R}^d)}.$$

By the Lipschitz continuity of ϕ, there holds

$$\|\phi - \phi(c_Q)\|_{L^\infty(Q;\mathbb{R}^d)} \leq L(\phi)\frac{\sqrt{d}}{2}h.$$

By the convexity of the mapping $x \mapsto \|\phi^h(x) - \phi(c_Q)\|_2$, there holds that

$$\begin{aligned}
\|\phi^h - \phi(c_Q)\|_{L^\infty(Q;\mathbb{R}^d)} &= \sup_{x\in Q}\|\phi^h(x) - \phi(c_Q)\|_2 \\
&= \max_{i\in\{1,\ldots,2^d\}}\|\phi^h(q_i) - \phi(c_Q)\|_2 \\
&\leq L(\phi)\frac{d}{2}h.
\end{aligned}$$

The last inequality follows from the fact that for each extreme point q_i of Q there holds for the components ϕ_j^h of ϕ^h and ϕ_j of ϕ, $j \in \{1,\ldots,d\}$, that

$$\phi_j^h(q_i) = \frac{1}{h}\int_{F_j^i}\phi_j(x)\,\mathrm{d}\mathcal{H}^{d-1}(x),$$

where F_j^i, $j \in \{1,\ldots,d\}$, are the appropriately numbered facets of Q adjoining q_i, which gives

$$|\phi_j^h(q_i) - \phi_j(c_Q)| \leq L(\phi)\|q_i - c_Q\|_2 = L(\phi)\frac{\sqrt{d}}{2}h$$

for $i \in \{1,\ldots,2^d\}$. In total, we obtain

$$\|\phi - I_{RT0^h}\phi\|_{L^\infty(Q;\mathbb{R}^d)} \leq L(\phi)\left(\frac{\sqrt{d}}{2} + \frac{d}{2}\right)h.$$

Hence, $C(\phi) := L(\phi)\left(\frac{d}{2} + \frac{\sqrt{d}}{2}\right)$ yields the claim. $\square$

Chapter 3

Trust-region algorithm

In this chapter, we derive first-order optimality conditions for problem (P) and provide a trust-region algorithm in function space to solve (P). The considerations in this chapter closely follow the article [77].

The trust-region algorithm extends the idea of the novel trust-region algorithm introduced in [70] for the one-dimensional case to the multi-dimensional case. Hence, we assume $d \geq 2$ within this chapter. For the one-dimensional case, we refer to [70]. A general overview on trust-region methods can be found in [39].

The subproblems that are associated to the trust-region algorithm have an objective function that linearizes F and keeps the total variation term $\alpha\mathrm{TV}$ exactly. The feasible set of the trust-region subproblem, the so-called trust region, is a closed L^1 ball with radius $\Delta > 0$ around the most recently accepted iterate. After discretization, this leads to mixed-integer linear programs as discussed in Chapter 6.

In the one-dimensional case, the trust-region subproblems can be solved efficiently with the strategies from [79, 97]. As noted in [70], optimizing (P) for fixed jump heights of the input function corresponds to switching point optimization, for which optimization techniques have been investigated in [45,80,91,102]. In contrast to that, a proximal-gradient method was proposed in [79].

The multi-dimensional case is not covered by the mentioned works. Since the geometric properties of the total variation on multi-dimensional domains fundamentally differs from the one-dimensional case, much effort is needed to obtain similar results for the multi-dimensional case.

This chapter is structured as follows. We start by introducing local variations in Section 3.1, which can be used to perturb the interfaces of the level sets of the input functions. In Section 3.2, we define local optimality and formulate a first-order optimality condition in terms of perturbations with local variations. We state the trust-region subproblems, prove the convergence of their objective functions in the sense of Γ-convergence with respect to convergence of the linearization point, and state the trust-region algorithm in Section 3.3. In Section 3.4, we prove that the iterates of the trust-region algorithm converge to stationary points.

3.1 Local variations

Feasible input functions $u \in \mathrm{BV_U}(\Omega)$ of the optimization problem (P) can be rewritten as

$$u = \sum_{i=1}^{N} \nu_i \chi_{E_i} \tag{3.1}$$

with a Caccioppoli partition $\{E_1, \ldots, E_N\}$ of Ω by Lemma 2.18. The total variation $\mathrm{TV}(u)$ of a function $u \in \mathrm{BV_U}(\Omega)$ is according to (2.5) the sum of the $\mathcal{H}^{d-1}$ measures of the intersections of the reduced boundaries $\partial^* E_i$ and $\partial^* E_j$ of two level sets E_i and E_j at each time multiplied by the jump height of u from E_i to E_j, that is,

$$\mathrm{TV}(u) = \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} |\nu_i - \nu_j| \mathcal{H}^{d-1}(\partial^* E_i \cap \partial^* E_j).$$

This motivates the idea to perturb the boundaries of the level sets E_i and to analyze the resulting effect on the objective function $J(u) = F(u) + \alpha \mathrm{TV}(u)$. These perturbations will be generated by means of local variations, which are defined leaning on [72].

Definition 3.1 ([77, Def. 3.1]). (a) A one parameter family of diffeomorphisms of $\mathbb{R}^d$ is a smooth function $f : (-\varepsilon, \varepsilon) \times \mathbb{R}^d \to \mathbb{R}^d$ for some $\varepsilon > 0$ such that for all $t \in (-\varepsilon, \varepsilon)$, the function $f_t(\cdot) \coloneqq f(t, \cdot) : \mathbb{R}^d \to \mathbb{R}^d$ is a diffeomorphism.

(b) Let $A \subset \mathbb{R}^d$ be open. Then the family $(f_t)_{t \in (-\varepsilon, \varepsilon)}$ is a local variation in A if additionally to (a) we have $f_0(x) = x$ for all $x \in \mathbb{R}^d$ and there is a compact set $K \subset A$ such that $\{x \in \mathbb{R}^d : f_t(x) \neq x\} \subset K$ for all $t \in (-\varepsilon, \varepsilon)$.

(c) For a local variation, we define its initial velocity by $\psi(x) \coloneqq \frac{\partial f}{\partial t}(0, x)$ for $x \in \mathbb{R}^d$.

We will focus on the following type of local variations.

Proposition 3.2 ([77, Prop. 3.2]). *Let $\psi \in C_c^\infty(\Omega, \mathbb{R}^d)$. Let $f_t \coloneqq I + t\psi$ for $t \in \mathbb{R}$. Then $(f_t)_{t \in (-\varepsilon, \varepsilon)}$ is a local variation in Ω with initial velocity ψ for some $\varepsilon > 0$.*

Proof. We denote the matrix norm by $\|A\|_2 \coloneqq \sup_{\|x\|_2 = 1} \|Ax\|_2$ for $A \in \mathbb{R}^{d \times d}$. There holds

$$\|\nabla f_t(x) - I\|_2 = \|t \nabla \psi(x)\|_2 = |t| \|\nabla \psi(x)\|_2 \leq |t| M$$

with $M \coloneqq \max_{x \in S} \|\nabla\psi(x)\|_2 < \infty$ with $S \coloneqq \operatorname{supp} \nabla\psi \subset \Omega$ due to $\psi \in C_c^\infty(\Omega; \mathbb{R}^d)$ and $\nabla\psi \in C_c^\infty(\Omega; \mathbb{R}^{d\times d})$. If we now choose $\delta \in (0,1)$ and $\varepsilon \le \frac{\delta}{M}$, there holds

$$\|\nabla f_t(x) - I\|_2 < \delta$$

for all $t \in (-\varepsilon, \varepsilon)$ such that Lemma A.2 yields that $(f_t)_{t\in(-\varepsilon,\varepsilon)}$ is a one parameter family of diffeomorphisms. We have $f_0(x) = x$ for all $x \in \mathbb{R}^d$ and

$$\{x \in \mathbb{R}^d : f_t(x) \neq x\} \subset \operatorname{supp} \psi \subset \Omega$$

for all $t \in \mathbb{R}$. Hence, $(f_t)_{t\in(-\varepsilon,\varepsilon)}$ is a local variation in Ω with initial velocity $\frac{\partial f}{\partial t}(0,x) = \psi(x)$ for $x \in \mathbb{R}^d$. □

If we now apply a local variation $(f_t)_{t\in(-\varepsilon,\varepsilon)}$ in Ω with initial velocity $\psi \in C_c^\infty(\Omega; \mathbb{R}^d)$ to the Caccioppoli partition $\{E_1, \ldots, E_N\}$ defining $u = \sum_{i=1}^N \nu_i \chi_{E_i} \in \mathrm{BV_U}(\Omega)$, the resulting sets $\{f_t(E_1), \ldots, f_t(E_N)\}$ are again a partition of Ω and induce the piecewise constant function

$$f_t^{\#} u \coloneqq \sum_{i=1}^{N} \nu_i \chi_{f_t(E_i)}.$$

For $\psi \in C_c^\infty(\Omega; \mathbb{R}^d)$ we define the boundary divergence $\operatorname{div}_E \psi : \partial^* E \to \mathbb{R}$ of ψ on a set E of finite perimeter in Ω according to (17.23) in [72] by

$$\operatorname{div}_E \psi(x) \coloneqq \operatorname{div} \psi(x) - n_E(x) \cdot \nabla\psi(x) n_E(x),$$

where n_E denotes the generalized unit outer normal vector on the reduced boundary of E.

The objective functions of the trust-region subproblems consist of a linear part $(g, u)_{L^2(\Omega)}$ and the exact total variation term $\alpha \mathrm{TV}(u)$, see (TR). In order to estimate the influence of the application of a local variation to the level sets of the input function, we state the Taylor expansions for $(g, f_t^{\#} u)_{L^2(\Omega)}$ and $\mathrm{TV}(f_t^{\#} u)$ with respect to t.

We denote the symmetric difference of two sets $E_1, E_2 \subset \Omega$ by $E_1 \Delta E_2$.

Lemma 3.3 ([77, Lem. 3.3], extension of [72, Thm. 17.5])**.** *Let $\{E_1, \ldots, E_N\}$ be a Caccioppoli partition of Ω and $u = \sum_{i=1}^N \nu_i \chi_{E_i}$. Let $(f_t)_{t\in(-\varepsilon,\varepsilon)}$ be a local variation in Ω with initial velocity $\psi \in C_c^\infty(\Omega; \mathbb{R}^d)$. Then there exists $\varepsilon_0 > 0$ such that*

$$\mathrm{TV}(f_t^{\#} u) = \mathrm{TV}(u) + t \sum_{i=1}^{N} \sum_{j=i+1}^{N} |\nu_i - \nu_j| \int_{\partial^* E_i \cap \partial^* E_j} \operatorname{div}_{E_i} \psi(x) \,\mathrm{d}\mathcal{H}^{d-1}(x) + O(t^2)$$

for all $t \in (-\varepsilon_0, \varepsilon_0)$.

Proof. We denote by $\mathcal{J}_x f_t$ the Jacobian determinant of the function f_t. By (2.5), there holds

$$\mathrm{TV}(f_t^{\#} v) = \sum_{i=1}^{N} \sum_{j=i+1}^{N} |\nu_i - \nu_j| \mathcal{H}^{d-1}(\partial^* f_t(E_i) \cap \partial^* f_t(E_j)). \tag{3.2}$$

By (17.4) from Proposition 17.1 in [72], there holds

$$\mathcal{H}^{d-1}(\partial^* f_t(E_j) \Delta f_t(\partial^* E_j)) = 0$$

such that

$$\mathcal{H}^{d-1}(\partial^* f_t(E_i) \cap \partial^* f_t(E_j)) = \mathcal{H}^{d-1}(\partial^* f_t(E_i) \cap f_t(\partial^* E_j)).$$

It follows from (17.6) in [72] that

$$\begin{aligned} &\mathcal{H}^{d-1}(\partial^* f_t(E_i) \cap f_t(\partial^* E_j)) \\ &\quad = \int_{\partial^* E_i \cap \partial^* E_j} \mathcal{J}_x f_t(x) \|((\nabla f_t^{-1} \circ f_t)(x))^T n_{E_i}(x)\|_2 \, \mathrm{d}\mathcal{H}^{d-1}(x). \end{aligned}$$

In the proof of Theorem 17.5 in [72], it was shown that there exists $0 < \varepsilon_0 \leq \varepsilon$ such that

$$\mathcal{J}_x f_t(x) \|((\nabla f_t^{-1} \circ f_t)(x))^T n_{E_i}(x)\|_2 = 1 + t \mathrm{div}_{E_i} \psi(x) + O(t^2)$$

for $t \in (-\varepsilon_0, \varepsilon_0)$ and $x \in \partial^* E_i$, which yields together with (2.5) that

$$\begin{aligned} \mathrm{TV}(f_t^{\#} u) &= \sum_{i=1}^{N} \sum_{j=i+1}^{N} |\nu_i - \nu_j| \int_{\partial^* E_i \cap \partial^* E_j} (1 + t \mathrm{div}_{E_i} \psi(x)) \, \mathrm{d}\mathcal{H}^{d-1}(x) + O(t^2) \\ &= \mathrm{TV}(u) + t \sum_{i=1}^{N} \sum_{j=i+1}^{N} |\nu_i - \nu_j| \int_{\partial^* E_i \cap \partial^* E_j} \mathrm{div}_{E_i} \psi(x) \, \mathrm{d}\mathcal{H}^{d-1}(x) + O(t^2) \end{aligned}$$

for $t \in (-\varepsilon_0, \varepsilon_0)$, similarly to (17.22) in [72]. $\square$

We can conclude from Lemma 3.3 that the functions

$$f_t^{\#} u := \sum_{i=1}^{N} \nu_i \chi_{f_t(E_i)}$$

that are obtained by the application of a local variation $(f_t)_{t \in (-\varepsilon, \varepsilon)}$ with initial velocity $\psi \in C_c^\infty(\Omega; \mathbb{R}^d)$ to the level sets of a function $u = \sum_{i=1}^N \nu_i \chi_{E_i} \in \mathrm{BV}_{\mathrm{U}}(\Omega)$ are again functions in $\mathrm{BV}_{\mathrm{U}}(\Omega)$ if we restrict t to a small enough interval $(-\varepsilon_0, \varepsilon_0)$ for some $0 < \varepsilon_0 < \varepsilon$.

Corollary 3.4 ([77, Cor. 3.4]). *Let $\{E_1, \ldots, E_N\}$ be a Caccioppoli partition of Ω. Let $(f_t)_{t\in(-\varepsilon,\varepsilon)}$ be a local variation with initial velocity $\psi \in C_c^\infty(\Omega;\mathbb{R}^d)$. Then there exists $0 < \varepsilon_0 < \varepsilon$ such that the partition $\{f_t(E_1), \ldots, f_t(E_N)\}$ of Ω is a Caccioppoli partition for all $t \in (-\varepsilon_0, \varepsilon_0)$.*

Proof. We define $u \in \mathrm{BV_U}(\Omega)$ by $u \coloneqq \sum_{i=1}^N \nu_i \chi_{E_i}$. By Lemma 3.3 there exists $\varepsilon_0 > 0$ such that $\mathrm{TV}(f_t^\# u) < \infty$ for all $t \in (-\varepsilon_0, \varepsilon_0)$. Since $2\mathrm{TV}(f_t^\# u) \geq \sum_{i=1}^N P(f_t(E_i), \Omega)$ by (2.6), there holds that $\{f_t(E_1), \ldots, f_t(E_N)\}$ is a Caccioppoli partition for all $t \in (-\varepsilon_0, \varepsilon_0)$. □

Lemma 3.5 ([77, Lem. 3.5], extension of [72, Prop. 17.8]). *Let $u = \sum_{i=1}^N \nu_i \chi_{E_i} \in \mathrm{BV_U}(\Omega)$ with a Caccioppoli partition $\{E_1, \ldots, E_N\}$ of Ω. Let $(f_t)_{t\in(-\varepsilon,\varepsilon)}$ be a local variation with initial velocity $\psi \in C_c^\infty(\Omega;\mathbb{R}^d)$. Let $g \in C(\bar{\Omega})$. Then it follows that*

$$\int_\Omega g(x)(f_t^\# u(x) - u(x))\,\mathrm{d}x = t\sum_{i=1}^N \nu_i \int_{\partial^* E_i\cap\Omega} g(x)\,(\psi(x)\cdot n_{E_i}(x))\,\mathrm{d}\mathcal{H}^{d-1}(x) + o(t).$$

Proof. We use Proposition 17.8 in [72] to obtain

$$\begin{aligned}
\int_\Omega g(x)(f_t^\# u(x) - u(x))\,\mathrm{d}x &= \sum_{i=1}^N \nu_i \int_\Omega g(x)(\chi_{f_t(E_i)}(x) - \chi_{E_i}(x))\,\mathrm{d}x \\
&= \sum_{i=1}^N \nu_i \left(\int_{f_t(E_i)} g(x)\,\mathrm{d}x - \int_{E_i} g(x)\,\mathrm{d}x\right) \\
&= t\sum_{i=1}^N \nu_i \int_{\partial^* E_i} g(x)\,(\psi(x)\cdot n_{E_i}(x))\,\mathrm{d}\mathcal{H}^{d-1}(x) + o(t) \\
&= t\sum_{i=1}^N \nu_i \int_{\partial^* E_i\cap\Omega} g(x)\,(\psi(x)\cdot n_{E_i}(x))\,\mathrm{d}\mathcal{H}^{d-1}(x) + o(t),
\end{aligned}$$

where the last equality is due to the compact support of ψ in Ω. □

Besides the estimation of the influence of the application of local variations to the level sets on the objective value of the trust-region subproblem, we need to estimate the L^1 norm in regards to feasibility for the trust-region subproblems. To this end, we first prove Lipschitz continuity with respect to t for the Lebesgue measure of the intersection of a Caccioppoli set F and a transformed Caccioppoli set $f_t(E)$, where $(f_t)_{t\in(-\varepsilon,\varepsilon)}$ is a local variation as in Proposition 3.2.

Lemma 3.6 ([77, Lem. 3.6]). *Let $\psi \in C_c^\infty(\Omega;\mathbb{R}^d)$ and let $(f_t)_{t\in(-\varepsilon,\varepsilon)}$ be the local variation defined by $f_t \coloneqq I + t\psi$ for $t \in (-\varepsilon, \varepsilon)$. Then there exist $0 < \varepsilon_0 < \varepsilon$ and $L > 0$ such that for all $s, t \in (-\varepsilon_0, \varepsilon_0)$ and all Caccioppoli sets E, F in Ω it holds* *that*

$$\big|\,|f_t(E)\cap F| - |f_s(E)\cap F|\,\big| \le L|t-s|P(E,\Omega).$$

Proof. We adapt the strategy of the proof of Lemma 17.9 in [72] that shows the bound

$$|f_t(E)\Delta E| \le L|t|P(E,\Omega).$$

We abbreviate $g_t \coloneqq f_t^{-1}$ for $t \in (-\varepsilon,\varepsilon)$. Let $s,\ t \in (-\varepsilon,\varepsilon)$ be given. In order to approximate χ_E with smooth functions, we define the functions $\{w_k\}_{k\in\mathbb{N}} \in \mathrm{BV}(\Omega)\cap C^\infty(\Omega)$ as in Theorem 5.3 in [50] so that

$$w_k \to \chi_E \text{ in } L^1(\Omega) \quad \text{as } k\to\infty \tag{3.3}$$

and

$$\int_\Omega \|\nabla w_k(x)\|\,\mathrm{d}x = |Dw_k|(\Omega) \to |D\chi_E|(\Omega) \quad \text{as } k\to\infty. \tag{3.4}$$

Then (3.3) together with the reverse triangle inequality implies

$$\begin{aligned}
\big|\,|f_t(E)\cap F| - |f_s(E)\cap F|\,\big| &= \left|\int_F \chi_{f_t(E)}(x)\,\mathrm{d}x - \int_F \chi_{f_s(E)}(x)\,\mathrm{d}x\right| \\
&= \left|\int_F |\chi_E(g_t(x))|\,\mathrm{d}x - \int_F |\chi_E(g_s(x))|\,\mathrm{d}x\right| \\
&\le \int_F |\chi_E(g_t(x)) - \chi_E(g_s(x))|\,\mathrm{d}x \\
&= \lim_{k\to\infty}\int_F |w_k(g_t(x)) - w_k(g_s(x))|\,\mathrm{d}x.
\end{aligned} \tag{3.5}$$

Before we continue to analyze the right hand side of this estimate, we need some preparations.

We define $G_{t,s,\alpha}(x) \coloneqq \alpha g_t(x) + (1-\alpha)g_s(x)$ for $\alpha\in[0,1]$ and $x\in\mathbb{R}^d$. For the derivative of $G_{t,s,\alpha}(x)$ with respect to α, we obtain

$$\frac{\partial}{\partial\alpha}G_{t,s,\alpha}(x) = g_t(x) - g_s(x)$$

and for the derivative of $G_{t,s,\alpha}$ with respect to x, we obtain

$$\nabla G_{t,s,\alpha}(x) = \alpha\nabla g_t(x) + (1-\alpha)\nabla g_s(x).$$

We further define for fixed $k,t,s,$ and x the functions $\tilde G : [0,1]\to\mathbb{R}^d$ by $\tilde G(\alpha) \coloneqq G_{t,s,\alpha}(x)$ and $h : [0,1]\to\mathbb{R}^d$ by $h(\alpha)\coloneqq w_k(\tilde G(\alpha))$. By the fundamental theorem of calculus, there holds

$$w_k(G_{t,s,1}(x)) - w_k(G_{t,s,0})(x) = h(1) - h(0) = \int_0^1 h'(\alpha)\,\mathrm{d}\alpha.$$

The chain rule yields

$$h'(\alpha) = (\nabla w_k(\tilde{G}(\alpha))) \cdot \tilde{G}'(\alpha) = \nabla w_k(G_{t,s,\alpha}(x)) \cdot (g_t(x) - g_s(x)).$$

This gives

$$\begin{aligned}
|w_k(g_t(x)) - w_k(g_s(x))| &= \int_0^1 |w_k(G_{t,s,1}(x)) - w_k(G_{t,s,0}(x))|\,\mathrm{d}\alpha \\
&= \int_0^1 |\nabla w_k(G_{t,s,\alpha}(x)) \cdot (g_t(x) - g_s(x))|\,\mathrm{d}\alpha \\
&\leq \int_0^1 \|\nabla w_k(G_{t,s,\alpha}(x))\|_2\,\mathrm{d}\alpha\,\|g_t(x) - g_s(x)\|_2.
\end{aligned}$$

We deduce

$$\begin{aligned}
d_k &:= \int_F |w_k(g_t(x)) - w_k(g_s(x))|\,\mathrm{d}x \\
&\leq \int_F \int_0^1 \|\nabla w_k(G_{t,s,\alpha}(x))\|_2\,\mathrm{d}\alpha\,\|g_t(x) - g_s(x)\|_2\,\mathrm{d}x.
\end{aligned}$$

Let $0 < \varepsilon_0 \leq \varepsilon$ such that the statements from Lemma A.3 hold for g_t with $t \in (-\varepsilon_0, \varepsilon_0)$. This yields for fixed $x \in \mathbb{R}^d$ that $g_t(x)$ is Lipschitz continuous with respect to $t \in (-\varepsilon_0, \varepsilon_0)$, that is, there exists $C_2 > 0$ such that $\|g_t(x) - g_s(x)\| \leq C_2|t-s|$ for all $s, t \in (-\varepsilon_0, \varepsilon_0)$. We insert this inequality and apply Fubini's theorem to obtain the estimate

$$d_k \leq C_2|t-s| \int_0^1 \int_F \|\nabla w_k(G_{t,s,\alpha}(x))\|_2\,\mathrm{d}x\,\mathrm{d}\alpha. \tag{3.6}$$

Lemma A.3 gives $\nabla g_t(x) \to I$ for $t \to 0$ uniformly for all $x \in \mathbb{R}^d$. For arbitrary but fixed $\delta \in (0,1)$, we can reduce ε_0 further if necessary such that for all $t \in (-\varepsilon_0, \varepsilon_0)$ there holds

$$\|I - \nabla g_t(x)\|_2 \leq \delta \quad \forall\, x \in \mathbb{R}^d.$$

This yields

$$\|I - \nabla G_{s,t,\alpha}(x)\|_2 \leq \alpha\|I - \nabla g_s(x)\|_2 + (1-\alpha)\|I - \nabla g_t(x)\|_2 \leq \delta \quad \forall\, x \in \mathbb{R}^d$$

for all $s, t \in (-\varepsilon_0, \varepsilon_0)$ and all $\alpha \in [0,1]$. By virtue of Lemma A.2 the function $G_{s,t,\alpha} : \mathbb{R}^d \to \mathbb{R}^d$ is invertible for all s, $t \in (-\varepsilon_0, \varepsilon_0)$ and all $\alpha \in [0,1]$.

Next, we observe

$$G_{t,s,\alpha}(\Omega) = \Omega \tag{3.7}$$

To see this, we note that $G_{t,s,\alpha}(\mathbb{R}^d \setminus \operatorname{supp}\psi) = \mathbb{R}^d \setminus \operatorname{supp}\psi$ and in particular $G_{t,s,\alpha}(\Omega \setminus \operatorname{supp}\psi) = \Omega \setminus \operatorname{supp}\psi$ because if $x \notin \operatorname{supp}\psi$, then $x = G_{t,s,\alpha}(x)$. Since $G_{t,s,\alpha} : \mathbb{R}^d \to \mathbb{R}^d$ is invertible for all $t, s \in (-\varepsilon_0, \varepsilon_0)$ and all $\alpha \in [0,1]$ there must hold that $G_{t,s,\alpha}(\operatorname{supp}\psi) = \operatorname{supp}\psi$ such that $G_{t,s,\alpha}(\Omega) = \Omega$. For arbitrary but fixed $\alpha \in [0,1]$, we compute the Jacobian determinant $\mathcal{J}_x G_{t,s,\alpha}$ of $G_{t,s,\alpha}$ with respect to x and obtain

$$\mathcal{J}_x G_{t,s,\alpha}(x) = \sqrt{\det\left((\alpha\nabla g_t(x) + (1-\alpha)\nabla g_s(x))(\alpha\nabla g_t(x) + (1-\alpha)\nabla g_s(x))^T\right)}.$$

Since $\nabla g_t \to I$ for $t \to 0$ uniformly for all $x \in \mathbb{R}^d$, we obtain by possibly reducing ε_0 further for all s, $t \in (-\varepsilon_0, \varepsilon_0)$ and all $x \in \mathbb{R}^d$ that

$$\mathcal{J}_x G_{t,s,\alpha}(x) \geq 0.5. \tag{3.8}$$

We apply the change of variables formula and the inverse function theorem to obtain

$$\begin{aligned}
\int_F \|\nabla w_k(G_{t,s,\alpha}(x))\|_2 \,\mathrm{d}x &= \int_F \frac{\|\nabla w_k(G_{t,s,\alpha}(x))\|_2}{\mathcal{J}_x G_{t,s,\alpha}(x)} \mathcal{J}_x G_{t,s,\alpha}(x) \,\mathrm{d}x && \cdot 1 \\
&= \int_{G_{t,s,\alpha}(F)} \frac{\|\nabla w_k(y)\|_2}{\mathcal{J}_x G_{t,s,\alpha}(G_{t,s,\alpha}^{-1}(y))} \,\mathrm{d}y && \text{Area formula} \\
&\leq 2 \int_\Omega \|\nabla w_k(y)\| \,\mathrm{d}y. && (3.8), (3.7)
\end{aligned} \tag{3.9}$$

We insert the estimate (3.9) into (3.6) and pass to the limit $k \to \infty$, which gives

$$d_k \leq 2C_2 \int_\Omega \|\nabla w_k(y)\| \,\mathrm{d}y|t-s| \underset{(3.4)}{\to} 2C_2|D\chi_E|(\Omega)|t-s| = 2C_2 P(E,\Omega)|t-s|.$$

The claim follows by combining these considerations with the estimate $\big||f_t(E) \cap F| - |f_s(E) \cap F|\big| \leq \lim_{k\to\infty} d_k$ and the choice $L \coloneqq 2C_2$. $\square$

Remark 3.7 ([77, Rem. 3.7])**.** From the proof of Lemma 3.6, it can also be derived that

$$|f_t(E)\Delta f_s(E)| \leq L|t-s|P(E,\Omega)$$

by using that $|f_t(E)\Delta f_s(E)| = \int_\Omega |\chi_E(g_t(x)) - \chi_E(g_s(x))| \,\mathrm{d}x$ and continuing the proof from (3.5) with the choice $F = \Omega$.

Lemma 3.6 implies the following estimation of the L^1 distance of a function $u \in \mathrm{BV}_{\mathrm{U}}(\Omega)$ to the functions that result from the application of a local variation $(f_t)_{t\in(-\varepsilon,\varepsilon)}$ to its level sets.

Lemma 3.8 ([77, Lem. 3.8])**.** *Let $\{E_1,\dots,E_N\}$ be a Caccioppoli partition of Ω and $u = \sum_{i=1}^N \nu_i\chi_{E_i}$. Let $(f_t)_{t\in(-\varepsilon,\varepsilon)}$ be the local variation defined by $f_t := I + t\psi$ with $\psi \in C_c^\infty(\Omega,\mathbb{R}^d)$. Then there exist $\varepsilon_0 > 0$ and $\bar{C} > 0$ such that for all $t \in (-\varepsilon_0,\varepsilon_0)$ it holds that*

$$\|{f_t}^{\#}u - u\|_{L^1(\Omega)} \leq \bar{C}|t|.$$

Proof. Corollary 3.4 and Lemma 3.6 give the existence of a constant $L > 0$ and some $\varepsilon_0 > 0$ such that $\{f_t(E_1),\dots,f_t(E_N)\}$ is a Caccioppoli partition for all $t \in (-\varepsilon_0,\varepsilon_0)$ and

$$|f_t(E_i)\cap E_j| = \big|\,|f_t(E_i)\cap E_j| - |f_0(E_i)\cap E_j|\,\big| \leq L|t|P(E_i,\Omega)$$

holds for all $t \in (-\varepsilon_0,\varepsilon_0)$ and all i, $j \in \{1,\dots,N\}$ with $i \neq j$, where we have used that $f_0(E_i) = E_i$ is disjoint to E_j for $i \neq j$ and hence $|f_0(E_i)\cap E_j| = 0$. Because $\{E_1,\dots,E_N\}$ and $\{f_t(E_1),\dots,f_t(E_N)\}$ are Caccioppoli partitions of Ω, we insert the estimate above to obtain

$$\begin{aligned}
\|f_t^{\#}u - u\|_{L^1(\Omega)} &= \int_\Omega \Big|\sum_{i=1}^N \nu_i\chi_{f_t(E_i)}(x) - \sum_{j=1}^N \nu_j\chi_{E_j}(x)\Big|\,\mathrm{d}x \\
&= \sum_{i=1}^N\sum_{j=1}^N \int_{f_t(E_i)\cap E_j} |\nu_i - \nu_j|\,\mathrm{d}x \\
&= \sum_{i=1}^N\sum_{j=1}^N |\nu_i - \nu_j|\,|f_t(E_i)\cap E_j| \\
&\leq L|t|\sum_{i=1}^N P(E_i,\Omega)\sum_{j=1}^N |\nu_i - \nu_j|
\end{aligned}$$

for all $t \in (-\varepsilon_0,\varepsilon_0)$. We note that $P(E_i,\Omega) = \mathrm{TV}(\chi_{E_i}) = \mathcal{H}^{d-1}(\partial^*E_i\cap\Omega) = \sum_{j=1,j\neq i}^N \mathcal{H}^{d-1}(\partial^*E_i\cap\partial^*E_j) \leq \mathrm{TV}(u) < \infty$ because of Theorem 2.14 and $|\nu_i - \nu_j| \geq 1$ for $i \neq j$ and (2.5), which yields

$$\|f_t^{\#}u - u\|_{L^1(\Omega)} \leq |t|L\mathrm{TV}(u)\sum_{i=1}^N\sum_{j=1}^N |\nu_i - \nu_j|.$$

□

3.2 Locally optimal solutions and first-order optimality conditions

We define r-optimality of feasible solutions for (P), which means that a feasible solution $u \in \mathrm{BV}_{\mathrm{U}}(\Omega)$ is optimal with respect to all feasible solutions within a closed L^1 ball around u with radius $r > 0$.

Definition 3.9 ([77, Def. 4.1])**.** Let $u \in \mathrm{BV}_{\mathrm{U}}(\Omega)$ be feasible for (P). Then u is r-optimal for some $r > 0$ if

$$F(u) + \alpha \mathrm{TV}(u) \leq F(\tilde{u}) + \alpha \mathrm{TV}(\tilde{u})$$

holds for all $\tilde{u} \in \mathrm{BV}_{\mathrm{U}}(\Omega)$ that are feasible for (P) and satisfy $\|u - \tilde{u}\|_{L^1(\Omega)} \leq r$.

It is clear that r-optimality is a necessary condition for global optimality. In finite dimension, r-optimality corresponds to discrete local minimizers for mixed-integer optimization problems as defined in [86]. It gives a sensible notion of local optimality for finite-dimensional integer optimization problems, in which a small enough neighborhood around a feasible solution $u \in \mathbb{Z}^d$ always contains no point other than u itself such that by the usual definition of local optimality each feasible point would be a local minimizer.

In the case of infinite-dimensional integer optimization problems, this is generally not the case because we can change the level sets on sets of arbitrarily small Lebesgue measure.

Example 3.10 ([77, Expl. 4.2])**.** Let $N \geq 2$, and $u \in \mathrm{BV}_{\mathrm{U}}(\Omega)$ with $U = \{\nu_1, \ldots, \nu_N\} = [a, b] \cap \mathbb{Z}$, $a \leq b - 1$. We may consider a ball $B \subset \Omega$ and construct $\hat{u} \in \mathrm{BV}_{\mathrm{U}}(\Omega)$ by setting

$$\hat{u}(x) \coloneqq \begin{cases} \nu_{i+1} & \text{if } x \in B \text{ and } u(x) = \nu_i \text{ for } i < N, \\ \nu_{N-1} & \text{if } x \in B \text{ and } u(x) = \nu_N, \\ u(x) & \text{if } x \in \Omega \setminus B. \end{cases}$$

Then $\|u - \hat{u}\|_{L^1(\Omega)} = |B|$, which tends to zero when driving the radius of B to zero.

In the one-dimensional case, a feasible solution $u \in \mathrm{BV}_{\mathrm{U}}(\Omega)$ has only finitely many switching points. If we apply a small perturbation to any of the switching points, that is, if we move its position slightly to the left or to the right, then the total variation $\mathrm{TV}(u)$ does not change because the number of switching points and the jump heights do not change. This yields in particular that for an optimal solution $\bar{u}$, the derivative $\nabla F(\bar{u})$ must be zero because otherwise we could attain a descent by applying small perturbations to the switching points of $\bar{u}$ that do not change the total variation of u.

In our multi-dimensional case with $d \geq 2$, the boundaries of the level sets of the functions in $\mathrm{BV}_{\mathrm{U}}(\Omega)$ are not single points. In contrast to shifting of a single point, the perturbation of the boundary of a level set by means of a local variation generally does change its $\mathcal{H}^{d-1}$-measure and therefore the total variation of the corresponding function as we can see in (2.5) and (3.2).

We prove a first-order optimality condition for (P) under the following assumption.

Assumption 3.11 ([77, Ass. 4.3])**.** Let $F : L^2(\Omega) \to \mathbb{R}$ be twice continuously Fréchet differentiable. For some $C > 0$ and all $\xi \in L^2(\Omega)$, let the bilinear form induced by the Hessian $\nabla^2 F(\xi) : L^2(\Omega) \times L^2(\Omega) \to \mathbb{R}$ satisfy

$$|\nabla^2 F(\xi)(u, w)| \leq C \|u\|_{L^1(\Omega)} \|w\|_{L^1(\Omega)} \tag{3.10}$$

for all u, $w \in L^2(\Omega)$.

As discussed in [70, 76], this assumption requires an improvement of the regularity of the input function during the application of F, for example by function composition with a PDE solution operator. The first-order Taylor expansion of $F : L^2(\Omega) \to \mathbb{R}$ at $u \in L^2(\Omega)$ reads

$$F(w) = F(u) + \nabla F(u)(w - u) + \frac{1}{2} \nabla^2 F(\xi_w)(w - u, w - u)$$

for $w \in L^2(\Omega)$ and some $\xi_w \in L^2$ in the line segment between u and w. Assumption 3.11 is required to be able to dominate the remainder term $\frac{1}{2}\nabla^2 F(\xi_w)(w - u, w - u)$ with the linear part $\nabla F(u)(w-u)$, which can be estimated by $\|w-u\|_{L^1(\Omega)}$. This is necessary to obtain a sufficient decrease in case that the current iterate is not stationary, see, for example, the proof of Lemma 3.21 and the proof of outcome 3 (4) of Theorem 3.23. Since for discrete-valued functions $u, w \in \mathrm{BV}_{\mathrm{U}}(\Omega)$ there holds $\|u - w\|_{L^1(\Omega)} \in \Theta\left(\|u - w\|^2_{L^2(\Omega)}\right)$, it might not be sufficient if (3.10) is only fulfilled with $\|u\|_{L^2(\Omega)}$ and $\|w\|_{L^2(\Omega)}$. Similar assumptions are present in other works on discrete-valued control functions, see for example (5) in Lemma 3 and (10) in Theorem 2 in [60] and Assumption 3.1 3 in [74]. We will state examples that fulfill Assumption 3.11 in Chapter 6.

We define the concept of L-stationarity in Definition 3.12 below.

Definition 3.12 ([77, Def. 4.4])**.** Let F satisfy Assumption 3.11. Let $\{E_1, \ldots, E_N\}$ be a Caccioppoli partition of Ω, $u = \sum_{i=1}^N \nu_i \chi_{E_i} \in \mathrm{BV}_{\mathrm{U}}(\Omega)$, and $\nabla F(u) \in C(\bar{\Omega})$.

Then we say that u is L-stationary if the identity

$$\begin{aligned}\sum_{i=1}^{N}\nu_i\int_{\partial^* E_i\cap\Omega}(-\nabla F(u))(x)(\psi(x)\cdot n_{E_i}(x))\,\mathrm{d}\mathcal{H}^{d-1}(x) =\\ \alpha\sum_{i=1}^{N}\sum_{j=i+1}^{N}|\nu_i-\nu_j|\int_{\partial^* E_i\cap\partial^* E_j}\operatorname{div}_{E_i}\psi(x)\,\mathrm{d}\mathcal{H}^{d-1}(x).\end{aligned}\tag{3.11}$$

holds for all $\psi\in C_c^\infty(\Omega;\mathbb{R}^d)$.

Remark 3.13 ([77, Rem. 4.5]). Definition 3.12 means that on the intersection of the reduced boundaries of E_i and E_j, the set E_i has distributional mean curvature of $\frac{\nu_i-\nu_j}{\alpha|\nu_i-\nu_j|}(-\nabla F(u))$, see Remark 17.7 in [4].

We prove that r-optimality implies L-stationarity, which yields that L-stationarity is a necessary optimality condition, too.

Theorem 3.14 ([77, Thm. 4.6]). *Let F satisfy Assumption 3.11. Let $\{E_1,\ldots,E_N\}$ be a Caccioppoli partition of Ω, and $u=\sum_{i=1}^N\nu_i\chi_{E_i}\in\mathrm{BV}_{\mathrm{U}}(\Omega)$. Let $\nabla F(u)\in C(\bar\Omega)$. If u is r-optimal for* (P) *for some $r>0$, then it is L-stationary. Moreover, for all $i,j\in\{1,\ldots,N\}$ with $i\neq j$ we have*

$$\begin{aligned}(\nu_i-\nu_j)\int_{\partial^* E_i\cap\partial^* E_j}(-\nabla F(u))(x)(\psi(x)\cdot n_{E_i}(x))\,\mathrm{d}\mathcal{H}^{d-1}(x)\\ =\alpha|\nu_i-\nu_j|\int_{\partial^* E_i\cap\partial^* E_j}\operatorname{div}_{E_i}\psi(x)\,\mathrm{d}\mathcal{H}^{d-1}(x).\end{aligned}\tag{3.12}$$

for all $\psi\in C_c^\infty(E_i\cup E_j;\mathbb{R}^d)$.

Proof. Let $(f_t)_{t\in(-\varepsilon,\varepsilon)}$ be the local variation defined by $f_t:=I+t\psi$ for some fixed $\psi\in C_c^\infty(\Omega;\mathbb{R}^d)$. We first prove that the function $t\mapsto F(f_t^\# u)$ is differentiable at $t=0$. Since $F:L^2(\Omega)\to\mathbb{R}$ is twice continuously Fréchet differentiable, we may apply Taylor's theorem to obtain

$$F(f_t^\# u)=F(u)+\nabla F(u)(f_t^\# u-u)+\frac{1}{2}\nabla^2F(\xi^t)(f_t^\# u-u,f_t^\# u-u)$$

for some $\xi^t\in L^2(\Omega)$ in the line segment between u and $f_t^\# u$. We compute the derivative of $t\mapsto F(f_t^\# u)$ at $t=0$ by evaluating

$$\lim_{t\to0}\frac{F(f_t^\# u)-F(u)}{t}=\lim_{t\to0}\frac{\nabla F(u)(f_t^\# u-u)-\frac{1}{2}\nabla^2F(\xi^t)(f_t^\# u-u,f_t^\# u-u)}{t}.$$

In order to determine this limit, we recall from Lemma 3.8 that there exist $\varepsilon_0 > 0$ and $\bar{C} > 0$ such that

$$\|f_t^{\#}u - u\|_{L^1(\Omega)} \leq \bar{C}|t|$$

holds for all $t \in (-\varepsilon_0, \varepsilon_0)$. Combining this with Assumption 3.11 gives

$$\left|\nabla^2 F(\xi^t)(f_t^{\#}u - u, f_t^{\#}u - u)\right| \leq C\|f_t^{\#}u - u\|^2_{L^1(\Omega)} \leq C\bar{C}^2 t^2$$

for all $t \in (-\varepsilon_0, \varepsilon_0)$ with the constant $C > 0$ from Assumption 3.11. This yields

$$\lim_{t\to 0} \frac{F(f_t^{\#}u) - F(u)}{t} = \lim_{t\to 0} \frac{\nabla F(u)(f_t^{\#}u - f_0^{\#}u)}{t} = \frac{\mathrm{d}}{\mathrm{d}t}(\nabla F(u), f_t^{\#}u)_{L^2(\Omega)}\Big|_{t=0}.$$

By Lemma 3.5, there holds

$$\frac{\mathrm{d}}{\mathrm{d}t}(\nabla F(u), f_t^{\#}u)_{L^2(\Omega)}\Big|_{t=0} = \sum_{i=1}^{N} \nu_i \int_{\partial^* E_i \cap \Omega} (\psi(x) \cdot n_{E_i}(x))\nabla F(u)(x) \,\mathrm{d}\mathcal{H}^{d-1}(x).$$

Next, we prove that $t \mapsto \alpha \mathrm{TV}(f_t^{\#}u)$ is differentiable at $t = 0$. This immediately follows from Lemma 3.3 with

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{TV}(f_t^{\#}u)\Big|_{t=0} = \sum_{i=1}^{N}\sum_{j=i+1}^{N} |\nu_i - \nu_j| \int_{\partial^* E_i \cap \partial^* E_j} \mathrm{div}_{E_i}\psi(x) \,\mathrm{d}\mathcal{H}^{d-1}(x).$$

Consequently, the function $t \mapsto F(f_t^{\#}u) + \alpha \mathrm{TV}(f_t^{\#}u)$ is differentiable at $t = 0$ and we obtain the first-order optimality condition

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(F(f_t^{\#}u) + \alpha\mathrm{TV}(f_t^{\#}u)\right)\Big|_{t=0} = 0$$

by virtue of Fermat's theorem and the r-optimality of u.

Combining these equations yields the identity

$$\sum_{i=1}^{N} \nu_i \int_{\partial^* E_i \cap \Omega} (-\nabla F(u)(x))(\psi(x) \cdot n_{E_i}(x)) \,\mathrm{d}\mathcal{H}^{d-1}(x) = \\ \alpha \sum_{i=1}^{N}\sum_{j=i+1}^{N} |\nu_i - \nu_j| \int_{\partial^* E_i \cap \partial^* E_j} \mathrm{div}_{E_i}\psi(x) \,\mathrm{d}\mathcal{H}^{d-1}(x)$$

for all $\psi \in C_c^\infty(\Omega; \mathbb{R}^d)$, which is L-stationarity of u. Restricting to $\psi \in C_c^\infty(E_i \cup E_j, \mathbb{R}^d)$ for $i \neq j$ and using $n_{E_i} = -n_{E_j}$ on $\partial^* E_i \cap \partial^* E_j$ gives the second claim. $\square$

3.3 Sequential linear integer programming algorithm

In this section, we introduce the function space trust-region algorithm to solve (P). The algorithm is adapted from [70] to the multi-dimensional case. We will start by stating and analyzing the trust-region subproblem in Section 3.3.1. In particular, we will prove Γ-convergence results with respect to convergence of the linearization point and L-stationarity of optimal solutions to the subproblems. In Section 3.3.2, we will state the trust-region algorithm.

3.3.1 Trust-region subproblem

The trust-region subproblem is defined by

$$\text{(TR)} \qquad \mathrm{TR}(\bar u, g, \Delta) := \begin{cases} \min\limits_{u \in L^2(\Omega)} & (g, u - \bar u)_{L^2(\Omega)} + \alpha \mathrm{TV}(u) - \alpha \mathrm{TV}(\bar u) \\ \quad \text{s.t.} & \|u - \bar u\|_{L^1(\Omega)} \le \Delta \text{ and } u(x) \in U \text{ for a.a. } x \in \Omega, \end{cases}$$

where $\Delta > 0$ is the radius of the trust region and $\bar u \in \mathrm{BV_U}(\Omega)$ is the linearization point. For the trust-region subproblem corresponding to (P), we choose $g = \nabla F(\bar u)$, that is, the function F is linearized and the total variation term TV is considered exactly. The trust-region problem $\mathrm{TR}(\bar u, g, \Delta)$ admits a minimizer, which follows similarly as for the superordinate problem (P) because both have the same structure except for the additional constraint concerning the trust region.

Proposition 3.15 ([77, Prop. 5.1])**.** *Let* $\bar u \in \mathrm{BV_U}(\Omega)$*,* $g \in L^2(\Omega)$*,* $\Delta \ge 0$*, and* $\mathrm{TV}(\bar u) < \infty$*. Then* $\mathrm{TR}(\bar u, g, \Delta)$ *admits a minimizer.*

Proof. The claim follows similarly to Theorem 2.21 with the choice $F(u) = (g, u - \bar u)_{L^2(\Omega)} - \alpha \mathrm{TV}(\bar u)$ for $u \in L^2(\Omega)$ which fulfills Assumption 1.1. The feasible set with the additional constraint $\|u - \bar u\|_{L^1(\Omega)} \le \Delta$ is still sequentially weakly-$*$ closed in $\mathrm{BV}(\Omega)$ because weak-$*$ convergence in $\mathrm{BV}(\Omega)$ implies convergence in $L^1(\Omega)$. □

The trust-region algorithm that will be introduced in Section 3.3.2 produces a sequence of iterates that has subsequences that converge strictly in $\mathrm{BV}(\Omega)$ as we will prove in Theorem 3.23. For that reason, we will prove the following Γ-convergence results for the trust-region subproblems with fixed trust-region radius Δ with respect to strict convergence of the linearization points in $\mathrm{BV}(\Omega)$.

Additionally, we will allow for an approximation of the functional g in the objective function of $\mathrm{TR}(\bar u, g, \Delta)$. This gives the possibility to approximate and regularize g in order to improve the solution process of the trust-region subproblems.

Theorem 3.16 ([77, Thm. 5.2])**.** *Let* $d \ge 2$*. Let* $u_n \to u$ *strictly in* $\mathrm{BV_U}(\Omega)$*. Let* $g_n \rightharpoonup g$ *in* $L^2(\Omega)$*. Let* $\Delta > 0$*. Then the functionals* $T_n : (\mathrm{BV_U}(\Omega), \text{weak-}^*) \to \mathbb{R}$*,* *defined as*

$$T_n(w) \coloneqq (g_n, w - u_n)_{L^2(\Omega)} + \alpha \mathrm{TV}(w) - \alpha \mathrm{TV}(u_n) + \delta_{[0,\Delta]}(\|w - u_n\|_{L^1(\Omega)})$$

for $w \in \mathrm{BV}_{\mathrm{U}}(\Omega)$*,* Γ*-converge to* $T : (\mathrm{BV}_{\mathrm{U}}(\Omega), \text{weak-}^*) \to \mathbb{R}$*, defined as*

$$T(w) \coloneqq (g, w - u)_{L^2(\Omega)} + \alpha \mathrm{TV}(w) - \alpha \mathrm{TV}(u) + \delta_{[0,\Delta]}(\|w - u\|_{L^1(\Omega)})$$

for $w \in \mathrm{BV}_{\mathrm{U}}(\Omega)$*, where* $\delta_{[0,\Delta]}$ *is the* $\{0, \infty\}$*-valued indicator function of* $[0, \Delta]$.

Proof. We start with the $\liminf$ inequality. To this end, let a sequence $\{w_n\} \subset \mathrm{BV}_{\mathrm{U}}(\Omega)$ with $w_n \overset{*}{\rightharpoonup} w$ in $\mathrm{BV}(\Omega)$ be given. We need to show

$$T(w) \le \liminf_{n \to \infty} T_n(w_n).$$

By Lemma 2.19, there holds $w_n \to w$ in $L^2(\Omega)$ and also $u_n \to u$ in $L^2(\Omega)$, implying $(g_n, w_n - u_n)_{L^2(\Omega)} \to (g, w - u)_{L^2(\Omega)}$. The strict convergence of $\{u_n\}_{n \in \mathbb{N}}$ and the weak-$*$ convergence of $\{w_n\}_{n \in \mathbb{N}}$ imply

$$\alpha \mathrm{TV}(w) - \alpha \mathrm{TV}(u) \le \liminf_{n \to \infty} \alpha \mathrm{TV}(w_n) - \alpha \mathrm{TV}(u_n)$$

because TV is lower semi-continuous with respect to weak-$*$ convergence in $\mathrm{BV}(\Omega)$. We may assume that $\|w_n - u_n\|_{L^1(\Omega)} \le \Delta$ for all $n \in \mathbb{N}$ by restricting to subsequences because the non-existence of such subsequences would imply the trivial case $\liminf T_n(w_n) = \infty$. Applying the triangle inequality yields

$$\|w - u\|_{L^1(\Omega)} \le \|w - w_n\|_{L^1(\Omega)} + \|w_n - u_n\|_{L^1(\Omega)} + \|u_n - u\|_{L^1(\Omega)}.$$

Driving $n \to \infty$ yields

$$\|w - u\|_{L^1(\Omega)} \le \Delta$$

because $w_n \to w$ in $L^1(\Omega)$ and $u_n \to u$ in $L^1(\Omega)$.

Next, we need to show the $\limsup$ inequality, that is, we need to prove that for each $w \in \mathrm{BV}_{\mathrm{U}}(\Omega)$ there exists a sequence $\{w_n\}_{n \in \mathbb{N}} \subset \mathrm{BV}_{\mathrm{U}}(\Omega)$ such that $w_n \overset{*}{\rightharpoonup} w$ in $\mathrm{BV}(\Omega)$ and

$$T(w) \ge \limsup_{n \to \infty} T_n(w_n).$$

We distinguish three cases for the value of $\|w - u\|_{L^1(\Omega)}$.

Case $\|w - u\|_{L^1(\Omega)} > \Delta$: Then $T(w) = \infty$ and $w_n \coloneqq w$ for all $n \in \mathbb{N}$ yields the claim.

Case $\|w - u\|_{L^1(\Omega)} < \Delta$: Let $w_n \coloneqq w$ for all $n \in \mathbb{N}$. By the triangle inequality and $u_n \to u$ in $L^1(\Omega)$, there holds

$$\|w_n - u_n\|_{L^1(\Omega)} = \|w - u_n\|_{L^1(\Omega)} \leq \|w - u\|_{L^1(\Omega)} + \|u - u_n\|_{L^1(\Omega)} < \Delta$$

for all $n \in \mathbb{N}$ large enough. Because $u_n \to u$ in $L^2(\Omega)$ by Lemma 2.19, we obtain

$$\begin{aligned}(g_n, w_n - u_n)_{L^2(\Omega)} &+ \alpha\mathrm{TV}(w_n) - \alpha\mathrm{TV}(u_n) + \delta_{[0,\Delta]}(\|w_n - u_n\|_{L^1(\Omega)}) \\ &\to (g, w - u)_{L^2(\Omega)} + \alpha\mathrm{TV}(w) - \alpha\mathrm{TV}(u) + \delta_{[0,\Delta]}(\|w - u\|_{L^1(\Omega)}),\end{aligned}$$

that is, $T_n(w_n) \to T(w)$.

Case $\|w - u\|_{L^1(\Omega)} = \Delta$: The fact $\Delta > 0$ implies that there exists a set

$$D \coloneqq \{x \in \Omega \,|\, u(x) = \nu_1 \neq \nu_2 = w(x)\}$$

with $|D| > 0$, where the specific control realizations ν_1 and ν_2 are without loss of generality because we may reorder the indices of the elements of U if necessary. Moreover, D is a set of finite perimeter because

$$\begin{aligned}P(D, \Omega) &= P(u^{-1}(\{\nu_1\}) \cap w^{-1}(\{\nu_2\}), \Omega) \\ &\leq P(u^{-1}(\{\nu_1\}), \Omega) + P(w^{-1}(\{\nu_2\}), \Omega) \underset{(2.6)}{\leq} 2\mathrm{TV}(u) + 2\mathrm{TV}(w) < \infty,\end{aligned}$$

where the first inequality follows from Proposition 3.38 in [4]. Because $|D| > 0$, there exists some point $\bar{x} \in D$ of density 1, that is,

$$\lim_{r \searrow 0} \frac{|D \cap B_r(\bar{x})|}{|B_r(\bar{x})|} = 1.$$

This implies that the existence of a monotonically decreasing sequence $\{r_k\}_{k\in\mathbb{N}}$ such that $0 < l_k \coloneqq |D \cap B_{r_k}(\bar{x})|$ defines a monotonically decreasing sequence $\{l_k\}_{k\in\mathbb{N}}$ with $l_k \searrow 0$ as $k \to \infty$. Since $u_n \to u$ in $L^1(\Omega)$, we deduce that there exist $N_k \in \mathbb{N}$ such that

$$\begin{aligned}\|u - u_n\|_{L^1(\Omega)} &\leq l_k \quad \text{for all } n \geq N_k \text{ and} \\ N_k &< N_{k+1}\end{aligned}$$

hold for all $k \in \mathbb{N}$. Let $n \in \mathbb{N}$ with $n \geq N_1$. Then $n \in [N_k, N_{k+1})$ for some $k \in \mathbb{N}$ and we define

$$w_n(x) \coloneqq \begin{cases} u(x) & \text{if } x \in D \cap B_{r_k}(\bar{x}), \\ w(x) & \text{else} \end{cases}$$

for all $x \in \Omega$. We deduce

$$\begin{aligned}\|w_n - u_n\|_{L^1(\Omega)} &\leq \|u - u_n\|_{L^1(\Omega)} + \|w_n - u\|_{L^1(\Omega)} \\ &\leq l_k + \Delta - |\nu_1 - \nu_2| l_k \leq \Delta,\end{aligned}$$

where we have used that $|\nu_1 - \nu_2| \geq 1$ due to $\nu_1, \nu_2 \in \mathbb{Z}$ with $\nu_1 \neq \nu_2$. The construction gives $\{w_n\}_{n\in\mathbb{N}}$ with $w_n \overset{*}{\rightharpoonup} w$ in $\mathrm{BV}(\Omega)$. It remains to show $\mathrm{TV}(w_n) \to \mathrm{TV}(w)$.

To see this let $E_i \coloneqq w^{-1}(\{\nu_i\})$, $E_i^n \coloneqq (w_n)^{-1}(\{\nu_i\})$ for $i \in \{1, \dots, N\}$ and $n \in \mathbb{N}$, and $D^n \coloneqq D \cap B_{r_k}(\bar{x})$ for $n \in \mathbb{N}$ and the corresponding k that depends on n as above. There hold $E_1^n = E_1 \cup D^n$, $E_2^n = E_2 \setminus D^n$, $E_i^n = E_i$ for $i \geq 3$, and therefore also

$$\partial^* E_1^n \subset \partial^e E_1 \cup \partial^e D^n$$

and

$$\partial^* E_2^n \subset \partial^e E_2 \cup \partial^e D^n,$$

where the inclusions follow from Lemma 2.15. This yields

$$\partial^* E_1^n \cap \partial^* E_2^n \subset (\partial^e E_1 \cup \partial^e D^n) \cap (\partial^e E_2 \cup \partial^e D^n) = (\partial^e E_1 \cap \partial^e E_2) \cup \partial^e D^n$$

and, analogously,

$$\partial^* E_1^n \cap \partial^* E_i^n \subset (\partial^e E_1 \cap \partial^e E_i) \cup \partial^e D^n$$

and

$$\partial^* E_2^n \cap \partial^* E_i^n \subset (\partial^e E_2 \cap \partial^e E_i) \cup \partial^e D^n$$

for $i \geq 3$. We deduce

$$\begin{aligned}\mathcal{H}^{d-1}(\partial^* E_1^n \cap \partial^* E_2^n) &\leq \mathcal{H}^{d-1}((\partial^e E_1 \cap \partial^e E_2) \cup \partial^e D^n) \\ &= \mathcal{H}^{d-1}((\partial^* E_1 \cap \partial^* E_2) \cup \partial^* D^n) && \text{[4, Thm. 3.61]} \\ &\leq \mathcal{H}^{d-1}(\partial^* E_1 \cap \partial^* E_2) + \mathcal{H}^{d-1}(\partial^* D^n)\end{aligned}$$

and, analogously,

$$\mathcal{H}^{d-1}(\partial^* E_1^n \cap \partial^* E_i^n) \leq \mathcal{H}^{d-1}(\partial^* E_1 \cap \partial^* E_i) + \mathcal{H}^{d-1}(\partial^* D^n)$$

and

$$\mathcal{H}^{d-1}(\partial^* E_2^n \cap \partial^* E_i^n) \leq \mathcal{H}^{d-1}(\partial^* E_2 \cap \partial^* E_i) + \mathcal{H}^{d-1}(\partial^* D^n).$$

Then

$$\begin{aligned}
\mathrm{TV}(w_n) \underset{(2.5)}{=} &\sum_{i=1}^{N}\sum_{\ell=i+1}^{N}|\nu_i-\nu_\ell|\mathcal{H}^{d-1}(\partial^* E_i^n\cap\partial^* E_\ell^n)\\
&\leq \sum_{i=1}^{2}\sum_{\ell=i+1}^{N}|\nu_i-\nu_\ell|(\mathcal{H}^{d-1}(\partial^* E_i\cap\partial^* E_\ell)+\mathcal{H}^{d-1}(\partial^* D^n))\\
&+\sum_{i=3}^{N}\sum_{\ell=i+1}^{N}|\nu_i-\nu_\ell|\mathcal{H}^{d-1}(\partial^* E_i\cap\partial^* E_\ell)\\
&= \mathrm{TV}(w)+\sum_{i=1}^{2}\sum_{\ell=i+1}^{N}|\nu_i-\nu_\ell|\mathcal{H}^{d-1}(\partial^* D^n)\\
&\leq \mathrm{TV}(w)+2N\nu_{\max}\,\mathcal{H}^{d-1}(\partial^* D^n)\\
&\leq \mathrm{TV}(w)+2N\nu_{\max}\,(\mathcal{H}^{d-1}(\partial B_{r_k}(\bar{x}))+\mathcal{H}^{d-1}\llcorner\,\partial^* D(B_{r_k}(\bar{x})))
\end{aligned}$$

with $\nu_{\max} \coloneqq \max_{i,k\in\{1,\dots,N\}}|\nu_i-\nu_k| \geq 1$ and where the last inequality follows from (15.15) in [72]. Because the measure $\mathcal{H}^{d-1}\llcorner\,\partial^* D$ is a finite Radon measure, the term $\mathcal{H}^{d-1}\llcorner\,\partial^* D(B_{r_k}(\bar{x}))$ tends to zero as n and the corresponding k tend to infinity. Moreover, $\mathcal{H}^{d-1}(\partial B_{r_k}(\bar{x}))$ tends to zero as well. Together with the lower semicontinuity of TV we obtain $\mathrm{TV}(w_n)\to\mathrm{TV}(w)$. □

Corollary 3.17 ([77, Cor. 5.3])**.** *Let $F: L^2(\Omega)\to\mathbb{R}$ be continuously Fréchet differentiable and $\{u_n\}_{n\in\mathbb{N}}\subset \mathrm{BV}_{\mathrm{U}}(\Omega)$ such that $u_n\to u$ strictly in $\mathrm{BV}(\Omega)$. Then the claimed Γ-convergence in Theorem 3.16 holds with the choice $g_n=\nabla F(u_n)$.*

Proof. Since $\{u_n\}_{n\in\mathbb{N}}\subset \mathrm{BV}_{\mathrm{U}}(\Omega)$ and $u_n\to u$ strictly in $\mathrm{BV}_{\mathrm{U}}(\Omega)$, Lemma 2.19 implies that $u_n\to u\in L^2(\Omega)$, which in turn implies $\nabla F(u_n)\to\nabla F(u_n)$ in $L^2(\Omega)$. □

The proof of the lim inf inequality of Theorem 3.16 applies for the case $d=1$ as well. The proof of the lim sup inequality uses that the perimeter $P(B_{r_k}(x)) = \mathcal{H}^{d-1}(\partial B_{r_k}(x))$ of a ball $B_{r_k}(x)$ converges to zero if its radius $r_k>0$ is driven to zero, that is, $\mathcal{H}^{d-1}(\partial B_{r_k}(x))\searrow 0$ as $r_k\searrow 0$. This requires $d>1$ because in the one-dimensional case, the boundary of a ball always consists of two points independent of the radius, which yields $P(B_{r_k}(x))=\mathcal{H}^0(\partial B_{r_k}(x))=2$ for all radii $r_k>0$. In fact, the claim is not true for $d=1$, as we demonstrate in the following example.

Example 3.18 ([77, Expl. 5.4])**.** Let $\Omega\coloneqq(-2,2)\subset\mathbb{R}$. Let $U\coloneqq\{0,1\}$ and $w\equiv 0\in \mathrm{BV}_{\mathrm{U}}(\Omega)$. Let $u\coloneqq\chi_{[-1,1]}\in\mathrm{BV}_{\mathrm{U}}(\Omega)$. Let $\Delta=2$. Let $u_n\coloneqq\chi_{[-1-\frac{1}{n},1+\frac{1}{n}]}\in\mathrm{BV}_{\mathrm{U}}(\Omega)$. Then $u_n\to u$ strictly in $\mathrm{BV}(\Omega)$ because $\|u-u_n\|_{L^1(\Omega)}=\frac{2}{n}\to 0$ as $n\to\infty$ and $\mathrm{TV}(u_n)=2=\mathrm{TV}(u)$ for all $n\in\mathbb{N}$. Let $g_n\rightharpoonup g$ in $L^2(\Omega)$. Let T_n for $n\in\mathbb{N}$ and T be defined as in Theorem 3.16. There hold $\|u-w\|_{L^1(\Omega)}=\Delta$ and $\|u_n-w\|_{L^1(\Omega)}=$

$2+\frac{2}{n} > \Delta$. We need to approximate w by a sequence $\{w_n\}_{n\in\mathbb{N}} \subset \mathrm{BV}_{\mathrm{U}}(\Omega)$ such that $\|u_n - w_n\|_{L^1(\Omega)} \le \Delta$. Since $\|w-1\|_{L^1(\Omega)} = 4$ and $\mathrm{TV}(v) \ge 1$ for all $v \in \mathrm{BV}_{\mathrm{U}}(\Omega)$ that are not constant almost everywhere on Ω, this yields that there must hold $\mathrm{TV}(w_n) \ge 1$ for all $n \in \mathbb{N}$ large enough. We obtain

$$\limsup_{n\to\infty} T_n(w_n) \ge \alpha > 0 = T(w)$$

for any sequence $\{w_n\}_{n\in\mathbb{N}} \subset \mathrm{BV}_{\mathrm{U}}(\Omega)$ with $w_n \stackrel{*}{\rightharpoonup} w$ in $\mathrm{BV}(\Omega)$ such that $\|u_n - w_n\|_{L^1(\Omega)} \le \Delta$. Thus, the lim sup inequality is violated.

Next, we apply Theorem 3.14 to $\mathrm{TR}(\bar{u}, g, \Delta)$. In particular, it follows that r-optimality of $\bar{u}$ for $\mathrm{TR}(\bar{u}, \nabla F(\bar{u}), \Delta)$ implies L-stationary for (P).

Proposition 3.19 ([77, Prop. 5.5])**.** *Let $g \in C(\bar{\Omega})$. Let $\Delta > 0$. Let $\{E_1, \dots, E_N\}$ be a Caccioppoli partition of Ω, and $\tilde{u} = \sum_{i=1}^N \nu_i \chi_{E_i}$ with $\|\bar{u} - \tilde{u}\|_{L^1(\Omega)} < \Delta$. If $\tilde{u}$ is r-optimal for $\mathrm{TR}(\bar{u}, g, \Delta)$ for some $r > 0$, then for all $\psi \in C_c^\infty(\Omega; \mathbb{R}^d)$ it holds that*

$$\begin{aligned}\sum_{i=1}^N \nu_i & \int_{\partial^* E_i \cap \Omega} (-g(x))\psi(x) \cdot n_{E_i}(x) \,\mathrm{d}\mathcal{H}^{d-1}(x) \\ &= \alpha \sum_{i=1}^N \sum_{j=i+1}^N |\nu_i - \nu_j| \int_{\partial^* E_i \cap \partial^* E_j} \operatorname{div}_{E_i} \psi(x) \,\mathrm{d}\mathcal{H}^{d-1}(x)\end{aligned}$$

In particular, it holds for all $i, j \in \{1, \dots, N\}$ with $i \ne j$ that

$$\begin{aligned}(\nu_i - \nu_j) & \int_{\partial^* E_i \cap \partial^* E_j} (-g(x))\psi(x) \cdot n_{E_i}(x) \,\mathrm{d}\mathcal{H}^{d-1}(x) \\ &= \alpha |\nu_i - \nu_j| \int_{\partial^* E_i \cap \partial^* E_j} \operatorname{div}_{E_i} \psi(x) \,\mathrm{d}\mathcal{H}^{d-1}(x)\end{aligned}$$

for all $\psi \in C_c^\infty(E_i \cup E_j, \mathbb{R}^d)$.

Proof. Let $\tilde{u}$ be r-optimal for $\mathrm{TR}(\bar{u}, g, \Delta)$ for some $r > 0$. Then $\tilde{u}$ is r_0-optimal for $0 < r_0 \le \min\{r, \Delta - \|\bar{u} - \tilde{u}\|_{L^1(\Omega)}\}$, such the constraint $\|u - \bar{u}\|_{L^1(\Omega)} \le \Delta$ from $\mathrm{TR}(\bar{u}, g, \Delta)$ is already included in $\|u - \tilde{u}\|_{L^1(\Omega)} \le r_0$ by

$$\begin{aligned}\|u - \bar{u}\|_{L^1(\Omega)} &\le \|u - \tilde{u}\|_{L^1(\Omega)} + \|\tilde{u} - \bar{u}\|_{L^1(\Omega)} \\ &\le \min\{r, \Delta - \|\bar{u} - \tilde{u}\|_{L^1(\Omega)}\} + \|\tilde{u} - \bar{u}\|_{L^1(\Omega)} \le \Delta\end{aligned}$$

for all $u \in \mathrm{BV}_{\mathrm{U}}(\Omega)$ with $\|u - \tilde{u}\|_{L^1(\Omega)} \le r_0$. This gives that $\tilde{u}$ is r_0-optimal for (P) with the choice $F(u) \coloneqq (g, u)_{L^2(\Omega)}$ for $u \in L^2(\Omega)$, which yields $\nabla F(\tilde{u}) = g$ and $\nabla^2 F(\tilde{u}) = 0$. In particular, Assumption 3.11 is satisfied. Then the claim follows from Theorem 3.14 with r_0 for r. □

3.3.2 Algorithm statement

We will adapt the trust-region algorithm from [70] to the multi-dimensional case and apply it to (P). In each iteration, the algorithm solves (TR) for a given trust-region radius $\Delta > 0$, $\bar{u} \in \mathrm{BV_U}(\Omega)$, and with $g = \nabla F(\bar{u})$, that is,

$$\mathrm{TR}(\bar{u}, \nabla F(\bar{u}), \Delta) = \begin{cases} \min\limits_{u \in L^2(\Omega)} & (\nabla F(\bar{u}), u - \bar{u})_{L^2(\Omega)} + \alpha \mathrm{TV}(u) - \alpha \mathrm{TV}(\bar{u}) \\ \text{s.t.} & \|u - \bar{u}\|_{L^1(\Omega)} \leq \Delta \text{ and } u(x) \in U \text{ for a.a. } x \in \Omega. \end{cases}$$

We denote an optimal solution by $\tilde{u} \in \mathrm{BV_U}(\Omega)$ and define the predicted reduction by

$$\mathrm{pred}(\bar{u}, \Delta) \coloneqq (\nabla F(\bar{u}), \bar{u} - \tilde{u})_{L^2(\Omega)} + \alpha \mathrm{TV}(\bar{u}) - \alpha \mathrm{TV}(\tilde{u}),$$

which corresponds to the negative optimal value of $\mathrm{TR}(\bar{u}, \nabla F(\bar{u}), \Delta)$. If the predicted reduction $\mathrm{pred}(\bar{u}, \Delta)$ is zero, then the algorithm terminates because $\bar{u}$ is an L-stationary point for (P). This follows from the fact that the predicted reduction is bounded from below by zero because $\bar{u}$ is always a feasible solution to $\mathrm{TR}(\bar{u}, \nabla F(\bar{u}), \Delta)$. If the optimal value of $\mathrm{TR}(\bar{u}, \nabla F(\bar{u}), \Delta)$ is 0, then $\bar{u}$ is an optimal solution to $\mathrm{TR}(\bar{u}, \nabla F(\bar{u}), \Delta)$ and the L-stationarity of $\bar{u}$ for (P) follows from Proposition 3.19.

If the predicted reduction $\mathrm{pred}(\bar{u}, \Delta)$ is strictly positive, the algorithm checks the following sufficient decrease condition from [74] that reads

$$\mathrm{ared}(\bar{u}, \tilde{u}) \geq \sigma \, \mathrm{pred}(\bar{u}, \Delta), \tag{3.13}$$

where $\sigma \in (0, 1)$ is a fixed value and

$$\mathrm{ared}(\bar{u}, \tilde{u}) \coloneqq F(\bar{u}) + \alpha \mathrm{TV}(\bar{u}) - F(\tilde{u}) - \alpha \mathrm{TV}(\tilde{u})$$

is the actual reduction that is achieved by $\tilde{u}$. We highlight that the actual reduction can be negative in contrast to the predicted reduction. If the sufficient decrease condition (3.13) is fulfilled, we update $\bar{u} \leftarrow \tilde{u}$ and reset the trust-region radius $\Delta \leftarrow \Delta_0$ to the fixed reset-radius $\Delta_0 > 0$.

If the sufficient decrease condition (3.13) is not fulfilled, the trust-region radius Δ is halved, that is, $\Delta \leftarrow \frac{\Delta}{2}$ and a new iteration is started by solving $\mathrm{TR}(\bar{u}, \nabla F(\bar{u}), \Delta)$ with the new trust-region radius Δ but with the same $\bar{u}$.

Similar reset strategies for the trust region radius are also used in [53, 68].

Algorithm 1 Sequential linear integer programming method (SLIP)

Input: F sufficiently regular, $\Delta_0 > 0$, $u_0 \in \mathrm{BV_U}(\Omega)$, $\sigma \in (0,1)$.

for $n = 1, \dots$ **do**
  $k \leftarrow 0$
  $\Delta_{n,0} \leftarrow \Delta_0$
  while not sufficient decrease according to (3.13) **do**
    $\tilde{u}_{n,k} \leftarrow$ minimizer of (TR) with $\Delta = \Delta_{n,k}$, $\bar{u} = \bar{u}_{n-1}$, and $g = \nabla F(\bar{u}_{n-1})$.
    $\mathrm{pred}(\bar{u}_{n-1}, \Delta_{n,k}) \leftarrow (\nabla F(\bar{u}_{n-1}), \bar{u}_{n-1} - \tilde{u}_{n,k})_{L^2} + \alpha \mathrm{TV}(\bar{u}_{n-1}) - \alpha \mathrm{TV}(\tilde{u}_{n,k})$
    $\mathrm{ared}(\bar{u}_{n-1}, \tilde{u}_{n,k}) \leftarrow F(\bar{u}_{n-1}) + \alpha \mathrm{TV}(\bar{u}_{n-1}) - F(\tilde{u}_{n,k}) - \alpha \mathrm{TV}(\tilde{u}_{n,k})$
    if $\mathrm{pred}(\bar{u}_{n-1}, \Delta_{n,k}) \le 0$ **then**
      Terminate. The predicted reduction for $\bar{u}_{n-1}$ is zero.
    else if not sufficient decrease according to (3.13) **then**
      $k \leftarrow k + 1$
      $\Delta_{n,k} \leftarrow \frac{\Delta_{n,k-1}}{2}$.
    else
      $\bar{u}_n \leftarrow \tilde{u}_{n,k}$
    end if
  end while
end for

3.4 Convergence of Algorithm 1

We analyze the convergence behavior of the function space Algorithm 1 under Assumption 3.11, which corresponds to Assumption 4.1 in [70]. The Hessian regularity assumed in Assumption 3.11 is already required for Theorem 3.14. We analyze the inner loop in Section 3.4.1 and the outer loop in Section 3.4.2, which gives that all accumulation points produced by Algorithm 1 are L-stationary.

3.4.1 Analysis of the inner loop of Algorithm 1

First we will prove in Lemma 3.20 that for each local variation $(f_t)_{t\in(-\varepsilon,\varepsilon)}$, we can restrict the parameter t to a sufficiently small interval $t \in (-\varepsilon_1, \varepsilon_1)$ such that we may apply the local variation to the level sets of $u \in \mathrm{BV_U}(\Omega)$ while preserving feasibility to the trust-region subproblem $\mathrm{TR}(\bar{u}, g, \Delta)$.

Lemma 3.20 ([77, Lem. 6.1]). *Let $g \in C(\bar{\Omega})$, and $\Delta > 0$. Let $\{E_1, \dots, E_N\}$ be a Caccioppoli partition of Ω, and $\bar{u} = \sum_{i=1}^N \nu_i \chi_{E_i}$ with $\mathrm{TV}(\bar{u}) < \infty$. Let $(f_t)_{t\in(-\varepsilon,\varepsilon)}$ be the local variation defined by $f_t \coloneqq I + t\psi$ for $\psi \in C_c^\infty(\Omega; \mathbb{R}^d)$. Then there exist $\varepsilon_1 > 0$ such that $f_t^\# \bar{u}$ is feasible for $\mathrm{TR}(\bar{u}, g, \Delta)$ for all $t \in (-\varepsilon_1, \varepsilon_1)$.*

Proof. Let $\varepsilon_0 > 0$ and $\bar{C} > 0$ denote the constants that are asserted in Lemma 3.8. We choose $\varepsilon_1 \coloneqq \min\{\varepsilon_0, {}^{\Delta}/_{\bar{C}}\}$. Then there holds

$$\|f_t^\# \bar{u} - \bar{u}\|_{L^1(\Omega)} \le \bar{C}|t| \le \Delta$$

for all $t \in (-\varepsilon_1, \varepsilon_1)$, that is, $f_t^{\#}\bar{u}$ is feasible for $\mathrm{TR}(\bar{u}, g, \Delta)$. $\square$

The following lemma states that the inner loop always terminates after finitely many iterations if the current iterate $\bar{u}_{n-1}$ is not L-stationary. We use local variations to obtain a sufficient decrease in the trust-region subproblem that eventually implies acceptance of a step in case of violation of L-stationarity. Since the outer iteration index n is fixed within the inner iteration, we leave it out for better clarity, that is we abbreviate $\bar{u} \coloneqq \bar{u}_{n-1}$, $\tilde{u}_k \coloneqq \tilde{u}_{n,k}$, and $\Delta_k \coloneqq \Delta_{n,k}$.

Lemma 3.21 ([77, Lem. 6.2]). *Let Assumption 3.11 hold and $\sigma \in (0,1)$ be given. Let $\bar{u} = \sum_{i=1}^N \nu_i \chi_{E_i}$ for a Caccioppoli partition $\{E_1, \ldots, E_N\}$ of Ω and $\nabla F(\bar{u}) \in C(\bar{\Omega})$. Let a sequence $\{\Delta_k\}_{k\in\mathbb{N}} \subset \mathbb{R}_{>0}$ of trust region radii be given with $\Delta_k \searrow 0$ as $k \to \infty$ and $\tilde{u}_k$ minimize $\mathrm{TR}(\bar{u}, \nabla F(\bar{u}), \Delta_k)$ for $k \in \mathbb{N}$. Then at least one of the following statements holds true.*

1. *The function $\bar{u}$ is L-stationary.*

2. *There exists $k_0 \in \mathbb{N}$ such that the optimal value of $\mathrm{TR}(\bar{u}, \nabla F(\bar{u}), \Delta_{k_0})$ is zero.*

3. *There exists $k_0 \in \mathbb{N}$ such that (3.13) holds, that is $\mathrm{ared}(\bar{u}, \tilde{u}_{k_0}) \geq \sigma\, \mathrm{pred}(\bar{u}, \Delta_{k_0})$.*

Proof. We first prove that Outcome 2 implies Outcome 1. To this end, we assume that Outcome 2 holds true, that is, there exists $k_0 \in \mathbb{N}$ such that the optimal objective of $\mathrm{TR}(\bar{u}, \nabla F(\bar{u}), \Delta_{k_0})$ is zero. Since $\bar{u}$ has objective value zero, it is optimal for $\mathrm{TR}(\bar{u}, \nabla F(\bar{u}), \Delta_{k_0})$ and therefore L-stationary by Proposition 3.19, which corresponds to Outcome 1.

It remains to show that Outcome 3 holds true if Outcome 1 does not. That means, we assume that Outcome 1 does not hold true, that is, we assume that $\bar{u}$ is not L-stationary. We abbreviate

$$a_k \coloneqq \mathrm{ared}(\bar{u}, \tilde{u}_k) \quad \text{and} \quad p_k \coloneqq \mathrm{pred}(\bar{u}, \Delta_k)$$

for $k \in \mathbb{N}$. We prove that there exists $k_0 \in \mathbb{N}$ such that the sufficient decrease condition $a_{k_0} \geq \sigma p_{k_0}$ is fulfilled. Since $\bar{u}$ is not L-stationary, there exists $\psi \in C_c^\infty(\Omega; \mathbb{R}^d)$ with $\operatorname{supp} \psi \neq \emptyset$ and $\eta > 0$ such that

$$\begin{aligned}\sum_{i=1}^N \nu_i \int_{\partial^* E_i \cap \Omega} &(-\nabla F(\bar{u}))(x)(\psi(x) \cdot n_{E_i}(x)) \,\mathrm{d}\mathcal{H}^{d-1}(x) \\ &- \alpha \sum_{i=1}^N \sum_{j=i+1}^N |\nu_i - \nu_j| \int_{\partial^* E_i \cap \partial^* E_j} \operatorname{div}_{E_i} \psi(x) \,\mathrm{d}\mathcal{H}^{d-1}(x) > \eta.\end{aligned}$$

Let $(f_t)_{t\in(-\varepsilon,\varepsilon)}$ denote the local variation defined by $f_t := I+t\psi$. By Lemma 3.20 there exists for each $k \in \mathbb{N}$ some $\varepsilon_k \in (0,\varepsilon)$ such that $f_t^{\#}\bar{u}$ is feasible for $\mathrm{TR}(\bar{u}, \nabla F(\bar{u}), \Delta_k)$ for all $t \in [0, \varepsilon_k]$.

Since the constants $\bar{C}$ and $\varepsilon > 0$ from Lemma 3.8 are fixed by the choice of ψ and $\bar{u}$, the construction of ε_1 in the proof of Lemma 3.20 gives that we may always choose $\varepsilon_k > 0$ so that $\|f_{\varepsilon_k}^{\#}\bar{u} - \bar{u}\|_{L^1(\Omega)} = \Delta_k$ holds for all k sufficiently large because $\min\{\varepsilon_0, {}^{\Delta_k}/_{\bar{C}}\} = {}^{\Delta_k}/_{\bar{C}}$ for k sufficiently large.

Assumption 3.11 allows us to apply Taylor's theorem to F, which yields

$$F(\tilde{u}_k) = F(\bar{u}) + \nabla F(\bar{u})(\tilde{u}_k - \bar{u}) + \frac{1}{2}\nabla^2 F(\xi_k)(\tilde{u}_k - \bar{u}, \tilde{u}_k - \bar{u})$$

with some ξ_k in the line segment between $\tilde{u}_k$ and $\bar{u}$ for $k \in \mathbb{N}$. This gives

$$a_k = p_k - \frac{1}{2}\nabla^2 F(\xi_k)(\tilde{u}_k - \bar{u}, \tilde{u}_k - \bar{u}) \geq p_k - \frac{C}{2}\|\tilde{u}_k - \bar{u}\|^2_{L^1(\Omega)}$$

with the constant $C > 0$ from Assumption 3.11. We deduce for $t \in (0, \varepsilon_k]$

$$\begin{aligned}
a_k &\geq p_k - \frac{C}{2}\|\tilde{u}_k - \bar{u}\|^2_{L^1(\Omega)} \\
&= \sigma p_k - (1-\sigma)\big((\nabla F(\bar{u}), \tilde{u}_k - \bar{u})_{L^2(\Omega)} + \alpha \mathrm{TV}(\tilde{u}_k) - \alpha \mathrm{TV}(\bar{u})\big) \\
&\quad - \frac{C}{2}\|\tilde{u}_k - \bar{u}\|^2_{L^1(\Omega)} \\
&\geq \sigma p_k - (1-\sigma)\Big((\nabla F(\bar{u}), f_{\varepsilon_k}^{\#}\bar{u} - \bar{u})_{L^2(\Omega)} + \alpha \mathrm{TV}(f_{\varepsilon_k}^{\#}\bar{u}) - \alpha \mathrm{TV}(\bar{u})\Big) \\
&\quad - \frac{C}{2}\|\tilde{u}_k - \bar{u}\|^2_{L^1(\Omega)} \\
&\geq \sigma p_k + (1-\sigma)\left(\varepsilon_k \eta + o(\varepsilon_k)\right) - \frac{C}{2}\|\tilde{u}_k - \bar{u}\|^2_{L^1(\Omega)} \\
&\geq \sigma p_k + (1-\sigma)\left(\varepsilon_k \eta + o(\varepsilon_k)\right) - \frac{C}{2}\kappa^2\varepsilon_k^2
\end{aligned}$$

for some $\kappa > 0$. In particular, the second inequality follows from the optimality of $\tilde{u}_k$ for $\mathrm{TR}(\bar{u}, \nabla F(\bar{u}), \Delta_k)$, the third inequality follows from Lemmas 3.3 and 3.5, and the fourth from

$$-\|\tilde{u}_k - \bar{u}\|^2_{L^1(\Omega)} \geq -\Delta_k^2 = -\|f_{\varepsilon_k}^{\#}\bar{u} - \bar{u}\|^2_{L^1(\Omega)} \geq -\kappa^2\varepsilon_k^2,$$

where the existence of κ follows from Lemma 3.8. Because $(1-\sigma)\varepsilon_k\eta > 0$ and $(1-\sigma)\varepsilon_k\eta$ eventually dominates the terms $(1-\sigma)o(\varepsilon_k)$ and $-\frac{C}{2}\kappa^2\varepsilon_k^2$, there exists $k_0 \in \mathbb{N}$ such that $a_{k_0} \geq \sigma p_{k_0}$, which corresponds to Outcome 2. $\square$

We can now deduce that the inner loop of Algorithm 1 terminates after finitely many iterations if the current iterate is not L-stationary.

Corollary 3.22 ([77, Cor. 6.3]). *Let Assumption 3.11 hold. Let $\bar{u}_{n-1} \in \mathrm{BV}_{\mathrm{U}}(\Omega)$ for some $n \in \mathbb{N}$ be an iterate produced by Algorithm 1 and let $\nabla F(\bar{u}_{n-1}) \in C(\bar{\Omega})$ be fulfilled. Then iteration n satisfies one of the following outcomes.*

1. *The inner loop terminates after finitely many iterations and*

   (a) the sufficient decrease condition (3.13) *is satisfied or*

   (b) the predicted reduction is zero and the iterate $\bar{u}_{n-1}$ is L-stationary.

2. *The inner loop does not terminate, and the iterate $\bar{u}_{n-1}$ is L-stationary.*

Proof. We apply Lemma 3.21 with the choices $\Delta_k = \Delta_{n,k}$ and $\bar{u} = \bar{u}_{n-1}$. □

3.4.2 Analysis of the outer loop of Algorithm 1

To complete our convergence analysis for Algorithm 1, we need to consider the case that Algorithm 1 does not terminate after finitely many iterations. For that case, we prove that the accumulation points of the sequence of iterates $\{\bar{u}_n\}_{n\in\mathbb{N}}$ produced by Algorithm 1 are L-stationary under Assumption 3.11.

Theorem 3.23 ([77, Thm. 6.4]). *Let F be bounded from below. Let Assumption 3.11 hold. Let the iterates $\bar{u}_n$ for $n \in \mathbb{N}$ be produced by Algorithm 1. Let $\nabla F(\bar{u}_n) \in C(\bar{\Omega})$ for all $n \in \mathbb{N}$. Then all iterates are feasible for* (P) *and the objective values $J(\bar{u}_n)$ for $n \in \mathbb{N}$ are monotonically decreasing. Moreover, one of the following mutually exclusive outcomes holds:*

1. *The number of iterates $\bar{u}_n$ is finite. The final element $\bar{u}_M$ solves the trust-region subproblem $\mathrm{TR}(\bar{u}_M, \nabla F(\bar{u}_M), \Delta)$ for some $\Delta > 0$ and is L-stationary.*

2. *The number of iterates $\bar{u}_n$ is finite and the inner loop does not terminate for the final element $\bar{u}_M$, which is L-stationary.*

3. *The sequence $\{\bar{u}_n\}_{n\in\mathbb{N}}$ has a weak-$*$ accumulation point in $\mathrm{BV}(\Omega)$ and every weak-$*$ accumulation point of $\{\bar{u}_n\}_{n\in\mathbb{N}}$ is feasible and strict. If $\bar{u}$ is a weak-$*$ accumulation point of $\{\bar{u}_n\}_{n\in\mathbb{N}}$ that satisfies $\nabla F(\bar{u}) \in C(\bar{\Omega})$, then it is L-stationary.*

   If the trust-region radii are bounded away from zero for a subsequence $\{\bar{u}_{n_\ell}\}_{\ell\in\mathbb{N}}$, that is, if

   $$0 < \underline{\Delta} := \liminf_{\ell\to\infty} \min_k \Delta_{n_\ell+1,k}$$

   and $\bar{u}$ is a weak-$*$ accumulation point of $\{\bar{u}_{n_\ell}\}_{\ell\in\mathbb{N}}$ with $\nabla F(\bar{u}) \in C(\bar{\Omega})$, then $\bar{u}$ solves $\mathrm{TR}(\bar{u}, \nabla F(\bar{u}), \underline{\Delta}/2)$.

Proof. Parts of this proof are identical to the proof of Theorem 4.23 in [70] for the case $d = 1$ without any change other than the definition of L-stationarity. For the sake of completeness, we repeat and adapt them for the case $d \geq 2$.

The iterates $\{\bar{u}_n\}_{n\in\mathbb{N}}$ produced by Algorithm 1 are the minimizers to $(\mathrm{TR})(\bar{u}_{n-1}, \nabla F(\bar{u}_{n-1}), \Delta_{n,k})$, which exist due to Proposition 3.15. Since the feasible set of $\mathrm{TR}(\bar{u}_{n-1}, \nabla F(\bar{u}_{n-1}), \Delta_{n,k})$ is included in the feasible set of (P), the minimizer is also feasible for (P).

Moreover, a minimizer $\tilde{u}_{n,k}$ is accepted as the next iterate $\bar{u}_n$ if $\mathrm{pred}(\bar{u}_{n-1}, \Delta_{n,k}) > 0$ and if it fulfills the sufficient decrease condition (3.13), that is,

$$\begin{aligned}\mathrm{ared}(\bar{u}_{n-1}, \tilde{u}_{n,k}) &= F(\bar{u}_{n-1}) + \alpha\mathrm{TV}(\bar{u}_{n-1}) - F(\tilde{u}_{n,k}) - \alpha\mathrm{TV}(\tilde{u}_{n,k})\\ &\geq \sigma\, \mathrm{pred}(\bar{u}_{n-1}, \Delta_{n,k}) > 0,\end{aligned}$$

which yields

$$J(\bar{u}_{n-1}) = F(\bar{u}_{n-1}) + \alpha\mathrm{TV}(\bar{u}_{n-1}) > F(\bar{u}_n) + \alpha\mathrm{TV}(\bar{u}_n) = J(\bar{u}_n),$$

that is, the objective values $J(\bar{u}_n)$ are monotonically decreasing for $n \in \mathbb{N}$.

As in the proof of Theorem 4.23 in [70], we may restrict to the case that Outcomes 1 and 2 do not hold true and prove Outcome 3 in this case. To this end, we substitute Lemma 4.19 in [70] by Lemma 3.21 in the respective argument. We split the proof that Outcome 3 holds into four parts.

Outcome 3 (1) existence and feasibility of weak-$*$ accumulation points: This follows exactly as in the proof of Theorem 4.23 [70]. We already proved that the objective values are monotonically decreasing. Since F is bounded from below by Assumption 1.1, this yields that the sequence $\{\mathrm{TV}(\bar{u}_n)\}_{n\in\mathbb{N}}$ is bounded. Moreover, since $\{\bar{u}_n\}_{n\in\mathbb{N}} \subset \mathrm{BV}_{\mathrm{U}}(\Omega)$, the sequence $\{\bar{u}_n\}_{n\in\mathbb{N}}$ is bounded in $L^\infty(\Omega)$ and therefore also in $L^1(\Omega)$. By Theorem 2.8, there exists a subsequence of $\{\bar{u}_n\}_{n\in\mathbb{N}}$ that converges weakly-$*$ in $\mathrm{BV}(\Omega)$. Using that $\mathrm{BV}_{\mathrm{U}}(\Omega)$ is sequentially closed in the weak-$*$ topology of $\mathrm{BV}(\Omega)$ by Lemma 2.20 yields that each accumulation point is an element of $\mathrm{BV}_{\mathrm{U}}(\Omega)$ and therefore feasible for (P).

Outcome 3 (2) weak-$*$ accumulation points are strict: We follow the idea of a contradictory argument from the proof of Theorem 4.23 in [70] and assume that there exists a weak-$*$ accumulation point $\bar{u}$ of $\{\bar{u}_n\}_{n\in\mathbb{N}}$ with corresponding subsequence $\bar{u}_{n_\ell} \overset{*}{\rightharpoonup} \bar{u}$ in $\mathrm{BV}(\Omega)$ as $\ell \to \infty$ such that $\mathrm{TV}(\bar{u}) < \liminf_{\ell\to\infty} \mathrm{TV}(\bar{u}_{n_\ell})$, that is, the convergence is not strict. We define $\delta \coloneqq \frac{1}{2}(\liminf_{\ell\to\infty} \mathrm{TV}(\bar{u}_{n_\ell}) - \mathrm{TV}(\bar{u}))$ but cannot assume the inequality $\delta \geq \frac{1}{2}$ as in the proof of Theorem 4.23 in [70] because $\mathrm{TV}(\bar{u}_{n_\ell}) - \mathrm{TV}(\bar{u}) \in \mathbb{Z}$ is generally not true for our case $d \geq 2$. By Assumption 3.11, F and ∇F are continuous with respect to convergence in $L^2(\Omega)$. Hence, they are also continuous with respect to convergence in $L^1(\Omega)$ in the subset $\mathrm{BV}_{\mathrm{U}}(\Omega)$ because Lemma 2.19 yields that convergence in $L^1(\Omega)$ implies convergence in $L^2(\Omega)$ due to the restriction to values in U. The fact that $\Delta_{n_\ell+1,k} \to 0$ for $k \to \infty$ independently of ℓ gives that we obtain the existence of some $\ell_0 \in \mathbb{N}$ and $k_0 \in \mathbb{N}$ such that for all $\ell \geq \ell_0$ and $k \geq k_0$

$$(3.14) \quad |(\nabla F(\bar{u}_{n_\ell}), \bar{u}_{n_\ell} - u)_{L^2(\Omega)}| \leq \frac{1-\sigma}{3-\sigma}\alpha\delta \quad \text{and} \quad |F(\bar{u}_{n_\ell}) - F(u)| \leq \frac{1-\sigma}{3-\sigma}\alpha\delta$$

hold for all u that are feasible for $\mathrm{TR}(\bar{u}_{n_\ell}, \nabla F(\bar{u}_{n_\ell}), \Delta_{n_\ell+1,k})$ due to the constraints $u(x) \in U$ for almost all $x \in \Omega$ and $\|u - \bar{u}_{n_\ell}\|_{L^1(\Omega)} \leq \Delta_{n_\ell+1,k}$.

If $\bar{u}$ is feasible for $\mathrm{TR}(\bar{u}_{n_\ell}, \nabla F(\bar{u}_{n_\ell}), \Delta_{n_\ell+1,k})$, the optimality of $\tilde{u}_{n_\ell+1,k}$ for $\mathrm{TR}(\bar{u}_{n_\ell}, \nabla F(\bar{u}_{n_\ell}), \Delta_{n_\ell+1,k})$, see line 5 in Algorithm 1, implies

$$\begin{aligned}
\mathrm{pred}(\bar{u}_{n_\ell}, \Delta_{n_\ell+1,k}) &= (\nabla F(\bar{u}_{n_\ell}), \bar{u}_{n_\ell} - \tilde{u}_{n_\ell+1,k})_{L_2(\Omega)} + \alpha \mathrm{TV}(\bar{u}_{n_\ell}) - \alpha \mathrm{TV}(\tilde{u}_{n_\ell+1,k}) \\
&\geq (\nabla F(\bar{u}_{n_\ell}), \bar{u}_{n_\ell} - \bar{u})_{L^2(\Omega)} + \alpha(\mathrm{TV}(\bar{u}_{n_\ell}) - \mathrm{TV}(\bar{u})).
\end{aligned}$$

By the definition of δ, there exists $\ell_1 \geq \ell_0$ such that for all $\ell \geq \ell_1$ there holds $\mathrm{TV}(\bar{u}_{n_\ell}) - \mathrm{TV}(\bar{u}) \geq \delta$. This gives together with (3.14) and $\sigma \in (0,1)$ that

$$\begin{aligned}
\mathrm{pred}(\bar{u}_{n_\ell}, \Delta_{n_\ell+1,k}) &\geq (\nabla F(\bar{u}_{n_\ell}), \bar{u}_{n_\ell} - \bar{u})_{L^2(\Omega)} + \alpha\delta \\
&\geq -\frac{1-\sigma}{3-\sigma}\alpha\delta + \alpha\delta \geq \frac{2}{3}\alpha\delta > 2\frac{1-\sigma}{3-\sigma}\alpha\delta
\end{aligned}$$

if $\bar{u}$ is feasible for $\mathrm{TR}(\bar{u}_{n_\ell}, \nabla F(\bar{u}_{n_\ell}), \Delta_{n_\ell+1,k})$. Because $\bar{u}_{n_\ell} \stackrel{*}{\rightharpoonup} \bar{u}$ in $\mathrm{BV}(\Omega)$ implies $\bar{u}_{n_\ell} \to \bar{u}$ in $L^1(\Omega)$, there exists $\ell_2 \geq \ell_1$ such that $\|\bar{u}_{n_\ell} - \bar{u}\|_{L^1(\Omega)} \leq \Delta_{n_\ell+1,k_0}$ for all $\ell \geq \ell_2$ such that the function $\bar{u}$ is feasible for $\mathrm{TR}(\bar{u}_{n_\ell}, \nabla F(\bar{u}_{n_\ell}), \Delta_{n_\ell+1,k_0})$ for all $\ell \geq \ell_2$ for fixed k_0.

By (3.14), there holds

$$\begin{aligned}
\mathrm{ared}(\bar{u}_{n_\ell}, \tilde{u}_{n_\ell+1,k_0}) &= F(\bar{u}_{n_\ell}) + \alpha \mathrm{TV}(\bar{u}_{n_\ell}) - F(\tilde{u}_{n_\ell+1,k_0}) - \alpha \mathrm{TV}(\tilde{u}_{n_\ell+1,k_0}) \\
&= (\nabla F(\bar{u}_{n_\ell}), \bar{u}_{n_\ell} - \tilde{u}_{n_\ell+1,k_0})_{L^2(\Omega)} + \alpha \mathrm{TV}(\bar{u}_{n_\ell}) - \alpha \mathrm{TV}(\tilde{u}_{n_\ell+1,k_0}) \\
&\quad + F(\bar{u}_{n_\ell}) - F(\tilde{u}_{n_\ell+1,k_0}) - (\nabla F(\bar{u}_{n_\ell}), \bar{u}_{n_\ell} - \tilde{u}_{n_\ell+1,k_0})_{L^2(\Omega)} \\
&\geq \mathrm{pred}(\bar{u}_{n_\ell}, \Delta_{n_\ell+1,k_0}) - 2\frac{1-\sigma}{3-\sigma}\alpha\delta.
\end{aligned}$$

Thus, if the inner loop reaches iteration k_0 for $\ell \geq \ell_2$, we obtain

$$\begin{aligned}
\frac{\mathrm{ared}(\bar{u}_{n_\ell}, \tilde{u}_{n_\ell+1,k_0})}{\mathrm{pred}(\bar{u}_{n_\ell}, \Delta_{n_\ell+1,k_0})} &\geq \frac{\mathrm{pred}(\bar{u}_{n_\ell}, \Delta_{n_\ell+1,k_0}) - 2\frac{1-\sigma}{3-\sigma}\alpha\delta}{\mathrm{pred}(\bar{u}_{n_\ell}, \Delta_{n_\ell+1,k_0})} \\
&\geq \frac{\alpha\delta - \frac{1-\sigma}{3-\sigma}\alpha\delta - 2\frac{1-\sigma}{3-\sigma}\alpha\delta}{\alpha\delta - \frac{1-\sigma}{3-\sigma}\alpha\delta} \\
&= \frac{1 - 3\frac{1-\sigma}{3-\sigma}}{1 - \frac{1-\sigma}{3-\sigma}} = \sigma,
\end{aligned}$$

where the second inequality is due to the fact that $p \mapsto \frac{p-c}{p}$ is monotone ($c = 2\frac{1-\sigma}{3-\sigma}$). Consequently,

$$\operatorname{ared}(\bar{u}_{n_\ell}, \tilde{u}_{n_\ell+1,k_0}) \geq \sigma \operatorname{pred}(\bar{u}_{n_\ell}, \Delta_{n_\ell+1,k_0})$$

for $\ell \geq \ell_2$ and the iterate is accepted not later than in iteration k_0. Because the predicted reduction decreases with shrinking trust-region radii, the actual reduction in iteration n_ℓ is greater than or equal to $\sigma \operatorname{pred}(\bar{u}_{n_\ell}, \Delta_{n_\ell+1,k_0}) \geq \frac{2}{3}\sigma\alpha\delta$ for all $\ell \geq \ell_2$. Because the sequence of objective function values for the accepted iterates is monotonically decreasing, we obtain $J(\bar{u}_{n_\ell+1}) \to -\infty$ as $\ell \to \infty$, which is a contradiction to the boundedness of J. We conclude that $\bar{u}_{n_\ell} \to \bar{u}$ strictly in $\mathrm{BV}(\Omega)$.

Outcome 3 (3) strict accumulation points are optimal for (TR) **if the trust-region radius is bounded away from zero:** Next, we assume that $\bar{u}$ with $\nabla F(\bar{u}) \in C(\bar{\Omega})$ is a weak-$*$ and strict limit of a subsequence $\{\bar{u}_{n_\ell}\}_{\ell\in\mathbb{N}}$. Moreover, we assume that the trust-region radius upon acceptance of the iterates $\bar{u}_{n_\ell+1}$ is bounded away from zero, that is $0 < \underline{\Delta} \coloneqq \inf_{\ell\in\mathbb{N}} \min_k \Delta_{n_\ell+1,k}$. Because $0 < \underline{\Delta}$ and $\Delta_{n_\ell+1,k} = \Delta_0 2^{-k}$ for all inner iterations k, we may restrict to an infinite subsequence of $\{\bar{u}_{n_\ell}\}_{\ell\in\mathbb{N}}$, which we denote by the same symbol for ease of notation, such that all iterates $\bar{u}_{n_\ell+1}$ are accepted in iteration k_0 with $\underline{\Delta} = \Delta_0 2^{-k_0}$. The Γ-convergence established in Theorem 3.16 gives that every cluster point of $\{\bar{u}_{n_\ell+1}\}_{\ell\in\mathbb{N}}$ minimizes $\operatorname{TR}(\bar{u}, \nabla F(\bar{u}), \underline{\Delta})$. Moreover, the optimal objective function values of the optimization problems $\operatorname{TR}(\bar{u}_{n_\ell}, \nabla F(\bar{u}_{n_\ell}), \underline{\Delta})$, which are the predicted reductions $\operatorname{pred}(\bar{u}_{n_\ell}, \underline{\Delta})$ upon acceptance, converge to zero because otherwise we would obtain the contradiction $J(\bar{u}_{n_\ell+1}) \to -\infty$ due to the fulfilled sufficient decrease condition (3.13) given by $\operatorname{ared}(\bar{u}_{n_\ell}, \bar{u}_{n_\ell+1}) \geq \sigma \operatorname{pred}(\bar{u}_{n_\ell}, \underline{\Delta})$. Thus the minimal objective of $\operatorname{TR}(\bar{u}, \nabla F(\bar{u}), \underline{\Delta})$ is zero, implying that $\bar{u}$ is optimal for $\operatorname{TR}(\bar{u}, \nabla F(\bar{u}), \underline{\Delta})$. In particular, $\bar{u}$ is L-stationary by virtue of Proposition 3.19.

Outcome 3 (4) strict accumulation points are L-stationary if the trust-region radius vanishes: We close the proof by proving that if $\bar{u}_{n_\ell} \overset{*}{\rightharpoonup} \bar{u}$ in $\mathrm{BV}(\Omega)$ as $\ell \to \infty$ and the trust-region radii upon acceptance of the iterates $\bar{u}_{n_\ell+1}$ vanish, then $\bar{u}$ is L-stationary. We argue by contraposition and assume that $\bar{u}$ is not L-stationary. We have to show that the trust-region radii upon acceptance of the iterates $\bar{u}_{n_\ell+1}$ are bounded away from zero. Let $\ell \in \mathbb{N}$, let $\Delta_* \in \{\Delta_0 2^{-j} \,|\, j \in \mathbb{N}\}$, and let $\tilde{u}$ minimize $\operatorname{TR}(\bar{u}_{n_\ell}, \nabla F(\bar{u}_{n_\ell}), \Delta_*)$. Assumption 3.11 allows us to apply Taylor's theorem to F, which yields

$$F(\tilde{u}) = F(\bar{u}_{n_\ell}) + \nabla F(\bar{u}_{n_\ell})(\tilde{u} - \bar{u}_{n_\ell}) + \frac{1}{2}\nabla^2 F(\xi)(\tilde{u} - \bar{u}_{n_\ell}, \tilde{u} - \bar{u}_{n_\ell})$$

with some ξ in the line segment between $\tilde{u}$ and $\bar{u}_{n_\ell}$. Then

$$
\begin{aligned}
\operatorname{ared}(\bar{u}_{n_\ell}, \tilde{u}) &= F(\bar{u}_{n_\ell}) + \alpha \mathrm{TV}(\bar{u}_{n_\ell}) - F(\tilde{u}) - \alpha \mathrm{TV}(\tilde{u}) \\
&= \nabla F(\bar{u}_{n_\ell})(\bar{u}_{n_\ell} - \tilde{u}) + \alpha \mathrm{TV}(\bar{u}_{n_\ell}) - \alpha \mathrm{TV}(\tilde{u}) \\
&\quad - \frac{1}{2} \nabla^2 F(\xi)(\bar{u}_{n_\ell} - \tilde{u}, \bar{u}_{n_\ell} - \tilde{u}) \\
&\geq \operatorname{pred}(\bar{u}_{n_\ell}, \Delta_*) - \frac{C}{2} \|\bar{u}_{n_\ell} - \tilde{u}\|_{L^1(\Omega)}^2 \\
&\geq \sigma \operatorname{pred}(\bar{u}_{n_\ell}, \Delta_*) + (1-\sigma) \operatorname{pred}(\bar{u}_{n_\ell}, \Delta_*) - \frac{C}{2} \Delta_*^2
\end{aligned}
$$

by virtue of the estimate from Assumption 3.11 and the feasibility of $\tilde{u}$ to $\mathrm{TR}(\bar{u}_{n_\ell}, \nabla F(\bar{u}_{n_\ell}), \Delta_*)$, which gives $\|\bar{u}_{n_\ell} - \tilde{u}\|_{L^1(\Omega)} \leq \Delta_*$. Thus it is sufficient to show that $(1-\sigma) \operatorname{pred}(\bar{u}_{n_\ell}, \Delta_*) - \frac{C}{2}(\Delta_*)^2 \geq 0$ holds for some Δ_* and $\ell \in \mathbb{N}$ that are large enough.

Because $\bar{u}$ is not L-stationary, there exist $\psi \in C_c^\infty(\Omega; \mathbb{R}^d)$ and $\eta > 0$ such that

$$
\begin{aligned}
&\sum_{i=1}^N \nu_i \int_{\partial^* E_i \cap \Omega} (-\nabla F(\bar{u}))(x)(\psi(x) \cdot n_{E_i}(x)) \,\mathrm{d}\mathcal{H}^{d-1}(x) \\
&\quad - \alpha \sum_{i=1}^N \sum_{j=i+1}^N |\nu_i - \nu_j| \int_{\partial^* E_i \cap \partial^* E_j} \operatorname{div}_{E_i} \psi(x) \,\mathrm{d}\mathcal{H}^{d-1}(x) > \eta,
\end{aligned}
$$

where $\{E_1, \ldots, E_N\}$ is a Caccioppoli partition of Ω such that $\bar{u} = \sum_{i=1}^N \nu_i \chi_{E_i}$. Let $(f_t)_{t \in (-\varepsilon, \varepsilon)}$ be the local variation defined by $f_t \coloneqq I + t\psi$. We obtain that

$$
-(\nabla F(\bar{u}), f_t^\# \bar{u} - \bar{u})_{L^2(\Omega)} - \alpha \mathrm{TV}(f_t^\# \bar{u}) + \alpha \mathrm{TV}(\bar{u}) \geq t\eta + g(t), \tag{3.15}
$$

where $g : [-\varepsilon, \varepsilon] \to \mathbb{R}$ is a function such that $g(t) \in o(t)$ by virtue of Lemmas 3.3 and 3.5. Lemma 3.8 implies that there exist $\kappa > 0$ and $\varepsilon_1 \in (0, \varepsilon)$ such that

$$
\|f_t^\# \bar{u} - \bar{u}\|_{L^1(\Omega)} \leq |t|\kappa
$$

holds for all $t \in (-\varepsilon_1, \varepsilon_1)$.

We choose $\Delta_* \in \{\Delta_0 2^{-j} \,|\, j \in \mathbb{N}\}$ small enough and corresponding $j_* \in \mathbb{N}$ large enough such that $\Delta_* = \Delta_0 2^{j_*}$, such that

(a) $\Delta_* \leq 2\varepsilon_1 \kappa$ and

(b) $(1-\sigma)\left(\eta \frac{\Delta_*}{2\kappa} + g\left(\frac{\Delta_*}{2\kappa}\right)\right) - (2(1-\sigma) + 0.5C)\,\Delta_*^2 \geq 0$

hold true. The second inequality can be satisfied because $g(t) \in o(t)$ and $\kappa > 0$ is constant. Then we choose $\ell_0 \in \mathbb{N}$ large enough such that for all $\ell \geq \ell_0$ we obtain

(c) $\|\bar{u}_{n_\ell} - \bar{u}\|_{L^1(\Omega)} \leq \frac{\Delta_*}{2}$.

Let $t \coloneqq \frac{\Delta_*}{2\kappa}$. Then (a) gives $t \le \varepsilon_1$ and $\|f_t^{\#}\bar u - \bar u\|_{L^1(\Omega)} \le |t|\kappa \le \frac{\Delta_*}{2}$ gives

$$\|f_t{}^{\#}\bar u - \bar u_{n_\ell}\|_{L^1(\Omega)} \le \|f_t{}^{\#}\bar u - \bar u\|_{L^1(\Omega)} + \|\bar u_{n_\ell} - \bar u\|_{L^1(\Omega)} \le \Delta_*,$$

which implies that $f_t^{\#}\bar u$ is feasible for $\mathrm{TR}(\bar u_{n_\ell}, \nabla F(\bar u_{n_\ell}), \Delta_*)$ for all $\ell \ge \ell_0$.

The strict convergence of $\{\bar u_{n_\ell}\}_{\ell\in\mathbb{N}}$ in $\mathrm{BV}(\Omega)$ and Lemma 2.19 yield that there is $\ell_1 \ge \ell_0$ such that for all $\ell \ge \ell_1$

$$\begin{aligned} |(\nabla F(\bar u), f_t^{\#}\bar u - \bar u)_{L^2(\Omega)} - (\nabla F(\bar u_{n_\ell}), f_t^{\#}\bar u - \bar u_{n_\ell})_{L^2(\Omega)}| &\le \Delta_*^2 \text{ and} \\ |\alpha \mathrm{TV}(\bar u) - \alpha\mathrm{TV}(\bar u_{n_\ell})| &\le \Delta_*^2. \end{aligned} \tag{3.16}$$

Then we can estimate

$$\begin{aligned}
&(1-\sigma)\,\mathrm{pred}(\bar u_{n_\ell}, \Delta_*) - 0.5C\Delta_*^2 \\
&\quad\ge -(1-\sigma)\left((\nabla F(\bar u_{n_\ell}), f_t^{\#}\bar u - \bar u_{n_\ell})_{L^2(\Omega)} + \alpha\mathrm{TV}(f_t^{\#}\bar u) - \alpha\mathrm{TV}(\bar u_{n_\ell})\right) - 0.5C\Delta_*^2 \\
&\quad\ge -(1-\sigma)\left((\nabla F(\bar u), f_t^{\#}\bar u - \bar u)_{L^2(\Omega)} + \alpha\mathrm{TV}(f_t^{\#}\bar u) - \alpha\mathrm{TV}(\bar u)\right) - C_1\Delta_*^2 \\
&\quad\ge (1-\sigma)\left(\eta\frac{\Delta_*}{2\kappa} + g\left(\frac{\Delta_*}{2\kappa}\right)\right) - C_1\Delta_*^2,
\end{aligned}$$

where the first inequality follows from the feasibility of $f_t^{\#}\bar u$ for $\mathrm{TR}(\bar u_{n_\ell}, \nabla F(\bar u_{n_\ell}), \Delta_*)$, the second from (3.16) with the choice $C_1 \coloneqq 2(1-\sigma) + 0.5C$, and the third from (3.15) with $t = \frac{\Delta_*}{2\kappa}$.

Because Δ_* has been chosen small enough such that (b) holds, we have shown $(1-\sigma)\,\mathrm{pred}(\bar u_{n_\ell}, \Delta_*) - 0.5C\Delta_*^2 \ge 0$ for all $\ell \ge \ell_1$, that is, each outer iteration n_ℓ, $\ell \in \mathbb{N}$, is accepted not later than in inner iteration j_* with radius Δ_* such that the trust-region radii do not vanish. □

We have proved that the iterates of trust-region Algorithm 1 have L-stationary accumulation points. In order to solve (P) to global optimality, the knowledge of lower bounds for the optimal value of (P) is crucial. Lower bounds for (P) can be obtained from the optimal values of relaxations of (P). We will deal with such relaxations in the next chapter.

For the numerical solution of (P) and the implementation of Algorithm 1, we will need appropriate discretizations of (P). These are introduced in Chapter 5.

Chapter 4

Relaxation

This chapter is dedicated to the following relaxation of the superordinate problem (P) that reads

$$
(\mathrm{P_R}) \qquad \begin{aligned} \min_{u \in L^2(\Omega)} \quad & F(u) + \alpha \mathrm{TV}(u) \\ \text{s.t.} \quad & \underline{\nu} \leq u(x) \leq \overline{\nu} \quad \text{for a.a. } x \in \Omega, \end{aligned}
$$

where $\underline{\nu} = \min\{\nu : \nu \in U\}$ and $\overline{\nu} = \max\{\nu : \nu \in U\}$. From this relaxation, we can derive lower bounds for the optimal value of (P) which is crucial for the determination of global optimal solutions to (P). This chapter uses similar concepts as [82], where an optimal control problem with a restriction of the total variation in the constraints is considered. For the solution of $(\mathrm{P_R})$, we will use a function space outer-approximation algorithm that is related to the outer-approximation algorithm from [82]. Outer-approximation algorithms are a well-known solution technique for finite-dimensional mixed-integer nonlinear programs [48, 52, 67, 84, 98] and also non-differentiable optimization problems [81] in finite dimension. Recently, outer-approximation algorithms have also been applied to integer optimal control problems [17, 20]. In order to solve $(\mathrm{P_R})$ with the outer-approximation algorithm, we introduce a regularization of $(\mathrm{P_R})$ that includes a regularized total variation and a Tikhonov regularization. The regularized total variation is the same as used in [82] and will be introduced in Section 4.2. The Tikhonov regularization will be needed to prove the convergence of the iterates of the outer-approximation algorithm in $L^2(\Omega)$ which then yields the feasibility of the limit. The regularized optimization problem and the outer-approximation algorithm will be stated in Section 4.3. In Section 4.4, we derive necessary and sufficient optimality conditions and use them for the construction an instance with known exact solution for our numerical experiments.

4.1 Existence of solutions

For some results within this chapter, we need to assume that $F : L^2(\Omega) \to \mathbb{R}$ fulfills the following assumption additionally to Assumption 1.1.

Assumption 4.1. The function $F : L^2(\Omega) \to \mathbb{R}$ is weakly lower semicontinuous and fulfills Assumption 1.1.

For example, $F : L^2(\Omega) \to \mathbb{R}$ is weakly lower semicontinuous if it is convex and lower semicontinuous.

Theorem 4.2. *Let $F : L^2(\Omega) \to \mathbb{R}$ fulfill Assumption 1.1. Then problem* $(\mathrm{P_R})$ *admits an optimal solution.*

Proof. The feasible set $Z_R \coloneqq \{u \in L^2(\Omega) : \underline{\nu} \le u(x) \le \overline{\nu} \text{ for a.a. } x \in \Omega\}$ is nonempty. Consider a minimizing sequence $\{u_k\}_{k\in\mathbb{N}} \subset Z_R$ of $(\mathrm{P_R})$. Since F is bounded from below, the sequence $\{\mathrm{TV}(u_k)\}_{k\in\mathbb{N}}$ is bounded. By Theorem 2.8, the sequence $\{u_k\}_{k\in\mathbb{N}}$ admits a subsequence $\{u_{k_\ell}\}_{\ell\in\mathbb{N}}$ that converges weakly-$*$ in $\mathrm{BV}(\Omega)$. This yields in particular that $u_{k_\ell} \to u$ in $L^1(\Omega)$ as $\ell \to \infty$ for some $u \in \mathrm{BV}(\Omega)$ and since there holds $\underline{\nu} \le u_{k_\ell}(x) \le \overline{\nu}$ for almost all $x \in \Omega$ and all $\ell \in \mathbb{N}$, Lemma 2.19 yields that $u_{k_\ell} \to u$ in $L^2(\Omega)$ as $\ell \to \infty$ and $u \in Z_R$. By the lower semicontinuity of F and TV, there holds

$$F(u) + \alpha\mathrm{TV}(u) \le \liminf_{\ell\to\infty} F(u_{k_\ell}) + \alpha\mathrm{TV}(u_{k_\ell}) = \lim_{k\to\infty} F(u_k) + \alpha\mathrm{TV}(u_k)$$

such that u is an optimal solution to $(\mathrm{P_R})$. □

We aim to solve problem $(\mathrm{P_R})$ by means of an outer-approximation algorithm. To clarify the idea of the algorithm, we rewrite $(\mathrm{P_R})$ as

$$(\bar{\mathrm{P}}_\mathrm{R}) \qquad \begin{aligned} \min_{(u,V)\in L^2(\Omega)\times\mathbb{R}} \quad & F(u) + \alpha V \\ \text{s.t.} \quad & \mathrm{TV}(u) \le V \\ & \underline{\nu} \le u(x) \le \overline{\nu} \quad \text{for a.a. } x \in \Omega. \end{aligned}$$

The fact that V is bounded from below by $\mathrm{TV}(u)$ together with the minimization imply that $\bar{u} \in L^2(\Omega)$ is an optimal solution to $(\mathrm{P_R})$ if and only if $(\bar{u}, \mathrm{TV}(\bar{u})) \in L^2(\Omega) \times \mathbb{R}$ is an optimal solution to $(\bar{\mathrm{P}}_\mathrm{R})$. Moreover, the optimal values are identical.

The constraint $\mathrm{TV}(u) \le V$ can be rewritten by infinitely many linear inequalities

$$(4.1) \qquad \int_\Omega u(x)\,\mathrm{div}\,\phi(x)\,\mathrm{d}x \le V \quad \forall\, \phi \in C_c^1(\Omega;\mathbb{R}^d) \text{ with } \|\phi\|_{L^\infty(\Omega;\mathbb{R}^d)} \le 1.$$

The idea of the outer-approximation algorithm is to consider the relaxation

$$\begin{aligned} \min_{(u,V)\in L^2(\Omega)\times\mathbb{R}_{\geq 0}} \quad & F(u)+\alpha V \\ \text{s.t.} \quad & \int_\Omega u(x)\operatorname{div}\phi(x)\,\mathrm{d}x \leq V \quad \forall\,\phi\in\Phi \\ & \underline{\nu} \leq u(x) \leq \overline{\nu} \quad \text{for a.a. } x\in\Omega, \end{aligned}$$

of ($\bar{\mathrm{P}}_{\mathrm{R}}$) with a (possibly empty) subset $\Phi \subset \{\phi \in C_c^1(\Omega;\mathbb{R}^d) : \|\phi\|_{L^\infty(\Omega;\mathbb{R}^d)} \leq 1\}$ and to iteratively add the inequalities from (4.1) as long as the current optimal solution (u,V) is infeasible for ($\bar{\mathrm{P}}_{\mathrm{R}}$). That is, if there holds $\mathrm{TV}(u) > V$ for (u,V), there exists a violated inequality in (4.1) that we add to the above relaxation of ($\bar{\mathrm{P}}_{\mathrm{R}}$). In order to guarantee the convergence of the iterates of the outer-approximation algorithm to an optimal solution to ($\bar{\mathrm{P}}_{\mathrm{R}}$), we want to add the most violated inequality from (4.1) that can be obtained for fixed $u \in L^2(\Omega)$ by the computation of the maximizer $\phi \in C_c^1(\Omega;\mathbb{R}^d)$ of the dual formulation

$$\mathrm{TV}(u) = \sup\left\{\int_\Omega u(x)\operatorname{div}\phi(x)\,\mathrm{d}x : \phi\in C_c^1(\Omega;\mathbb{R}^d), \|\phi\|_{L^\infty(\Omega;\mathbb{R}^d)} \leq 1\right\}$$

but such a maximizer might not exist because the set $\{\operatorname{div}\phi \in L^2(\Omega) : \phi \in C_c^1(\Omega;\mathbb{R}^d), \|\phi\|_{L^\infty(\Omega;\mathbb{R}^d)} \leq 1\}$ is neither bounded nor closed in $L^2(\Omega)$ and if a maximizer exists, it might be not unique. To solve this issue, we replace the total variation TV by a regularized total variation whose dual formulation admits a unique maximizer for each $u \in L^2(\Omega)$.

4.2 Regularized total variation

We will introduce the regularized total variation TV_ε in this section. The regularization will be done by adding the negative of a bounded and coercive bilinear form to the objective in the dual formulation (TV) multiplied with the regularization parameter $\varepsilon > 0$. This gives the regularized total variation some desirable properties like the representation by a unique maximizer that realizes the regularized total variation for a given input function $u \in L^2(\Omega)$. A similar regularization of the total variation by adding an L^2 term to the dual formulation can be found in [32]. In order to define the regularized total variation TV_ε, we first replace the space $C_c^1(\Omega;\mathbb{R}^d)$ in (TV) by a Hilbert space H with $C_c^1(\Omega;\mathbb{R}^d) \subset H \subset H_0(\mathrm{div};\Omega)$ that embeds continuously into $H_0(\mathrm{div};\Omega)$, that is, there exists a constant $C_H > 0$ such that

$$\|\phi\|_{H(\mathrm{div};\Omega)} \leq C_H\|\phi\|_H \quad \forall\phi\in H.$$

This does not change the supremum of (TV).

Lemma 4.3. *Let $u \in L^2(\Omega)$. There holds*

$$\text{(TV}_H) \qquad \mathrm{TV}(u) = \sup\left\{ \int_\Omega u(x) \operatorname{div} \phi(x) \, \mathrm{d}x : \phi \in H, \ \|\phi\|_{L^\infty(\Omega;\mathbb{R}^d)} \leq 1 \right\}.$$

Proof. It follows from Theorem 1.1 in [103] or Theorem 1 and Corollary 3 in [62] that for $u \in L^2(\Omega)$, there holds

$$\mathrm{TV}(u) = \sup\left\{ \int_\Omega u(x) \operatorname{div} \phi(x) \, \mathrm{d}x : \phi \in H_0(\operatorname{div};\Omega), \ \|\phi\|_{L^\infty(\Omega;\mathbb{R}^d)} \leq 1 \right\}.$$

The claim follows since $C_c^1(\Omega;\mathbb{R}^d) \subset H \subset H_0(\operatorname{div};\Omega)$. □

Now we introduce the bounded and coercive bilinear form $a : H \times H \to \mathbb{R}$, that is, there exist constants $\beta, \gamma > 0$ such that

$$a[\phi, \phi] \geq \beta \|\phi\|_H^2$$

for all $\phi \in H$ and

$$a[\phi_1, \phi_2] \leq \gamma \|\phi_1\|_H \|\phi_2\|_H$$

for all $\phi_1, \phi_2 \in H$. The dual regularization of TV is now obtained by adding $-\frac{\varepsilon}{2} a[\phi, \phi]$ to the objective in the maximization problem (TV$_H$) with the regularization parameter $\varepsilon > 0$, which yields

$$(\mathrm{TV}_\varepsilon) \qquad \mathrm{TV}_\varepsilon(u) := \sup\left\{ R_\varepsilon(u, \phi) : \phi \in H, \ \|\phi\|_{L^\infty(\Omega;\mathbb{R}^d)} \leq 1 \right\},$$

where $R_\varepsilon : L^2(\Omega) \times H \to \mathbb{R}$ is defined by

$$R_\varepsilon(u, \phi) := -\frac{\varepsilon}{2} \, a[\phi, \phi] + \int_\Omega u(x) \operatorname{div} \phi(x) \, \mathrm{d}x.$$

This means that we have to solve the following optimization problem to compute $\mathrm{TV}_\varepsilon(u)$ for a given function $u \in L^2(\Omega)$ that reads

$$(\mathrm{Q}_\varepsilon) \qquad \begin{aligned} \sup_{\phi \in H} \quad & R_\varepsilon(u, \phi) \\ \text{s.t.} \quad & \|\phi\|_{L^\infty(\Omega;\mathbb{R}^d)} \leq 1. \end{aligned}$$

Example 4.4. Two examples for suitable choices for H are $H = H_0^1(\Omega;\mathbb{R}^d)$ and $H = H_0(\operatorname{div};\Omega)$. In [82], the choices $H = H_0^1(\Omega;\mathbb{R}^d)$ and

$$a[\phi_1, \phi_2] := \int_\Omega (\epsilon(\phi_1))(x) : \mathbb{C}(\epsilon(\phi_2))(x) \, \mathrm{d}x$$

for $\phi_1, \phi_2 \in H_0^1(\Omega; \mathbb{R}^d)$ with $\epsilon(\phi) = \frac{1}{2}(\nabla\phi + (\nabla\phi)^T)$ for $\phi \in H_0^1(\Omega; \mathbb{R}^d)$ and a linear elasticity tensor $\mathbb{C} \in \mathbb{R}^{d\times d}_{sym}$ were used for the numerical experiments. This choice for the bilinear form a gives an additional control over the L^2 norm of $\phi \in H_0^1(\Omega; \mathbb{R}^d)$, similar to Friedrichs' inequality in [104]. In this thesis, we will later make the choice $H = H_0(\operatorname{div}; \Omega)$. With that choice for H, the space of Raviart–Thomas functions is conforming. This is beneficial because the discretization introduced in Chapter 5 is based on the discretization of ϕ by means of lowest-order Raviart–Thomas functions.

In contrast to $\mathrm{TV}(u)$, the regularized total variation $\mathrm{TV}_\varepsilon(u)$ always admits a unique maximizer for each $u \in L^2(\Omega)$, that is, $\mathrm{TV}_\varepsilon(u) = R_\varepsilon(u, \phi) < \infty$ for a unique $\phi \in H$ with $\|\phi\|_{L^\infty(\Omega;\mathbb{R}^d)} \leq 1$ depending on $u \in L^2(\Omega)$.

Lemma 4.5. *The optimization problem* (Q_ε) *admits a unique maximizer for each* $u \in L^2(\Omega)$.

Proof. The coercivity of the bilinear form a yields strict concavity of $R_\varepsilon(u, \cdot) : H \to \mathbb{R}$ because for fixed $u \in L^2(\Omega)$ and with the notation $R(\phi) = R_\varepsilon(u, \phi)$ for $\phi \in H$, there holds for $\phi_1, \phi_2 \in H$ with $\phi_1 \neq \phi_2$ that

$$R(\phi_1) - R(\phi_2) - R'(\phi_2)(\phi_1 - \phi_2) = -\frac{\varepsilon}{2} a[\phi_1 - \phi_2, \phi_1 - \phi_2] \leq -\beta\frac{\varepsilon}{2}\|\phi_1 - \phi_2\|_H^2 < 0.$$

The feasible set $Z_Q = \{\phi \in H : \|\phi\|_{L^\infty(\Omega;\mathbb{R}^d)} \leq 1\}$ of (Q_ε) is non-empty and convex. Moreover, it is closed in H because each sequence $\{\phi_k\}_{k\in\mathbb{N}} \subset Z_Q$ that converges in H, that is, $\phi_k \to \bar{\phi}$ in H as $k \to \infty$ with $\bar{\phi} \in H$, also converges in $L^1(\Omega; \mathbb{R}^d)$ and therefore admits a subsequence $\{\phi_{k_\ell}\}_{\ell\in\mathbb{N}}$ that converges pointwise almost everywhere to $\bar{\phi}$ by Lemma 3.22 in [3]. This yields that $\|\bar{\phi}\|_{L^\infty(\Omega;\mathbb{R}^d)} \leq 1$ because $\|\phi_{k_\ell}\|_{L^\infty(\Omega;\mathbb{R}^d)} \leq 1$ for all $\ell \in \mathbb{N}$. Therefore, Z_Q sequentially weakly closed in H.

By the coercivity of a and the continuous embedding of H into $H_0(\operatorname{div}; \Omega)$, there holds for $u \in L^2(\Omega)$ and $\phi \in H$ that

$$\begin{aligned} R_\varepsilon(u, \phi) &= -\frac{\varepsilon}{2} a[\phi, \phi] + \int_\Omega u(x) \operatorname{div}\phi(x)\,\mathrm{d}x \leq -\beta\frac{\varepsilon}{2}\|\phi\|_H^2 + \|u\|_{L^2(\Omega)}\|\operatorname{div}\phi\|_{L^2(\Omega)} \\ &\leq -\beta\frac{\varepsilon}{2}\|\phi\|_H^2 + \|u\|_{L^2(\Omega)}\|\phi\|_{H(\operatorname{div};\Omega)} \leq -\beta\frac{\varepsilon}{2}\|\phi\|_H^2 + C_H\|u\|_{L^2(\Omega)}\|\phi\|_H, \end{aligned}$$

which implies $R_\varepsilon(u, \phi) \to -\infty$ as $\|\phi\|_H \to \infty$. Since $R_\varepsilon(u, \cdot) : H \to \mathbb{R}$ is continuous and concave, the above radial unboundedness yields the existence of an optimal solution to (Q_ε), because maximizing sequences $\{\phi_k\}_{k\in\mathbb{N}} \subset Z_Q$ are bounded in H and therefore admit weakly convergent subsequences $\{\phi_{k_\ell}\}_{\ell\in\mathbb{N}}$ in H with $\phi_{k_\ell} \rightharpoonup \bar{\phi}$ in H and $\bar{\phi} \in Z_Q$, such that

$$R_\varepsilon(u, \bar{\phi}) \geq \limsup_{\ell\to\infty} R_\varepsilon(u, \phi_{k_\ell}) = \lim_{k\to\infty} R_\varepsilon(u, \phi_k).$$

The strict concavity of $R_\varepsilon(u, \cdot)$ gives the uniqueness of the optimal solution. $\square$

The existence of a maximizer to (Q_ε) immediately gives the following corollary.

Corollary 4.6. *For all $u \in L^2(\Omega)$ and all $\varepsilon > 0$ there holds $\mathrm{TV}_\varepsilon(u) < \infty$.*

In contrast to $\mathrm{TV}_\varepsilon(u)$, the total variation $\mathrm{TV}(u)$ is not necessarily finite for all $u \in L^2(\Omega)$.

Lemma 4.7. *Let $u \in L^2(\Omega)$ and $\varepsilon > 0$. Then there holds $\mathrm{TV}_\varepsilon(u) \leq \mathrm{TV}(u)$.*

Proof. According to Lemma 4.5, the optimization problem (Q_ε) admits for each $\varepsilon > 0$ a unique maximizer $\phi_\varepsilon \in H$ with $\|\phi_\varepsilon\|_{L^\infty(\Omega;\mathbb{R}^d)} \leq 1$. There holds

$$\mathrm{TV}(u) \geq \int_\Omega u(x) \operatorname{div} \phi_\varepsilon(x) \, \mathrm{d}x$$

for all $\varepsilon > 0$ and hence

$$\begin{aligned}
\mathrm{TV}_\varepsilon(u) - \mathrm{TV}(u) &= -\frac{\varepsilon}{2} a[\phi_\varepsilon, \phi_\varepsilon] + \int_\Omega u(x) \operatorname{div} \phi_\varepsilon(x) \, \mathrm{d}x - \mathrm{TV}(u) \\
&\leq -\frac{\varepsilon}{2} a[\phi_\varepsilon, \phi_\varepsilon] \leq -\frac{\varepsilon}{2} \beta \|\phi_\varepsilon\|_H^2 \leq 0
\end{aligned}$$

due to the coercivity of a. □

Lemma 4.8. *Let $u \in L^2(\Omega)$ be fixed. The mapping $\mathrm{TV}_\bullet(u) : (0, \infty) \to \mathbb{R}$, $\varepsilon \mapsto \mathrm{TV}_\varepsilon(u)$ is strictly monotonically decreasing.*

Proof. Let $\varepsilon_1, \varepsilon_2 > 0$ with $\varepsilon_1 < \varepsilon_2$ be given. Denote the unique optimal solutions to $(\mathrm{Q}_{\varepsilon_1})$ and $(\mathrm{Q}_{\varepsilon_2})$ by $\phi_1 \in H$ and $\phi_2 \in H$, respectively. Since ϕ_2 is feasible for $(\mathrm{Q}_{\varepsilon_1})$, there holds

$$\begin{aligned}
\mathrm{TV}_{\varepsilon_1}(u) &= -\frac{\varepsilon_1}{2} a[\phi_1, \phi_1] + \int_\Omega u(x) \operatorname{div} \phi_1(x) \, \mathrm{d}x \\
&> -\frac{\varepsilon_1}{2} a[\phi_2, \phi_2] + \int_\Omega u(x) \operatorname{div} \phi_2(x) \, \mathrm{d}x \\
&> -\frac{\varepsilon_2}{2} a[\phi_2, \phi_2] + \int_\Omega u(x) \operatorname{div} \phi_2(x) \, \mathrm{d}x = \mathrm{TV}_{\varepsilon_2}(u).
\end{aligned}$$

□

The regularized total variation $\mathrm{TV}_\varepsilon(u)$ of $u \in L^2(\Omega)$ converges to its actual total variation $\mathrm{TV}(u)$ as the regularization parameter ε is driven to zero.

Lemma 4.9. *Let $u \in L^2(\Omega)$ be fixed. Then $\mathrm{TV}_\varepsilon(u) \to \mathrm{TV}(u)$ as $\varepsilon \searrow 0$.*

Proof. Denote by $\{\phi_k\}_{k \in \mathbb{N}} \subset H$ a maximizing sequence of (TV_H), that is, there holds $\phi_k \in H$ and $\|\phi_k\|_{L^\infty(\Omega;\mathbb{R}^d)} \leq 1$ for all $k \in \mathbb{N}$ and $\int_\Omega u(x) \operatorname{div} \phi_k(x) \, \mathrm{d}x \to \mathrm{TV}(u)$

as $k \to \infty$. Since each ϕ_k, $k \in \mathbb{N}$, is also feasible for $(\mathrm{TV}_\varepsilon)$, there holds

$$\mathrm{TV}_\varepsilon(u) \geq R_\varepsilon(u, \phi_k) \geq -\frac{\varepsilon\gamma}{2}\|\phi_k\|_H^2 + \int_\Omega u(x) \operatorname{div} \phi_k(x) \, \mathrm{d}x$$

for all $\varepsilon > 0$ by the boundedness of a. Consider the monotonically decreasing null sequence $\{\varepsilon_k\}_{k\in\mathbb{N}} \subset \mathbb{R}_{>0}$ defined by

$$\varepsilon_k := \frac{1}{k \max_{j\in\mathbb{N}, j\leq k} \|\phi_j\|_H^2}.$$

Then there holds

$$\begin{aligned}
\mathrm{TV}(u) &\geq \limsup_{k\to\infty} \mathrm{TV}_{\varepsilon_k}(u) \geq \liminf_{k\to\infty} \mathrm{TV}_{\varepsilon_k}(u) \\
&\geq \lim_{k\to\infty} -\frac{\gamma}{2k} + \int_\Omega u(x) \operatorname{div} \phi_k(x) \, \mathrm{d}x = \mathrm{TV}(u).
\end{aligned}$$

By the monotonicity of the mapping $\mathrm{TV}_\bullet(u) : (0, \infty) \to \mathbb{R}$, $\varepsilon \mapsto \mathrm{TV}_\varepsilon(u)$ for fixed $u \in L^2(\Omega)$ by Lemma 4.8, there holds $\mathrm{TV}_\varepsilon(u) \to \mathrm{TV}(u)$ for arbitrary $\varepsilon \searrow 0$. $\square$

Since we pursue to replace the total variation in $(\mathrm{P_R})$ by its regularized version, we want to ensure that we retain a weakly lower semicontinuous objective function.

Lemma 4.10. *For each $\varepsilon > 0$, the mapping $\mathrm{TV}_\varepsilon : L^2(\Omega) \to \mathbb{R}$, $u \mapsto \mathrm{TV}_\varepsilon(u)$ is convex and continuous.*

Proof. Let $\varepsilon > 0$ be arbitrary and $\lambda \in [0, 1]$. For each u, $v \in L^2(\Omega)$, there holds

$$\begin{aligned}
\mathrm{TV}_\varepsilon(\lambda u + (1-\lambda)v) &= \sup\Big\{R_\varepsilon(\lambda u + (1-\lambda)v, \phi) : \phi \in H, \|\phi\|_{L^\infty(\Omega;\mathbb{R}^d)} \leq 1\Big\} \\
&= \sup\Big\{\lambda R_\varepsilon(u, \phi) + (1-\lambda) R_\varepsilon(v, \phi) : \phi \in H, \|\phi\|_{L^\infty(\Omega;\mathbb{R}^d)} \leq 1\Big\} \\
&\leq \lambda \sup\Big\{R_\varepsilon(u, \phi) : \phi \in H, \|\phi\|_{L^\infty(\Omega;\mathbb{R}^d)} \leq 1\Big\} \\
&\quad + (1-\lambda) \sup\Big\{R_\varepsilon(v, \phi) : \phi \in H, \|\phi\|_{L^\infty(\Omega;\mathbb{R}^d)} \leq 1\Big\} \\
&= \lambda \mathrm{TV}_\varepsilon(u) + (1-\lambda)\mathrm{TV}_\varepsilon(v)
\end{aligned}$$

due to the linearity of the integral and λ, $1 - \lambda \geq 0$. Hence, the mapping is convex.

To prove the continuity, we employ the optimality conditions associated with (Q_ε). For this purpose, let $u_1, u_2 \in L^2(\Omega)$ be arbitrary and denote corresponding the solutions to (Q_ε) by $\phi_1, \phi_2 \in H$. The necessary and sufficient optimality conditions of (Q_ε) read

$$\frac{\varepsilon}{2}\left(a[\phi_i, h - \phi_i] + a[h - \phi_i, \phi_i]\right) \geq \int_\Omega u_i(x) \operatorname{div}(h - \phi_i)(x) \, \mathrm{d}x$$

for all $h \in H$ with $\|h\|_{L^\infty(\Omega;\mathbb{R}^d)} \le 1$ for $i = 1, 2$. Inserting $h = \phi_2$ in the inequality for $i = 1$ and $h = \phi_1$ for $i = 2$ and adding the arising inequalities give

$$\begin{aligned}
\varepsilon\, a[\phi_1 - \phi_2, \phi_1 - \phi_2] &\le \int_\Omega (u_1(x) - u_2(x)) \operatorname{div}(\phi_1 - \phi_2)(x)\, \mathrm{d}x \\
&\le \|u_1 - u_2\|_{L^2(\Omega)}\, \|\operatorname{div}(\phi_1 - \phi_2)\|_{L^2(\Omega)} \\
&\le \|u_1 - u_2\|_{L^2(\Omega)}\, \|\phi_1 - \phi_2\|_{H(\mathrm{div};\Omega)} \\
&\le C_H \|u_1 - u_2\|_{L^2(\Omega)}\, \|\phi_1 - \phi_2\|_H.
\end{aligned}$$

The coercivity of a thus implies that

$$\varepsilon\beta\|\phi_1 - \phi_2\|_H^2 \le \varepsilon a[\phi_1 - \phi_2, \phi_1 - \phi_2] \le C_H\|u_1 - u_2\|_{L^2(\Omega)}\|\phi_1 - \phi_2\|_H$$

and hence

$$\|\phi_1 - \phi_2\|_H \le \frac{C_H}{\varepsilon\beta}\, \|u_1 - u_2\|_{L^2(\Omega)},$$

that is, the solution mapping of (Q_ε) is globally Lipschitz continuous with a Lipschitz constant proportional to $\frac{1}{\varepsilon}$. The continuity of $R_\varepsilon : L^2(\Omega) \times H \to \mathbb{R}$, $(u, \phi) \mapsto R_\varepsilon(u, \phi)$ then gives the claimed continuity of TV_ε. □

Corollary 4.11. *For each $\varepsilon > 0$, the mapping $\mathrm{TV}_\varepsilon : L^2(\Omega) \to \mathbb{R}$, $u \mapsto \mathrm{TV}_\varepsilon(u)$ is weakly lower semicontinuous.*

Proof. Follows directly from Lemma 4.10 because convexity and continuity imply weak lower semicontinuity. □

4.3 Regularization and outer approximation

Associated with the regularized total variation, we consider the following regularization of (P) with regularization parameters $\delta > 0$ for the L^2 regularization and $\varepsilon > 0$ for the regularized total variation TV_ε that reads

$$(\mathrm{P}_{\delta,\varepsilon}) \qquad \begin{aligned} \min_{u \in L^2(\Omega)} \quad & F(u) + \frac{\delta}{2}\|u\|_{L^2(\Omega)}^2 + \alpha \mathrm{TV}_\varepsilon(u) \\ \text{s.t.} \quad & \underline{\nu} \le u(x) \le \overline{\nu} \quad \text{for a.a. } x \in \Omega. \end{aligned}$$

We abbreviate the objective function by

$$J_{\delta,\varepsilon} : L^2(\Omega) \to \mathbb{R}, \quad J_{\delta,\varepsilon}(u) := F(u) + \frac{\delta}{2}\|u\|_{L^2(\Omega)}^2 + \alpha \mathrm{TV}_\varepsilon(u).$$

The above regularization of $(\mathrm{P_R})$ contains an additional L^2 regularization term $\frac{\delta}{2}\|u\|_{L^2(\Omega)}^2$. This regularization term is also known as Tikhonov regularization and is a common regularization technique, for example for inverse problems [49, 55, 65]. In our case, we use the L^2 regularization to prove convergence of a subsequence of the iterates produced by the outer-approximation algorithm in $L^2(\Omega)$ which we need to prove feasibility of its limit.

Theorem 4.12. *Let $F : L^2(\Omega) \to \mathbb{R}$ fulfill Assumption 4.1. The optimization problem* $(\mathrm{P}_{\delta,\varepsilon})$ *admits an optimal solution. If F is additionally convex, the optimal solution is unique.*

Proof. The objective function $J_{\delta,\varepsilon}$ is weakly lower semicontinuous and bounded from below and the feasible set of $(\mathrm{P}_{\delta,\varepsilon})$ is non-empty, bounded, convex, and closed in $L^2(\Omega)$ by Lemma 2.19. Hence, there exists a minimizer of $(\mathrm{P}_{\delta,\varepsilon})$. If F is additionally convex, then $J_{\delta,\varepsilon}$ is strictly convex due to the term $\frac{\delta}{2}\|u\|^2_{L^2(\Omega)}$ which yields the uniqueness of the optimal solution. □

We obtain lower bounds for (P_R) from the optimal value of the regularized problem $(\mathrm{P}_{\delta,\varepsilon})$.

Theorem 4.13. *Let $u_{\delta,\varepsilon} \in L^2(\Omega)$ be an optimal solution to* $(\mathrm{P}_{\delta,\varepsilon})$ *and $\bar{u} \in L^2(\Omega)$ be an optimal solution to* (P_R). *Then there exists some constant $M = M(U, \Omega) \geq 0$ depending only on U and Ω such that*

$$J(\bar{u}) \geq J_{\delta,\varepsilon}(u_{\delta,\varepsilon}) - \frac{\delta}{2}M,$$

that is, $J_{\delta,\varepsilon}(u_{\delta,\varepsilon}) - \frac{\delta}{2}M$ is a lower bound for the optimal value of (P_R).

Proof. It holds that $\underline{\nu} \leq \bar{u}(x) \leq \overline{\nu}$ for almost all $x \in \Omega$ so that we can estimate $0 \leq \|\bar{u}\|^2_{L^2(\Omega)} \leq \nu^2_{\max}|\Omega| =: M$ with $\nu_{\max} := \max_{\nu \in U} |\nu|$. Since $\bar{u}$ is feasible for $(\mathrm{P}_{\delta,\varepsilon})$ for all $\delta, \varepsilon > 0$, there holds with Lemma 4.7 that

$$\begin{aligned} J_{\delta,\varepsilon}(u_{\delta,\varepsilon}) \leq J_{\delta,\varepsilon}(\bar{u}) &= F(\bar{u}) + \frac{\delta}{2}\|\bar{u}\|^2_{L^2(\Omega)} + \mathrm{TV}_\varepsilon(\bar{u}) \\ &\leq F(\bar{u}) + \frac{\delta}{2}M + \mathrm{TV}(\bar{u}) \\ &= J(\bar{u}) + \frac{\delta}{2}M \end{aligned}$$

for all $\delta, \varepsilon > 0$, which yields the claim. □

4.3.1 Outer-approximation algorithm for the regularized problem

Within this subsection, the regularization parameters $\delta > 0$ and $\varepsilon > 0$ are fixed. To make the outer-approximation approach applicable to $(\mathrm{P}_{\delta,\varepsilon})$, we reformulate it similarly to ($\bar{\mathrm{P}}_{\mathrm{R}}$) as

$$(\bar{\mathrm{P}}_{\delta,\varepsilon})\qquad \begin{aligned} \min_{(u,V)\in L^2(\Omega)\times\mathbb{R}} \quad & F(u)+\frac{\delta}{2}\|u\|^2_{L^2(\Omega)}+\alpha V \\ \text{s.t.} \quad & \mathrm{TV}_\varepsilon(u)\le V \\ & \underline{\nu}\le u(x)\le\overline{\nu} \quad \text{for a.a. } x\in\Omega. \end{aligned}$$

We denote the objective function of ($\bar{\mathrm{P}}_{\delta,\varepsilon}$) for fixed $\delta>0$ and $\varepsilon>0$ by

$$\bar{J}: L^2(\Omega)\times\mathbb{R}\to\mathbb{R},\quad \bar{J}(u,V)=F(u)+\frac{\delta}{2}\|u\|^2_{L^2(\Omega)}+\alpha V.$$

Since V is bounded from below by $\mathrm{TV}_\varepsilon(u)$ and due to the minimization, there holds that $\bar{u}\in L^2(\Omega)$ is an optimal solution to ($\mathrm{P}_{\delta,\varepsilon}$) if and only if $(\bar{u},\mathrm{TV}_\varepsilon(\bar{u}))\in L^2(\Omega)\times\mathbb{R}$ is an optimal solution to ($\bar{\mathrm{P}}_{\delta,\varepsilon}$). Moreover, both optimization problems have the same optimal value.

Analogously to (4.1), we equivalently reformulate the constraint $\mathrm{TV}_\varepsilon(u)\le V$ with infinitely many linear constraints, that is, by

$$R_\varepsilon(u,\phi)\le V \quad \forall\,\phi\in H \text{ with } \|\phi\|_{L^\infty(\Omega;\mathbb{R}^d)}\le 1. \tag{4.2}$$

The outer-approximation algorithm now works as follows. We consider a relaxation of ($\bar{\mathrm{P}}_{\delta,\varepsilon}$) with $k\in\mathbb{N}_0$ constraints from (4.2) that reads

$$(\mathrm{P}_k)\qquad \begin{aligned} \min_{(u,V)\in L^2(\Omega)\times\mathbb{R}_{\ge 0}} \quad & \bar{J}(u,V) \\ \text{s.t.} \quad & R_\varepsilon(u,\phi_i)\le V \quad \forall\, i\in\mathbb{N} \text{ with } i\le k \\ & \underline{\nu}\le u(x)\le\overline{\nu} \quad \text{for a.a. } x\in\Omega, \end{aligned}$$

and compute a minimizer $(u_k,V_k)\in L^2(\Omega)\times\mathbb{R}$. Subsequently, we compute $\mathrm{TV}_\varepsilon(u_k)$ by solving the maximization problem (Q_ε) for $u=u_k$, that is, we solve

$$(\mathrm{Q}_k)\qquad \begin{aligned} \max_{\phi\in H} \quad & R_\varepsilon(u_k,\phi) \\ \text{s.t.} \quad & \|\phi\|_{L^\infty(\Omega;\mathbb{R}^d)}\le 1 \end{aligned}$$

which admits a unique maximizer $\phi_{k+1}\in H$. If $\mathrm{TV}_\varepsilon(u_k)=R_\varepsilon(u_k,\phi_{k+1})\le V_k$, the solution u_k is optimal for ($\mathrm{P}_{\delta,\varepsilon}$). If $\mathrm{TV}_\varepsilon(u_k)=R_\varepsilon(u_k,\phi_{k+1})>V_k$, we add the linear constraint $R_\varepsilon(u,\phi_{k+1})\le V$ from (4.2) corresponding to ϕ_{k+1} to (P_k) and denote the resulting problem by (P_{k+1}). This completes the iteration and we start the next one by solving (P_{k+1}). We provide the complete algorithm in Algorithm 2.

We first prove that accumulation points of the sequence of iterates produced by Algorithm 2 are feasible for ($\bar{\mathrm{P}}_{\delta,\varepsilon}$).

Algorithm 2 Outer-approximation algorithm for $(\mathrm{P}_{\delta,\varepsilon})$

Input: F sufficiently regular, $\alpha > 0$, $\varepsilon > 0$, $\delta > 0$.

1: Set $k = 0$

2: Solve

$$(\mathrm{P}_k) \qquad \begin{aligned} \min_{(u,V)\in L^2(\Omega)\times\mathbb{R}_{\geq 0}} \quad & F(u) + \frac{\delta}{2}\|u\|^2_{L^2(\Omega)} + \alpha V \\ \text{s.t.} \quad & R_\varepsilon(u,\phi_i) \leq V \quad \forall\, i \in \mathbb{N} \text{ with } i \leq k \\ & \underline{\nu} \leq u(x) \leq \overline{\nu} \quad \text{for a.a. } x \in \Omega \end{aligned}$$

and denote an optimal solution by (u_k, V_k).

3: Compute $\mathrm{TV}_\varepsilon(u_k)$ by solving

$$(\mathrm{Q}_k) \qquad \begin{aligned} \max_{\phi\in H} \quad & R_\varepsilon(u_k,\phi) \\ \text{s.t.} \quad & \|\phi\|_{L^\infty(\Omega;\mathbb{R}^d)} \leq 1 \end{aligned}$$

and denote the unique optimal solution by ϕ_{k+1}.

4: **if** $R_\varepsilon(u_k, \phi_{k+1}) \leq V_k$ **then**

5: **return** u_k as an optimal solution to $(\mathrm{P}_{\delta,\varepsilon})$.

6: **end if**

7: Set $k \leftarrow k+1$ and go to step 2.

Lemma 4.14. *Let $F : L^2(\Omega) \to \mathbb{R}$ fulfill Assumption 4.1. Denote by $\{(u_k, V_k)\}_{k\in\mathbb{N}} \subset L^2(\Omega)\times\mathbb{R}_{\geq 0}$ the sequence of optimal solutions to (P_k) produced by Algorithm 2. Each weakly converging subsequence of $\{(u_k, V_k)\}_{k\in\mathbb{N}}$ in $L^2(\Omega)\times\mathbb{R}$ also converges strongly in $L^2(\Omega)\times\mathbb{R}$. For each accumulation point $(\bar{u}, \bar{V}) \in L^2(\Omega)\times\mathbb{R}$ of $\{(u_k, V_k)\}_{k\in\mathbb{N}}$, there holds $\mathrm{TV}_\varepsilon(\bar{u}) \leq \bar{V}$, that is, $(\bar{u}, \bar{V})$ is feasible for $(\bar{\mathrm{P}}_{\delta,\varepsilon})$.*

Proof. We first prove that each weakly converging subsequence in $L^2(\Omega)\times\mathbb{R}$ of the sequence $\{(u_k, V_k)\}_{k\in\mathbb{N}}$ actually converges strongly in $L^2(\Omega)\times\mathbb{R}$. To this end, we consider a subsequence of $\{(u_{k_\ell}, V_{k_\ell})\}_{\ell\in\mathbb{N}}$ that converges weakly to some $(\bar{u}, \bar{V}) \in L^2(\Omega)\times\mathbb{R}$, that is, $u_{k_\ell} \rightharpoonup \bar{u}$ in $L^2(\Omega)$ and $V_{k_\ell} \to \bar{V}$ in $\mathbb{R}$ as $\ell \to \infty$. Since $\{u \in L^2(\Omega) : \underline{\nu} \leq u(x) \leq \overline{\nu}$ for a.a. $x \in \Omega\}$ is sequentially weakly closed in $L^2(\Omega)$, the weak limit $\bar{u}$ fulfills $\underline{\nu} \leq \bar{u}(x) \leq \overline{\nu}$ for almost all $x \in \Omega$ and since $V_k \geq 0$ for all $k \in \mathbb{N}$, there holds $\bar{V} \geq 0$. The limit $(\bar{u}, \bar{V})$ is feasible for (P_i) for all $i \in \mathbb{N}$ because for each fixed but arbitrary $i \in \mathbb{N}$, there exists some $\ell_0 \in \mathbb{N}$ large enough such that $k_\ell \geq i$ for all $\ell \geq \ell_0$, which implies

$$R_\varepsilon(u_{k_\ell}, \phi_i) = -\frac{\varepsilon}{2}a[\phi_i, \phi_i] + \int_\Omega u_{k_\ell}(x) \operatorname{div} \phi_i(x)\,\mathrm{d}x \leq V_{k_\ell}$$

for all $\ell \geq \ell_0$. By passing $\ell \to \infty$, we obtain for each $i \in \mathbb{N}$ that

$$\begin{aligned} R_\varepsilon(\bar{u}, \phi_i) &= -\frac{\varepsilon}{2} a[\phi_i, \phi_i] + \int_\Omega \bar{u}(x) \operatorname{div} \phi_i(x) \,\mathrm{d}x \\ &= \lim_{\ell\to\infty} -\frac{\varepsilon}{2} a[\phi_i, \phi_i] + \int_\Omega u_{k_\ell}(x) \operatorname{div} \phi_i(x) \,\mathrm{d}x \\ &\leq \lim_{\ell\to\infty} V_{k_\ell} = \bar{V}. \end{aligned} \tag{4.3}$$

Since (u_k, V_k) is optimal for (P_k), there holds $\bar{J}(u_k, V_k) \leq \bar{J}(\bar{u}, \bar{V})$ for all $k \in \mathbb{N}$. Together with the weak lower semicontinuity of $\bar{J}$, there holds

$$\bar{J}(\bar{u}, \bar{V}) \leq \liminf_{\ell\to\infty} \bar{J}(u_{k_\ell}, V_{k_\ell}) \leq \limsup_{\ell\to\infty} \bar{J}(u_{k_\ell}, V_{k_\ell}) \leq \bar{J}(\bar{u}, \bar{V})$$

and hence $\bar{J}(\bar{u}, \bar{V}) = \lim_{\ell\to\infty} \bar{J}(u_{k_\ell}, V_{k_\ell})$. Since F is weakly lower semicontinuous by Assumption 4.1 and $\bar{V} = \lim_{\ell\to\infty} V_{k_\ell}$, there holds

$$-F(\bar{u}) - \alpha\bar{V} \geq \limsup_{\ell\to\infty} -F(u_{k_\ell}) - \alpha V_{k_\ell}.$$

Since $\frac{\delta}{2}\|\cdot\|^2_{L^2(\Omega)}$ is weakly lower semicontinuous, it follows

$$\begin{aligned} \frac{\delta}{2}\|\bar{u}\|^2_{L^2(\Omega)} &\leq \liminf_{\ell\to\infty} \frac{\delta}{2}\|u_{k_\ell}\|^2_{L^2(\Omega)} \\ &\leq \limsup_{\ell\to\infty} \frac{\delta}{2}\|u_{k_\ell}\|^2_{L^2(\Omega)} \\ &= \limsup_{\ell\to\infty} \left(\frac{\delta}{2}\|u_{k_\ell}\|^2_{L^2(\Omega)} + F(u_{k_\ell}) + \alpha V_{k_\ell} - F(u_{k_\ell}) - \alpha V_{k_\ell}\right) \\ &\leq \limsup_{\ell\to\infty} \left(\frac{\delta}{2}\|u_{k_\ell}\|^2_{L^2(\Omega)} + F(u_{k_\ell}) + \alpha V_{k_\ell}\right) + \limsup_{\ell\to\infty} \left(-F(u_{k_\ell}) - \alpha V_{k_\ell}\right) \\ &\leq J(\bar{u}, \bar{V}) - F(\bar{u}) - \alpha\bar{V} \\ &= \frac{\delta}{2}\|\bar{u}\|^2_{L^2(\Omega)}, \end{aligned}$$

that is, $\lim_{\ell\to\infty} \|u_{k_\ell}\|_{L^2(\Omega)} = \|\bar{u}\|_{L^2(\Omega)}$. Together with the weak convergence $u_{k_\ell} \rightharpoonup \bar{u}$ in $L^2(\Omega)$ as $\ell \to \infty$, it follows that the subsequence $\{u_{k_\ell}\}_{\ell\in\mathbb{N}}$ converges strongly to $\bar{u}$ in $L^2(\Omega)$, which proves the first claim.

Next, we prove that $\mathrm{TV}_\varepsilon(\bar{u}) \leq \bar{V}$. To this end, we restrict the subsequence $\{(u_{k_\ell}, V_{k_\ell})\}_{\ell\in\mathbb{N}}$ and the corresponding maximizers $\{\phi_{k_\ell+1}\}_{\ell\in\mathbb{N}}$ further to subsubsequences $\{(u_{k_{\ell_m}}, V_{k_{\ell_m}})\}_{m\in\mathbb{N}}$ and $\{\phi_{k_{\ell_m}+1}\}_{m\in\mathbb{N}}$, such that $u_{k_{\ell_m}} \to \bar{u}$ in $L^2(\Omega)$ as $m \to \infty$, $\phi_{k_{\ell_m}+1} \rightharpoonup \tilde{\phi}$ in $L^2(\Omega; \mathbb{R}^d)$ as $m \to \infty$, and $\operatorname{div} \phi_{k_{\ell_m}+1} \rightharpoonup \operatorname{div} \tilde{\phi}$ in $L^2(\Omega)$ as $m \to \infty$. Such subsubsequences exist due to Lemma A.4. This yields in particular $\phi_{k_{\ell_m}} \rightharpoonup \tilde{\phi}$ in H as $m \to \infty$. Since the feasible set $\{\phi \in H : \|\phi\|_{L^\infty(\Omega;\mathbb{R}^d)} \leq 1\}$ of (Q_ε) and (Q_k) is non-empty, convex, and closed in H, it is sequentially weakly closed in

H, which yields that the weak limit $\tilde{\phi}$ is feasible for (Q_ε) and (Q_k). We denote the unique optimal solution to (Q_ε) for $u = \bar{u}$ by $\bar{\phi} \in H$, which exists due to Lemma 4.5. Since the mapping $H \ni \phi \mapsto a[\phi, \phi] \in \mathbb{R}$ is convex and continuous, it is weakly lower semicontinuous, which yields

$$-\frac{\varepsilon}{2} a[\tilde{\phi}, \tilde{\phi}] \geq \limsup_{m\to\infty} -\frac{\varepsilon}{2} a[\phi_{k_{\ell_m}+1}, \phi_{k_{\ell_m}+1}].$$

By the strong convergence of the subsubsequence $u_{k_{\ell_m}} \to \bar{u}$ in $L^2(\Omega)$ and the optimality of $\phi_{k_{\ell_m}+1}$ for (Q_k), we obtain

$$\begin{aligned}
R_\varepsilon(\bar{u}, \bar{\phi}) &\geq R_\varepsilon(\bar{u}, \tilde{\phi}) \\
&= -\frac{\varepsilon}{2} a[\tilde{\phi}, \tilde{\phi}] + \int_\Omega \bar{u}(x) \operatorname{div} \tilde{\phi}(x) \,\mathrm{d}x \\
&\geq \limsup_{m\to\infty} \Big(-\frac{\varepsilon}{2} a[\phi_{k_{\ell_m}+1}, \phi_{k_{\ell_m}+1}] \Big) + \lim_{m\to\infty} \int_\Omega u_{k_{\ell_m}}(x) \operatorname{div} \phi_{k_{\ell_m}+1}(x) \,\mathrm{d}x \\
&\geq \limsup_{m\to\infty} R_\varepsilon(u_{k_{\ell_m}}, \phi_{k_{\ell_m}+1}) \\
&\geq \liminf_{m\to\infty} R_\varepsilon(u_{k_{\ell_m}}, \phi_{k_{\ell_m}+1}) \\
&\geq \liminf_{m\to\infty} R_\varepsilon(u_{k_{\ell_m}}, \bar{\phi}) \\
&= R_\varepsilon(\bar{u}, \bar{\phi}).
\end{aligned}$$

Hence, there holds

$$\lim_{m\to\infty} R_\varepsilon(u_{k_{\ell_m}}, \phi_{k_{\ell_m}+1}) = R_\varepsilon(\bar{u}, \tilde{\phi}) = R_\varepsilon(\bar{u}, \bar{\phi}). \tag{4.4}$$

and since (Q_ε) is uniquely solvable by Lemma 4.5, it follows that $\tilde{\phi} = \bar{\phi}$ because $\bar{\phi}$ is feasible and $\tilde{\phi}$ is optimal for (Q_ε) with $u = \bar{u}$. Moreover, (4.4) implies

$$\begin{aligned}
\frac{\varepsilon}{2}\, a[\bar{\phi}, \bar{\phi}] &\leq \liminf_{m\to\infty} \frac{\varepsilon}{2}\, a[\phi_{k_{\ell_m}+1}, \phi_{k_{\ell_m}+1}] \\
&\leq \limsup_{m\to\infty} \frac{\varepsilon}{2}\, a[\phi_{k_{\ell_m}+1}, \phi_{k_{\ell_m}+1}] \\
&\leq \limsup_{m\to\infty} -R_\varepsilon(u_{k_{\ell_m}}, \phi_{k_{\ell_m}+1}) + \limsup_{m\to\infty} \int_\Omega u_{k_{\ell_m}}(x) \operatorname{div} \phi_{k_{\ell_m}+1}(x) \,\mathrm{d}x \\
&= -R_\varepsilon(\bar{u}, \bar{\phi}) + \int_\Omega \bar{u}(x) \operatorname{div} \bar{\phi}(x) \,\mathrm{d}x = \frac{\varepsilon}{2}\, a[\bar{\phi}, \bar{\phi}]
\end{aligned}$$

and therefore $\lim_{m\to\infty} a[\phi_{k_{\ell_m}+1}, \phi_{k_{\ell_m}+1}] = a[\bar{\phi}, \bar{\phi}]$. By (4.3), there holds with $i = k_{\ell_m} + 1$ that

$$R_\varepsilon(\bar{u}, \phi_{k_{\ell_m}+1}) \leq \bar{V}$$

for all $m \in \mathbb{N}$. Together with the strong convergence of $u_{k_{\ell_m}} \to \bar{u}$ in $L^2(\Omega)$ and the weak convergence of $\operatorname{div} \phi_{k_{\ell_m}+1} \rightharpoonup \operatorname{div} \bar{\phi}$ in $L^2(\Omega)$ as $m \to \infty$, this leads to

$$\begin{aligned}
\bar{V} &\geq R_\varepsilon(\bar{u}, \phi_{k_{\ell_m}+1}) \\
&= -\frac{\varepsilon}{2} a[\phi_{k_{\ell_m}+1}, \phi_{k_{\ell_m}+1}] + \int_\Omega \bar{u}(x) \operatorname{div} \phi_{k_{\ell_m}+1}(x) \,\mathrm{d}x \\
&\to -\frac{\varepsilon}{2} a[\bar{\phi}, \bar{\phi}] + \int_\Omega \bar{u}(x) \operatorname{div} \bar{\phi}(x) \,\mathrm{d}x \\
&= R_\varepsilon(\bar{u}, \bar{\phi}) = \mathrm{TV}_\varepsilon(\bar{u})
\end{aligned}$$

as $m \to \infty$. This proves that each accumulation point of $\{(u_k, V_k)\}_{k\in\mathbb{N}}$ is feasible for $(\bar{\mathrm{P}}_{\delta,\varepsilon})$. □

Now we prove that the accumulation points of the sequence of iterates produced by Algorithm 2 are optimal for $(\bar{\mathrm{P}}_{\delta,\varepsilon})$.

Theorem 4.15. *Let $F : L^2(\Omega) \to \mathbb{R}$ fulfill Assumption 4.1. Assume that Algorithm 2 does not stop after a finite number of iterations. Denote the sequence of optimal solutions to (P_k) generated by Algorithm 2 by $\{(u_k, V_k)\}_{k\in\mathbb{N}} \subset L^2(\Omega)\times\mathbb{R}_{\geq 0}$. The sequence $\{(u_k, V_k)\}_{k\in\mathbb{N}}$ admits a converging subsequence in $L^2(\Omega) \times \mathbb{R}$ and each accumulation point $(\bar{u}, \bar{V})$ of $\{(u_k, V_k)\}_{k\in\mathbb{N}}$ is an optimal solution to $(\bar{\mathrm{P}}_{\delta,\varepsilon})$ with $\mathrm{TV}_\varepsilon(\bar{u}) = \bar{V}$. If F is additionally convex, the whole sequence $\{(u_k, V_k)\}_{k\in\mathbb{N}}$ converges in $L^2(\Omega) \times \mathbb{R}$ to the unique minimizer $(\bar{u}, \mathrm{TV}_\varepsilon(\bar{u}))$ of $(\bar{\mathrm{P}}_{\delta,\varepsilon})$.*

Proof. For each $k \in \mathbb{N}$, there holds $\underline{\nu} \leq u_k(x) \leq \overline{\nu}$ for almost all $x \in \Omega$ such that the sequence $\{u_k\}_{k\in\mathbb{N}}$ is bounded in $L^\infty(\Omega)$. Define $w \in L^2(\Omega)$ by $w \equiv \underline{\nu}$. Because $(w, 0) \in L^2(\Omega) \times \mathbb{R}_{\geq 0}$ is a feasible solution to (P_k) for all $k \in \mathbb{N}$ with finite objective value, $V_k \geq 0$, and $F(u_k) \geq B$ by (1.1), there holds

$$0 \leq V_k \leq \frac{1}{\alpha}\left(F(w) - B + \frac{\delta}{2}\|w\|^2_{L^2(\Omega)}\right).$$

This yields that we can extract a weakly converging subsequence of $\{(u_k, V_k)\} \subset L^2(\Omega) \times \mathbb{R}$ that actually converges strongly in $L^2(\Omega) \times \mathbb{R}$ by Lemma 4.14.

The feasibility of an accumulation point $(\bar{u}, \bar{V})$ of $\{(u_k, V_k)\}_{k\in\mathbb{N}}$ for $(\bar{\mathrm{P}}_{\delta,\varepsilon})$ is proven in Lemma 4.14. Denote by $\{(u_{k_\ell}, V_{k_\ell})\}_{\ell\in\mathbb{N}}$ a subsequence of $\{(u_k, V_k)\}_{k\in\mathbb{N}}$ that converges to $(\bar{u}, \bar{V})$ in $L^2(\Omega)$. To show the optimality of $(\bar{u}, \bar{V})$, consider arbitrary $(u, V) \in L^2(\Omega) \times \mathbb{R}_{\geq 0}$ that is feasible for $(\bar{\mathrm{P}}_{\delta,\varepsilon})$. Then, by construction, (u, V) is also feasible for (P_k) for each $k \in \mathbb{N}$ and the optimality of (u_k, V_k) implies $\bar{J}(u_k, V_k) \leq \bar{J}(u, V)$ for all $k \in \mathbb{N}$. The lower semicontinuity of $\bar{J}$ thus gives

$$\bar{J}(\bar{u}, \bar{V}) \leq \liminf_{\ell\to\infty} \bar{J}(u_{k_\ell}, V_{k_\ell}) \leq \limsup_{\ell\to\infty} \bar{J}(u_{k_\ell}, V_{k_\ell}) \leq \bar{J}(u, V)$$

and, since (u, V) was an arbitrary feasible solution to $(\bar{\mathrm{P}}_{\delta,\varepsilon})$, this yields the optimality of $(\bar{u}, \bar{V})$ for $(\bar{\mathrm{P}}_{\delta,\varepsilon})$. Since $\bar{V}$ is bounded from below by $\mathrm{TV}_\varepsilon(\bar{u})$, there holds $\mathrm{TV}_\varepsilon(\bar{u}) = \bar{V}$ due to the minimization.

Now let F be convex, then the optimal solution $(\bar{u}, \bar{V})$ to $(\bar{\mathrm{P}}_{\delta,\varepsilon})$ is unique by Theorem 4.12. It follows that each subsequence of $\{u_k\}_{k\in\mathbb{N}}$ has subsequence that converges strongly in $L^2(\Omega) \times \mathbb{R}$ to the unique minimizer $(\bar{u}, \bar{V})$. The Urysohn subsequence principle yields that the whole sequence $\{(u_k, V_k)\}_{k\in\mathbb{N}}$ converges strongly in $L^2(\Omega) \times \mathbb{R}$ to the unique minimizer $(\bar{u}, \bar{V})$. $\square$

4.3.2 Convergence of the minimizers of the regularized problems

It remains to be shown that the optimal solutions to the regularized problems $(\mathrm{P}_{\delta,\varepsilon})$ converge to an optimal solution to (P_R) as the regularization parameters $\varepsilon > 0$ and $\delta > 0$ are driven to zero.

Theorem 4.16. *Let $F : L^2(\Omega) \to \mathbb{R}$ fulfill Assumption 4.1. Assign to each $\varepsilon > 0$ a $\delta(\varepsilon) > 0$ such that $\delta(\varepsilon) \searrow 0$ as $\varepsilon \searrow 0$ and denote an optimal solution to $(\mathrm{P}_{\delta,\varepsilon})$ with $\delta = \delta(\varepsilon)$ by $u_\varepsilon \in L^2(\Omega)$. There exists a weakly convergent subsequence in $L^2(\Omega)$ of $\{u_\varepsilon\}_{\varepsilon>0}$ and each weak accumulation point in $L^2(\Omega)$ is a minimizer of (P_R). If F is strictly convex, the whole sequence converges weakly to the unique minimizer of (P_R).*

Proof. Due to the boundedness $\underline{\nu} \le u_\varepsilon(x) \le \overline{\nu}$ for almost all $x \in \Omega$ for all $\varepsilon > 0$, there exists a weakly converging subsequence of $\{u_\varepsilon\}_{\varepsilon>0}$ in $L^2(\Omega)$.

Now let $\{u_\varepsilon\}_{\varepsilon>0}$ converge weakly in $L^2(\Omega)$ to some $\bar{u} \in L^2(\Omega)$, that is, $u_\varepsilon \rightharpoonup \bar{u}$ in $L^2(\Omega)$ as $\varepsilon \searrow 0$. Since $\{u \in L^2(\Omega) : \underline{\nu} \le u(x) \le \overline{\nu}$ for a.a. $x \in \Omega\}$ is weakly sequentially closed in $L^2(\Omega)$, the limit $\bar{u}$ is feasible for (P_R). There holds for each $\phi \in H$ with $\|\phi\|_{L^\infty(\Omega;\mathbb{R}^d)} \le 1$ that

$$\int_\Omega u_\varepsilon(x) \operatorname{div} \phi(x) \,\mathrm{d}x \le \mathrm{TV}_\varepsilon(u_\varepsilon) + \frac{\varepsilon}{2} a[\phi, \phi].$$

This yields that

$$\int_\Omega \bar{u}(x) \operatorname{div} \phi(x) \,\mathrm{d}x = \lim_{\varepsilon\searrow 0} \int_\Omega u_\varepsilon(x) \operatorname{div} \phi(x) \,\mathrm{d}x \le \liminf_{\varepsilon\searrow 0} \mathrm{TV}_\varepsilon(u_\varepsilon)$$

for all $\phi \in H$. Supremizing over all $\phi \in H$ yields

$$\mathrm{TV}(\bar{u}) \le \liminf_{\varepsilon\searrow 0} \mathrm{TV}_\varepsilon(u_\varepsilon).$$

Let $u \in L^2(\Omega)$ be feasible for $(\mathrm{P_R})$. Since F is weakly lower semicontinuous, there holds

$$\begin{aligned} F(\bar{u}) + \alpha \mathrm{TV}(\bar{u}) &\leq \liminf_{\varepsilon \searrow 0} F(u_\varepsilon) + \frac{\delta(\varepsilon)}{2} \|u_\varepsilon\|^2_{L^2(\Omega)} + \alpha \mathrm{TV}_\varepsilon(u_\varepsilon) \\ &\leq \liminf_{\varepsilon \searrow 0} F(u) + \frac{\delta(\varepsilon)}{2} \|u\|^2_{L^2(\Omega)} + \alpha \mathrm{TV}_\varepsilon(u) \\ &= F(u) + \alpha \mathrm{TV}(u) \end{aligned}$$

by Lemma 4.9. Since $\bar{u}$ is feasible for $(\mathrm{P_R})$, this yields the optimality of $\bar{u}$ for $(\mathrm{P_R})$.

If F is strictly convex, then the minimizer $\bar{u}$ of $(\mathrm{P_R})$ is unique. This yields that each subsequence of $\{u_\varepsilon\}_{\varepsilon>0}$ has a subsequence that converges weakly in $L^2(\Omega)$ to $\bar{u}$. By the Urysohn subsequence principle, the whole sequence $\{u_\varepsilon\}_{\varepsilon>0}$ converges weakly in $L^2(\Omega)$ to $\bar{u}$ as $\varepsilon \searrow 0$. □

4.4 Optimality conditions and construction of an exact solution

In this Section 4.4, we aim to construct instances of $(\mathrm{P_R})$ and (P) with known exact optimal solution for our numerical experiments in Chapter 6. To this end, we state the optimality conditions to $(\mathrm{P_R})$ from [85] and the optimality conditions to the unrestricted relaxation of $(\mathrm{P_R})$ from [25, 40].

4.4.1 Optimality conditions

To meet all assumptions and to keep the notation simple, we restrict to the case $F(u) = \frac{1}{2}\|S(u+f) - y_d\|^2_{L^2(\Omega)}$ with $f, y_d \in L^2(\Omega)$ and where $S : L^2(\Omega) \to H^1_0(\Omega)$, $w \mapsto y$, is the linear and continuous solution operator that maps $w \in L^2(\Omega)$ to the unique solution $y \in H^1_0(\Omega)$ to

$$\text{(PDE)} \qquad -\kappa \Delta y + y = w \text{ in } \Omega, \quad y = 0 \text{ on } \partial\Omega$$

with fixed $\kappa > 0$. We highlight that the solution operator $S : L^2(\Omega) \to H^1_0(\Omega)$ is also injective such that $F : L^2(\Omega) \to \mathbb{R}$ is strictly convex. This yields the uniqueness of the optimal solution to $(\mathrm{P_R})$. The optimality conditions corresponding to $(\mathrm{P_R})$ are stated in [85] and read

$$
\begin{aligned}
&(4.5a)\quad \bar u \in \mathrm{BV}(\Omega), \quad \bar y \in H_0^1(\Omega), \quad \bar p \in H_0^1(\Omega), \quad \bar\lambda \in \mathcal{M}(\Omega;\mathbb{R}^d)^*, \quad \bar\lambda_\ell, \bar\lambda_u \in L^2(\Omega)\\
&(4.5b)\quad \underline{\nu} \le \bar u \le \overline{\nu}, \quad -\kappa\Delta\bar y + \bar y = \bar u + f, \quad -\kappa\Delta\bar p + \bar p = \bar y - y_d\\
&(4.5c)\quad -\alpha \operatorname{div}\bar\lambda - \bar\lambda_\ell + \bar\lambda_u = \bar p\\
&(4.5d)\quad -\operatorname{div}\bar\lambda \in \partial\mathrm{TV}(\bar u)\\
&(4.5e)\quad \bar\lambda_\ell, \bar\lambda_u \ge 0, \quad \int_\Omega \bar\lambda_\ell(x)(\underline{\nu} - \bar u(x))\,\mathrm{d}x = 0, \quad \int_\Omega \bar\lambda_u(x)(\bar u(x) - \overline{\nu})\,\mathrm{d}x = 0,
\end{aligned}
$$

where the operator

$$-\operatorname{div} : \mathcal{M}(\Omega;\mathbb{R}^d)^* \to \mathrm{BV}(\Omega)^*$$

is the adjoint operator to

$$\nabla : \mathrm{BV}(\Omega) \to \mathcal{M}(\Omega;\mathbb{R}^d)$$

that is defined by

$$\langle \operatorname{div}\Phi, v\rangle_{\mathrm{BV}^*,\mathrm{BV}} = -\langle \Phi, \nabla v\rangle_{\mathcal{M}^*,\mathcal{M}} \tag{4.6}$$

for $\Phi \in \mathcal{M}(\Omega;\mathbb{R}^d)^*$ and $v \in \mathrm{BV}(\Omega)$. To construct an exact solution to $(\mathrm{P_R})$, we restrict to the case $d = 2$, which yields that the embedding $\mathrm{BV}(\Omega) \hookrightarrow L^2(\Omega)$ is continuous by Corollary 3.49 in [4]. To simplify the optimality conditions further, we consider the following relaxation of $(\mathrm{P_R})$ that reads

$$\min_{u\in L^2(\Omega)} \quad F(u) + \alpha\mathrm{TV}(u) \tag{$\mathrm{P_{R_R}}$}$$

and follow [25,40] to state the corresponding optimality conditions. In Section 4.4.2, we use the optimality conditions to construct an optimal solution $\bar u$ to $(\mathrm{P_{R_R}})$. If we then choose $\underline{\nu}, \overline{\nu} \in \mathbb{R}$ such that $\underline{\nu} \le \bar u(x) \le \overline{\nu}$ for almost all $x \in \Omega$, this immediately yields the optimality of $\bar u$ for $(\mathrm{P_R})$. By [25], the necessary optimality conditions to $(\mathrm{P_{R_R}})$ read

$$
\begin{aligned}
&(4.7a)\quad \bar u \in \mathrm{BV}(\Omega), \quad \bar y \in H_0^1(\Omega), \quad \bar p \in H_0^1(\Omega), \quad \bar\Phi \in C_0(\Omega;\mathbb{R}^2)\\
&(4.7b)\quad -\kappa\Delta\bar y + \bar y = \bar u + f, \quad -\kappa\Delta\bar p + \bar p = \bar y - y_d\\
&(4.7c)\quad \alpha\langle\nabla v, \bar\Phi\rangle_{\mathcal{M},C_0} + \int_\Omega \bar p(x) v(x)\,\mathrm{d}x = 0 \quad \forall\, v \in \mathrm{BV}(\Omega)\\
&(4.7d)\quad \langle\nabla\bar u, \bar\Phi\rangle_{\mathcal{M},C_0} = \mathrm{TV}(\bar u), \quad \langle\nabla v, \bar\Phi\rangle_{\mathcal{M},C_0} \le \mathrm{TV}(v) \quad \forall\, v \in \mathrm{BV}(\Omega)
\end{aligned}
$$

By the embedding $C_0(\Omega;\mathbb{R}^d) \hookrightarrow \mathcal{M}(\Omega;\mathbb{R}^d)^*$, we may use (4.6) to reformulate (4.7c) as $\alpha \operatorname{div} \bar{\Phi} = \bar{p}$. If we insert this into (4.7d) and use that

$$\int_\Omega \bar{p}(x)\,\mathrm{d}x = \alpha \int_\Omega \operatorname{div} \bar{\Phi}(x)\,\mathrm{d}x = \alpha \int_{\partial\Omega} \bar{\Phi}(x) \cdot n(x)\,\mathrm{d}\mathcal{H}^1(x) = 0$$

due to $\bar{\Phi} \in C_0(\Omega;\mathbb{R}^d)$ and where n denotes the outer normal of Ω, we arrive at

$$\bar{u} \in \mathrm{BV}(\Omega), \quad \bar{y} \in H_0^1(\Omega), \quad \bar{p} \in H_0^1(\Omega) \tag{4.8a}$$

$$-\kappa\Delta\bar{y} + \bar{y} = \bar{u} + f, \quad -\kappa\Delta\bar{p} + \bar{p} = \bar{y} - y_d \tag{4.8b}$$

$$\int_\Omega \bar{p}(x)\,\mathrm{d}x = 0 \tag{4.8c}$$

$$-\int_\Omega \bar{p}(x)\bar{u}(x)\,\mathrm{d}x = \alpha\mathrm{TV}(\bar{u}), \quad -\int_\Omega \bar{p}(x)v(x)\,\mathrm{d}x \le \alpha\mathrm{TV}(v) \quad \forall\, v \in \mathrm{BV}(\Omega) \tag{4.8d}$$

which corresponds to the optimality conditions from [40]. Since the optimality conditions from [40] are sufficient, this gives the equivalence of both optimality systems (4.7) and (4.8).

4.4.2 Construction of an exact optimal solution

For our numerical experiments in Chapter 6, we construct an instance with known unique optimal solution. To this end, let $F(u) = \frac{1}{2}\|S(u+f) - y_d\|^2_{L^2(\Omega)}$ be defined as above. We define $\Omega = (0,2)^2$ and $B \coloneqq B_{0.5}((1,1)^T) \subset \Omega$ as the ball around $(1,1)^T \in \mathbb{R}^2$ with radius 0.5 and perimeter $P(B) = \pi$. Moreover, we specify the optimal solution to $(\mathrm{P_R})$ by $\bar{u} \in \mathrm{BV}(\Omega)$ with

$$\bar{u}(x) = \begin{cases} 1 & \text{if } x \in B \\ 0 & \text{if } x \in \Omega \setminus B. \end{cases}$$

We then have $\nabla\bar{u} = -n_B\mathcal{H}^1 \llcorner \partial B$, where n_B denotes the outer unit normal to B. We aim to find functions $\bar{y}, \bar{p} \in C_0^2(\Omega)$, $\bar{\Phi} \in C_c^3(\Omega;\mathbb{R}^d)$, $y_d \in L^2(\Omega)$, and $f \in L^2(\Omega)$ such that the optimality system (4.7) is fulfilled.

We choose the function $\bar{\Phi}$ such that $\bar{\Phi}|_{\partial B} = -n_B$. To this end, we define $\bar{\Phi}(x) \coloneqq \Phi_0(x - (1,1)^T)$ with

$$\Phi_0(x) = \begin{cases} -\Psi(\|x\|_2)\frac{x}{\|x\|_2} & \text{if } \|x\|_2 \in (\rho_1, \rho_2) \\ 0 & \text{else} \end{cases}$$

with $0 < \rho_1 < 0.5 < \rho_2 < 1$ and $\Psi \in C_c^3((0,1))$ such that $\operatorname{supp}\Psi \subset [\rho_1, \rho_2]$, $\Psi(\frac{1}{2}) = 1$, and $|\Psi(r)| \le 1$ for all $r \in (0,1)$. This can be done by using the ansatz $\Psi(r) =$

$\sum_{k=0}^{8} a_k r^k$ and solving the linear equation system resulting from the constraints

$$\begin{aligned} \Psi(\rho_1) = \Psi(\rho_2) = 0, \quad \Psi(0.5) = 1,\\ \Psi'(\rho_1) = \Psi'(\rho_2) = 0,\\ \Psi''(\rho_1) = \Psi''(\rho_2) = 0,\\ \Psi'''(\rho_1) = \Psi'''(\rho_2) = 0 \end{aligned}$$

to determine the coefficients $a_k \in \mathbb{R}$ for $k = 0, \ldots, 8$. This yields

$$\begin{aligned} \langle \nabla \bar{u}, \bar{\Phi} \rangle_{\mathcal{M},C_0} &= \int_\Omega \bar{\Phi}(x) \, \mathrm{d}\nabla \bar{u}(x) = -\int_{\partial B} \bar{\Phi}(x) \cdot n_B(x) \, \mathrm{d}\mathcal{H}^1(x)\\ &= \int_{\partial B} 1 \, \mathrm{d}\mathcal{H}^1(x) = P(B) = \mathrm{TV}(\bar{u}), \end{aligned}$$

where n_B denotes the outer unit normal vector of B. Moreover, since $\bar{\Phi} \in C_0^1(\Omega; \mathbb{R}^d)$ with $\|\bar{\Phi}\|_{L^\infty(\Omega;\mathbb{R}^d)} \leq 1$, there holds with (4.6) that

$$\langle \nabla v, \bar{\Phi} \rangle_{\mathcal{M},C_0} = -\int_\Omega \operatorname{div} \bar{\Phi}(x) v(x) \, \mathrm{d}x \leq \mathrm{TV}(v),$$

for all $v \in \mathrm{BV}(\Omega)$ which yields (4.7d). To fulfill (4.7c), we define $\bar{p} = \alpha \operatorname{div} \bar{\Phi} \in C_0^2(\Omega)$. Moreover, we define $\bar{y} \in C_0^2(\Omega)$ by $\bar{y}(x) \coloneqq -2x_1^2(2-x_1)^2 x_2^2 (2-x_2)^2$. To fulfill (4.7b), we define $y_d \coloneqq \bar{y} + \kappa \Delta \bar{p} - \bar{p}$ and $f \coloneqq -\kappa \Delta \bar{y} + \bar{y} - \bar{u}$. In particular, $\bar{u}$ is an optimal solution to ($\mathrm{P}_{\mathrm{R}_\mathrm{R}}$).

If we now choose $U = \{0, 1\}$ for (P) and keep $\Omega = (0, 2)^2$ and F as above, then $\underline{\nu} = 0$ and $\overline{\nu} = 1$ for (P_R) such that $\bar{u}$ is the unique optimal solution to (P) and (P_R). We will use these instances in our numerical experiments in Section 6.3. With the knowledge of the unique optimal solution, we can compute the errors between the solutions that are obtained by the application of the algorithms to the regularized and discretized problems corresponding to (P) and (P_R). The discretizations of (P) and (P_R) will be introduced in the following chapter.

Chapter 5

Discretization

This chapter is dedicated to the discretization of problem (P), which was given by

$$
\text{(P)} \qquad \begin{aligned} \min_{u \in L^2(\Omega)} \quad & F(u) + \alpha \mathrm{TV}(u) \\ \text{s.t.} \quad & u(x) \in U \subset \mathbb{Z} \text{ for a.a. } x \in \Omega, \end{aligned}
$$

where $U = \{\nu_1, \ldots, \nu_N\} \subset \mathbb{Z}$ is a finite set of integers and $\alpha > 0$ and $\underline{\nu} = \min\{\nu : \nu \in U\}$ and $\overline{\nu} = \max\{\nu : \nu \in U\}$. The considerations in this chapter closely follow the article [95].

Additional to Assumption 1.1, we assume the following regarding Ω and $F : L^2(\Omega) \to \mathbb{R}$ throughout this chapter.

Assumption 5.1. For the bounded Lipschitz domain Ω and the function $F : L^2(\Omega) \to \mathbb{R}$, we assume the following:

(i) $\Omega \subset \mathbb{R}^d$ with $d \in \{1, 2, 3\}$.

(ii) Ω is a finite union of bounded intervals, axis-aligned squares, or axis-aligned cubes.

(iii) $F : L^p(\Omega) \to \mathbb{R}$ fulfills Assumption 1.1 and is continuous for some $1 \le p < \infty$.

We will discretize (P) such that we retain the integrality condition on the input functions while being able to recover the total variation of solutions to (P). This is challenging because the underlying mesh prescribes the geometry of the discretized input functions. One approach, in particular for $U = \{0, 1\}$, to handle the total variation in combination with the integrality condition would be to use a Modica–Mortola energy functional [83] and relax the integrality condition. Then, one drives a parameter to zero that controls the non-binarity and may recover the total variation in the limit. In this case, the difficulty of the integrality is replaced by the non-convexity of the Modica–Mortola energy, however.

Many discretizations of the total variation term can already be found in literature but focus rarely on restrictions to integers and we refer to [30] for an overview. Approaches that rely on the primal formulation of the total variation are finite-differences discretizations and variants [1, 27, 69, 105], $P1$ discretizations [8, 9], and non-conforming discretizations with Crouzeix–Raviart functions [29]. A discretization of the dual formulation by discretizing the dual fields with Raviart–Thomas functions is presented in [23]. Related approaches based on discretized dual fields are [31, 38, 63]. While a $P1$ or Crouzeix–Raviart input function ansatz does not permit the integrality condition, the other approaches allow for piecewise constant ansatz functions. However, if there are Γ-convergence results for the discretized total variation, the recovery sequences generally need to attain non-integer values to recover the total variation of the limit functions, even if the limit functions themselves are integer-valued, see the proofs of Theorem 4 in [32] and Theorem 1.2 in [23]. The latter proves an error rate for the discretized optimization problems by means of strong duality of the continuous problems which does not hold in our integer setting.

We will solve this issue by introducing two coupled discretizations with a fine mesh for the input functions that is embedded into a coarser mesh for the total variation term. A superlinear coupling of the fine mesh size to the coarser one allows to consider integer-valued and piecewise constant discretized input functions and still be able to recover the total variation of limit functions due to an averaging effect on the coarser mesh. This makes it possible to prove convergence of the discretized total variation in the sense of Γ-convergence despite the restriction to integrality. Moreover, we are able to prove an error bound for the discretized total variation of the recovery sequence with respect to the total variation of its limit.

The discretization of the considered optimization problems is accompanied with the loss of compactness of the sequence of minimizers, which is due to possible chattering enabled by the enlarged null space of the discretized total variation caused by the coupled meshes. We compensate for this lack by adding a constraint to the discretized problems that enforces compactness on the sequence of minimizers and vanishes in the limit. This constraint contains a degree of freedom whose admissible range we determine. Even though its concrete choice is irrelevant when the mesh sizes are driven to zero, it may impact on the solutions of the discretized problems in practice. Together with the approximation properties of the discretized total variation, we prove the convergence of minimizers of the discretized problems to a minimizer of the original problem. To solve the discretized problems, we will introduce an outer-approximation algorithm.

In addition, we will derive a corresponding discretization for the relaxation ($\mathrm{P_R}$) and its regularization ($\mathrm{P}_{\delta,\varepsilon}$). Since the integrality constraint is relaxed to box constraints, discretizations known from literature like [23] apply. In order to be able to compute lower bounds from the discretizations of ($\mathrm{P_R}$) and ($\mathrm{P}_{\delta,\varepsilon}$), the focus lies on the conformity of the discretizations with respect to the discretization of (P).

(a) Mesh size $h = \frac{1}{4}$: $\mathrm{TV}(u_h) = 1.5$. (b) Mesh size $h = \frac{1}{8}$: $\mathrm{TV}(u_h) = 1.75$. (c) Mesh size $h = \frac{1}{16}$: $\mathrm{TV}(u_h) = 1.875$. (d) Limit function u: $\mathrm{TV}(u) = \sqrt{2}$.

Figure 5.1: Approximation of a limit function $u = \chi_{\{(-1,1)^T x > 0\}}$ with integer-valued functions u_h on quadrilateral meshes with mesh size h on the domain $\Omega = (0,1)$ with the corresponding values of the total variation.

This chapter is structured as follows. We start by introducing the discretized total variation and analyzing its approximation properties in Section 5.1. Section 5.2 is dedicated to the discretization of the considered optimization problems and the convergence of their minimizers to a minimizer of the original problem. In Section 5.3, we state and analyze an outer-approximation algorithm to solve the discretized problems. In Section 5.4, we introduce the discretization for $(\mathrm{P_R})$ and $(\mathrm{P}_{\delta,\varepsilon})$.

5.1 Discretized total variation

In order to discretize (P) appropriately, it is generally not sufficient to only discretize the input function $u \in \mathrm{BV_U}(\Omega)$ while keeping the usual total variation TV in the objective of (P) because it is generally not possible to recover the total variation of a limit function with the total variation of integer-valued discretized functions on prescribed cubic meshes. This can be seen by the following simple example which addresses the same issue as Figure 2 in [35].

Example 5.2. Let $d = 2$, $\Omega = (0,1)^2$, $U = \{0,1\}$, and $u \in \mathrm{BV_U}(\Omega)$ be defined by $u = \chi_{\{(-1,1)^T x > 0\}}$ as in Figure 5.1d. We discretize Ω by means of quadratic finite-element meshes $\{\mathcal{Q}_h\}_{h>0}$, where h denotes the height of the squares in $\mathcal{Q}_h$. We approximate the function u by binary-valued functions u_h that are piecewise constant on the mesh cells of $\mathcal{Q}_h$ as exemplarily demonstrated in Figure 5.1 for the mesh sizes $h = \frac{1}{4}, \frac{1}{8}, \frac{1}{16}$. Then there holds $\lim_{h \searrow 0} \mathrm{TV}(u_h) = 2 \neq \sqrt{2} = \mathrm{TV}(u)$, that is, we cannot recover the total variation of the limit function u with the total variation of such binary-valued discretized functions u_h on the cubic meshes $\mathcal{Q}_h$.

To solve this issue, we will introduce a discretized total variation and a discretization of (P) that includes two coupled finite-element meshes, where the finer mesh is used for the discretization of the input function u and the coarser mesh is used for the discretized total variation. To this end, we make the following assumptions regarding the finite-element meshes on Ω.

Assumption 5.3 ([95, Ass. 2.3])**.**

1. TV meshes: For $h > 0$, we consider a partition $\mathcal{Q}_h$ of Ω into intervals or axis-aligned squares or cubes $Q \in \mathcal{Q}_h$ of height $h > 0$, that is, $\Omega = \bigcup_{Q \in \mathcal{Q}_h} Q$.

2. Input meshes: For each $h > 0$, we consider a mesh $\mathcal{Q}_{\tau_h}$ of Ω of intervals or axis-aligned squares or cubes $Q \in \mathcal{Q}_{\tau_h}$ of height $\tau_h \in (0, h]$ that is embedded into the TV mesh $\mathcal{Q}_h$. Specifically, for each $\tilde{Q} \in \mathcal{Q}_h$ there exists $\mathcal{Q}_{\tilde{Q}} \subset \mathcal{Q}_{\tau_h}$ such that $\bigcup_{Q \in \mathcal{Q}_{\tilde{Q}}} Q = \tilde{Q}$.

We introduce the discretized total variation in Section 5.1.1. We prove its approximation properties in a Γ-convergence sense, where Section 5.1.2 is dedicated to the lim inf inequality and Section 5.1.3 provides the lim sup inequality and an error estimate for the discretized total variation of the recovery sequence.

5.1.1 Discretization of the total variation

By Lemma 4.3, we may replace the test function space $C_c^1(\Omega; \mathbb{R}^d)$ for the computation of the total variation term by $H_0(\mathrm{div}; \Omega)$, that is,

$$(\mathrm{TV}_{H_0(\mathrm{div})}) \quad \mathrm{TV}(u) = \sup \left\{ \int_\Omega u(x) \operatorname{div} \phi(x) \, \mathrm{d}x : \phi \in H_0(\mathrm{div}; \Omega), \|\phi\|_{L^\infty(\Omega; \mathbb{R}^d)} \leq 1 \right\}$$

for $u \in L^2(\Omega)$. For a mesh $\mathcal{Q}_h$ with $h > 0$ fulfilling Assumption 5.3.1 and $u \in L^1(\Omega)$, we define the following discretization of the total variation term TV as in [23] by replacing the test functions for the computation of TV by lowest-order Raviart–Thomas functions on the mesh $\mathcal{Q}_h$, that is,

$$\mathrm{TV}^{\mathrm{h}}(u) := \sup \left\{ \int_\Omega u(x) \operatorname{div} \phi(x) \, \mathrm{d}x \; : \; \phi \in RT0_0^h, \; \|\phi\|_{L^\infty(\Omega, \mathbb{R}^d)} \leq 1 \right\},$$

where $RT0_0^h := \left\{ \phi \in RT0^h : (\phi \cdot n)|_{\partial\Omega} \equiv 0 \right\}$ and n denotes the outer unit normal of Ω. We refer to Section 2.5 for a complete definition of Raviart–Thomas functions.

Lemma 5.4 ([95, Lem. 2.4])**.** *Let $\phi \in RT0^h$ for some $h > 0$ satisfy $\|\phi\|_{L^\infty(\Omega; \mathbb{R}^d)} \leq 1$. Then $\|\operatorname{div} \phi\|_{L^\infty(\Omega)} \leq \frac{2d}{h}$.*

Proof. We use the notations from Section 2.5. Then for each $Q \in \mathcal{Q}_h$ there holds $\phi|_Q \in P_1(Q)$ (for the case $d = 1$), $\phi|_Q \in P_{1,0}(Q) \times P_{0,1}(Q)$ (for the case $d = 2$), or $\phi|_Q \in P_{1,0,0}(Q) \times P_{0,1,0}(Q) \times P_{0,0,1}(Q)$ (for the case $d = 3$). Consider $\phi|_{\bar{Q}}$ on the element $\bar{Q} = \ell + [0, h]^d$ with $\ell \in \mathbb{R}^d$. Then $\phi|_{\bar{Q}}(x) = a + \sum_{i=1}^d c_i e^i x$ for $x \in \bar{Q}$ with $a, c \in \mathbb{R}^d$ and $e^i \in \mathbb{R}^d$, $i \in \{1, \ldots, d\}$, the canonical unit vector basis.

Since $\|\phi\|_{L^\infty(\Omega;\mathbb{R}^d)} \leq 1$, we have for each $i \in \{1,...,d\}$ that $|a_i + c_i x_i| \leq 1$ for all $x_i \in \ell_i + [0,h]$, in particular $|a_i + c_i\ell_i + c_i h| \leq 1$ and $|a_i + c_i\ell_i| \leq 1$. This yields

$$|\operatorname{div}\phi|_{\bar{Q}}| \leq \sum_{i=1}^d |c_i| \leq \sum_{i=1}^d \frac{1}{h}(|a_i + c_i\ell_i + c_i h| + |a_i + c_i\ell_i|) \leq \sum_{i=1}^d \frac{2}{h} = \frac{2d}{h}.$$

□

In contrast to TV, the discretized total variation $\mathrm{TV^h}$ always admits a maximizer and is thus always finite. Moreover, TV is always an upper bound for $\mathrm{TV^h}$.

Lemma 5.5 ([95, Lem. 2.5]). *Let $u \in L^2(\Omega)$. Then there is some $\phi \in RT0_0^h$ with $\|\phi\|_{L^\infty(\Omega;\mathbb{R}^d)} \leq 1$ such that $\mathrm{TV^h}(u) = \int_\Omega u(x)\operatorname{div}\phi(x)\,\mathrm{d}x < \infty$ and $\mathrm{TV^h}(u) \leq \mathrm{TV}(u)$.*

Proof. $\mathrm{TV^h}(u) \leq \mathrm{TV}(u)$ follows from ($\mathrm{TV}_{H_0(\mathrm{div})}$) since $RT0_0^h \subset H_0(\mathrm{div};\Omega)$. The existence of a maximizer of $\mathrm{TV^h}$ follows from the fact that the corresponding maximization problem can be rewritten as a finite-dimensional optimization problem with linear objective function and compact feasible set. This is because $RT0_0^h$ is finite-dimensional, $\operatorname{div}\phi$ for $\phi \in RT0_0^h$ is constant on the grid cells $Q \in \mathcal{Q}_h$, and the feasible set of $\mathrm{TV^h}(u)$ can be described by finitely many convex inequalities by bounding the Euclidean norm of the point evaluations of ϕ in each node of the mesh $\mathcal{Q}_h$ by 1. □

In order to discretize (P), we replace the total variation term $\mathrm{TV}(u)$ by the discretized total variation term $\mathrm{TV^h}(u)$ in the objective. As a result, we can no longer guarantee the existence of minimizers, as stated in the following example, because $\mathrm{TV^h}$ has a greater null space than TV. We will fix this issue in Section 5.2.

Example 5.6 (cf. [95, Rem. 2.6])**.** If we replace TV by $\mathrm{TV^h}$ in problem (P), we can not guarantee the existence of minimizers anymore because $\mathrm{TV^h}$ has a greater null space than TV. This is due to the fact that for each $u \in L^1(\Omega)$, we have that

$$\int_\Omega u(x)\operatorname{div}\phi(x)\,\mathrm{d}x = \sum_{Q\in\mathcal{Q}_h} d_Q \int_Q u(x)\,\mathrm{d}x = \int_\Omega \tilde{u}(x)\operatorname{div}\phi(x)\,\mathrm{d}x \tag{5.1}$$

for all $\phi \in RT0^h$ with $\operatorname{div}\phi|_Q =: d_Q \in \mathbb{R}$ for all $Q \in \mathcal{Q}_h$ and all $\tilde{u} \in L^1(\Omega)$ with $\int_Q \tilde{u}(x)\,\mathrm{d}x = \int_Q u(x)\,\mathrm{d}x$ for all $Q \in \mathcal{Q}_h$. This in particular yields $\mathrm{TV^h}(u) = \mathrm{TV^h}(\tilde{u})$. Hence, there may exist a minimizing sequence $\{u_k\}_{k\in\mathbb{N}} \subset \mathrm{BV_U}(\Omega)$ consisting of chattering functions (for example with checkerboard structure) with $\mathrm{TV^h}(u_k) = 0$ for all $k \in \mathbb{N}$ that converges weakly-$*$ in $L^\infty(\Omega)$ to some limit function that is not in $\mathrm{BV_U}(\Omega)$.

A concrete example can be constructed in the following way. Define $\Omega := (0,1)^2$ and let $h := \frac{1}{m}$ for some fixed $m \in \mathbb{N}$. We further define

$$\mathcal{Q}_h := \left\{ Q_{ij} := \left(\frac{i-1}{m}, \frac{j-1}{m}\right) \times \left(\frac{i}{m}, \frac{j}{m}\right) : i,j \in \{1,\dots,m\} \right\},$$

$U \coloneqq \{-1, 1\}$, and $F(u) \coloneqq \|K(u)\|_{L^1(\Omega)}$, where $K : L^1(\Omega) \to C^\infty(\Omega)$, $K(u) \coloneqq \eta * u$, and η denotes the standard mollifier. Note that $K(u) \neq 0$ for all $u \in \mathrm{BV_U}(\Omega)$ and therefore $F(u) > 0$ for all $u \in \mathrm{BV_U}(\Omega)$. Hence, we have $F(u) + \mathrm{TV^h}(u) > 0$ for all $u \in \mathrm{BV_U}(\Omega)$. We construct the minimizing sequence $\{u_k\}_{k\in\mathbb{N}} \subset \mathrm{BV_U}(\Omega)$ such $F(u_k) + \mathrm{TV^h}(u_k) \to 0$. To this end, we define for $k \in \mathbb{N}$

$$\tilde{\mathcal{Q}}_k \coloneqq \left\{ \tilde{Q}^k_{ij} \coloneqq \left(\frac{i-1}{2km}, \frac{j-1}{2km} \right) \times \left(\frac{i}{2km}, \frac{j}{2km} \right) : i, j \in \{1, \dots, 2km\} \right\}.$$

and the sequence $\{u_k\}_{k\in\mathbb{N}}$ by the following checkerboard structure

$$u_k(x) = \begin{cases} 1 & \text{if } x \in \tilde{Q}^k_{ij} \text{ with } i + j \in 2\mathbb{N} \\ -1 & \text{if } x \in \tilde{Q}^k_{ij} \text{ with } i + j - 1 \in 2\mathbb{N}. \end{cases}$$

Then for all $\phi \in RT0^h$, it holds that

$$\int_\Omega u_k(x) \operatorname{div} \phi(x) \,\mathrm{d}x = 0$$

and therefore $\mathrm{TV^h}(u_k) = 0$ for all $k \in \mathbb{N}$. Moreover, $u_k \overset{*}{\rightharpoonup} 0$ in $L^\infty(\Omega)$ as $k \to \infty$ and since K is a compact operator, this yields $K(u_k) \to K(u) = 0$ in $L^1(\Omega)$ as $k \to \infty$ and consequently, there holds

$$F(u_k) + \underbrace{\mathrm{TV^h}(u_k)}_{=0} \to 0$$

as $k \to \infty$. On the other hand, as mentioned above, we have that $F(u) + \mathrm{TV^h}(u) > 0$ for all $u \in \mathrm{BV_U}(\Omega)$ such that no minimizer exists in that case.

5.1.2 Lim inf inequality for $\mathrm{TV^h}$

Let a mesh $\mathcal{Q}_\tau$ with $\tau > 0$ fulfill Assumption 5.3. As introduced in Section 2.5, we define the space of functions that are piecewise constant on the mesh cells $Q \in \mathcal{Q}_\tau$ by $P0^\tau$. Let $\Pi_{P0^\tau} : L^1(\Omega) \to P0^\tau$ denote the projection onto $P0^\tau$.

Lemma 5.7 ([95, Lem. 2.7])**.** *Let $u \in L^\infty(\Omega)$. Then $\|u - \Pi_{P0^\tau} u\|_{L^1(\Omega)} \to 0$ as $\tau \searrow 0$. Moreover, for all $u \in \mathrm{BV}(\Omega)$ there holds $\|u - \Pi_{P0^\tau} u\|_{L^1(\Omega)} \leq \sqrt{d}\tau \mathrm{TV}(u)$.*

Proof. Let $u \in L^\infty(\Omega)$. Lebesgue's differentiation theorem [101, Chap. 3, Cor. 1.6 and 1.7] gives $\Pi_{P0^\tau} u \to u$ pointwise almost everywhere in Ω as $\tau \searrow 0$. Moreover, $\|\Pi_{P0^\tau} u\|_{L^\infty(\Omega)} \leq \|u\|_{L^\infty(\Omega)}$ such that Lebesgue's dominated convergence theorem yields $\|u - \Pi_{P0^\tau} u\|_{L^1(\Omega)} \to 0$ as $\tau \searrow 0$.

For the second claim let $u \in \mathrm{BV}(\Omega)$. Since Ω is in particular of finite perimeter, the proof of the second claim runs along the lines of the proof of Theorem 12.26 in [72].

In step one they proved that for a continuously differentiable function $w \in C^1(\Omega)$ there holds for each $Q \in \mathcal{Q}_\tau$ that

$$\int_Q |w(x) - \Pi_{P0^\tau} w(x)| \,\mathrm{d}x \leq \sqrt{d}\tau \int_Q \|\nabla w(x)\|_2 \,\mathrm{d}x.$$

If we now insert $w = u_\varepsilon := u * \eta_\varepsilon \in C^\infty(\Omega)$ for $\varepsilon > 0$ with η_ε denoting the standard mollifier, we obtain by following step two of the proof of Theorem 12.26 in [72] that

$$\sqrt{d}\tau \int_\Omega \|\nabla u_\varepsilon(x)\|_2 \,\mathrm{d}x \geq \sum_{Q \in \mathcal{Q}_\tau} \int_Q |u_\varepsilon(x) - \Pi_{P0^\tau} u_\varepsilon(x)|$$

such that driving $\varepsilon \searrow 0$ yields

$$\sqrt{d}\tau \mathrm{TV}(u) \geq \sum_{Q \in \mathcal{Q}_\tau} \int_Q |u(x) - \Pi_{P0^\tau} u(x)| \,\mathrm{d}x = \|u - \Pi_{P0^\tau} u\|_{L^1(\Omega)}.$$

A similar proof is given in Lemma 3.2 in [23]. □

Lemma 5.8 ([95, Lem. 2.8])**.** *Let* $\{u_h\}_{h>0} \subset L^1(\Omega)$ *with* $u_h \to u$ *in* $L^1(\Omega)$ *and* $u \in L^2(\Omega)$. *Let* $\phi \in C_c^1(\Omega; \mathbb{R}^d)$ *satisfy* $\|\phi\|_{L^\infty(\Omega;\mathbb{R}^d)} \leq 1$. *Then*

$$\int_\Omega \operatorname{div} \phi(x) u(x) \,\mathrm{d}x \leq \liminf_{h \searrow 0} \mathrm{TV}^{\mathrm{h}}(u_h). \tag{5.2}$$

Proof. Let $\varepsilon \in (0, 1)$ be arbitrary but fixed. Then we approximate ϕ with $\hat{\phi} := (1-\varepsilon)\phi \in C_c^1(\Omega; \mathbb{R}^d)$ such that $\|\hat{\phi}\|_{L^\infty(\Omega;\mathbb{R}^d)} = (1-\varepsilon)\|\phi\|_{L^\infty(\Omega;\mathbb{R}^d)} \leq 1 - \varepsilon$ and

$$\begin{aligned} \int_\Omega \operatorname{div} \phi(x) u(x) \,\mathrm{d}x &= \int_\Omega \operatorname{div} \hat{\phi}(x) u(x) \,\mathrm{d}x + \varepsilon \int_\Omega \operatorname{div} \phi(x) u(x) \,\mathrm{d}x \\ &\leq \int_\Omega \operatorname{div} \hat{\phi}(x) u(x) \,\mathrm{d}x + \underbrace{\|\operatorname{div} \phi\|_{L^\infty(\Omega)} \|u\|_{L^1(\Omega)}}_{c_1:=} \varepsilon. \end{aligned}$$

By virtue of Proposition 2.26, we obtain that $\|I_{RT0^h}\hat{\phi}\|_{L^\infty(\Omega;\mathbb{R}^d)} \leq 1$ holds for all $h \leq \frac{\varepsilon}{C(\hat{\phi})}$, where $I_{RT0^h} : W^{1,\infty}(\Omega; \mathbb{R}^d) \to RT0^h$ denotes the interpolation operator for $RT0^h$ as defined in Section 2.5 and $C(\hat{\phi}) > 0$ is a constant depending on $\hat{\phi}$. Moreover, since $\hat{\phi}$ has compact support in Ω, there holds $\hat{\phi} \cdot n \equiv 0$ on $\partial\Omega$, where n denotes the outer unit normal of Ω, so that $I_{RT0^h}\hat{\phi} \in RT0_0^h$. Since $\hat{\phi} \in C_c^1(\Omega; \mathbb{R}^d)$, there exists a constant $M(\hat{\phi}) > 0$ only depending on $\hat{\phi}$ such that $\|\operatorname{div} \hat{\phi}\|_{L^\infty(\Omega)} \leq M(\hat{\phi})$. This implies that $\|\Pi_{P0^h} \operatorname{div} \hat{\phi}\|_{L^\infty(\Omega)} \leq M(\hat{\phi})$ for all $h > 0$ such that for all $0 < h \leq \frac{\varepsilon}{C(\hat{\phi})}$,

there holds

$$
\begin{aligned}
&\int_\Omega \operatorname{div} I_{RT0^h}\hat{\phi}(x)u(x)\,\mathrm{d}x\\
&\qquad \le \int_\Omega \operatorname{div} I_{RT0^h}\hat{\phi}(x)u_h(x)\,\mathrm{d}x + \|\operatorname{div} I_{RT0^h}\hat{\phi}\|_{L^\infty(\Omega)}\|u_h - u\|_{L^1(\Omega)}\\
&\qquad \le \mathrm{TV^h}(u_h) + \|\operatorname{div} I_{RT0^h}\hat{\phi}\|_{L^\infty(\Omega)}\|u_h - u\|_{L^1(\Omega)}\\
&\qquad = \mathrm{TV^h}(u_h) + \|\Pi_{P0^h}\operatorname{div}\hat{\phi}\|_{L^\infty(\Omega)}\|u_h - u\|_{L^1(\Omega)}\\
&\qquad \le \mathrm{TV^h}(u_h) + M(\hat{\phi})\|u_h - u\|_{L^1(\Omega)},
\end{aligned}
$$

where we used $\operatorname{div} I_{RT0^h}\hat{\phi} = \Pi_{P0^h}\operatorname{div}\hat{\phi}$ by Lemma 2.24, yielding

$$
\begin{aligned}
\int_\Omega \operatorname{div}\phi(x)u(x)\,\mathrm{d}x &\le \int_\Omega \operatorname{div}\hat{\phi}(x)u(x)\,\mathrm{d}x + c_1\varepsilon\\
&= \int_\Omega \operatorname{div} I_{RT0^h}\hat{\phi}(x)u(x)\,\mathrm{d}x + \int_\Omega \operatorname{div}(\hat{\phi} - I_{RT0^h}\hat{\phi})(x)u(x)\,\mathrm{d}x + c_1\varepsilon\\
&\le \mathrm{TV^h}(u_h) + M(\hat{\phi})\|u_h - u\|_{L^1(\Omega)} + \|\operatorname{div}(\hat{\phi} - I_{RT0^h}\hat{\phi})\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)} + c_1\varepsilon.
\end{aligned}
$$

By (2.14), there holds $\|\operatorname{div}(\hat{\phi} - I_{RT0^h}\hat{\phi})\|_{L^2(\Omega)} \to 0$ as $h \searrow 0$. Since $u \in L^2(\Omega)$, driving $h \searrow 0$ implies that

$$
\int_\Omega \operatorname{div}\phi(x)u(x)\,\mathrm{d}x \le c_1\varepsilon + \liminf_{h\searrow 0}\mathrm{TV^h}(u_h).
$$

The claim follows because $\varepsilon \in (0,1)$ was chosen arbitrarily. □

Theorem 5.9 ([95, Thm. 2.9]). *Let $\{u_h\}_{h>0} \subset L^1(\Omega)$ with $u_h \to u$ in $L^1(\Omega)$ as $h \searrow 0$ and $u \in L^2(\Omega)$. Then*

$$
\mathrm{TV}(u) \le \liminf_{h\searrow 0}\mathrm{TV^h}(u_h).
$$

Proof. We supremize over all $\phi \in C_c^1(\Omega;\mathbb{R}^d)$ with $\|\phi\|_{L^\infty(\Omega;\mathbb{R}^d)} \le 1$ in (5.2). □

Up to this point the discretization of the input function u has not been relevant. Indeed, the previous statements hold for arbitrary sequences with $u_h \to u$ in $L^1(\Omega)$ with $u \in L^2(\Omega)$. Instead, the discretized total variation $\mathrm{TV^h}$ discretizes its inputs u implicitly in the sense that $\mathrm{TV^h}(u) = \mathrm{TV^h}(\Pi_{P0^h}u)$, see (5.1). That means if we would waive the integrality constraint for the discretized problems, then we could approximate the total variation of a given function $u \in \mathrm{BV_U}(\Omega)$ with the discretized total variation of the projections $\Pi_{P0^h}u$. In particular, we refer to the results in [23]. Since we additionally—and in particular differing from [23]—require that the discretized input functions only attain values in the discrete set U, we can not work with the projections $\Pi_{P0^h}u$ as in [23]. Due to the given geometry of the mesh, it is

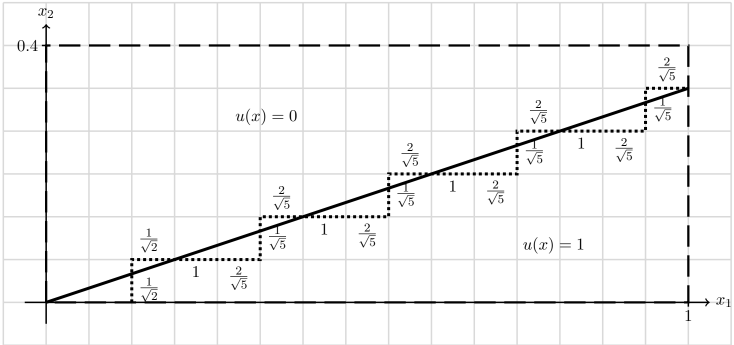


Figure 5.2: Example for the construction in Example 5.10 with $k = 1$. The level sets of the limit function u are separated by the solid line and the level sets of the rounding of $\Pi_{P0^h} u$ are separated by the dotted line. Nonzero normal traces of ϕ are indicated next to the corresponding edges of the grid cells.

generally not possible to approximate the term $\mathrm{TV}(u)$ for $u \in \mathrm{BV}_{\mathrm{U}}(\Omega)$ with TV^{h} of functions in $P0^h \cap \mathrm{BV}_{\mathrm{U}}(\Omega)$, as the following example shows.

Example 5.10 ([95, Expl. 2.10])**.** Consider $\Omega = (0,1) \times (0, 0.4)$ and the mesh $\mathcal{Q}_h$ consisting of squares of size $h = \frac{1}{15k}$ for $k \in \mathbb{N}$. Define $u = \chi_{\{(\frac{1}{3},-1)^T x \geq 0\}}$ and consider $u_h \in P0^h \cap \mathrm{BV}_{\mathrm{U}}(\Omega)$ obtained by rounding the projection $\Pi_{P0^h} u$ on each square to the nearest value in u, see Figure 5.2 for an example for $k = 1$. Let $\phi \in RT0^h$ be the Raviart–Thomas function that is defined by the normal traces over the edges of the grid cells as exemplarily illustrated in Figure 5.2. It is immediate that $\|\phi\|_{L^\infty(\Omega;\mathbb{R}^d)} \leq 1$. Denote by $\mathcal{E}_h$ the interior edges of the mesh $\mathcal{Q}_h$ and let n_E be a unit normal vector to an edge $E \in \mathcal{E}_h$. Let $u_h^{E^+}$ and $u_h^{E^-}$ denote the values of u_h on the two cubes adjoining E. Then,

$$\begin{aligned}
\mathrm{TV}^{\mathrm{h}}(u_h) &\geq \int_\Omega \operatorname{div} \phi(x) u_h(x) \,\mathrm{d}x = \sum_{E \in \mathcal{E}_h} |u_h^{E^+} - u_h^{E^-}| \int_E \phi(x) \cdot n_E(x) \,\mathrm{d}\mathcal{H}^1(x) \\
&= \frac{1}{15k} \left((5k-1) \left(1 + 2 \cdot \frac{2}{\sqrt{5}} + \frac{1}{\sqrt{5}} \right) + 2 \cdot \frac{1}{\sqrt{2}} \right) \\
&= \frac{1+\sqrt{5}}{3} + \frac{\sqrt{2} - 1 - \sqrt{5}}{15k} \to \frac{1+\sqrt{5}}{3} \approx 1.078 \quad \text{as } k \to \infty.
\end{aligned}$$

On the other hand, $\mathrm{TV}(u) = \frac{\sqrt{10}}{3} \approx 1.054$ such that $\mathrm{TV}^{\mathrm{h}}(u_h) \not\to \mathrm{TV}(u)$ as $h \searrow 0$. Instead, there holds $\mathrm{TV}^{\mathrm{h}}(u_h) > \mathrm{TV}(u)$ for $k \geq 5$.

5.1.3 Lim sup inequality for TV^{h}

To achieve an averaging effect by other means than the projection Π_{P0^h}, we discretize the functions $u \in \mathrm{BV}_{\mathrm{U}}(\Omega)$ on a finer mesh $\mathcal{Q}_{\tau_h}$ embedded into the mesh $\mathcal{Q}_h$ for TV^{h} such that Assumption 5.3 is fulfilled. This allows to recover $\mathrm{TV}(u)$ with $\mathrm{TV}^{\mathrm{h}}(u_{\tau_h})$

with functions $u_{\tau_h} \in P0^{\tau_h} \cap \mathrm{BV_U}(\Omega)$ when the mesh sizes are superlinearly coupled, that is, $\frac{\tau_h}{h} \searrow 0$ as $h \searrow 0$.

Definition 5.11 ([95, Def. 2.11])**.** We define the operator $R^U_{P0^\tau} : \mathrm{BV_U}(\Omega) \to \mathrm{BV_U}(\Omega) \cap P0^\tau$ for $\tau > 0$ and a corresponding mesh $\mathcal{Q}_\tau$ as follows. For $u \in \mathrm{BV_U}(\Omega)$ and $Q \in \mathcal{Q}_\tau$, let

$$R^U_{P0^\tau}(u)|_Q \in \arg\min \left\{ \left| \frac{1}{|Q|} \int_Q u(x)\,\mathrm{d}x - \nu \right| \; : \; \nu \in U \text{ and } \left| u|_Q^{-1}(\nu) \right| > 0 \right\}.$$

If the minimizer is not unique, we choose the smallest one.

Similar to Lemma 5.7, we obtain convergence results for the sequence $\{R^U_{P0^\tau}(u)\}_{\tau>0}$ for $\tau \searrow 0$ with a convergence rate depending on $\mathrm{TV}(u)$ if $u \in \mathrm{BV_U}(\Omega)$.

Lemma 5.12 ([95, Lem. 2.12])**.** *Let $u \in L^1_U(\Omega)$. Then*

$$\|u - R^U_{P0^\tau}(u)\|_{L^1(\Omega)} \le 2\|u - \Pi_{P0^\tau} u\|_{L^1(\Omega)} \to 0 \text{ as } \tau \searrow 0.$$

Moreover, for all $u \in \mathrm{BV_U}(\Omega)$ the following estimates hold:

$$\|u - R^U_{P0^\tau}(u)\|_{L^1(\Omega)} \le 2\sqrt{d}\tau \mathrm{TV}(u) \;\text{ and }\; \|R^U_{P0^\tau}(u) - \Pi_{P0^\tau} u\|_{L^1(\Omega)} \le \sqrt{d}\tau \mathrm{TV}(u).$$

Proof. For the first claim, let $u \in L^1_U(\Omega)$. By Definition 5.11, there holds

$$\|\Pi_{P0^\tau} u - R^U_{P0^\tau}(u)\|_{L^1(\Omega)} \le \|u - \Pi_{P0^\tau} u\|_{L^1(\Omega)}$$

due to

$$|(\Pi_{P0^\tau} u)(x) - (R^U_{P0^\tau}(u))(x)| \le |(\Pi_{P0^\tau} u)(x) - u(x)|$$

for almost all $x \in \Omega$. Hence,

$$\begin{aligned} \|u - R^U_{P0^\tau}(u)\|_{L^1(\Omega)} &\le \|u - \Pi_{P0^\tau} u\|_{L^1(\Omega)} + \|\Pi_{P0^\tau} u - R^U_{P0^\tau}(u)\|_{L^1(\Omega)} \\ &\le 2\|u - \Pi_{P0^\tau} u\|_{L^1(\Omega)} \to 0 \text{ as } \tau \searrow 0. \end{aligned}$$

The second claim follows then by Lemma 5.7. □

With these preparations, we are now able to prove a result corresponding to Lemma 3.1 in [23], where it is stated that $\mathrm{TV^h}(\Pi_{P0^h} u) \le \mathrm{TV}(u)$ for $u \in L^2(\Omega)$.

Proposition 5.13 ([95, Prop. 2.13])**.** *Let $u \in \mathrm{BV_U}(\Omega)$ and a tuple (h, τ_h) with $h > 0$ be given such that Assumption 5.3 is fulfilled. Then*

$$\mathrm{TV^h}(R^U_{P0^{\tau_h}}(u)) \le \mathrm{TV^h}(u) + \varepsilon(\tau_h, h, u) \le \mathrm{TV}(u) + \varepsilon(\tau_h, h, u) \tag{5.3}$$

with $\varepsilon(\tau_h, h, u) \coloneqq \sqrt{d}\mathrm{TV}(u)\frac{2d\tau_h}{h}$.

Proof. Let $u \in \mathrm{BV_U}(\Omega)$. We define $u_{\tau_h} := R^U_{P0^{\tau_h}}(u) \in \mathrm{BV_U}(\Omega)$ and $\bar{u}_{\tau_h} := \Pi_{P0^{\tau_h}} u \in \mathrm{BV}(\Omega)$. Let $\phi \in RT0^h_0$ with $\|\phi\|_{L^\infty(\Omega;\mathbb{R}^d)} \leq 1$. Then

$$\begin{aligned}\int_\Omega u_{\tau_h}(x) \operatorname{div} \phi(x) \,\mathrm{d}x &= \int_\Omega \bar{u}_{\tau_h}(x) \operatorname{div} \phi(x) \,\mathrm{d}x + \int_\Omega (u_{\tau_h}(x) - \bar{u}_{\tau_h}(x)) \operatorname{div} \phi(x) \,\mathrm{d}x \\ &\leq \mathrm{TV^h}(u) + \int_\Omega (u_{\tau_h}(x) - \bar{u}_{\tau_h}(x)) \operatorname{div} \phi(x) \,\mathrm{d}x \\ &\leq \mathrm{TV^h}(u) + \|u_{\tau_h} - \bar{u}_{\tau_h}\|_{L^1(\Omega)} \|\operatorname{div} \phi\|_{L^\infty(\Omega)},\end{aligned}$$

where we have used that the meshes $\mathcal{Q}_h$ and $\mathcal{Q}_{\tau_h}$ fulfill Assumption 5.3 and that $\operatorname{div} \phi$ is constant on each $Q \in \mathcal{Q}_h$ to deduce the inequality. By Lemma 5.4, it follows that $\|\operatorname{div} \phi\|_{L^\infty(\Omega)} \leq \frac{2d}{h}$, and Lemma 5.12 yields $\|u_{\tau_h} - \bar{u}_{\tau_h}\|_{L^1(\Omega)} \leq \sqrt{d}\tau_h \mathrm{TV}(u)$ which implies $\int_\Omega u_{\tau_h}(x) \operatorname{div} \phi(x) \,\mathrm{d}x \leq \mathrm{TV^h}(u) + \varepsilon(\tau_h, h, u)$. □

We are now able to prove the lim sup inequality for $\mathrm{TV^h}$.

Theorem 5.14 ([95, Thm. 2.14])**.** *Let $u \in L^1_U(\Omega)$ and tuples $\{(h, \tau_h)\}_{h>0}$ be given such that Assumption 5.3 is fulfilled and $\frac{\tau_h}{h} \searrow 0$ as $h \searrow 0$. Then there holds*

$$\mathrm{TV}(u) \geq \limsup_{h \searrow 0} \mathrm{TV^h}(R^U_{P0^{\tau_h}}(u)).$$

Proof. The claim follows from Proposition 5.13 by applying $\limsup_{h\searrow 0}$ to both sides of (5.3) and using the assumption $\frac{\tau_h}{h} \searrow 0$. □

The embedding of the fine meshes $\mathcal{Q}_{\tau_h}$ into the coarse meshes $\mathcal{Q}_h$ and the superlinear coupling of their mesh sizes yield a lower bound estimate for the discretized total variation of the recovery sequence from Theorem 5.14 using techniques from Proposition 3.7 in [23].

Proposition 5.15 ([95, Prop. 2.15])**.** *Let $u \in \mathrm{BV_U}(\Omega)$ such that $\mathrm{TV}(u) = -\int_\Omega \operatorname{div} \phi(x) u(x) \,\mathrm{d}x$ with $\phi \in W^{1,\infty}_0(\Omega; \mathbb{R}^d)$ and $\|\phi\|_{L^\infty(\Omega;\mathbb{R}^d)} \leq 1$. Let $\{(h, \tau_h)\}_{h>0}$ be coupled such that Assumption 5.3 is fulfilled. Then*

$$\mathrm{TV^h}(R^U_{P0^{\tau_h}}(u)) \geq \mathrm{TV}(u) - c(u, \phi) \left(h + \frac{\tau_h}{h}\right) - \frac{1}{2} \|\operatorname{div}(I_{RT0^h}\phi - \phi)\|^2_{L^2(\Omega)}$$

holds with some constant $c(u, \phi) > 0$. If additionally $\operatorname{div} \phi \in \mathrm{BV}(\Omega)$*, then*

$$\mathrm{TV^h}(R^U_{P0^{\tau_h}}(u)) \geq \mathrm{TV}(u) - \tilde{c}(u, \phi) \left(h + \frac{\tau_h}{h}\right)$$

with some constant $\tilde{c}(u, \phi) > 0$. If additionally $\operatorname{div} \phi \in H^1(\Omega)$*, then*

$$\mathrm{TV^h}(R^U_{P0^{\tau_h}}(u)) \geq \mathrm{TV}(u) - \hat{c}(u, \phi) \left(h^2 + h + \frac{\tau_h}{h}\right)$$

with some constant $\hat{c}(u, \phi) > 0$.

Proof. Let $\phi_h \coloneqq I_{RT0^h}\phi \in RT0_0^h$ for $h > 0$. By Proposition 2.26, there holds $\|\phi_h\|_{L^\infty(\Omega;\mathbb{R}^d)} \le 1 + Ch$ with $C = C(\phi) > 0$. We define $\tilde{\phi}_h \coloneqq \frac{1}{1+Ch}\phi_h \in RT0_0^h$ for $h > 0$, which fulfills $\|\tilde{\phi}_h\|_{L^\infty(\Omega;\mathbb{R}^d)} \le 1$. Moreover, we define $u_h = \Pi_{P0^h} u$ for $\tau_h > 0$, which fulfills $\int_\Omega u(x)u_h(x)\,\mathrm{d}x = \|u_h\|^2_{L^2(\Omega)}$ due to the optimality condition of the projection. By Lemma 2.24, there holds $\operatorname{div} I_{RT0^h}\phi = \Pi_{P0^h}\operatorname{div}\phi$ so that also

$$\int_\Omega \operatorname{div}\phi(x)\operatorname{div}\phi_h(x)\,\mathrm{d}x = \|\operatorname{div}\phi_h\|^2_{L^2(\Omega)}.$$

Hence, there holds

$$\begin{aligned}
&\mathrm{TV}(u) = -\frac{1}{2}\|\operatorname{div}\phi + u\|^2_{L^2(\Omega)} + \frac{1}{2}\|u\|^2_{L^2(\Omega)} + \frac{1}{2}\|\operatorname{div}\phi\|^2_{L^2(\Omega)}\\
&\le -\frac{1}{2}\|\operatorname{div}\phi_h + u_h\|^2_{L^2(\Omega)} + \frac{1}{2}\|u\|^2_{L^2(\Omega)} + \frac{1}{2}\|\operatorname{div}\phi\|^2_{L^2(\Omega)}\\
&= \frac{1}{2}\big(\|\operatorname{div}\phi\|^2_{L^2(\Omega)} - \|\operatorname{div}\phi_h\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\Omega)} - \|u_h\|^2_{L^2(\Omega)}\big) - \int_\Omega \operatorname{div}\phi_h(x)u_h(x)\,\mathrm{d}x\\
&= \frac{1}{2}\|\operatorname{div}(\phi_h - \phi)\|^2_{L^2(\Omega)} + \frac{1}{2}\|u - u_h\|^2_{L^2(\Omega)} - (1 + Ch)\int_\Omega \operatorname{div}\tilde{\phi}_h(x)u_h(x)\,\mathrm{d}x\\
&\le \frac{1}{2}\|\operatorname{div}(\phi_h - \phi)\|^2_{L^2(\Omega)} + \frac{1}{2}\|u - u_h\|_{L^1(\Omega)}\|u - u_h\|_{L^\infty(\Omega)} + (1 + Ch)\,\mathrm{TV^h}(u_h).
\end{aligned}$$

Lemma 5.7 gives $\|u - u_h\|_{L^1(\Omega)} \le \sqrt{d}h\mathrm{TV}(u)$. Since $u, u_h \in \mathrm{BV_U}(\Omega)$, there holds

$$\|u - u_h\|_{L^\infty(\Omega)} \le \max_{\nu_1,\nu_2\in U}|\nu_1 - \nu_2| \eqqcolon U_{\max}.$$

There holds $\mathrm{TV^h}(u_h) = \mathrm{TV^h}(\Pi_{P0^{\tau_h}}u)$ due to $\int_Q u_h(x)\,\mathrm{d}x = \int_Q \Pi_{P0^{\tau_h}}u(x)\,\mathrm{d}x$ for all $Q \in \mathcal{Q}_h$ by Assumption 5.3. By Lemma 5.5, there exists $\hat{\phi}_h \in RT0_0^h$ with $\|\hat{\phi}_h\|_{L^\infty(\Omega)} \le 1$ so that

$$\begin{aligned}
\mathrm{TV^h}(\Pi_{P0^{\tau_h}}u) &= \int_\Omega \operatorname{div}\hat{\phi}_h(x)\Pi_{P0^{\tau_h}}u(x)\,\mathrm{d}x\\
&= \int_\Omega \operatorname{div}\hat{\phi}_h(x)(\Pi_{P0^{\tau_h}}u(x) - R^U_{P0^{\tau_h}}(u)(x))\,\mathrm{d}x + \int_\Omega \operatorname{div}\hat{\phi}_h(x)R^U_{P0^{\tau_h}}(u)(x)\,\mathrm{d}x\\
&\le \|\operatorname{div}\hat{\phi}_h\|_{L^\infty(\Omega)}\|\Pi_{P0^{\tau_h}}u - R^U_{P0^{\tau_h}}(u)\|_{L^1(\Omega)} + \mathrm{TV^h}(R^U_{P0^{\tau_h}}(u))\\
&\le \frac{4d\sqrt{d}\tau_h}{h}\mathrm{TV}(u) + \mathrm{TV^h}(R^U_{P0^{\tau_h}}(u)),
\end{aligned}$$

where the last inequality follows from Lemmas 5.4 and 5.12. In total, we obtain

$$\begin{aligned}
&\mathrm{TV^h}(R^U_{P0^{\tau_h}}(u)) - \mathrm{TV}(u)\\
&\qquad \ge -\frac{\Big(Ch + \frac{1}{2}U_{\max}\sqrt{d}h + (1 + Ch)\frac{4d\sqrt{d}\tau_h}{h}\Big)\mathrm{TV}(u) + \frac{1}{2}\|\operatorname{div}(\phi_h - \phi)\|^2_{L^2(\Omega)}}{1 + Ch}.
\end{aligned}$$

If additionally $\operatorname{div}\phi \in \mathrm{BV}(\Omega)$, we may apply Lemmas 2.24 and 5.7 to obtain

$$\begin{aligned}\|\operatorname{div}(\phi_h - \phi)\|_{L^2(\Omega)}^2 &= \|\operatorname{div}\phi - \Pi_{P0^h}\operatorname{div}\phi\|_{L^2(\Omega)}^2 \\ &\le \|\operatorname{div}\phi - \Pi_{P0^h}\operatorname{div}\phi\|_{L^\infty(\Omega)}\|\operatorname{div}\phi - \Pi_{P0^h}\operatorname{div}\phi\|_{L^1(\Omega)} \\ &\le \|\operatorname{div}\phi - \Pi_{P0^h}\operatorname{div}\phi\|_{L^\infty(\Omega)}\sqrt{d}\mathrm{TV}(\operatorname{div}\phi)h\end{aligned}$$

and define $\tilde{b} \coloneqq 2\|\operatorname{div}\phi\|_{L^\infty(\Omega)}\sqrt{d}\mathrm{TV}(\operatorname{div}\phi)$ to arrive at

$$\begin{aligned}&\mathrm{TV^h}(R^U_{P0^{\tau_h}}(u)) - \mathrm{TV}(u) \\ &\qquad \ge -\frac{\left(Ch + \frac{1}{2}U_{\max}\sqrt{d}h + (1+Ch)\frac{4d\sqrt{d}\tau_h}{h}\right)\mathrm{TV}(u) + \frac{\tilde{b}h}{2}}{1+Ch}.\end{aligned}$$

If additionally $\operatorname{div}\phi \in H^1(\Omega)$, then we may apply Lemma 2.25 to obtain

$$\|\operatorname{div}(\phi_h - \phi)\|_{L^2(\Omega)}^2 \le c^2h^2\|\nabla\operatorname{div}\phi\|_{L^2(\Omega;\mathbb{R}^d)}^2$$

such that

$$\begin{aligned}&\mathrm{TV^h}(R^U_{P0^{\tau_h}}(u)) - \mathrm{TV}(u) \\ &\qquad \ge -\frac{\left(Ch + \frac{1}{2}U_{\max}\sqrt{d}h + (1+Ch)\frac{4d\sqrt{d}\tau_h}{h}\right)\mathrm{TV}(u) + \frac{\hat{b}h^2}{2}}{1+Ch}\end{aligned}$$

with $\hat{b} \coloneqq c^2\|\nabla\operatorname{div}\phi\|_{L^2(\Omega;\mathbb{R}^d)}^2$. □

5.2 Discretization of problem (P)

We define the two optimization problems

$$(\mathrm{P}_c) \qquad \min_{u\in L^2(\Omega)} G(u) \coloneqq F(u) + \alpha\mathrm{TV}(u) + \mathcal{I}_Z(u)$$

and

$$(\mathrm{P}_c^h) \qquad \min_{(u,V)\in L^2(\Omega)\times\mathbb{R}} G^h(u,V) \coloneqq F(u) + \alpha V + \mathcal{I}_{Z^h}(u,V).$$

The feasible sets are respectively given by $Z \coloneqq \left\{u \in L^1_U(\Omega) : \mathrm{TV}(u) \le c\mathrm{TV}(u)\right\}$ and $Z^h \coloneqq \left\{(u,V) \in (P0^{\tau_h} \cap L^1_U(\Omega)) \times \mathbb{R} : \mathrm{TV}(u) \le cV,\ \mathrm{TV^h}(u) \le V\right\}$ with a constant $c \ge 1$ and with $\tau_h > 0$ coupled to $h > 0$ such that Assumption 5.3 is fulfilled. The feasibility in (P_c) and (P_c^h) is ensured by the $\{0,\infty\}$-valued indicator functionals $\mathcal{I}_Z$ and $\mathcal{I}_{Z^h}$. The purpose of the constraint $\mathrm{TV}(u) \le cV$ in (P_c^h) is to obtain boundedness of the sequence of solutions to (P_c^h) in $\mathrm{BV}(\Omega)$ which yields weak-* sequential compactness in $\mathrm{BV}(\Omega)$. This also yields the existence of minimizers and will be analyzed in Section 5.2.1. Compared to (P), the problem (P_c) has the additional constraint $\mathrm{TV}(u) \leq c\mathrm{TV}(u)$, which does not change the feasible set.

Lemma 5.16 ([95, Lem. 3.1])**.** (P_c) *is equivalent to* (P).

Proof. Because $c \geq 1$ and $\mathrm{TV}(u) \geq 0$ for all $u \in L^1(\Omega)$, the constraint $\mathrm{TV}(u) \leq c\mathrm{TV}(u)$ is trivially fulfilled for all $u \in L^1(\Omega)$. □

Instead, the constraint $\mathrm{TV}(u) \leq c\mathrm{TV}(u)$ is used to illustrate that (P_c) is the limit problem to the family of problems (P_c^h) when driving $h \searrow 0$ in a Γ-convergence sense. Specifically, we obtain the following results.

Theorem 5.17 (Lim inf inequality, [95, Thm. 3.2])**.** *Let $F : L^2(\Omega) \to \mathbb{R}$ fulfill Assumption 5.1. Let G and G^h be defined as above with $c \geq 1$. Consider tuples $\{(h, \tau_h)\}_{h>0}$ such that Assumption 5.3 is fulfilled. Let $\{(u_{\tau_h}, V_h)\}_{h>0} \subset L^2(\Omega) \times \mathbb{R}$ be a sequence with $u_{\tau_h} \to u$ in $L^1(\Omega)$ as $h \searrow 0$. Then there holds*

$$G(u) \leq \liminf_{h \searrow 0} G^h(u_{\tau_h}, V_h).$$

Proof. Without loss of generality, we may assume $(u_{\tau_h}, V_h) \in Z^h$ for all $h > 0$ because $(u_{\tau_h}, V_h) \notin Z^h$ implies the trivial case $G^h(u_{\tau_h}, V_h) = \infty$. That is, in particular there holds $u_{\tau_h} \in L^1_U(\Omega)$ and $\mathrm{TV^h}(u_{\tau_h}) \leq V_h$ for all $h > 0$. By Lemma 2.19, there holds $u_{\tau_h} \to u$ in $L^p(\Omega)$ for all $1 \leq p < \infty$ and $u \in L^1_U(\Omega)$. Together with Theorem 5.9, there holds $\alpha\mathrm{TV}(u) \leq \liminf_{h\searrow 0} \alpha\, \mathrm{TV^h}(u_{\tau_h}) \leq \liminf_{h\searrow 0} \alpha V_h$.

Again without loss of generality, we may assume that $\liminf_{h\searrow 0} \alpha V_h < \infty$, because that again would imply the trivial case $\liminf_{h\searrow 0} G^h(u_{\tau_h}, V_h) = \infty$ since F is bounded from below by assumption. Hence $\mathrm{TV}(u) < \infty$ and we obtain by Lemma 2.19 that $u \in \mathrm{BV_U}(\Omega)$ and therefore $u \in Z$, which implies $\mathcal{I}_{Z^h}(u_{\tau_h}) = \mathcal{I}_Z(u)$ for all $h > 0$. Because F fulfills Assumption 5.1, there holds $F(u) = \lim_{h\searrow 0} F(u_{\tau_h})$. In total, we obtain $G(u) \leq \liminf_{h\searrow 0} G^h(u_{\tau_h}, V_h)$. □

Theorem 5.18 (Lim sup inequality, [95, Thm. 3.3])**.** *Let $d \in \{1, 2, 3\}$ and $F : L^2(\Omega) \to \mathbb{R}$ fulfill Assumption 5.1. Let $c \geq 1$ in the case $d = 1$, $c \geq \sqrt{2}$ in the case $d = 2$, and $c \geq 13\sqrt{3}$ in the case $d = 3$. Let tuples $\{(h, \tau_h)\}_{h>0}$ be given such that Assumption 5.3 is fulfilled and $\frac{\tau_h}{h} \searrow 0$ as $h \searrow 0$. Let $u \in \mathrm{BV_U}(\Omega)$. Then there exists a sequence $\{(u_{\tau_h}, V_h)\}_{h>0} \subset \mathrm{BV_U}(\Omega) \times \mathbb{R}$ such that $(u_{\tau_h}, V_h) \in Z^h$, $u_{\tau_h} \stackrel{*}{\rightharpoonup} u$ in $\mathrm{BV}(\Omega)$, and*

$$G(u) \geq \limsup_{h \searrow 0} G^h(u_{\tau_h}, V_h).$$

The proof of Theorem 5.18 is provided in Section 5.2.2. In combination with the aforementioned compactness arguments, Theorems 5.17 and 5.18 yield that minimizers of (P_c^h) converge to a minimizer of (P_c) when driving $h \searrow 0$, which is the main result of this section.

Theorem 5.19 ([95, Thm. 3.4])**.** *Let $d \in \{1,2,3\}$ and $F : L^2(\Omega) \to \mathbb{R}$ fulfill Assumption 5.1. Let $c \geq 1$ for the case $d = 1$, $c \geq \sqrt{2}$ for the case $d = 2$, and $c \geq 13\sqrt{3}$ for the case $d = 3$. Let tuples $\{(h, \tau_h)\}_{h>0}$ be coupled such that Assumption 5.3 is fulfilled and $\frac{\tau_h}{h} \searrow 0$ as $h \searrow 0$. Denote by $\{(u_{\tau_h}, V_h)\}_{h>0} \subset \mathrm{BV}_{\mathrm{U}}(\Omega) \times \mathbb{R}$ with $u_{\tau_h} \in P0^{\tau_h}$ a sequence of optimal solutions to (P_c^h). Then $\{u_{\tau_h}\}_{h>0}$ admits a subsequence that converges weakly-$*$ in $\mathrm{BV}(\Omega)$ and each accumulation point of $\{u_{\tau_h}\}_{h>0}$ is a minimizer of (P_c).*

Next we provide the compactness arguments and the proof of Theorem 5.19.

5.2.1 Existence of minimizers and compactness

In order to guarantee the existence of a subsequence of the sequences of optimal solutions to the problems (P_c^h) that converges in $L^1_U(\Omega)$ as $h \searrow 0$, we have implemented compactness with the help of the constraint $\mathrm{TV}^{\mathrm{h}}(u) \leq cV$ in the sense of Theorem 2.8. Before continuing with the statements and proofs, we illustrate the effect of absence and presence of the constraints $\mathrm{TV}(u) \leq cV$ in (P_c^h) as $h \searrow 0$ in an extension of Example 5.6.

Example 5.20 (cf. [95, Rem. 3.6])**.** In Example 5.6, we showed that the replacement of TV by TV^{h} causes that the existence of minimizers is no longer guaranteed since the null space of TV^{h} is greater than the null space of TV. If we consider the problem

$$(\bar{P}^h) \qquad \begin{aligned} \min_{u \in P0^{\tau_h}} \quad & F(u) + \alpha\,\mathrm{TV}^{\mathrm{h}}(u) \\ \text{s.t.} \quad & u(x) \in U \text{ for a.a. } x \in \Omega, \end{aligned}$$

we can now guarantee the existence of minimizers of $(\bar{P}^h)$ due to its finite dimension. However, we now have the issue that minimizers of $(\bar{P}^h)$ might not converge to a minimizer of (P) if we couple τ_h and h superlinearly as required for the recovery sequence according to Theorem 5.14 and drive h to zero. This coupling allows for chattering of the sequence of minimizers without affecting the discretized total variation on the coarser mesh. To see this, we take up Example 5.6 with the choice $k = m$. That is, we define $F(u) = \|\eta * u\|_{L^1(\Omega)}$ and for $k \in \mathbb{N}$,

$$\mathcal{Q}_{h_k} \coloneqq \left\{ Q_{ij} \coloneqq \left(\frac{i-1}{k}, \frac{j-1}{k}\right) \times \left(\frac{i}{k}, \frac{j}{k}\right) : i,j \in \{1,\ldots,k\} \right\}$$

with $h_k = \frac{1}{k}$, and $\mathcal{Q}_{\tau_{h_k}} \coloneqq \tilde{Q}_k$ with

$$\tilde{\mathcal{Q}}_k \coloneqq \left\{ \tilde{Q}^k_{ij} \coloneqq \left(\frac{i-1}{2k^2}, \frac{j-1}{2k^2}\right) \times \left(\frac{i}{2k^2}, \frac{j}{2k^2}\right) : i,j \in \{1,\ldots,2k^2\} \right\}.$$

There holds $\tau_{h_k} = \frac{1}{2k^2}$ and therefore $\frac{\tau_{h_k}}{h_k} = \frac{1}{2k} \to 0$ as $k \to \infty$. Define the sequence $\{u_k\}_{k\in\mathbb{N}}$ by

$$u_k(x) = \begin{cases} 1 & \text{if } x \in \tilde{Q}_{ij}^k \text{ with } i+j \in 2\mathbb{N} \\ -1 & \text{if } x \in \tilde{Q}_{ij}^k \text{ with } i+j-1 \in 2\mathbb{N}. \end{cases}$$

Then u_k is feasible for $(\bar{P}^h)$ with $h = h_k$ and $\tau_h = \tau_{h_k}$. From Example 5.6, we already know $u_k \stackrel{*}{\rightharpoonup} 0$ in $L^\infty(\Omega)$ and

$$F(u_k) + \alpha \underbrace{\mathrm{TV}^{h_k}(u_k)}_{=0} \to 0$$

as $k \to \infty$. Now consider the sequence $\{\bar{u}_k\}_{k\in\mathbb{N}}$, where $\bar{u}_k$ denotes the minimizer of $(\bar{P}^h)$ with $h = h_k$ and $\tau_h = \tau_{h_k}$. Assume that the minimizers converge to some $\bar{u} \in L^1_U(\Omega)$, that is $\bar{u}_k \to \bar{u}$ in $L^1(\Omega)$ as $k \to \infty$. By Theorem 5.9, there holds $\mathrm{TV}(\bar{u}) \le \liminf_{k\to\infty} \mathrm{TV}^{h_k}(\bar{u}_k)$ and together with the continuity of F there holds

$$0 \le F(\bar{u}) + \mathrm{TV}(\bar{u}) \le \liminf_{k\to\infty} F(\bar{u}_k) + \mathrm{TV}^{h_k}(\bar{u}_k) \le \lim_{k\to\infty} F(u_k) + \mathrm{TV}^{h_k}(u_k) = 0.$$

In particular, this yields $\mathrm{TV}(\bar{u}) = 0$ and hence $\bar{u} \equiv 1$ or $\bar{u} \equiv -1$ in contradiction to $F(\bar{u}) = 0$ so that the sequence $\{\bar{u}_k\}_{k\in\mathbb{N}}$ of minimizers of $(\bar{P}^h)$ can not converge to a minimizer of (P) in $L^1(\Omega)$.

If we now add the constraint $\mathrm{TV}(u) \le cV$ to $(\bar{P}^h)$ to obtain (P_c^h), the constraint $8k^2 - 4 = \mathrm{TV}(u_k) \le V_k$ would imply $V_k \to \infty$ as $k \to \infty$ and since F is bounded from below by zero, this would imply for the corresponding objective values $F(u_k) + \alpha V_k \to \infty$ as $k \to \infty$. Hence, the constraint $\mathrm{TV}(u) \le cV$ bounds the total variation of the minimizers since the function F is bounded from below and therefore yields weak-$*$ convergence in $\mathrm{BV}(\Omega)$ of a subsequence as stated in Theorem 2.8.

Theorem 5.21 ([95, Thm. 3.7])**.** *Let $F : L^2(\Omega) \to \mathbb{R}$ fulfill Assumption 5.1 and let a tuple (h, τ_h) be given such that Assumption 5.3 is fulfilled. Then problem (P_c^h) admits a solution.*

Proof. Problem (P_c^h) has the same optimal value as the optimization problem

$$(\tilde{\mathrm{P}}^{\mathrm{h}}) \qquad \min_{u \in \mathrm{BV_U}(\Omega) \cap P0^{\tau_h}} F(u) + \alpha \max\left\{\frac{1}{c}\mathrm{TV}(u), \mathrm{TV}^{\mathrm{h}}(u)\right\}.$$

Since the number of elements in $\mathrm{BV_U}(\Omega) \cap P0^{\tau_h}$ is finite, problem $(\tilde{\mathrm{P}}^{\mathrm{h}})$ admits an optimal solution which we denote by $\bar{u}$. Define $\bar{V} := \max\left\{\frac{1}{c}\mathrm{TV}(\bar{u}), \mathrm{TV}^{\mathrm{h}}(\bar{u})\right\}$, then $(\bar{u}, \bar{V})$ is optimal for (P_c^h). □

Remark 5.22 ([95, Rem. 3.8])**.** We could also prove the existence of minimizers of (P_c^h) without the discretization of u, that is, with

$$Z^h := \left\{u \in L^1_U(\Omega) : \mathrm{TV}(u) \leq cV,\ \mathrm{TV^h}(u) \leq V\right\}$$

because the constraint $\mathrm{TV}(u) \leq cV$ bounds minimizing sequences in $\mathrm{BV}(\Omega)$. This yields the existence of a subsequence that converges weakly-$*$ in $\mathrm{BV}(\Omega)$ by Theorem 2.8.

Lemma 5.23 ([95, Lem. 3.9])**.** *Let $F : L^2(\Omega) \to \mathbb{R}$ fulfill Assumption 5.1. Let $\{(h, \tau_h)\}_{h>0}$ be coupled such that Assumption 5.3 is fulfilled and let (u_{τ_h}, V_h) denote an optimal solution to* (P_c^h). *Then $\{(u_{\tau_h}, V_h)\}_{h>0}$ is bounded in* $\mathrm{BV}(\Omega) \times \mathbb{R}$.

Proof. The functional F is bounded from below by B such that $F(u) \geq B$ for all $u \in L^1(\Omega)$ with $\underline{\nu} \leq u(x) \leq \overline{\nu}$ for almost all $x \in \Omega$. Define $\tilde{u} \equiv \nu$ for some $\nu \in U$. Then $(\tilde{u}, 0)$ is feasible for (P_c^h) for all $h > 0$ with objective value $G^h(\tilde{u}, 0) = F(\tilde{u})$ since $\mathrm{TV^h}(\tilde{u}) = \mathrm{TV}(\tilde{u}) = 0$. Now let (u_{τ_h}, V_h) be optimal for (P_c^h) for $h > 0$. Then $F(u_{\tau_h}) + \alpha V_h \leq F(\tilde{u})$ and therefore $V_h \leq \frac{1}{\alpha}(F(\tilde{u}) - B)$ for all $h > 0$. This yields that $\mathrm{TV}(u_{\tau_h}) \leq cV_h \leq \frac{c}{\alpha}(F(\tilde{u}) - B)$. The boundedness in $L^1(\Omega)$ follows directly from the boundedness in $L^\infty(\Omega)$, which is due to $u_{\tau_h}(x) \in U$ for almost all $x \in \Omega$. □

Proof of Theorem 5.19. By Lemma 5.23 and Theorem 2.8, the sequence $\{u_{\tau_h}\}_{h>0}$ admits a subsequence that converges weakly-$*$ in $\mathrm{BV}(\Omega)$ to some $u \in \mathrm{BV}(\Omega)$ which we denote by the same symbol, that is, $u_{\tau_h} \stackrel{*}{\rightharpoonup} u$ in $\mathrm{BV}(\Omega)$. By Theorem 5.17, there holds $G(u) \leq \liminf_{h\searrow 0} G^h(u_{\tau_h}, V_h)$. By Theorem 5.18, there exists for each $(v, W) \in \mathrm{BV_U}(\Omega) \times \mathbb{R}$ a sequence $\{(v_h, W_h)\}_{h>0} \subset \mathrm{BV_U}(\Omega) \times \mathbb{R}$ such that $v_h \in P0^{\tau_h}$, $v_h \stackrel{*}{\rightharpoonup} v$ in $\mathrm{BV}(\Omega)$, and $G(v) \geq \limsup_{h\searrow 0} G^h(v_h, W_h)$. Together we then have

$$G(u) \leq \liminf_{h\searrow 0} G^h(u_{\tau_h}, V_h) \leq \liminf_{h\searrow 0} G^h(v_h, W_h) \leq \limsup_{h\searrow 0} G^h(v_h, W_h) \leq G(v)$$

for each $v \in \mathrm{BV_U}(\Omega)$. This yields the optimality of u for (P_c). □

Remark 5.24 ([95, Rem. 3.10])**.** Theorem 5.19 implicitly yields the existence of a minimizer for (P_c) and therefore for (P).

5.2.2 Lim sup inequality

In this section, we prove the lim sup inequality for the functions G and G^h, which is stated in Theorem 5.18. We need to find a constant c so that the inequality in the set Z^h can be satisfied by a recovery sequence. The admissible range of the constant c depends on the dimension d so that we provide different proofs depending on d. As an intermediate step, we first prove the inequality $\mathrm{TV}(R^U_{P0^\tau}(u)) \leq c\mathrm{TV}(u)$ in each case for a respective constant $c \geq 1$ depending on d. For the case $d = 1$ it is immediate that any constant $c \geq 1$ is permissible.

Lemma 5.25 ([95, Lem. 3.11])**.** *Let $d = 1$ and $\mathcal{Q}_\tau$ be a partition of Ω into intervals of length $\tau > 0$. Let $u \in \mathrm{BV_U}(\Omega)$. There holds $\mathrm{TV}(R^U_{P0^\tau}(u)) \le \mathrm{TV}(u)$.*

Proof. Let $\Omega = (\underline{a}, \overline{a})$ with $\mathcal{Q}_\tau = \{Q_1, \dots, Q_n\}$ with $Q_i = (a_i, a_{i+1}]$ for $i = 1, \dots, n-1$ and $Q_n = (a_n, a_{n+1})$ with $\underline{a} = a_1 < a_2 < \cdots < a_n < a_{n+1} = \overline{a}$. Let $R^U_{P0^\tau}|_{Q_i} \equiv \bar{\nu}_i$ with $\bar{\nu}_i \in U$ for $i = 1, \dots, n$. Definition 5.11 yields that $\left| u|^{-1}_{Q_i}(\bar{\nu}_i) \right| > 0$ for each $i = 1, \dots, n$, so that $\mathrm{TV}(R^U_{P0^\tau}(u)) = \sum_{i=1}^n |\bar{\nu}_i - \bar{\nu}_{i+1}| \le \mathrm{TV}(u)$. □

For the cases $d = 2$ and $d = 3$, we provide a constant that is not sharp but can be proven straightforwardly in Theorem 5.26. In Theorem 5.27, we will improve upon this in the case $d = 2$ and prove that $\sqrt{2}$ is a sharp lower bound on c in that case. Since the proof of Theorem 5.27 needs technical arguments, we only provide a sketch of the proof and postpone the detailed proof and the preparatory results to Section 5.2.3.

Theorem 5.26 ([95, Thm. 3.12])**.** *Let $d \in \{2, 3\}$ and $\mathcal{Q}_\tau$ be a partition of Ω into axis-aligned squares or cubes $Q \in \mathcal{Q}_\tau$ of height $\tau > 0$. Let $u \in \mathrm{BV_U}(\Omega)$. There holds*

$$\mathrm{TV}(R^U_{P0^\tau}(u)) \le c_d \mathrm{TV}(u)$$

with $c_d := (4d + 1)\sqrt{d}$.

Proof. We construct a function $\phi \in H_0(\mathrm{div}; \Omega)$ to realize the exact value of $\mathrm{TV}(R^U_{P0^\tau}(u))$ following the construction of Raviart–Thomas basis functions in [5]. Afterwards, we apply $(\mathrm{TV}_{H_0(\mathrm{div})})$ and use ϕ as a test function for $\mathrm{TV}(u)$ to estimate $\mathrm{TV}(R^U_{P0^\tau}(u))$ against $\mathrm{TV}(u)$. We denote the collection of facets in the interior of Ω by $\mathcal{E}$ and $\bar{u} := R^U_{P0^\tau}(u)$. Consider a facet $E \in \mathcal{E}$. Then $E = \overline{Q^E_1} \cap \overline{Q^E_2}$ for $Q^E_1, Q^E_2 \in \mathcal{Q}_\tau$ with $Q^E_1 \ne Q^E_2$. Denote by $q^E_1, q^E_2 \in \mathbb{R}^d$ the respective centers of the cubes Q^E_1 and Q^E_2 and define $T^E_i := \mathrm{conv}(E \cup \{q^E_i\})$ for $i \in \{1, 2\}$. We define the function ϕ_E by

$$\phi_E(x) = \begin{cases} (-1)^i s_E \frac{2}{\tau}(x - q^E_i) & \text{for } x \in T^E_i,\ i \in \{1, 2\} \\ 0 & \text{elsewhere,} \end{cases}$$

where $s_E := \mathrm{sgn}(\bar{u}|_{Q^E_2} - \bar{u}|_{Q^E_1}) \in \{-1, 0, 1\}$. There holds that $\phi_E \in H_0(\mathrm{div}; \Omega)$ because $\mathrm{div}\, \phi_E = (-1)^i d s_E \frac{2}{\tau}$ on T^E_i, $i \in \{1, 2\}$, and $\mathrm{div}\, \phi_E = 0$ elsewhere as well as $\phi_E \cdot n_E = s_E$ on E, $\phi_E \cdot n_{T^E_i} = 0$ on $\partial T^E_i \setminus E$ as in [5], where $n_{T^E_i}$ denotes the outer normal to T^E_i, $i \in \{1, 2\}$, and $\phi_E \cdot n_E = 0$ on $H \in \mathcal{E} \setminus E$, where n_E denotes the unit normal vector of E pointing from Q^E_2 to Q^E_1. Moreover, there holds $\|\phi_E\|_{L^\infty(\Omega;\mathbb{R}^d)} \le \sqrt{d}$. Note that the supports of all ϕ_E only intersect on a set of Lebesgue measure zero. We define $\phi := \frac{1}{\sqrt{d}} \sum_{E \in \mathcal{E}} \phi_E$, then $\phi \in H_0(\mathrm{div}; \Omega)$ with

$\|\phi\|_{L^\infty(\Omega;\mathbb{R}^d)} \leq 1$. There holds

$$\int_\Omega \bar{u}(x) \operatorname{div} \phi(x) \,\mathrm{d}x = \frac{1}{\sqrt{d}} \sum_{E\in\mathcal{E}} \int_E (\bar{u}|_{Q_2^E} - \bar{u}|_{Q_1^E})\, \phi_E(x) \cdot n_E(x) \,\mathrm{d}\mathcal{H}^{d-1}(x)$$
$$= \frac{1}{\sqrt{d}} \sum_{E\in\mathcal{E}} \int_E |\bar{u}|_{Q_2^E} - \bar{u}|_{Q_1^E}| \,\mathrm{d}\mathcal{H}^{d-1}(x) = \frac{1}{\sqrt{d}} \mathrm{TV}(\bar{u}),$$

where the last equality follows from Lemma 2.18, and therefore

$$\mathrm{TV}(\bar{u}) = \sqrt{d} \int_\Omega (\bar{u}(x) - u(x)) \operatorname{div} \phi(x) \,\mathrm{d}x + \sqrt{d} \int_\Omega u(x) \operatorname{div} \phi(x) \,\mathrm{d}x$$
$$\leq \sqrt{d} \|\bar{u} - u\|_{L^1(\Omega)} \| \operatorname{div} \phi\|_{L^\infty(\Omega)} + \sqrt{d}\mathrm{TV}(u) \leq (4d+1)\sqrt{d}\mathrm{TV}(u),$$

where we have used Lemma 5.12 for the second inequality. □

For the case $d = 2$, we improve the constant c.

Theorem 5.27 ([95, Thm. 3.13]). *Let $d = 2$ and Assumption 5.1 be fulfilled. Consider tuples $\{(h, \tau_h)\}_{h>0}$ such that Assumption 5.3 is fulfilled and $\frac{\tau_h}{h} \searrow 0$ as $h \searrow 0$. Let $u \in \mathrm{BV_U}(\Omega)$ be given. There is a sequence $\{u_h\}_{h>0}$ with $u_h \in P0^{\tau_h} \cap \mathrm{BV_U}(\Omega)$ as $h \searrow 0$ such that $u_h \stackrel{*}{\rightharpoonup} u$ in $\mathrm{BV}(\Omega)$,*

$$\limsup_{h\searrow 0} \mathrm{TV}(u_h) \leq \sqrt{2}\mathrm{TV}(u), \quad \textit{and} \quad \limsup_{h\searrow 0} \mathrm{TV}^{\mathrm{h}}(u_h) \leq \mathrm{TV}(u).$$

Sketch of proof. The proof is based on a diagonal sequence selection argument that chains several approximations. First, we approximate u strictly by functions with polygonal jump sets by means of Lemma 5.31. Then we apply the rounding operator $R^U_{P0^{\tau_h}}$ to the elements of the sequence to obtain functions in $P0^{\tau_h}$ with values in U. Next, we establish Theorem 5.32 to estimate the total variation of a rounding of a function with polygonal jump sets against the total variation of its preimage for vanishing grid size τ_h, which yields the constant $\sqrt{2}$. From this, a diagonal sequence is selected by balancing the approximation by functions with polygonal jump sets with the approximation of the rounding operator. □

The following example illustrates that the constant $\sqrt{2}$ in Theorem 5.27 is sharp, that is, there cannot exist $c < \sqrt{2}$ that satisfies $\limsup_{h\searrow 0} \mathrm{TV}(R^U_{P0^{\tau_h}}(u)) \leq c\mathrm{TV}(u)$ uniformly for all $u \in \mathrm{BV_U}(\Omega)$ if $d = 2$.

Example 5.28 ([95, Expl. 3.14]). We return to Example 5.2, that is, let $d = 2$, $\Omega = (0,1)^2$, $U = \{0,1\}$, and $u \in \mathrm{BV_U}(\Omega)$ be defined by $u = \chi_{\{(-1,1)^T x > 0\}}$. Then $\mathrm{TV}(u) = \sqrt{2}$. Consider the family of meshes $\{\mathcal{Q}_{\tau_k}\}_{k\in\mathbb{N}}$ with $\tau_k = \frac{1}{k}$ for $k \in \mathbb{N}$. Then $\mathrm{TV}(R^U_{P0^{\tau_k}}(u)) = \frac{2(k-1)}{k} \to 2$ as $k \to \infty$. This yields that the constant $\sqrt{2}$ in Theorem 5.32 is sharp.

Remark 5.29 ([95, Rem. 3.15])**.** The constant $\sqrt{2}$ in Theorem 5.27 is sharp for the case $d = 2$ as we proved in Example 5.28. We are convinced that a similar result holds for $d = 3$ that can be obtained with a similar proof strategy. The necessary technical effort goes beyond the scope of this thesis, however.

With the help of the former statements, we can now prove Theorem 5.18, which is stated in the beginning of the section.

Proof of Theorem 5.18. To prove the lim sup inequality, we may assume $u \in Z$ and $\mathrm{TV}(u) < \infty$ because otherwise the inequality follows immediately because in this case there holds $G(u) = \infty$. We consider the components of the objective functions G and G^h separately and start by proving $\limsup_{h\searrow 0} \alpha V_h \leq \alpha \mathrm{TV}(u)$. For the case $d = 2$, we consider the sequence $\{u_{\mathcal{T}_h}\}_{h>0}$ from Theorem 5.27 and for the cases $d = 1$ and $d = 3$ we choose $u_{\mathcal{T}_h} \coloneqq R^U_{P0^{\mathcal{T}_h}}(u)$ for $h > 0$. Then there holds

$$\limsup_{h\searrow 0} \mathrm{TV}(u_{\mathcal{T}_h}) \leq c\mathrm{TV}(u) \tag{5.4}$$

for the respective $c \geq 1$ in the case $d = 1$, $c \geq \sqrt{2}$ in the case $d = 2$, and $c \geq 13\sqrt{3}$ in the case $d = 3$. We define $\overline{T} \coloneqq \limsup_{h\searrow 0} \mathrm{TV}(u_{\mathcal{T}_h}) \leq c\mathrm{TV}(u) < \infty$ and $\underline{T} \coloneqq \liminf_{h\searrow 0} \mathrm{TV}^{\mathrm{h}}(u_{\mathcal{T}_h}) \geq 0$. Then we define the following two sequences $\delta^1_h \coloneqq \max\{0, \mathrm{TV}(u_{\mathcal{T}_h}) - \overline{T}\}$ and $\delta^2_h \coloneqq \max\{0, \underline{T} - \mathrm{TV}^{\mathrm{h}}(u_{\mathcal{T}_h})\}$, which fulfill $\frac{\delta^1_h}{c} + \delta^2_h \to 0$ as $h \searrow 0$. We define $V_h \coloneqq \mathrm{TV}^{\mathrm{h}}(u_{\mathcal{T}_h}) + \frac{\delta^1_h}{c} + \delta^2_h$. There holds

$$\limsup_{h\searrow 0} \mathrm{TV}^{\mathrm{h}}(u_{\mathcal{T}_h}) + \frac{\delta^1_h}{c} + \delta^2_h \leq \limsup_{h\searrow 0} \mathrm{TV}^{\mathrm{h}}(u_{\mathcal{T}_h}) + \lim_{h\searrow 0} \frac{\delta^1_h}{c} + \delta^2_h \leq \mathrm{TV}(u)$$

by Proposition 5.13 and Theorem 5.27 and therefore $\limsup_{h\searrow 0} \alpha V_h \leq \alpha\mathrm{TV}(u)$. Next, we consider the indicator functionals $\mathcal{I}_{Z^h}$ and $\mathcal{I}_Z$. Since $\mathrm{TV}(u) \leq c\mathrm{TV}(u)$ is trivially fulfilled due to $c \geq 1$, there holds $\mathcal{I}_Z(u) = 0$. We prove that $\mathcal{I}_{Z^h}(u_{\mathcal{T}_h}, V_h) = 0$ for all $h > 0$, that is we need to prove that $\mathrm{TV}^{\mathrm{h}}(u_{\mathcal{T}_h}) \leq V_h$ and $\mathrm{TV}(u_{\mathcal{T}_h}) \leq cV_h$ for all $h > 0$. The inequality $\mathrm{TV}^{\mathrm{h}}(u_{\mathcal{T}_h}) \leq V_h = \mathrm{TV}^{\mathrm{h}}(u_{\mathcal{T}_h}) + \frac{\delta^1_h}{c} + \delta^2_h$ is trivially fulfilled due to $\frac{\delta^1_h}{c}, \delta^2_h \geq 0$. For the second inequality, there holds

$$\begin{aligned} \mathrm{TV}(u_{\mathcal{T}_h}) - \delta^1_h &\leq \limsup_{h\searrow 0} \mathrm{TV}(u_{\mathcal{T}_h}) \leq c\mathrm{TV}(u) \\ &\leq c \liminf_{h\searrow 0} \mathrm{TV}^{\mathrm{h}}(u_{\mathcal{T}_h}) \leq c(\mathrm{TV}^{\mathrm{h}}(u_{\mathcal{T}_h}) + \delta^2_h), \end{aligned}$$

where the second inequality is due to (5.4) and the third follows from Theorem 5.9, which yields $\mathrm{TV}(u_{\mathcal{T}_h}) \leq cV_h$.

Finally, since the function F fulfills Assumption 5.1, there holds $F(u) = \lim_{h\searrow 0} F(u_{\mathcal{T}_h})$ because $u_{\mathcal{T}_h} \to u$ in $L^p(\Omega)$ for all $1 \leq p < \infty$ by Lemma 2.19. In total, we have

$$G(u) = F(u) + \alpha \mathrm{TV}(u) + \mathcal{I}_Z(u) \geq \lim_{h \searrow 0} F(u_{\tau_h}) + \limsup_{h \searrow 0} \alpha V_h + \lim_{h \searrow 0} \mathcal{I}_{Z^h}(u_{\tau_h}, V_h)$$
$$\geq \limsup_{h \searrow 0} F(u_{\tau_h}) + \alpha V_h + \mathcal{I}_{Z^h}(u_{\tau_h}, V_h) = \limsup_{h \searrow 0} G^h(u_{\tau_h}, V_h).$$

□

5.2.3 Proof of Theorem 5.27

We present the arguments that are needed for the proof of Theorem 5.27 and eventually the detailed proof of Theorem 5.27. The first result is due to [13] and states that each function in $\mathrm{BV}_{\mathrm{U}}(\Omega)$ can be approximated by a strictly converging sequence in $\mathrm{BV}_{\mathrm{U}}(\Omega)$ of functions with polygonal level sets. The approximation argument in [13] relies on the rectifiability of the reduced boundaries of the level sets of functions in $\mathrm{BV}_{\mathrm{U}}(\Omega)$. We recall that the level sets of functions in $\mathrm{BV}_{\mathrm{U}}(\Omega)$ are Caccioppoli partitions of Ω by Lemma 2.18 whose reduced boundaries are rectifiable by the theorem of De Giorgi, see Section 2.2. In line with [13], we first define polygonal sets for $d = 2$.

Definition 5.30 ([95, Def. A.1])**.** Let $d = 2$. We say that a set $\Sigma \subset \Omega$ is *polygonal* if there is a finite number of closed line segments $S_1, \ldots, S_n \subset \mathbb{R}^2$ of strictly positive $\mathcal{H}^1$-measure such that Σ coincides, up to $\mathcal{H}^1$-null sets, with $\bigcup_{j=1}^{n} S_j \cap \Omega$. We call a point in which at least two line segments intersect or a line segment ends a *vertex.*

Lemma 5.31 ([95, Lem. A.2])**.** *Let $u \in \mathrm{BV}_{\mathrm{U}}(\Omega)$. There is a sequence $\{u_k\}_{k \in \mathbb{N}} \subset \mathrm{BV}_{\mathrm{U}}(\Omega)$ of functions with level sets with polygonal boundaries such that $u_k \to u$ in $L^1(\Omega)$ and $\mathrm{TV}(u_k) \to \mathrm{TV}(u)$.*

Proof. Follows from Theorem 2.1 and Corollary 2.5 in [13]. Note that Ω is bounded in our setting and we do not need to work with $L^1_{loc}(\Omega)$ here. □

For $u \in \mathrm{BV}_{\mathrm{U}}(\Omega)$ with jump sets with polygonal boundary, we are able to bound the limit of $\mathrm{TV}(R^U_{P0^\tau}(u))$ for $\tau \searrow 0$ from above by $\sqrt{2}\mathrm{TV}(u)$.

Theorem 5.32 ([95, Thm. A.3])**.** *Let $d = 2$. Let $\{\mathcal{Q}_\tau\}_{\tau > 0}$ be partitions of Ω fulfilling Assumption 5.3 and $u \in \mathrm{BV}_{\mathrm{U}}(\Omega)$ be given such that the level sets of u have polygonal boundaries. Then there holds*

$$\limsup_{\tau \searrow 0} \mathrm{TV}(R^U_{P0^\tau}(u)) \leq \sqrt{2}\mathrm{TV}(u).$$

Proof. Let $u \in \mathrm{BV}_{\mathrm{U}}(\Omega)$ have level sets with polygonal boundaries consisting of $n \in \mathbb{N}$ line segments. We denote the set of these line segments by $\mathcal{S}$ and the associated set of vertices by $\mathcal{V}$. We may assume without loss of generality that the line segments in

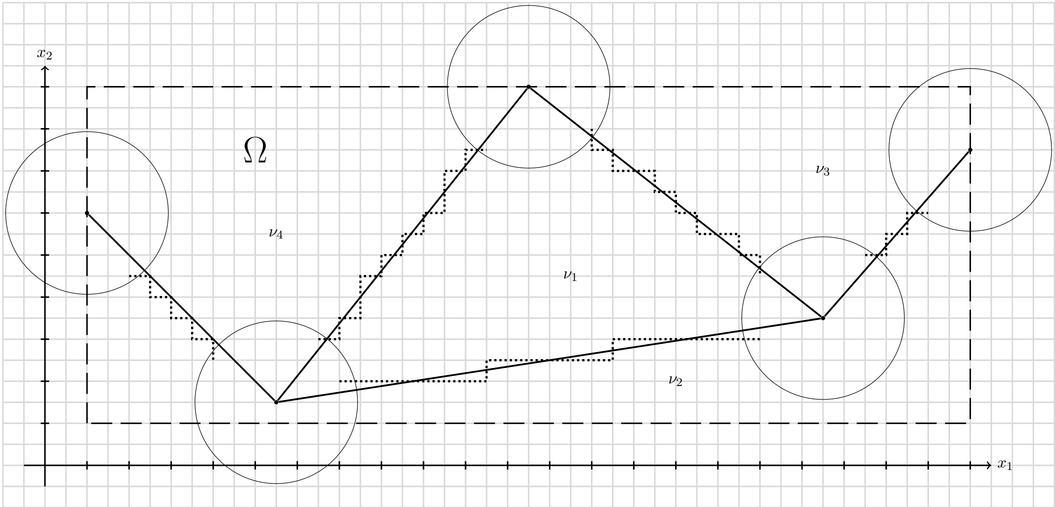


Figure 5.3: Example for a function $u \in \mathrm{BV_U}(\Omega)$ with level sets that have polygonal boundaries with the notations from the proof of Theorem 5.32.

$\mathcal{S}$ do not contain vertices in their relative interior, that is, one line segment connects exactly two adjacent vertices. This yields that the number of vertices is also bounded by twice the number of line segments n. Moreover, a line segment $S \in \mathcal{S}$ separates exactly two level sets inside Ω and we may restrict the line segments to $\bar{\Omega}$.

We make the following preliminary considerations: consider a ball $B_\varepsilon(v)$ with radius $\varepsilon > 0$ centered at a vertex $v \in \mathcal{V}$. Denote by $\alpha \in (0, \pi]$ the angle between two segments $S_1, S_2 \in \mathcal{S}$ with $S_1 \neq S_2$ meeting in v. The distance $\delta \geq 0$ between the two points $\partial B_\varepsilon(v) \cap S_1$ and $\partial B_\varepsilon(v) \cap S_2$ is then given by $\delta = \varepsilon s$, where we define $s \coloneqq 2 \sin\left(\frac{\alpha}{2}\right)$. Let the mesh size $\tau > 0$ be small enough such that there hold $\partial B_\varepsilon(v) \cap S_1 \neq \emptyset$ and $\partial B_\varepsilon(v) \cap S_2 \neq \emptyset$ with $\varepsilon \coloneqq \left(1 + \frac{1}{s}\right) \sqrt{2}\tau$. Then there holds $\delta > \sqrt{2}\tau$, that is $\partial B_\varepsilon(v) \cap S_1$ and $\partial B_\varepsilon(v) \cap S_2$ do not lie in the same cube $Q \in \mathcal{Q}_\tau$. The number of cubes of a given mesh $\mathcal{Q}_\tau$ that intersect the ball $B_\varepsilon(v)$ can be bounded from above by $\frac{4(\varepsilon+\tau)^2}{\tau^2}$ so that we obtain for our choice of ε the upper bound $b \coloneqq 8\left(\frac{1}{\sqrt{2}} + 1 + \frac{1}{s}\right)^2$ which is constant and independent of τ and ε.

Now let $\alpha \in (0, \pi]$ denote the minimum angle between two segments meeting in a vertex $v \in \mathcal{V}$ and define ε as above. We consider the line segments $\mathcal{S}^\varepsilon \coloneqq \{S \setminus \cup_{v \in \mathcal{V}} B_\varepsilon(v) : S \in \mathcal{S}\}$ outside the balls $B_\varepsilon(v)$. The number of segments in $\mathcal{S}^\varepsilon$ is then for τ small enough equal to n. Moreover, by the choice of ε, there is no cube $Q \in \mathcal{Q}_\tau$ that contains more than one line segment from $\mathcal{S}^\varepsilon$.

We make the following observations. The reduced boundary of the level sets of $R^U_{P0^\tau}(u)$ is by Definition 5.11 a subset of the boundaries of the cells of $\mathcal{Q}_\tau$. Because the total variation of a U-valued function is the sum of the interface lengths weighted by their respective jump heights, we can split $\mathrm{TV}(R^U_{P0^\tau}(u))$ into a contribution along segments from $\mathcal{S}^\varepsilon$ and a contribution inside the balls $B_\varepsilon(v)$.

We estimate the contribution inside the balls conservatively by multiplying the number of cubes inside the balls with their perimeter, that is in total by $4bn\tau$.

Now let a segment $S = \operatorname{conv}\{v, w\} \in \mathcal{S}^\varepsilon$ with $v, w \in \Omega$ be given. Then we can bound the contribution to $\mathrm{TV}(R^U_{P0^\tau}(u))$ along S from above by

$$(\|v-w\|_1 + 4\tau)|\nu_1 - \nu_2| \leq (\sqrt{2}\|v-w\|_2 + 4\tau)|\nu_1 - \nu_2|,$$

where $\nu_1, \nu_2 \in U$ denote the two values of u within the two level sets separated by S. The estimate holds because the value of $R^U_{P0^\tau}(u)$ on a cube along S is determined solely by the value of u in the center point of the cube and by the monotonicity of the line segment S. Since the total variation of u along S is given by $|\nu_1 - \nu_2|\|v-w\|_2$, we conclude

$$\mathrm{TV}(R^U_{P0^\tau}(u)) \leq \sqrt{2}\mathrm{TV}(u) + 4n(U_{\max} + b)\tau$$

with $U_{\max} \coloneqq \max_{\nu_1,\nu_2 \in U} |\nu_1 - \nu_2|$ so that

$$\limsup_{\tau \searrow 0} \mathrm{TV}(R^U_{P0^\tau}(u)) \leq \sqrt{2}\mathrm{TV}(u).$$

$\square$

Finally, we provide the proof of Theorem 5.27.

Proof of Theorem 5.27. By Lemma 5.31, there is a sequence $\{v_j\}_{j\in\mathbb{N}} \subset \mathrm{BV}_{\mathrm{U}}(\Omega)$ of functions with jump sets with polygonal boundaries such that $v_j \to u$ in $L^1(\Omega)$ and $\mathrm{TV}(v_j) \to \mathrm{TV}(u)$ as $j \to \infty$. We pick a subsequence $\{v_{j_m}\}_{m\in\mathbb{N}}$ such that we have $\|v_{j_m} - u\|_{L^1(\Omega)} \leq \frac{1}{m}$ and

$$\mathrm{TV}(v_{j_m}) \leq \mathrm{TV}(u) + \frac{1}{m}.$$

By Lemma 5.12, there holds $R^U_{P0^{\tau_h}}(v_{j_m}) \to v_{j_m}$ as $h \searrow 0$ in $L^1(\Omega)$. Since the functions $\{v_{j_m}\}_{m\in\mathbb{N}}$ have jump sets with polygonal boundaries, we may apply Theorem 5.32 such that there holds $\limsup_{h\searrow 0} \mathrm{TV}(R^U_{P0^{\tau_h}}(v_{j_m})) \leq \sqrt{2}\mathrm{TV}(v_{j_m})$. This yields that for each $m \in \mathbb{N}$ there is some $h_m > 0$ such that

$$\|R^U_{P0^{\tau_h}}(v_{j_m}) - v_{j_m}\|_{L^1(\Omega)} \leq \frac{1}{m}$$

and

$$\mathrm{TV}(R^U_{P0^{\tau_h}}(v_{j_m})) \leq \sqrt{2}\mathrm{TV}(v_{j_m}) + \frac{1}{m}$$

for all $h \leq h_m$. We additionally choose the sequence $\{h_m\}_{m\in\mathbb{N}}$ such that it decreases strictly monotonically. We define $\{\tilde{u}_h\}_{h>0}$ by $\tilde{u}_h \coloneqq v_{j_m}$ for $h \in (h_{m+1}, h_m]$ and

$u_h := R^U_{P0^{\tau_h}}(\tilde{u}_h)$ for $h > 0$. There holds for $h \in (h_{m+1}, h_m]$ that

$$
\begin{aligned}
\|u_h - u\|_{L^1(\Omega)} &= \|R^U_{P0^{\tau_h}}(\tilde{u}_h) - u\|_{L^1(\Omega)} = \|R^U_{P0^{\tau_h}}(v_{j_m}) - u\|_{L^1(\Omega)} \\
&\le \|R^U_{P0^{\tau_h}}(v_{j_m}) - v_{j_m}\|_{L^1(\Omega)} + \|v_{j_m} - u\|_{L^1(\Omega)} \le \frac{2}{m},
\end{aligned}
$$

and

$$
\begin{aligned}
\mathrm{TV}(u_h) &= \mathrm{TV}(R^U_{P0^{\tau_h}}(\tilde{u}_h)) = \mathrm{TV}(R^U_{P0^{\tau_h}}(v_{j_m})) \\
&\le \sqrt{2}\mathrm{TV}(v_{j_m}) + \frac{1}{m} \le \sqrt{2}\mathrm{TV}(u) + \frac{1+\sqrt{2}}{m}.
\end{aligned}
$$

That is, for each $m \in \mathbb{N}$ there hold $\|u_h - u\|_{L^1(\Omega)} \le \frac{2}{m}$ and

$$
\mathrm{TV}(u_h) \le \sqrt{2}\mathrm{TV}(u) + \frac{1+\sqrt{2}}{m}
$$

for all $h \le h_m$, which yields $u_h \to u$ in $L^1(\Omega)$ as $h \searrow 0$ and

$$
\limsup_{h \searrow 0} \mathrm{TV}(u_h) \le \sqrt{2}\mathrm{TV}(u).
$$

In particular, the sequence $\{u_h\}_{h>0}$ is bounded in $\mathrm{BV}(\Omega)$, which together with the convergence in $L^1(\Omega)$ yields that $u_h \overset{*}{\rightharpoonup} u$ as $h \searrow 0$ in $\mathrm{BV}(\Omega)$.

Moreover, by Proposition 5.13, for each $m \in \mathbb{N}$ there holds for all $h \in (h_{m+1}, h_m]$

$$
\begin{aligned}
\mathrm{TV}^{\mathrm{h}}(u_h) &= \mathrm{TV}^{\mathrm{h}}(R^U_{P0^{\tau_h}}(v_{j_m})) \\
&\le \left(1 + \frac{4\sqrt{2}\tau_h}{h}\right) \mathrm{TV}(v_{j_m}) \le \left(1 + \frac{4\sqrt{2}\tau_h}{h}\right)\left(\mathrm{TV}(u) + \frac{1}{m}\right),
\end{aligned}
$$

such that $\limsup_{h \searrow 0} \mathrm{TV}^{\mathrm{h}}(u_h) \le \mathrm{TV}(u)$ due to $\frac{\tau_h}{h} \searrow 0$ as $h \searrow 0$. ☐

5.3 Outer approximation

In this section, we provide an outer-approximation algorithm for solving (P^h_c). The inequality $\mathrm{TV}^{\mathrm{h}}(u) \le V$ in (P^h_c) is equivalent to

$$
\int_\Omega \operatorname{div} \phi(x) u(x) \,\mathrm{d}x \le V \quad \forall\, \phi \in RT0^h_0 \text{ with } \|\phi\|_{L^\infty(\Omega;\mathbb{R}^d)} \le 1. \tag{5.5}
$$

The idea of the algorithm is to start by solving (P^h_c) without the constraint $\mathrm{TV}^{\mathrm{h}}(u) \le V$ and iteratively add the inequalities in (5.5) to cut off infeasible solutions until the remaining violation of (5.5) vanishes. The algorithm is stated in Algorithm 3.

Algorithm 3 Outer-approximation algorithm for (P_c^h)

Input: F sufficiently regular, $\alpha > 0$, $\tau > 0$, $h > 0$.

1: Set $k = 0$.

2: Compute an optimal solution (u_k, V_k) to

$$
\text{(MIP)} \qquad \begin{aligned}
\min_{(u,V)\in P0^\tau \times \mathbb{R}} \quad & F(u) + \alpha V \\
\text{s.t.} \quad & \mathrm{TV}(u) \le cV \\
& \int_\Omega \operatorname{div} \phi_i(x) u(x) \,\mathrm{d}x \le V \quad \forall\, i \in \mathbb{N}, i \le k \\
& u(x) \in U \text{ for a.a. } x \in \Omega.
\end{aligned}
$$

3: Compute $\mathrm{TV^h}(u_k)$ by solving

$$
\text{(QP)} \qquad \begin{aligned}
\max_{\phi \in RT0_0^h} \quad & \int_\Omega \operatorname{div} \phi(x) u_k(x) \,\mathrm{d}x \\
\text{s.t.} \quad & \|\phi\|_{L^\infty(\Omega;\mathbb{R}^d)} \le 1
\end{aligned}
$$

with optimal solution ϕ_{k+1}.

4: **if** $\int_\Omega \operatorname{div} \phi_{k+1}(x) u_k(x) \,\mathrm{d}x - V_k \le 0$ **then**

5: **return** u_k as optimal solution

6: **end if**

7: Set $k \leftarrow k + 1$ and go to step 2.

Theorem 5.33 ([95, Thm. 4.2])**.** *Algorithm 3 is well-defined, stops after finitely many iterations, and returns an optimal solution to* (P_c^h)*.*

Proof. Algorithm 3 is well-defined because problem (MIP) admits an optimal solution due to its finite and non-empty feasible set and the boundedness of the objective function from below and problem (QP) admits an optimal solution by Lemma 5.5.

Assume that the algorithm finds (u_i, V_i) in step 2 in iteration i and (u_j, V_j) in step 2 of iteration j with $i < j$ and $u_i = u_j$. Since $\mathrm{TV^h}(u_j) = \mathrm{TV^h}(u_i) = \int_\Omega \operatorname{div} \phi_{i+1}(x) u_i(x) \,\mathrm{d}x \le V_j$ and $\mathrm{TV}(u_j) = \mathrm{TV}(u_i) \le cV_j$, the algorithm terminates and $u_j = u_i$ is the optimal solution to (P_c^h). This yields that the algorithm always stops after finitely many iterations since the number of functions in $\mathrm{BV_U}(\Omega) \cap P0^\tau$ is finite. $\square$

5.4 Discretization of the relaxation

We will now introduce discretizations of the relaxation $(\mathrm{P_R})$ and the regularized relaxation $(\mathrm{P}_{\delta,\varepsilon})$ of (P). In order to discretize those problems and to prove Γ-convergence results, we require that Assumptions 4.1 and 5.1 hold within this section. As already discussed in Section 5.1.2, we are able to recover the total variation of a limit function $u \in L^\infty(\Omega)$ with the total variation of the projections onto the finite-element space $P0^{\tau_h}$ if we are not restricted to values in U. This can be used for the discretization of the relaxation ($\mathrm{P_R}$) of (P) since we are not restricted to values in U but only have box constraints $\underline{\nu} \le u(x) \le \overline{\nu}$ for almost all $x \in \Omega$. This was also elaborated in [23], so that we can deduce the following lemma.

Lemma 5.34. *Let tuples $\{(h, \tau_h)\}_{h>0}$ be given such that Assumption 5.3 is fulfilled and $h \searrow 0$. Let $u \in L^\infty(\Omega)$ be given. Then there holds*

$$\mathrm{TV}^{\mathrm{h}}(\Pi_{P0^{\tau_h}} u) \to \mathrm{TV}(u) \quad \text{as} \quad h \searrow 0.$$

Proof. By Assumption 5.3, there holds $\mathrm{TV}^{\mathrm{h}}(\Pi_{P0^{\tau_h}} u) = \mathrm{TV}^{\mathrm{h}}(\Pi_{P0^h} u)$ because the divergence $\operatorname{div}\phi$ of a lowest-order Raviart–Thomas function $\phi \in RT0^h$ is constant on the mesh cells $\mathcal{Q}_h$. According to Lemma 3.1 in [23], there holds $\mathrm{TV}^{\mathrm{h}}(\Pi_{P0^h} u) = \mathrm{TV}^{\mathrm{h}}(u) \le \mathrm{TV}(u)$. Since $h \searrow 0$ also implies $\tau_h \searrow 0$ due to Assumption 5.3, there holds $\Pi_{P0^{\tau_h}} u \to u$ in $L^1(\Omega)$ as $h \searrow 0$ by Lemma 5.7. We then obtain by Theorem 5.9 that

$$\begin{aligned}
\mathrm{TV}(u) &\le \liminf_{h \searrow 0} \mathrm{TV}^{\mathrm{h}}(\Pi_{P0^{\tau_h}} u) \le \limsup_{h \searrow 0} \mathrm{TV}^{\mathrm{h}}(\Pi_{P0^{\tau_h}} u) \\
&\le \limsup_{h \searrow 0} \mathrm{TV}^{\mathrm{h}}(\Pi_{P0^h} u) \le \mathrm{TV}(u),
\end{aligned}$$

which yields $\mathrm{TV}(u) = \lim_{h \searrow 0} \mathrm{TV}^{\mathrm{h}}(\Pi_{P0^{\tau_h}} u)$. □

We highlight that tuples $\{(h, \tau_h)\}_{h>0}$ with $\tau_h = h$ are valid for Lemma 5.34.

The feasible set of the relaxation ($\mathrm{P_R}$) is sequentially weakly closed in $L^2(\Omega)$. We therefore do not need to add an additional constraint that bounds the total variation as in (P_c^h) to guarantee the convergence of optimal solutions to the discretized problems to a feasible solution to ($\mathrm{P_R}$) if we replace the total variation TV by the discretized total variation TV^{h}. However, in that case we can only prove the existence of weakly convergent subsequences in $L^2(\Omega)$ due to the box constraints. We therefore need to prove the statements regarding the lim inf inequalities with respect to weak convergence in $L^2(\Omega)$. The following lemma is a variant of Lemma 5.8 and Theorem 5.9 for weakly convergent sequences in $L^2(\Omega)$.

Lemma 5.35 (Variant of [95, Lem. 2.8])**.** *Let $\{u_h\}_{h>0} \subset L^2(\Omega)$ with $u_h \rightharpoonup u$ in $L^2(\Omega)$ as $h \searrow 0$. There holds $\mathrm{TV}(u) \le \liminf_{h \searrow 0} \mathrm{TV}^{\mathrm{h}}(u_h)$.*

Proof. The proof closely follows the proofs of Lemma 5.8 and Theorem 5.9. Let $\phi \in C_c^1(\Omega; \mathbb{R}^d)$ satisfy $\|\phi\|_{L^\infty(\Omega;\mathbb{R}^d)} \le 1$. Let $\rho \in (0,1)$ be arbitrary but fixed. Then we approximate ϕ with $\hat{\phi} \coloneqq (1-\rho)\phi \in C_c^1(\Omega; \mathbb{R}^d)$ such that

$$\|\hat{\phi}\|_{L^\infty(\Omega;\mathbb{R}^d)} = (1-\rho)\|\phi\|_{L^\infty(\Omega;\mathbb{R}^d)} \le 1 - \rho$$

and

$$\begin{aligned}\int_\Omega \operatorname{div}\phi(x)u(x)\,\mathrm{d}x &= \int_\Omega \operatorname{div}\hat{\phi}(x)u(x)\,\mathrm{d}x + \rho\int_\Omega \operatorname{div}\phi(x)u(x)\,\mathrm{d}x \\ &\leq \int_\Omega \operatorname{div}\hat{\phi}(x)u(x)\,\mathrm{d}x + \underbrace{\|\operatorname{div}\phi\|_{L^\infty(\Omega)}\|u\|_{L^1(\Omega)}}_{c_1:=}\rho.\end{aligned}$$

By virtue of Proposition 2.26, we obtain that $\|I_{RT0^h}\hat{\phi}\|_{L^\infty(\Omega;\mathbb{R}^d)} \leq 1$ holds for all $0 < h \leq \frac{\rho}{C(\hat{\phi})}$, where $I_{RT0^h} : W^{1,\infty}(\Omega;\mathbb{R}^d) \to RT0^h$ denotes the interpolation operator for $RT0^h$ as defined in Section 2.5 and $C(\hat{\phi}) > 0$ is a constant depending on $\hat{\phi}$. Moreover, since $\hat{\phi}$ has compact support in Ω, there holds $\hat{\phi}\cdot n \equiv 0$ on $\partial\Omega$, where n denotes the outer unit normal of Ω, so that $I_{RT0^h}\hat{\phi} \in RT0_0^h$.

Moreover, there holds for all $0 < h \leq \frac{\rho}{C(\hat{\phi})}$ that

$$\begin{aligned}\int_\Omega \operatorname{div}\hat{\phi}(x)u_h(x)\,\mathrm{d}x &= \int_\Omega \operatorname{div} I_{RT0^h}\hat{\phi}(x)u_h(x)\,\mathrm{d}x + \int_\Omega \operatorname{div}(\hat{\phi} - I_{RT0^h}\hat{\phi})(x)u_h(x)\,\mathrm{d}x \\ &\leq \mathrm{TV^h}(u_h) + \|\operatorname{div}(\hat{\phi} - I_{RT0^h}\hat{\phi})\|_{L^2(\Omega)}\|u_h\|_{L^2(\Omega)}.\end{aligned}$$

Since $\|u_h\|_{L^2(\Omega)}$ is bounded and $\|\operatorname{div}(\hat{\phi} - I_{RT0^h}\hat{\phi})\|_{L^2(\Omega)} \to 0$ by (2.14), this yields

$$\int_\Omega \operatorname{div}\hat{\phi}(x)u(x)\,\mathrm{d}x = \lim_{h\searrow 0}\int_\Omega \operatorname{div}\hat{\phi}(x)u_h(x)\,\mathrm{d}x \leq \liminf_{h\searrow 0}\mathrm{TV^h}(u_h).$$

Together, we obtain

$$\int_\Omega \operatorname{div}\phi(x)u(x)\,\mathrm{d}x \leq \liminf_{h\searrow 0}\mathrm{TV^h}(u_h) + \rho c_1.$$

Since $\rho \in (0,1)$ was arbitrary, this gives

$$\int_\Omega \operatorname{div}\phi(x)u(x)\,\mathrm{d}x \leq \liminf_{h\searrow 0}\mathrm{TV^h}(u_h).$$

Supremizing over all $\phi \in C_c^1(\Omega;\mathbb{R}^d)$ with $\|\phi\|_{L^\infty(\Omega;\mathbb{R}^d)} \leq 1$ then yields

$$\mathrm{TV}(u) \leq \liminf_{h\searrow 0}\mathrm{TV^h}(u_h).$$

□

In order to discretize ($\mathrm{P_R}$), we propose the following discretization with mesh sizes (h, τ_h) that fulfill Assumption 5.3 that reads

$$(\mathrm{P_R^h}) \qquad \begin{aligned} \min_{u\in P0^{\tau_h}} &\quad F(u) + \alpha\,\mathrm{TV^h}(u) \\ \text{s.t.} &\quad \underline{\nu} \leq u(x) \leq \overline{\nu} \quad \text{for a.a. } x \in \Omega.\end{aligned}$$

We define the two functions $G_R : L^2(\Omega) \to \mathbb{R}$ and $G_R^h : L^2(\Omega) \to \mathbb{R}$ by

$$G_R(u) \coloneqq F(u) + \alpha \mathrm{TV}(u) + \mathcal{I}_{Z_R}(u)$$

and

$$G_R^h(u) \coloneqq F(u) + \alpha\, \mathrm{TV^h}(u) + \mathcal{I}_{Z_R^h}(u)$$

for $u \in L^2(\Omega)$ and where $\mathcal{I}_{Z_R}$ and $\mathcal{I}_{Z_R^h}$ denote the $\{0, \infty\}$-valued indicator functionals to the sets

$$Z_R \coloneqq \{u \in L^2(\Omega) : \underline{\nu} \le u(x) \le \overline{\nu} \text{ for a.a. } x \in \Omega\}$$

and

$$Z_R^h \coloneqq \{u \in P0^{\mathcal{T}_h} : \underline{\nu} \le u(x) \le \overline{\nu} \text{ for a.a. } x \in \Omega\}.$$

Since Z_R^h is finite-dimensional, the existence of a minimizer to $(\mathrm{P^h_R})$ follows immediately.

Theorem 5.36. *Let $F : L^2(\Omega) \to \mathbb{R}$ fulfill Assumption 4.1. Let G_R and G_R^h be defined as above. Consider tuples $\{(h, \mathcal{T}_h)\}_{h>0}$ such that Assumption 5.3 is fulfilled. Let $\{u_{\mathcal{T}_h}\}_{h>0} \subset L^2(\Omega)$ be a sequence with $u_{\mathcal{T}_h} \rightharpoonup u$ in $L^2(\Omega)$ as $h \searrow 0$. Then there holds*

$$G_R(u) \le \liminf_{h \searrow 0} G_R^h(u_{\mathcal{T}_h}).$$

Proof. Without loss of generality, we may assume that $u_{\mathcal{T}_h} \in Z_R^h$ for all $h > 0$ because $u_{\mathcal{T}_h} \notin Z_R^h$ implies the trivial case $G_R^h(u_{\mathcal{T}_h}) = \infty$. That is, in particular there holds $u_{\mathcal{T}_h} \in P0^{\mathcal{T}_h}$ and $\underline{\nu} \le u_{\mathcal{T}_h}(x) \le \overline{\nu}$ for almost all $x \in \Omega$ for all $h > 0$. Since the set Z_R is convex and closed in $L^2(\Omega)$ and therefore sequentially weakly closed in $L^2(\Omega)$, there hold $\underline{\nu} \le u(x) \le \overline{\nu}$ for almost all $x \in \Omega$ which gives $\mathcal{I}_{Z_R}(u) = 0$. Together with Lemma 5.35, there holds

$$\alpha \mathrm{TV}(u) \le \liminf_{h \searrow 0} \alpha\, \mathrm{TV^h}(u_{\mathcal{T}_h}).$$

Because F fulfills Assumption 4.1, there holds $F(u) \le \liminf_{h \searrow 0} F(u_{\mathcal{T}_h})$. In total, we obtain

$$G_R(u) \le \liminf_{h \searrow 0} G_R^h(u_{\mathcal{T}_h}).$$

$\square$

Theorem 5.37. *Let $F : L^2(\Omega) \to \mathbb{R}$ fulfill Assumption 5.1 and tuples $\{(h, \tau_h)\}_{h>0}$ be given such that Assumption 5.3 is fulfilled. Let $u \in L^1(\Omega)$ be given. Let G_R and G_R^h be defined as above. Then there holds*

$$G_R(u) \geq \limsup_{h \searrow 0} G_R^h(\Pi_{P0^{\tau_h}} u).$$

Proof. If $u \notin Z_R$, then $G_R(u) = \infty$, which implies the trivial case. Therefore, we may assume $u \in Z_R$, that is, $\underline{\nu} \leq u(x) \leq \overline{\nu}$ for almost all $x \in \Omega$. This yields that also $\underline{\nu} \leq \Pi_{P0^{\tau_h}} u(x) \leq \overline{\nu}$ for almost all $x \in \Omega$ for all $h > 0$ such that $\mathcal{I}_{Z_R^h}(\Pi_{P0^{\tau_h}} u) = 0$ for all $h > 0$. By Lemma 5.34, it follows that

$$\alpha \mathrm{TV}(u) = \lim_{h \searrow 0} \alpha\, \mathrm{TV}^{\mathrm{h}}(\Pi_{P0^{\tau_h}} u).$$

As in the proof of Lemma 5.34, there holds $\Pi_{P0^{\tau_h}} u \to u$ in $L^1(\Omega)$ as $h \searrow 0$ by Assumption 5.3 and Lemma 5.7. Hence, Lemma 2.19 yields $\Pi_{P0^{\tau_h}} u \to u$ in $L^p(\Omega)$ as $h \searrow 0$ for all $1 \leq p < \infty$. Since F fulfills Assumption 5.1, this yields

$$F(u) = \lim_{h \searrow 0} F(\Pi_{P0^{\tau_h}} u).$$

In total, we obtain

$$G_R(u) = \lim_{h \searrow 0} G_R^h(\Pi_{P0^{\tau_h}} u).$$

□

Theorem 5.38. *Let $F : L^2(\Omega) \to \mathbb{R}$ fulfill Assumptions 4.1 and 5.1. Let tuples $\{(h, \tau_h)\}_{h>0}$ be coupled such that Assumption 5.3 is fulfilled. Denote for $h > 0$ by $u_{\tau_h} \in P0^{\tau_h}$ a minimizer of* ($\mathrm{P}_{\mathrm{R}}^{\mathrm{h}}$)*. The sequence $\{u_{\tau_h}\}_{h>0}$ admits a weakly converging subsequence in $L^2(\Omega)$ and each weak accumulation point is optimal for* (P_{R})*. If F is strictly convex, then the whole sequence $\{u_{\tau_h}\}_{h>0}$ converges weakly to the unique minimizer of* (P_{R})*.*

Proof. The sequence $\{u_{\tau_h}\}_{h>0}$ is bounded due to $\underline{\nu} \leq u_{\tau_h}(x) \leq \overline{\nu}$ for almost all $x \in \Omega$ and all $h > 0$. This yields the existence of a subsequence of $\{u_{\tau_h}\}_{h>0}$ that converges weakly in $L^2(\Omega)$.

Now let $u_{\tau_h} \rightharpoonup u$ in $L^2(\Omega)$ as $h \searrow 0$ and $\bar{u}$ be an optimal solution to (P_{R}), which exists due to Theorem 4.2. By Theorems 5.36 and 5.37, u is feasible for (P_{R}) and

there holds

$$
\begin{aligned}
F(\bar u) + \alpha \mathrm{TV}(\bar u) &\le F(u) + \alpha \mathrm{TV}(u) \\
&\le \liminf_{h \searrow 0} F(u_{\tau_h}) + \alpha\, \mathrm{TV}^{\mathrm{h}}(u_{\tau_h}) \\
&\le \limsup_{h \searrow 0} F(u_{\tau_h}) + \alpha\, \mathrm{TV}^{\mathrm{h}}(u_{\tau_h}) \\
&\le \limsup_{h \searrow 0} F(\Pi_{P0^{\tau_h}} \bar u) + \alpha\, \mathrm{TV}^{\mathrm{h}}(\Pi_{P0^{\tau_h}} \bar u) \\
&\le F(\bar u) + \alpha \mathrm{TV}(\bar u).
\end{aligned}
$$

Hence, u is optimal for $(\mathrm{P_R})$ with

$$
F(u) + \alpha \mathrm{TV}(u) = \lim_{h \searrow 0} F(u_{\tau_h}) + \alpha\, \mathrm{TV}^{\mathrm{h}}(u_{\tau_h}).
$$

If F is strictly convex, then the minimizer of $(\mathrm{P_R})$ is unique by Theorem 4.2 such that each subsequence of $\{u_{\tau_h}\}_{h>0}$ has a subsequence that converges weakly in $L^2(\Omega)$ to the unique minimizer. The Urysohn subsequence principle then yields that the whole sequence $\{u_{\tau_h}\}_{h>0}$ converges weakly in $L^2(\Omega)$ to the unique minimizer of $(\mathrm{P_R})$. $\square$

We discretize the regularized problems $(\mathrm{P}_{\delta,\varepsilon})$ with fixed $\delta, \varepsilon > 0$ for mesh sizes (h, τ_h) that fulfill Assumption 5.3 analogously with the following discretized problem that reads

$$
(\mathrm{P}^h_{\delta,\varepsilon}) \qquad \begin{aligned} \min_{u \in P0^{\tau_h}} \quad & F(u) + \frac{\delta}{2}\|u\|^2_{L^2(\Omega)} + \alpha \mathrm{TV}^h_\varepsilon(u) \\ \text{s.t.} \quad & \underline{\nu} \le u(x) \le \overline{\nu} \quad \text{for a.a. } x \in \Omega. \end{aligned}
$$

We define the discretized and regularized total variation $\mathrm{TV}^h_\varepsilon : L^1(\Omega) \to \mathbb{R}$ by

$$
\mathrm{TV}^h_\varepsilon(u) \coloneqq \sup\left\{ -\frac{\varepsilon}{2} a[\phi, \phi] + \int_\Omega u(x) \operatorname{div} \phi(x) \,\mathrm{d}x : \phi \in RT0^h_0, \|\phi\|_{L^\infty(\Omega;\mathbb{R}^d)} \le 1 \right\}
$$

with a bounded and coercive bilinear form $a : H_0(\mathrm{div};\Omega) \times H_0(\mathrm{div};\Omega) \to \mathbb{R}$ as in $(\mathrm{TV}_\varepsilon)$. For the regularized total variation TV_ε, we assume $H = H_0(\mathrm{div};\Omega)$ such that $RT0^h_0 \subset H$. Moreover, we choose the same bounded and coercive bilinear form $a : H_0(\mathrm{div};\Omega) \times H_0(\mathrm{div};\Omega) \to \mathbb{R}$ for TV_ε as for $\mathrm{TV}^h_\varepsilon$. Again, the existence of a minimizer for $(\mathrm{P}^h_{\delta,\varepsilon})$ follows immediately because the feasible set is finite-dimensional. Under these assumptions, the following lemma similar to Lemma 5.8 and Theorem 5.9 holds.

Lemma 5.39. *Let $\{u_h\}_{h>0} \subset L^2(\Omega)$ with $u_h \rightharpoonup u$ in $L^2(\Omega)$ as $h \searrow 0$. Then*

$$
\mathrm{TV}_\varepsilon(u) \le \liminf_{h \searrow 0} \mathrm{TV}^h_\varepsilon(u_h).
$$

Proof. The proof closely follows the proofs of Lemma 5.8 and Theorem 5.9. By Theorem 1 in [62], we can approximate $\phi \in H(\mathrm{div};\Omega)$ by a sequence $\{\phi_k\}_{k\in\mathbb{N}} \subset C_c^\infty(\Omega;\mathbb{R}^d)$ with $\|\phi_k\|_{L^\infty(\Omega;\mathbb{R}^d)} \leq 1$ for all $k \in \mathbb{N}$ with respect to the norm $\|\cdot\|_{H(\mathrm{div};\Omega)}$, that is, $\|\phi - \phi_k\|_{H(\mathrm{div};\Omega)} \to 0$, such that

$$\begin{aligned}
&\mathrm{TV}_\varepsilon(u) = \\
&\qquad \sup\left\{-\frac{\varepsilon}{2}a[\psi,\psi] + \int_\Omega u(x)\,\mathrm{div}\,\psi(x)\,\mathrm{d}x : \psi \in C_c^\infty(\Omega;\mathbb{R}^d), \|\psi\|_{L^\infty(\Omega;\mathbb{R}^d)} \leq 1\right\}.
\end{aligned}$$

Let $\psi \in C_c^\infty(\Omega;\mathbb{R}^d)$ satisfy $\|\psi\|_{L^\infty(\Omega;\mathbb{R}^d)} \leq 1$ and let $\rho \in (0,1)$ arbitrary but fixed. We approximate ψ with $\hat{\psi} \coloneqq (1-\rho)\psi \in C_c^\infty(\Omega;\mathbb{R}^d)$ such that

$$\|\hat{\psi}\|_{L^\infty(\Omega;\mathbb{R}^d)} \leq (1-\rho)\|\psi\|_{L^\infty(\Omega;\mathbb{R}^d)} \leq 1-\rho$$

and

$$\begin{aligned}
&\int_\Omega \mathrm{div}\,\psi(x)u(x)\,\mathrm{d}x - \frac{\varepsilon}{2}a[\psi,\psi] \\
&= \int_\Omega \mathrm{div}\,\hat{\psi}(x)u(x)\,\mathrm{d}x - \frac{\varepsilon}{2}a[\hat{\psi},\hat{\psi}] + \rho\int_\Omega \mathrm{div}\,\psi(x)u(x)\,\mathrm{d}x \\
&\quad - \frac{\varepsilon}{2}\rho\left(a[\hat{\psi},\psi] + a[\psi,\hat{\psi}] + \rho a[\psi,\psi]\right) \\
&= \int_\Omega \mathrm{div}\,\hat{\psi}(x)u(x)\,\mathrm{d}x - \frac{\varepsilon}{2}a[\hat{\psi},\hat{\psi}] + \rho\int_\Omega \mathrm{div}\,\psi(x)u(x)\,\mathrm{d}x - \frac{\varepsilon}{2}\rho(2-\rho)a[\psi,\psi] \\
&\leq \int_\Omega \mathrm{div}\,\hat{\psi}(x)u(x)\,\mathrm{d}x - \frac{\varepsilon}{2}a[\hat{\psi},\hat{\psi}] + \rho\underbrace{\|\mathrm{div}\,\psi\|_{L^\infty(\Omega)}\|u\|_{L^1(\Omega)}}_{c_1\coloneqq}.
\end{aligned}$$

By virtue of Proposition 2.26, we obtain that $\|I_{RT0^h}\hat{\psi}\|_{L^\infty(\Omega;\mathbb{R}^d)} \leq 1$ holds for all $0 < h \leq \frac{\rho}{C(\hat{\psi})}$. Moreover, since $\hat{\psi}$ has compact support in Ω, there holds $\hat{\psi}\cdot n \equiv 0$ on $\partial\Omega$, where n denotes the outer unit normal of Ω, so that $I_{RT0^h}\hat{\psi} \in RT0_0^h$. Moreover, there holds for all $0 < h \leq \frac{\rho}{C(\hat{\psi})}$ that

$$\begin{aligned}
&\int_\Omega \mathrm{div}\,\hat{\psi}(x)u_h(x)\,\mathrm{d}x - \frac{\varepsilon}{2}a[\hat{\psi},\hat{\psi}] \\
&\quad = \int_\Omega \mathrm{div}\,I_{RT0^h}\hat{\psi}(x)u_h(x)\,\mathrm{d}x - \frac{\varepsilon}{2}a[I_{RT0^h}\hat{\psi}, I_{RT0^h}\hat{\psi}] \\
&\qquad + \int_\Omega \mathrm{div}(\hat{\psi} - I_{RT0^h}\hat{\psi})(x)u_h(x)\,\mathrm{d}x + \frac{\varepsilon}{2}a[I_{RT0^h}\hat{\psi}, I_{RT0^h}\hat{\psi}] - \frac{\varepsilon}{2}a[\hat{\psi},\hat{\psi}] \\
&\quad \leq \mathrm{TV}_\varepsilon^h(u_h) + \|\mathrm{div}(\hat{\phi} - I_{RT0^h}\hat{\phi})\|_{L^2(\Omega)}\|u_h\|_{L^2(\Omega)} \\
&\qquad + \frac{\varepsilon}{2}a[I_{RT0^h}\hat{\psi}, I_{RT0^h}\hat{\psi}] - \frac{\varepsilon}{2}a[\hat{\psi},\hat{\psi}]
\end{aligned}$$

Since $\|u_h\|_{L^2(\Omega)}$ is bounded, $\|\hat{\psi} - I_{RT0^h}\hat{\psi}\|_{H(\mathrm{div};\Omega)} \to 0$ by (2.14) and Lemma 2.23, and $H(\mathrm{div};\Omega) \ni \phi \mapsto a[\phi,\phi] \in \mathbb{R}$ is continuous, this yields

$$\begin{aligned}\int_\Omega \mathrm{div}\,\hat{\psi}(x)u(x)\,\mathrm{d}x - \frac{\varepsilon}{2}a[\hat{\psi},\hat{\psi}] &= \lim_{h\searrow 0}\int_\Omega \mathrm{div}\,\hat{\psi}(x)u_h(x)\,\mathrm{d}x - \frac{\varepsilon}{2}a[\hat{\psi},\hat{\psi}] \\ &\leq \liminf_{h\searrow 0}\mathrm{TV}_\varepsilon^h(u_h).\end{aligned}$$

Together, we obtain

$$\int_\Omega \mathrm{div}\,\psi(x)u(x)\,\mathrm{d}x - \frac{\varepsilon}{2}a[\psi,\psi] \leq \liminf_{h\searrow 0}\mathrm{TV}_\varepsilon^h(u_h) + \rho c_1.$$

Since $\rho \in (0,1)$ was arbitrary, this gives

$$\int_\Omega \mathrm{div}\,\psi(x)u(x)\,\mathrm{d}x - \frac{\varepsilon}{2}a[\psi,\psi] \leq \liminf_{h\searrow 0}\mathrm{TV}_\varepsilon^h(u_h).$$

Supremizing over all $\psi \in C_c^\infty(\Omega;\mathbb{R}^d)$ with $\|\psi\|_{L^\infty(\Omega;\mathbb{R}^d)} \leq 1$ then yields

$$\mathrm{TV}_\varepsilon(u) \leq \liminf_{h\searrow 0}\mathrm{TV}_\varepsilon^h(u_h).$$

□

Moreover, the following lemma similar to Lemma 5.34 holds.

Lemma 5.40. *Let tuples $\{(h,\tau_h)\}_{h>0}$ be given such that Assumption 5.3 is fulfilled. Let $u \in L^\infty(\Omega)$ be given. Then there holds*

$$\mathrm{TV}_\varepsilon^h(\Pi_{P0^{\tau_h}}u) \to \mathrm{TV}_\varepsilon(u) \quad \text{as} \quad h \searrow 0.$$

Proof. The proof is analogous to the proof of Lemma 5.34. By Assumption 5.3, there holds $\mathrm{TV}_\varepsilon^h(\Pi_{P0^{\tau_h}}u) = \mathrm{TV}_\varepsilon^h(\Pi_{P0^h}u) = \mathrm{TV}_\varepsilon^h(u)$ because the divergence $\mathrm{div}\,\phi$ of a lowest-order Raviart–Thomas function $\phi \in RT0^h$ is constant on the mesh cells $\mathcal{Q}_h$. Because $RT0_0^h \subset H_0(\mathrm{div};\Omega)$, there holds $\mathrm{TV}_\varepsilon^h(u) \leq \mathrm{TV}_\varepsilon(u)$. Since $h \searrow 0$ also implies $\tau_h \searrow 0$ due to Assumption 5.3, there holds $\Pi_{P0^{\tau_h}}u \to u$ in $L^1(\Omega)$ as $h \searrow 0$ by Lemma 5.7 and therefore $\Pi_{P0^{\tau_h}}u \to u$ in $L^2(\Omega)$ as $h \searrow 0$ by Lemma 2.19 due to $u \in L^\infty(\Omega)$. We then obtain by Lemma 5.39 that

$$\begin{aligned}\mathrm{TV}_\varepsilon(u) &\leq \liminf_{h\searrow 0}\mathrm{TV}_\varepsilon^h(\Pi_{P0^{\tau_h}}u) \leq \limsup_{h\searrow 0}\mathrm{TV}_\varepsilon^h(\Pi_{P0^{\tau_h}}u) \\ &\leq \limsup_{h\searrow 0}\mathrm{TV}_\varepsilon^h(u) \leq \mathrm{TV}_\varepsilon(u),\end{aligned}$$

which yields $\mathrm{TV}_\varepsilon(u) = \lim_{h\searrow 0}\mathrm{TV}_\varepsilon^h(\Pi_{P0^{\tau_h}}u)$. □

We define the two functions $G_\varepsilon : L^2(\Omega) \to \mathbb{R}$ and $G_\varepsilon^h : L^2(\Omega) \to \mathbb{R}$ by

$$G_\varepsilon(u) := F(u) + \frac{\delta}{2}\|u\|_{L^2(\Omega)}^2 + \alpha \mathrm{TV}_\varepsilon(u) + \mathcal{I}_{Z_R}(u)$$

and

$$G_\varepsilon^h(u) := F(u) + \frac{\delta}{2}\|u\|_{L^2(\Omega)}^2 + \alpha \mathrm{TV}_\varepsilon^h(u) + \mathcal{I}_{Z_R^h}(u)$$

for $u \in L^2(\Omega)$ and where $\mathcal{I}_{Z_R}$ and $\mathcal{I}_{Z_R^h}$ are defined as above. We obtain the same Γ-convergence results as for ($\mathrm{P_R}$).

Theorem 5.41. *Let $F : L^2(\Omega) \to \mathbb{R}$ fulfill Assumption 4.1. Let G_ε and G_ε^h be defined as above. Consider tuples $\{(h, \tau_h)\}_{h>0}$ such that Assumption 5.3 is fulfilled. Let $\{u_{\tau_h}\}_{h>0} \subset L^2(\Omega)$ be a sequence with $u_{\tau_h} \rightharpoonup u$ in $L^2(\Omega)$ as $h \searrow 0$. Then there holds*

$$G_\varepsilon(u) \le \liminf_{h\searrow 0} G_\varepsilon^h(u_{\tau_h}).$$

Proof. Without loss of generality, we may assume that $u_{\tau_h} \in Z_R^h$ for all $h > 0$ because $u_{\tau_h} \notin Z_R^h$ implies the trivial case $G_\varepsilon^h(u_{\tau_h}) = \infty$. That is, in particular there holds $u_{\tau_h} \in P0^{\tau_h}$ and $\underline{\nu} \le u_{\tau_h}(x) \le \overline{\nu}$ for almost all $x \in \Omega$ for all $h > 0$. Since the set Z_R is convex and closed in $L^2(\Omega)$ and therefore sequentially weakly closed, there holds $\underline{\nu} \le u(x) \le \overline{\nu}$ for almost all $x \in \Omega$ which gives $\mathcal{I}_{Z_R}(u) = 0$. Together with Lemma 5.39, there holds

$$\alpha \mathrm{TV}_\varepsilon(u) \le \liminf_{h\searrow 0} \alpha \mathrm{TV}_\varepsilon^h(u_{\tau_h}).$$

Because F fulfills Assumption 4.1, there holds $F(u) \le \liminf_{h\searrow 0} F(u_{\tau_h})$. Moreover, there holds $\frac{\delta}{2}\|u\|_{L^2(\Omega)}^2 \le \liminf_{h\searrow 0} \frac{\delta}{2}\|u_{\tau_h}\|_{L^2(\Omega)}^2$. In total, we obtain

$$G_\varepsilon(u) \le \liminf_{h\searrow 0} G_\varepsilon^h(u_{\tau_h}).$$

□

Theorem 5.42. *Let $F : L^2(\Omega) \to \mathbb{R}$ fulfill Assumption 5.1 and tuples $\{(h, \tau_h)\}_{h>0}$ be given such that Assumption 5.3 is fulfilled and $h \searrow 0$. Let G_ε and G_ε^h be defined as above. Let $u \in L^2(\Omega)$ be given. Then there holds*

$$G_\varepsilon(u) \ge \limsup_{h\searrow 0} G_\varepsilon^h(\Pi_{P0^{\tau_h}} u).$$

Proof. If $u \notin Z_R$, then $G_\varepsilon(u) = \infty$, which implies the trivial case. Therefore, we may assume $u \in Z_R$, that is, $\underline{\nu} \le u(x) \le \overline{\nu}$ for almost all $x \in \Omega$. This yields that also

$\underline{\nu} \leq \Pi_{P0^{\tau_h}} u(x) \leq \overline{\nu}$ for almost all $x \in \Omega$ for all $h > 0$ such that $\mathcal{I}_{Z_R^h}(\Pi_{P0^{\tau_h}} u) = 0$ for all $h > 0$. By Lemma 5.40, it follows that

$$\alpha \mathrm{TV}_\varepsilon(u) = \lim_{h \searrow 0} \alpha \mathrm{TV}_\varepsilon^h(\Pi_{P0^{\tau_h}} u).$$

As in the proof of Lemma 5.40, there holds $\Pi_{P0^{\tau_h}} u \to u$ in $L^1(\Omega)$ as $h \searrow 0$ by Assumption 5.3 and Lemma 5.7. Hence, $\Pi_{P0^{\tau_h}} u \to u$ in $L^p(\Omega)$ as $h \searrow 0$ for all $1 \leq p < \infty$ by Lemma 2.19. Since F fulfills Assumption 5.1, this yields

$$F(u) = \lim_{h \searrow 0} F(\Pi_{P0^{\tau_h}} u).$$

Moreover, there holds

$$\frac{\delta}{2} \|u\|_{L^2(\Omega)}^2 = \lim_{h \searrow 0} \frac{\delta}{2} \|\Pi_{P0^{\tau_h}} u\|_{L^2(\Omega)}^2.$$

In total, we obtain

$$G_\varepsilon(u) = \lim_{h \searrow 0} G_\varepsilon^h(\Pi_{P0^{\tau_h}} u).$$

□

The following theorem is analogous to Theorem 5.38. Due to the Tikhonov term, we obtain strong convergence in $L^2(\Omega)$ instead of just weak convergence.

Theorem 5.43. *Let $F : L^2(\Omega) \to \mathbb{R}$ fulfill Assumptions 4.1 and 5.1. Let tuples $\{(h, \tau_h)\}_{h>0}$ be coupled such that Assumption 5.3 is fulfilled. Denote for fixed $\varepsilon, \delta > 0$ by $u_{\tau_h} \in P0^{\tau_h}$ a minimizer of $(\mathrm{P}_{\delta,\varepsilon}^h)$ for $h > 0$. The sequence $\{u_{\tau_h}\}_{h>0}$ admits a converging subsequence in $L^2(\Omega)$ and each accumulation point is optimal for $(\mathrm{P}_{\delta,\varepsilon})$. If F is convex, then the whole sequence $\{u_{\tau_h}\}_{h>0}$ converges in $L^2(\Omega)$ to the unique minimizer of $(\mathrm{P}_{\delta,\varepsilon})$.*

Proof. The sequence $\{u_{\tau_h}\}_{h>0}$ is bounded due to $\underline{\nu} \leq u_{\tau_h}(x) \leq \overline{\nu}$ for almost all $x \in \Omega$ and all $h > 0$. This yields the existence of a subsequence of $\{u_{\tau_h}\}_{h>0}$ that converges weakly in $L^2(\Omega)$.

Now let $u_{\tau_h} \rightharpoonup u$ in $L^2(\Omega)$ as $h \searrow 0$ and $\bar{u}$ be an optimal solution to $(\mathrm{P}_{\delta,\varepsilon})$, which exists due to Theorem 4.12. By Theorems 5.41 and 5.42, u is feasible for $(\mathrm{P}_{\delta,\varepsilon})$ and

there holds

$$
\begin{aligned}
F(\bar{u}) &+ \frac{\delta}{2}\|\bar{u}\|^2_{L^2(\Omega)} + \alpha \mathrm{TV}_\varepsilon(\bar{u}) \\
&\leq F(u) + \frac{\delta}{2}\|u\|^2_{L^2(\Omega)} + \alpha \mathrm{TV}_\varepsilon(u) \\
&\leq \liminf_{h \searrow 0} F(u_{\tau_h}) + \frac{\delta}{2}\|u_{\tau_h}\|^2_{L^2(\Omega)} + \alpha \mathrm{TV}^h_\varepsilon(u_{\tau_h}) \\
&\leq \limsup_{h \searrow 0} F(u_{\tau_h}) + \frac{\delta}{2}\|u_{\tau_h}\|^2_{L^2(\Omega)} + \alpha \mathrm{TV}^h_\varepsilon(u_{\tau_h}) \\
&\leq \limsup_{h \searrow 0} F(\Pi_{P0^{\tau_h}}\bar{u}) + \frac{\delta}{2}\|\Pi_{P0^{\tau_h}}\bar{u}\|^2_{L^2(\Omega)} + \alpha \mathrm{TV}^h_\varepsilon(\Pi_{P0^{\tau_h}}\bar{u}) \\
&\leq F(\bar{u}) + \frac{\delta}{2}\|\bar{u}\|^2_{L^2(\Omega)} + \alpha \mathrm{TV}_\varepsilon(\bar{u}).
\end{aligned}
$$

Since F is weakly lower semicontinuous by Assumption 4.1 and $\mathrm{TV}_\varepsilon(u) \leq \liminf_{h \searrow 0} \mathrm{TV}^h_\varepsilon(u_{\tau_h})$, there holds

$$
-F(u) - \alpha \mathrm{TV}_\varepsilon(u) \geq \limsup_{h \searrow 0} -F(u_{\tau_h}) - \alpha \mathrm{TV}^h_\varepsilon(u_{\tau_h}).
$$

Since $\frac{\delta}{2}\|\cdot\|^2_{L^2(\Omega)}$ is weakly lower semicontinuous, it follows

$$
\begin{aligned}
\frac{\delta}{2}\|u\|^2_{L^2(\Omega)} &\leq \liminf_{h \searrow 0} \frac{\delta}{2}\|u_{\tau_h}\|^2_{L^2(\Omega)} \\
&\leq \limsup_{h \searrow 0} \frac{\delta}{2}\|u_{\tau_h}\|^2_{L^2(\Omega)} \\
&= \limsup_{h \searrow 0} \left(\frac{\delta}{2}\|u_{\tau_h}\|^2_{L^2(\Omega)} + F(u_{\tau_h}) + \alpha \mathrm{TV}^h_\varepsilon(u_{\tau_h}) - F(u_{\tau_h}) - \alpha \mathrm{TV}^h_\varepsilon(u_{\tau_h}) \right) \\
&\leq \limsup_{h \searrow 0} \left(\frac{\delta}{2}\|u_{\tau_h}\|^2_{L^2(\Omega)} + F(u_{\tau_h}) + \alpha \mathrm{TV}^h_\varepsilon(u_{\tau_h}) \right) \\
&\quad + \limsup_{h \searrow 0} \left(-F(u_{\tau_h}) - \alpha \mathrm{TV}^h_\varepsilon(u_{\tau_h}) \right) \\
&\leq F(u) + \frac{\delta}{2}\|u\|^2_{L^2(\Omega)} + \alpha \mathrm{TV}_\varepsilon(u) - F(u) - \alpha \mathrm{TV}_\varepsilon(u) \\
&= \frac{\delta}{2}\|u\|^2_{L^2(\Omega)},
\end{aligned}
$$

that is, $\lim_{h \searrow 0} \|u_{\tau_h}\|_{L^2(\Omega)} = \|u\|_{L^2(\Omega)}$. Together with the weak convergence $u_{\tau_h} \rightharpoonup u$ in $L^2(\Omega)$ as $h \searrow 0$, it follows that $u_{\tau_h} \to u$ in $L^2(\Omega)$ as $h \searrow 0$.

If F is convex, then the minimizer of $(\mathrm{P}_{\delta,\varepsilon})$ is unique by Theorem 4.12 such that each subsequence of $\{u_{\tau_h}\}_{h>0}$ has a subsequence that converges in $L^2(\Omega)$ to the unique minimizer. The Urysohn subsequence principle then yields that the whole sequence $\{u_{\tau_h}\}_{h>0}$ converges in $L^2(\Omega)$ to the unique minimizer of $(\mathrm{P}_{\delta,\varepsilon})$. □

Chapter 6

Numerical examples

In this chapter, we provide numerical examples that combine the findings from this thesis. In the course of this, we will check the algorithms for practicability, examine the approximation properties of the discretized total variation TV^{h} and the corresponding discretized optimization problems (P_c^h) in practice, compare the discretization (P_c^h) to an anisotropic discretization of (P), and evaluate the lower bounds obtained by the relaxation (P_{R}).

Sections 6.1 and 6.2 are based on the numerical experiments from [95] and closely follow the observations made therein.

In Section 6.1, we compare the approximation behavior of the discretized total variation TV^{h} with and without coupled meshes for the input function using Example 5.10 and compute the errors with respect to the total variation of the limit function.

In Section 6.2, we examine the influence of the constant c on the discretization (P_c^h) from Chapter 5. To this end, we will consider an instance from imaging with an L^1 fidelity term and use outer-approximation Algorithm 3 to solve the corresponding discretizations (P_c^h) with different choices for the constant c.

In Section 6.3, we take up the instance from Section 4.4.2, which was designed so that we know its exact optimal solution in function space. We discretize the instance in two different ways. The first one is the discretization (P_c^h) that was introduced in Chapter 5 in which both the input function and the total variation term are discretized. In the second discretization, we only discretize the input function and keep the total variation term TV. We apply trust-region Algorithm 1 to both variants of discretized problems and compare the results. To solve the trust-region subproblems that arise from the discretization (P_c^h), we use outer-approximation Algorithm 3 which was introduced in Section 5.3. The trust-region subproblems that arise from the discretization with the usual total variation term TV can be reformulated as integer linear programs and solved by standard off-the-shelf solvers. We also compute the errors to the known optimal solution in function space. Moreover, we consider the relaxation ($\mathrm{P_R}$) as introduced in Chapter 4, discretize it following Section 5.4, and solve it with Algorithm 2 to compute lower bounds for (P_c^h).

In Section 6.4, we examine an instance from imaging that contains an L^2 fidelity term and a PDE constraint. Here we also apply the discretization (P_c^h) and the anisotropic discretization that only discretizes the input function and keeps the total variation term TV. Again, we solve the discretized problems with trust-region Algorithm 1. We use outer-approximation Algorithm 3 to solve the trust-region subproblems that arise from (P_c^h) and reformulate the trust-region subproblems that arise from the discretization with the usual total variation term TV as integer linear programs to solve them with a standard off-the-shelf solver. Subsequently, we compare the results that arise from the differing discretizations.

All numerical experiments were run on a single node of the Linux HPC605 cluster LiDO3 with two AMD EPYC 7542 32-Core CPUs and 64 GB RAM (computations were restricted to one CPU). We have used DOLFINx 0.7.2 [7] for the finite-element discretization and Gurobi 10.0.3 [59] to solve the occurring optimization problems.

6.1 Approximation of TV with $\mathrm{TV^h}$

The first numerical experiment is concerned with the approximation of TV with $\mathrm{TV^h}$. Example 5.10 demonstrated that there are functions $u \in \mathrm{BV_U}(\Omega)$ such that $\mathrm{TV}(u)$ cannot be approximated with $\mathrm{TV}^\tau(R^U_{P0^\tau}(u))$. We solved this issue by discretizing u on a finer mesh than the mesh for the total variation and proved in Theorems 5.9 and 5.14 that we can approximate $\mathrm{TV}(u)$ with $\mathrm{TV^h}(R^U_{P0^\tau}(u))$ when the meshes $\mathcal{Q}_h$ and $\mathcal{Q}_\tau$ are coupled such that $\frac{\tau}{h} \searrow 0$ as $h \searrow 0$ and Assumption 5.3 is fulfilled.

6.1.1 Experiment description

We take up Example 5.10 and consider the function $u = \chi_{\{(\frac{1}{3},-1)^T x \geq 0\}} \in \mathrm{BV_U}(\Omega)$ with $\Omega = (0,1)^2$ and $U = \{0,1\}$. We discretize Ω into meshes $\mathcal{Q}_h$ and $\mathcal{Q}_\tau$ of axis-aligned squares for decreasing h and τ fulfilling Assumption 5.3 and compute the values $\mathrm{TV}^\tau(u_\tau)$ and $\mathrm{TV^h}(u_\tau)$ with $u_\tau \coloneqq R^U_{P0^\tau}(u)$.

Practical implementation

For each tuple (h,τ), we first compute $u_\tau = R^U_{P0^\tau}(u)$. We formulate the optimization problem for the computation of $\mathrm{TV^h}(u_\tau)$ as a maximization problem with linear objective function and quadratic constraints. For the linear formulation of the objective function, we use that $\mathrm{TV^h}(u_\tau) = \mathrm{TV^h}(\Pi_{P0^h} u_\tau)$ by Assumption 5.3 and that the divergence $\operatorname{div}\phi$ of a lowest-order Raviart–Thomas function $\phi \in RT0^h$ is constant on the mesh cells $\mathcal{Q}_h$. For the formulation of the constraint $\|\phi\|_{L^\infty(\Omega;\mathbb{R}^d)} \leq 1$ for $\phi \in RT0^h_0$ as quadratic constraints, we use that lowest-order Raviart–Thomas

Table 6.1: Values of h^{-1}, τ^{-1}, $\mathrm{TV}^{\tau}(u_\tau)$, $\mathrm{TV}^{\mathrm{h}}(u_\tau)$, $\mathrm{TV}(u)$, and $\mathrm{TV}(u_\tau)$ with $u_\tau = R^{U}_{P0^\tau}(u)$.

h^{-1}	τ^{-1}	$\mathrm{TV}^{\tau}(u_\tau)$	$\mathrm{TV}^{\mathrm{h}}(u_\tau)$	$\mathrm{TV}(u_\tau)$	$\mathrm{TV}(u)$	$\frac{\mathrm{TV}(u_\tau)}{\mathrm{TV}^{\mathrm{h}}(u_\tau)}$	$(\mathrm{TV}(u) - \mathrm{TV}^{\mathrm{h}}(u_\tau))\frac{h}{\tau}$
2	$2 \cdot 9$	0.97748	0.36456	1.22222	1.05409	3.35259	8.513×10^{-3}
4	$4 \cdot 10$	1.03118	0.67116	1.27500	1.05409	1.89970	2.292×10^{-3}
8	$8 \cdot 11$	1.05709	0.86291	1.30682	1.05409	1.51443	2.727×10^{-4}
16	$16 \cdot 12$	1.0692	0.95678	1.32292	1.05409	1.38268	1.259×10^{-3}
32	$32 \cdot 13$	1.07393	1.00593	1.32692	1.05409	1.31910	1.526×10^{-3}
64	$64 \cdot 14$	1.07648	1.02999	1.33036	1.05409	1.29162	1.599×10^{-3}
128	$128 \cdot 15$	1.07774	1.04208	1.33229	1.05409	1.27849	1.577×10^{-3}
256	$256 \cdot 16$	1.07822	1.04805	1.33276	1.05409	1.27166	1.508×10^{-3}

functions are linear on the mesh cells $\mathcal{Q}_h$ and the convexity of the Euclidean norm such that

$$\|\phi\|_{L^\infty(Q;\mathbb{R}^d)} = \sup_{x \in Q} \|\phi(x)\|_2 = \max_{i \in \{1,\ldots,4\}} \|\phi(q_i)\|_2,$$

where q_i, $i = 1, \ldots, 4$, denote the four vertices of a square $Q \in \mathcal{Q}_h$, such that

$$\|\phi\|_{L^\infty(Q;\mathbb{R}^d)} \leq 1$$

is equivalent to

$$\|\phi(q_i)\|_2^2 \leq 1 \quad \forall\, i \in \{1, \ldots, 4\}.$$

We solve the quadratic problems for the computation of $\mathrm{TV}^{\mathrm{h}}(u_\tau)$ and $\mathrm{TV}^{\tau}(u_\tau)$ with Gurobi [59].

6.1.2 Results

We obtained the following results from the experiments described above.

Table and figure description

In Table 6.1, we list all considered mesh sizes (h, τ), the computed values $\mathrm{TV}^{\tau}(u_\tau)$, $\mathrm{TV}^{\mathrm{h}}(u_\tau)$, and $\mathrm{TV}(u_\tau)$, the total variation $\mathrm{TV}(u)$ of the limit function u, the ratio $\frac{\mathrm{TV}(u_\tau)}{\mathrm{TV}^{\mathrm{h}}(u_\tau)}$, and the rate $\left(\mathrm{TV}(u) - \mathrm{TV}^{\mathrm{h}}(u_\tau)\right)\frac{h}{\tau}$. We plotted the errors $|\mathrm{TV}(u) - \mathrm{TV}^{\tau}(u_\tau)|$ and $|\mathrm{TV}(u) - \mathrm{TV}^{\mathrm{h}}(u_\tau)|$ for the tuples (h, τ) from Table 6.1 in Figure 6.1.

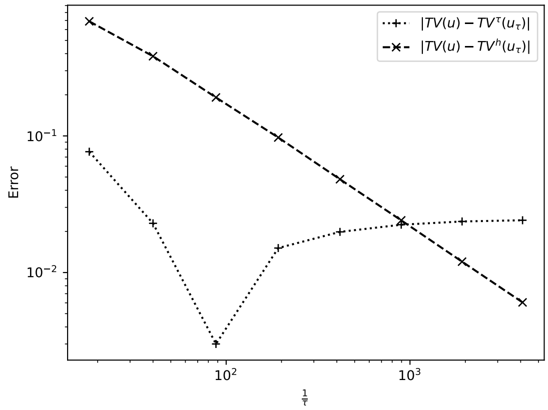


Figure 6.1: Errors $|\mathrm{TV}(u) - \mathrm{TV}^{\tau}(u_{\tau})|$ and $|\mathrm{TV}(u) - \mathrm{TV}^{\mathrm{h}}(u_{\tau})|$ for the tuples (h, τ) listed in Table 6.1 with $u_{\tau} = R^{U}_{P0^{\tau}}(u)$.

Result description

In Table 6.1, we perceive that $\mathrm{TV}^{\tau}(u_{\tau}) \to \frac{1+\sqrt{5}}{3} \approx 1.07869 > \mathrm{TV}(u)$ as $\tau \searrow 0$ and $\mathrm{TV}^{\mathrm{h}}(u_{\tau}) \to \mathrm{TV}(u) = \frac{\sqrt{10}}{3} \approx 1.05409$ as $h \searrow 0$. Moreover, we observe an experimental error $\mathrm{TV}(u) - \mathrm{TV}^{\mathrm{h}}(u_{\tau}) \leq r\,\frac{\tau}{h}$ with $r \approx 8.513 \cdot 10^{-3}$.

In Figure 6.1, we can observe that the error $|\mathrm{TV}(u) - \mathrm{TV}^{\tau}(u_{\tau})|$ is not vanishing while the error $|\mathrm{TV}(u) - \mathrm{TV}^{\mathrm{h}}(u_{\tau})|$ is vanishing for $\frac{\tau}{h} \searrow 0$, $h \searrow 0$.

Interpretation

The results in Table 6.1 and Figure 6.1 confirm our theoretical results as the values $\mathrm{TV}^{\mathrm{h}}(u_{\tau})$ approximate the total variation $\mathrm{TV}(u)$ of the limit function u as $\frac{\tau}{h} \searrow 0$ and $h \searrow 0$. As predicted, the discretization without coupled meshes by $\mathrm{TV}^{\tau}(u_{\tau})$ or without a discretization of the total variation by $\mathrm{TV}(u_{\tau})$ are not appropriate for the discretization of TV because they do not approximate $\mathrm{TV}(u)$.

In accordance with our theoretical error estimates in Propositions 5.13 and 5.15, we can observe in Table 6.1 and Figure 6.1 that an error rate with respect to $\frac{\tau}{h}$ applies to our numerically computed errors $|\mathrm{TV}(u) - \mathrm{TV}^{\mathrm{h}}(u_{\tau})|$.

Another advantage of using two superlinearly coupled meshes for the input function and the total variation besides the approximation is that the computation of $\mathrm{TV}^{\mathrm{h}}(u_{\tau})$ is faster than the computation of $\mathrm{TV}^{\tau}(u_{\tau})$ for the same $\tau > 0$, which is due to the smaller size of the optimization problem that needs to be solved to compute $\mathrm{TV}^{\mathrm{h}}(u_{\tau})$ compared to the size of the optimization problem for the computation of $\mathrm{TV}^{\tau}(u_{\tau})$. For $h^{-1} = 256$ and $\tau^{-1} = 256 \cdot 16$, the runtimes have been 831.16 seconds to compute $\mathrm{TV}^{\mathrm{h}}(u_{\tau})$ and 30933.33 seconds to compute $\mathrm{TV}^{\tau}(u_{\tau})$.

In Example 5.28, we have provided an example in which the constant $c = \sqrt{2}$ is sharp for the case $d = 2$. Nevertheless, there are instances in which the constant c could be chosen within the interval $[1, \sqrt{2})$. For example, if the limit function is a square whose sides lie on the grid lines of the mesh, then even the choice $c = 1$ is valid. In Table 6.1, we can estimate the asymptotically valid constant experimentally by considering the limit $\lim_{\frac{\tau}{h}\searrow 0} \frac{\mathrm{TV}(u_\tau)}{\mathrm{TV}^{\mathrm{h}}(u_\tau)} \approx 1.27$. In general, the choice $c \in [1, \sqrt{2})$ can only be validated if the solution to the limit problem is known while the choice $c \geq \sqrt{2}$ is always valid by Theorem 5.27. Due to this certainty, we believe that $c \geq \sqrt{2}$ is generally the preferred choice in optimization contexts, where the solution to the limit problem is unknown.

6.2 Influence of the constant c in (P_c^h)

In our second numerical experiment, we will examine the influence of the choice of the constant c for the discretization (P_c^h) of (P) on the numerical results. To this end, we consider (P) with the choices $F(u) = \|u - u_d\|_{L^1(\Omega)}$ and $\alpha = 5 \cdot 10^{-3}$, that is, we consider the imaging optimization problem

$$
(\mathrm{P}_I) \qquad \begin{aligned} \min_{u\in L^2(\Omega)} \quad & \|u - u_d\|_{L^1(\Omega)} + 0.005\mathrm{TV}(u) \\ \text{s.t.} \quad & u(x) \in U \subset \mathbb{Z} \text{ for a.a. } x \in \Omega, \end{aligned}
$$

where $u_d \in L^1(\Omega)$ represents a noisy image. The original picture is shown in Figure 6.2a and the noisy version is shown in Figure 6.2b. To obtain u_d, we have added Gaussian noise with standard deviation $\sigma = 5\%$ to the original picture and scaled the gray scale values to the interval $[0, 5]$. We set $U = \{0, \ldots, 5\}$ which represents six evenly distributed gray scale values. In order to keep the focus on the influence of the constant c and to keep the problem computationally manageable, we have measured the distance between u and u_d in the L^1 norm instead of the more common L^2 norm and refer to [33] for a discussion regarding this choice. A similar example with an L^2 tracking term is provided in Section 6.4.

6.2.1 Discretized optimization problems

For the meshes $\mathcal{Q}_h$ and $\mathcal{Q}_\tau$, we assume that Assumption 5.3 holds. The discretization of (P_I) according to (P_c^h) reads

$$
(\mathrm{P}_I^h) \qquad \begin{aligned} \min_{(u,V)\in P0^\tau\times\mathbb{R}} \quad & \|u - u_d\|_{L^1(\Omega)} + 0.005V \\ \text{s.t.} \quad & \mathrm{TV}(u) \leq cV \\ & \mathrm{TV}^{\mathrm{h}}(u) \leq V \\ & u(x) \in U \text{ for a.a. } x \in \Omega. \end{aligned}
$$

6.2.2 Experiment description

As discussed in Chapter 5, the constant c in the discretization (P^h_c) influences the level of chattering that is permitted by the averaging effect of the discretized total variation TV^{h} for fixed meshes. We examine the influence of the choice of the constant c on the numerical results by applying outer-approximation Algorithm 3 to (P^h_I) with the mesh sizes $(h,\tau) = (\frac{1}{32}, \frac{1}{128})$ and $(h,\tau) = (\frac{1}{64}, \frac{1}{512})$ and the constants $c = 3^i\sqrt{2}$ with $i = 0,\dots,4$. We have applied Algorithm 3 with an iteration limit of 25 iterations and a tolerance of 10^{-3} for the gap $\frac{\mathrm{TV}^{\mathrm{h}}(u)-V}{\mathrm{TV}^{\mathrm{h}}(u)}$. We have set the time limit for Gurobi [59] to solve the mixed-integer linear programs (MIP) to 48 hours and the acceptable optimality gap to the default value 10^{-4}.

Practical implementation

We reformulate the optimization problems of the form (MIP) that occur during the application of Algorithm 3 to (P^h_I) as mixed-integer linear programs. To this end, we need to reformulate the L^1 norm in the objective and the constraint $\mathrm{TV}(u) \le cV$ as linear terms. To this end, let us denote $\mathcal{Q}_\tau = \{Q_1,\dots,Q_m\}$ for some $m \in \mathbb{N}$. By Lemma 2.18, the total variation of a function $u \in P0^\tau$ can be rewritten as

$$\mathrm{TV}(u) = \sum_{j=1}^{m-1} \sum_{\ell=j+1}^{m} |u_j - u_\ell| H_{j,\ell},$$

where u_j denotes the the value of u on square Q_j, $j \in \{1,\dots,m\}$, and where we denote $H_{j,\ell} \coloneqq \mathcal{H}^{d-1}(\partial^* Q_j \cap \partial^* Q_\ell)$ for $j,\ell \in \{1,\dots,m\}$. We introduce the variables $y_{j,\ell} \in \mathbb{R}$ that represent the jump heights between the squares Q_j, Q_ℓ for $j,\ell \in \{1,\dots,m\}$. We model this with the following constraints

$$\begin{aligned} u_j - u_\ell &\le y_{j,\ell} \\ u_\ell - u_j &\le y_{j,\ell} \end{aligned}$$

with $j,\ell \in \{1,\dots,m\}$. We replace $\mathrm{TV}(u)$ by

$$\sum_{j=1}^{m-1} \sum_{\ell=j+1}^{m} y_{j,\ell} H_{j,\ell}.$$

Similarly, we reformulate the L^1 norm in the objective by adding the constraints

$$\begin{aligned} u_j - (u_d)_j &\le z_j \\ (u_d)_j - u_j &\le z_j \end{aligned}$$

for $j \in \{1,\dots,m\}$ and where $(u_d)_j$ denotes the value of u_d on the square Q_j, $j \in \{1,\dots,m\}$. We replace $\|u - u_d\|_{L^1(\Omega)}$ in the objective by $\sum_{j=1}^m z_j$. In total, we reformulate (MIP) for (P_I^h) with $k \in \mathbb{N}_0$ as

$$
(\mathrm{MIP}_I)\qquad
\begin{aligned}
\min_{(u,V,z,y)\in\mathbb{R}^m\times\mathbb{R}\times\mathbb{R}^m\times\mathbb{R}^{m\times m}} \quad & \sum_{j=1}^{m} z_j + 0.005V \\
\text{s.t.} \quad & \sum_{j=1}^{m-1}\sum_{\ell=j+1}^{m} y_{j,\ell}H_{j,\ell} \le cV \\
& \sum_{j=1}^{m} d_{i,j}u_j \le V \quad \forall\, i \in \mathbb{N}, i \le k \\
& u_j - u_\ell \le y_{j,\ell} \quad \forall\, j,\ell \in \{1,\dots,m\} \\
& u_\ell - u_j \le y_{j,\ell} \quad \forall\, j,\ell \in \{1,\dots,m\} \\
& u_j - (u_d)_j \le z_j \quad \forall\, j \in \{1,\dots,m\} \\
& (u_d)_j - u_j \le z_j \quad \forall\, j \in \{1,\dots,m\} \\
& u_j \in U \quad \forall\, j \in \{1,\dots,m\},
\end{aligned}
$$

where we may denote $d_{i,j} \coloneqq (\operatorname{div}\phi_i)|_{Q_j} \in \mathbb{R}$ for $i \in \mathbb{N}$, $i \le k$, and $j \in \{1,\dots,m\}$ because the divergence $\operatorname{div}\phi$ of a lowest-order Raviart–Thomas function $\phi \in RT0^h$ is constant on the squares of $\mathcal{Q}_\tau$ due to Assumption 5.3.

Moreover, we have reformulated (QP) as an optimization problem with linear objective and quadratic constraints following Section 6.1.1. Within the application of outer-approximation Algorithm 2, we have solved the reformulated problems (MIP) and (QP) with Gurobi [59].

6.2.3 Results

We obtained the following results from the experiments described above.

Table and figure description

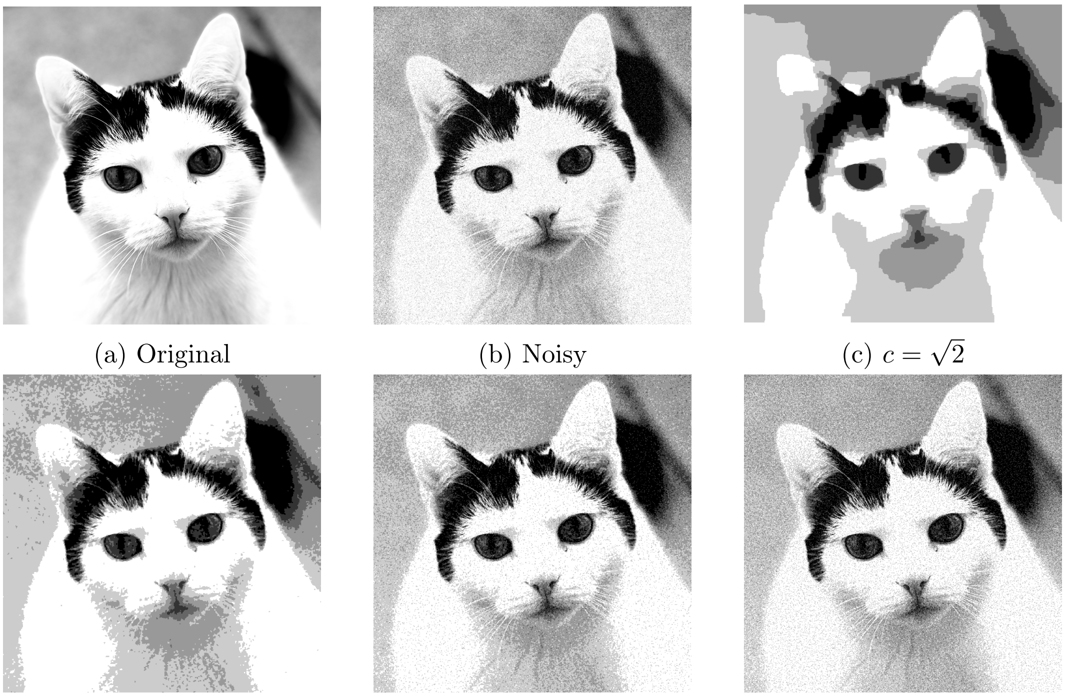
We present the results for the case $(h,\tau) = (\frac{1}{32}, \frac{1}{128})$ in Table 6.2 and for the case $(h,\tau) = (\frac{1}{64}, \frac{1}{512})$ in Table 6.3. If we denote the solution that is returned by Algorithm 3 by (u^*, V^*), then c represents the value of the constant c for the discretization (P_c^h), *Term.* gives the information why the algorithm terminated, *It.* represents the number of iterations until termination, *Obj. val.* represents the objective value $F(u^*) + 0.005V^*$, TV represents $\mathrm{TV}(u^*)$, $\mathrm{TV^h}$ represents $\mathrm{TV^h}(u^*)$, V represents V^*, *Gap* represents $\frac{\mathrm{TV^h}(u^*)-V^*}{\mathrm{TV^h}(u^*)}$, and *Time (s)* represents the running time in seconds. In column 2 of Tables 6.2 and 6.3, the abbreviations indicate the reason for the termination of the algorithm, where *Opt* means that we have found an optimal solution, *Tol* means that $\frac{\mathrm{TV^h}(u)-V}{\mathrm{TV^h}(u)} \le 10^{-3}$, *MaxIter* means that Algorithm 3 reached the iteration maximum, and *GrbTime* means that Gurobi reached the time limit while solving (MIP). The resulting images for the case $(h,\tau) = (\frac{1}{64}, \frac{1}{512})$ for the different

Table 6.2: Results from the application of Algorithm 3 to (P_c^h) resulting from (P_I) with $(h, \tau) = (\frac{1}{32}, \frac{1}{128})$.

c	Term.	It.	Obj. val.	TV	TV$^\mathrm{h}$	V	Gap	Time (s)
$\sqrt{2}$	Opt	1	0.453581	34.546875	19.985456	24.428330	-2.223×10^{-1}	28
$3\sqrt{2}$	Tol	24	0.328303	89.679688	21.143576	21.137705	2.777×10^{-4}	7613
$9\sqrt{2}$	MaxIter	25	0.297379	135.218750	18.054645	17.843617	1.169×10^{-2}	2154
$27\sqrt{2}$	MaxIter	25	0.297393	135.132813	17.967840	17.798740	9.411×10^{-3}	1502
$81\sqrt{2}$	MaxIter	25	0.297377	135.140625	17.973196	17.782075	1.063×10^{-2}	1786

Table 6.3: Results from the application of Algorithm 3 to (P_c^h) resulting from (P_I) with $(h, \tau) = (\frac{1}{64}, \frac{1}{512})$.

c	Term.	It.	Obj. val.	TV	TV$^\mathrm{h}$	V	Gap	Time (s)
$\sqrt{2}$	GrbTime	9	0.498596	29.820312	21.295309	21.086300	9.815×10^{-3}	343893
$3\sqrt{2}$	GrbTime	13	0.437718	97.527344	24.093675	22.987415	4.591×10^{-2}	384429
$9\sqrt{2}$	MaxIter	25	0.343889	301.912109	24.076626	23.721174	1.476×10^{-2}	539392
$27\sqrt{2}$	MaxIter	25	0.307785	464.355469	21.445615	20.954688	2.289×10^{-2}	515671
$81\sqrt{2}$	Tol	5	0.312697	491.566406	22.357736	22.357736	0.000	35436



(a) Original (b) Noisy (c) $c = \sqrt{2}$

(d) $c = 3\sqrt{2}$ (e) $c = 9\sqrt{2}$ (f) $c = 27\sqrt{2}$

Figure 6.2: Original image, noisy image, and resulting images obtained by applying Algorithm 3 to the discretization of (P_I) with $(h, \tau) = (\frac{1}{64}, \frac{1}{512})$ and different values for the constant c.

choices for the constant c as well as the original and the noisy image are shown in Figure 6.2.

Result description

In Figure 6.2, we can observe that the chattering in the resulting images increases as the value of c increases. This can also be observed in Tables 6.2 and 6.3, where the total variation generally increases for increasing c. In Table 6.2 we can moreover see that this effect flatters for $c \geq 9\sqrt{2}$. In our numerical experiments, we can observe that Algorithm 3 was mostly able to close the gap between V and $\mathrm{TV}^{\mathrm{h}}(u)$ within the first few iterations to an accuracy of order 10^{-2} but it was not able to close the gap to the desired accuracy of 10^{-3} within the prescribed iteration limit of 25 iterations. Further experiments suggest that a moderate increase of the iteration limit is not sufficient to achieve an accuracy of 10^{-3} in this example.

Interpretation

We highlight the impact of the choice of the constant c in Figure 6.2 on the results obtained by the application of outer-approximation Algorithm 3 to the discretized problems (P_c^h), even if the specific choice of $c \geq \sqrt{2}$ does not make a difference in the limit from a theoretical point of view. The constant $c = \sqrt{2}$ from Theorem 5.27 in Figure 6.2c and the constant $c = 9\sqrt{2}$ from Theorem 5.26 in Figure 6.2e lead to significantly different results. This underlines the importance of the knowing of the sharp bound for the constant c. Since in general the purpose of the total variation regularization is the avoidance of chattering, we believe that the choice of the smallest possible constant c is favorable.

Numerical difficulties

The solution of mixed-integer linear programs is in general computationally expensive. Since in each iteration of outer-approximation Algorithm 3 the mixed-integer linear program (MIP_I) has to be solved, this leads to very long runtimes of Algorithm 3. In order to reduce the runtime of Algorithm 3, it is sensible to search for efficient algorithms for a faster solution of (MIP_I) and further cutting planes that can be added to (MIP_I) for a faster convergence of outer-approximation Algorithm 3.

6.3 Numerical example with known optimal solution

For our third numerical example, we consider (P) with the choice $F : L^2(\Omega) \to \mathbb{R}$ defined by $F(u) = \frac{1}{2}\|S(u+f) - y_d\|_{L^2(\Omega)}^2$ with $f \in L^2(\Omega)$ and $y_d \in L^2(\Omega)$ as introduced in Section 4.4, where $S : L^2(\Omega) \to H_0^1(\Omega)$ is the solution operator that maps $w \in L^2(\Omega)$ to the unique solution $y \in H_0^1(\Omega)$ to

$$\text{(PDE)} \qquad -\kappa\Delta y + y = w \text{ in } \Omega, \quad y = 0 \text{ on } \partial\Omega$$

with fixed $\kappa > 0$. We highlight that $S : L^2(\Omega) \to H_0^1(\Omega)$ is injective such that $F : L^2(\Omega) \to \mathbb{R}$ is strictly convex. We define $\Omega := (0,2)^2$, $\rho_1 = \frac{1}{32}$, $\rho_2 = \frac{31}{32}$, $y_d \in L^2(\Omega)$, and $f \in L^2(\Omega)$ as in Section 4.4.2 such that $\bar{u} \in \mathrm{BV_U}(\Omega)$ defined by

$$\bar{u}(x) = \begin{cases} 1 & \text{if } x \in B \\ 0 & \text{if } x \in \Omega \setminus B \end{cases}$$

with $B = B_{0.5}((1,1)^T) \subset \Omega$ is the unique optimal solution to

$$\begin{aligned} &\min_{u \in L^2(\Omega)} && \frac{1}{2}\|y - y_d\|^2_{L^2(\Omega)} + \alpha \mathrm{TV}(u) \\ &\text{s.t.} && -\kappa\Delta y + y = u + f \text{ in } \Omega, \quad y = 0 \text{ on } \partial\Omega \\ & && u(x) \in U = \{0,1\} \text{ for a.a. } x \in \Omega \end{aligned} \tag{6.1}$$

with $\alpha > 0$.

Since $F : L^2(\Omega) \to \mathbb{R}$ is continuous, convex, and bounded from below by 0, Assumptions 1.1, 4.1 and 5.1 are fulfilled. We use the same arguments as in Section 5.1 in [76] to prove that $F : L^2(\Omega) \to \mathbb{R}$ fulfills Assumption 3.11. By Lemma A.3 in [36], the bounded Lipschitz domain $\Omega = (0,2)^2$ is a regular set in the sense of Gröger [58]. By Theorems 1 and 3 in [58], there exist $q' > 2$ and some constant $c_0 > 0$ such that $\|Su\|_{W^{1,q'}(\Omega)} \le c_0\|u\|_{W^{-1,q'}(\Omega)}$ for $u \in W^{-1,q'}(\Omega)$. By analyzing the adjoint, we obtain for some constant $c_1 > 0$ that $\|Su\|_{W^{1,q}(\Omega)} \le c_1\|u\|_{W^{-1,q}(\Omega)}$ holds for $u \in W^{-1,q}(\Omega)$, where $q < 2$ is the Hölder conjugate of q'. The continuous embeddings $W^{1,q}(\Omega) \hookrightarrow L^2(\Omega)$, $W^{1,q'}(\Omega) \hookrightarrow C(\bar{\Omega}) \hookrightarrow L^\infty(\Omega)$, and $L^1(\Omega) \hookrightarrow W^{-1,q}(\Omega)$, see §7.1 in [104], yield the existence of two constants $c_2, c_3 > 0$ such that

$$\|y\|_{L^2(\Omega)} \le c_2\|y\|_{W^{1,q}(\Omega)}$$

for $y \in W^{1,q}(\Omega)$ and

$$\|u\|_{W^{-1,q}(\Omega)} \le c_3\|u\|_{L^1(\Omega)}$$

for $u \in L^1(\Omega)$. Hence, there holds for $u, v, w \in L^2(\Omega)$ that

$$\begin{aligned} |\nabla^2 F(u)(v,w)| &= (Sv, Sw)_{L^2(\Omega)} \le \|Sv\|_{L^2(\Omega)}\|Sw\|_{L^2(\Omega)} \\ &\le c_2^2\|Sv\|_{W^{1,q}(\Omega)}\|Sw\|_{W^{1,q}(\Omega)} \le c_1^2c_2^2\|v\|_{W^{-1,q}(\Omega)}\|w\|_{W^{-1,q}(\Omega)} \\ &\le c_1^2c_2^2c_3^2\|v\|_{L^1(\Omega)}\|w\|_{L^1(\Omega)}. \end{aligned}$$

Moreover, since also $\nabla F(u) = S(S(u+f) - y_d) \in W^{1,q'}(\Omega)$ for $u \in L^2(\Omega)$, the continuous embedding $W^{1,q'}(\Omega) \hookrightarrow C(\bar{\Omega})$ yields that $\nabla F(u) \in C(\bar{\Omega})$ for $u \in L^2(\Omega)$ as required for Theorems 3.14 and 3.23, Lemma 3.21, and Corollary 3.22.

6.3.1 Discretized optimization problems

In order to discretize (6.1), we discretize the control $u \in L^2(\Omega)$ in (6.1) by piecewise constant functions $P0^{\mathcal{T}_h}$ on the cubic mesh $\mathcal{Q}_{\mathcal{T}_h}$. We denote the space of continuous Lagrange elements of order 1 on the mesh $\mathcal{Q}_{\mathcal{T}_h}$ by $CG1^{\mathcal{T}_h} \subset H^1(\Omega)$ and denote $CG1_0^{\mathcal{T}_h} = CG1^{\mathcal{T}_h} \cap H_0^1(\Omega)$. In order to discretize (PDE), we consider the weak formulation of (PDE), which reads: find $y \in H_0^1(\Omega)$ that fulfills

$$\kappa \int_\Omega \nabla y(x) \cdot \nabla v(x) \,\mathrm{d}x + \int_\Omega y(x)v(x)\,\mathrm{d}x = \int_\Omega w(x)v(x)\,\mathrm{d}x \quad \forall\, v \in H_0^1(\Omega).$$

The corresponding discretized PDE then reads: find $y \in CG1_0^{\mathcal{T}_h}$ that fulfills

$$(\mathrm{PDE}_{\mathcal{T}_h}) \quad \kappa \int_\Omega \nabla y(x) \cdot \nabla v(x)\,\mathrm{d}x + \int_\Omega y(x)v(x)\,\mathrm{d}x = \int_\Omega w(x)v(x)\,\mathrm{d}x \quad \forall\, v \in CG1_0^{\mathcal{T}_h}.$$

We denote the corresponding discretized solution operator that maps $w \in L^2(\Omega)$ to the unique solution $y \in CG1_0^{\mathcal{T}_h}$ to $(\mathrm{PDE}_{\mathcal{T}_h})$ by $S^{\mathcal{T}_h} : L^2(\Omega) \to CG1_0^{\mathcal{T}_h}$. Accordingly, we define $F^{\mathcal{T}_h} : L^2(\Omega) \to \mathbb{R}$ by $F^{\mathcal{T}_h}(u) \coloneqq \frac{1}{2}\|S^{\mathcal{T}_h}(u + I_{P0^{\mathcal{T}_h}} f) - I_{CG1^{\mathcal{T}_h}} y_d\|_{L^2(\Omega)}^2$, where $I_{P0^{\mathcal{T}_h}} : L^2(\Omega) \to P0^{\mathcal{T}_h}$ denotes the interpolation operator for $P0^{\mathcal{T}_h}$ and $I_{CG1^{\mathcal{T}_h}} : L^2(\Omega) \to CG1^{\mathcal{T}_h}$ denotes the interpolation operator for $CG1^{\mathcal{T}_h}$. We consider two discretizations of (6.1). The first one implements the discretization (P_c^h) introduced in Chapter 5 and reads

$$(6.2) \qquad \begin{aligned} \min_{(u,V) \in P0^{\mathcal{T}_h} \times \mathbb{R}} \quad & F^{\mathcal{T}_h}(u) + \alpha V \\ \text{s.t.} \quad & \mathrm{TV}(u) \le cV \\ & \mathrm{TV^h}(u) \le V \\ & u(x) \in U = \{0,1\} \text{ for a.a. } x \in \Omega \end{aligned}$$

with $c \ge \sqrt{2}$. The second discretization is without the application of the discretized total variation $\mathrm{TV^h}$ and discretizes only the control u, that is,

$$(6.3) \qquad \begin{aligned} \min_{u \in P0^{\mathcal{T}_h}} \; & F^{\mathcal{T}_h}(u) + \alpha \mathrm{TV}(u) \\ \text{s.t.} \; & u(x) \in U = \{0,1\} \text{ for a.a. } x \in \Omega. \end{aligned}$$

We aim to apply trust-region Algorithm 1 to (6.1) or rather to its discretizations (6.2) and (6.3). The corresponding trust-region subproblem to (6.1) reads

$$
\begin{aligned}
\min_{u\in L^2(\Omega)} \quad & (\bar p_{n-1}, u - \bar u_{n-1})_{L^2(\Omega)} + \alpha \mathrm{TV}(u) - \alpha \mathrm{TV}(\bar u_{n-1}) \\
\text{s.t.} \quad & \|u - \bar u_{n-1}\|_{L^1(\Omega)} \le \Delta \\
& u(x) \in U = \{0,1\} \text{ for a.a. } x \in \Omega,
\end{aligned}
\tag{6.4}
$$

where $\bar u_{n-1} \in \mathrm{BV_U}(\Omega)$ is the solution most recently accepted by trust-region Algorithm 1, $\bar y_{n-1} = S(\bar u_{n-1} + f) \in H_0^1(\Omega)$ is the corresponding state, and $\nabla F(\bar u_{n-1}) = \bar p_{n-1} = S(\bar y_{n-1} - y_d) \in H_0^1(\Omega)$ is the adjoint state. The discretized trust-region subproblem corresponding to (6.2) reads

$$
\begin{aligned}
\min_{(u,V)\in P0^{\mathcal{T}_h}\times\mathbb{R}} \quad & (\bar p_{n-1}, u - \bar u_{n-1})_{L^2(\Omega)} + \alpha V - \alpha \bar V_{n-1} \\
\text{s.t.} \quad & \mathrm{TV}(u) \le cV \\
& \mathrm{TV^h}(u) \le V \\
& \|u - \bar u_{n-1}\|_{L^1(\Omega)} \le \Delta \\
& u(x) \in U = \{0,1\} \text{ for a.a. } x \in \Omega,
\end{aligned}
\tag{6.5}
$$

where $(\bar u_{n-1}, \bar V_{n-1}) \in P0^{\mathcal{T}_h} \times \mathbb{R}$ is the solution most recently accepted by trust-region Algorithm 1, $\bar y_{n-1} = S^{\mathcal{T}_h}(\bar u_{n-1} + I_{P0^{\mathcal{T}_h}} f) \in CG1_0^{\mathcal{T}_h}$ is the corresponding state, and $\nabla F^{\mathcal{T}_h}(\bar u_{n-1}) = \bar p_{n-1} = S^{\mathcal{T}_h}(\bar y_{n-1} - I_{CG1^{\mathcal{T}_h}} y_d) \in CG1_0^{\mathcal{T}_h}$ is the adjoint state. Similarly, the discretized trust-region subproblem corresponding to (6.3) reads

$$
\begin{aligned}
\min_{u\in P0^{\mathcal{T}_h}} \quad & (\bar p_{n-1}, u - \bar u_{n-1})_{L^2(\Omega)} + \alpha \mathrm{TV}(u) - \alpha \mathrm{TV}(\bar u_{n-1}) \\
\text{s.t.} \quad & \|u - \bar u_{n-1}\|_{L^1(\Omega)} \le \Delta \\
& u(x) \in U = \{0,1\} \text{ for a.a. } x \in \Omega,
\end{aligned}
\tag{6.6}
$$

where again $\bar u_{n-1} \in P0^{\mathcal{T}_h}$ is the solution most recently accepted by trust-region Algorithm 1, $\bar y_{n-1} = S^{\mathcal{T}_h}(\bar u_{n-1} + I_{P0^{\mathcal{T}_h}} f) \in CG1_0^{\mathcal{T}_h}$ is the corresponding state, and $\nabla F^{\mathcal{T}_h}(\bar u_{n-1}) = \bar p_{n-1} = S^{\mathcal{T}_h}(\bar y_{n-1} - I_{CG1^{\mathcal{T}_h}} y_d) \in CG1_0^{\mathcal{T}_h}$ is the adjoint state.

Additionally, we consider the relaxation ($\mathrm{P_R}$) of (6.1) according to Chapter 4 that reads

$$
\begin{aligned}
\min_{u\in L^2(\Omega)} \quad & F(u) + \alpha \mathrm{TV}(u) \\
\text{s.t.} \quad & 0 \le u(x) \le 1 \text{ for a.a. } x \in \Omega.
\end{aligned}
\tag{6.7}
$$

According to Section 4.4, the function $\bar u$ as defined above is also the unique optimal solution to (6.7). In order to apply outer-approximation Algorithm 2, we introduce the regularization ($\bar{\mathrm{P}}_{\delta,\varepsilon}$) of (6.7) that reads

$$
\begin{aligned}
\min_{u\in L^2(\Omega)} \quad & F(u)+\frac{\delta}{2}\|u\|^2_{L^2(\Omega)}+\alpha \mathrm{TV}_\varepsilon(u) \\
\text{s.t.} \quad & 0\le u(x)\le 1 \text{ for a.a. } x\in\Omega
\end{aligned}
\tag{6.8}
$$

with $\varepsilon>0$ and $\delta>0$. For the numerical solution of (6.8), we apply the finite-element discretizations from Section 5.4 that read

$$
\begin{aligned}
\min_{u\in P0^{\mathcal{T}_h}} \quad & F^{\mathcal{T}_h}(u)+\frac{\delta}{2}\|u\|^2_{L^2(\Omega)}+\alpha \mathrm{TV}^h_\varepsilon(u) \\
\text{s.t.} \quad & 0\le u(x)\le 1 \text{ for a.a. } x\in\Omega,
\end{aligned}
\tag{6.9}
$$

corresponding to discretized problem (6.2) and

$$
\begin{aligned}
\min_{u\in P0^{\mathcal{T}_h}} \quad & F^{\mathcal{T}_h}(u)+\frac{\delta}{2}\|u\|^2_{L^2(\Omega)}+\alpha \mathrm{TV}^{\mathcal{T}_h}_\varepsilon(u) \\
\text{s.t.} \quad & 0\le u(x)\le 1 \text{ for a.a. } x\in\Omega,
\end{aligned}
\tag{6.10}
$$

corresponding to discretized problem (6.3), where we define for fixed $u\in L^2(\Omega)$ the discretized and regularized total variation $\mathrm{TV}^h_\varepsilon : L^2(\Omega)\to\mathbb{R}$ by

$$
\mathrm{TV}^h_\varepsilon(u) \coloneqq \sup\left\{-\frac{\varepsilon}{2}a[\phi,\phi]+\int_\Omega u(x)\operatorname{div}\phi(x)\,\mathrm{d}x : \phi\in RT0^h_0,\ \|\phi\|_{L^\infty(\Omega;\mathbb{R}^d)}\le 1\right\}.
$$

We choose $H = H_0(\mathrm{div};\Omega)$ and the bounded bilinear form $a : H_0(\mathrm{div};\Omega)\times H_0(\mathrm{div},\Omega)\to\mathbb{R}$ defined by

$$
a[\phi,\phi]=\int_\Omega \operatorname{div}\phi(x)\operatorname{div}\phi(x)\,\mathrm{d}x.
$$

Even if this specific choice for a is not coercive with respect to $\|\cdot\|_{H(\mathrm{div};\Omega)}$, all results from Chapter 4 and Section 5.4 still hold except for the uniqueness of the maximizer of (Q_ε). In particular, (Q_ε) still admits a maximizer, $\mathrm{TV}_\varepsilon : L^2(\Omega)\to\mathbb{R}$ is still convex and continuous, and the convergence proof for Algorithm 2 in Lemma 4.14 still holds, because we can just continue after (4.4) with $\tilde{\phi}$ instead of $\bar{\phi}$. We highlight that there holds $\mathrm{TV}^h_\varepsilon(u)\le\mathrm{TV}^{\mathrm{h}}(u)$ due to $a[\phi,\phi]\ge 0$ and $\mathrm{TV}^h_\varepsilon(u)\le\mathrm{TV}_\varepsilon(u)\le\mathrm{TV}(u)$ due to $RT0^h_0\subset H_0(\mathrm{div};\Omega)$. Similar to Theorem 4.13, we obtain a lower bound for (6.2) and (6.3) from the optimal values of (6.9) and (6.10). To see this, let $\tilde{u}_1\in P0^{\mathcal{T}_h}$ denote the optimal solution to (6.9), $\tilde{u}_2\in P0^{\mathcal{T}_h}$ denote the optimal solution to (6.10), $(\bar{u}_1,\bar{V}_1)\in P0^{\mathcal{T}_h}\times\mathbb{R}$ denote the optimal solution to (6.2), and $\bar{u}_2\in P0^{\mathcal{T}_h}$ denote the optimal solution to (6.3). We denote $M=\nu^2_{\max}|\Omega|$ and $\nu_{\max}=\max_{\nu\in U}|\nu|$ as in

Theorem 4.13. By the feasibility of $\bar{u}_1$ for (6.9), there holds

$$
\begin{aligned}
&F^{\tau_h}(\tilde{u}_1) + \frac{\delta}{2}\|\tilde{u}_1\|^2_{L^2(\Omega)} + \alpha \mathrm{TV}^h_\varepsilon(\tilde{u}_1) - \frac{\delta}{2}M \\
&\le F^{\tau_h}(\bar{u}_1) + \frac{\delta}{2}\|\bar{u}_1\|^2_{L^2(\Omega)} + \alpha \mathrm{TV}^h_\varepsilon(\bar{u}_1) - \frac{\delta}{2}M \\
&\le F^{\tau_h}(\bar{u}_1) + \alpha \bar{V}_1
\end{aligned}
$$

and analogously by the feasibility of $\bar{u}_2$ for (6.10), there holds

$$
\begin{aligned}
&F^{\tau_h}(\tilde{u}_2) + \frac{\delta}{2}\|\tilde{u}_2\|^2_{L^2(\Omega)} + \alpha \mathrm{TV}^{\tau_h}_\varepsilon(\tilde{u}_2) - \frac{\delta}{2}M \\
&\le F^{\tau_h}(\bar{u}_2) + \frac{\delta}{2}\|\bar{u}_2\|^2_{L^2(\Omega)} + \alpha \mathrm{TV}^{\tau_h}_\varepsilon(\bar{u}_2) - \frac{\delta}{2}M \\
&\le F^{\tau_h}(\bar{u}_2) + \alpha \mathrm{TV}(\bar{u}_2).
\end{aligned}
$$

Additionally, we also consider the variant

$$
\tag{6.11}
\begin{aligned}
\min_{u \in P0^{\tau_h}} \quad & F^{\tau_h}(u) + \frac{\delta}{2}\|u - I_{P0^{\tau_h}}\bar{u}\|^2_{L^2(\Omega)} + \alpha \mathrm{TV}^h_\varepsilon(u) \\
\text{s.t.} \quad & 0 \le u(x) \le 1 \text{ for a.a. } x \in \Omega,
\end{aligned}
$$

of (6.9) and the variant

$$
\tag{6.12}
\begin{aligned}
\min_{u \in P0^{\tau_h}} \quad & F^{\tau_h}(u) + \frac{\delta}{2}\|u - I_{P0^{\tau_h}}\bar{u}\|^2_{L^2(\Omega)} + \alpha \mathrm{TV}^{\tau_h}_\varepsilon(u) \\
\text{s.t.} \quad & 0 \le u(x) \le 1 \text{ for a.a. } x \in \Omega,
\end{aligned}
$$

of (6.10). The optimization problems (6.11) and (6.12) are the discretizations of the regularization

$$
\begin{aligned}
\min_{u \in L^2(\Omega)} \quad & F(u) + \frac{\delta}{2}\|u - \bar{u}\|^2_{L^2(\Omega)} + \alpha \mathrm{TV}_\varepsilon(u) \\
\text{s.t.} \quad & 0 \le u(x) \le 1 \text{ for a.a. } x \in \Omega
\end{aligned}
$$

of (6.1). We highlight that the regularization with the alternative Tikhonov term $\frac{\delta}{2}\|u - \bar{u}\|^2_{L^2(\Omega)}$ does neither change the limit of the minimizers nor the limit of the optimal objective values for vanishing regularization parameters $\varepsilon, \delta \searrow 0$. It might even accelerate the convergence because it vanishes for the optimal control $\bar{u}$ of the limit problem (6.1).

6.3.2 Experiment description

We have proved in Chapter 5 that the optimal solutions to (6.2) recover the optimal value of (6.1) when the coupled mesh sizes (h, τ_h) are driven to zero such that $\frac{\tau_h}{h} \searrow 0$.

Moreover, we have proved that it is generally not possible recover the optimal value with the anisotropic discretization (6.3). We apply trust-region Algorithm 1 to solve the discretized optimization problems (6.2) and (6.3) to examine the influence of the both discretizations on the numerical results.

As introduced in Chapter 4, we obtain lower bounds for the discretized optimization problems (6.2) and (6.3) from the optimal values of the discretized and regularized problems (6.9) and (6.10). We apply Algorithm 2 to solve (6.9) and (6.10) to compute the lower bounds for (6.2) and (6.3). Additionally, we apply Algorithm 2 to the optimization problems (6.11) and (6.12) with the alternative Tikhonov term to obtain approximate lower bounds for (6.2) and (6.3).

Since we know the optimal solution to the limit problems (6.1) and (6.7), we additionally compute the relative errors of the solutions to the discretized and regularized problems to the optimal solution in function space in order to evaluate the convergence behavior when the mesh sizes and the regularization parameters are driven to zero.

We list all instances including the corresponding parameters, limits, and settings for the application of Algorithms 1 to 3 in Tables 6.4 to 6.6.

Table 6.4 lists the experiments in which Algorithm 1 is applied to the discretizations (6.2) and (6.3). Therein, Δ_0 denotes the reset radius for trust-region Algorithm 1, σ denotes the parameter σ for the sufficient decrease condition (3.13), *MaxIt OA* denotes the iteration maximum for outer-approximation Algorithm 3, *TOL OA* denotes the chosen tolerance for outer-approximation Algorithm 3, *TOL GRB* denotes the chosen tolerance for Gurobi, and *Time GRB* denotes the time limit for Gurobi.

Table 6.5 lists the experiments in which Algorithm 2 is applied to the discretizations (6.9) and (6.10) and Table 6.6 lists the experiments in which Algorithm 2 is applied to the discretizations (6.11) and (6.12). Therein, *TOL* denotes the chosen tolerance for outer-approximation Algorithm 2, *MaxIt* denotes the iteration maximum for outer-approximation Algorithm 2, and *Time limit* denotes the time for Algorithm 2.

Practical implementation

When we apply Algorithm 1 to discretization (6.2), we use outer-approximation Algorithm 3 to solve the occurring subproblems of type (6.5). We reformulate (MIP) as a mixed-integer linear optimization problem following Section 6.2.2 and (QP) as a quadratic optimization problem following Section 6.1.1. For the solution of (MIP) and (QP), we use Gurobi [59]. For outer-approximation Algorithm 3, we use a tolerance TOL for the gap between $\mathrm{TV}^{\mathrm{h}}(u_k)$ and V_k, that is, Algorithm 3 terminates if

$$\int_\Omega \operatorname{div} \phi_{k+1}(x) u_k(x) \,\mathrm{d}x - V_k \le \mathrm{TOL} \int_\Omega \operatorname{div} \phi_{k+1}(x) u_k(x) \,\mathrm{d}x. \tag{6.13}$$

Table 6.4: Instances, parameters, and settings for the application of trust-region Algorithm 1 to (6.2) and (6.3).

No.	$(\frac{1}{h}, \frac{1}{\tau_h})$	Discr.	α	κ	c	Δ_0	σ	MaxIt OA	TOL OA	TOL GRB	Time GRB
E1	$(16, 64)$	(6.2)	1×10^{-3}	1×10^{-2}	$\sqrt{2}$	$1/8$	1×10^{-4}	25	1×10^{-2}	1×10^{-3}	48 h
E2	$(16, 64)$	(6.2)	1×10^{-3}	1×10^{-2}	$3\sqrt{2}$	$1/8$	1×10^{-4}	25	1×10^{-2}	1×10^{-3}	48 h
E3	$(64, 64)$	(6.3)	1×10^{-3}	1×10^{-2}	-	$1/8$	1×10^{-4}	-	-	1×10^{-4}	48 h
E4	$(32, 128)$	(6.2)	1×10^{-3}	1×10^{-2}	$\sqrt{2}$	$1/8$	1×10^{-4}	10	1×10^{-2}	1×10^{-3}	48 h
E5	$(128, 128)$	(6.3)	1×10^{-3}	1×10^{-2}	-	$1/8$	1×10^{-4}	-	-	1×10^{-4}	48 h
E6	$(64, 512)$	(6.2)	1×10^{-3}	1×10^{-2}	$\sqrt{2}$	$1/512$	1×10^{-4}	10	1×10^{-2}	1×10^{-3}	4 h
E7	$(512, 512)$	(6.3)	1×10^{-3}	1×10^{-2}	-	$1/512$	1×10^{-4}	-	-	1×10^{-3}	4 h

Table 6.5: Instances, parameters, and settings for the application of outer-approximation Algorithm 2 to (6.9) and (6.10).

No.	$(\frac{1}{h}, \frac{1}{\tau_h})$	Discr.	α	κ	ε	δ	TOL	MaxIt	Time limit
R1	(16,64)	(6.9)	1×10^{-3}	1×10^{-2}	1×10^{-4}	1×10^{-4}	1×10^{-2}	1001	168 h
R2	(64,64)	(6.10)	1×10^{-3}	1×10^{-2}	1×10^{-4}	1×10^{-4}	1×10^{-2}	1001	168 h
R3	(32,128)	(6.9)	1×10^{-3}	1×10^{-2}	1×10^{-4}	1×10^{-4}	1×10^{-2}	1001	168 h
R4	(128,128)	(6.10)	1×10^{-3}	1×10^{-2}	1×10^{-4}	1×10^{-4}	1×10^{-2}	1001	168 h
R5	(16,64)	(6.9)	1×10^{-3}	1×10^{-2}	1×10^{-4}	1×10^{-5}	1×10^{-2}	1001	168 h
R6	(64,64)	(6.10)	1×10^{-3}	1×10^{-2}	1×10^{-4}	1×10^{-5}	1×10^{-2}	1001	168 h
R7	(32,128)	(6.9)	1×10^{-3}	1×10^{-2}	1×10^{-4}	1×10^{-5}	1×10^{-2}	1001	168 h
R8	(128,128)	(6.10)	1×10^{-3}	1×10^{-2}	1×10^{-4}	1×10^{-5}	1×10^{-2}	1001	168 h

Table 6.6: Instances, parameters, and settings for the application of outer-approximation Algorithm 2 to (6.11) and (6.12).

No.	$(\frac{1}{h}, \frac{1}{\tau_h})$	Discr.	α	κ	ε	δ	TOL	MaxIt	Time limit
RV1	(16,64)	(6.11)	1×10^{-3}	1×10^{-2}	1×10^{-3}	1×10^{-2}	1×10^{-2}	1001	168 h
RV2	(64,64)	(6.12)	1×10^{-3}	1×10^{-2}	1×10^{-3}	1×10^{-2}	1×10^{-2}	1001	168 h
RV3	(32,128)	(6.11)	1×10^{-3}	1×10^{-2}	1×10^{-3}	1×10^{-2}	1×10^{-2}	1001	168 h
RV4	(128,128)	(6.12)	1×10^{-3}	1×10^{-2}	1×10^{-3}	1×10^{-2}	1×10^{-2}	1001	168 h
RV5	(16,64)	(6.11)	1×10^{-3}	1×10^{-2}	5×10^{-4}	5×10^{-3}	1×10^{-2}	1001	168 h
RV6	(64,64)	(6.12)	1×10^{-3}	1×10^{-2}	5×10^{-4}	5×10^{-3}	1×10^{-2}	1001	168 h
RV7	(32,128)	(6.11)	1×10^{-3}	1×10^{-2}	5×10^{-4}	5×10^{-3}	1×10^{-2}	1001	168 h
RV8	(128,128)	(6.12)	1×10^{-3}	1×10^{-2}	5×10^{-4}	5×10^{-3}	1×10^{-2}	1001	168 h

In order to limit the runtime of Algorithm 1, we specify an iteration maximum for Algorithm 3. If Algorithm 3 is not able to close this gap up to the tolerance TOL with the current iterate (u_k, V_k) within the iteration maximum, we return $(\tilde{u}_{n,k}, \tilde{V}_{n,k}) = (u_k, \max\{\frac{1}{c}\mathrm{TV}(u_k), \mathrm{TV}^{\mathrm{h}}(u_k)\})$ as the solution to (6.5) and continue with trust-region Algorithm 1. If the suboptimal solution results in a non-positive predicted reduction, we do not let the algorithm terminate and reduce the trust-region radius Δ in spite of this.

We highlight that a negative predicted reduction can occur in our numerical experiments due to the acceptance tolerances. This can be explained by the gap between $\bar{V}_{n-1}$ and $\mathrm{TV}^{\mathrm{h}}(\bar{u}_{n-1})$ due to the outer approximation which leads to an underestimation of the optimal value of (6.5). In particular, the solution $(\bar{u}_{n-1}, \bar{V}_{n-1})$ obtained by the application of outer-approximation Algorithm 3 with a tolerance might not be feasible for (6.5) but only $(\bar{u}_{n-1}, \max\{\mathrm{TV}^{\mathrm{h}}(\bar{u}_{n-1}), \frac{1}{c}\mathrm{TV}(\bar{u}_{n-1})\})$ which generally has not objective value 0 but $\alpha \max\{\mathrm{TV}^{\mathrm{h}}(\bar{u}_{n-1}), \frac{1}{c}\mathrm{TV}(\bar{u}_{n-1})\} - \alpha\bar{V}_{n-1}$.

When we apply Algorithm 1 to discretization (6.3), we reformulate the occurring subproblems of type (6.6) as mixed-integer linear optimization problems similar as in Section 6.2.2 and solve them with Gurobi [59].

When we apply Algorithm 2 to the discretized and regularized problems (6.9), (6.11), (6.10), and (6.12), we choose a tolerance TOL for the termination criterion of Algorithm 2, that is, Algorithm 2 terminates if

$$R_\varepsilon(u_k, \phi_{k+1}) - V_k \leq \mathrm{TOL} \cdot R_\varepsilon(u_k, \phi_{k+1})$$

for the current iterate u_k and the corresponding $\phi_{k+1} \in RT0^h$ that realizes $\mathrm{TV}^h_\varepsilon(u_k) = R_\varepsilon(u_k, \phi_{k+1})$. We formulate (P_k) as an optimization problem with quadratic objective function and linear constraints and (Q_k) as an optimization problem with linear objective function and quadratic constraints following Section 6.1.1. For the solution of both, we again use Gurobi [59].

6.3.3 Results

We obtained the following results from the experiments described above.

Table and figure description

In Table 6.7, we list the results that are obtained by the application of trust-region Algorithm 1 to the both discretizations (6.2) and (6.3). If we denote the solution returned by trust-region Algorithm 1 applied to (6.2) and (6.3) by (u^*, V^*) and u^*, respectively, the corresponding state by y^*, and the corresponding adjoint state by p^*, then *Obj.* represents the objective value $F^{\mathcal{T}_h}(u^*) + \alpha V^*$ or $F^{\mathcal{T}_h}(u^*) + \alpha\mathrm{TV}(u^*)$, *Tracking term* represents the tracking term $F^{\mathcal{T}_h}(u^*)$, TV *term* represents the total variation term αV^* or $\alpha\mathrm{TV}(u^*)$, V represents the value of V^*, TV represents $\mathrm{TV}(u^*)$,

Table 6.7: Results obtained by the application of trust-region Algorithm 1 to (6.2) and (6.3) with $\alpha = 1 \times 10^{-3}$ and $\kappa = 1 \times 10^{-2}$.

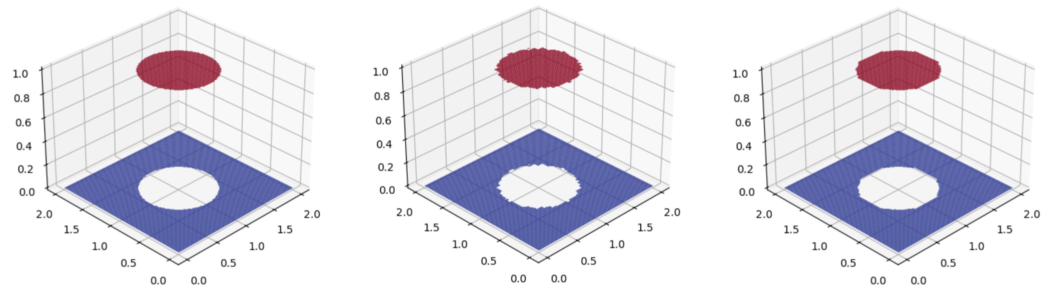
No.	$(\frac{1}{h}, \frac{1}{\tau_h})$	TV	c	Obj.	Tracking term	TV term	V	TV	$\mathrm{TV^h}$	TV ex.	Err. u	Err. y	Err. p	Err. TV	It.	Term.	Time (h)
E1	$(16, 64)$	$\mathrm{TV^h}$	$\sqrt{2}$	3.055×10^{-3}	7.791×10^{-5}	2.977×10^{-3}	2.977	4.188	3.005	3.142	1.72×10^{-1}	5.66×10^{-2}	8.01×10^{-1}	4.34×10^{-2}	13	pred	87.03
E2	$(16, 64)$	$\mathrm{TV^h}$	$3\sqrt{2}$	3.043×10^{-3}	7.789×10^{-5}	2.966×10^{-3}	2.966	4.875	2.989	3.142	2.08×10^{-1}	5.34×10^{-2}	7.00×10^{-1}	4.84×10^{-2}	14	Δ	6.19
E3	$(64, 64)$	TV	-	4.042×10^{-3}	4.223×10^{-5}	4.000×10^{-3}	-	4.000	3.218	3.142	1.22×10^{-1}	3.08×10^{-2}	3.97×10^{-1}	2.73×10^{-1}	19	Δ	0.97
E4	$(32, 128)$	$\mathrm{TV^h}$	$\sqrt{2}$	3.124×10^{-3}	6.749×10^{-5}	3.056×10^{-3}	3.056	4.313	3.086	3.142	1.98×10^{-1}	4.88×10^{-2}	6.63×10^{-1}	1.77×10^{-2}	9	pred	34.21
E5	$(128, 128)$	TV	-	3.994×10^{-3}	1.195×10^{-4}	3.875×10^{-3}	-	3.875	3.209	3.142	1.71×10^{-1}	5.91×10^{-2}	1.40	2.33×10^{-1}	47	pred	13.33
E6	$(64, 512)$	$\mathrm{TV^h}$	$\sqrt{2}$	3.405×10^{-3}	2.862×10^{-4}	3.118×10^{-3}	3.118	4.398	3.149	3.142	2.12×10^{-1}	6.77×10^{-2}	1.85	2.30×10^{-3}	137	pred	334.66
E7	$(512, 512)$	TV	-	3.967×10^{-3}	6.103×10^{-5}	3.906×10^{-3}	-	3.906	3.210	3.142	1.43×10^{-1}	3.96×10^{-2}	7.76×10^{-1}	2.43×10^{-1}	156	pred	132.37

Table 6.8: Results obtained by the application of outer-approximation Algorithm 2 to (6.9) and (6.10) with $\alpha = 1 \times 10^{-3}$, $\kappa = 1 \times 10^{-2}$, $\varepsilon = 1 \times 10^{-4}$, and $\delta = 1 \times 10^{-4}$ (R1-R4) or $\delta = 1 \times 10^{-5}$ (R5-R8).

No.	$(\frac{1}{h}, \frac{1}{\tau_h})$	Bound	Obj.	Tracking term	Tikhonov term	V	$\mathrm{TV}^h_\varepsilon$	$\mathrm{TV^h}$	TV	TV ex.	Err. u	Err. y	Err. p	Err. TV	It.	Term.	Time (h)
R1	(16,64)	2.848×10^{-3}	3.048×10^{-3}	4.839×10^{-5}	3.913×10^{-5}	2.961	2.990	3.011	4.686	3.142	9.25×10^{-2}	1.81×10^{-2}	2.92×10^{-1}	4.82×10^{-2}	44	TOL	0.47
R2	(64,64)	2.952×10^{-3}	3.152×10^{-3}	5.449×10^{-5}	3.844×10^{-5}	3.059	3.208	3.253	4.230	3.142	1.05×10^{-1}	2.43×10^{-2}	3.54×10^{-1}	2.11×10^{-2}	1001	MaxIt	53.66
R3	(32,128)	2.925×10^{-3}	3.125×10^{-3}	4.166×10^{-5}	3.907×10^{-5}	3.045	3.072	3.130	5.413	3.142	1.04×10^{-1}	1.41×10^{-2}	1.64×10^{-1}	2.22×10^{-2}	370	TOL	33.35
R4	(128,128)	2.796×10^{-3}	2.996×10^{-3}	7.009×10^{-5}	3.792×10^{-5}	2.888	3.740	3.888	5.322	3.142	1.74×10^{-1}	3.57×10^{-2}	5.35×10^{-1}	1.90×10^{-1}	781	Time	168
R5	(16,64)	2.993×10^{-3}	3.013×10^{-3}	4.850×10^{-5}	3.914×10^{-6}	2.961	2.991	3.001	4.732	3.142	8.69×10^{-2}	1.72×10^{-2}	2.87×10^{-1}	4.80×10^{-2}	61	TOL	0.67
R6	(64,64)	3.044×10^{-3}	3.064×10^{-3}	6.098×10^{-5}	3.830×10^{-6}	2.999	3.259	3.298	4.368	3.142	1.34×10^{-1}	2.98×10^{-2}	4.33×10^{-1}	3.73×10^{-2}	1001	MaxIt	56.00
R7	(32,128)	3.070×10^{-3}	3.090×10^{-3}	4.140×10^{-5}	3.911×10^{-6}	3.045	3.075	3.134	5.523	3.142	1.03×10^{-1}	1.38×10^{-2}	1.59×10^{-1}	2.13×10^{-2}	451	TOL	49.52
R8	(128,128)	2.955×10^{-3}	2.975×10^{-3}	1.499×10^{-4}	3.783×10^{-6}	2.821	3.873	3.971	5.478	3.142	1.83×10^{-1}	3.92×10^{-2}	1.24	2.33×10^{-1}	748	Time	168

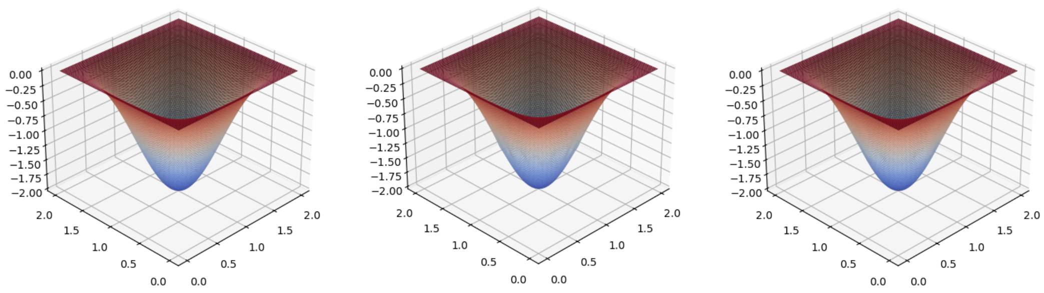
Table 6.9: Results obtained by the application of outer-approximation Algorithm 2 to (6.11) and (6.12) with $\alpha = 1 \times 10^{-3}$, $\kappa = 1 \times 10^{-2}$, $\varepsilon = 1 \times 10^{-3}$ (RV1-RV4) or $\varepsilon = 5 \times 10^{-4}$ (RV5-RV8), and $\delta = 1 \times 10^{-2}$ (RV1-RV4) or $\delta = 5 \times 10^{-3}$ (RV5-RV8).

No.	$(\frac{1}{h}, \frac{1}{\tau_h})$	Obj.	Tracking term	Tikhonov term	V	$\mathrm{TV}^h_\varepsilon$	$\mathrm{TV^h}$	TV	TV ex.	Err. u	Err. y	Err. p	Err. TV	It.	Term.	Time (h)
RV1	(16,64)	3.004×10^{-3}	4.451×10^{-5}	3.865×10^{-6}	2.956	2.980	3.051	4.625	3.142	3.12×10^{-2}	1.03×10^{-2}	3.31×10^{-1}	5.15×10^{-2}	4	TOL	0.03
RV2	(64,64)	3.148×10^{-3}	4.070×10^{-5}	9.865×10^{-6}	3.097	3.128	3.378	4.369	3.142	4.99×10^{-2}	1.11×10^{-2}	2.14×10^{-1}	4.31×10^{-3}	43	TOL	0.94
RV3	(32,128)	3.085×10^{-3}	4.334×10^{-5}	3.606×10^{-6}	3.038	3.063	3.186	5.188	3.142	3.03×10^{-2}	1.02×10^{-2}	2.90×10^{-1}	2.49×10^{-2}	5	TOL	0.17
RV4	(128,128)	3.150×10^{-3}	3.688×10^{-5}	6.952×10^{-6}	3.107	3.129	3.565	4.621	3.142	4.20×10^{-2}	6.39×10^{-3}	1.58×10^{-1}	3.96×10^{-3}	59	TOL	5.89
RV5	(16,64)	3.009×10^{-3}	4.696×10^{-5}	3.018×10^{-6}	2.959	2.984	3.036	4.729	3.142	3.90×10^{-2}	1.22×10^{-2}	3.52×10^{-1}	5.02×10^{-2}	5	TOL	0.04
RV6	(64,64)	3.147×10^{-3}	4.069×10^{-5}	8.887×10^{-6}	3.097	3.128	3.299	4.265	3.142	6.70×10^{-2}	1.37×10^{-2}	1.84×10^{-1}	4.18×10^{-3}	193	TOL	4.52
RV7	(32,128)	3.088×10^{-3}	4.179×10^{-5}	2.712×10^{-6}	3.044	3.071	3.153	5.194	3.142	3.71×10^{-2}	1.01×10^{-2}	2.32×10^{-1}	2.26×10^{-2}	13	TOL	0.49
RV8	(128,128)	3.147×10^{-3}	3.682×10^{-5}	6.973×10^{-6}	3.103	3.134	3.485	4.520	3.142	5.95×10^{-2}	8.27×10^{-3}	1.30×10^{-1}	2.40×10^{-3}	240	TOL	26.92



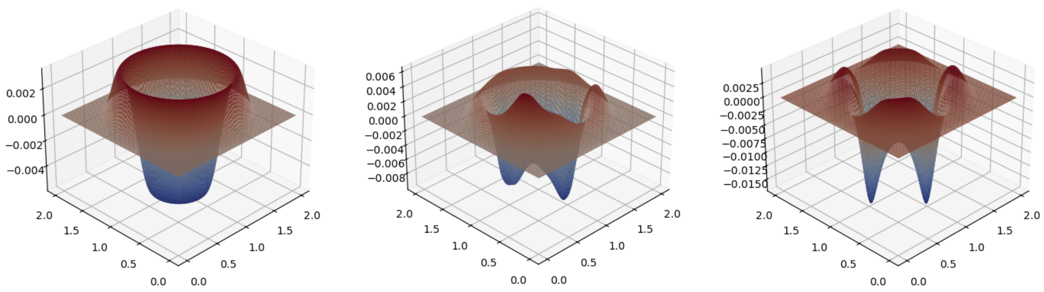
(a) Optimal control $\bar{u}$ (b) Exp. E4: (6.2), $c = \sqrt{2}$ (c) Exp. E5: (6.3)

Figure 6.3: Control u returned by Algorithm 1 applied to (6.2) and (6.3) with mesh size $(h, \tau_h) = \left(\frac{1}{32}, \frac{1}{128}\right)$.



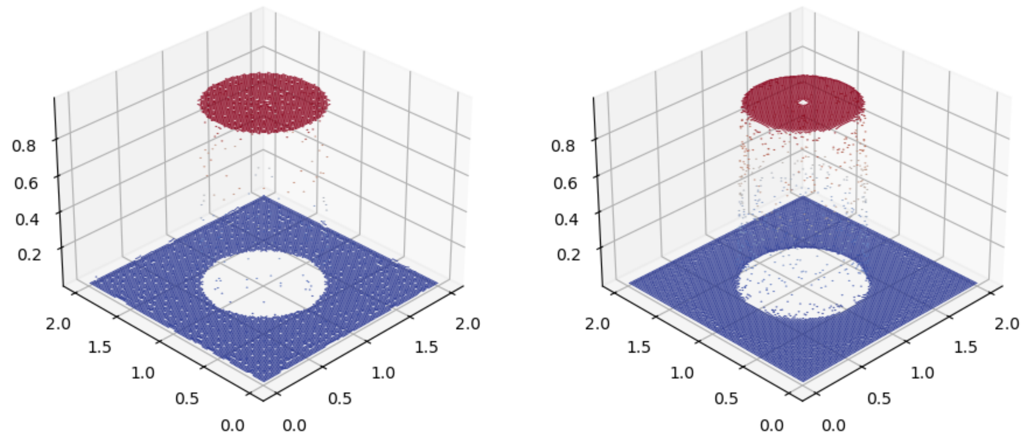
(a) Optimal state $\bar{y}$ (b) Exp. E4: (6.2), $c = \sqrt{2}$ (c) Exp. E5: (6.3)

Figure 6.4: State y returned by Algorithm 1 applied to (6.2) and (6.3) with mesh size $(h, \tau_h) = \left(\frac{1}{32}, \frac{1}{128}\right)$.



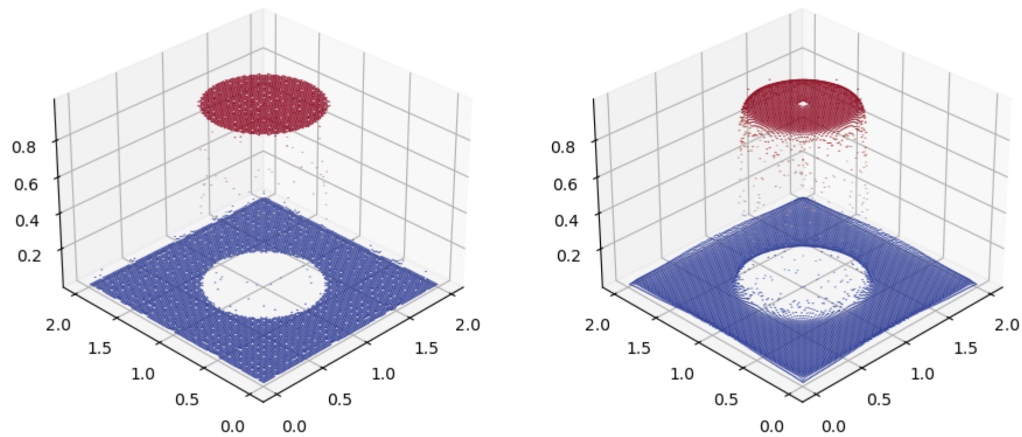
(a) Optimal adjoint state $\bar{p}$ (b) Exp. E4: (6.2), $c = \sqrt{2}$ (c) Exp. E5: (6.3)

Figure 6.5: Adjoint state p returned by Algorithm 1 applied to (6.2) and (6.3) with mesh size $(h, \tau_h) = \left(\frac{1}{32}, \frac{1}{128}\right)$.



(a) Exp. R3: (6.9) with $\varepsilon = 1 \times 10^{-4}$, $\delta = 1 \times 10^{-4}$

(b) Exp. R4: (6.10) with $\varepsilon = 1 \times 10^{-4}$, $\delta = 1 \times 10^{-4}$



(c) Exp. R7: (6.9) with $\varepsilon = 1 \times 10^{-4}$, $\delta = 1 \times 10^{-5}$

(d) Exp. R8: (6.10) with $\varepsilon = 1 \times 10^{-4}$, $\delta = 1 \times 10^{-5}$

Figure 6.6: Resulting u obtained by the application of Algorithm 2 to (6.9) and (6.10) with the mesh sizes $(h, \tau_h) = (\frac{1}{32}, \frac{1}{128})$.

TV^{h} represents $\mathrm{TV}^{\mathrm{h}}(u^*)$, TV *ex.* represents the total variation $\mathrm{TV}(\bar{u})$ of the optimal solution $\bar{u}$ in function space, *Err. u* represents the relative error $\frac{\|\bar{u}-u^*\|_{L^2(\Omega)}}{\|\bar{u}\|_{L^2(\Omega)}}$, *Err. y* represents the relative error $\frac{\|\nabla\bar{y}-\nabla y^*\|_{L^2(\Omega;\mathbb{R}^d)}}{\|\nabla\bar{y}\|_{L^2(\Omega;\mathbb{R}^d)}}$, *Err. p* represents the relative error $\frac{\|\nabla\bar{p}-\nabla p^*\|_{L^2(\Omega)}}{\|\nabla\bar{p}\|_{L^2(\Omega)}}$, *Err.* TV represents the relative error $\frac{|\mathrm{TV}(\bar{u})-\mathrm{TV}^{\mathrm{h}}(u^*)|}{\mathrm{TV}(\bar{u})}$ or $\frac{|\mathrm{TV}(\bar{u})-\mathrm{TV}(u^*)|}{\mathrm{TV}(\bar{u})}$, *It.* represents the number of outer iterations that trust-region Algorithm 1 carried out until termination, *Term.* represents the reason for the termination of Algorithm 1, and *Time (h)* represents the runtime of trust-region Algorithm 1 measured in hours. In column *Term.*, pred means that the predicted reduction was non-positive and Δ means that the trust-region radius contracted.

We plotted the returned control u, the corresponding state y, and the corresponding adjoint state p returned by trust-region Algorithm 1 applied to (6.2) with constant $c = \sqrt{2}$ and mesh size $(h, \tau_h) = \left(\frac{1}{32}, \frac{1}{128}\right)$ and (6.3) with mesh size $(h, \tau_h) = \left(\frac{1}{128}, \frac{1}{128}\right)$ as well as the optimal solution $\bar{u}$ in function space, the corresponding state $\bar{y}$, and the corresponding adjoint state $\bar{p}$ in Figures 6.3 to 6.5.

We list the results obtained by the application of outer-approximation Algorithm 2 to (6.9) and (6.10) in Table 6.8 and the results obtained by the application of outer-approximation Algorithm 2 to (6.11) and (6.12) in Table 6.9. If we denote the solution returned by outer-approximation Algorithm 2 applied to (6.11) or (6.12) by (u^*, V^*), then *Bound* represents the lower bound $F^{\tau_h}(u^*) + \frac{\delta}{2}\|u^*\|^2_{L^2(\Omega)} + \alpha V^* - \frac{\delta}{2}M$ for (6.2) and (6.3), *Obj.* represents the objective value $F^{\tau_h}(u^*) + \frac{\delta}{2}\|u^*\|^2_{L^2(\Omega)} + \alpha V^*$ or $F^{\tau_h}(u^*) + \frac{\delta}{2}\|u^* - I_{P0^{\tau_h}}\bar{u}\|^2_{L^2(\Omega)} + \alpha V^*$, *Tracking term* represents the value $F^{\tau_h}(u^*)$, *Tikhonov term* represents the value $\frac{\delta}{2}\|u^*\|^2_{L^2(\Omega)}$ or $\frac{\delta}{2}\|u^* - I_{P0^{\tau_h}}\bar{u}\|^2_{L^2(\Omega)}$, V represents the value of V^*, $\mathrm{TV}^h_\varepsilon$ represents the value $\mathrm{TV}^h_\varepsilon(u^*)$, TV^{h} represents the value $\mathrm{TV}^{\mathrm{h}}(u^*)$, TV represents the value $\mathrm{TV}(u^*)$, TV *ex.* represents $\mathrm{TV}(\bar{u})$, *Err. u* represents the relative error $\frac{\|\bar{u}-u^*\|_{L^2(\Omega)}}{\|\bar{u}\|_{L^2(\Omega)}}$, *Err. y* represents the relative error $\frac{\|\nabla\bar{y}-\nabla y^*\|_{L^2(\Omega;\mathbb{R}^d)}}{\|\nabla\bar{y}\|_{L^2(\Omega;\mathbb{R}^d)}}$, *Err. p* represents the relative error $\frac{\|\nabla\bar{p}-\nabla p^*\|_{L^2(\Omega;\mathbb{R}^d)}}{\|\nabla\bar{p}\|_{L^2(\Omega;\mathbb{R}^d)}}$, *Err.* TV denotes the relative error $\frac{|\mathrm{TV}(\bar{u})-\mathrm{TV}^h_\varepsilon(u^*)|}{\mathrm{TV}(\bar{u})}$, *It.* represents the number of outer iterations that outer-approximation Algorithm 2 carried out until termination, *Term.* represents the reason why Algorithm 2 terminated, where *TOL* means that the termination criterion (6.13) is fulfilled, *MaxIt* means that the iteration maximum was reached, and *Time* means that the time limit was reached, and *Time (h)* presents the runtime of outer-approximation Algorithm 2 measured in hours.

The solutions $u \in L^2(\Omega)$ returned by Algorithm 2 applied to (6.9) and (6.10) with mesh sizes $(h, \tau_h) = \left(\frac{1}{32}, \frac{1}{128}\right)$ are plotted in Figure 6.6.

Result description

We can observe that the discretization (6.3) generally leads to slightly smaller relative errors with respect to the optimal control $\bar{u}$ in function space, the corresponding state $\bar{y}$, and the corresponding adjoint state $\bar{p}$ in our experiments. In contrast to that, the relative error with respect to the total variation $\mathrm{TV}(\bar{u})$ of the optimal control $\bar{u}$ is significantly smaller with the discretization (6.2) from Chapter 5. Both constants $c \in \{\sqrt{2}, 3\sqrt{2}\}$ for the discretization (6.2) lead for the mesh sizes $(h, \tau_h) = \left(\frac{1}{16}, \frac{1}{64}\right)$ to similar results with a moderately smaller relative error for the total variation term and a significantly larger running time for the constant $c = \sqrt{2}$.

The influence of the total variation term can also be observed visually in Figure 6.3, where we can see that the usual total variation TV leads to level sets with axis-aligned interfaces whereas the discretized total variation TV^{h} leads to solutions with more chattering along the interfaces of the level sets.

From Tables 6.7 to 6.9, we can observe that the gap between the lower bound obtained by Algorithm 2 and the optimal value returned by Algorithm 1 is significantly smaller for the cases in which we have coupled mesh sizes with $h > \tau_h$ than for the cases $h = \tau_h$. Moreover, the runtime of Algorithm 2 is much shorter in the case $h > \tau_h$ than in the case $h = \tau_h$ while it was the opposite case for Algorithm 1.

Interpretation

As expected from the theory in Chapter 5, the discretized total variation TV^{h} of the solutions to the discretized problems (6.2) indeed converges to the total variation of the optimal solution $\bar{u}$ in function space as the superlinearly coupled mesh sizes (h, τ_h) are driven to zero, while the total variation TV of the solutions to the discretized problems (6.3) does not recover the total variation of the optimal solution $\bar{u}$.

As exemplified in Figure 6.3, the total variation TV prefers axis-aligned interfaces of the level sets which differ from the round interfaces of the limit function $\bar{u}$. The total variation TV^{h} in contrast tolerates a certain degree of chattering depending on the constant c due to its averaging effect on the coarse mesh cells $\mathcal{Q}_h$. This chattering can be reduced by choosing finer mesh sizes (h, τ_h).

We were able to close the gap between the lower bound obtained by the relaxation (6.9) and the objective value of the superordinate mixed-integer problem (6.2) up to 1.73 %. We highlight that the gap between the lower bounds obtained by (6.10) and the optimal value for (6.3) cannot be closed because (6.3) approximates a different problem than (6.1) when the mesh size is driven to zero. Also in terms of running time, the computation of lower bounds for (6.2) is much more sensible than the computation of lower bounds for (6.3).

In Figure 6.6, we can observe that the discretizations with coupled mesh sizes with $h > \tau_h$ lead to results Figure 6.6a and Figure 6.6c with sharper jumps between the level sets than the results Figure 6.6b and Figure 6.6d that arise from discretizations with $h = \tau_h$. This is due to the fact that the approximation property of the discretized total variation with superlinearly coupled meshes is not reliant on the approximation by means of averaging. This shows that the use of coupled meshes can be beneficial also for cases in which averages are feasible like in ($\mathrm{P_R}$).

Numerical difficulties

We point out that the constructed optimal solution is numerically difficult to handle, which is mainly due to the steepness of the optimal adjoint state $\bar{p}$, which causes large values in $\Delta\bar{p}$ that influence the desired state $y_d = \bar{y} + \kappa\Delta\bar{p} - \bar{p}$. We have taken measures to reduce the influence by enlarging the domain Ω and by choosing small values for κ.

Our experiments were mainly restricted by the runtime that was consumed for the solution of the mixed-integer problems. One option to reduce the runtime is the choice of a small reset radius Δ_0 which reduces the number of elements in the discrete set

$$\{u \in P0^{\mathcal{T}_h} : u(x) \in \{0, 1\} \text{ for a.a. } x \in \Omega, \|u - \bar{u}^{n-1}\|_{L^1(\Omega)} \leq \Delta\}.$$

We have used such a small reset radius Δ_0 in Experiments E6 and E7 from Table 6.4, which has caused that the trust-region radius Δ has not been reduced in almost any iteration, which indicates that the reset-radius was chosen too small to exhaust the maximum descent. Although the small reset-radius results in a large number of iterations, see Table 6.7, it was necessary to solve the subproblems with an acceptable runtime.

Since a larger choice of the constant $c \geq \sqrt{2}$ for the discretization (6.2) generally leads the outer-approximation Algorithm 3 to need more iterations to solve the subproblems (6.5), see also the results in Section 6.2, we refrained from testing other choices than $\sqrt{2}$ for the instances on finer meshes in order to keep the running time of Algorithm 1 within reasonable limits.

We point out that several mixed-integer linear programs must be solved during the application of trust-region Algorithm 1 to the discretizations of (P). The solution of mixed-integer linear optimization problems is generally computationally expensive such that this has been mostly the limiting factor in our experiments. This shows the importance of the research on the fast solution of the arising mixed-integer linear optimization problems. Works that deal with the complexity and the fast solution of the discretized subproblems that arise from the application of trust-region algorithms to optimization problems like (P) are for example [78, 79, 97].

In order to obtain meaningful lower bounds, we need to choose the regularization parameters $\delta > 0$ and $\varepsilon > 0$ small enough. In our numerical experiments it has turned out that the choice of small parameters δ and ε caused that Algorithm 2 needs significantly more iterations to fulfill the termination criterion. A benefit of the regularization with the Tikhonov term $\frac{\delta}{2}\|u - I_{P0^{\tau_h}}\bar{u}\|^2_{L^2(\Omega)}$ is that already moderate choices for the values of the regularization parameters $\delta > 0$ and $\varepsilon > 0$ lead to optimal objective values of the relaxation that are close to the optimal values of the corresponding mixed-integer problems and to relatively small errors while also keeping the number of iterations small. Since the optimal solution $\bar{u}$ is generally not known in practice, this variant is generally no realistic setting but there may also be other choices $\frac{\delta}{2}\|u - u_d\|^2_{L^2(\Omega)}$ with some appropriate $u_d \in L^2(\Omega)$ that could improve the convergence.

The experiments have shown that the runtime of Algorithm 2 needs to be improved. One promising approach might be to adjust the regularization parameters δ and ε during the application of Algorithm 2. Initial improvements have already been achieved in [82], where the regularization parameter ε was halved in each iteration of a similar outer approximation algorithm to Algorithm 2 until a desired value was reached. The search for more sophisticated strategies for the adjustment of the regularization parameters gives rise to future research.

6.4 Numerical example from imaging

As a fourth numerical example, we consider the optimization problem from imaging with L^2 fidelity term that reads

$$
\begin{aligned}
&\min_{u \in L^2(\Omega)} && \frac{1}{2}\|y - y_d\|^2_{L^2(\Omega)} + \alpha \mathrm{TV}(u) \\
&\quad\text{s.t.} && -\kappa\Delta y + y = u \text{ in } \Omega, \quad \partial_n y = 0 \text{ on } \partial\Omega \\
& && u(x) \in U = \{0, \dots, 5\} \subset \mathbb{Z} \text{ for a.a. } x \in \Omega = (0,1)^2
\end{aligned}
\tag{6.14}
$$

with $\Omega = (0,1)^2$, $\alpha > 0$, and $\kappa > 0$, and where $\partial_n y$ denotes the normal derivative of y and n is the outer unit normal of Ω. The contained PDE is related to (PDE) from Section 6.3 but has homogeneous Neumann boundary conditions instead of Dirichlet boundary conditions. We denote by $S_N : L^2(\Omega) \to H^1(\Omega)$ the linear solution operator that maps $u \in L^2(\Omega)$ onto the unique solution $y \in H^1(\Omega)$ to

$$
-\kappa\Delta y + y = u \text{ in } \Omega, \quad \partial_n y = 0 \text{ on } \partial\Omega. \tag{PDE-N}
$$

We highlight that (PDE-N) has a unique solution and that $S_N : L^2(\Omega) \to H^1(\Omega)$ is injective so that $F : L^2(\Omega) \to \mathbb{R}$ defined by $F(u) \coloneqq \frac{1}{2}\|S_N u - y_d\|^2_{L^2(\Omega)}$ for $u \in L^2(\Omega)$ is strictly convex. Since $F : L^2(\Omega) \to \mathbb{R}$ is continuous, convex, and bounded from below by 0, Assumptions 1.1, 4.1 and 5.1 are fulfilled. Also, F fulfills Assumption 3.11 and $\nabla F(u) \in C(\bar{\Omega})$ for $u \in L^2(\Omega)$ because the arguments in Section 6.3 still hold for the case of homogeneous Neumann boundary conditions instead of Dirichlet boundary conditions because Theorems 1 and 3 in [58] also apply for homogeneous Neumann boundary conditions.

In (6.14), $y_d = S_N(u_d) \in H^1(\Omega)$ represents a blurred picture that is obtained as the solution to (PDE-N) for $u_d \in L^2(\Omega)$, which contains the gray scale values of the original picture rescaled to the interval $[0,5]$. The corresponding weak formulation to (PDE-N) reads: find $y \in H^1(\Omega)$ that fulfills

$$\kappa \int_\Omega \nabla y(x) \cdot \nabla v(x) \,\mathrm{d}x + \int_\Omega y(x)v(x)\,\mathrm{d}x = \int_\Omega u(x)v(x)\,\mathrm{d}x \quad \forall\, v \in H^1(\Omega).$$

6.4.1 Discretized optimization problems

Similar to Section 6.3, we will discretize the input function $u \in L^2(\Omega)$ by piecewise constant functions $P0^{\tau_h}$ on the cubic mesh $\mathcal{Q}_{\tau_h}$. The corresponding discretized PDE now reads: find $y \in P1^{\tau_h}$ that fulfills

$$(\text{PDE-N}_{\tau_h}) \quad \kappa \int_\Omega \nabla y(x) \cdot \nabla v(x) \,\mathrm{d}x + \int_\Omega y(x)v(x)\,\mathrm{d}x = \int_\Omega u(x)v(x)\,\mathrm{d}x \;\forall\, v \in CG1^{\tau_h}.$$

We denote the solution operator that maps $u \in L^2(\Omega)$ to the unique solution $y \in CG1^{\tau_h}$ to (PDE-N$_{\tau_h}$) by $S_N^{\tau_h} : L^2(\Omega) \to CG1^{\tau_h}$. We define $F^{\tau_h} : P0^{\tau_h} \to \mathbb{R}$ by $F^{\tau_h}(u) := \frac{1}{2}\|S_N^{\tau_h} u - I_{CG1^{\tau_h}} y_d\|^2_{L^2(\Omega)}$, where $I_{CG1^{\tau_h}} : L^2(\Omega) \to CG1^{\tau_h}$ denotes the interpolation operator for $CG1^{\tau_h}$. Again, we consider two discretizations of (6.14). The first is again the discretization that was introduced in Chapter 5 and reads

$$\tag{6.15} \begin{aligned} \min_{(u,V)\in P0^{\tau_h}\times\mathbb{R}} \quad & F^{\tau_h}(u) + \alpha V \\ \text{s.t.} \quad & \mathrm{TV}(u) \le cV \\ & \mathrm{TV^h}(u) \le V \\ & u(x) \in U = \{0,\dots,5\} \text{ for a.a. } x \in \Omega \end{aligned}$$

with $c \ge \sqrt{2}$. The second discretization uses again the original total variation TV and reads

$$\tag{6.16} \begin{aligned} \min_{u\in P0^{\tau_h}} \quad & F^{\tau_h}(u) + \alpha \mathrm{TV}(u) \\ \text{s.t.} \quad & u(x) \in U = \{0,\dots,5\} \text{ for a.a. } x \in \Omega. \end{aligned}$$

As in Section 6.3, we aim to apply trust-region Algorithm 1 to (6.14) or rather to its discretizations (6.15) and (6.15). The corresponding trust-region subproblems

(TR) to (6.14) reads

$$\begin{aligned}
&\min_{u\in L^2(\Omega)} && (\bar{p}_{n-1}, u - \bar{u}_{n-1})_{L^2(\Omega)} + \alpha \mathrm{TV}(u) - \alpha \mathrm{TV}(\bar{u}_{n-1}) \\
&\text{s.t.} && \|u - \bar{u}_{n-1}\|_{L^1(\Omega)} \le \Delta \\
& && u(x) \in U = \{0, \dots, 5\} \text{ for a.a. } x \in \Omega,
\end{aligned} \tag{6.17}$$

where $\bar{u}_{n-1} \in \mathrm{BV_U}(\Omega)$ is the solution most recently accepted by trust-region Algorithm 1, $\bar{y}_{n-1} = S_N(\bar{u}_{n-1}) \in H^1(\Omega)$ is the corresponding state, and $\nabla F(\bar{u}_{n-1}) = \bar{p}_{n-1} = S_N(\bar{y}_{n-1} - y_d) \in H^1(\Omega)$ is the adjoint state. The discretized trust-region subproblem corresponding to (6.15) reads

$$\begin{aligned}
&\min_{(u,V)\in P0^{\tau_h}\times\mathbb{R}} && (\bar{p}_{n-1}, u - \bar{u}_{n-1})_{L^2(\Omega)} + \alpha V - \alpha \bar{V}_{n-1} \\
&\text{s.t.} && \mathrm{TV}(u) \le cV \\
& && \mathrm{TV^h}(u) \le V \\
& && \|u - \bar{u}_{n-1}\|_{L^1(\Omega)} \le \Delta \\
& && u(x) \in U = \{0, \dots, 5\} \text{ for a.a. } x \in \Omega,
\end{aligned} \tag{6.18}$$

where $(\bar{u}_{n-1}, \bar{V}_{n-1}) \in P0^{\tau_h} \times \mathbb{R}$ is the solution most recently accepted by trust-region Algorithm 1, $\bar{y}_{n-1} = S_N^{\tau_h}(\bar{u}_{n-1}) \in CG1^{\tau_h}$ is the corresponding state, and $\nabla F^{\tau_h}(\bar{u}_{n-1}) = \bar{p}_{n-1} = S_N^{\tau_h}(\bar{y}_{n-1} - I_{CG1^{\tau_h}} y_d) \in CG1^{\tau_h}$ is the adjoint state. Similarly, the discretized trust-region subproblem corresponding to (6.16) reads

$$\begin{aligned}
&\min_{u\in P0^{\tau_h}} && (\bar{p}_{n-1}, u - \bar{u}_{n-1})_{L^2(\Omega)} + \alpha \mathrm{TV}(u) - \alpha \mathrm{TV}(\bar{u}_{n-1}) \\
&\text{s.t.} && \|u - \bar{u}_{n-1}\|_{L^1(\Omega)} \le \Delta \\
& && u(x) \in U = \{0, \dots, 5\} \text{ for a.a. } x \in \Omega,
\end{aligned} \tag{6.19}$$

where again $\bar{u}_{n-1} \in P0^{\tau_h}$ is the solution most recently accepted by trust-region Algorithm 1, $\bar{y}_{n-1} = S_N^{\tau_h}(\bar{u}_{n-1}) \in CG1^{\tau_h}$ is the corresponding state, and $\nabla F^{\tau_h}(\bar{u}_{n-1}) = \bar{p}_{n-1} = S_N^{\tau_h}(\bar{y}_{n-1} - I_{CG1^{\tau_h}} y_d) \in CG1^{\tau_h}$ is the adjoint state.

6.4.2 Experiment description

We have proved in Chapter 5 that the optimal solutions to (6.15) approximate the optimal solution to (6.14) for vanishing coupled mesh sizes (h, τ_h) such that $\frac{\tau_h}{h} \searrow 0$, while the optimal solutions to (6.16) generally do not. We examine the influence of the two different discretizations in the context of imaging by applying trust-region Algorithm 1 to them.

Table 6.10 lists the experiments in which Algorithm 1 is applied to the discretizations (6.15) and (6.16). Therein, Δ_0 denotes the reset radius for trust-region Algorithm 1, σ denotes the parameter σ for the sufficient decrease condition (3.13), *MaxIt*

Table 6.10: Parameters and settings for the application of trust-region Algorithm 1 to (6.15) and (6.16).

No.	$(\frac{1}{h}, \frac{1}{\tau_h})$	Discretization	α	κ	c	Δ_0	σ	MaxIt OA	TOL OA	TOL GRB	time GRB	Result
I1	$(32, 128)$	(6.15)	5×10^{-3}	1×10^{-3}	$\sqrt{2}$	$\frac{1}{32}$	1×10^{-4}	15	1×10^{-2}	1×10^{-3}	300 s	Figure 6.7
I2	$(128, 128)$	(6.16)	5×10^{-3}	1×10^{-3}	-	$\frac{1}{32}$	1×10^{-4}	-	-	1×10^{-3}	300 s	Figure 6.8

OA denotes the iteration maximum for outer-approximation Algorithm 3, *TOL OA* denotes the chosen tolerance for outer-approximation Algorithm 3, *TOL GRB* denotes the chosen tolerance for Gurobi, and *Time GRB* denotes the time limit for Gurobi.

Practical implementation

We implemented trust-region Algorithm 1 as described in Section 6.3.2.

6.4.3 Results

We obtained the following results from the experiments described above.

Table and figure description

In Table 6.11, we list the results that are obtained by the application of trust-region Algorithm 1 to the both discretizations (6.15) and (6.16). If we denote the solution returned by trust-region Algorithm 1 applied to (6.15) and (6.16) by (u^*, V^*) and u^*, respectively, the corresponding state by y^*, and the corresponding adjoint state by p^*, then *Obj.* represents the objective value $F^{\tau_h}(u^*) + \alpha V^*$ or $F^{\tau_h}(u^*) + \alpha \mathrm{TV}(u^*)$, *Tracking term* represents the tracking term $F^{\tau_h}(u^*)$, TV *term* represents the total variation term αV^* or $\alpha \mathrm{TV}(u^*)$, V represents the value of V^*, TV represents $\mathrm{TV}(u^*)$, TV^{h} represents $\mathrm{TV}^{\mathrm{h}}(u^*)$, *It.* represents the number of outer iterations that trust-region Algorithm 1 carried out until termination, *Term.* represents the reason why Algorithm 1 terminated, where pred means that the predicted reduction was non-positive and Δ means that the trust-region radius contracted, and *Time (h)* represents the runtime of trust region Algorithm 1 measured in hours.

In Table 6.12, we exemplarily present all intermediate values that were produced by Algorithm 1 applied to (6.15). If we denote the solution obtained by the application of Algorithm 3 to (6.18) with $\Delta = \Delta_{n,k}$ by $(\tilde{u}_{n,k}, \tilde{V}_{n,k})$, then n represents the number of the outer iteration, k represents the number of the inner iteration within outer iteration n, Δ represents the trust-region radius $\Delta_{n,k}$ of subproblem (6.18), $\|\bar{u}_{n-1} - \tilde{u}_{n,k}\|_{L^1}$ represents the value $\|\bar{u}_{n-1} - \tilde{u}_{n,k}\|_{L^1(\Omega)}$, *ared* represents the actual reduction $F^{\tau_h}(\bar{u}_{n-1}) + \alpha \bar{V}_{n-1} - F^{\tau_h}(\tilde{u}_{n,k}) - \alpha \tilde{V}_{n,k}$, *pred* represents the predicted reduction $(\nabla F^{\tau_h}(\bar{u}_{n-1}), \bar{u}_{n-1} - \tilde{u}_{n,k})_{L^2(\Omega)} + \alpha \bar{V}_{n-1} - \alpha \tilde{V}_{n,k}$, *Obj.* represents the objective value $F^{\tau_h}(\tilde{u}_{n,k}) + \alpha \tilde{V}_{n,k}$, *Tracking term* represents the tracking term $F^{\tau_h}(\tilde{u}_{n,k})$, TV *term* represents the total variation term $\alpha \tilde{V}_{n,k}$, V represents the value

Table 6.11: Results obtained by the application of trust-region Algorithm 1 to (6.15) and (6.16).

No.	$(\frac{1}{h}, \frac{1}{\tau_h})$	TV	c	Obj.	Tracking term	TV term	V	TV	TV^{h}	It.	Term.	Time (h)
I1	$(32, 128)$	TV^{h}	$\sqrt{2}$	8.298×10^{-2}	2.617×10^{-2}	5.680×10^{-2}	1.136×10^{1}	1.605×10^{1}	1.147×10^{1}	7	pred	16.7
I2	$(128, 128)$	TV	-	9.423×10^{-2}	2.564×10^{-2}	6.859×10^{-2}	1.372×10^{1}	1.372×10^{1}	1.247×10^{1}	64	pred	34.2

of $\tilde{V}_{n,k}$, TV represents $\mathrm{TV}(\tilde{u}_{n,k})$, TV^{h} represents $\mathrm{TV}^{\mathrm{h}}(\tilde{u}_{n,k})$, *It. Sub.* represents the number of outer iterations that outer-approximation Algorithm 3 carried out until termination, *Term.* represents the reason why Algorithm 3 terminates, where TOL means that the termination criterion (6.13) is fulfilled and *MaxIt* means that the iteration maximum is reached, and *Time (s)* represents the runtime of trust region Algorithm 1 measured in seconds.

In Figures 6.7 and 6.8, we plotted the original image, the blurred image, and the results obtained by the application of trust-region Algorithm 1 to the both discretizations (6.15) and (6.16).

Result description

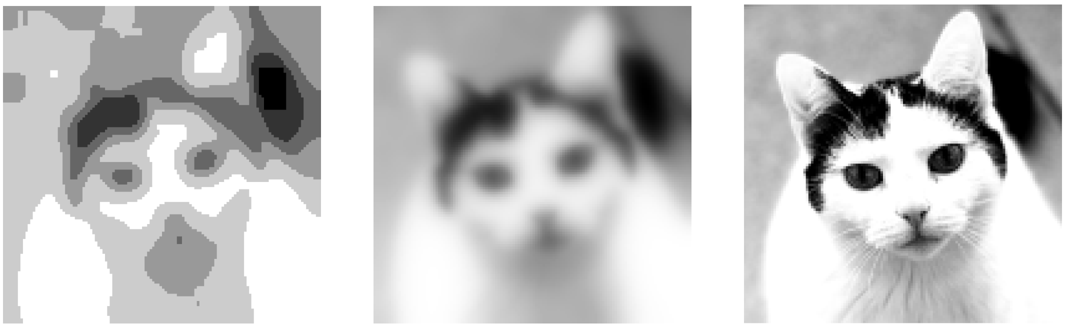
In Table 6.11, we can observe that trust-region Algorithm 1 performed less iterations for the discretization (6.15) than for the discretization (6.16). Also the running time for the application of Algorithm 1 to (6.15) is shorter than the running time for the application to (6.16), even though the average running time per outer iteration is shorter for (6.16).

In Figures 6.7a and 6.8a, we can observe that the results obtained by the application of trust-region Algorithm 1 to (6.15) and (6.16) both have large one-colored level sets. The interfaces of the result that is obtained by the application of Algorithm 1 to (6.15) in Figure 6.7a are mostly round, whereas the interfaces of the result in Figure 6.8a that is obtained by the application of Algorithm 1 to (6.16) has level sets that are mostly axis-aligned.

In Table 6.12, we can observe that the trust-region radii of the subproblems which could not be solved by Algorithm 3 within the iteration maximum up to the given tolerance have the values Δ_0 or $\frac{\Delta_0}{2}$. In addition, the number of iterations that Algorithm 3 has performed until termination generally reduces when the trust-region radius Δ is reduced. Algorithm 1 terminates in outer iteration $n = 7$ and inner iteration $k = 6$ because the predicted reduction is non-positive.

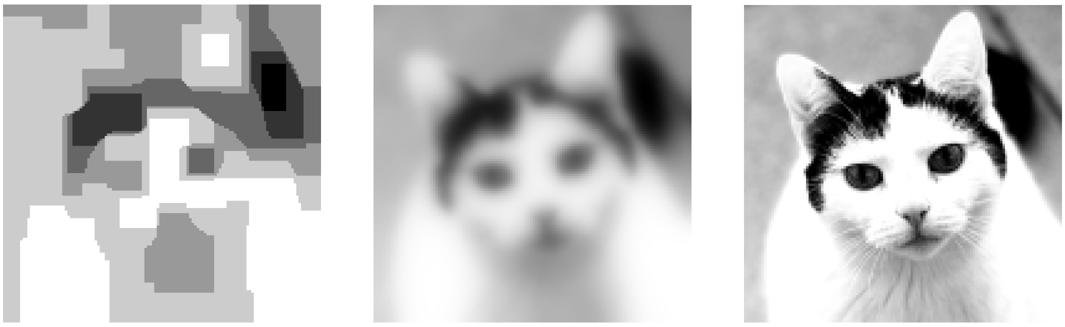
Interpretation

The result in Figure 6.7a that arises from the discretization (6.15) confirms our theoretical results in Chapter 5 in the sense that it recovers the interfaces of the original picture in Figure 6.7c while being able to restore the level sets from the blurred picture Figure 6.7b. In contrast to that, the interfaces of the level sets of



(a) Result (b) Blurred (c) Original

Figure 6.7: Results experiment I1 with TV^{h} and $(h, \tau) = (\frac{1}{32}, \frac{1}{128})$.



(a) Result (b) Blurred (c) Original

Figure 6.8: Results experiment I2 with exact TV term and $\tau_h = \frac{1}{128}$.

the result Figure 6.8a that is obtained from the anisotropic discretization (6.16) do not match the interfaces of the level sets of the original picture Figure 6.8c. This underlines the importance of the introduced discretization (P_c^h) in Chapter 5 in the context of imaging in order to recover the shapes of the original images correctly.

The course of Algorithm 1 presented in Table 6.12 is representative for the application of Algorithm 1 to the problems that are discretized following Chapter 5. We want to justify the negative predicted reduction that can occur due to our numerical implementation of trust-region Algorithm 1. For example, in outer iteration $n = 7$ and inner iteration $k = 6$, we have the predicted reduction $\mathrm{pred} = -5.586 \times 10^{-5}$. In the mentioned iteration, we have that

$$\alpha \max \left\{ \mathrm{TV}^{\mathrm{h}}(\bar{u}_{n-1}), \frac{1}{c}\mathrm{TV}(\bar{u}_{n-1}) \right\} - \alpha \bar{V}_{n-1} = \alpha(\mathrm{TV}^{\mathrm{h}}(\bar{u}_{n-1}) - \bar{V}_{n-1}) \approx 5.5 \times 10^{-4}$$

is an upper bound for the optimal value of (6.19) such that -5.5×10^{-4} is a lower bound for the predicted reduction which is consistent with the theory because $-5.5 \times 10^{-4} < \mathrm{pred} = -5.586 \times 10^{-5}$.

Numerical difficulties

The numerical experiments were mainly restricted by the runtime that was consumed to solve the mixed-integer subproblems so that we were limited to the rather coarse mesh size $(h, \tau_h) = (\frac{1}{32}, \frac{1}{128})$. Even though we were able to reduce the runtime slightly by choosing a relatively small reset radius Δ_0, this strengthens the need for proceeding research on the efficient solution of the subproblems that arise from the application of Algorithm 1 as already mentioned in Section 6.3.3.

Within our numerical experiments we made the observation that already a moderate increase in the constant c, for example $c = 3\sqrt{2}$, leads to strong chattering of the iterates. Since a larger constant c also generally increases the number of iterations that Algorithm 3 needs to solve the subproblems (6.18), see also the results in Section 6.2, we have decided to restrict ourselves to the constant $c = \sqrt{2}$.

Table 6.12: Cat experiment I1 with$(h,\tau) = (\frac{1}{32}, \frac{1}{128})$.

n	k	Δ	$\Vert\bar{u}_{n-1} - \tilde{u}_{n,k}\Vert_{L^1}$	ared	pred	Obj.	Tracking term	TV term	V	TV	TV$^{\text{h}}$	It. Sub.	Term.	Time (s)
1	1	3.125×10^{-2}	3.119×10^{-2}	1.146×10^{-3}	7.882×10^{-3}	8.570×10^{-2}	2.911×10^{-2}	5.659×10^{-2}	1.132×10^{1}	1.598×10^{1}	1.138×10^{1}	4	TOL	1183.1
2	1	3.125×10^{-2}	3.125×10^{-2}	-5.958×10^{-3}	9.935×10^{-3}	9.165×10^{-2}	3.576×10^{-2}	5.589×10^{-2}	1.118×10^{1}	1.580×10^{1}	1.127×10^{1}	11	TOL	3394.8
2	2	1.563×10^{-2}	1.563×10^{-2}	1.058×10^{-3}	5.519×10^{-3}	8.464×10^{-2}	2.878×10^{-2}	5.586×10^{-2}	1.117×10^{1}	1.580×10^{1}	1.128×10^{1}	4	TOL	1148.0
3	1	3.125×10^{-2}	3.125×10^{-2}	-2.228×10^{-3}	7.028×10^{-3}	8.687×10^{-2}	2.971×10^{-2}	5.716×10^{-2}	1.143×10^{1}	1.594×10^{1}	1.143×10^{1}	15	MaxIt	4539.5
3	2	1.563×10^{-2}	1.563×10^{-2}	-3.269×10^{-4}	3.946×10^{-3}	8.497×10^{-2}	2.868×10^{-2}	5.628×10^{-2}	1.126×10^{1}	1.591×10^{1}	1.135×10^{1}	11	TOL	3269.1
3	3	7.813×10^{-3}	7.813×10^{-3}	-3.641×10^{-5}	1.798×10^{-3}	8.468×10^{-2}	2.813×10^{-2}	5.655×10^{-2}	1.131×10^{1}	1.598×10^{1}	1.140×10^{1}	12	TOL	3366.0
3	4	3.906×10^{-3}	3.906×10^{-3}	6.637×10^{-5}	7.454×10^{-4}	8.457×10^{-2}	2.816×10^{-2}	5.641×10^{-2}	1.128×10^{1}	1.595×10^{1}	1.134×10^{1}	9	TOL	1987.7
4	1	3.125×10^{-2}	3.125×10^{-2}	-1.180×10^{-3}	7.742×10^{-3}	8.575×10^{-2}	2.911×10^{-2}	5.664×10^{-2}	1.133×10^{1}	1.601×10^{1}	1.142×10^{1}	10	TOL	3059.1
4	2	1.563×10^{-2}	1.563×10^{-2}	-4.666×10^{-4}	3.005×10^{-3}	8.504×10^{-2}	2.782×10^{-2}	5.722×10^{-2}	1.144×10^{1}	1.596×10^{1}	1.144×10^{1}	15	MaxIt	4837.7
4	3	7.813×10^{-3}	7.813×10^{-3}	3.156×10^{-4}	1.662×10^{-3}	8.426×10^{-2}	2.752×10^{-2}	5.674×10^{-2}	1.135×10^{1}	1.605×10^{1}	1.145×10^{1}	6	TOL	1940.0
5	1	3.125×10^{-2}	3.125×10^{-2}	-5.058×10^{-4}	5.760×10^{-3}	8.476×10^{-2}	2.762×10^{-2}	5.714×10^{-2}	1.143×10^{1}	1.592×10^{1}	1.143×10^{1}	15	MaxIt	4147.2
5	2	1.563×10^{-2}	1.563×10^{-2}	3.354×10^{-7}	2.447×10^{-3}	8.426×10^{-2}	2.576×10^{-2}	5.850×10^{-2}	1.170×10^{1}	1.633×10^{1}	1.170×10^{1}	15	MaxIt	4566.8
6	1	3.125×10^{-2}	3.125×10^{-2}	-1.225×10^{-4}	5.451×10^{-3}	8.438×10^{-2}	2.541×10^{-2}	5.897×10^{-2}	1.179×10^{1}	1.620×10^{1}	1.179×10^{1}	15	MaxIt	4541.4
6	2	1.563×10^{-2}	1.563×10^{-2}	1.279×10^{-3}	3.747×10^{-3}	8.298×10^{-2}	2.617×10^{-2}	5.680×10^{-2}	1.136×10^{1}	1.605×10^{1}	1.147×10^{1}	15	TOL	4420.9
7	1	3.125×10^{-2}	3.125×10^{-2}	-1.786×10^{-3}	5.295×10^{-3}	8.476×10^{-2}	2.687×10^{-2}	5.790×10^{-2}	1.158×10^{1}	1.606×10^{1}	1.158×10^{1}	15	MaxIt	4817.1
7	2	1.563×10^{-2}	1.563×10^{-2}	-1.013×10^{-3}	3.035×10^{-3}	8.399×10^{-2}	2.657×10^{-2}	5.742×10^{-2}	1.148×10^{1}	1.623×10^{1}	1.159×10^{1}	8	TOL	2399.3
7	3	7.813×10^{-3}	7.813×10^{-3}	-2.730×10^{-4}	1.235×10^{-3}	8.325×10^{-2}	2.545×10^{-2}	5.780×10^{-2}	1.156×10^{1}	1.632×10^{1}	1.167×10^{1}	9	TOL	2604.8
7	4	3.906×10^{-3}	3.906×10^{-3}	-2.029×10^{-8}	4.854×10^{-4}	8.298×10^{-2}	2.550×10^{-2}	5.748×10^{-2}	1.150×10^{1}	1.626×10^{1}	1.158×10^{1}	8	TOL	2284.3
7	5	1.953×10^{-3}	1.953×10^{-3}	-5.734×10^{-5}	8.083×10^{-5}	8.304×10^{-2}	2.574×10^{-2}	5.729×10^{-2}	1.146×10^{1}	1.620×10^{1}	1.155×10^{1}	6	TOL	1411.5
7	6	9.766×10^{-4}	9.766×10^{-4}	-7.509×10^{-5}	-5.586×10^{-5}	8.305×10^{-2}	2.599×10^{-2}	5.707×10^{-2}	1.141×10^{1}	1.614×10^{1}	1.153×10^{1}	2	TOL	189.7

Chapter 7

Conclusion and outlook

We have introduced solution techniques for integer optimization problems with total variation regularization (P) and their relaxations ($\mathrm{P_R}$) in function space and demonstrated their practicability. For the numerical solution of (P), we have introduced a novel discretization scheme (P_c^h) with superlinearly coupled meshes. Due to the restriction of the input function to integer values, the known discretizations from literature act anisotropic and are therefore not suitable for the discretization of (P) as we have demonstrated theoretically and numerically. With the novel discretization (P_c^h), however, we are able to recover the objective values and the total variation of optimal solutions to (P) with optimal solutions to (P_c^h). The constant c in the constraints of (P_c^h) has a significant impact on the results for fixed mesh sizes as we have pointed out in our numerical experiments. We have determined the exact admissible range for c in the two-dimensional case. We believe that it is worthwhile to extend the corresponding proof of the two-dimensional case to the three-dimensional case.

As the trust-region Algorithm 1 and the outer-approximation Algorithm 3 need to solve several integer optimization problems exactly, long runtimes can hardly be avoided. Nevertheless, it is an important research field to accelerate the solution of the occurring subproblems as it is done in current research like [78, 79, 97]. But also improvements of the underlying trust-region Algorithm 1 for example by domain decomposition are subject to current research [6].

The outer-approximation Algorithm 2 is applied to the regularization ($\mathrm{P}_{\delta,\varepsilon}$) of the relaxation ($\mathrm{P_R}$). We have proven convergence of the optimal solutions to ($\mathrm{P}_{\delta,\varepsilon}$) to an optimal solution to ($\mathrm{P_R}$). As future research, one can determine a convergence rate for this convergence as it is done in [82] for a related problem with total variation restrictions in the constraints. In order to improve the runtime of outer-approximation Algorithm 2, the adjustment of the regularization parameters δ and ε during the execution of Algorithm 2 is a promising improvement as even simple strategies as in [82] lead to a reduction of the number of iterations required.

Appendix A

Auxiliary results

For a matrix $M \in \mathbb{R}^{d\times d}$, we define the matrix norm

$$\|M\|_2 \coloneqq \max_{\|x\|_2=1} \|Mx\|_2.$$

We denote the identity matrix by $I \in \mathbb{R}^{d\times d}$.

Lemma A.1. *Let $M \in \mathbb{R}^{d\times d}$ fulfill $\|M - I\|_2 < 1$. Then M is invertible.*

Proof. Assume that M is not invertible, that is, there exist $x, y \in \mathbb{R}^d$ with $x \neq y$ such that $Mx = My$. We define $z \in \mathbb{R}^d$ by

$$z \coloneqq \frac{1}{\|x-y\|_2}(x-y).$$

Then $Mz = 0$ and $\|z\|_2 = 1$ and hence

$$\|M - I\|_2 = \max_{\|x\|_2=1} \|Mx - Ix\|_2 \geq \|Mz - z\|_2 = \|z\|_2 = 1$$

in contradiction to $\|M - I\|_2 < 1$. Therefore, M must be invertible. $\square$

The following two lemmas closely follow the considerations in [77].

Lemma A.2 ([77, Lem. A.1])**.** *Let $g : \mathbb{R}^d \to \mathbb{R}^d$ be differentiable and $\delta \in (0, 1)$. If there holds*

$$\|\nabla g(x) - I\|_2 \leq \delta \quad \forall\, x \in \mathbb{R}^d,$$

then the function g is a diffeomorphism.

Proof. For $y \in \mathbb{R}^d$, we define the function $G_y : \mathbb{R}^d \to \mathbb{R}^d$ by $G_y(x) \coloneqq y + (x - g(x))$ with $\nabla G_y(x) = I - \nabla g(x)$. By the mean value theorem [34, Thm. 7.2.1], there holds for all $x_1, x_2 \in \mathbb{R}^d$ that

$$\|G_y(x_1) - G_y(x_2)\|_2 \leq \delta \|x_1 - x_2\|_2$$

and since $\delta \in (0,1)$, we obtain the existence of a unique fixed point by virtue of Banach's fixed point theorem. This yields in particular, $x = G_y(x)$ if and only if $y = g(x)$. By the uniqueness of the fixed point and since $y \in \mathbb{R}^d$ was arbitrary, this yields that $g : \mathbb{R}^d \to \mathbb{R}^d$ is invertible. Moreover, by Lemma A.1, $\nabla g(x)$ is invertible for all $x \in \mathbb{R}^d$ and thus the inverse function theorem gives that g^{-1} is continuously differentiable. Hence g is a diffeomorphism. $\square$

Lemma A.3 ([77, Lem. A.2]). *Let $f_t \coloneqq I + t\psi$ for $\psi \in C_c^\infty(\Omega; \mathbb{R}^d)$. Then there is $\varepsilon > 0$ such that $g_t \coloneqq f_t^{-1} = I - t\psi \circ g_t$ for $t \in (-\varepsilon, \varepsilon)$. Moreover, the mapping $(-\varepsilon, \varepsilon) \ni t \mapsto g_t(y) \in \mathbb{R}^d$ is Lipschitz continuous for each $y \in \mathbb{R}^d$ and $\nabla g_t(y) \to I$ as $t \to 0$ uniformly for $y \in \mathbb{R}^d$.*

Proof. Let $\psi \in C_c^\infty(\Omega; \mathbb{R}^d)$. We denote the Lipschitz constant of ψ by L_ψ. Let $\varepsilon < \frac{1}{L_\psi}$. Define $T_y : \mathbb{R}^d \to \mathbb{R}^d$, $T_y(x) \coloneqq y - t\psi(x)$, for fixed $t \in (-\varepsilon, \varepsilon)$ and $y \in \mathbb{R}^d$. Then T_y is a contraction mapping since for arbitrary $x, z \in \mathbb{R}^d$ it holds that

$$\|T_y(x) - T_y(z)\|_2 = |t| \|\psi(x) - \psi(z)\|_2 \leq |t| L_\psi \|x - z\|_2.$$

By Banach's fixed point theorem, there exists a unique fixed point $\tilde{x}_y \in \mathbb{R}^d$ with $T_y(\tilde{x}_y) = \tilde{x}_y$. We define $g_t(y) \coloneqq \tilde{x}_y$ and prove that $g_t = f_t^{-1}$. It holds that $g_t(y) = T_y(\tilde{x}_y) = y - t\psi(\tilde{x}_y) = y - t\psi(g_t(y))$ for $y \in \mathbb{R}^d$ and therefore

$$\begin{aligned} \|g_t(f_t(y)) - y\|_2 &= \|f_t(y) - t\psi(g_t(f_t(y))) - y\|_2 \\ &= \|t\psi(y) - t\psi(g_t(f_t(y)))\|_2 \leq |t| L_\psi \|g_t(f_t(y)) - y\|_2. \end{aligned}$$

Since $|t| L_\psi < 1$, it must hold that $\|g_t(f_t(y)) - y\|_2 = 0$ such that $g_t = f_t^{-1}$.

We now prove the Lipschitz continuity of $g_t(y)$ in $t \in (-\varepsilon, \varepsilon)$ for fixed $y \in \mathbb{R}^d$. To this end, let $t, s \in (-\varepsilon, \varepsilon)$. Then

$$\begin{aligned} \|g_t(y) - g_s(y)\|_2 &= \|s\psi(g_s(x)) - t\psi(g_t(y))\|_2 \\ &= \|(s-t)\psi(g_s(x)) + t(\psi(g_s(y)) - \psi(g_t(y)))\|_2 \\ &\leq |s-t| \|\psi(g_s(y))\|_2 + |t| L_\psi \|g_s(y) - g_t(y)\|_2 \end{aligned}$$

and therefore

$$\|g_t(y) - g_s(y)\|_2 \leq |t-s| \frac{\|\psi(g_s(y))\|_2}{1 - |t| L_\psi} < |t-s| \frac{\|\psi(g_s(y))\|_2}{1 - \varepsilon L_\psi} \leq |t-s| C$$

with

$$C = \frac{1}{1-\varepsilon L_\psi} \max_{x \in \operatorname{supp} \psi} \|\psi(x)\|_2 < \infty$$

only depending on ψ but not on y.

In order to prove the convergence of $\nabla g_t(y)$, we want to apply Lemma A.2 to f_t. There holds

$$\|\nabla f_t(x) - I\|_2 = \|I + t\nabla\psi(x) - I\|_2 = |t| \|\nabla\psi(x)\|_2 \leq |t|M$$

with $M \coloneqq \max_{x\in S} \|\nabla\psi(x)\|_2 < \infty$ with $S \coloneqq \operatorname{supp} \|\nabla\psi(x)\|_2 \subset \Omega$ because $\psi \in C_c^\infty(\Omega;\mathbb{R}^d)$ and therefore $\nabla\psi \in C_c^\infty(\Omega;\mathbb{R}^{d\times d})$. Now we choose arbitrary $\delta \in (0,1)$ and reduce ε further if necessary to obtain $0 < \varepsilon \leq \frac{\delta}{M}$ such that

$$\|\nabla f_t(x) - I\|_2 \leq |t|M < \varepsilon M \leq \delta$$

for all $t \in (-\varepsilon, \varepsilon)$. Hence Lemma A.2 yields that f_t is a diffeomorphism for all $t \in (-\varepsilon,\varepsilon)$. In particular, there exists $(\nabla f_t(x))^{-1}$ for all $x \in \Omega$, which we write as its Neumann series $(\nabla f_t(x))^{-1} = I + \sum_{k=1}^\infty (-1)^k t^k \nabla\psi(x)^k$. By the inverse function theorem we have

$$\nabla g_t(y) = (\nabla f_t(g_t(y)))^{-1} = I + \sum_{k=1}^\infty (-1)^k t^k \nabla\psi(g_t(y))^k.$$

Since $|tM| \leq \delta < 1$, we obtain

$$\begin{aligned}
\|I - \nabla g_t(y)\|_2 &= \left\| \sum_{k=1}^\infty (-1)^k t^k \nabla\psi(g_t(y))^k \right\|_2 \\
&\leq \sum_{k=1}^\infty |t|^k \|\nabla\psi(g_t(y))\|_2^k \leq \sum_{k=1}^\infty |t|^k M^k = \frac{|t|M}{1-|t|M} \to 0
\end{aligned}$$

as $|t| \to 0$ uniformly for all $y \in \mathbb{R}^d$. □

Let $C_c^1(\Omega;\mathbb{R}^d) \subset H \subset H_0(\operatorname{div};\Omega)$ be defined as in Section 4.2.

Lemma A.4. *Let $\{\phi_{k+1}\}_{k\in\mathbb{N}} \subset H$ be a sequence of maximizers of (Q_k) produced by Algorithm 2. There exists $\bar\phi \in H$ and a subsequence $\{\phi_{k_\ell+1}\}_{\ell\in\mathbb{N}}$ such that $\phi_{k_\ell+1} \rightharpoonup \bar\phi$ and $\operatorname{div}\phi_{k_\ell+1} \to \operatorname{div}\bar\phi$ as $\ell \to \infty$.*

Proof. By the optimality of ϕ_{k+1} for (Q_k) and the feasibility of $\phi \equiv 0$ for (Q_k), there holds

$$-\frac{\varepsilon}{2} a[\phi_{k+1},\phi_{k+1}] + \int_\Omega u_k(x) \operatorname{div}\phi_{k+1}(x)\,\mathrm{d}x \geq 0$$

for all $k \in \mathbb{N}$. This gives

$$\begin{aligned}
\frac{\varepsilon}{2} a[\phi_{k+1}, \phi_{k+1}] &\leq \int_\Omega u_k(x) \operatorname{div} \phi_{k+1}(x) \,\mathrm{d}x \\
&\leq \|u_k\|_{L^2(\Omega)} \|\operatorname{div} \phi_{k+1}\|_{L^2(\Omega)} \\
&\leq \|u_k\|_{L^2(\Omega)} \|\phi_{k+1}\|_{H(\operatorname{div};\Omega)} \\
&\leq C_H \|u_k\|_{L^2(\Omega)} \|\phi_{k+1}\|_H
\end{aligned}$$

for all $k \in \mathbb{N}$. Together with the coercivity of a, there holds

$$\begin{aligned}
\beta \frac{\varepsilon}{2} \|\phi_{k+1}\|_H^2 &\leq \frac{\varepsilon}{2} a[\phi_{k+1}, \phi_{k+1}] \\
&\leq C_H \|u_k\|_{L^2(\Omega)} \|\phi_{k+1}\|_H,
\end{aligned}$$

which gives

$$\|\phi_{k+1}\|_H \leq \frac{2C_H}{\beta\varepsilon} \|u_k\|_{L^2(\Omega)}.$$

Together, we obtain

$$\|\operatorname{div} \phi_{k+1}\|_{L^2(\Omega)} \leq \|\phi_{k+1}\|_{H(\operatorname{div};\Omega)} \leq C_H \|\phi_{k+1}\|_H \leq \frac{2C_H^2}{\beta\varepsilon} \|u_k\|_{L^2(\Omega)}.$$

Because u_k is optimal for (P_k), there holds $\underline{\nu} \leq u_k(x) \leq \overline{\nu}$ for almost all $x \in \Omega$ such that $\|u_k\|_{L^2(\Omega)}$ is bounded. This yields the boundedness of $\|\operatorname{div} \phi_{k+1}\|_{L^2(\Omega)}$. Since ϕ_{k+1} is feasible for (Q_k), there holds $\|\phi_{k+1}\|_{L^\infty(\Omega;\mathbb{R}^d)} \leq 1$ for all $k \in \mathbb{N}$. Since both sequences $\{\phi_{k+1}\}_{k\in\mathbb{N}}$ and $\{\operatorname{div} \phi_{k+1}\}_{k\in\mathbb{N}}$ are bounded in $L^2(\Omega)$, we can restrict them to subsequences $\{\phi_{k_\ell+1}\}_{\ell\in\mathbb{N}}$ and $\{\operatorname{div} \phi_{k_\ell+1}\}_{\ell\in\mathbb{N}}$ such that both converge weakly in $L^2(\Omega)$, that is, $\phi_{k_\ell+1} \rightharpoonup \bar{\phi}$ and $\operatorname{div} \phi_{k_\ell+1} \rightharpoonup d$ for some functions $\bar{\phi} \in L^2(\Omega; \mathbb{R}^d)$ and $d \in L^2(\Omega)$. We need to prove $\operatorname{div} \bar{\phi} = d$. To this end, let $w \in H_0^1(\Omega; \mathbb{R}^d)$. There holds

$$\begin{aligned}
\langle d - \operatorname{div} \bar{\phi}, w \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} &= (d, w)_{L^2(\Omega)} + (\bar{\phi}, \nabla w)_{L^2(\Omega)} \\
&= \lim_{\ell\to\infty} (\operatorname{div} \phi_{k_\ell+1}, w)_{L^2(\Omega)} + (\phi_{k_\ell+1}, \nabla w)_{L^2(\Omega)} \\
&= \lim_{\ell\to\infty} (\operatorname{div} \phi_{k_\ell+1}, w)_{L^2(\Omega)} - (\operatorname{div} \phi_{k_\ell+1}, w)_{L^2(\Omega)} = 0.
\end{aligned}$$

Hence, there holds $\operatorname{div} \bar{\phi} = d$. In total, we have $\phi_{k_\ell+1} \rightharpoonup \bar{\phi}$ and $\operatorname{div} \phi_{k_\ell+1} \rightharpoonup \operatorname{div} \bar{\phi}$ in $L^2(\Omega)$ as $\ell \to \infty$. $\square$

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