TESTING FOR STRUCTURAL CHANGE IN THE PRESENCE OF LONG MEMORY[1]

by

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Version June 2000

Abstract

We derive the limiting null distributions of the standard and OLS–based CUSUM-tests for structural change of the coefficients of a linear regression model in the context of long memory disturbances. We show that both tests behave fundamentally different in a long memory environment, as compared to short memory, and that long memory is easily mistaken for structural change when standard critical values are employed.

1 Introduction and Summary

It is by now well known that long memory and structural change are easily confused (Lobato and Sawin 1997, Engle and Smith 1999, Granger and Hyung 1999, Diebold and Inoue 1999 and many others). Therefore it is of interest to know about both the stochastic properties of procedures for detecting and measuring long memory when there is only structural change, and of the performance of tests for structural change when there is only long memory.

[1]Research supported by Deutsche Forschungsgemeinschaft; we are grateful to A. Zeileis for expert computational assistance.

While the former problem has attracted considerable attention, there has been rather little work on the latter (Hidalgo and Robinson 1996, Wright 1998). Below we consider the behaviour of the standard and the OLS-based CUSUM-tests, whose limiting distributions are well understood in the context of various regressor-sequences and iid- or short memory disturbances (Krämer et al. 1988, Ploberger and Krämer 1992, 1996). As shown by Wright (1998) for the OLS-based CUSUM-test and the special case of polynomial regressors, these limiting distributions are not robust to departures from short memory - in fact, the OLS-based CUSUM-test has an asymptotic size of unity. The present paper allows for more general regressor sequences also covers the conventional CUSUM-test based on recursive residuals as well. We show that Wright's results concerning the behaviour under H_0 essentially go through with more general regressors, and that similar results hold for the standard CUSUM-test. This is a rather negative result which confirms related theorems from the structural-change-mistaken-for-long-memory-literature: Similar to structural change being mistaken for long memory, long memory is likewise easily mistaken for structural change, and it is remains an open problem to efficiently discriminate between the two[2].

2 Two unpleasant theorems

We consider the standard linear regression model

$$y_t = \beta' x_t + \varepsilon_t, \quad (t = 1, \ldots, T) \tag{1}$$

with nonstochastic, fixed regressors x_t and stationary mean zero disturbances ε_t. We assume that

$$\frac{1}{T}\sum_{t=1}^{T} x_t \;\; \rightarrow \;\; c < \infty \quad \text{and} \tag{2}$$

[2] There do exist solutions for some special cases, such as Künsch's (1986) procedure to discriminate between long memory and monotonic trends, but a general treatment of this problem is still missing.

$$\frac{1}{T}\sum_{t=1}^{T} x_t x_t' \quad \rightarrow \quad Q \text{ (finite, nonsingular).} \tag{3}$$

These are standard assumptions in linear regression large sample asymptotics; they exclude trending data, which require separate treatment and proofs which differ from the ones below.

We are concerned with testing the model (1) against the alternative of unspecified structural change in the regression coefficients β. We consider first the OLS–based CUSUM–test, as proposed by Ploberger and Krämer (1992). This test rejects the null hypothesis of no structural change for large values of

$$TS \quad := \quad \sup_{0<\lambda<1} |C_T(\lambda)|, \quad \text{where} \tag{4}$$

$$C_T(\lambda) \quad := \quad T^{-\frac{1}{2}}\hat{\sigma}_\varepsilon^{-1}\sum_{t=1}^{[T\lambda]} e_t, \tag{5}$$

and where $e_t := y_t - x_t'\hat{\beta}$ are the OLS–residuals from (1).

The limiting null distribution of TS is well known for white noise and short memory disturbances. Our first theorem extends these results to stationary long memory disturbances, where the ε_t follow a stationary ARFIMA(p,d,q)–process:

$$E(\varepsilon_t \varepsilon_{t-k}) = L(k)k^{-d}, \tag{6}$$

$L(k)$ slowly varying, $0 < d < 1/2$.

Theorem 1 *In the regression model (1), with disturbances as in (6) we have*

$$T^{-d}C_T(\lambda) \rightarrow B_d(\lambda) - c'Q^{-1}\xi(\lambda), \tag{7}$$

where $B_d(\lambda)$ is fractional Brownian Motion with self-similarity parameter d and $\xi(\lambda) \sim N(0, \lambda\sigma_\varepsilon^2 Q)$.

PROOF: We have

$$
\begin{aligned}
C_T(\lambda) &= T^{-\frac{1}{2}}\hat{\sigma}_\varepsilon^{-1}\left\{\sum_{t=1}^{[T\lambda]}\varepsilon_t - \sum_{t=1}^{[T\lambda]} x_t'\left(\sum_{t=1}^{T} x_t x_t'\right)^{-1}\sum_{t=1}^{T} x_t\varepsilon_t\right\}, \text{ so} \qquad (8)\\
T^{-d}C_T(\lambda) &= \Bigg\{T^{-d+\frac{1}{2}}\, z_{[T\lambda]}\\
&\quad -T^{-d+\frac{1}{2}}\sum_{t=1}^{[T\lambda]} x_t'\left(\sum_{1}^{T} x_t x_t'\right)^{-1}\sum_{1}^{T} x_t\varepsilon_t\Bigg\}/\hat{\sigma}_\varepsilon, \qquad (9)
\end{aligned}
$$

where $z_t = z_{t-1} + \varepsilon_t$, $z_0 = 0$. In view of

$$
\begin{aligned}
T^{-d-\frac{1}{2}}\, z_{[T\lambda]} &\to \sigma_\varepsilon B(\lambda) \quad \text{(see e.g. Marmol 1995)}, \qquad (10)\\
\frac{1}{T}\sum_{t=1}^{[T\lambda]} x_t &\to \lambda c, \qquad (11)\\
\left(\frac{1}{T}\sum_{t=1}^{[T\lambda]} x_t x_t'\right)^{-1} &\to \lambda^{-1}Q^{-1}, \qquad (12)\\
T^{-d-\frac{1}{2}}\sum_{t=1}^{[T\lambda]} x_t\varepsilon_t &\to \xi(\lambda) \quad \text{(see Giraitis and Taqqu 1998)}, \qquad (13)
\end{aligned}
$$

and

$$
\hat{\sigma}_\varepsilon^2 = \sum_{t=1}^{T}\frac{e_t^2}{T} = \sum_{t=1}^{T}\frac{\varepsilon^2}{T} + o_P(1) \to \sigma_\varepsilon^2
$$

the limiting relationship (7) follows. □

¿From (7), it is immediately seen that $TS \xrightarrow{P} \infty$ under H_0, so the OLS-based CUSUM–test is extremely non–robust to long–memory disturbances, in the sense that long memory is easily mistaken for structural change when conventional critical values are employed.

Next we consider the standard CUSUM-test based on recursive residuals

$$
\begin{aligned}
\tilde{e}_t &= \frac{y_t - x_t'\hat{\beta}^{(t-1)}}{f_t}, \quad \hat{\beta}^{(t-1)} = \left(X^{(t-1)'}X^{(t-1)}\right)^{-1}X^{(t-1)'}y^{(t-1)} \qquad (14)\\
f_t &= \left(1 + x_t'(X^{(t-1)'}X^{(t-1)})^{-1}x_t\right)^{\frac{1}{2}} \ (t = K+1,\ldots,T), \qquad (15)
\end{aligned}
$$

where the superscript $t-1$ means that only observations $1,\ldots,t-1$ are used. It rejects for large values of

$$S_T = \sup_{0<\lambda<1} W_T(\lambda)/(1+2\lambda). \tag{16}$$

where

$$W_T(\lambda) := T^{-\frac{1}{2}}\hat{\sigma}_\varepsilon^{-1} \sum_{t=K+1}^{[T\lambda]} \tilde{e}_t. \tag{17}$$

Theorem 2 *In the regression model (1), with disturbances as in (6) we have*

$$T^{-d} W_T(\lambda) \to B_d(\lambda), \tag{18}$$

where again $B_d(\lambda)$ is fractional Brownian Motion with self-similarity parameter d.

PROOF: Following Krämer et al. (1988), we write $W_T(\lambda)$ as

$$W_T(\lambda) = \frac{1}{\sqrt{T}} \sum_{t=K+1}^{[T\lambda]} \varepsilon_t - \sum_{t=K+1}^{[T\lambda]} \left(\hat{\beta}^{(t-1)} - \beta\right)' x_t. \tag{19}$$

Let $Q_j := \frac{1}{T}\sum_{i=1}^{j} x_i x_i'$. First we show that

$$\max_{K\le t\le T} || \left(\hat{\beta}^{(t)} - \beta\right) - \sum_{j=K}^{t} [(y_j - x_j'\beta)x_j] Q_j^{-1} || = o_p\left(T^{d+\frac{1}{2}}\right)(\ln\ \ln\ T)^{\frac{1}{2}}. \tag{20}$$

Let $S_t := \sum_{j=1}^{t}(y_j - x_j'\beta)x_j$. By the law of the iterated logarithm for the sums of long memory Gaussian random variables we have for some slowly varying function $L(T)$

$$\max_{1\le t\le T} \frac{S_t}{\left[\frac{2}{(d+\frac{1}{2})2(d+\frac{1}{2})-1} T^{2(d+\frac{1}{2})} L(T)\ \ln\ \ln(T)\right]^{\frac{1}{2}}} = O_p(1), \tag{21}$$

so (20) follows directly from Lemma 3.1 of Jureckova and Sen (1984).

Combining (19) and (20) gives

$$\begin{aligned}
&\frac{1}{\sqrt{T}}\sum_{t=K+1}^{[T\lambda]}(y_t - x_t'\hat{\beta}^{(t-1)}) \\
= &\quad \frac{1}{\sqrt{T}}\sum_{t=1}^{[T\lambda]}\sum_{j=1}^{t} c_{ij}(y_j - x_j'\beta) + o(T^{(d+\frac{1}{2})}\ln\ \ln\ T)^{\frac{1}{2}},
\end{aligned} \tag{22}$$

where

$$c_{ij} = \begin{cases} -x_j' Q_{(i-1)}^{-1} x_i & i > j \\ 1 & i = j \\ 0 & i < j \end{cases} \tag{23}$$

(see also Sibbertsen, 2000). In view of a result by Sen (1984) that

$$\sum_{j \leq i} c_{ij}^2 = 1 + 0(\frac{1}{i}) \tag{24}$$

and theorem 5.1 of Taqqu (1975), the theorem now follows from (22). □

Theorem 2 shows that the null distribution of the standard CUSUM-test tends to infinity as well, so the standard CUSUM-test has likewise an asymptotic size of unity.

3 Some finite sample Monte Carlo evidence

Figure 1 below gives the empirical rejection rates, using 1000 runs and standard critical values form the iid-disturbance case, for the OLS-based CUSUM-test. When the disturbances are in fact ARFIMA(0,d,0). It confirmes our theoretical results: rejection rates increase with d and sample size, and produce misleading evidence even for small d and T.

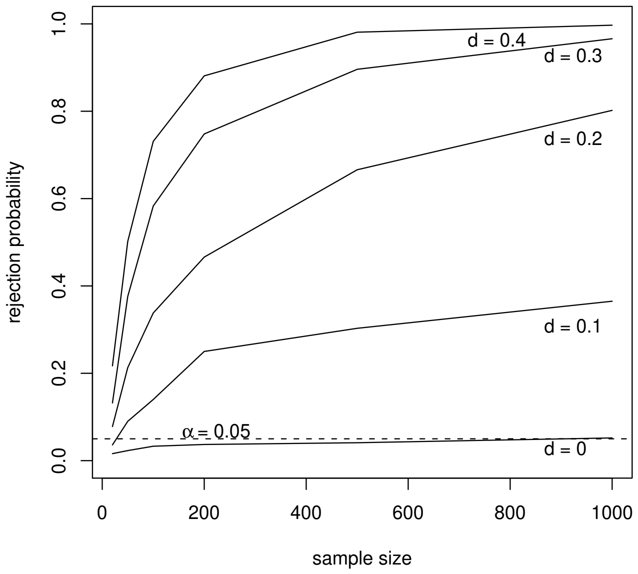

Figure 1: Empirical rejection probability of OLS-based CUSUM-test

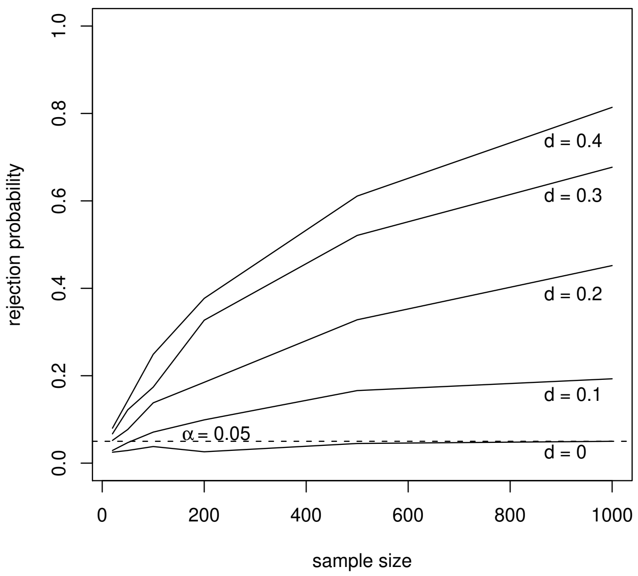

Figure 2: Empirical rejection probability of standard CUSUM-test

Figure 2 gives the corresponding empirical rejection rates for the standard CUSUM-test. Not surprisingly, the empirical size is not as far off the mark as for the OLS-based CUSUM-test, but the test is misleading here as well.

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