# The robustness of the F-test to spatial autocorrelation among regression disturbances[1]

by

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**Abstract**

It is shown that the null distribution of the F-test in a linear regression is rather non-robust to spatial autocorrelation among the regression disturbances. In particular, the true size of the test tends to either zero or unity when the spatial autocorrelation coefficient approaches the boundary of the parameter space.

Key words: F-test, size, spatial autocorrelation.

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## 1 Introduction and summary

The null distribution of the F-test under nonspherical errors has concerned applied statisticians and econometricians for many decades. Let

$$y = X\beta + u = X^{(1)}\beta^{(1)} + X^{(2)}\beta^{(2)} + u \tag{1}$$

be the model under test, where $y$ and $u$ are $T \times 1$, $X$ is $T \times K$ and nonstochastic of rank $K < T$, $\beta$ is $K \times 1$, and the disturbance vector $u$ is multivariate normal with mean zero and (possibly) nonscalar covariance matrix $V$. The design matrix is partioned into $X^{(1)}(T \times q)$ and $X^{(2)}(T \times (K-q))$ and the null hypothesis to be tested is $Ho : \beta^{(1)} = b^{(1)}$.

The standard F-test rejects for large values of

$$F = \frac{(\tilde{u}'\tilde{u} - \hat{u}'\hat{u})/q}{\hat{u}'\hat{u}/(T-K)}, \tag{2}$$

where $\hat{u} = y - X\hat{\beta}$, $\hat{\beta} = (X'X)^{-1}X'y$, $\tilde{u} = y - X^{(1)}b^{(1)} - X^{(2)}\tilde{\beta}^{(2)}$, $\tilde{\beta}^{(2)} = (X^{(2)\prime}X^{(2)})^{-1}X^{(2)\prime}(y - X^{(1)}b^{(1)})$. It has a central F-distribution with $q$ and $T-K$ degrees of freedom, given $H_0$ and $V = \sigma^2 I$, and the problem to be studied here is the robustness of this null distribution to deviations from $V = \sigma^2 I$.

So far, this problem has been addressed mainly for given specific forms of $V$, with various bounds for the size of the test being derived as the design matrix $X$ is allowed to vary across all $T \times K$-matrices of rank $K$ (Vinod 1976, Kiviet 1980). Below we take a different approach, following Krämer (1989) and Krämer et al. (1990), by fixing $X$ and letting $V$ vary across a certain range of disturbance covariance matrices. In particular, we allow the disturbance vector $u$ to be generated by the spatial autoregressive scheme

$$u = \rho W u + \varepsilon, \tag{3}$$

where $\varepsilon$ is a $T \times 1$ normal random vector with mean zero and scalar covariance matrix $\sigma_\varepsilon^2 I$, and $W$ is some known $T \times T$-matrix of nonnegative spatial weights with $w_{ii} = 0$ $(i = 1, \ldots, T)$. Such patterns of dependence are often entertained when the objects under study are positioned in some "space", whether geographical or sociological (in some social network, say) and account for spillovers from one unit to its neighbors, whichever way "neighborhood" may be defined. They date back to Whittle (1954) and has become quite popular in econometric recently. See Anselin and Florax (1995), Kelejian and Prucha (2001) or Anselin (2001) for a convenient survey of this literature.

The coefficient $\rho$ in (3) measures the degree of correlation, which can be both positive and negative. Below we focus on the empirically more relevant case of positive disturbance correlation, where

$$0 \leq \rho < \frac{1}{\lambda_{max}}$$

and where $\lambda_{max}$ is the Frobenius-root of $W$ (i.e. the unique positive real eigenvalue such that $\lambda_{max} \geq |\lambda_i|$ for arbitrary eigenvalues $\lambda_i$). The disturbances are then given by

$$u = (I - \rho W)^{-1}\varepsilon, \tag{4}$$

so $V := Cov(u) = \sigma_\varepsilon^2 \left[(I - \rho W)(I - \rho W)'\right]^{-1}$ and $V = \sigma_\varepsilon^2 I$ whenever $\rho = 0$.

Below we consider the null distribution of the F-test for $\rho \neq 0$. This is shown to be extremely non-robust, with the size of the test tending to either zero or unity as $\rho \to 1/\lambda_{max}$. The same limits are also obtained for the power of the test. Which of these limits obtains is easily seen from $X$ and $W$, which are both observed and known. Therefore, our result provides an easy guide to the interpretation of both a significant and insignificant F-test when there is possible spatial correlation among the regression disturbances: If $H_0$ is rejected, and $X$ and $W$ are such that the size of the test tends to unity, an error of the first kind has to be suspected. And if $H_0$ is not rejected, and $X$ and $W$ are such that the size and the power of the test tends to zero, one has to beware of an error of the second kind.

## 2 The null distribution of the F-test as autocorrelation increases

We first rewrite the test statistic (2) as

$$F = \frac{u'(M^{(2)} - M)u/q}{u'Mu/(T-K)}, \tag{5}$$

where $M = I - X(X'X)^{-1}X'$ and $M^{(2)} = I - X^{(2)}(X^{(2)\prime}X^{(2)})^{-1}X^{(2)\prime}$. Let $F^{\alpha}_{q,T-K}$ be the $(1-\alpha)$ quantile of the central F-distribution with $q$ and $T-K$ degrees of freedom, respectively, where $\alpha$ is the nominal size of the test. Then

$$\begin{aligned}
P(F \geq F^{\alpha}_{q,T-K}) &= P(u'(M^{(2)} - M)u - \frac{q}{T-K}F^{\alpha}_{q,T-K}u'Mu \geq 0) \\
&= P(u'(M^{(2)} - dM)u \geq 0) \\
&\quad (\text{where } d = 1 + \frac{q}{T-K}F^{\alpha}_{q,T-K}) \\
&= P(\varepsilon' V^{1/2}(M^{(2)} - dM)V^{1/2}\varepsilon \geq 0) \\
&\quad (\text{where } \varepsilon = V^{-1/2}u \sim N(0, I)) \\
&= P(\sum_{i=1}^{T} \lambda_i \xi_i^2 \geq 0) \\
&= P((1-\rho\lambda_{max})^2 \sum_{i=1}^{T} \lambda_i \xi_i^2 \geq 0), 
\end{aligned} \tag{6}$$

where the $\xi_i^2$ are iid $\chi^2_{(1)}$ and the $\lambda_i$ are the eigenvalues of $V^{1/2}(M^{(2)} - dM)V^{1/2}$, and therefore also of $V(M^{(2)} - dM)$.

The limiting rejection probability as $\rho \to 1/\lambda_{max}$ depends upon the limiting behavior of $(1-\rho\lambda_{max})^2 V$. Let

$$W = \sum_{i=1}^{T} \lambda_i \omega_i {\omega_i}' \tag{7}$$

be the spectral decomposition of $W$, with the eigenvalues $\lambda_i$ in increasing order. Then

$$V = \sum_{i=1}^{T} \frac{1}{(1-\rho\lambda_i)^2} \omega_i {\omega_i}' \tag{8}$$

is the spectral decomposition of $V$, and

$$lim_{\rho\to 1/\lambda_{max}}(1-\rho\lambda_i)^2 V = \omega_T'\omega_T, \tag{9}$$

a matrix of rank 1. Therefore, all limiting eigenvalues of $(1-\rho\lambda_{max})^2 V(M^{(2)} - dM)$ are zero except one, which is given by

$$tr(\omega_T{\omega_T}'(M^{(2)} - dM)) = {\omega_T}'(M^{(2)} - dM)\omega_T. \tag{10}$$

If ${\omega_T}'(M^{(2)} - dM)\omega_T$ is positive, the rejection probability tends to one. If ${\omega_T}'(M^{(2)} - dM)\omega_T$ is negative, the rejection probability tends to zero.

As ${\omega_T}'(M^{(2)} - dM)\omega_T$ is known, it is easy to determine in practice which of these cases obtains. For illustration, figure 1 shows both an example where rejection probability tends to one, and an example where the rejection probability tends to zero. The weight matrix is $25 \times 25$ and is derived from a regular $5 \times 5$ lattice using the queen criterion, which assigns a weight of one to all cells immediately surrounding a given cell, and zero otherwise. The case where the rejection probability tends to zero corresponds to a $25 \times 2$ $X$-matrix with a first column of ones, and a second column given by $(1, 2, 3, \ldots 25)'$, where we test wether the coefficient of the second regressor is zero. The case where the rejection probability tends to one corresponds to an $X$-matrix where the second column was generated as $nid(0, 1)$ variables.

(figure 1 here)

The figure shows that convergence to zero of the rejection probability need not be monotone and that an $X$-matrix which induces a limiting rejection probability of zero might, for certain regions of the parameter space, engender higher rejection probabilities than $X$-matrices where the rejection probability eventually tends to one.

Whether a limit of the rejection probability of zero or one obtains depends on the interplay of the design matrix $X$, the weight matrix $W$, and the nominal size of the test. For $T = 25$, a nominal size of 5%, a weight matrix defined by the queen criterion, and an $X$ matrix given by a first column of ones, and a second column of $nid(0,1)$ variables (whose significance is to be tested), a Monte Carlo experiment was performed with 1000 independent runs. We obtained a limiting rejection probability of one in about 10% of all cases. Ceteris paribus the incidence of this irregular behavior diminishes as sample size increases, but it is possible to find examples where the rejection probability tends to one also for larger samples and for different types of spatial weights.

# 3 Spatial autocorrelation and the power of the test

Let $g := X^{(1)}\beta^{(1)} - X^{(1)}b^{(1)}$. Under the alternative, $g \neq 0$, and the expression (5) becomes

$$F = \frac{[u'(M^{(2)} - M)u + 2g'M^{(2)}u + g'M^{(2)}g]/q}{u'Mu(T-K)}, \tag{11}$$

with rejection occurring if and only if

$$\begin{aligned} & (1-\rho\lambda_{max})^2 u'[M^{(2)} - (1 + \frac{q}{T-K}F^{\alpha}_{q,T-k})M]u \\ + \quad & (1-\rho\lambda_{max})^2[2g'M^{(2)}u] \\ + \quad & (1-\rho\lambda_{max})^2 g'M^{(2)}g \geq 0. \end{aligned} \tag{12}$$

Since the last two terms in expression (12) are easily seen to tend to zero as $\rho \rightarrow 1/\lambda_{max}$, the power of the test has the same limiting behavior as the size.

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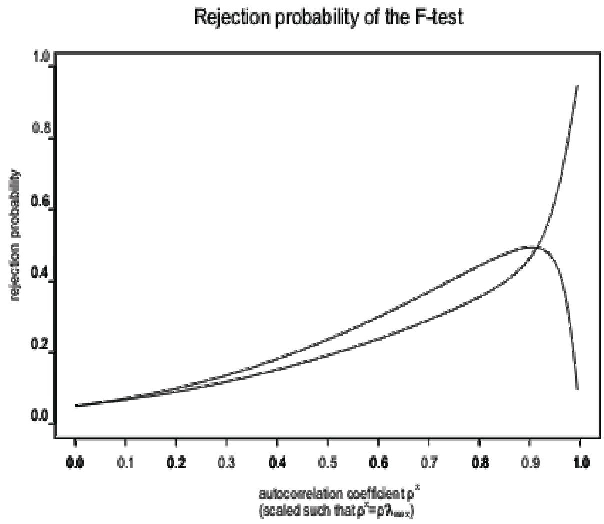


Figure 1: Rejection probability of the F-Test