Spin(7)-manifolds of cohomogeneity one

DISSERTATION

zur Erlangung des Grades
eines Doktors der Naturwissenschaften

Der Fakultät für Mathematik
der Technischen Universität Dortmund
vorgelegt von

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Dortmund, Juni 2008

Acknowledgments

In the first place I am grateful to my thesis advisor Prof. Dr. Lorenz Schwachhöfer for providing me this interesting topic as well as for the time he spent on discussions and his valuable suggestions for improvement.

I also want to thank Dr. Yaroslav Bazaikin, Prof. Dr. Jost-Hinrich Eschenburg, Dr. Mario Listing, M.Sc. Eugene Malkovich, Dr. Martin Panák, Prof. Dr. Uwe Semmelmann, Prof. Dr. Wilderich Tuschmann, and Prof. Dr. Wolfgang Ziller for helpful discussions and advice.

Moreover, I would like to gratefully acknowledge my colleague Dr. Frank Klinker for debating many mathematical and other questions during my research on this thesis. I am also grateful to the other staff members of the Lehrstuhl VII (Differentialgeometrie), including Prof. em. Dr. Rolf Walter, Dr. Günter Skoruppa, and Sabine Willrich.

Furthermore, I want to express my thanks to my parents and sister as well as to Melanie Kramer for her love and support during the last years.

Finally, it is a pleasure to thank all the other people who have helped and encouraged me while I was working on my dissertation project.

Contents

Chapter 1

Introduction

The subject of this thesis are cohomogeneity-one metrics whose holonomy is contained in Spin(7). Our aim is to find new examples and to prove a partial classification result. The motivation for this work is the paper "Cohomogeneity-one G_2-structures" of Richard Cleyton and Andrew Swann [20]. In that paper the authors construct G_2-structures with and without torsion which admit a cohomogeneity-one action. We apply the methods which were developed by Cleyton and Swann to the Spin(7)-case. It turns out that our case is more intricate than the G_2-case. We only consider Spin(7)-structures without torsion, since this task already is sufficiently extensive.

We shortly recall what Spin(7)-structures are and why they are interesting for mathematicians and physicists. Let M be an eight-dimensional manifold. A Spin(7)-structure on M is a reduction of the coframe bundle to a principal bundle with structure group Spin(7). Equivalently, a Spin(7)-structure is determined by a four-form Ω with the property that at every $p \in M$ there is a basis of T_pM such that Ω can be identified via that basis with a special four-form on $\mathbb{R}^8$. We call a pair of an eight-dimensional manifold and a Spin(7)-structure a Spin(7)-manifold. On any Spin(7)-manifold, there exists a canonical orientation, which is induced by $\Omega \wedge \Omega$, and a canonical metric g, which depends non-linearly on Ω. With respect to g and the volume form $\Omega \wedge \Omega$, the four-form Ω is self-dual. A Spin(7)-structure is called parallel or torsion-free if $\nabla^g \Omega = 0$ or equivalently $d\Omega = 0$. In this situation, g is Ricci-flat and the holonomy of the Levi-Civita connection is a subgroup of Spin(7). The most interesting case is where the holonomy is all of Spin(7). If this is the case, then there exists up to constant multiples a unique parallel spinor on M. This fact makes manifolds with holonomy Spin(7) interesting for physicists. In recent years, manifolds with holonomy G_2 and Spin(7) have been studied in the context of M-theory (see Atiyah, Witten [3] and Gukov, Sparks [39]).

The equation $d\Omega = 0$ should be considered as a non-linear partial differential equation. Although the operator d is linear, the condition that Ω determines a Spin(7)-structure is a non-linear restriction. For this reason, it is difficult to construct examples of metrics with holonomy Spin(7). That there are indeed local metrics of that kind has first been proven by Bryant [14]. The next breakthrough was the construction of complete metrics with holonomy G_2 and Spin(7) by Bryant and Salamon [15]. The authors give an explicit description of their examples and it turns out that they are of cohomogeneity one. In particular, the metric with holonomy Spin(7) has the sphere S^7 as principal orbit. The first compact manifolds with exceptional holonomy have been constructed by Joyce [47]. Since the techniques of Bryant

[14] and Joyce [47] yield no explicit description of their metrics, it would be nice to have further explicit examples, apart from those which were constructed in [15].

The problem of finding new examples can be simplified by assuming that M admits an action of a Lie group G which preserves Ω such that the principal orbits of that action have codimension one. If this is the case, the G-action is called a cohomogeneity-one action and (M, Ω) is called a Spin(7)-manifold of cohomogeneity one. In the literature, there are many examples of such manifolds. Calabi [18] has constructed a complete metric with holonomy $Sp(2)$ on $T^*\mathbb{CP}^2$. This metric is left invariant by a cohomogeneity-one action of $SU(3)$ whose principal orbit is the exceptional Aloff-Wallach space $N^{1,1}$ (see Cvetič, Gibbons, Lü, Pope [22] and Kanno, Yasui [49]). Explicit cohomogeneity-one metrics with holonomy $SU(4)$ can be found in Cvetič et al. [21], [25] and in Herzog, Klebanov [41]. The work of these authors is based on earlier papers by Berard-Bergery [8], Page, Pope [58], and Stenzel [64]. Cohomogeneity-one metrics with holonomy Spin(7) have been described in a series of papers by Cvetič et al. [21], [22], [23], [24], [25], [26], [27] and by Kanno, Yasui [48], [49]. The results of these papers are often based on power series expansions up to a finite order and on numeric arguments. The reason behind this is that in the context of these papers $d\Omega = 0$ has an explicit solution in special cases only. Recently, Bazaikin [5] has rigorously proven the existence of two one-parameter families of non-homothetic cohomogeneity-one metrics with holonomy Spin(7), which are asymptotically locally conical. The principal orbit is $N^{1,1}/\mathbb{Z}_2$ for the first and $S^7/\mathbb{Z}_4$ for the second family.

In the cohomogeneity-one case, the equation $d\Omega = 0$ is equivalent to a system of ordinary differential equations. There are several methods to deduce these equations. In [48], a superpotential for the equations for the Ricci-flatness is constructed, which yields a sufficient condition for the holonomy reduction. Bazaikin [5] obtained the differential equations which he studied by deforming a cone with holonomy $Sp(2)$. In a forthcoming paper [6], Bazaikin and Malkovich will deform cones with holonomy $SU(4)$. This deformation will yield a more general system of differential equations. In this thesis, we will work with a third method, which is motivated by a paper of Hitchin [42] and has also been applied in [48]. If (M, Ω) is of cohomogeneity one, the principal orbits are a one-parameter family of equidistant hypersurfaces. The union of all principal orbits is diffeomorphic to a product $N \times I$, where I is an interval. On any hypersurface inside a Spin(7)-manifold, there exists a canonical G_2-structure. The G_2-structure is, analogously to the Spin(7)-structure, determined by a three-form, which we denote by ω. Let $\frac{\partial}{\partial t}$ be the Lie derivative in the direction of the coordinate t which parameterizes I. With this notation, the equation $d\Omega = 0$ is equivalent to:

$$\frac{\partial}{\partial t} * \omega = d\omega \quad \text{and} \quad d * \omega = 0\,.$$

If we have $d * \omega = 0$ for a single t, the equation holds for all t and $\frac{\partial}{\partial t} * \omega = d\omega$ has a unique short-time solution. The first of the above two equations is equivalent to a system of non-linear ordinary differential equations, since the principal orbit is a homogeneous space. The second one is equivalent to a system of polynomial equations, which is in many cases automatically satisfied. If we are able to solve the differential equations, we have found a metric with holonomy $\subseteq$ Spin(7).

Before we can study the equation $\frac{\partial}{\partial t} * \omega = d\omega$ in a concrete situation, we first have to choose a principal orbit G/H. After that, we have to find a sufficiently large space of coclosed homogeneous G_2-structures on G/H which is preserved by $\frac{\partial}{\partial t} * \omega = d\omega$. By parameterizing that space, we are finally able to deduce a system of ordinary differential equations which is equivalent to $\frac{\partial}{\partial t} * \omega = d\omega$.

The seven-dimensional coset spaces G/H which admit a G-invariant G_2-structure can be classified by algebraic methods. The group H acts on the tangent space of G/H by its isotropy representation. G_2 also acts on the tangent space as the stabilizer of the three-form ω. In this situation, $H \subseteq G_2$ is necessary and sufficient for G/H to admit a G-invariant G_2-structure. Therefore, we have to classify all connected Lie subgroups $H \subseteq G_2$ and then all Lie groups G with $H \subseteq G$ and $\dim G - \dim H = 7$. With these methods, we are able to classify all possible principal orbits under some mild restrictions:

Theorem 1.1. *Let G/H be a compact, connected coset space which admits a G-invariant G_2-structure. We assume that G is compact and connected, too, and acts almost effectively on G/H. If G/H is not covered by a Cartesian product of lower-dimensional homogeneous spaces, G, H, and G/H are up to finite coverings one of the following:*

G	H	G/H
$SU(3)$	$U(1)$	$N^{k,l}$ *with* $k,l \in \mathbb{Z}$
$SO(5)$	$SO(3)$	$V^{5,2}$
$Sp(2)$	$Sp(1)$	S^7
$SO(5)$	$SO(3)$	B^7
$SU(2)^3$	$U(1)^2$	$Q^{1,1,1}$
$SU(3) \times SU(2)$	$SU(2) \times U(1)$	$M^{1,1,0}$

If G/H is covered by a product, at least one of the factors is a circle.

In the above table, the indices of $N^{k,l}$ ($Q^{1,1,1}$, $M^{1,1,0}$) denote a special embedding of $U(1)$ ($U(1)^2$, $SU(2) \times U(1)$) into $SU(3)$ ($SU(2)^3$, $SU(3) \times SU(2)$). Those embeddings are described explicitly in Section 5.4 (5.2, 5.3). $V^{5,2}$ denotes the Stiefel-manifold of all orthonormal pairs in $\mathbb{R}^5$ and B^7 is the Berger space, which is described in more detail in Section 5.1. Some of the above spaces admit a transitive action by a bigger group, which also preserves the G_2-structure. These group actions are not listed in the table of the theorem, but are described in Chapter 4. The principal orbits which are not a product of lower-dimensional homogeneous spaces we call irreducible and the other ones we call reducible. The irreducible principal orbits coincide with the homogeneous spaces which admit a nearly parallel G_2-structure. Those spaces are classified in a paper of Friedrich, Kath, Moroianu, and Semmelmann [37]. Since any nearly parallel G_2-structure ω also satisfies $d * \omega = 0$, all of those spaces admit a coclosed G_2-structure. If G/H is a reducible principal orbit, the group G splits into $G' \times U(1)$ and G/H splits into $G'/H \times U(1)$ up to a finite cover. The six-dimensional space G'/H admits a homogeneous $SU(3)$-structure. Spaces of that kind are possible principal orbits for cohomogeneity-one G_2-structures. For this reason, six-dimensional spaces admitting a homogeneous $SU(3)$-structure are studied by Cleyton and Swann [20]. Their list coincides with our list with the single exception of a space of type $SU(2)^2/U(1) \times U(1)$, which is included in our list, but seems to be missing in [20]. Many of the reducible principal orbits admit a homogeneous coclosed G_2-structure, but we will not prove this fact for all cases.

Any Riemannian product of a parallel G_2-structure and a circle is also a parallel Spin(7)-structure. We will shortly consider the question if there are any further examples of parallel cohomogeneity-one Spin(7)-structures with a reducible principal orbit, for example warped products. In the general case, there seems to be nothing which obstructs the existence of such examples. We investigate one particular principal orbit, namely $SU(3)/U(1)^2 \times U(1)$, in detail. In that case, we obtain a negative result only: Any parallel cohomogeneity-one Spin(7)-manifold with that principal orbit is a Riemannian product of a circle and a parallel cohomogeneity-one G_2-manifold with principal orbit $SU(3)/U(1)^2$. Since that kind of manifolds has been studied by Cleyton and Swann [20], we will not consider this issue further and focus on the irreducible principal orbits.

As we have mentioned above, we have to find a sufficiently large space of coclosed G_2-structures on any of the orbits G/H which we consider. We split this problem into two subproblems. First, we classify all G-invariant metrics on G/H and then we classify all invariant G_2-structures which have a fixed associated metric and orientation. The classification of the metrics can be done with help of representation theory. We fix a normal background metric q on G/H. Any other G-invariant metric on G/H can be identified by q with an H-equivariant endomorphism of a fixed tangent space. The set of those endomorphisms can be described with help of Schur's lemma. If the tangent space splits into pairwise inequivalent H-submodules, any G-invariant metric is diagonal with respect to any basis which is adapted to the splitting. This condition will be satisfied in many of the cases which we consider. In the other cases, we will also assume that the metric is diagonal in order to keep our calculations simple. The space of all G-invariant G_2-structures which have a fixed associated metric and orientation can be described by $\mathrm{Norm}_{SO(7)}H/\mathrm{Norm}_{G_2}H$. Since that space is independent of the choice of the metric and orientation, we also call it the space of all G_2-structures with the same associated metric and orientation. In many cases, we are able to describe that space explicitly. In the other cases, we restrict ourselves to a submanifold such that the set of all G_2-structures which we consider is preserved by $\frac{\partial}{\partial t} * \omega = d\omega$. We are now able to make an ansatz for ω. After having checked if $d * \omega = 0$, we can finally deduce a system of ordinary differential equations which is sufficient for the holonomy reduction.

The space M of cohomogeneity one on which we want to define the Spin(7)-structure has to have at least one singular orbit G/K, where $H \subseteq K \subseteq G$. Otherwise, the metric would be either non-complete or a product metric. The quotient K/H has to be a sphere in order to make M a manifold. In some cases, K/H is a quotient of a sphere by a discrete group. We investigate these cases, too, since M is an orbifold bundle over G/K in that situation.

There are several problems which have to be solved before we finally obtain a cohomogeneity-one metric with reduced holonomy. The first problem is that not necessarily all solutions of our differential equations correspond to metrics which can be smoothly extended to the singular orbit. There are certain smoothness conditions, which have to be satisfied. In the case where the metric is analytic, these conditions translate into conditions on the coefficients of the Taylor expansion of the metric. Since any metric whose holonomy is contained in Spin(7) is also Ricci-flat, it follows by a well-known theorem of DeTurck and Kazdan [30] that the metric is analytic. We therefore may assume that the metric is described by a power series. In a paper of Eschenburg and Wang [32], the smoothness conditions for analytic cohomogeneity-one metrics are deduced with help of representation theoretical arguments. With help of the methods from that paper, we are able to deduce the smoothness conditions for our situation. These conditions usually state that some of the functions which describe the metric are even and some are odd. Sometimes there are also relations between different functions or other restrictions on the metric, which could not be easily seen without applying the methods of [32]. After having deduced the smoothness conditions, we can check them with help of symmetry arguments. If the metric is smooth, its holonomy is contained in Spin(7). Therefore, there has to exist one, or if the holonomy is not all of Spin(7), a certain family of smooth parallel four-forms. With help of this argument, the smoothness of Ω follows.

The second problem is that the differential equations degenerate at the singular orbit. In general, we have to divide by zero on the right hand side of our equations. This may cause singularities, but in all except one case we always find metrics on the singular orbit such that the zero in the denominator is compensated by another zero in the numerator. There are two subcases which we have to discuss: In the first case, there are no restrictions which we have to impose on the initial values at the singular orbit and we are in the situation of the Picard-Lindelöf theorem. The existence and uniqueness of the solutions of our initial value problem thus immediately follows. Moreover, we will see that the differential equations which we obtain in this case are explicitly solvable. In the second case, we have to impose certain extra conditions on the initial values in order to compensate the zeroes in the denominator. If this is the case, the right hand side of the differential equations is defined on a non-open set only and we cannot apply the Picard-Lindelöf theorem. In particular, we do neither know if there exists a solution of our initial value problem nor if this solution is unique. There are indeed cases where the solution depends on initial conditions of higher order which we can freely prescribe. For the cohomogeneity-one Einstein condition, the number and order of those free parameters can be calculated by means of representation theory. This result is proven in the paper of Eschenburg and Wang [32]. We therefore make the following plan: Let m be the maximal order of the free parameters for the cohomogeneity-one Einstein condition. Any of the free parameters for the equations for the holonomy reduction also has to be one of the parameters of Eschenburg and Wang. We make a power series expansion up to m^{th} order and check which of those degrees of freedom actually remain in our situation. Then we prove by an explicit calculation that for any choice of the free parameters there exists a formal power series solution of our initial value problem. The convergence of the power series follows by another result of Eschenburg and Wang [32]. After that, we finally have classified all solutions of our initial value problem.

In the first of the above two cases, we have to consider certain spaces of K-equivariant maps in order to deduce the smoothness conditions. This information is already sufficient to apply the theorem of Eschenburg and Wang [32]. As a by-product of our calculations, we thus can prove in both cases the existence of cohomogeneity-one Einstein metrics on a neighborhood of the singular orbit.

We remark that most results of Eschenburg and Wang hold only if the tangent and the normal space of the singular orbit contain no equivalent H-submodules. This is a general assumption which was made in [32]. Eschenburg and Wang [32] suppose that all of their results can be carried over to the general case. In some of the cases which we consider the above assumption is violated. Nevertheless, we show by a modification of the proof of the theorem of Eschenburg and Wang that their theorem applies to our cases as well.

It is already a demanding task to carry out the program which we have outlined above for a single principal orbit. Therefore, we will not consider all of the reducible principal orbits in this thesis. More concretely, we will not investigate cohomogeneity-one metrics whose principal orbit is the Stiefel-manifold $V^{5,2}$ or covered by the sphere S^7. One of the reasons why we omit these principal orbits is that they are intensively studied in the literature. Metrics with principal orbit $V^{5,2}$ and holonomy $SU(4)$ are constructed by Stenzel [64] and Cvetič et al. [25]. On parallel cohomogeneity-one Spin(7)-structures whose principal orbit is covered by S^7 there exists much literature. The most simple example of such a structure is the flat Spin(7)-structure on $\mathbb{R}^8$. Spin(7) acts by its eight-dimensional representation on $\mathbb{R}^8$. The orbits of that action are spheres and the action thus is of cohomogeneity one. As we have mentioned above, the parallel Spin(7)-structure of Bryant and Salamon [15] is also of cohomogeneity one and has S^7 as principal orbit. Furthermore, Cvetič et. al. investigate parallel Spin(7)-structures of that kind in [23], [24], [26], and [27]. Among their examples are manifolds with principal orbit S^7 and singular orbit S^4 as well as manifolds with principal orbit $S^7/\mathbb{Z}_4$ and singular orbit $\mathbb{CP}^3$. Finally, the cohomogeneity-one metrics of Bazaikin [5] with principal orbit $S^7/\mathbb{Z}_4$ fit in this context, too. Those metrics are a special case of the numerical examples of Cvetič et al. and also have $\mathbb{CP}^3$ as singular orbit. All of the above metrics are diagonal. On both $V^{5,2}$ and S^7 there exist non-diagonal homogeneous metrics. Since we can transform any of the non-diagonal metrics into a diagonal one by the action of the normalizer, it seems unlikely that there are many further examples apart from those in the literature. This argument will be explained in more detail in Section 7.

The first principal orbit which we consider in detail is the Berger space $B^7 \cong SO(5)/SO(3)$. The embedding of $SO(3)$ into $SO(5)$ is given by the five-dimensional irreducible representation of $SO(3)$. Since B^7 is isotropy-irreducible, there is up to constant multiples only one $SO(5)$-invariant metric and one $SO(5)$-invariant G_2-structure on B^7. With help of this fact, we can conclude that the only parallel cohomogeneity-one Spin(7)-structures with principal orbit B^7 are cones over that space. Next, we consider principal orbits of type $Q^{k,l,m} := SU(2)^3/U(1)^2_{k,l,m}$, where the three indices describe the embedding of $U(1)^2_{k,l,m}$ into $SU(2)^3$. We will see that only $Q^{1,1,1}$ admits a homogeneous G_2-structure. Since the tangent space of $Q^{1,1,1}$ splits into pairwise inequivalent $U(1)^2_{1,1,1}$-modules, any homogeneous metric on that space is diagonal. For any fixed metric and orientation on $Q^{1,1,1}$, there exists a one-parameter family of $SU(2)^3$-invariant G_2-structures whose associated metric and orientation coincides with the chosen ones. Moreover, there are no further G_2-structures of that kind and the one-parameter family can be generated by an action of $U(1)$ by isometries. We fix one of those isometries and let it act on all of the principal orbits simultaneously. This construction yields an isometry of the whole cohomogeneity-one manifold. If we find a single parallel Spin(7)-structure, we obtain by the $U(1)$-action a whole family of parallel Spin(7)-structures, which all have the same associated metric. From this it follows that the holonomy is $SU(4)$. In the situation which we study at the moment, the equations for the holonomy reduction are explicitly solvable. There are solutions with $S^2 \times S^2$ and with $S^2 \times S^2 \times S^2$ as singular orbit. The metrics of the first kind are smooth at singular orbit. In the second case, the length of the collapsing circle $K/U(1)^2$ is $4\pi t + O(t^2)$ for small t. If the metric was smooth, this length would have to be $2\pi t + O(t^2)$. Our metric therefore has a singularity at the singular orbit. We nevertheless suppose that it is possible to make the metric smooth by replacing the principal orbit $Q^{1,1,1}$ by $Q^{1,1,1}/\mathbb{Z}_2$ for a suitable $\mathbb{Z}_2$-action. We will not carry out this construction explicitly, since later on there will be another principal orbit for which this will be explained in detail. All in all, we have proven the following theorem:

Theorem 1.2. *Let (M, Ω) be a parallel cohomogeneity-one Spin(7)-manifold whose principal orbits are of type $Q^{k,l,m} := SU(2)^3/U(1)^2_{k,l,m}$. In this situation, the following statements are true:*

1. *The principal orbits are $SU(2)^3$-equivariantly diffeomorphic to $Q^{1,1,1}$.*

2. *The metric g which is associated to Ω has holonomy $SU(4)$.*

3. *If M has a singular orbit, it has to be $S^2 \times S^2$ or $S^2 \times S^2 \times S^2$. In the first case, g can be extended to a smooth, complete metric and M is non-compact. In the second case, M is non-compact, too, but g cannot be smooth at the singular orbit. However, g can be extended such that any geodesic which does not intersect the singular orbit is defined on all of $\mathbb{R}$.*

The above metrics have already been described in another context (see Cvetič et al. [21], [25] and Herzog, Klebanov [41]). Our proof that there are no further Spin(7)-structures on a cohomogeneity-one manifold with principal orbit $Q^{k,l,m}$ is new. Moreover, we have introduced a new point of view on these metrics.

The next kind of principal orbits which we consider are spaces of type $(SU(3)\times SU(2))/(SU(2) \times U(1))$ where the first factor of $SU(2) \times U(1)$ is embedded into $SU(3)$. In this case, we obtain similar results as in the previous one. The embedding of $U(1)$ into $SU(3) \times SU(2)$ has to be special in order to make $(SU(3) \times SU(2))/(SU(2) \times U(1))$ a space which admits a $SU(3) \times SU(2)$-invariant G_2-structure. On a cohomogeneity-one manifold (or orbifold) with that space, which we denote by $M^{1,1,0}$, as principal orbit, the differential equations for the holonomy reduction are explicitly solvable. There are three possible singular orbits, namely S^2, $\mathbb{CP}^2$, and $S^2 \times \mathbb{CP}^2$. If the singular orbit is S^2, our space of cohomogeneity one is an orbifold but not a manifold, since $M^{1,1,0}$ is a $S^5/\mathbb{Z}_3$-bundle over the singular orbit. In Section 5.3, we will prove the following theorem:

Theorem 1.3. *Let (M,Ω) be a parallel cohomogeneity-one Spin(7)-manifold whose principal orbit is of type $M^{k,l,m} := (SU(3) \times SU(2))/(SU(2) \times U(1))_{k,l,m}$. The indices k, l, and m describe the embedding of the abelian factor of $(SU(2) \times U(1))_{k,l,m}$ into $SU(3) \times SU(2)$ and the embedding of the semisimple factor shall be as above. In this situation, the following statements are true:*

1. *The principal orbit is $SU(3) \times SU(2)$-equivariantly diffeomorphic to $M^{1,1,0}$.*

2. *The metric g which is associated to Ω has holonomy $SU(4)$.*

3. *If M has a singular orbit, it has to be S^2, $\mathbb{CP}^2$, or $S^2 \times \mathbb{CP}^2$. In all three cases, M is non-compact and g can be extended such that any geodesic which does not intersect the singular orbit is defined on all of $\mathbb{R}$. This means in particular that if g is smooth, (M,g) is complete. If the singular orbit is S^2, g cannot be a smooth orbifold metric. In the second case, the metric is smooth and in the third case the metric cannot be smooth at the singular orbit.*

Although the metric is not smooth if the singular orbit is $S^2 \times \mathbb{CP}^2$, it may be possible to replace the principal orbit by $M^{1,1,0}/\mathbb{Z}_2$ in such a way that the metric becomes a smooth one. As in the previous case, all of the above metrics are mentioned in [21], [25], and [41]. Nevertheless, our classification result is new.

Finally, we consider the Aloff-Wallach spaces as principal orbits. An Aloff-Wallach space is a coset space of type $N^{k,l} := SU(3)/U(1)_{k,l}$, where $k,l \in \mathbb{Z}$ and $U(1)_{k,l}$ is the subgroup of $SU(3)$ which is generated by $\mathrm{diag}(ik, il, -i(k+l))$. The Aloff-Wallach spaces which are $SU(3)$-equivariantly diffeomorphic to $N^{1,0}$ or $N^{1,1}$ are called exceptional and the other ones are called generic. If $N^{k,l}$ is an exceptional Aloff-Wallach space, the tangent space $T_{eU(1)_{k,l}}N^{k,l}$ contains a pair of equivalent $U(1)_{k,l}$-submodules. Therefore, there exist non-diagonal $SU(3)$-invariant metrics on those spaces no matter how we fix the basis of the tangent space. As we have announced above, we nevertheless assume that the metric is diagonal in order to keep our considerations simple. For any diagonal metric g on an Aloff-Wallach space $N^{k,l}$, there exists a special coclosed G_2-structure such that its associated metric is g. As in the $Q^{1,1,1}$- and $M^{1,1,0}$-case, there is a $U(1)$-action on this G_2-structure which generates a whole one-parameter family of $SU(3)$-invariant G_2-structures with the same associated metric and orientation. However, there are two differences to the previous cases. First, the $U(1)$-action is not induced by a subgroup of the normalizer $\mathrm{Norm}_{SU(3)}U(1)_{k,l}$. Second, the set of all coclosed G_2-structures in the above family is usually discrete. Only in the case where $N^{k,l}$ is $SU(3)$-equivariantly diffeomorphic to $N^{1,1}$ and the initial G_2-structure is of a special type all G_2-structures in the family are coclosed. In the other cases, it is even possible to restrict ourselves without loss of generality to the initial coclosed G_2-structure. Unfortunately, we cannot exclude that the space of all $SU(3)$-invariant G_2-structures with the same associated metric and orientation is larger than the $U(1)$-orbit. If $N^{k,l}$ is generic or $SU(3)$-equivariantly diffeomorphic to $N^{1,0}$, it may have further connected components, which are diffeomorphic to a circle. If $N^{k,l}$ is $SU(3)$-equivariantly diffeomorphic to $N^{1,1}$, the space is three-dimensional and its connected components are generated by a $SO(3)$-action. Since we suppose that there are not many coclosed G_2-structures on the Aloff-Wallach spaces, we restrict ourselves to G_2-structures which belong to the $U(1)$-orbit. With help of the above facts, we can prove that the holonomy of a parallel cohomogeneity-one Spin(7)-manifold whose principal orbit is a generic Aloff-Wallach space is all of Spin(7). If the principal orbit is $SU(3)$-equivariantly to $N^{1,0}$, the holonomy is the same. In the $N^{1,1}$-case, we do not investigate all G_2-structures in the $SO(3)$-orbit. Therefore, we can conclude only that the holonomy is contained in Spin(7). If the G_2-structure on the principal orbit is of the special type which we have mentioned above, we can moreover prove that the holonomy is contained in $SU(4)$. In Kanno, Yasui [49], further results on the holonomy of cohomogeneity-one metrics with principal orbit $N^{1,1}$ are proven. By classifying all closed connected groups K with $U(1)_{k,l} \subseteq K \subseteq SU(3)$, we see that there are four possible singular orbits, namely $SU(3)/U(1)^2$, S^5, $SU(3)/SO(3)$, and $\mathbb{CP}^2$. If the singular orbit is $\mathbb{CP}^2$, the space on which $SU(3)$ is acting on is usually an orbifold but not a manifold, since $K/U(1)_{k,l}$ is a lens space of type $S^3/\mathbb{Z}_{|k+l|}$. If furthermore $k+l=0$, the space has a singularity which is not an orbifold singularity. S^5 or $SU(3)/SO(3)$ as a singular orbit is possible only if the principal orbit is of type $N^{1,0}$.

After having classified the possible singular orbits, we consider solutions of the equations for the holonomy reduction which are defined on a tubular neighborhood of one of those orbits. These equations have an explicit solution in special cases only. We therefore have to work with the power series methods which we have described earlier in this introduction. Since these methods yield local results only, we do not check the completeness of our metrics. The first singular orbit which we investigate is $SU(3)/U(1)^2$. If the principal orbit is a generic Aloff-Wallach space or $N^{1,0}$, there exist no solutions of our differential equations near the singular orbit. If the principal orbit is $N^{1,1}$, there exists a short-time solution, which depends on no parameters except the metric on $SU(3)/U(1)^2$. Unfortunately, the metric cannot be smoothly extended to the singular orbit. Nevertheless, this becomes possible if we replace the principal orbit by a suitable quotient $N^{1,1}/\mathbb{Z}_2$. This time, we explicitly carry out that procedure and obtain another cohomogeneity-one manifold with singular orbit $SU(3)/U(1)^2$. The next singular orbit which we consider is S^5. We will prove that any $SU(3)$-invariant metric on $S^5 \cong SU(3)/SU(2)$ can be uniquely extended to a smooth cohomogeneity-one metric which solves the equations for the holonomy reduction. If the singular orbit is $SU(3)/SO(3)$, we find no parallel $SU(3)$-invariant Spin(7)-structures. In the cases where the singular orbit is $\mathbb{CP}^2$, the metrics are no longer unique, but depend on additional free parameters of second or third order. We will explain these parameters by the results of Eschenburg and Wang [32]. To sum up, we have proven the following result:

Theorem 1.4. *Let (M, Ω) be a parallel cohomogeneity-one Spin(7)-manifold (or -orbifold) whose principal orbit is an Aloff-Wallach space. We assume that the metric associated to Ω is diagonal with respect to a natural basis and that the G_2-structure on the principal orbit is always contained in the one-parameter family which we have mentioned above. In this situation, the following statements are true:*

1. *If the principal orbit is $N^{1,1}$ and the coefficients of Ω satisfy certain relations, the holonomy of the metric associated to Ω is contained in $SU(4)$. In the other cases, it is contained in Spin(7). If the principal orbit is a generic Aloff-Wallach space or $N^{1,0}$, the holonomy is all of Spin(7).*

2. *If the principal orbit is $N^{1,1}$ and the singular orbit is $SU(3)/U(1)^2$, the metric associated to Ω can never be smoothly extended to the singular orbit. If we divide the principal orbit by a suitable $\mathbb{Z}_2$-action which leaves Ω invariant, the metric can always be smoothly extended. Moreover, it is uniquely determined by the metric on the singular orbit.*

3. *If the singular orbit is S^5, any $SU(3)$-invariant metric on the singular orbit can be uniquely extended to a smooth cohomogeneity-one metric with holonomy Spin(7) such that the principal orbit is $N^{1,0}$ and the G_2-structure on the principal orbit satisfies our restrictions.*

4. *If the singular orbit is $\mathbb{CP}^2$, there are initial conditions of higher order which we can prescribe. Their number and order depend on further details of the principal orbit and the initial values of 0^{th} order. There are several subcases, in which there are two initial conditions of third order, one initial condition of third order, or one initial condition of second order.*

5. *Under some mild restrictions, the above examples are all parallel cohomogeneity-one Spin(7)-manifolds which satisfy the assumptions on the metric and the G_2-structure which we have made.*

The metrics with singular orbit $\mathbb{CP}^2$ which depend on two parameters of third order are new contributions of the author. The other metrics from the above theorem have already been mentioned in the literature [5], [24], [39], [48], [49]. Since those results are often numerical, many of our proofs on the local existence and smoothness of our metrics are new. The same applies to the fact that there are no further free parameters apart from those in the theorem.

With help of the theorem of Eschenburg and Wang [32], we find many examples of cohomogeneity-one Einstein metrics with principal orbit $Q^{1,1,1}$, $M^{1,1,0}$, or $N^{k,l}$. Earlier in this introduction, we have remarked that there is a technical assumption which is necessary for the theorem. If the principal orbit is an exceptional Aloff-Wallach space, that assumption is violated in some cases. As we have announced above, we can nevertheless apply the theorem of Eschenburg and Wang to those cases. We state the results which we obtain this way as a separate theorem. On the quaternionic projective space $\mathbb{HP}^2$ there is an Einstein metric of cohomogeneity one with principal orbit $N^{1,0}$ (see Püttmann, Rigas [59]). The examples of cohomogeneity-one Einstein metrics which are neither $\mathbb{HP}^2$ nor have reduced holonomy are further contributions of the author.

This thesis is organized as follows: In Chapter 2, we recall some facts about G_2- and Spin(7)-structures and about the two groups themselves. In the course of our considerations, we need many results on manifolds with a large isometry group, which we collect in Chapter 3. In Section 3.1, we introduce some methods which simplify our calculations on homogeneous spaces. Section 3.2 deals with manifolds of cohomogeneity one. In particular, we resume the paper of Eschenburg and Wang [32] and introduce the flow equation of Hitchin [42]. In the fourth chapter, we classify the spaces which admit a homogeneous G_2-structure. In order to do this, we need a list of all connected Lie subgroups of G_2. Therefore, we classify them, too. Chapter 5 is the most extensive one of this thesis. In that chapter, we investigate the irreducible principal orbits which we have chosen in detail. We start with the Berger space B^7, which is studied in Section 5.1. In Section 5.2 (5.3), we explore the issue of cohomogeneity-one metrics with special holonomy whose principal orbit is $Q^{1,1,1}$ ($M^{1,1,0}$). Parallel Spin(7)-manifolds of cohomogeneity one with an Aloff-Wallach space as principal orbit are investigated in Section 5.4. This section is the most comprehensive section of Chapter 5. The reasons for this are that we have to consider the generic and the two exceptional Aloff-Wallach spaces separately, that the calculations for the exceptional Aloff-Wallach spaces are more difficult than those for the other principal orbits, and that the equations for the holonomy reduction do not have a general explicit solution. The sixth chapter is devoted to cohomogeneity-one Spin(7)-manifolds with a reducible principal orbit. In Section 6.1, we deal with the equation $d\Omega = 0$ in the general situation. The case where the principal orbit is $SU(3)/U(1)^2 \times U(1)$ is treated in the second section of that chapter. The seventh chapter is called "Conclusion and outlook". It begins with a short summary of our results. Furthermore, we motivate why we have not considered S^7, $S^7/\mathbb{Z}_4$, or the Stiefel-manifold $V^{5,2}$ as a principal orbit. The calculations which we were necessary to derive the equations for the holonomy reduction and for the Einstein condition are included in the appendix.

Chapter 2

Exceptional holonomies

2.1 The groups G_2 and $\mathrm{Spin}(7)$

Before we introduce the most important facts on metrics with exceptional holonomy, we first have to take a closer look at the groups G_2 and Spin(7). These groups can be described with help of the octonions. Therefore, we have to give a short introduction to normed division algebras, too. This introduction is in part based on the paper of John Baez [4] on the octonions. Since most of the results of this section are well-known, we usually will omit their proofs and refer to the literature instead.

Definition 2.1.1. A *normed division algebra* is a (not necessarily associative) real algebra A with a unit element and a scalar product $\langle .,. \rangle$ satisfying

$$\langle xy, xy \rangle = \langle x, x \rangle \langle y, y \rangle \quad \forall x, y \in A\,.$$

Let A be a normed division algebra:

1. A non-zero linear map $\Phi : A \to A$ satisfying

$$\Phi(xy) = \Phi(x)\Phi(y) \quad \forall x, y \in A$$

   is called an *automorphism of* A.

2. A linear map $\varphi : A \to A$ with

$$\varphi(xy) = x\varphi(y) + \varphi(x)y \quad \forall x, y \in A$$

   is called a *derivation of* A.

3. Let x be an element of A. The map $L_x : A \to A$ with $L_x(y) = xy$ is called the *left multiplication by* x. Analogously, the map $R_x : A \to A$ with $R_x(y) = yx$ is called the *right multiplication by* x.

Example 2.1.2. $\mathbb{R}$ and $\mathbb{C}$ together with the scalar products $\langle x, y\rangle := xy \quad \forall x, y \in \mathbb{R}$ and $\langle x, y\rangle := \mathrm{Re}(x\bar{y}) \quad \forall x, y \in \mathbb{C}$ are normed division algebras. On the quaternions $\mathbb{H}$ there exists a scalar product, too, which makes $\mathbb{H}$ a normed division algebra and $(1, i, j, k)$ an orthonormal basis of $\mathbb{H}$.

The real numbers $\mathbb{R}$ can be embedded as $\mathbb{R} \cdot 1$ into any normed division algebra A. This fact allows us to define the notion of imaginary numbers in A:

Definition 2.1.3. Let A be a normed division algebra. The orthogonal complement of $\mathbb{R} \subseteq A$ is called the *imaginary space Im*(A) *of* A. Its elements are the *imaginary numbers of* A. Let x be an element of A. The projection of x onto $\mathbb{R}$ is called the *real part Re*(x) *of* x and its projection onto $\mathrm{Im}(A)$ is called the *imaginary part Im*(x) *of* x. $\mathrm{Re}(x) - \mathrm{Im}(x)$ we call the *conjugate* $\bar{x}$ *of* x.

We collect some useful facts on normed division algebras:

Lemma 2.1.4. *Let A be a normed division algebra. In this situation, the following statements are true:*

1. *Let $x, y \in Im(A)$. Then $xy + yx = -2\langle x, y\rangle$ and, in particular, $x^2 = -\|x\|^2$.*
2. *Any $x \in A\backslash\{0\}$ has an inverse $x^{-1} = \frac{\bar{x}}{\|x\|^2}$. This fact justifies the name "division algebra" and shows that the maps L_x and R_x are bijective.*
3. *Let $\|x\| = 1$. Then, we have $\langle yx, zx\rangle = \langle xy, xz\rangle = \langle y, z\rangle$. In this situation, L_x and R_x are therefore both orthogonal.*
4. *Since A is not necessarily associative, we do not have $L_x \circ L_y = L_{xy}$. The set of all L_x (or R_x) with $\|x\| = 1$ therefore is not necessarily a group, but there is still a group generated by all L_x (or R_x). That group turns out to be a Lie subgroup of $O(\dim A)$.*

Remark 2.1.5. Let A be an algebra with a unit element. If for all $x \in A\backslash\{0\}$ the maps R_x and L_x are bijective, A is called a *division algebra*. Any normed division algebra is a division algebra. The converse is not necessarily true.

The automorphisms of a normed division algebra have the following properties:

Lemma 2.1.6. *Let A be a normed division algebra and Φ be an automorphism of A. Then, we have:*

1. *$\Phi(1) = 1$.*
2. *If A is finite-dimensional, Φ is bijective. This fact simply follows from the equation $\Phi(x)\Phi(x^{-1}) = 1$. Now, we have justified why we call Φ an automorphism.*
3. *Let x be in $\mathbb{R}$. Then we have $\Phi(x) = x$. An automorphism therefore is determined by its restriction to Im(A).*
4. *Φ is an orthogonal map, i.e., $\langle\Phi(x), \Phi(y)\rangle = \langle x, y\rangle \quad \forall x, y \in A$.*
5. *Let x be in Im(A). Then $\Phi(x)$ is in Im(A), too. We therefore can identify any automorphism of A with an element of $O(Im(A))$.*

6. *The automorphisms of A form a group, the so called automorphism group Aut(A). It is possible to show that $Aut(A)$ is a Lie subgroup of $O(\dim A - 1)$.*

We can describe the Lie algebra of this Lie group explicitly:

Lemma 2.1.7. *Let A be a normed division algebra. Furthermore, let φ and ψ be derivations of A. Then, we have:*

1. *For any $x \in \mathbb{R}$, we have $\varphi(x) = 0$. Therefore, φ is determined by its restriction to $Im(A)$.*
2. *Let x be in $Im(A)$. Then $\varphi(x)$ is in $Im(A)$, too. We therefore can identify any derivation with a linear map $Im(A) \to Im(A)$.*
3. *φ is skew-symmetric, i.e., $\langle \varphi(x), y \rangle + \langle x, \varphi(y) \rangle = 0$ for all $x, y \in A$.*
4. *The commutator $\varphi \circ \psi - \psi \circ \varphi$ is a derivation, too. The set of all derivations of A therefore is a Lie algebra $\mathfrak{der}(A)$ which is a subalgebra of $\mathfrak{so}(n-1)$.*
5. *$\mathfrak{der}(A)$ is the Lie algebra of the Lie group $Aut(A)$.*

On any normed division algebra, there is a canonical 3-form and a canonical 4-form:

Definition 2.1.8. Let A be a normed division algebra. The map

$$\begin{aligned} [.,.] &: A \times A \to A \\ [x, y] &:= xy - yx \end{aligned}$$

is called the *commutator of A*. The map

$$\begin{aligned} [.,.,.] &: A \times A \times A \to A \\ [x, y, z] &:= (xy)z - x(yz) \end{aligned}$$

is called the *associator of A*.

Lemma 2.1.9. *Let A be a normed division algebra and let $x, y, z \in Im(A)$. Then $[x, y]$ and $[x, y, z]$ are in $Im(A)$, too.*

We can use the above maps to construct our forms:

Lemma 2.1.10. *Let A be a normed division algebra.*

1. *The map*

$$\begin{aligned} \omega &: A \times A \times A \to \mathbb{R} \\ \omega(x, y, z) &:= \frac{1}{2} \langle [x, y], z \rangle \end{aligned}$$

   is a three-form, the canonical three-form of A.

2. The map

$$\omega^* : A \times A \times A \times A \to \mathbb{R}$$
$$\omega^*(x, y, z, w) := -\frac{1}{2}\langle [x, y, z], w\rangle$$

is a four-form, the canonical four-form of A.

Remark 2.1.11. 1. We have inserted the minus in the definition of the canonical four-form in order to make it consistent with later conventions for the signs.

2. Since we have $xy + yx = -2\langle x, y\rangle$ for imaginary x and y, the restriction of ω to the imaginary space satisfies $\omega|_{\mathrm{Im}(A)^3}(x, y, z) = \langle xy, z\rangle$.

3. If x, y or z, (or w) is real, we have $\omega(x, y, z) = 0$ ($\omega^*(x, y, z, w) = 0$).

4. From the antisymmetry of ω^*, it follows that $(x^2)y = x(xy)$, $(xy)x = x(yx)$, and $(yx)x = y(x^2)$ for all $x, y \in A$. Therefore, any subalgebra of A which is generated by two elements is associative. An algebra with this property is called *alternative*.

5. It is easily possible to determine the objects we have defined above for the quaternions. The automorphism group of $\mathbb{H}$ is isomorphic to $SO(3)$. The group generated by the left-multiplication and the group generated by the right-multiplication with unit quaternions are both isomorphic to $SU(2)$. Together they generate all of $SO(4)$. The canonical three-form ω restricted to $\mathrm{Im}(\mathbb{H})$ is the volume form which satisfies $\omega(i, j, k) = 1$. The canonical four-form of $\mathbb{H}$ vanishes, since $\mathbb{H}$ is associative.

The normed division algebras have been classified long ago:

Theorem 2.1.12. *(See Hurwitz [44].) There are exactly four normed division algebras, namely $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and the octonions $\mathbb{O}$.*

The octonions are determined up to an isomorphism by the above theorem. $\mathbb{O}$ has a subalgebra which is isomorphic to $\mathbb{H}$. Let ϵ be an element of $\mathbb{O}$ with $\|\epsilon\| = 1$ and $\epsilon \perp \mathbb{H} \subseteq \mathbb{O}$. Then $(1, i, j, k, \epsilon, i\epsilon, j\epsilon, k\epsilon)$ is an orthonormal basis of $\mathbb{O}$, which we call the *standard basis of* $\mathbb{O}$. For this basis, we obtain the following multiplication table:

·	i	j	k	ϵ	iϵ	jϵ	kϵ
i	-1	k	-j	iϵ	-ϵ	-kϵ	jϵ
j	-k	-1	i	jϵ	kϵ	-ϵ	-iϵ
k	j	-i	-1	kϵ	-jϵ	iϵ	-ϵ
ϵ	-iϵ	-jϵ	-kϵ	-1	i	j	k
iϵ	ϵ	-kϵ	jϵ	-i	-1	-k	j
jϵ	kϵ	ϵ	-iϵ	-j	k	-1	-i
kϵ	-jϵ	iϵ	ϵ	-k	-j	i	-1

In the above table, we multiply a x from the left column with a y from the upper row and obtain $x \cdot y$. The multiplication table is independent of ϵ and the subalgebra which isomorphic to $\mathbb{H}$.

Remark 2.1.13. The octonions are neither commutative nor associative, since, for example, $ij = k = -ji$ and $(ij)\epsilon = k\epsilon = -i(j\epsilon)$.

The derivations of $\mathbb{O}$ are isomorphic to one of the exceptional Lie algebras:

Proposition 2.1.14. *The Lie algebra $\mathfrak{der}(\mathbb{O})$ is isomorphic to the compact real form of the complex simple Lie algebra $\mathfrak{g}_2$.*

Proof : First, we determine the dimension of $\mathfrak{der}(\mathbb{O})$. Each element of the standard basis can be written in the form $(i^{\alpha_1} j^{\alpha_2})\epsilon^{\alpha_3}$ with $\alpha_1, \alpha_2, \alpha_3 \in \{0,1\}$. Let Φ be an automorphism of $\mathbb{O}$. Since $\Phi(xy)$ is determined by $\Phi(x)$ and $\Phi(y)$, Φ is determined by $\Phi(i)$, $\Phi(j)$, and $\Phi(\epsilon)$. Let $\Psi : \mathbb{O} \to \mathbb{O}$ be a $\mathbb{R}$-linear map. The following conditions are necessary for Ψ to be an automorphism:

- $\Psi(i),\ \Psi(j),\ \Psi(\epsilon) \in \mathrm{Im}(\mathbb{O})$
- $\|\Psi(i)\| = 1,\ \|\Psi(j)\| = 1,\ \|\Psi(\epsilon)\| = 1$
- $\Psi(j) \perp \Psi(i)$
- $\Psi(\epsilon) \perp \mathrm{span}(\Psi(i),\ \Psi(j),\ \Psi(k))$

A triple which satisfies the same conditions as $(\Psi(i), \Psi(j), \Psi(\epsilon))$ is called a *basis triple.* It is possible to show that for all basis triples (x, y, z) there exists a unique automorphism $\Phi \in \mathrm{Aut}(\mathbb{O})$ with $\Phi(i) = x$, $\Phi(j) = y$, $\Phi(\epsilon) = z$. We can choose the value of $\Phi(i)$ freely out of $S^6 \subseteq \mathrm{Im}(\mathbb{O})$. Since $\Phi(i) \perp \Phi(j)$, we can choose for any $\Phi(i)$ the value of $\Phi(j)$ out of $S^5 \subseteq \mathrm{span}(1, \Phi(i))^\perp$. Analogously, we can choose for any $(\Phi(i), \Phi(j))$ the value of $\Phi(\epsilon)$ out of $S^3 \subseteq \Phi(\mathbb{H})^\perp$. Therefore, $\mathrm{Aut}(\mathbb{O})$ is a S^3-bundle over the space of all orthonormal 2-frames in $\mathrm{Im}(\mathbb{O})$. That space is called the *Stiefel-manifold* $V_{2,7}$ and is a S^5-bundle over S^6. Therefore, $\dim V_{2,7} = \dim S^6 + \dim S^5 = 11$ and $\dim \mathfrak{der}(\mathbb{O}) = \dim \mathrm{Aut}(\mathbb{O}) = \dim V_{2,7} + \dim S^3 = 14$.

Next, we determine the rank of $\mathfrak{der}(\mathbb{O})$. In order to do this, we consider the following Cartan subalgebra of $\mathfrak{so}(7)$:

$$\mathfrak{t} := \left\{ \left(\begin{array}{c|cc|cc|cc} 0 & & & & & & \\ \hline & 0 & \lambda_1 & & & & \\ & -\lambda_1 & 0 & & & & \\ \hline & & & 0 & \lambda_2 & & \\ & & & -\lambda_2 & 0 & & \\ \hline & & & & & 0 & \lambda_3 \\ & & & & & -\lambda_3 & 0 \end{array} \right) \middle|\ \lambda_1, \lambda_2, \lambda_3 \in \mathbb{R} \right\}$$

The above matrices should be considered as matrix representations of $\mathbb{R}$-linear maps with respect to the standard basis of $\mathrm{Im}(\mathbb{O})$. $\mathfrak{der}(\mathbb{O}) \cap \mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{der}(\mathbb{O})$. We therefore have to check which of the above matrices are derivations of $\mathbb{O}$. After a short calculation, we see that

$$\mathfrak{der}(\mathbb{O}) \cap \mathfrak{t} = \left\{ \left(\begin{array}{ccccccc} 0 & & & & & & \\ & 0 & \lambda_1 & & & & \\ & -\lambda_1 & 0 & & & & \\ & & & 0 & \lambda_2 & & \\ & & & -\lambda_2 & 0 & & \\ & & & & & 0 & \lambda_1+\lambda_2 \\ & & & & & -\lambda_1-\lambda_2 & 0 \end{array} \right) \middle| \; \lambda_1, \lambda_2 \in \mathbb{R} \right\} \tag{2.1}$$

Since this is a two-dimensional Lie algebra, the rank of $\mathfrak{der}(\mathbb{O})$ is 2. $\mathrm{Aut}(\mathbb{O}) \subseteq O(7)$ is a compact Lie group. It is a well known fact that the Lie algebra of a compact Lie group has to be the direct sum of an abelian and a semisimple Lie algebra. With help of the classification of the complex simple Lie algebras we see that there is only one 14-dimensional complex Lie algebra of rank 2 which is of this type, namely $\mathfrak{g}_2$. Since $\mathrm{Aut}(\mathbb{O})$ is compact, $\mathfrak{der}(\mathbb{O})$ has to be the compact real form of $\mathfrak{g}_2$. We will denote that real form shortly by $\mathfrak{g}_2$, too.

$\square$

Remark 2.1.15. Any $\varphi \in \mathfrak{g}_2$ is determined by its values on (i, j, ϵ) for the same reasons as automorphisms of $\mathbb{O}$.

Before we proceed, we will mention a few facts on the structure of $\mathfrak{g}_2$. The Dynkin diagram of $\mathfrak{g}_2$ consists of two nodes and a threefold directed arrow connecting the two nodes:

$$\circ \Rrightarrow \circ \ .$$

Its root system is generated by a short and a long root and has 12 Weyl chambers. The Weyl group of $\mathfrak{g}_2$ is the Dieder group D_6 (see Humphreys [43]). We will now prove some properties of $\mathrm{Aut}(\mathbb{O})$:

Lemma 2.1.16. *$\mathrm{Aut}(\mathbb{O})$ is a simply connected Lie subgroup of $SO(7)$. Moreover, it is isomorphic to the simply connected Lie group with Lie algebra $\mathfrak{g}_2$, which is denoted by G_2.*

Proof : $V_{2,7}$ is connected, since it is a S^5-bundle over S^6. The S^3-bundle $\mathrm{Aut}(\mathbb{O})$ over $V_{2,7}$ is connected for the same reasons. $\mathrm{Aut}(\mathbb{O})$ therefore is not only a Lie subgroup of $O(7)$, but of $SO(7)$, too.

It is known that the center of the Lie group G_2 is trivial. Therefore, there are no connected Lie groups covered by G_2, except G_2 itself. Since G_2 is the only connected Lie group with Lie algebra $\mathfrak{g}_2$, we have shown that $\mathrm{Aut}(\mathbb{O})$ is isomorphic to G_2 and therefore simply connected.

$\square$

We now consider the group which is generated by all L_x with the property that $x \in \mathbb{O}$ is of unit length. Until we have found an explicit description of this group, we denote it by K. The following result will be useful in order to find that description:

Lemma 2.1.17. *(See Cacciatori, Cerchiai, Della Vedova, Ortenzi, and Scotti [17].) The automorphism group G_2 of $\mathbb{O}$ is a Lie subgroup of K. Moreover, it consists of precisely those elements of K which leave $1 \in \mathbb{O}$ invariant.*

Corollary 2.1.18. *Let $\mathfrak{k}$ be the Lie algebra of K. $\mathfrak{g}_2$ consists of exactly those $\psi \in \mathfrak{k}$ with $\psi(1) = 0$.*

Since K acts transitively on the unit sphere in $\mathbb{O}$, we obtain a second corollary from the above lemma:

Corollary 2.1.19. *The coset space K/G_2 is diffeomorphic to the 7-sphere. Moreover, K is connected and a Lie subgroup of $SO(8)$.*

The last part of the corollary can be shown by similar arguments as in the proof of Lemma 2.1.16. We are now able to describe K explicitly:

Proposition 2.1.20. *The group generated by the left multiplication with unit octonions is isomorphic to Spin(7) and acts on $\mathbb{O} \cong \mathbb{R}^8$ by the spinor representation.*

Proof : Since $K/G_2 \cong S^7$, K is a G_2-bundle over S^7. We have the following exact sequence:

$$\ldots \to \pi_1(G_2) \to \pi_1(K) \to \pi_1(S^7) \to \ldots .$$

G_2 and S^7 are both simply connected. The above sequence therefore becomes:

$$\ldots \to \{0\} \to \pi_1(K) \to \{0\} \to \ldots$$

and we have shown that K is simply connected. Let $\mathfrak{k}$ be the Lie algebra of K and q be a biinvariant metric on K. The q-orthogonal complement $\mathfrak{m}$ of $\mathfrak{g}_2 \subseteq \mathfrak{k}$ is a $\mathfrak{g}_2$-module. On page 25, we will show that the non-trivial $\mathfrak{g}_2$-module with the lowest dimension is seven-dimensional. Therefore, $\mathfrak{m}$ is either a trivial module or equivalent to $\mathbb{R}^7$ with $\mathfrak{g}_2 \subseteq \mathfrak{gl}(7)$ acting by matrix multiplication. In the first case, G_2 would be a normal subgroup of K. If this was the case, $L_i G_2 L_{-i}$ would be contained in G_2. Let Φ be an arbitrary automorphism of $\mathbb{O}$. If our assumption was true, $\Psi : \mathbb{O} \to \mathbb{O}$ with $\Psi(x) := i\Phi(-ix)$ would have to fix 1. Since there are Φ with $\Phi(-i) \neq -i$, G_2 is not a normal subgroup of K. Since K is compact, $\mathfrak{k}$ is the direct sum of an abelian and a semisimple Lie algebra. $\dim \mathfrak{k} = 21$ and the only possibilities for $\mathfrak{k}$ therefore are:

1. $\mathfrak{so}(6) \oplus \mathfrak{h}$, where $\mathfrak{h}$ is a six-dimensional Lie algebra,
2. $\mathfrak{so}(7)$,
3. $\mathfrak{sp}(3)$.

In the first case, we had shown the existence of a six-dimensional faithful representation of $\mathfrak{g}_2$, since $\mathfrak{g}_2 \subseteq \mathfrak{so}(6) \subseteq \mathfrak{gl}(6)$. Since such a representation does not exist, we can exclude that case. In the third case, we had $\mathfrak{g}_2 \subseteq \mathfrak{sp}(3) \subseteq \mathfrak{gl}(6, \mathbb{C})$. Therefore, that case can be excluded for the same reasons as the first one. Since K is simply connected, we have shown that $K = \mathrm{Spin}(7)$. The orbits of the action of K on $\mathbb{O}$ are seven-spheres. Therefore, K acts irreducibly on $\mathbb{O}$ and the action of K has to be given by the spinor representation. □

Remark 2.1.21. We can describe G_2 as $\mathrm{Spin}(7) \cap SO(7)$, where the first group acts by the spinor representation on $\mathbb{O}$ and the second one acts on $\mathrm{Im}(\mathbb{O})$.

Convention 2.1.22. We will sometimes denote the Lie algebra of Spin(7) by $\mathfrak{spin}(7)$, when we want to emphasize the fact that it acts on $\mathbb{O} \cong \mathbb{R}^8$ by the spinor representation.

We will now construct certain forms with stabilizer G_2 or Spin(7). Later on, we will see that we can describe any G_2- or Spin(7)-structure with help of those forms. We identify the basis $(i, j, k, \epsilon, i\epsilon, j\epsilon, k\epsilon)$ of $\mathrm{Im}(\mathbb{O})$ with the standard basis $(e_1, \dots, e_7)$ of $\mathbb{R}^7$. The canonical three-form $\omega \in \bigwedge^3(\mathrm{Im}(\mathbb{O}))^*$ can also be identified with a three-form on $\mathbb{R}^7$, which we denote by ω, too. We obtain:

$$\omega = dx^{123} + dx^{145} - dx^{167} + dx^{246} + dx^{257} + dx^{347} - dx^{356}\,. \tag{2.2}$$

The dx^i are as usual defined by $dx^i(e_j) := \delta^i_j$ and we further have defined $dx^{ijk} := dx^i \wedge dx^j \wedge dx^k$. We can describe the canonical four-form on $\mathbb{O}$ in the same way and obtain:

$$\omega^* = -dx^{1247} + dx^{1256} + dx^{1346} + dx^{1357} - dx^{2345} + dx^{2367} + dx^{4567} \tag{2.3}$$

where $dx^{ijkl} := dx^i \wedge dx^j \wedge dx^k \wedge dx^l$. We equip $\mathbb{R}^7$ with the standard scalar product and the volume form vol with $\mathrm{vol}(e_1, e_2, \dots, e_7) = 1$. It is easy to see that

$$\omega^* = *\omega$$

where $* : \bigwedge^3(\mathbb{R}^7)^* \to \bigwedge^4(\mathbb{R}^7)^*$ is the Hodge star operator. We denote the standard basis of $\mathbb{R}^8$ by $(e_0, \dots, e_7)$. By mapping this basis to $(1, i, j, k, \epsilon, i\epsilon, j\epsilon, k\epsilon)$, we can identify $\mathbb{O}$ and $\mathbb{R}^8$. We choose an orientation on $\mathbb{O}$ such that $(1, i, j, k, \epsilon, i\epsilon, j\epsilon, k\epsilon)$ is positive oriented. The four-form

$$\Omega := *\omega + dx^0 \wedge \omega$$

is self-dual. We can write down Ω explicitly:

$$\begin{aligned}\Omega = {} & dx^{0123} + dx^{0145} - dx^{0167} + dx^{0246} + dx^{0257} + dx^{0347} - dx^{0356} \\ & - dx^{1247} + dx^{1256} + dx^{1346} + dx^{1357} - dx^{2345} + dx^{2367} + dx^{4567}\,.\end{aligned} \tag{2.4}$$

Convention 2.1.23. From now on, we will often identify $\mathbb{R}^7$ with $\mathrm{Im}(\mathbb{O})$ and $\mathbb{R}^8$ with $\mathbb{O}$ by the above identifications of the bases.

The forms ω, ω^*, and Ω are clearly G_2-invariant. In the following, we will determine their stabilizer and annihilator exactly:

Definition 2.1.24. Let V be a vector space and $\alpha : V^n \to \mathbb{R}$ a multilinear map. The *stabilizer* $\mathrm{Stab}(\alpha)$ *of* α is the set of all linear maps $\Phi : V \to V$ with $\Phi^*\alpha = \alpha$, i.e.,

$$\alpha(\Phi(v_1), \dots, \Phi(v_n)) = \alpha(v_1, \dots, v_n) \quad \forall v_1, \dots, v_n \in V\,.$$

The stabilizer obviously is a closed group, hence:

Lemma 2.1.25. *The stabilizer of a multilinear map $\alpha : V^n \to \mathbb{R}$ is a Lie subgroup of $GL(V)$.*

Definition 2.1.26. Let V be a vector space and $\alpha : V^n \to \mathbb{R}$ a multilinear map. The *annihilator* $\mathrm{Ann}(\alpha)$ *of* α is the set of all linear maps $\phi : V \to V$ with

$$\alpha(\phi(x_1), x_2, \ldots, x_n) + \alpha(x_1, \phi(x_2), \ldots, x_n) + \ldots + \alpha(x_1, x_2, \ldots, \phi(x_n)) = 0\,.$$

Lemma 2.1.27. *The annihilator of a multilinear map together with the bracket $[\phi, \psi] := \phi \circ \psi - \psi \circ \phi$ is a Lie algebra. Moreover, it is the Lie algebra of the stabilizer group.*

We are now able to formulate our theorem:

Theorem 2.1.28. *Let $\omega \in \bigwedge^3 Im(\mathbb{O})^*$, $\omega^* \in \bigwedge^4 Im(\mathbb{O})^*$, and $\Omega \in \bigwedge^4 \mathbb{O}^*$ be the forms we have constructed above. For these forms the following statements are true:*

1. *The stabilizer of ω is G_2.*
2. *The stabilizer of ω^* is $G_2 \times \{-id, id\}$.*
3. *The stabilizer of Ω is $Spin(7)$.*
4. *The annihilator of ω and ω^* is $\mathfrak{g}_2$.*
5. *The annihilator of Ω is $\mathfrak{spin}(7)$ acting on $\mathbb{O}$ by the spinor representation.*

Proof : The last two statements directly follow from the first three ones. A proof of the first statement can be found in a paper of Robert Bryant [14]. The stabilizer of Ω has been determined in [14], too. In the diploma thesis of the author [61], it has been shown that $\mathrm{Stab}(\omega^*) = G_2 \times \{-\mathrm{id}, \mathrm{id}\}$.

□

Remark 2.1.29. The orbit of $\omega \in \bigwedge^3(\mathbb{R}^7)^*$ with respect to the canonical action of $GL(7)$ is an open set. This can be easily shown by the following calculation:

$$\dim GL(\mathbb{R}^7) - \dim \mathrm{Stab}(\omega) = 49 - 14 = 35 = \dim \bigwedge^3 (\mathbb{R}^7)^*\,.$$

Moreover: On $\mathbb{R}^7$ there is another three-form ω', which is stabilized by the split real form of the complex Lie group G_2. The union of the orbits of ω and ω' is an open, dense subset of $\bigwedge^3(\mathbb{R}^7)^*$ (see Reichel [60] and Schouten [62]).

At the end of this section, we will briefly investigate the representations of $\mathfrak{g}_2$ and $\mathfrak{spin}(7)$. Furthermore, we will decompose some modules, which will become interesting for us, into irreducible submodules. We start with the representations of $\mathfrak{g}_2$. Since $\mathfrak{g}_2$ is a simple Lie algebra, all representations of $\mathfrak{g}_2$ are either faithful or trivial. The center of G_2 contains only the neutral element. Therefore, the representations of G_2 are either faithful or trivial, too. We denote the irreducible representation of $\mathfrak{g}_2$ with highest weight (a, b) by $\mathbb{V}_{a,b}$. The fundamental representations $\mathbb{V}_{1,0}$ and $\mathbb{V}_{0,1}$ of $\mathfrak{g}_2$ have the following description:

1. $\mathbb{V}_{1,0}$ is $\mathfrak{g}_2$-equivariantly isomorphic to $\mathbb{R}^7$, where $\mathfrak{g}_2 \subseteq \mathfrak{gl}(7)$ acts on $\mathbb{R}^7$ by matrix multiplication.

2. $\mathbb{V}_{0,1}$ is the adjoint representation of $\mathfrak{g}_2$.

The dimensions of the smallest representations of $\mathfrak{g}_2$ are:

$\dim \mathbb{V}_{a,b}$	b=0	1	2
a= 0	1	14	77
1	7	64	286
2	27	189	729
3	77	448	1547

Before we consider the representations of $\mathfrak{spin}(7)$, we will decompose certain $\mathfrak{g}_2$-modules into submodules. The $\mathfrak{g}_2$-module $\mathfrak{gl}(7) = \mathbb{R}^7 \otimes (\mathbb{R}^7)^*$ decomposes into four irreducible $\mathfrak{g}_2$-submodules:

$$\mathfrak{gl}(7) = \mathbb{V}_{0,0} \oplus \mathbb{V}_{1,0} \oplus \mathbb{V}_{0,1} \oplus \mathbb{V}_{2,0} \,, \tag{2.5}$$

where

- $\mathbb{V}_{0,0}$ consists of the identity map on $\mathbb{R}^7$ and its multiples,
- $\mathbb{V}_{1,0}$ consists of the maps $L_x - R_x$, where $x \in \mathrm{Im}(\mathbb{O}) \cong \mathbb{R}^7$,
- $\mathbb{V}_{0,1}$ is $\mathfrak{g}_2 \subseteq \mathfrak{gl}(7)$, and
- $\mathbb{V}_{2,0}$ are the trace-free, symmetric 7×7-matrices.

The decomposition of the spaces $\bigwedge^k(\mathbb{R}^7)^*$ is useful to know in the context of G_2-structures. There are the following splittings which also can be found in Bryant [14]:

$$\begin{aligned}
\bigwedge^0(\mathbb{R}^7)^* &\cong \mathbb{V}_{0,0}\\
\bigwedge^1(\mathbb{R}^7)^* &\cong \mathbb{V}_{1,0}\\
\bigwedge^2(\mathbb{R}^7)^* &\cong \mathfrak{so}(7) \cong \mathbb{V}_{1,0} \oplus \mathbb{V}_{0,1}\\
\bigwedge^3(\mathbb{R}^7)^* &\cong \mathbb{V}_{0,0} \oplus \mathbb{V}_{1,0} \oplus \mathbb{V}_{2,0}
\end{aligned}$$

The trivial summand of $\bigwedge^3(\mathbb{R}^7)^*$ simply is $\mathrm{span}(\omega)$. Since the Hodge star operator is $\mathfrak{so}(7)$- and therefore $\mathfrak{g}_2$-equivariant, the spaces $\bigwedge^k(\mathbb{R}^7)^*$ and $\bigwedge^{7-k}(\mathbb{R}^7)^*$ are $\mathfrak{g}_2$-equivariantly isomorphic. For this reason, the decomposition of $\bigwedge^*(\mathbb{R}^7)^*$ is determined by the above splittings. In the following chapter, we need to know the *normalizer of G_2 in $SO(7)$*, i.e., the group:

$$\mathrm{Norm}_{SO(7)} G_2 := \{g \in SO(7) | g G_2 g^{-1} = G_2\} \,.$$

Let $g \in SO(7)$ be arbitrary. We want to check if $ghg^{-1} \in G_2$ for all $h \in G_2$. This statement is equivalent to:

$$\omega(ghg^{-1}(x), ghg^{-1}(y), ghg^{-1}(z)) = \omega(x, y, z) \quad \forall x, y, z \in \mathbb{R}^7, h \in G_2 .$$

We define $u := g^{-1}(x)$, $v := g^{-1}(y)$, and $w := g^{-1}(z)$. The above equation now becomes:

$$h^* (g^* \omega) = g^* \omega .$$

Therefore, we have to check for which $g \in SO(7)$ the pull-back $g^*\omega$ is G_2-invariant. Since the maximal trivial $\mathfrak{g}_2$-submodule of $\bigwedge^3(\mathbb{R}^7)^*$ is one-dimensional, the only G_2-invariant three-forms are the multiples of ω. For this reason, there has to exist a $\lambda \in \mathbb{R}$ with $g^*\omega = \lambda\omega$. We conclude that g is an element of the group $\sqrt[3]{\lambda} \cdot G_2$. Since $g \in SO(7)$, λ has to equal 1. We obviously have $G_2 \subseteq \mathrm{Norm}_{SO(7)} G_2$ and therefore have proven the following lemma:

Lemma 2.1.30. *The normalizer group $Norm_{SO(7)} G_2$ of G_2 in $SO(7)$ is G_2.*

Before we proceed to the representations of $\mathfrak{spin}(7)$, we decompose $\mathfrak{spin}(7)$ into irreducible $\mathfrak{g}_2$-submodules:

Lemma 2.1.31. *Let $\mathfrak{spin}(7)$ be the Lie algebra of Spin(7), which acts on $\mathbb{O}$ by the spinor representation. In this situation, we have:*

$$\mathfrak{spin}(7) = \mathfrak{g}_2 \oplus L_{Im(\mathbb{O})} ,$$

where $L_{Im(\mathbb{O})}$ is the vector space of all left multiplications by imaginary octonions. Moreover, $L_{Im(\mathbb{O})}$ is a $\mathfrak{g}_2$-module, which is equivalent to $\mathbb{R}^7$.

Proof : In Corollary 2.1.19 we have shown that $\mathrm{Spin}(7)/G_2 = S^7$, where S^7 can be identified with the unit sphere in $\mathbb{O}$. More precisely, any $x \in S^7$ corresponds to the coset $L_x G_2 \subseteq \mathrm{Spin}(7)$. We consider Spin(7) as a G_2-bundle over S^7 and split the tangent space $T_e\mathrm{Spin}(7) \cong \mathfrak{spin}(7)$ into its horizontal and vertical part. The lemma easily follows from the observation that the tangent space of S^7 at $1 \in \mathbb{O}$ coincides with $\mathrm{Im}(\mathbb{O})$.

$\square$

Finally, we briefly consider the representations of the Lie algebra $\mathfrak{spin}(7)$, which is isomorphic to $\mathfrak{so}(7)$. Since $\mathfrak{so}(7)$ is a simple Lie algebra, the representations of $\mathfrak{so}(7)$ are either faithful or trivial. The center of Spin(7) is isomorphic to $\mathbb{Z}_2$. Therefore, the non-trivial representations of Spin(7) either are faithful or have $\mathbb{Z}_2$ as kernel. The latter can be considered as representations of $SO(7)$. The rank of $\mathfrak{so}(7)$ is 3 and its Dynkin diagram is B_3. We denote the irreducible representation of $\mathfrak{so}(7)$ with highest weight (a, b, c) by $\mathbb{V}_{a,b,c}$. The fundamental representations of $\mathfrak{so}(7)$ can be described as follows:

1. $\mathbb{V}_{1,0,0}$ is the vector representation of $\mathfrak{so}(7)$.
2. $\mathbb{V}_{0,1,0}$ is the representation of $\mathfrak{so}(7)$ on $\bigwedge^2(\mathbb{R}^7)^*$ or equivalently the adjoint representation.

3. $\mathbb{V}_{0,0,1}$ is the spinor representation of $\mathfrak{so}(7)$.

As in the $\mathfrak{g}_2$-case, it is possible to decompose the spaces $\bigwedge^k(\mathbb{R}^8)^* \cong \bigwedge^k \mathbb{V}_{0,0,1}$ into irreducible $\mathfrak{so}(7)$-modules. This work has been done in [14] and the following decompositions have been obtained:

$$\begin{aligned}
\bigwedge^0(\mathbb{R}^8)^* &\cong \mathbb{V}_{0,0,0} \\
\bigwedge^1(\mathbb{R}^8)^* &\cong \mathbb{V}_{0,0,1} \\
\bigwedge^2(\mathbb{R}^8)^* &\cong \mathbb{V}_{0,1,0} \oplus \mathbb{V}_{1,0,0} \\
\bigwedge^3(\mathbb{R}^8)^* &\cong \mathbb{V}_{1,0,1} \oplus \mathbb{V}_{0,0,1} \\
\bigwedge^4(\mathbb{R}^8)^* &\cong \mathbb{V}_{2,0,0} \oplus \mathbb{V}_{0,0,2} \oplus \mathbb{V}_{1,0,0} \oplus \mathbb{V}_{0,0,0}
\end{aligned}$$

Since the Hodge star operator on $\bigwedge^*(\mathbb{R}^8)^*$ is $\mathfrak{spin}(7)$-equivariant, we are able to decompose all of $\bigwedge^*(\mathbb{R}^8)^*$. The irreducible submodules of that space can be described explicitly (see Bryant [14]). For example, we have:

1. $\bigwedge^3(\mathbb{R}^8)^* \supseteq \mathbb{V}_{0,0,1} = \{*(\Omega \wedge \alpha) | \alpha \in (\mathbb{R}^8)^*\}$
2. $\bigwedge^3(\mathbb{R}^8)^* \supseteq \mathbb{V}_{1,0,1} = \{\beta \in \bigwedge^3(\mathbb{R}^8)^* | \Omega \wedge \beta = 0\}$
3. $\bigwedge^4(\mathbb{R}^8)^* \supseteq \mathbb{V}_{0,0,0} = \{c\,\Omega | c \in \mathbb{R}\}$
4. $\bigwedge^4(\mathbb{R}^8)^* \supseteq \mathbb{V}_{0,0,2} = \{\gamma \in \bigwedge^4(\mathbb{R}^8)^* | *\gamma = -\gamma\}$

The dimensions of the irreducible submodules of $\bigwedge^*(\mathbb{R}^8)^*$, which are no fundamental representations of $\mathfrak{so}(7)$, are:

1. $\dim \mathbb{V}_{1,0,1} = 48$
2. $\dim \mathbb{V}_{2,0,0} = 27$
3. $\dim \mathbb{V}_{0,0,2} = 35$

2.2 G_2-structures

In this section, we will introduce the most elementary facts on G_2-structures. Before we start, we first have to make some definitions:

Definition 2.2.1.
1. Let M be a n-dimensional smooth manifold. The *coframe bundle of* M is the set of all bijective linear maps $u : T_pM \to \mathbb{R}^n$, where $p \in M$ is arbitrary. This set can be given the structure of a $GL(n)$-principal bundle over M.
2. Let G be a Lie subgroup of $GL(n)$. A *G-structure on M* is a reduction of the coframe bundle to a principal bundle with structure group G.
3. Whenever we write about G_2-structures, we consider G_2 as a Lie subgroup of $GL(7)$. The underlying manifold M therefore has to be seven-dimensional.
4. A seven-dimensional manifold equipped with a fixed G_2-structure we will call a *G_2-manifold*.

In Theorem 2.1.28, we have seen that the stabilizer of the three-form $\omega \in \bigwedge^3(\mathbb{R}^7)^*$ coincides with G_2. Therefore, we can describe any G_2-structure by a three-form:

Lemma 2.2.2. *Let M be a seven-dimensional manifold and let $\widetilde{\omega}$ be a three-form on M with the following property: For each $p \in M$, there exist an open neighborhood U of p and linearly independent vector fields $X_1, \ldots, X_7$ on U with*

$$\widetilde{\omega}|_U = X^{123} + X^{145} - X^{167} + X^{246} + X^{257} + X^{347} - X^{356}\,.$$

In the above formula, X^i is defined by $X^i(X_j) := \delta^i_j$ and X^{ijk} by $X^i \wedge X^j \wedge X^k$. If this is the case

$$\begin{aligned}\mathfrak{G}_{G_2} := \{u : T_pM \to \mathbb{R}^7 |\; u \text{ linear},\; p \in M,\; &\widetilde{\omega}_p(X,Y,Z) = \omega(u(X),u(Y),u(Z))\\ &\forall\, X,Y,Z \in T_pM\}\,,\end{aligned}$$

where $\omega \in \bigwedge^3(\mathbb{R}^7)^*$ denotes the three-form (2.2), is a G_2-structure on M. Conversely, let $\mathfrak{G}_{G_2}$ be a G_2-structure on M. Then

$$\widetilde{\omega}_p(X,Y,Z) := \omega(u(X),u(Y),u(Z))\,, \quad \text{where } \mathfrak{G}_{G_2} \ni u : T_pM \to \mathbb{R}^7$$

determines a well-defined, smooth three-form on M with the above properties.

In the following, we will often denote G_2-manifolds as pairs of a manifold and a three-form. With help of the decompositions we have found in Section 2.1, we are able to construct G_2-invariant objects on a G_2-manifold:

Lemma 2.2.3. *Let M be a G_2-manifold with a G_2-structure $\mathfrak{G}_{G_2}$ and let $\phi : G_2 \to GL(V)$ be a representation. We denote the maximal trivial submodule of V by W. The G_2-invariant sections of the vector bundle $\mathfrak{G}_{G_2} \times_{G_2} V$ are exactly the sections of $\mathfrak{G}_{G_2} \times_{G_2} W$.*

In particular, we find the following invariant objects:

Lemma 2.2.4. *Let (M, ω) be a G_2-manifold. The following objects on M are G_2-invariant:*

1. *The three-form ω.*
2. *The metric g and the volume form vol which are defined by*

$$g(X,Y)vol := -\frac{1}{6}\,(X\lfloor\omega) \wedge (Y\lfloor\omega) \wedge \omega\,,$$

where $X\lfloor\omega := \omega(X,\cdot,\cdot)$ denotes the interior product of X and ω. Let $u : T_pM \to \mathbb{R}^7$ be an element of the G_2-structure. We can obtain the value of $g_p(X,Y)$ by inserting $u^{-1}(e_1),\ldots,u^{-1}(e_7)$ in the right hand side of the above equation. Analogously, we can obtain $vol_p(Z_1,\ldots,Z_7)$ by setting $X_p := u^{-1}(e_1)$ and $Y_p := u^{-1}(e_2)$.

3. *The Hodge star operator $*$ associated to g and vol is obviously G_2-equivariant. Therefore, the four-form $*\omega$ is G_2-invariant, too. For any $p \in M$, $*\omega_p$ can be identified by an element of the G_2-structure with $\omega^* \in \bigwedge^4(\mathbb{R}^7)^*$.*
4. *The spinor representation of $\mathfrak{so}(7)$ is eight-dimensional and can be identified with $\mathbb{O}$. Since $G_2 \subseteq Spin(7)$, it is a representation of G_2, too. The associated vector bundle of this representation is the spinor bundle of M. Since $\mathbb{O}$ splits into $\mathbb{R} \oplus Im(\mathbb{O})$ with respect to the action of G_2, the spinor bundle splits into a seven-dimensional and a one-dimensional G_2-invariant subbundle. Therefore, there exist G_2-invariant spinors on M.*

From this lemma, we immediately obtain the following important corollary:

Corollary 2.2.5. *Let (M, ω) be a G_2-manifold. Then M is orientable and admits a canonical spin structure.*

Convention 2.2.6. Let (M, ω) be a G_2-manifold. We call the tensor field ω $(*\omega, g, \mathrm{vol})$ the *canonical 3-form (4-form, metric, volume form)* on M. The orientation of M induced by vol we will call the *canonical orientation* of M. Analogously, the spin-structure induced by the inclusion $G_2 \subseteq \mathrm{Spin}(7)$ will be called the *canonical spin structure*. Alternatively, we often call these objects the *objects associated to the G_2-structure*.

The converse of Corollary 2.2.5 is also true:

Lemma 2.2.7. *(See Lawson, Michelsohn [53], pp. 348f.) Let M be a seven-dimensional orientable manifold which admits a spin structure. Then M admits a G_2-structure, too.*

Proof : Let N be a manifold. It is known that if the dimension of the real spinor bundle is greater than $\dim N$, there exists a nowhere vanishing spinor on N (see Isham, Pope, Warner [45]). Let η be a nowhere vanishing spinor on M, which has to exist, since $8 > 7$. A G_2-structure on M can be defined by:

$$\omega^{ijk} := \eta^\dagger \gamma^{ijk} \eta$$

where the γ^{ijk} are the gamma matrices.

$\square$

We call a connection ∇ on the tangent bundle *compatible with the G_2-structure* if $\nabla\omega = 0$. The G_2-invariant objects which we have defined above are parallel with respect to any compatible connection. Moreover:

Lemma 2.2.8. *Let M be a seven-dimensional manifold with a G_2-structure $\mathfrak{G}_{G_2}$ and let ∇ be a connection on TM which is compatible with the G_2-structure. Furthermore, let V be a G_2-module. We can extend ∇ to a connection on the vector bundle $\mathfrak{G}_{G_2} \times_{G_2} V$. The dimension of the maximal trivial submodule of V we denote by n. In this situation, the set of all G_2-invariant parallel sections of $\mathfrak{G}_{G_2} \times_{G_2} V$ is a n-dimensional vector space.*

In particular, the tensor fields ω, g, vol, $*\omega$ are parallel and G_2-invariant. Moreover, they are up to constant multiples the only elements of $\bigwedge^3 T^*M$, $S^2(T^*M)$, $\bigwedge^7 T^*M$, and $\bigwedge^4 T^*M$ with these properties. Finally, we have up to constant multiples exactly one G_2-invariant spinor on M which is parallel with respect to ∇.

Remark 2.2.9.

1. From the decomposition of $\bigwedge^*(\mathbb{R}^7)^*$ we have found in Section 2.1, we can conclude that the constant functions, ω, $*\omega$, and vol are up to constant multiples the only G_2-invariant parallel forms on M.

2. If the holonomy of ∇ is all of G_2, the above sections are the only parallel sections of the considered bundles. If the holonomy is smaller, more parallel sections, which are not necessarily G_2-invariant, may exist. As an example we consider the canonical G_2-structure on the flat $\mathbb{R}^7$. The space of all parallel three-forms on $\mathbb{R}^7$ has dimension 35. Since $\bigwedge^3(\mathbb{R}^7)^*$ contains exactly one trivial summand, the space of all G_2-invariant parallel three-forms on $\mathbb{R}^7$ is only one-dimensional.

Since $\mathfrak{g}_2 \subsetneq \mathfrak{so}(7)$, a torsion-free compatible connection on a given G_2-manifold does not necessarily exist. The obstruction for the existence of such a connection is the intrinsic torsion:

Definition 2.2.10. Let G be a Lie subgroup of $O(n)$ and $\mathfrak{G}_G$ a G-structure on a n-dimensional manifold M. We denote the Lie algebra of G by $\mathfrak{g}$.

1. The vector bundle over M associated to the adjoint representation of G, is called the *adjoint vector bundle* $\mathfrak{g}M$.

2. A connection ∇ on TM is called *compatible with the G-structure* $\mathfrak{G}_G$ if ∇_X is a section of $\mathfrak{g}M$ for all vector fields X.

3. We define the map

$$\begin{aligned} \delta : \mathfrak{g}M \otimes T^*M &\to TM \otimes \bigwedge^2 T^*M \\ \delta(\alpha)(X,Y) &:= \alpha(X)(Y) - \alpha(Y)(X) \end{aligned}$$

and the G-invariant vector bundle

$$H^{0,2}M := (TM \otimes \bigwedge^2 T^*M)/\delta(\mathfrak{g}M \otimes T^*M)\,.$$

4. Let Θ be the torsion of an arbitrary compatible connection. The *intrinsic torsion of the G-structure* $\mathfrak{G}_G$ is defined as $\Theta^{int} := \Theta + \delta(\mathfrak{g}M \otimes T^*M)$ and is a section of $H^{0,2}M$.

5. $\mathfrak{G}_G$ is called *parallel* or *torsion-free* if $\Theta^{int} = 0$ on all of M.

Remark 2.2.11. 1. The adjoint vector bundle is a subbundle of the endomorphism bundle $TM \otimes T^*M$. This fact explains the notation in the definition of δ.

2. Both notions of a compatible connection ∇ on a G_2-manifold coincide, i.e., $\nabla\omega = 0$ is equivalent to $\nabla_X \in \mathfrak{g}_2 M$.

3. Let VM be the vector bundle associated to the orthogonal complement of $\mathfrak{g} \subseteq \mathfrak{so}(n)$ with respect to a G-invariant inner product on $\mathfrak{so}(7)$. $H^{0,2}M$ is G-equivariantly isomorphic to $VM \otimes T^*M$.

4. Θ^{int} is independent of the choice of the compatible connection and therefore well-defined.

We return to the case, where G coincides with $G_2 \subseteq O(7)$. It is possible to show that the intrinsic torsion is determined by $d\omega$ and $d * \omega$:

Lemma 2.2.12. *Let (M, ω) be a G_2-manifold with intrinsic torsion Θ^{int}. There is an injective, linear, and G_2-equivariant map*

$$\varphi : H^{0,2}M \to \bigwedge^4 T^*M \oplus \bigwedge^5 T^*M$$

such that $\varphi(\Theta^{int}) = (d\omega, d * \omega)$.

In our situation, $H^{0,2}M$ can be identified with $TM \otimes T^*M$. The fibers of $H^{0,2}M$ therefore decompose into $\mathbb{V}_{0,0} \oplus \mathbb{V}_{1,0} \oplus \mathbb{V}_{0,1} \oplus \mathbb{V}_{2,0}$. Depending on the subbundle of which the intrinsic torsion is a section there are 16 types of G_2-structures. Because of Lemma 2.2.12, each of the 16 types can be described by an equation which $d\omega$ and $d * \omega$ have to satisfy. Those equations have been determined by Fernandez and Gray [34]. For our considerations, we are only interested in the following types of G_2-structures:

Definition 2.2.13. A G_2-manifold (M, ω) with intrinsic torsion Θ^{int} is called

1. *symplectic* or *closed* if $d\omega = 0$. This condition is equivalent to $\Theta^{int} \in \mathbb{V}_{0,1}$.

2. *cosymplectic* or *coclosed* if $d * \omega = 0$. This condition is equivalent to $\Theta^{int} \in \mathbb{V}_{0,0} \oplus \mathbb{V}_{2,0}$.

3. *nearly parallel* if $d\omega = \lambda * \omega$ and $d * \omega = 0$ for a $\lambda : M \to \mathbb{R}$. This condition is equivalent to $\Theta^{int} \in \mathbb{V}_{0,0}$.

4. *parallel* or *torsion-free* if the intrinsic torsion vanishes.

Remark 2.2.14. We can conclude from $d\omega = \lambda * \omega$ the equation $d * \omega = 0$ only if λ does not vanish. Since there are G_2-manifolds with $d\omega = 0$ but $d*\omega \neq 0$, we have to include the second condition in the definition of nearly parallel.

We now collect some properties of nearly parallel G_2-manifolds:

Proposition 2.2.15. *(See Friedrich, Kath, Moroianu, and Semmelmann [37].) Let (M, ω) be a nearly parallel G_2-manifold. The canonical metric g on M is an Einstein metric. Its Einstein constant is equal to $\frac{3}{8}\lambda^2$, where λ is taken from the definition of "nearly parallel". In particular, λ has to be constant. Moreover, there exists a G_2-invariant Killing spinor η on M, i.e.:*

$$\nabla^g_X \eta = \mu X \cdot \eta \quad \forall \text{ vector fields } X \, ,$$

where the dot denotes the Clifford multiplication, ∇^g the Levi-Civita connection, and $\mu = -\frac{1}{8}\lambda$ is a constant.

There are the following equivalent conditions for the torsion-freeness of a G_2-manifold:

Theorem 2.2.16. *(See Fernandez, Gray [34].) Let (M, ω) be a G_2-manifold. We denote the canonical metric on M by g and the Levi-Civita connection of g by ∇^g. In this situation, the following statements are equivalent:*

1. *The intrinsic torsion of the G_2-structure vanishes.*
2. *The holonomy of g is a subgroup of G_2.*
3. *$\nabla^g \omega = 0$.*
4. *$d\omega = 0$ and $d * \omega = 0$.*

In this context, the following fact should be mentioned:

Lemma 2.2.17. *Let (M, g) be a seven-dimensional oriented Riemannian manifold whose holonomy is contained in G_2. Then, there exists a parallel G_2-structure on M whose associated metric is g and whose associated orientation is the same as of M. If the holonomy is all of G_2, the G_2-structure is moreover unique.*

We finally state the following facts about parallel G_2-manifolds:

Proposition 2.2.18. *(See Bonan [12].) The canonical metric on a parallel G_2-manifold is Ricci-flat.*

Proposition 2.2.19. *(See Joyce [47], p. 245.) The restricted holonomy group of the Levi-Civita connection on a parallel G_2-manifold is one of the following:*

1. *G_2,*
2. *$SU(3)$, which acts irreducibly on a six-dimensional subspace of the tangent space and trivially on its orthogonal complement,*
3. *$SU(2)$, which acts irreducibly on a four-dimensional subspace and trivially on its orthogonal complement,*
4. *the trivial group $\{1\}$.*

Proposition 2.2.20. *(See Wang [66].) Let (M, g) be a seven-dimensional Riemannian manifold with holonomy G_2. The space of all spinors which are parallel with respect to the Levi-Civita connection is one-dimensional.*

2.3 Spin(7)-structures

In this section, we will present some facts about Spin(7)-structures. Many of this facts will be similar to the corresponding results in Section 2.2. Analogously to that case, we call an eight-dimensional manifold with a fixed Spin(7)-structure a *Spin*(7)*-manifold*. Since the stabilizer of the four-form (2.4) is Spin(7), we can describe any Spin(7)-structure by a four-form:

Lemma 2.3.1. *Let M be an eight-dimensional manifold and let $\widetilde{\Omega}$ be a four-form on M with the following property: For each $p \in M$ there exists an open neighborhood U of p and linearly independent vector fields $X_0, \ldots, X_7$ on U with*

$$\begin{aligned}\widetilde{\Omega}|_U = {} & X^{0123} + X^{0145} - X^{0167} + X^{0246} + X^{0257} + X^{0347} - X^{0356} \\ & - X^{1247} + X^{1256} + X^{1346} + X^{1357} - X^{2345} + X^{2367} + X^{4567}\,.\end{aligned}$$

In the above formula, X^i is defined by $X^i(X_j) := \delta^i_j$ and X^{ijkl} by $X^i \wedge X^j \wedge X^k \wedge X^l$. If this is the case,

$$\begin{aligned}\mathfrak{G}_{Spin(7)} := \{u : T_pM \to \mathbb{R}^8 \,|\, & u \text{ linear},\ \ p \in M,\ \ \widetilde{\Omega}_p(W,X,Y,Z) = \Omega(u(W),u(X),u(Y),u(Z)) \\ & \forall\, W,X,Y,Z \in T_pM\}\,,\end{aligned}$$

where $\Omega \in \bigwedge^4(\mathbb{R}^8)^*$ is the four-form (2.4), is a Spin(7)-structure on M. Conversely, let $\mathfrak{G}_{Spin(7)}$ be a Spin(7)-structure on M. Then

$$\widetilde{\Omega}_p(W,X,Y,Z) := \Omega(u(W),u(X),u(Y),u(Z))\,, \quad \textit{where } \ \mathfrak{G}_{Spin(7)} \ni u : T_pM \to \mathbb{R}^8$$

determines a well-defined, smooth four-form on M with the above properties.

Because of the above lemma, we will denote Spin(7)-manifolds as pairs of a manifold and a four-form. The Spin(7)-invariant objects on a Spin(7)-manifold can be obtained by a construction analogous to Lemma 2.2.3. The most important of them are given by the following lemma:

Lemma 2.3.2. *Let (M,Ω) be a Spin(7)-manifold. The following objects on M are Spin(7)-invariant:*

1. *$\Omega \wedge \Omega$, which is a volume form on M. We call $\Omega \wedge \Omega$ the canonical volume form vol.*
2. *There is a canonical metric g on (M,Ω) which is determined by the following relation (see Karigiannis [50]):*

   $$6\,|X \wedge Y|^2\, vol = (X \lfloor Y \lfloor \Omega) \wedge (X \lfloor Y \lfloor \Omega) \wedge \Omega\,.$$

   In the above formula, $|\,.\,|$ denotes the norm on the fibers of $\bigwedge^2 TM$ which is induced by g.

Corollary 2.3.3. *Any Spin(7)-manifold is orientable.*

Remark 2.3.4. 1. As in Section 2.2, we often call the canonical volume form and metric the *volume form and metric associated to the Spin(7)-structure.*

2. It is easy to prove that Ω is self-dual with respect to the metric g and the orientation induced by $\Omega \wedge \Omega$.

3. With help of the decompositions we have found in Section 2.1, it could be easily proved that the constant function 1, Ω, and $\Omega \wedge \Omega$ are a basis of the space of all Spin(7)-invariant forms on M.

4. We can lift the inclusion Spin(7) $\subseteq SO(8)$ to an inclusion of Spin(7) into Spin(8). Therefore, any Spin(7)-manifold admits a spin-structure. Since the dimension of a Spin(7)-manifold is even, the spinor bundle splits into a positive and a negative part. The positive spinor bundle splits into a one-dimensional and an irreducible part and the negative spinor bundle is irreducible (see Joyce [47] p.256 for details). Therefore, we have up to a multiplication by a scalar function exactly one Spin(7)-invariant spinor on any Spin(7)-manifold (see Wang [66]).

The existence of a spin-structure on an eight-dimensional manifold is not sufficient for the existence of a Spin(7)-structure on that manifold. Moreover, the Euler characteristic of the positive or the negative part of the spinor bundle has to vanish (see Lawson, Michelsohn [53]).

A connection ∇ on the tangent bundle of a Spin(7)-manifold (M, Ω) is compatible with the Spin(7)-structure if and only if $\nabla\Omega = 0$. Analogously to Section 2.2, we have the following lemma:

Lemma 2.3.5. *Let (M, Ω) be a Spin(7)-manifold and let ∇ be a connection on TM which is compatible with the Spin(7)-structure. In this situation, there is up to a constant multiple exactly one positive, parallel, and Spin(7)-invariant spinor and no negative, non-zero, and Spin(7)-invariant spinor on M.*

Our next theme is the intrinsic torsion of a Spin(7)-structure. It is possible to prove by means of representation theory that the fiber $H^{0,2}_p M$ is isomorphic to $\mathbb{V}_{1,0,1} \oplus \mathbb{V}_{0,0,1}$. This is exactly the decomposition of $\bigwedge^5 T^*_p M$ into irreducible Spin(7)-modules. Therefore, it is not surprising that the intrinsic torsion is measured by $d\Omega$ and that we have an analogon to Lemma 2.2.12:

Lemma 2.3.6. *Let (M, Ω) be a Spin(7)-manifold with intrinsic torsion Θ^{int}. There is a bijective, linear, and Spin(7)-equivariant map*

$$\varphi : H^{0,2}M \to \bigwedge^5 T^*M \,,$$

such that $\varphi(\Theta^{int}) = d\Omega$.

There are four types of Spin(7)-structures, which are distinguished by the invariant subbundle of $H^{0,2}M$ of which Θ^{int} is a section. In order to give simple descriptions of each of this types, we define the Lee-form:

Definition 2.3.7. Let (M, Ω) be a Spin(7)-manifold. The one-form

$$\theta := *(*d\Omega \wedge \Omega)$$

is called the *Lee-form* of (M, Ω).

We are now able to describe the four types:

Definition 2.3.8. A Spin(7)-manifold (M, Ω), whose intrinsic torsion and Lee-form we denote by Θ^{int} and θ, is called

1. *parallel* or *torsion-free* if the intrinsic torsion vanishes,
2. *locally conformally parallel* if $\Theta^{int} \in \mathbb{V}_{0,0,1}$ or equivalently if $d\Omega = \theta \wedge \Omega$,
3. *balanced* if $\Theta^{int} \in \mathbb{V}_{1,0,1}$ or equivalently if $\theta = 0$,
4. *generic* if none of the above is the case.

Remark 2.3.9. If $\Theta^{int} \in \mathbb{V}_{0,0,1}$, then there exists for each $p \in M$ an open neighborhood U of p and a conformal change of the Spin(7)-structure on U such that the new Spin(7)-structure is parallel (see Cabrera [16]). This fact justifies the name "locally conformal parallel".

We are first of all interested in parallel Spin(7)-structures. The most important criteria for the torsion-freeness of a Spin(7)-structure are the following:

Theorem 2.3.10. *(See Bryant [14].) Let (M, Ω) be a Spin(7)-manifold. We denote the canonical metric on M by g and its Levi-Civita connection by ∇^g. In this situation, the following statements are equivalent:*

1. *The intrinsic torsion vanishes.*
2. *The holonomy of g is a subgroup of Spin(7).*
3. *$\nabla^g \Omega = 0$.*
4. *$d\Omega = 0$.*

Analogously to Section 2.2, the following two statements can be proven:

Lemma 2.3.11. *Let (M, g) be an eight-dimensional oriented Riemannian manifold whose holonomy is contained in Spin(7). Then, there exists a parallel Spin(7)-structure on M whose associated metric is g and whose associated orientation is the same as of M. If the holonomy is all of Spin(7), the Spin(7)-structure is moreover unique.*

Proposition 2.3.12. *(See Bonan [12].) The canonical metric of a parallel Spin(7)-manifold is Ricci-flat.*

If Ω is parallel, then the holonomy of the canonical metric is contained in Spin(7), but not necessarily equal to Spin(7). According to Joyce [47], there are the following possibilities for the holonomy group:

Proposition 2.3.13. *Let (M, Ω) be a parallel Spin(7)-manifold. The possible restricted holonomy groups of the Levi-Civita connection and their inclusions are given by the diagram below:*

$$\begin{tikzcd}
\{1\} \arrow[r] & SU(2) \arrow[r, equal] \arrow[d] & SU(2) \arrow[r] \arrow[d] & SU(3) \arrow[r] \arrow[d] & G_2 \arrow[d] \\
& SU(2) \times SU(2) \arrow[r] & Sp(2) \arrow[r] & SU(4) \arrow[r] & Spin(7)
\end{tikzcd}$$

The last three groups in the lower row of this diagram act irreducibly on the tangent space. If the restricted holonomy is $SU(2) \times SU(2)$, the first factor acts irreducibly on a four-dimensional subspace of the tangent space and the second one acts irreducibly on its orthogonal complement. The other groups act irreducibly on a certain subspace of the tangent space and trivially on its orthogonal complement, which in each case has a positive dimension. In particular, we have:

1. *$SU(2)$ acts irreducibly on a four-dimensional subspace.*
2. *$SU(3)$ acts irreducibly on a six-dimensional subspace.*
3. *G_2 acts irreducibly on a seven-dimensional subspace.*

Remark 2.3.14. The inclusions $SU(3) \hookrightarrow G_2$ and $SU(4) \hookrightarrow \mathrm{Spin}(7)$ from the above diagram are not given by the identity, but by conjugation by an element of $SO(8)$.

As in the G_2-case, we have:

Proposition 2.3.15. *(See Wang [66].) Let (M, g) be an eight-dimensional Riemannian manifold with holonomy Spin(7). The space of all spinors which are parallel with respect to the Levi-Civita connection is one-dimensional.*

There are several methods to determine the holonomy of a parallel Spin(7)-manifold (M, Ω). We first show how it can be checked if the holonomy is a subset of G_2. If this is the case, the holonomy acts trivially on a non-trivial subspace of the tangent space. Therefore, there exists a parallel vector field on the manifold. It is usually easy to decide if such a vector field exists. If the metric associated to the Spin(7)-structure is complete, we can show by the de Rham decomposition theorem that M is covered by a product with a flat factor. If M even is compact, M has to be finitely covered by a Riemannian product of a flat torus and another manifold. The converse of this statement is also true:

Lemma 2.3.16. *Let (M, Ω) be a compact parallel Spin(7)-manifold whose associated metric we denote by g. We assume that the restricted holonomy of g is one of $SU(2) \times SU(2)$, $Sp(2)$, $SU(4)$, or Spin(7). Then $\pi_1(M)$ is finite.*

Proof : We assume that $\pi_1(M)$ is infinite. Since g is Ricci-flat, it follows from the Cheeger-Gromoll splitting theorem that M has a finite cover, which is globally isometric to $N \times T^k$, where N is a simply connected, compact manifold and T^k is a flat k-dimensional torus. Since $|\pi_1(M)| = \infty$, we have $k \geqslant 1$. In this situation, the restricted holonomy acts trivially on a at least one-dimensional subspace of the tangent space. Therefore, the restricted holonomy cannot be one of the above groups.

□

The following lemma will help us decide if the holonomy is a subgroup of $SU(4)$:

Lemma 2.3.17. *Let (M,Ω) be a parallel Spin(7)-manifold. There exists a unique $SO(8)$-structure on M, i.e. a Riemannian metric and an orientation, which is the extension of the Spin(7)-structure. We assume that the holonomy of the associated metric is a subgroup of $SU(4)$. The holonomy bundle or its extension defines a parallel $SU(4)$-structure on M. Let $\mathcal{S}$ be the space of all parallel Spin(7)-structures on M, which are extensions of the $SU(4)$-structure and induce the same Riemannian metric and orientation. In this situation, any connected component of $\mathcal{S}$ is diffeomorphic to a circle.*

Conversely, let M be an eight-dimensional manifold with a one-parameter family of parallel Spin(7)-structures which is diffeomorphic to a circle and whose members all induce the same metric and orientation. Then, there exists a parallel $SU(4)$-structure on M.

Proof : First, we consider the situation on $\mathbb{R}^8$. This restriction is justified, since we will see below that a change of the parallel Spin(7)-structure can be described by conjugation of Spin(7) by an $h \in GL(8)$.

Since we want the $SO(8)$-structure to be preserved, h has to leave $SO(8)$ invariant. The normalizer $\mathrm{Norm}_{GL(8)}SO(8)$ is $SO(8) \times \mathbb{R}\backslash\{0\}$, where the second factor is given by $\{\lambda\,\mathrm{Id}_{\mathbb{R}^8} \mid \lambda \in \mathbb{R}\backslash\{0\}\ \}$. Since conjugation by $\lambda\,\mathrm{Id}_{\mathbb{R}^8}$ leaves any matrix invariant, we will assume that $h \in SO(8)$. In the statement of the lemma, we assume that the $SU(4)$-structure is fixed. Therefore, $SU(4)$ has to be preserved by h, too, and we have $h \in \mathrm{Norm}_{SO(8)}SU(4)$. It is known that $\mathrm{Norm}_{SO(8)}SU(4) = U(4) \rtimes \mathbb{Z}_2$.

Next, we search for the group K of all $h \in \mathrm{Norm}_{SO(8)}SU(4)$ which preserve Spin(7), i.e., $K = \mathrm{Norm}_{SO(8)}SU(4) \cap \mathrm{Norm}_{SO(8)}\mathrm{Spin}(7)$. Since $SU(4) \subseteq \mathrm{Spin}(7) \subseteq \mathrm{Norm}_{SO(8)}\mathrm{Spin}(7)$, we have $SU(4) \subseteq K \subseteq U(4) \rtimes \mathbb{Z}_2$. Thus, the identity component of K is either $SU(4)$ or $U(4)$. In order to determine the identity component, we consider the group

$$\left\{ \begin{pmatrix} \boxed{e^{i\varphi}} & & & \\ & \boxed{e^{i\varphi}} & & \\ & & \boxed{e^{i\varphi}} & \\ & & & \boxed{e^{i\varphi}} \end{pmatrix} \middle|\ \varphi \in \mathbb{R} \right\}$$

which we shortly denote by $U(1)$. $U(4)$ obviously is finitely covered by the direct product of $SU(4)$ and $U(1)$. We have to answer the question, if $U(1)\mathrm{Spin}(7)U(1)^{-1} = \mathrm{Spin}(7)$. Let $k \in U(1)$ be a matrix with $k\mathrm{Spin}(7)k^{-1} = \mathrm{Spin}(7)$. Since the stabilizer of the four-form Ω is exactly Spin(7), we have to consider the following equation:

$$\left(kgk^{-1}\right)^* \Omega = \Omega \quad \forall g \in \mathrm{Spin}(7)\ .$$

As usual, the asterisk denotes the pull-back. The above equation is equivalent to:

$$g^*(k^*\Omega) = k^*\Omega \quad \forall g \in \mathrm{Spin}(7)\ .$$

Since Ω and its multiples are the only Spin(7)-invariant four-forms on $\mathbb{R}^8$, we have:

$$k^*\Omega = \lambda\Omega \quad \text{for a } \lambda \in \mathbb{R}\ .$$

Therefore, k has to be a multiple of an element of Spin(7). Since $\det k = 1$, we even know that $k \in \mathrm{Spin}(7) \times \mathbb{Z}_2$, where $\mathbb{Z}_2$ is generated by $-\mathrm{Id}_{\mathbb{R}^8}$. By a straightforward calculation, we can show that $\mathfrak{u}(1) \cap \mathfrak{spin}(7) = \{0\}$. Therefore, $U(1) \cap \mathrm{Spin}(7)$ is discrete. We now have shown that $U(1)$ is not contained in $\mathrm{Norm}_{SO(8)}\mathrm{Spin}(7)$ and that therefore the identity component of K is $SU(4)$. We can conclude that $U(4)/K = U(1)/\mathbb{Z}_k \cong U(1)$ for a $k \in \mathbb{N}$.

Let $h \in \mathrm{Norm}_{SO(8)}SU(4)$ be arbitrary. We consider the given Spin(7)-structure as a principal bundle $\mathfrak{G}_{\mathrm{Spin}(7)}$. The following set defines a $h\,\mathrm{Spin}(7)\,h^{-1}$- or equivalently a new Spin(7)-structure on M:

$$\{h \circ u : T_pM \to \mathbb{R}^8 | u \in \mathfrak{G}_{\mathrm{Spin}(7)}\} .$$

This construction describes an action of $\mathrm{Norm}_{SO(8)}SU(4)$ on the set of all Spin(7)-structures on M. The considerations we have made prove that its isotropy group is K. The action of $\mathrm{Norm}_{SO(8)}SU(4)$ leaves the $SU(4)$-structure invariant, but its action on the Spin(7)-structure generates a set of new Spin(7)-structures. Since all of them are extensions of the parallel $SU(4)$-structure, they are parallel, too. The Spin(7)-structures can be extended to the same $SO(8)$-structure, since $\mathrm{Norm}_{SO(8)}SU(4) \subseteq SO(8)$. We have shown above that $\mathrm{Norm}_{SO(8)}SU(4)/K$ is a homogeneous space whose connected components are all diffeomorphic to a circle. Therefore, $\mathcal{S}$ has the same property, too.

We finally have to prove the second part of the lemma. Since any parallel object is determined by its value at a single point, it suffices to prove our statement at a fixed $p \in M$. We denote the members of the one-parameter family of Spin(7)-structures by $\Omega_\theta \in \bigwedge^4 T_p^*M$, where $\theta \in S^1$. Let $\mathrm{Stab}(\Omega_\theta)$ be the stabilizer group of the four-form Ω_θ and let

$$G = \bigcap_{\theta \in S^1} \mathrm{Stab}(\Omega_\theta) .$$

It is easy to see that there exists a parallel G-structure on M. If G was Spin(7), all Ω_θ would be stabilized by Spin(7). Since our one-parameter family is diffeomorphic to a circle, the Ω_θ are not contained in a one-dimensional subspace of $\bigwedge^4 T_p^*M$. Therefore, the space of all Spin(7)-invariant four-forms has to be at least two-dimensional. Since we have proven that this space is in fact one-dimensional, we have obtained a contradiction and the holonomy is $\subsetneq$ Spin(7).

If the holonomy was not contained in $SU(4)$, it would have to be G_2. The action of G_2 splits the tangent space into a seven-dimensional and a one-dimensional irreducible submodule. There is up to a sign only one unit vector X which is orthogonal to the space on which G_2 acts non-trivially. Let X^* be the dual of X. The only parallel Spin(7)-structures which are an extension of the G_2-structure ω and induce the same metric as Ω are $*\omega \pm X^* \wedge \omega$. Our assumption on the one-parameter family of parallel Spin(7)-structures is thus not satisfied and we can exclude this case.

□

Remark 2.3.18. 1. The above lemma describes the connected components of $\mathcal{S}$. It makes no statement on the number of those components. Since we will not need that number, the above lemma is sufficient for our considerations.

2. In the situation of the lemma, there are two possibilities: Either, there exists a parallel vector field on M. In this case, the holonomy is contained in $SU(3)$. Or, a parallel vector field does not exist. If this is the case, the restricted holonomy is one of the groups $SU(4)$, $Sp(2)$, or $SU(2) \times SU(2)$.

For the metrics we consider in Chapter 5, we determine the set of all parallel Spin(7)-structures which have the same associated metric and orientation. If we do not find a one-parameter family in this set, the holonomy is not a subgroup of $SU(4)$. Conversely, we know that the holonomy is contained in $SU(4)$ if there exists such a family.

If we want to discuss the holonomy group itself, we have to search for certain parallel forms on the manifold. Since we will consider spaces with a large isometry group, we have geometric and algebraic methods at hand to decide if such forms exist. Those methods will be explained in more detail when we need them.

Chapter 3

Spaces with a large isometry group

3.1 Homogeneous spaces

Many equations in differential geometry, for example, the Einstein equation $\mathrm{Ric} = \lambda g$ and the equation $d\Omega = 0$ on a Spin(7)-manifold become simpler if we assume that the metric has a large isometry group. The most symmetric case is that the isometry group acts transitively on the manifold. In that case, we have a so called homogeneous space. It will turn out that there are no homogeneous spaces with exceptional holonomy. For that reason we will assume later on that the metric is of cohomogeneity one, i.e., most of the orbits have codimension one. Nevertheless, we will first give a short introduction to the issue of homogeneous spaces. The reason for this is that in the non-homogeneous case the orbits of an isometric action are still homogeneous. If we want to make calculations on manifolds with a large isometry group, it will therefore often be useful to first consider the geometry on the orbit spaces. We start with some basic definitions:

Definition 3.1.1. Let (M, g) be a Riemannian manifold. The set $\mathrm{Isom}(M, g)$ of all isometries of (M, g) is called the *symmetry* or *isometry group of* (M, g).

The isometry group is clearly a group. Moreover, it is a Lie group:

Proposition 3.1.2. *Let (M, g) be a Riemannian manifold. Isom(M, g) has a canonical differentiable structure such that Isom(M, g) becomes a Lie group and the action Isom$(M, g) \times M \to M$ becomes differentiable.*

We are now able to define the concept of a homogeneous space:

Definition 3.1.3. Let (M, g) be a Riemannian manifold and let $G \subseteq \mathrm{Isom}(M, g)$ be a closed subgroup of the isometry group. If G acts transitively on the manifold M, we call (M, g) a *(G-)homogeneous space* and g a *(G-)homogeneous metric.*

Remark 3.1.4. 1. The idea behind the concept of a homogeneous space is that there is no way to distinguish one point of the homogeneous space from another.

2. A Riemannian manifold can be G-homogeneous with respect to different groups G. For example, let $S^{2n-1} \subseteq \mathbb{R}^{2n} \cong \mathbb{C}^n$ be the $(2n-1)$-dimensional sphere with the round

metric. $O(2n)$ and $U(n)$ act both transitively on S^{2n-1}, although $U(n) \subsetneq O(2n)$. As long as G is known from the context, it will not be mentioned.

Homogeneous spaces have the following useful property:

Proposition 3.1.5. *Any homogeneous space is complete.*

It is possible to describe homogeneous spaces as coset spaces:

Proposition 3.1.6. *Let (M, g) be a G-homogeneous space, where $G \subseteq Isom(M, g)$. We fix an arbitrary $p \in M$ and denote the isotropy group of the G-action at p by H_p. The map*

$$\begin{aligned} \varphi : M &\to G/H_p \\ \varphi(q) &:= gH_p \, , \end{aligned}$$

where g is an isometry in G mapping p to q, is a well-defined G-equivariant diffeomorphism. Furthermore, there exists a unique G-invariant metric on G/H_p such that φ becomes an isometry.

Conversely, if we equip any coset space G/H with a G-invariant metric g, $(G/H,g)$ is a homogeneous space.

Because of the above proposition, we can restrict ourselves to coset spaces G/H while studying homogeneous spaces. It is useful to make some restrictions on the pair (G, H):

Convention 3.1.7. A homogeneous space G/H can also be written as $(G \times K)/(H \times K)$. $H \times K$ shall be embedded in such a way into $G \times K$ that the second factor of $H \times K$ coincides with the second factor of $G \times K$. Since we want to keep the list of possible groups acting on G/H short, we will assume from now on that G acts *almost effectively* on G/H, i.e., the elements of G which act trivially are a discrete set. We also require that H is a closed subgroup of G, which is equivalent to G/H being a Hausdorff space.

Remark 3.1.8. 1. For the rest of this thesis, we define G/H as the space of all left cosets gH. When we write "group action" (or "invariant object"), we therefore mean "left group action" (or "left-invariant object"). If we consider a right action, it will explicitly mentioned.

2. Let g be an element of the normalizer $\mathrm{Norm}_G H$. Since we have $(kH)g = (kg)H$, the conjugation map $kH \mapsto gkg^{-1}H$ is well-defined and fixes eH. Those kind of maps will be important later on.

3. Let H_p be the isotropy group of an isometric action at p and let g be an isometry which maps p to q. The isotropy group at q is gH_pg^{-1}. It therefore makes sense to speak of the isotropy group of a homogeneous space without specifying a point.

4. Let H be a closed subgroup of G and let $H' = gHg^{-1}$ be a conjugate of H. Then there exists a G-equivariant diffeomorphism $\phi : G/H \to G/H'$ which is described by $\phi(kH) = gkg^{-1}H'$. We can therefore identify embeddings of H into G which are equal up to a conjugation by an element of G.

5. Nonconjugate embeddings of H into G will often give topologically different spaces G/H. There are, for example, infinitely many non-homotopic spaces of type $SU(3)/U(1)$. (See Section 5.4.) Moreover, if we embed $H \times K$ in a non-canonical way into $G \times K$ we do not necessarily have $(G \times K)/(H \times K) = G/H$. (See Lemma 3.1.12.) We will therefore specify the embedding of H into G if it is not clear from the context.

The following lemma gives another useful restriction for the shape of G and H:

Lemma 3.1.9. *Let G be a (not necessarily connected) Lie group and H be a (not necessarily connected) closed subgroup of G. Furthermore, let g be a G-invariant metric on G/H. Then, there exists a simply connected Lie group G', a closed connected subgroup H' of G', and a G'-invariant metric g' on G'/H' such that G'/H' is the universal cover of each connected component of G/H and the covering map is a local isometry. In particular, G'/H' is connected and simply connected.*

Proof : Let G_e be the identity component of G, and H_e the identity component of H. We choose G' as the universal cover of G_e. Let $\pi : G' \to G_e$ be the covering map. H' can be chosen as the identity component of $\pi^{-1}(H_e)$ and g' as the lift of the metric on $G_e/(H \cap G_e)$.

□

Corollary 3.1.10. *Let G/H be an arbitrary coset space, $K \subseteq GL(\dim G - \dim H)$ a closed subgroup, and $\mathfrak{K}$ a G-invariant K-structure on G/H. Furthermore, let G'/H' be the simply connected coset space from the lemma above. Then G'/H' carries a G'-invariant K-structure which is the lift of $\mathfrak{K}$*

Remark 3.1.11. Because of the above lemma and its corollary we will restrict ourselves from now on to the case where G is simply connected and H is connected.

The next result will be needed for the detailed description of the orbifold singularities which we have to consider in Section 5.3 and 5.4.

Lemma 3.1.12. *Let G be a Lie group and H be a closed subgroup of G. We consider the space $(G \times U(1))/(H \times U(1)')$ where $U(1)' \subseteq G \times U(1)$ is not a subgroup of G and does not necessarily coincide with the abelian factor of $G \times U(1)$. In this situation, G/H G-equivariantly covers $(G \times U(1))/(H \times U(1)')$ and the preimage of $e(H \times U(1)')$ in G/H is given by the discrete group $G \cap U(1)'$.*

Proof : $U(1)'$ is a group of the following type:

$$U(1)' = \{(\psi(e^{im\varphi}), e^{in\varphi}) \in G \times U(1) | \varphi \in \mathbb{R}\} ,$$

where $m, n \in \mathbb{Z}$, $n \neq 0$ and $\psi : U(1) \to G$ is an injective group homomorphism. Since the group by which we divide $G \times U(1)$ is isomorphic to $H \times U(1)$, we have $h\psi(e^{im\varphi}) = \psi(e^{im\varphi})h$. We define the following map:

$$\begin{aligned} \Phi : G/H &\to (G \times U(1))/(H \times U(1)') \\ gH &\mapsto (g, e)\{(h\psi(e^{im\varphi}), e^{in\varphi}) | h \in H, \varphi \in \mathbb{R}\} . \end{aligned}$$

This map is a G-equivariant covering map of $(G \times U(1))/(H \times U(1)')$:

1. Φ is well-defined: Let $k \in H$ be arbitrary. We have:

$$\begin{aligned}\Phi(gkH) &= (gk,e)\{(h\psi(e^{im\varphi}),e^{in\varphi})|h \in H, \varphi \in \mathbb{R}\}\\ &= (g,e)\{(kh\psi(e^{im\varphi}),e^{in\varphi})|h \in H, \varphi \in \mathbb{R}\}\\ &= (g,e)\{(h\psi(e^{im\varphi}),e^{in\varphi})|h \in H, \varphi \in \mathbb{R}\}\\ &= \Phi(gH)\end{aligned}$$

2. Φ is G-equivariant: We choose an arbitrary $g' \in G$ and obtain:

$$\begin{aligned}\Phi(g'gH) &= (g'g,e)\{(h\psi(e^{im\varphi}),e^{in\varphi})|h \in H, \varphi \in \mathbb{R}\}\\ &= (g',e)(g,e)\{(h\psi(e^{im\varphi}),e^{in\varphi})|h \in H, \varphi \in \mathbb{R}\}\\ &= (g',e)\Phi(gH)\end{aligned}$$

3. Φ is surjective: Let $(g,e^{i\phi}) \in G \times U(1)$ be arbitrary. We have:

$$\begin{aligned}\Phi(g\psi(e^{-i\frac{m}{n}\phi})H) &= (g\psi(e^{-i\frac{m}{n}\phi}),e)\{(h\psi(e^{im\varphi}),e^{in\varphi})|h \in H, \varphi \in \mathbb{R}\}\\ &= (g,e)\{\psi(e^{-i\frac{m}{n}\phi})h\psi(e^{im\varphi}),e^{in\varphi})|h \in H, \varphi \in \mathbb{R}\}\\ &= (g,e)\{(h\psi(e^{im(\varphi-\frac{\phi}{n})}),e^{in\varphi})|h \in H, \varphi \in \mathbb{R}\}\\ &= (g,e)\{(h\psi(e^{im\varphi}),e^{in(\varphi+\frac{\phi}{n})})|h \in H, \varphi \in \mathbb{R}\}\\ &= (g,e^{i\phi})\{(h\psi(e^{im\varphi}),e^{in\varphi})|h \in H, \varphi \in \mathbb{R}\}\end{aligned}$$

4. The preimage of $e(H \times U(1)')$ is discrete: An element $gH \in G/H$ is mapped by Φ to $e(H \times U(1)') \in (G \times U(1))/(H \times U(1)')$ if and only if

$$(g,e) \in \{(h\psi(e^{im\varphi}),e^{in\varphi})|h \in H, \varphi \in \mathbb{R}\}\ .$$

Since $n \neq 0$, the preimage of $H \times U(1)'$ is ΓH, where

$$\Gamma = G \cap U(1)' = \{\psi(e^{2\pi ik\frac{m}{n}})|k \in \mathbb{Z}\} \cong \mathbb{Z}_{\frac{n}{\gcd(n,m)}}\ .$$

This group is clearly discrete.

□

Remark 3.1.13. In the situation of the above lemma, we can rewrite $(G \times U(1))/(H \times U(1)')$ as $G/(H \times \Gamma)$ or $(G/H)/\Gamma$ respectively. The action of Γ we divide out is defined by the right-multiplication of cosets gH with elements of Γ. This action is well-defined, since the groups H and Γ commute. Since Γ is abelian, it is possible to consider this action as a left action.

There is a theorem on the topology of coset spaces, which will not explicitly needed in this thesis, but nevertheless is useful to know:

Theorem 3.1.14. *Let G be a compact Lie group and let G/H be a Riemannian homogeneous space with an infinite fundamental group. Then G/H is covered by $G'/H \times T$ where T is a flat torus and G'/H is a homogeneous space with a finite fundamental group.*

Proof : We assume without loss of generality that G and H are both connected. The Lie algebra of G we denote by $\mathfrak{g}$ and the Lie algebra of H by $\mathfrak{h}$. If there is a $x \in \mathfrak{g}$ which is not contained in $\mathfrak{h}$ and satisfies $[x, y] = 0$ for all $y \in \mathfrak{g}$, the Lie algebra $\mathfrak{g}$ is a direct sum of type $\mathfrak{g}' \oplus \mathfrak{u}(1)$. Since G is compact, we have $G/H = G'/H \times S^1$. By repeating this construction as often as possible, we obtain $G/H = K/H \times T$, where T is a flat torus. Since it is not possible to repeat our construction again, the Lie algebra $\mathfrak{k}$ of K contains no x with $x \in Z(\mathfrak{k})$ and $x \notin \mathfrak{h}$. It remains to show that in this situation the fundamental group of K/H is finite.

We equip K with the negative of the Killing form as a metric. As we will state below, there exists a unique K-invariant metric q on K/H such that the projection $\pi : K \to K/H$ becomes a Riemannian submersion. The fact that a $x \in \mathfrak{k}$ with the properties described above does not exist guarantees that q is positive definite. It is possible to show by a short calculation that the Ricci-curvature of q is $\frac{1}{4}q$. It follows from the Bonnet-Myers theorem that the fundamental group of K/H is finite.

□

Our next aim is to introduce certain techniques which we will need in order to carry out explicit calculations on homogeneous spaces. Let G/H be a homogeneous space and let $X \in T_eG$ be arbitrary. For any $t \in \mathbb{R}$ and any G-invariant metric on G/H the map

$$gH \mapsto (\exp(tX)g)H$$

is an isometry of G/H. Therefore, the vector field

$$gH \mapsto \frac{\partial}{\partial t}|_{t=0}(\exp(tX)g)H$$

is a Killing vector field on G/H, which we will shortly denote by X^*. The following simple lemmas will be useful later on:

Lemma 3.1.15. *Let G/H be a coset space and $X, Y \in \mathfrak{g} := T_eG$ be arbitrary. Then the following identities hold:*

1. $(X + Y)^* = X^* + Y^*$,
2. $(\lambda X)^* = \lambda X^*$ *for all* $\lambda \in \mathbb{R}$.

In the following, the right-invariant vector fields on Lie groups will become important. We therefore introduce the following notation:

Convention 3.1.16. Let G be a Lie group and $X \in \mathfrak{g}$. We denote the right-invariant vector field on G which coincides with X at e by X'.

The projection of X' simply is the Killing vector field X^*:

Lemma 3.1.17. *Let G/H be a coset space and $\pi : G \to G/H$ be the projection map. For any $X \in \mathfrak{g}$ we have:*

$$(d\pi)(X'_g) = X^*_{gH} .$$

We want to know how the Killing vector field X^* behaves under the action of G on G/H. The following lemma answers this question:

Lemma 3.1.18. *Let G be a Lie group and H be a closed subgroup of G. Furthermore, let $g, h \in G$ and $X \in \mathfrak{g}$ be arbitrary. The Killing vector field X^* on G/H obeys the following equation:*

$$(dL_h)X^*_{gH} = (Ad_h(X))^*_{hgH} ,$$

where $L_h : G/H \to G/H$ denotes the left-multiplication by h and Ad is the adjoint action of G on $\mathfrak{g}$.

The fact that $[X', Y']_e = -[X, Y]$ for all $X, Y \in \mathfrak{g}$ has the following consequence:

Lemma 3.1.19. *For any $X, Y \in \mathfrak{g}$, the Lie bracket of the Killing vector fields X^* and Y^* on G/H is given by the following formula:*

$$[X^*, Y^*] = -[X, Y]^* .$$

The group H acts from the left on the tangent space $T_{eH}G/H$ by $(h, X) \mapsto (dL_h)(X)$. Therefore, $T_{eH}G/H$ is not only a vector space but also a H-module. The H-module structure on $T_{eH}G/H$ can be described by the following proposition, which follows directly from the above lemmas:

Proposition 3.1.20. *Let G be a Lie group and H be a closed subgroup of G. We define the following map:*

$$\begin{aligned} \varphi : \mathfrak{g} &\to T_{eH}G/H \\ X &\mapsto X^*_{eH} . \end{aligned}$$

In this situation, the following statements are true:

1. *φ is linear and surjective.*
2. *$ker(\varphi) = T_eH =: \mathfrak{h}$.*
3. *$T_{eH}G/H$ is H-equivariantly isomorphic to the quotient $\mathfrak{g}/\mathfrak{h}$, where the action of H on $\mathfrak{g}/\mathfrak{h}$ is induced by the restriction of the adjoint action $Ad : G \to GL(\mathfrak{g})$ to H.*

Remark 3.1.21. Let $g \in G$ be arbitrary. The isotropy group of the G-action at $gH \in G/H$ is gHg^{-1}. Therefore, $T_{gH}G/H$ together with the action $(h, X) \mapsto (dL_{ghg^{-1}})(X)$ is a H-module, too. If we replace φ by the more general map

$$\begin{aligned} \varphi_g : \mathfrak{g} &\to T_{gH}G/H \\ X &\mapsto (\mathrm{Ad}_g(X))^*_{gH} \ , \end{aligned}$$

we obtain the same result on the H-module structure of $T_{gH}G/H$ as on the structure of $T_{eH}G/H$.

Convention 3.1.22. From now on, we will assume that G is a compact Lie group. In this case, there exists a biinvariant metric on G. We will fix one of those metrics and denote it by q. In the following, we will often have to consider the q-orthogonal complement of $\mathfrak{h}$ in $\mathfrak{g}$, which we will denote by $\mathfrak{m}$.

Remark 3.1.23. Let Ad be the adjoint action of G. From the biinvariance of q, it follows that $\mathrm{Ad}_h(\mathfrak{m}) \subseteq \mathfrak{m}$ for all $h \in H$. $\mathfrak{m}$ therefore is a H-module. It is easy to see that $\mathfrak{m}$ is H-equivariantly isomorphic to $\mathfrak{g}/\mathfrak{h}$ and therefore to any tangent space $T_{gH}G/H$.

When we do explicit calculations on a homogeneous space, we need to fix a basis of the tangent space. The choice of that basis should in a certain way respect the splitting of the tangent space into H-submodules:

Definition 3.1.24. Let G/H be a n-dimensional homogeneous space. As above, we identify the tangent space at a point $p \in G/H$ with the H-module $\mathfrak{m}$. Let

$$\mathfrak{m} = \mathfrak{m}_1 \oplus \ldots \oplus \mathfrak{m}_k$$

be a splitting of $\mathfrak{m}$ into (not necessarily irreducible) H-submodules. We denote the dimensions of $\mathfrak{m}_1, \ldots, \mathfrak{m}_k$ by $m_1, \ldots, m_k$. A basis $(e_i)_{i=1\ldots n}$ of $\mathfrak{m}$ is called *adapted to the splitting* $\mathfrak{m}_1 \oplus \ldots \oplus \mathfrak{m}_k$ if

$$\begin{aligned} \mathfrak{m}_1 &= \mathrm{span}(e_1, \ldots, e_{m_1}) \\ \mathfrak{m}_2 &= \mathrm{span}(e_{m_1+1}, \ldots, e_{m_1+m_2}) \\ &\ldots \\ \mathfrak{m}_k &= \mathrm{span}(e_{m_1+\ldots+m_{k-1}+1}, \ldots, e_n) \end{aligned}$$

Remark 3.1.25.

1. Later on, when we consider concrete homogeneous spaces, we will mostly choose a basis of $\mathfrak{m}$ which is adapted to a fixed splitting. Furthermore, the splitting $\mathfrak{m}_1 \oplus \ldots \oplus \mathfrak{m}_k$ and the basis $(e_i)_{i=1\ldots n}$ will usually be q-orthogonal and all vectors of a family $(e_{m_1+\ldots+m_l+1}, \ldots, e_{m_1+\ldots+m_{l+1}})$ will be of the same length with respect to q. It turns out that our calculations will become more simple if we describe our objects with respect to such a basis.

2. Since it often will be necessary to describe the action of $\mathfrak{h}$ on $\mathfrak{m}$ explicitly, we will often extend our basis of $\mathfrak{m}$ by a q-orthogonal basis of $\mathfrak{h}$ to a basis of all of $\mathfrak{g}$.

Our next aim is to describe the set of G-invariant metrics on a homogeneous space G/H. In order to describe an arbitrary G-invariant metric, we first need a background metric on G/H:

Proposition 3.1.26. *Let G be a compact Lie group equipped with a biinvariant metric q and let H be a closed subgroup of G. There is a unique G-invariant metric on G/H such that the projection map $\pi : G \to G/H$ becomes a Riemannian submersion.*

Convention 3.1.27. The metric on G/H which is induced by q we will denote by q, too. In the literature, metrics of this kind are often called *normal metrics.*

Let g be an arbitrary G-invariant metric on G/H. Since G/H is a homogeneous space, g is determined by its values on the tangent space $T_{eH}G/H$. Because of Remark 3.1.23, g can be considered as an element of $S^2(\mathfrak{m})$. In order to have a well-defined G-invariant extension to all of G/H, $g \in S^2(\mathfrak{m})$ has to be H-invariant. These considerations motivate the following lemma:

Lemma 3.1.28. *Let g be a G-invariant metric on a homogeneous space G/H. As above, we identify g with an element of $S^2(\mathfrak{m})$. The endomorphism $\varphi : \mathfrak{m} \to \mathfrak{m}$ defined by:*

$$q(\varphi(X), Y) := g(X, Y) \quad \forall X, Y \in \mathfrak{m}$$

is H-equivariant, q-symmetric and positive definite with respect to q. Conversely, any linear $\varphi : \mathfrak{m} \to \mathfrak{m}$ with these three properties defines by the above formula a G-invariant metric on G/H.

The possible linear maps $\varphi : \mathfrak{m} \to \mathfrak{m}$ describing a G-invariant metric on G/H can be classified with help of Schur's lemma. Therefore, the above result is useful for practical purposes. It will often be convenient to restrict ourselves to diagonal metrics, since this will simplify many calculations:

Definition 3.1.29. Let G/H be a n-dimensional homogeneous space and let $(e_i)_{i=1,\ldots,n}$ be a q-orthogonal basis of $\mathfrak{m}$. A G-invariant metric g on G/H is called *diagonal with respect to* $(e_i)_{i=1,\ldots,n}$ if the matrix representation of $g \in S^2(\mathfrak{m})$ with respect to $(e_i)_{i=1,\ldots,n}$ is a diagonal matrix.

Since $(e_i)_{i=1,\ldots,n}$ is q-orthogonal, g is diagonal if and only if φ is represented by a diagonal matrix.

Remark 3.1.30. Let G/H be a n-dimensional homogeneous space with the property that $\mathfrak{m}$ splits into pairwise inequivalent irreducible H-submodules. Furthermore, let $(e_i)_{i=1,\ldots,n}$ be a q-orthogonal basis of $\mathfrak{m}$ which is adapted to the splitting of $\mathfrak{m}$. In this situation, it follows from Schur's lemma that any G-invariant metric on G/H is diagonal with respect to $(e_i)_{i=1,\ldots,n}$. Therefore, the restriction to diagonal metrics is in many cases not a real one. In the other cases, we will carefully discuss what we will loose by this restriction.

Next, we will present formulas for the sectional and the Ricci-curvature of a homogeneous space. These formulas will be needed later on when we express the Einstein condition $\mathrm{Ric} = \lambda g$ for a cohomogeneity-one metric as an explicit system of ordinary differential equations. The following results, except the formula from Corollary 3.1.37, can also be found in Besse [10], pp. 183ff. Before we are able to determine the curvature tensor of a homogeneous space, we first have to consider the Levi-Civita connection:

Proposition 3.1.31. *Let g be a G-invariant metric on a homogeneous space G/H, whose Levi-Civita connection we denote by ∇ and let X, Y, and Z be in $\mathfrak{g}$. Then, we have:*

$$2g(\nabla_{X^*}Y^*, Z^*) = g([X^*, Y^*], Z^*) + g([X^*, Z^*], Y^*) + g([Y^*, Z^*], X^*)\,.$$

Proof : Since X^* is a Killing vector field, the Lie derivative $\mathcal{L}_{X^*}g$ vanishes. Therefore, we have the following equation, which is the key to the proof:

$$\begin{aligned} X^*g(Y^*, Z^*) &= \mathcal{L}_{X^*}g(Y^*, Z^*) \\ &= g(\mathcal{L}_{X^*}Y^*, Z^*) + g(Y^*, \mathcal{L}_{X^*}Z^*) \\ &= g([X^*, Y^*], Z^*) + g([X^*, Z^*], Y^*)\,. \end{aligned}$$

The statement of the proposition can now be obtained from the equations for the torsion-freeness of ∇ and its compatibility with the metric.

□

Our next step is to express $(\nabla_{X^*}Y^*)_{eH}$ in terms of the Lie bracket of $\mathfrak{g}$. This can be done by the following corollary:

Corollary 3.1.32. *Let G/H be a homogeneous space equipped with a G-invariant metric g. Furthermore, let $X, Y \in \mathfrak{m}$ be arbitrary. Then, the Levi-Civita connection ∇ of g at the point eH is given by:*

$$(\nabla_{X^*}Y^*)_{eH} = -\frac{1}{2}[X, Y]_{\mathfrak{m}} + U(X, Y)\,,$$

where $U : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ is defined by:

$$2g_{eH}(U(X, Y), Z) := g_{eH}([Z, X]_{\mathfrak{m}}, Y) + g_{eH}([Z, Y]_{\mathfrak{m}}, X)$$

and $[.,.]_{\mathfrak{m}}$ denotes the orthogonal projection of the Lie bracket of $\mathfrak{g}$ onto $\mathfrak{m}$.

Remark 3.1.33. $\nabla_{X^*}Y^*$ is not necessarily a Killing vector field. Therefore, we have to be careful if we want to consider $\nabla_{X^*}Y^*$ at another point as eH or to determine the second derivatives of a vector field.

We are now able to prove a formula for the sectional curvature: Let $X, Y \in T_{eH}G/H$ be an orthonormal pair of tangent vectors. In this case, $\mathrm{Sec}_{eH}(X, Y) = g_{eH}(R_{eH}(X, Y)Y, X)$. We extend X and Y to Killing vector fields X^* and Y^* on G/H and obtain after a straightforward calculation:

Proposition 3.1.34. *Let G/H be a homogeneous space and g be a G-invariant metric on G/H. Furthermore, let (X, Y) be a g-orthonormal pair in $\mathfrak{m}\times\mathfrak{m}$. Then, the sectional curvature of g satisfies:*

$$sec_p(X, Y) = g_p([X^*, [Y^*, X^*]], Y^*) - g_p(\nabla_{Y^*}Y^*, \nabla_{X^*}X^*) + \|\nabla_{X^*}Y^*\|_p^2 - \|[X^*, Y^*]\|_p^2\,,$$

where $p \in G/H$ can be chosen arbitrarily.

We want to have a formula for $\sec(X, Y)$ in terms of the Lie bracket of $\mathfrak{g}$. This can be done by applying Corollary 3.1.32. Since that corollary is only valid at $eH \in G/H$, the formula we obtain is only valid at eH, too:

Corollary 3.1.35. *In the situation of the proposition above, the sectional curvature of the metric g at the point eH is given by:*

$$\begin{aligned} sec_{eH}(X,Y) &= \frac{1}{2}g_{eH}([[X,Y]_{\mathfrak{g}},X]_{\mathfrak{m}},Y) - \frac{1}{2}g_{eH}([[X,Y]_{\mathfrak{g}},Y]_{\mathfrak{m}},X) - g_{eH}(U(X,X),U(Y,Y)) \\ &\quad + \|U(X,Y)\|^2_{eH} - \frac{3}{4}\|[X,Y]_{\mathfrak{m}}\|^2_{eH} \end{aligned}$$

Since the Ricci curvature is the average of certain sectional curvatures, we can use the above expression to derive a formula for the Ricci-curvature of a homogeneous space:

Proposition 3.1.36. *Let G/H be a n-dimensional homogeneous space. Furthermore, let g be a G-invariant metric on G/H. Then we have for any tangent vector $X \in T_{eH}G/H \cong \mathfrak{m}$ and any g-orthonormal basis $(X_i)_{1\leqslant i\leqslant n}$ of $T_{eH}G/H$:*

$$\begin{aligned} Ric_{eH}(X,X) &= -\frac{1}{2}\sum_{i=1}^{n} g_{eH}([X,X_i]_{\mathfrak{m}},[X,X_i]_{\mathfrak{m}}) - \frac{1}{2}\kappa(X,X) + \frac{1}{4}\sum_{i,j=1}^{n} g_{eH}([X_i,X_j]_{\mathfrak{m}},X)^2 \\ &\quad - g_{eH}([\sum_{i=1}^{n} U(X_i,X_i),X]_{\mathfrak{m}},X)\ , \end{aligned}$$

where κ denotes the Killing form of the Lie algebra $\mathfrak{g}$ of G.

We now prove a formula for the Ricci-curvature which is suitable for explicit calculations. Before we start, we have to fix some notations. As usual, we consider a G-invariant metric g on a n-dimensional homogeneous space G/H and fix a g-orthonormal basis $(X_i)_{1\leqslant i\leqslant n}$ of the tangent space $\mathfrak{m}$. For the following considerations, we have to extend g to an H-invariant inner product on $\mathfrak{g}$ and $(X_i)_{1\leqslant i\leqslant n}$ to a g-orthonormal basis $(X_i)_{1\leqslant i\leqslant \dim\mathfrak{g}}$ of $\mathfrak{g}$. Since g is H-invariant, the spaces $\mathfrak{h}$ and $\mathfrak{m}$ are orthogonal. The structure constants of the Lie algebra $\mathfrak{g}$ we define by:

$$[X_i,X_j] = \sum_{k=1}^{\dim\mathfrak{g}} c_{ij}^k X_k\ .$$

With these notations $U = \sum_{i,j,k=1}^{n} U_{ij}^k X^i \otimes X^j \otimes X_k$ can be written as:

$$U_{ij}^k = \frac{1}{2}(c_{ki}^j + c_{kj}^i)\ .$$

In particular, we have $U_{ii}^j = c_{ji}^i$. Now we are able to compute the Ricci tensor:

$$
\begin{aligned}
\mathrm{Ric}_{eH}(X_i,X_i) &= -\frac{1}{2}\sum_{j=1}^{n} g([X_i,X_j]_{\mathfrak{m}},[X_i,X_j]_{\mathfrak{m}}) - \frac{1}{2}\kappa(X_i,X_i) + \frac{1}{4}\sum_{j,k=1}^{n} g([X_j,X_k]_{\mathfrak{m}},X_i)^2 \\
&\quad - g([\sum_{j=1}^{n} U(X_j,X_j),X_i]_{\mathfrak{m}},X_i) \\
&= -\frac{1}{2}\sum_{j,k=1}^{n} g(c_{ij}^k X_k, c_{ij}^k X_k) - \frac{1}{2}\sum_{j=1}^{\dim\mathfrak{g}} g([X_i,[X_i,X_j]],X_j) + \frac{1}{4}\sum_{j,k,l=1}^{n} g(c_{jk}^l X_l,X_i)^2 \\
&\quad - g([\sum_{j,k=1}^{n} c_{kj}^j X_k, X_i]_{\mathfrak{m}},X_i) \\
&= -\frac{1}{2}\sum_{j,k=1}^{n}(c_{ij}^k)^2 - \frac{1}{2}\sum_{j,k=1}^{\dim\mathfrak{g}} g([X_i,c_{ij}^k X_k],X_j) + \frac{1}{4}\sum_{j,k=1}^{n}(c_{jk}^i)^2 \\
&\quad - g(\sum_{j,k,l=1}^{n} c_{kj}^j c_{ki}^l X_l, X_i) \\
&= -\frac{1}{2}\sum_{j,k=1}^{n}(c_{ij}^k)^2 - \frac{1}{2}\sum_{j,k,l=1}^{\dim\mathfrak{g}} g(c_{ij}^k c_{ik}^l X_l,X_j) + \frac{1}{4}\sum_{j,k=1}^{n}(c_{jk}^i)^2 \\
&\quad - \sum_{j,k=1}^{n} c_{kj}^j c_{ki}^i \\
&= \sum_{j,k=1}^{n}\left(-\frac{1}{2}(c_{ij}^k)^2 + \frac{1}{4}(c_{jk}^i)^2 - c_{kj}^j c_{ki}^i\right) - \frac{1}{2}\sum_{j,k=1}^{\dim\mathfrak{g}} c_{ij}^k c_{ik}^j
\end{aligned}
\tag{3.1}
$$

As a by-product of the above calculation, we have shown the following formula for the Killing form:

$$\kappa(X_i,X_i) = \sum_{j,k=1}^{\dim\mathfrak{g}} c_{ij}^k c_{ik}^j \,. \tag{3.2}$$

If we assume that the basis $(X_i)_{1\leqslant i\leqslant \dim\mathfrak{g}}$ is not only orthogonal with respect to g but also with respect to κ, we can simplify the above equation. In this situation, we have:

$$0 = \kappa(X_i,[X_j,X_j]) = \kappa([X_i,X_j],X_j) = \sum_{k=1}^{\dim\mathfrak{g}} c_{ij}^k \kappa(X_k,X_j) = c_{ij}^j \kappa(X_j,X_j)$$

and therefore $c_{ij}^j = 0$. By polarizing the equation (3.1) we finally have shown the following corollary:

Corollary 3.1.37. *Let G/H be a n-dimensional homogeneous space equipped with a G-invariant metric g. Furthermore, let $(X_i)_{1\leqslant i\leqslant n}$ be a g-orthonormal basis of $\mathfrak{m}$ which is orthogonal with respect to the Killing form κ of $\mathfrak{g}$. We extend g to an H-invariant inner* *product on $\mathfrak{g}$ and $(X_i)_{1\leqslant i\leqslant n}$ to a g-orthonormal basis $(X_i)_{1\leqslant i\leqslant \dim\mathfrak{g}}$, such that the extended basis is still κ-orthogonal. In this situation, the Ricci-curvature of g is determined by:*

$$Ric_{eH}(X_i,X_j) = \sum_{k,l=1}^{n}\left(-\frac{1}{2}c_{ik}^{l}c_{jk}^{l}+\frac{1}{4}c_{kl}^{i}c_{kl}^{j}\right)-\frac{1}{4}\sum_{k,l=1}^{m}(c_{ik}^{l}c_{jl}^{k}+c_{jk}^{l}c_{il}^{k})\,,$$

where the coefficients c_{ij}^k are defined by $[X_i,X_j]=\sum_{k=1}^{\dim\mathfrak{g}}c_{ij}^kX_k$ and i, j can chosen arbitrarily from $\{1,\dots,n\}$.

Remark 3.1.38.

1. Since the Ricci-curvature is invariant under isometries, the Ricci-tensor of a G-invariant metric on a homogeneous space G/H is G-invariant, too. Unfortunately, the Ricci-tensor of a diagonal homogeneous metric is not necessarily diagonal.
2. In practice, the condition that $(X_i)_{1\leqslant i\leqslant \dim\mathfrak{g}}$ is κ-orthogonal will often be satisfied. The bases of $\mathfrak{g}$ we choose in Chapter 5 are orthogonal with respect to the biinvariant metric q we have fixed. Since the restriction of the biinvariant metric to a simple factor of a semisimple Lie group is a multiple of the Killing form, our bases therefore are orthogonal with respect to the Killing form, too.

In Chapter 5, we will often have to compute the exterior derivatives of G-invariant differential forms on a homogeneous space G/H. Therefore, we will briefly describe how this can be done explicitly. Since we have to consider many different geometric objects on the next pages, we will work with a fixed basis $(e_i)_{1\leqslant i\leqslant \dim\mathfrak{g}}$ of $\mathfrak{g}$. This allows us to distinguish the different objects by the position of their indices. We also fix a biinvariant metric q on G and denote the corresponding normal metric on G/H by q, too. We remark that the following considerations could also be carried out without referring to a basis or a normal metric. Since in Chapter 5 we have to work with a fixed basis and normal metric anyway, we will not do this. We start our considerations by introducing certain right-invariant one-forms on G:

Definition 3.1.39. Let G be a Lie group, whose Lie algebra we denote by $\mathfrak{g}$, and $(e_i)_{1\leqslant i\leqslant \dim\mathfrak{g}}$ be a basis of G. As above, we denote the right-invariant vector field on G associated to e_i by e_i'. We define $e^{i'}$ as the unique right-invariant one-form satisfying:

$$e^{i'}(e_j') = q(e_i',e_j')\quad \forall i,j\in\{1,\dots,\dim\mathfrak{g}\}\,.$$

Remark 3.1.40. The term "right-invariant" means in this context that the following equation holds:

$$(R_{h^{-1}})^*(e^{i'})_k = (e^{i'})_{kh}\quad \forall h,k\in G, i\in\{1,\dots,\dim\mathfrak{g}\}\,.$$

Since the action of G by $(h,\alpha)\mapsto(R_{h^{-1}})^*\alpha$ is a right action on the cotangent bundle, it is justified to speak of right-invariance in the above definition.

On the space G/H there exist certain one-forms which are dual to the Killing vector fields e_i^*:

Definition 3.1.41. Let G/H be a coset space and $(e_i)_{1\leqslant i\leqslant \dim\mathfrak{g}}$ be a basis of the Lie algebra $\mathfrak{g}$. We define the one-form e^{i^*} on G/H by the equation:

$$e^{i^*}(e_j^*) = q(e_i^*, e_j^*) \quad \forall i, j \in \{1, \ldots, \dim \mathfrak{g}\}\ .$$

Remark 3.1.42. At any point $p \in G/H$, $e^{i^*}_p$, which is an element of the $(\dim \mathfrak{g} - \dim \mathfrak{h})$-dimensional vector space $T_p^* G/H$, has to satisfy $\dim \mathfrak{g}$ equations. Nevertheless, the one-form e^{i^*} is uniquely defined, since the kernel of the map

$$\begin{aligned} \varphi : \mathfrak{g} &\to T_p G/H \\ \varphi(X) &:= X_p^* \end{aligned}$$

is a conjugate of $\mathfrak{h}$.

The one-forms $e^{i'}$ and e^{i^*} are related by a lemma analogous to Lemma 3.1.17:

Lemma 3.1.43. *Let G/H be a coset space and let $e^{i'}$ and e^{i^*} be defined as above. We denote the projection of G onto G/H by π. Furthermore, let $pr_V : \mathfrak{g} \to \mathfrak{g}$ be the projection onto a subspace of $\mathfrak{g}$ with respect to q. In this situation, we have for all $i \in \{1, \ldots, \dim \mathfrak{g}\}$ and $k \in G$:*

$$\pi^*(e^{i^*})_k = (e^i - pr_{Ad_k(\mathfrak{h})}(e^i))'_k\ .$$

Proof : We show the lemma by evaluating the above equation on an arbitrary e'_j. The point $k \in G$ will be omitted from our formulas for reasons of simplicity. We have:

$$\begin{aligned} &\pi^*(e^{i^*})(e'_j) \\ =& e^{i^*}(d\pi(e'_j)) \\ =& e^{i^*}(e_j^*) \\ =& q(e_i^*, e_j^*) \end{aligned}$$

Since π is a Riemannian submersion, whose fiber at $kH \in G/H$ is $kH \subseteq G$, we can transform the above term into:

$$\begin{aligned} =& q((e_i - \mathrm{pr}_{\mathrm{Ad}_k(\mathfrak{h})}(e_i))', (e_j - \mathrm{pr}_{\mathrm{Ad}_k(\mathfrak{h})}(e_j))') \\ =& q((e_i - \mathrm{pr}_{\mathrm{Ad}_k(\mathfrak{h})}(e_i))', e'_j) - q((e_i - \mathrm{pr}_{\mathrm{Ad}_k(\mathfrak{h})}(e_i))', \mathrm{pr}_{\mathrm{Ad}_k(\mathfrak{h})}(e_j))') \\ =& (e^i - \mathrm{pr}_{\mathrm{Ad}_k \mathfrak{h}}(e^i))'(e'_j) \end{aligned}$$

□

The exterior derivatives of the one-forms $e^{i'}$ can be directly determined with help of the definition of d:

$$\begin{aligned}
de^{i'}(e'_j, e'_k) &= e'_j(e^{i'}(e'_k)) - e'_k(e^{i'}(e'_j)) - e^{i'}([e'_j, e'_k]) \\
&= e'_j(\underbrace{q(e'_i, e'_k)}_{const.}) - e'_k(\underbrace{q(e'_i, e'_j)}_{const.}) - e^{i'}([e'_j, e'_k]) \\
&= -e^{i'}([e'_j, e'_k]) \\
&= e^{i'}([e_j, e_k]') \\
&= q(e_i, [e_j, e_k])
\end{aligned}$$

We now assume that the basis $(e_i)_{1 \leqslant i \leqslant \dim \mathfrak{g}}$ is q-orthogonal. In this situation, we obtain the following explicit formula for de'_i:

$$de^{i'} = \sum_{1 \leqslant j < k \leqslant \dim \mathfrak{g}} \frac{q(e_i, [e_j, e_k])}{q(e_i, e_i)} e^{jk'} \,,$$

where $e^{jk'}$ denotes $e^{j'} \wedge e^{k'}$. Since the pull-back π^* commutes with the exterior differential, we are now able to deduce a formula for de^{i^*}. In order to make our formula easy to manage, we assume that there is a $n \in \{1, \ldots, \dim \mathfrak{g}\}$ such that $(e_i)_{1 \leqslant i \leqslant n}$ spans $\mathfrak{m}$ and $(e_i)_{n+1, \ldots, \dim \mathfrak{g}}$ spans $\mathfrak{h}$. The pull-back of e^{i^*} is not $e_i{}'$. Instead we have the following relations:

$$\begin{array}{rcll}
\pi^*(e^{i^*})_e &=& e^{i'}_{\ e} & \forall i \in \{1, \ldots, n\} \\
\pi^*(e^{i^*})_e &=& 0 & \forall i \in \{n+1, \ldots, \dim \mathfrak{g}\}
\end{array}$$

Because of the formula in Lemma 3.1.43, which is more complicated than Lemma 3.1.17, these relations are not necessarily true at another point of G than e. With help of the above relations and the formula for de^{i^*}, we obtain:

Lemma 3.1.44. *Let G/H be a coset space. We denote the Lie algebra of G by $\mathfrak{g}$. Let $(e_i)_{1 \leqslant i \leqslant \dim \mathfrak{g}}$ be a basis of $\mathfrak{g}$ which is orthogonal with respect to a biinvariant metric q. Furthermore, we assume that there exists a $n \in \{1, \ldots, \dim \mathfrak{g}\}$ with $\mathfrak{m} = span(e_i)_{1 \leqslant i \leqslant n}$ and $\mathfrak{h} = span(e_i)_{n+1, \ldots, \dim \mathfrak{g}}$. In this situation, we have:*

$$(de^{i^*})_{eH} = \sum_{1 \leqslant j < k \leqslant n} \frac{q_e(e_i, [e_j, e_k])}{q_e(e_i, e_i)} (e^{jk^*})_{eH} \,,$$

where e^{jk^*} denotes $e^{j^*} \wedge e^{k^*}$.

By applying the anti-derivation property of d, we could compute the exterior derivatives of arbitrary k-forms of type:

$$c_{i_1 \ldots i_k} e^{i_1 \ldots i_k{}^*} := c_{i_1 \ldots i_k} e^{i_1{}^*} \wedge \ldots \wedge e^{i_k{}^*} \tag{3.3}$$

at the point eH. Unfortunately, k-forms of that kind are in general not G-invariant. Nevertheless, the considerations we have made can be used for determining the exterior derivatives of G-invariant forms.

If we restrict a G-invariant k-form on G/H to the point $eH \in G/H$, we obtain an H-invariant tensor in $\bigwedge^k T^*_{eH}G/H$. Conversely, any H-invariant $\alpha \in \bigwedge^k T^*_{eH}G/H$ can be extended by the action of G to a well-defined G-invariant differential form on G/H, which we denote by α, too.

There exists a unique k-form of type (3.3) which coincides at eH with α. That form we will denote by α^*. Our next aim is to determine $d\alpha_{eH}$ and to compare it with $d\alpha^*_{eH}$. We recall that the exterior derivative of a differential form is defined by:

$$\begin{aligned} d\alpha(X_1^*, \ldots, X_{k+1}^*) = &\sum_{i=1}^{k+1} (-1)^{i+1} X_i^* \alpha(X_1^*, \ldots, \hat{X}_i^*, \ldots, X_{k+1}^*) \\ &+ \sum_{1 \leqslant i < j \leqslant k+1} (-1)^{i+j} \alpha([X_i^*, X_j^*], X_1^*, \ldots, \hat{X}_i^*, \ldots, \hat{X}_j^*, \ldots, X_{k+1}^*) \,. \end{aligned}$$

The vector fields we have inserted into $d\alpha$ in the above formula could be chosen arbitrarily. For our calculations, we assume that they are Killing vector fields. We obtain for the first kind of summands:

$$\begin{aligned} &X_i^* \alpha_{eH}(X_1^*, \ldots, \hat{X}_i^*, \ldots, X_{k+1}^*) \\ &= \frac{\partial}{\partial t}|_{t=0}\, \alpha_{(\exp tX_i)H}(X_1^*, \ldots, \hat{X}_i^*, \ldots, X_{k+1}^*) \\ &= \frac{\partial}{\partial t}|_{t=0}\, \alpha_{eH}(dL_{\exp(-tX_i)}(X_1^*), \ldots, \widehat{dL_{\exp(-tX_i)}(X_i^*)}, \ldots, dL_{\exp(-tX_i)}(X_{k+1}^*)) \\ &= \frac{\partial}{\partial t}|_{t=0}\, \alpha_{eH}((\mathrm{Ad}_{\exp(-tX_i)} X_1)^*, \ldots, \widehat{(\mathrm{Ad}_{\exp(-tX_i)} X_i)^*}, \ldots, (\mathrm{Ad}_{\exp(-tX_i)} X_{k+1})^*) \\ &= \sum_{j=1,\, j \neq i}^{k+1} \alpha_{eH}(X_1^*, \ldots, \hat{X}_i^*, \ldots, \underbrace{(\mathrm{ad}_{-X_i} X_j)^*}_{\text{entry of } X_j^*}, \ldots, X_{k+1}^*) \end{aligned}$$

Therefore, we have:

$$
\begin{aligned}
& d\alpha_{eH}(X_1^*, \ldots, X_{k+1}^*) \\
= & \sum_{1 \leqslant i < j \leqslant k+1} (-1)^{i+1} \alpha_{eH}(X_1^*, \ldots, \hat{X}_i^*, \ldots, \underbrace{(\mathrm{ad}_{-X_i} X_j)^*}_{\text{entry of } X_j^*}, \ldots, X_{k+1}^*) \\
& + \sum_{1 \leqslant i < j \leqslant k+1} (-1)^{j+1} \alpha_{eH}(X_1^*, \ldots, \underbrace{(\mathrm{ad}_{-X_j} X_i)^*}_{\text{entry of } X_i^*}, \ldots, \hat{X}_j^*, \ldots, X_{k+1}^*) \\
& + \sum_{1 \leqslant i < j \leqslant k+1} (-1)^{i+j} \alpha_{eH}([X_i^*, X_j^*], X_1^*, \ldots, \hat{X}_i^*, \ldots, \hat{X}_j^*, \ldots, X_{k+1}^*) \\
= & \sum_{1 \leqslant i < j \leqslant k+1} ((-1)^{(i+1)+(j-2)} - (-1)^{(j+1)+(i-1)}) \\
& \alpha_{eH}((\mathrm{ad}_{-X_i} X_j)^*, X_1^*, \ldots, \hat{X}_i^*, \ldots, \hat{X}_j^*, \ldots, X_{k+1}^*) \\
& + \sum_{1 \leqslant i < j \leqslant k+1} (-1)^{i+j+1} \alpha_{eH}([X_i, X_j]^*, X_1^*, \ldots, \hat{X}_i^*, \ldots, \hat{X}_j^*, \ldots, X_{k+1}^*) \\
= & \sum_{1 \leqslant i < j \leqslant k+1} (-1)^{i+j} \alpha_{eH}([X_i, X_j]^*, X_1^*, \ldots, \hat{X}_i^*, \ldots, \hat{X}_j^*, \ldots, X_{k+1}^*)
\end{aligned}
\tag{3.4}
$$

We could have made a similar calculation for $d\alpha_{gH}$. Since α is homogeneous, $d\alpha$ is determined by $d\alpha_{eH}$. Therefore, there is no need to compute $d\alpha_{gH}$. We now assume that α is a one-form. In this case, we have:

$$d\alpha_{eH}(X_1^*, X_2^*) = -\alpha_{eH}([X_1, X_2]^*) \,.$$

For the one-form α^*, which is dual to a Killing vector field, we have:

$$d\alpha^*_{eH}(X_1^*, X_2^*) = \alpha^*_{eH}([X_1, X_2]^*) \,.$$

Up to the sign, the values of $d\alpha_{eH}$ and $d\alpha^*_{eH}$ coincide. With help of the anti-derivation property of d, this observation can be generalized to forms of higher order:

Proposition 3.1.45. *Let α be a G-invariant differential form on a coset space G/H. Furthermore, let α^* be the unique differential form of type (3.3) which coincides with α at the point eH. Then, we have:*

$$d\alpha_{eH} = -d\alpha^*_{eH} \,.$$

Remark 3.1.46. 1. Let $(e_i)_{1 \leqslant i \leqslant n}$ be a basis of $\mathfrak{m}$. An H-invariant $\alpha \in \bigwedge^k T^*_{eH} G/H$ can be written as

$$\alpha = \alpha_{i_1 \ldots i_k} e^{i_1 \ldots i_k} \,. \tag{3.5}$$

The extension of α by the action of G to all of G/H will be denoted by (3.5), too, although there may not exist global coordinate vector fields $e_1, \ldots, e_n$. In most cases, even linearly independent vector fields $e_1, \ldots, e_n$ will not exist on G/H.

2. When we want to determine the exterior derivative of a G-invariant differential form α at eH, we will do this by the following procedure: We formally apply the product rule for the exterior differential to (3.5) and replace each e^{i_j} we have to derivate by $de^{i_j}{}^*_{eH}$. After that, we multiply the result by -1 and finally have obtained $d\alpha_{eH}$.

At the end of this section, we will collect some useful results on homogeneous G-structures. Let $\mathfrak{F}^{K/H}$ be the coframe bundle of a coset space K/H. K acts from the right on $\mathfrak{F}^{K/H}$ by $(k,u) \mapsto u \circ (dL_k)$, where $u : T_pM \to \mathbb{R}^{\dim K/H}$ and $k \in K$. It therefore makes sense to speak of K-invariant G-structures on K/H. If we have a K-invariant G_2- (Spin(7)-)structure on K/H, the associated three- (four-)form is K-invariant, too. Conversely, any K-invariant three- (four-)form stabilized by G_2 (Spin(7)) defines a K-invariant G_2- (Spin(7)-)structure.

In Chapter 4, we will classify the possible principal orbits for cohomogeneity-one Spin(7)-structures. We will show in Section 3.2, that there exists a canonical homogeneous G_2-structure on those orbits. Conversely, any G-invariant G_2-structure on a coset space G/H can be extended to a cohomogeneity-one Spin(7)-structure on $G/H \times I$, where I is an interval. We therefore need a necessary and sufficient criterion for the existence of a G-invariant G_2-structure on G/H:

Lemma 3.1.47. *Let G/H be a seven-dimensional coset space admitting a G-invariant G_2-structure $\mathfrak{G}$. H acts on the tangent space $T_{eH}G/H$ by $(h,v) \mapsto (dL_h)(v)$. Therefore, we can consider H as a subgroup of $GL(T_{eH}G/H)$. G_2 acts on $T_{eH}G/H$, too, where $G_2 \subseteq GL(T_{eH}G/H)$ is given by:*

$$\{u^{-1} \circ v | \mathfrak{G} \ni u, v : T_{eH}G/H \to \mathbb{R}^7\} .$$

In this situation, we have $H \subseteq G_2$.

Proof : Let $h \in H$ and $\mathfrak{G} \ni u : T_{eH}G/H \to \mathbb{R}^7$ be arbitrary. Since $\mathfrak{G}$ is G-invariant, $u \circ (dL_h) : T_{eH}G/H \to \mathbb{R}^7$ is an element of the G_2-structure, too. The endomorphism $u^{-1} \circ v$ is for any $\mathfrak{G} \ni v : T_{eH}G/H \to \mathbb{R}^7$ an element of G_2. By setting $v = u \circ (dL_h)$ we obtain the statement of the lemma.

□

Remark 3.1.48. 1. In the above lemma, we could define $G_2 \subseteq GL(T_{eH}G/H)$ also as the stabilizer of the associated three-form.

2. It is possible to prove a similar lemma for G-invariant K-structures on G/H, where $K \subseteq GL(\dim G/H)$ is an arbitrary Lie subgroup.

3. On a homogeneous space, there may exist more than one transitive group action. It is possible that a G_2-structure is invariant under one of those actions, but not under another one. The flat G_2-structure on $\mathbb{R}^7$, for example, is invariant under translations but not under the whole Euclidean group.

Lemma 3.1.49. *Let G/H be a seven-dimensional coset space. As above, we consider H as a subgroup of $GL(T_{eH}G/H)$. If there exists a linear $\varphi : T_{eH}G/H \to \mathbb{R}^7$ with $H \subseteq \varphi^{-1}G_2\varphi$, G/H admits a G-invariant G_2-structure.*

Proof : The set

$$\{\varphi \circ (dL_h) \circ (dL_{g^{-1}}) : T_{gH}G/H \to \mathbb{R}^7 | h \in H, g \in G\}$$

defines a G-invariant $\varphi H \varphi^{-1}$-structure on G/H. Since the extension of this set to a G_2-structure is G-invariant, too, the statement of the lemma is true.

□

We assume that we have found a fixed G-invariant G_2-structure on a coset space G/H. Furthermore, we assume without loss of generality that G and H are both connected. The G_2-structure is given by a principal bundle with fiber G_2, which we denote as above by $\mathfrak{G}$. $\mathfrak{G}$ can be extended to a $SO(7)$-structure, which induces an invariant metric and an orientation on G/H. Our next aim is to classify the invariant G_2-structures on G/H which have the same extension to a $SO(7)$-structure as $\mathfrak{G}$.

First, we will make some general observations. The projection map of $\mathfrak{G}$ we denote by $\pi : \mathfrak{G} \to G/H$. We fix a point $p \in G/H$. Since $\mathfrak{G}$ is G-invariant, any element of the fiber $\pi^{-1}(p)$ determines the whole G_2-structure. Let $\pi^{-1}(p) \ni u : T_pG/H \to \mathbb{R}^7$. We have:

$$\pi^{-1}(p) = \{h \circ u | h \in G_2\} \ .$$

Furthermore, we have:

$$\mathfrak{G} = \{h \circ u \circ (dL_{k^{-1}})_{L_k(p)} | h \in G_2, k \in G\} \ .$$

In Chapter 5, where we have to construct invariant G_2-structures, it will therefore suffice to find a $u \in \pi^{-1}(p)$ or equivalently a frame $(u^{-1}(e_1), \ldots, u^{-1}(e_7))$.

We choose another G_2-structure $\mathfrak{H}$ on G/H with the same properties as $\mathfrak{G}$. Its projection map is denoted by $\pi' : \mathfrak{H} \to G/H$. Let $v \in \pi'^{-1}(p)$. Since v determines $\mathfrak{H}$, and $\mathfrak{G}$ is fixed, the map $\phi := v \circ u^{-1} : \mathbb{R}^7 \to \mathbb{R}^7$ determines $\mathfrak{H}$. We can therefore solve our problem by classifying the possible endomorphisms ϕ.

Our next step is to describe the set of all ϕ. The maps $u^{-1} : \mathbb{R}^7 \to T_pG/H$ and $v : T_pG/H \to \mathbb{R}^7$ preserve the orientation and the scalar product. Therefore, we have $\phi \in SO(7)$. We remark that

$$\{t \circ u^{-1} | t \in \pi^{-1}(p)\} = G_2 \ .$$

Since H acts on T_pG/H, we consider H as a subgroup of $GL(T_pG/H)$. We can identify T_pG/H by u with $\mathbb{R}^7$. Therefore, we can take H as a subgroup of $GL(7)$. We have shown in Lemma 3.1.47 that with this identifications $H \subseteq G_2$. The set

$$\{w \circ v^{-1} | w \in \pi'^{-1}(p)\}$$

is given by G_2, too. As above, we identify T_pG/H by v with $\mathbb{R}^7$ and use this identification to define an action of H on $\mathbb{R}^7$. It is easy to see that $h \in H$ acts by matrix multiplication with

$\phi h\phi^{-1}$. For the same reasons as above, we have $\phi H\phi^{-1} \subseteq G_2$. All in all, we have shown that ϕ is an element of the set

$$M := \{\phi \in SO(7) | \phi H\phi^{-1} \subseteq G_2\} .$$

Conversely, each $\phi \in M$ describes a G_2-structure with the desired properties.

If $\phi H\phi^{-1} \subseteq G_2$ and $\psi H\psi^{-1} \subseteq G_2$, we do not necessarily have $(\phi\psi)H(\phi\psi)^{-1} \subseteq G_2$. Therefore, M is not necessarily a group, but an invariant set with respect to the adjoint action of H. H and $\phi H\phi^{-1}$ are both connected subgroups of G_2. In Lemma 4.1, we will show that connected subgroups of G_2 which act with the same weights on $\mathbb{R}^7$ are conjugate to each other not only by an element of $SO(7)$ but by a further $\varphi \in G_2$, too. Therefore, we have:

$$\varphi^{-1}\phi H\phi^{-1}\varphi = H \subseteq G_2 .$$

Since both ϕ and $\varphi^{-1}\phi$ describe the same G_2-structure, we can consider instead of M the normalizer of H in $SO(7)$:

$$\mathrm{Norm}_{SO(7)}H = \{\phi \in SO(7) | \phi H\phi^{-1} = H\} .$$

We remark that the space of all G-invariant G_2-structures with a given associated metric and orientation does not depend on the choice of the $SO(7)$-structure. Let $\mathfrak{K}$ be the extension of $\mathfrak{G}$ to a $SO(7)$-structure and let $\mathfrak{K}'$ be a further invariant $SO(7)$-structure on G/H. We choose an orientation-preserving isometry $\psi : T_pG/H \to T_pG/H$ where the former G/H is equipped with $\mathfrak{K}$ and the latter with $\mathfrak{K}'$. The set

$$\{h \circ u \circ \psi \circ (dL_{k^{-1}})_{L_k(p)} | h \in G_2, k \in G\}$$

is an invariant G_2-structure which can be extended to $\mathfrak{K}'$. Conversely, let $\mathfrak{G}'$ be an invariant G_2-structure with $\mathfrak{K}'$ as its extension and $\mathfrak{G}' \ni v : T_pG/H \to \mathbb{R}^7$. Then

$$\{h \circ v \circ \psi^{-1} \circ (dL_{k^{-1}})_{L_k(p)} | h \in G_2, k \in G\}$$

is a G_2-structure with $\mathfrak{K}$ as its extension. In order to describe the set of all G_2-structures with the desired properties we have to divide the normalizer by the action of those elements of $SO(7)$ which leave G_2 invariant. We obtain:

$$\mathrm{Norm}_{SO(7)}H/(\mathrm{Norm}_{SO(7)}H \cap \mathrm{Norm}_{SO(7)}G_2) .$$

In Lemma 2.1.30, we have shown that $\mathrm{Norm}_{SO(7)}G_2 = G_2$. Furthermore, we have: $\mathrm{Norm}_{SO(7)}H \cap G_2 = \mathrm{Norm}_{G_2}H$. All in all, we have proven the following lemma:

Lemma 3.1.50. *Let G/H, where H is connected, be a seven-dimensional homogeneous space admitting a G-invariant G_2-structure. The space of all G-invariant G_2-structures which have a fixed associated metric and orientation is H-equivariantly diffeomorphic to:*

$$Norm_{SO(7)}H/Norm_{G_2}H .$$

In particular, this space does not depend on the choice of the metric and orientation.

If one searches for metrics with exceptional holonomy, homogeneous metrics seem to be a good starting point. Unfortunately, there are no non-trivial homogeneous parallel G_2- or Spin(7)-structures. The reason for this is the following theorem:

Theorem 3.1.51. *(See Besse [10] p.191 or Alekseevskii, Kimelfeld [1].) Any homogeneous Ricci-flat metric is necessary flat.*

Proof : Let G/H be a homogeneous space equipped with a Ricci-flat metric g. Since the Ricci curvature of G/H is non-negative, we can apply the Cheeger-Gromoll splitting theorem. The universal cover of G/H therefore is the Riemannian product of a simply connected homogeneous space G'/H' and a flat $\mathbb{R}^d$. If we choose d maximal, G'/H' contains no line. It is known that any non-compact homogeneous space contains a line. Therefore G'/H' has to be compact. A theorem of Bochner [11] states that if the Ricci curvature of a compact Riemannian manifold is non-positive, the identity component of its isometry group is a torus. Since G'/H' is Ricci-flat, G' therefore is a Lie subgroup of a Lie group whose identity component is a torus. Since, we can assume that G' and H' are connected, G' is a torus, too. If $\dim G' > 0$, $\pi_1(G') = \mathbb{Z}^{\dim G'}$. Since we want G'/H' to be simply connected, $\pi_1(H')$ has to be isomorphic to $\pi_1(G')$. This is only possible, if $G' = H'$. Therefore G'/H' is a point and we have shown that the universal cover of G/H is a flat Euclidean space.

$\square$

Since there are no non-flat homogeneous metrics of exceptional holonomy, it is natural to search for metrics of that kind on spaces whose isometry group is large but not acting transitively on the manifold. In the next section, we will therefore introduce the most important results on metrics of cohomogeneity one.

3.2 Metrics of cohomogeneity one

Since there are no homogeneous metrics with exceptional holonomy, we have to consider metrics whose isometry group is smaller. First, we make the following assumption:

Convention 3.2.1. For reasons of simplicity, we only consider group actions by compact, connected Lie groups. This assumption guarantees that all orbits are closed subsets of the manifold M on which the group acts on. If M is a Riemannian manifold (a Spin(7)-manifold), we assume that the group action preserves the metric (the Spin(7)-structure) on M. Furthermore, we assume that M is connected. All of the above assumptions are maintained throughout this thesis unless otherwise stated.

In the following, we will often have to consider the orbit space M/G of an isometric group action:

Lemma 3.2.2. *Let (M, g) be a connected Riemannian manifold with an isometric action of a Lie group G. Furthermore, let $\pi : M \to M/G$ be the projection map and let $d : M \times M \to [0, \infty)$ be the distance function on M induced by g. The quotient M/G together with the function*

$$\begin{aligned} d' &: M/G \times M/G \to [0, \infty) \\ d'(x', y') &:= \inf \{ d(x, y) | \pi(x) = x', \pi(y) = y' \} \end{aligned}$$

is a metric space.

Example 3.2.3. The orbit space M/G is not necessarily a manifold. We consider the Euclidean space $\mathbb{R}^n$ with $SO(n)$ acting on it by matrix multiplication. The orbits of this action are the spheres around the origin and the origin itself. The space $\mathbb{R}^n/SO(n)$ is isometric to the interval $[0, \infty)$ with the standard metric, which is not a manifold.

By considering the above example, we see that an isometric action can have different types of orbits. The orbits whose neighborhood in M/G can be identified with an open set in $\mathbb{R}^n$ are of special interest for us:

Definition 3.2.4. Let (M, g) be a connected Riemannian manifold with an isometric action by a Lie group G. An orbit $\mathcal{O}$ of this action is called a *principal orbit* if there is an open subset U of M with the following properties: $\mathcal{O} \subseteq U$ and U is G-equivariantly diffeomorphic to $\mathcal{O} \times V$, where $V \subseteq \mathbb{R}^n$ is an open subset.

Since we are interested in spaces with a large isometry group, we have to find a measure for the size of G:

Definition 3.2.5. In the situation of the above definition, $\dim V$ (or equivalently $\dim M - \dim \mathcal{O}$) is called the *cohomogeneity of the G-action on (M, g)*.

If M is connected, any two principal orbits are G-equivariantly diffeomorphic. Therefore, the above definition is justified. Furthermore, we can identify any principal orbit with a coset space G/H, where $H \subseteq G$ is fixed.

Convention 3.2.6. From now on, we assume that we have a Lie group G acting isometrically on a Riemannian manifold (M, g), such that the cohomogeneity of the G-action is 1. In this situation, the subset U from Definition 3.2.4 is G-equivariantly diffeomorphic to $\mathcal{O} \times I$, where $I \subseteq \mathbb{R}$ is an open interval. The isotropy group of the G-action on the principal orbit we denote by H.

It is possible to show the following lemma:

Lemma 3.2.7. *In the situation of Convention 3.2.6, the following statements are true:*

1. *The union of all principal orbits of the G-action is an open, dense subset of M.*
2. *Any orbit of the cohomogeneity-one action is G-equivariantly diffeomorphic to a coset space G/K, where K is a Lie subgroup of G. We can conjugate K in such a way that $H \subseteq K \subseteq G$.*

The union of all principal orbits will often be denoted by M^o. We now describe the non-principal orbits in more detail. There are two kinds of these orbits, which are distinguished by the dimension of K/H:

Definition 3.2.8. In the situation of the above lemma, let G/K be a non-principal orbit. G/K is called a

- *singular orbit* if $\dim K > \dim H$.
- *exceptional orbit* if $\dim K = \dim H$, i.e., the quotient K/H is discrete.

The Lie group K is not arbitrary, but has to satisfy a certain condition:

Proposition 3.2.9. *(See Mostert [55].) Let (M, g) be a Riemannian manifold with a cohomogeneity-one action by a Lie group G. Furthermore, let the principal orbits of this action be G-equivariantly diffeomorphic to G/H and let G/K be a non-principal orbit. In this situation, the quotient K/H is diffeomorphic to a sphere.*

Remark 3.2.10.

- If we do not require M to be a smooth manifold, but still an orbifold, K/H is diffeomorphic to S^k/Γ, where $k := \dim K - \dim H$ and Γ is a discrete subgroup of $O(k+1)$ acting on S^k.
- Any non-principal orbit has a neighborhood in M/G which consists solely of principal orbits. The above proposition therefore guarantees that there is a neighborhood of the singular orbit which is a disc bundle over that orbit. We call such a neighborhood a *tubular neighborhood.* In the orbifold case, we have a S^k/Γ-bundle instead of a disc bundle. We will speak in that situation of a tubular neighborhood, too.

Convention 3.2.11.

- In the following, we assume for reasons of simplicity that M is a manifold. It is easily possible to modify the results of this section in such a way that they are true for orbifolds, too.
- Motivated by the above observation, we will often speak of a *collapse of the sphere S^k at the singular orbit* or simply of a *S^k-collapse* when we approach a singular orbit.

For our next theorem we need the following lemma:

Lemma 3.2.12. *On a connected Riemannian manifold* (M, g) *with a* G*-action of cohomogeneity one there exists a geodesic* $\gamma : I \to M$ *which intersects all orbits perpendicularly. Let* $\pi : M \to M/G$ *be the projection map. In this situation, we obviously have* $\pi \circ \gamma(I) = M/G$*.*

With help of this lemma, it is possible to show the following result on the topology of M/G:

Theorem 3.2.13. *(See Mostert [55].) Let* G *be a group acting with cohomogeneity one on a manifold* M*. The quotient* M/G *is homeomorphic to one of the following:*

1. $M/G \cong \mathbb{R}$,
2. $M/G \cong S^1$,
3. $M/G \cong [0, \infty)$,
4. $M/G \cong [0, 1]$.

All points in the interior of M/G *correspond to principal orbits. The points on the boundary, namely* 0 *in the third and* 0*,* 1 *in the fourth case correspond to non-principal orbits. In the situation of the fourth case, we denote the isotropy group of the* G*-action at* 0 *by* K^- *and the isotropy group at* 1 *by* K^+*. If we have two non-principal orbits, the following diagram has to commute:*

$$\begin{tikzcd}
& G & \\
K^- \arrow[ur] & & K^+ \arrow[ul] \\
& H \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

Let $K \subseteq G$ be the isotropy group of a non-principal orbit. The action of K on the sphere $S^k \cong K/H$ is transitive. Since we can fill $\mathbb{R}^{k+1}$ by spheres of nonnegative radii, we can extend this action to an action on $\mathbb{R}^{k+1}$. It is known that any transitive action by a Lie group on a sphere is the restriction of a linear action (see Montgomery, Samelson [54]). Therefore, the action on $\mathbb{R}^{k+1}$ we consider is linear, too. This action is the same as the action of K on the normal space of a $p \in G/K$. It should be noted that this action of K on $\mathbb{R}^{k+1}$ is solely determined by the choice of G, H, and K. These considerations together with the above theorem make it possible to show the following result on the global shape of M:

Corollary 3.2.14. *Let* (M, g) *be a Riemannian manifold with an isometric cohomogeneity-one action of a Lie group* G*. Furthermore, let* H *be the isotropy group of the* G*-action on a principal orbit. If the* G*-action has exactly one non-principal orbit, we denote it by* K*. If there are two non-principal orbits, we denote them by* K^- *and* K^+*. In this situation,* M *is, depending on the topology of* M/G*,* G*-equivariantly diffeomorphic to one of the following:*

1. *If* $M/G \cong \mathbb{R}$*, then* $M \cong G/H \times \mathbb{R}$*.*

2. *If $M/G \cong S^1$, then $M \cong G/H \times_{\mathbb{Z}} \mathbb{R}$. The action of $\mathbb{Z}$ on $G/H \times \mathbb{R}$ we divide out is the following: $\mathbb{Z}$ acts on $\mathbb{R}$ by translation and the action of $\mathbb{Z}$ on G/H is generated by the canonical right action of an element of $Norm_G H$.*

3. *If $M/G \cong [0, \infty)$, then M is G-equivariantly diffeomorphic to the disc bundle $G \times_K D$ over G/K, where D is the unit disc of dimension $\dim K - \dim H + 1$. Since D is diffeomorphic to $\mathbb{R}^{\dim D}$, we can also describe M as a vector bundle over the non-principal orbit. The projection of this bundle onto G/K is a Riemannian submersion.*

4. *If $M/G \cong [0, 1]$, then $M \cong (G \times_{K^-} D^-) \cup (G \times_{K^+} D^+)$ where $G \times_{K^-} D^-$ and $G \times_{K^+} D^+$ have the same description as above. The two spaces are glued along a copy of G/H by a G-equivariant diffeomorphism of G/H.*

In particular, the shape of M/G together with the groups H and $K^{(\pm)}$ determines the global shape of M.

We are able to exclude some cases from our considerations. First, we will show that it is possible to restrict ourselves to manifolds without an exceptional orbit. We assume that M is a cohomogeneity-one manifold with at least one exceptional orbit. Since the quotient K/H has to be a sphere, we have $K/H = \mathbb{Z}_2$. There are three possible cases:

1. If M has only one non-principal orbit, which of course has to be exceptional, it is possible to construct a two-fold cover of M which is G-equivariantly diffeomorphic to $G/H \times \mathbb{R}$.

2. Next, we assume that M has one exceptional and one singular orbit. We assume without loss of generality that G/K^- is exceptional. In this case, M is covered by a manifold of type $(G \times_{K^+} D^+) \cup (G \times_{K^+} D^+)$ which has two singular orbits of type G/K^+. This cover is two-fold, too. In the course of our considerations, we will not consider manifolds of this and the previous type anymore, since we can avoid them by passing to a cover.

3. We finally assume that M has two exceptional orbits. In this case, we can repeat the construction from the second case infinitely often. M is therefore a compact manifold with infinite fundamental group. Since we are interested in parallel Spin(7)-manifolds of cohomogeneity one, we assume that M carries a parallel Spin(7)-structure. It follows from Lemma 2.3.16 that the holonomy group acts trivially on a non-trivial subspace of the tangent space and thus is a subgroup of G_2.

Since we are only interested in examples with an irreducible holonomy representation, we will assume from now on that M has no exceptional orbits. The case where M has no singular orbit can be excluded, too: If $M/G \cong S^1$, then M is the total space of a homogeneous bundle over S^1 and hence has infinite fundamental group. Furthermore, M is compact. We can therefore apply the same arguments as above and exclude that case. Next, we consider the case where (M, g) is complete and $M/G \cong \mathbb{R}$. In this situation, the geodesic which intersects all orbits perpendicularly is a line. By applying the Cheeger-Gromoll splitting theorem, we can show that (M, g) is a Riemannian product of $\mathbb{R}$ and G/H. Since we are first of all interested in complete examples which are not locally a Riemannian product, we will assume from now on that M has at least one singular orbit. In this situation, it is very easy to decide if M is complete or compact:

Lemma 3.2.15. *Let (M, g) be a cohomogeneity-one manifold with a singular orbit. In this situation, we have:*

1. *M is compact if and only if it has two singular orbits.*
2. *M is complete and non-compact if and only if M has exactly one singular orbit, and there exists a geodesic γ which intersects all orbits perpendicularly and has infinite length.*
3. *M is non-complete if and only if M has exactly one singular orbit, and there exists a γ which cannot infinitely be extended.*

Proof : We will prove that in the situation of the second case the manifold is complete. The other statements of the lemma then are obviously true. Let $(p_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in M. The projection map $M \to M/G$ we denote as usual by π. Since

$$d(\pi(x), \pi(y)) \leqslant d(x, y) \quad \forall x, y \in M ,$$

$(\pi(p_n))_{n\in\mathbb{N}}$ is a Cauchy sequence, too. We equip $[0, \infty)$ with the standard metric. $\gamma : [0, \infty) \to M$ is an injective map. Therefore, M/G can isometrically be identified with $[0, \infty)$, which is complete. We define $x \in [0, \infty)$ as $\lim_{n\to\infty} \pi(p_n)$. Since all orbits of the G-action are compact, π is a proper map, i.e., all compact subsets $N \subseteq [0, \infty)$ have a compact preimage $\pi^{-1}(N)$. The set $\{\pi(p_n)|n \in \mathbb{N}\} \cup \{x\}$ a compact one. Therefore, $\pi^{-1}(\{p_n|n \in \mathbb{N}\} \cup \{x\})$ is compact, too. p_n is a sequence in that set. There exists a subsequence of p_n which converges against a $p \in M$. The Cauchy sequence $(p_n)_{n\in\mathbb{N}}$ therefore has to converge against p, too.

$\square$

The subject of the following pages are cohomogeneity-one Einstein metrics. This issue is important for us, since metrics with holonomy $\subseteq$ Spin(7) are Ricci-flat and therefore Einstein. Our considerations will be in large part a synopsis of the paper of Eschenburg and Wang [32]. Before we start, we fix some notation, which we will maintain throughout this Section:

Convention 3.2.16. In the following, let (M, g) be a Riemannian manifold with an isometric cohomogeneity-one action of a Lie group G. As usual, we assume that G is compact and connected. Furthermore, let G/H be the principal orbit of this action and G/K be a singular orbit. The Lie algebra of G we denote by $\mathfrak{g}$ and the Lie algebra of H by $\mathfrak{h}$. As in Section 3.1, we fix a biinvariant metric q on $\mathfrak{g}$. The q-orthogonal complement of $\mathfrak{h} \subseteq \mathfrak{g}$ is called $\mathfrak{m}$. Furthermore, we denote the Lie algebra of $\mathfrak{k}$ by K and the q-orthogonal complement of $\mathfrak{k} \subseteq \mathfrak{g}$ by $\mathfrak{p}$. The normal space of $G/K \subseteq M$ will be denoted by $\mathfrak{p}^\perp$. Finally, let the dimension of the sphere K/H be k.

The notation "$\mathfrak{p}^\perp$" is motivated by the following lemma which follows from our earlier considerations:

Lemma 3.2.17. *The tangent space of the singular orbit G/K is K-equivariantly isomorphic to $\mathfrak{p}$. The normal space of G/K at p is the orthogonal complement of $\mathfrak{p} \subseteq T_pM$ and a K-module, too. K acts on $\mathfrak{p}^\perp$ by cohomogeneity one. Furthermore, its action on $\mathfrak{p}^\perp$ is determined by the choice of G, H, and K.*

Let β be a G-invariant tensor field defined on a tubular neighborhood of a singular orbit G/K. As we have seen, the tubular neighborhood is a disc bundle over G/K, which we will shortly denote by D. For the theorem on the cohomogeneity-one Einstein equation which we will introduce, we have to consider the question how we can describe β by a power series and which power series of that type can smoothly be extended to the singular orbit.

The vector bundle over D of which β is a section we denote by $\mathcal{B}$. Since the tangent space of $p \in G/K \subseteq D$ splits into the K-modules $\mathfrak{p}$ and $\mathfrak{p}^\perp$, we can consider the fiber $\mathcal{B}_p$ as a K-module, too. If p is outside the singular orbit, $\mathcal{B}_p$ should be considered as an H-module.

The fiber of the disc bundle D at a $p \in G/K$ will be denoted by D_p. Since β is G-invariant, it is determined by its restriction to a fixed D_p. We choose a vertical unit vector $v \in \mathfrak{p}^\perp$. Since the following considerations hold for any choice of v, we assume without loss of generality that we have chosen v in such a way that it is stabilized by H. Let $\gamma : [0, r) \to D_p$, where r is the radius of D_p, be the geodesic with $\gamma(0) = p$ and $\gamma'(0) = v$. Since K acts transitively on the sphere K/H, β is determined by $\beta \circ \gamma$.

We can identify the open disc in $\mathfrak{p}^\perp$ of radius r by the exponential map with D_p. Therefore, we can identify $\beta \circ \gamma$ with a map $\widetilde{b} : \{vt | t \in [0, r)\} \to \mathcal{B}$. By letting K act on vt, we can extend $\widetilde{b}(vt)$ to a map $S_t^k \to \mathcal{B}$, where S_t^k denotes the sphere of radius t in $\mathfrak{p}^\perp$. Moreover, we can extend $\widetilde{b}(vt)$ by linearity to a map $\mathfrak{p}^\perp \to \mathcal{B}$. In the following, we will make use of both points of view and sometimes consider $\widetilde{b}(vt)$ as an element of $\mathcal{B}$ and sometimes as a map $\mathfrak{p}^\perp \to \mathcal{B}$. We finally define a map b on $[0, r)$ by $b(t) := \widetilde{b}(vt)$. b satisfies:

$$b : [0, r) \to \mathrm{Hom}_K(\mathfrak{p}^\perp, \mathcal{B}) \,, \tag{3.6}$$

where $\mathrm{Hom}_K(V, W)$ denotes the space of K-equivariant homomorphisms from V to W. Alternatively, we have:

$$b : [0, r) \to \mathcal{B} \,, \tag{3.7}$$

where $b(t)$ has to be invariant with respect to the isotropy group H.

The point $t = 0$ plays a special role. At that point the isotropy group of the G-action is K. Therefore, $b(t)$ has to turn into a K-invariant tensor when t approaches 0. If we consider the b described by (3.6), $t = 0$ does not play a special role, since $b(t)$ always has to be a K-equivariant homomorphism. This observation will be the key for the Theorem 3.2.18 we will introduce below. In the following, we restrict b to $(0, r)$ and consider the question if we can extend b to 0.

We assume that b is an analytic function. This assumption is justified, since Einstein metrics of dimension $\geqslant 3$ are real analytic in geodesic normal coordinates (see Besse [10], Section 5.F. and DeTurck, Kazdan [30]). In this situation, we can express the b from (3.7) as a power series

$$b(t) = \sum_{m=0}^{\infty} b_m t^m \,.$$

Since we can extend $b(t)$ to an element of $\mathrm{Hom}_K(\mathfrak{p}^\perp, \mathcal{B})$, the summand $b_m t^m$ has to be the restriction of a homogeneous polynomial P_m to $\mathrm{span}(v)$. More precisely, P_m is a m^{th}-order polynomial defined on $\mathfrak{p}^\perp$ with values in the fiber $\mathcal{B}_p$. Furthermore, P_m has to be K-invariant. It is possible to show that any choice of K-invariant homogeneous polynomials P_m defines a tensor field β, which can be smoothly extended to the singular orbit. This is the statement of our theorem. In order to properly state the theorem, we define the following spaces of homogeneous polynomials:

$$W_m := \{P : S^m(\mathfrak{p}^\perp) \to \mathcal{B}_p | P \text{ is linear and } K\text{-equivariant}\} .$$

Since K acts orthogonally on $\mathfrak{p}^\perp$, we can identify $\mathfrak{p}^\perp$ K-equivariantly with its dual space. Next, we define the evaluation map:

$$\begin{aligned} \epsilon_m &: W_m \to \mathcal{B}_p \\ \epsilon_m(P) &:= P(v) . \end{aligned}$$

We remark that $\epsilon_m(P)$ is an H-invariant polynomial, since H stabilizes v. Now, we are able to state our theorem:

Theorem 3.2.18. *(See Eschenburg and Wang [32].) Let (M, g) be Riemannian manifold with an isometric action of cohomogeneity one by a Lie group G. We assume that this action has a singular orbit. The isotropy group of the G-action at the singular orbit will be denoted by K. Let $\mathcal{B} \subseteq \bigotimes^r TM \otimes \bigotimes^s T^*M$ be a tensor bundle over M whose fibers at the singular orbit are K-equivariantly isomorphic to the K-module B. Let $b : (0, r) \to B$, where $r > 0$, be a real analytic map with Taylor expansion $\sum_{m=1}^{\infty} b_m t^m$. By the construction we have described above, we can identify b with a tensor field β. This tensor field is defined on a tubular neighborhood of the singular orbit, but not on the singular orbit itself. β is well-defined and has a smooth extension to the singular orbit if and only if*

$$b_m \in \epsilon_m(W_m) \quad \forall m \in \mathbb{N}_0 ,$$

where W_m and ϵ are defined as above.

We are in particular interested in the case where β is a metric tensor. It is useful to make the following assumption:

Assumption 3.2.19. For the following considerations on the cohomogeneity-one Einstein equation, we assume that $\mathfrak{p}$ and $\mathfrak{p}^\perp$ have no equivalent H-submodules in common. This assumption will be satisfied in most cases we consider. In this situation, we have the following splitting:

$$S^2(\mathfrak{p} \oplus \mathfrak{p}^\perp)^H = S^2(\mathfrak{p})^H \oplus S^2(\mathfrak{p}^\perp)^H . \tag{3.8}$$

The space V^H shall be defined as the set of all H-invariant elements of the H-module V.

We consider the space

$$\{P : S^m(\mathfrak{p}^\perp) \to S^2(\mathfrak{p} \oplus \mathfrak{p}^\perp) | P \text{ is linear and } K\text{-equivariant}\}$$

which is a special space of type W_m. Any P in that space can be considered as a K-invariant m^{th}-order polynomial with values in $S^2(\mathfrak{p} \oplus \mathfrak{p}^\perp)$. $eH \in K/H \subseteq \mathfrak{p}^\perp$ is invariant with respect to H. Therefore, $P(eH)$ has to be H-invariant, too. If Assumption 3.2.19 is satisfied, we can conclude that $P(eH) \in S^2(\mathfrak{p})^H \oplus S^2(\mathfrak{p}^\perp)^H$. Since $S^2(\mathfrak{p}) \oplus S^2(\mathfrak{p}^\perp)$ is a K-module, we have $P(v) \in S^2(\mathfrak{p}) \oplus S^2(\mathfrak{p}^\perp)$ for any $v \in \mathfrak{p}^\perp$. All in all, we have shown that if 3.2.19 is true, we have:

$$\begin{aligned} &\{S^m(\mathfrak{p}^\perp) \to S^2(\mathfrak{p} \oplus \mathfrak{p}^\perp) | P \text{ is linear and } K\text{-equivariant}\} \\ =&\{S^m(\mathfrak{p}^\perp) \to S^2(\mathfrak{p}) \oplus S^2(\mathfrak{p}^\perp) | P \text{ is linear and } K\text{-equivariant}\} \\ =&\{S^m(\mathfrak{p}^\perp) \to S^2(\mathfrak{p}) | P \text{ is linear and } K\text{-equivariant}\} \\ &\oplus \{S^m(\mathfrak{p}^\perp) \to S^2(\mathfrak{p}^\perp) | P \text{ is linear and } K\text{-equivariant}\} \end{aligned} \tag{3.9}$$

The existence of the splittings (3.8) and (3.9) will simplify some of the following representation theoretical arguments. If Assumption 3.2.19 is not satisfied, it is sometimes possible to prove results which are analogous to those we will introduce. Nevertheless, only the case where $\mathfrak{p}$ and $\mathfrak{p}^\perp$ have no H-submodules in common has been investigated in [32].

Remark 3.2.20. On the disc bundle D, we can introduce a radial coordinate t such that the cohomogeneity-one metric g becomes:

$$g = g_t + dt^2 ,$$

where g_t is a t-dependent G-invariant metric on the principal orbit. In Chapter 5, where we do concrete calculations, we will often describe our metric in that way. g_t should be considered as an element of $S^2(\mathfrak{m})$ rather than of $S^2(\mathfrak{p} \oplus \mathfrak{p}^\perp)$. Nevertheless, both points of view on the metric are equivalent.

We now apply Theorem 3.2.18 in order to find smoothness conditions on metrics near a singular orbit. The cases $m = 0$ and $m = 1$ are of special interest, since the smoothness conditions we obtain have a simple geometric interpretation. First, we investigate the case $m = 0$. In order to do this, we have to consider the space

$$W_0 = \{P : \mathbb{R} \to S^2(\mathfrak{p}) \oplus S^2(\mathfrak{p}^\perp) | P \text{ is linear and } K\text{-equivariant}\} ,$$

which describes the metric at a point of the singular orbit. Of course, W_0 also contains elements which correspond to symmetric forms which are not positive definite. We nevertheless call W_0 the space of possible metrics at the singular orbit, since this detail will be irrelevant for our representation theoretical considerations. The only restriction from Theorem 3.2.18 is that the metric at a point of the singular orbit has to be K-invariant. Next, we consider the case $m = 1$. The first derivative of the metric can be considered as an element of

$$W_1 = \{P : \mathfrak{p}^\perp \to S^2(\mathfrak{p}) \oplus S^2(\mathfrak{p}^\perp) | P \text{ is linear and } K\text{-equivariant}\} .$$

Eschenburg and Wang [32] have shown that

$$\{P : S^{2l+1}(\mathfrak{p}^\perp) \to S^2(\mathfrak{p}^\perp) | P \text{ is linear and } K\text{-equivariant}\} \cong \{0\} \quad \forall l \in \mathbb{N}_0 .$$

For $l = 0$ this translates into the condition that the $S^2(\mathfrak{p}^\perp)$-part of the metric has to approach the flat metric on D_p. Equivalently, the metric on K/H has to approach the round metric with sectional curvature $\frac{1}{t^2}$ sufficiently fast when t approaches 0. For the smoothness conditions of higher order there are similar interpretations, which we will not make explicit.

Remark 3.2.21. 1. Let g be a smooth cohomogeneity-one metric with a singular orbit. The collapsing spheres are distance spheres with respect to g. For different choices of g, the distance spheres are in general not the same. When we write g as $g_t + dt^2$, we choose $\frac{\partial}{\partial t}$ as a unit normal vector field on the collapsing spheres. This vector field of course depends on the choice of g, too. It is possible to choose different g such that the $g_t \in S^2(\mathfrak{m})$ are the same but $\frac{\partial}{\partial t}$ is different. In that situation, the metrics g are isometric to each other. We will therefore fix a vector in $\mathfrak{p}^\perp$, which describes the direction of the geodesic intersecting all orbits perpendicularly, and restrict ourselves to bilinear forms in $S^2(\mathfrak{p} \oplus \mathfrak{p}^\perp)$ with the property that our vector is of unit length and orthogonal to the tangent space of G/H. Since metrics which are isometric in the above sense will be identified in Chapter 5, the number of possible g_t which we have to consider is smaller than the number of metrics which satisfy the conditions from Theorem 3.2.18.

2. Let q be a K-invariant background metric on $\mathfrak{p}^\perp$. Since K acts transitively on a sphere in $\mathfrak{p}^\perp$, q is unique up to multiplication by a constant. In Chapter 5, we describe our cohomogeneity-one metrics by $g_t + dt^2$. For $t \to 0$, the restriction of g_t to $S^2(\mathfrak{p}^\perp)$ describes the metric on the collapsing sphere. It is easy to see that the length of any tangent vector of G/H with respect to q is $ct + O(t^2)$ for a $c \in \mathbb{R}$. Therefore, the smoothness condition of m^{th} order for the vertical part of the metric describes the m^{th} derivative of the restriction of $\frac{1}{t^2} g_t$ to the sphere K/H.

3. In the orbifold case, we can make similar considerations as in the manifold case. Nevertheless, the smoothness conditions for this case may be not the same as for manifolds. For example, let $\mathbb{Z}_k$ be a discrete subgroup of $SO(2)$. $\mathbb{R}^2/\mathbb{Z}_k$ is a cone whose vertex corresponds to $0 \in \mathbb{R}^2$. Let g be the metric on $\mathbb{R}^2/\mathbb{Z}_k$ which is induced by the Euclidean metric on $\mathbb{R}^2$. If we multiply the part of g which is directed along the circles around the origin of $\mathbb{R}^2/\mathbb{Z}_k$ by k, we again obtain a metric which is isometric to the Euclidean metric on $\mathbb{R}^2$. Therefore, we have to modify the vertical part of the metric at the singular orbit if our orbifold has such a conical singularity. If $\dim K/H \geqslant 2$, the smoothness conditions for the restriction of $\frac{\partial}{\partial t} g_t$ to K/H remain the same, since K/H still has to be a space whose sectional curvature is approximately $\frac{1}{t^2}$. We will deal with the further influences of the orbifold singularities on the smoothness conditions when we encounter those singularities.

We now describe the cohomogeneity-one Einstein condition explicitly. On the union of the principal orbits M^o, this condition can be described as a system of ordinary differential equations of second order. Since M^o is a dense subset of M, we only have to search for the initial values on the singular orbit such that the metric becomes smooth. We therefore obtain an initial value problem which we have to solve. Since the differential equations degenerate at the singular orbit, its solutions may depend on additional parameters apart from the initial metric and its first derivative.

We again introduce a radial coordinate t on a tubular neighborhood of a singular orbit. Any cohomogeneity-one metric g on that neighborhood can be described by:

$$g = g_t + dt^2 ,$$

where $(g_t)_{t\in[0,r)}$ is a t-dependent family of G-invariant metrics on the orbits. For $t = 0$, g_t becomes a degenerate bilinear form.

Before we can find an explicit description of the Einstein condition, we have to determine the shape operator $L = \nabla \frac{\partial}{\partial t}$ of the principal orbits. As in the section before, we identify the restriction of the metric to the principal orbit with an endomorphism φ of $\mathfrak{m}$. It is easy to see that

$$L = \frac{1}{2}\varphi^{-1}\left(\tfrac{\partial}{\partial t}\varphi\right) .$$

Let $(e_i)_{i=1,\ldots,n}$ be a basis of $\mathfrak{m}$. We assume from now on that g_t is for all t a diagonal metric with respect to this basis. If the matrix representation of g_t with respect to $(e_i)_{i=1,\ldots,n}$ is $\mathrm{diag}(f_1^2(t),\ldots,f_n^2(t))$, we have:

$$L = \mathrm{diag}\left(\frac{f_1'}{f_1},\ldots,\frac{f_n'}{f_n}\right) .$$

After some calculations involving the Gauss-, the Codazzi-, and the Riccatti-equation, we finally can express the cohomogeneity-one Einstein condition as a system of ordinary differential equations of second order:

Theorem 3.2.22. *(See Eschenburg and Wang [32].) Let (M, g) be a Riemannian manifold admitting an isometric cohomogeneity-one action by a Lie group G. The Ricci-curvature of g and its Ricci-endomorphism we denote both by Ric. We identify the union of all principal orbits by a G-equivariant diffeomorphism ϕ with $G/H \times I$, where I is an interval and H is the isotropy group of the G-action on the principal orbit. We choose ϕ in such a way that $\{eH\} \times I$ becomes a geodesic which intersects all orbits perpendicularly and is parameterized by arclength. In this situation, we have*

$$g = g_t + dt^2 ,$$

where t is the coordinate directed along I and $(g_t)_{t\in I}$ is a family of metrics on G/H. We denote the shape operator of the principal orbit by L_t and its Ricci-curvature by Ric_t. As above, we will denote the Ricci-endomorphism of the principal orbit by the same symbol. The Einstein condition $Ric = \lambda g$ is in this situation equivalent to:

$$-\tfrac{\partial}{\partial t}L_t - (trL_t)L_t + Ric_t = \lambda g_t \tag{3.10}$$
$$-tr\left(\tfrac{\partial}{\partial t}L_t\right) - tr(L_t^2) = \lambda \tag{3.11}$$
$$tr(\imath_X\, d^\nabla L_t) = 0 \quad \forall \text{ vector fields } X \perp \tfrac{\partial}{\partial t} \tag{3.12}$$

If g_t is for all $t \in I$ diagonal with respect to a basis $(e_i)_{i=1,\ldots,n}$ of $\mathfrak{m}$, we have

$$g_t = diag(f_1(t)^2,\ldots,f_n(t)^2)$$

for some functions $f_1, \ldots, f_n : I \to \mathbb{R}$. Furthermore, the first two equations of the cohomogeneity-one Einstein condition become:

$$
\begin{aligned}
-\tfrac{f_i''(t)}{f_i(t)} + \left(\tfrac{f_i'(t)}{f_i(t)}\right)^2 - \tfrac{f_i'(t)}{f_i(t)} \left(\sum_{j=1}^{n} \tfrac{f_j'(t)}{f_j(t)} \right) + Ric_t\left(\tfrac{1}{f_i(t)} e_i, \tfrac{1}{f_i(t)} e_i\right) &= \lambda \quad \forall\, i = 1, \ldots, n \\
Ric_t(e_i, e_j) &= 0 \quad \forall\, 1 \leqslant i < j \leqslant n \qquad (3.13) \\
-\sum_{i=1}^{n-1} \tfrac{f_i''(t)}{f_i(t)} &= \lambda
\end{aligned}
$$

Remark 3.2.23.

1. In equation (3.12), we consider L_t as a one-form on G/H with values in TG/H. $d^\nabla \alpha$ denotes the covariant exterior derivative of a vector-valued k-form α which is defined by:

$$
\begin{aligned}
(d^\nabla \alpha)(X_1, \ldots, X_{k+1}) = &\sum_{i=1}^{k+1} (-1)^{i+1} \nabla_{X_i}(\alpha(X_1, \ldots, \hat{X}_i, \ldots, X_{k+1})) \\
&+ \sum_{1 \leqslant i < j \leqslant k+1} (-1)^{i+j} \alpha([X_i, X_j], X_i, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_{k+1}) \,.
\end{aligned}
$$

   In (3.12), $\imath$ denotes the interior product of a vector with a vector-valued differential form. The equation (3.12) is equivalent to $\mathrm{Ric}(X, \frac{\partial}{\partial t}) = 0$, which has to be satisfied for any cohomogeneity-one Einstein metric. It turns out that (3.12) is automatically satisfied if $\dim K/H \geqslant 1$ (see Corollary 2.5. and 2.6. in [32]). Since all cohomogeneity-one manifolds which we consider are of that kind, we only have to take care of (3.10) and (3.11).

2. We assume that g is a cohomogeneity-one metric such that (3.12) is satisfied. The system consisting of (3.10) and (3.11) contains $n^2 + 1$ equations for the n^2 functions $g_t(e_i, e_j)$. Nevertheless, that system is not overdetermined. It has been proven in [32] that it has a unique short-time solution for any initial value of g_t and L_t on a principal orbit.

3. We consider the case that g_t is a diagonal metric but Ric_t is non-diagonal. The existence of metrics of that kind cannot be a priori ruled out. In that situation, the equation $\mathrm{Ric}(e_i, e_j) = 0$ is a rational equation for the metric functions $f_1, \ldots, f_n$. We therefore have a non-trivial restriction which the metric functions have to satisfy. This seems to contradict our remark on the existence of short-time solutions. The explanation of this fact is as follows: If we have $\mathrm{Ric}(e_i, e_j) \neq 0$, the assumption that g_t is always diagonal is false and the condition $\mathrm{Ric} = \lambda g$ forces the metric to become non-diagonal as soon as we leave the initial principal orbit.

4. In Grove, Ziller [38] and Schwachhöfer, Tuschmann [63], similar formulas for the Ricci-curvature of a cohomogeneity-one metric have been obtained. The notation we make use of in this thesis is in part motivated by those papers.

We consider the equations (3.13) for the diagonal case. When we approach a singular orbit, some of the f_i have to converge to zero. Since L_t contains the term $\frac{f_i'}{f_i}$, our equations degenerate at the singular orbit. In that situation, the existence and number of short-time solutions does not directly follow from Picard-Lindelöf's or a similar theorem. In the non-diagonal case, we face similar difficulties. Nevertheless, Eschenburg and Wang [32] have been able to prove a theorem on the local existence of cohomogeneity-one Einstein metrics. The cohomogeneity-one Einstein-equations can be rewritten as

$$\frac{\partial}{\partial t}L_t = \frac{1}{t^2}A(g_t) + \frac{1}{t}B(g_t, L_t) + C(t, g_t, L_t) ,$$

where A, B, and C are some functions which could be explicitly determined and have a smooth extension to $t = 0$. A necessary condition for the existence of local solutions is that the right-hand side of the above equation can be smoothly extended to $t = 0$ after we have inserted the initial values. Fortunately, this is always the case. Eschenburg and Wang [32] have made a power series ansatz for the metric g. We assume that the singular orbit is at $t = 0$ and obtain:

$$g_t = \sum_{m=0}^{\infty} g_m t^m .$$

On the previous pages, we have seen that the coefficients g_m can be considered as elements of $S^2(\mathfrak{p} \oplus \mathfrak{p}^\perp)$ and have to satisfy the smoothness conditions described in Theorem 3.2.18. Since the G-invariant metric g_t on G/H has to correspond to an element of $S^2(\mathfrak{p} \oplus \mathfrak{p}^\perp)^H = S^2(\mathfrak{p})^H \oplus S^2(\mathfrak{p}^\perp)^H$, we have $g_m \in S^2(\mathfrak{p})^H \oplus S^2(\mathfrak{p}^\perp)^H$, too. It is possible to show that g_{m+2} can be chosen as an arbitrary solution of a recursion equation of type

$$\mathcal{L}_m g_{m+2} = D_m(g_0, \ldots, g_{m+1}) , \tag{3.14}$$

where $\mathcal{L}_m$ is a H-equivariant endomorphism of $S^2(\mathfrak{p})^H \oplus S^2(\mathfrak{p}^\perp)^H$ and D_m is a function which we will not consider in detail. We have to show that there are solutions of (3.14) which satisfy the conditions of Theorem 3.2.18. That theorem states that g_{m+2} has to be in $\epsilon_{m+2}(W_{m+2})$. We can therefore restrict $\mathcal{L}_m$ to a map $\epsilon_{m+2}(W_{m+2}) \to S^2(\mathfrak{p})^H \oplus S^2(\mathfrak{p}^\perp)^H$. In [32], it is shown that $\mathcal{L}_m(\epsilon_{m+2}(W_{m+2})) \subseteq \epsilon_m(W_m)$ and $\mathcal{L}_m : \epsilon_{m+2}(W_{m+2}) \to \epsilon_m(W_m)$ is surjective. Furthermore, we have $D_m(g_0, \ldots, g_{m+1}) \in \epsilon_m(W_m)$. Therefore, a formal power series solution of our system of ordinary differential equations which satisfies the smoothness conditions does exist. The number of free parameters of $(m+2)^{nd}$ order is given by $\dim \ker \mathcal{L}_m = \dim \epsilon_{m+2}(W_{m+2}) - \dim \epsilon_m(W_m)$. Since K acts transitively on the spheres around 0 in $\mathfrak{p}^\perp$, any polynomial $P \in W_m$ can be recovered from $P(v)$, where $v \in \mathfrak{p}^\perp$ is the vector we have chosen on page 66. Therefore, the evaluation map ϵ_m is injective and the number of free parameters equals $\dim W_{m+2} - \dim W_m$. We are able to extend our considerations to $m \in \{-1, -2\}$. The possible initial values of 0^{th} and 1^{st} order are simply the K-invariant inner products on $\mathfrak{p} \oplus \mathfrak{p}^\perp$ and the G-invariant tensors in $(TG/H)^\perp \otimes T^*M$. Therefore, we define:

$$\widetilde{W}_m := \begin{cases} \{P : S^m(\mathfrak{p}^\perp) \to S^2(\mathfrak{p} \oplus \mathfrak{p}^\perp) | P \text{ is linear and } K\text{-equivariant}\} & m \geqslant 0 \\ \{0\} & m \leqslant 0 \end{cases} \tag{3.15}$$

With this notation, the number of free parameters of m^{th} order becomes $\dim \widetilde{W}_m - \dim \widetilde{W}_{m-2}$, which equals $\dim \ker \mathcal{L}_{m-2}$ for $m \geqslant 2$. On page 68, we have shown that for $m \geqslant 0$:

$$\begin{aligned}\widetilde{W}_m = &\{P : S^m(\mathfrak{p}^\perp) \to S^2(\mathfrak{p}) | P \text{ is linear and } K\text{-equivariant}\} \\ &\oplus \{P : S^m(\mathfrak{p}^\perp) \to S^2(\mathfrak{p}^\perp) | P \text{ is linear and } K\text{-equivariant}\}\,.\end{aligned}$$

We denote the first summand by $\widetilde{W}^h_m$ and the second one by $\widetilde{W}^v_m$. There are some facts about the modules $\widetilde{W}^v_m$ and $\widetilde{W}^h_m$ which should be mentioned. We start with $\widetilde{W}^v_m$. As we have mentioned before, $\mathrm{Hom}_K(S^m(\mathfrak{p}^\perp), S^2(\mathfrak{p}^\perp))$ is $\{0\}$ if m is odd. In the paper of Eschenburg and Wang [32], it has been shown that $\mathrm{Hom}_K(S^m(\mathfrak{p}^\perp), S^2(\mathfrak{p}^\perp)) \cong \mathrm{Hom}_K(S^2(\mathfrak{p}^\perp), S^2(\mathfrak{p}^\perp))$ if m is positive and even. For $m = 0$, we have $\dim \widetilde{W}^h_m = 1$, since the K-invariant inner product on $\mathfrak{p}^\perp$ is unique up to multiplication by a constant. The above considerations show that only in the 0^{th} and 2^{nd} order there are initial conditions for the vertical directions we can prescribe. The initial condition of 0^{th} order is omitted in [32], since we can rescale the metric on the normal space of G/K by changing $\frac{\partial}{\partial t}$ into $\lambda \frac{\partial}{\partial t}$. The number of 2^{nd} order initial conditions in the vertical direction is $\dim \mathrm{Hom}_K(S^2(\mathfrak{p}^\perp), S^2(\mathfrak{p}^\perp)) - 1$.

Next, we consider $\widetilde{W}^h_m$. The function defined by $q(tv) := t^2$ can be extended by the action of K to a K-invariant quadratic form on all of $\mathfrak{p}^\perp$. This form we denote by q, too. It is easy to see that there is up to multiplication by a positive constant only one positive definite K-invariant quadratic form on $\mathfrak{p}^\perp$. We have the following K-equivariant embedding

$$\begin{aligned}\imath : \widetilde{W}^h_m &\to \widetilde{W}^h_{m+2} \\ \imath(P) &:= q \vee P\,,\end{aligned}$$

where $\vee$ denotes the symmetric product. Therefore, we can consider $\widetilde{W}^h_m$ as a subspace of $\widetilde{W}^h_{m+2}$ and obtain the following chains of K-modules:

$$\begin{aligned}&\widetilde{W}^h_0 \subseteq \widetilde{W}^h_2 \subseteq \widetilde{W}^h_4 \subseteq \ldots \subseteq \widetilde{W}^h_{2m} \subseteq \ldots \\ &\widetilde{W}^h_1 \subseteq \widetilde{W}^h_3 \subseteq \widetilde{W}^h_5 \subseteq \ldots \subseteq \widetilde{W}^h_{2m+1} \subseteq \ldots\end{aligned} \tag{3.16}$$

We will show that both of these chains stabilize at a sufficiently large m. Let $\mathcal{S}$ be the sphere in $\mathfrak{p}^\perp$. Any polynomial $P : \mathfrak{p}^\perp \to S^2(\mathfrak{p})$ can be identified with its restriction to $\mathcal{S}$. Therefore, the space of all K-invariant polynomials $\bigoplus_{m\in\mathbb{N}_0} \widetilde{W}^h_m$ can be considered as a subspace of the greater space $\widetilde{W}^h$ of all K-equivariant maps $\mathcal{S} \to S^2(\mathfrak{p})$. Analogously to above, we define an evaluation map:

$$\begin{aligned}\epsilon : \widetilde{W}^h &\to S^2(\mathfrak{p}) \\ \epsilon(P) &:= P(v)\,.\end{aligned}$$

Since any $P \in \widetilde{W}^h$ is determined by $P(v)$, ϵ is injective. Therefore, $\dim \widetilde{W}^h$ is finite. We conclude from the fact that the dimensions of the spaces in (3.16) are non-decreasing that they have to stabilize at a certain m. All in all, we have shown that the solutions of the cohomogeneity-one Einstein equation depend only on a finite number of parameters.

Until now, we only have found formal power series as solutions. It still has to be checked if they converge. Let

$$\widetilde{g}_m := \sum_{k=0}^{m} g_k t^k$$

be the metric we obtain by summing up the first $m+1$ summands of our power series. We take $\widetilde{g}_m$ as the starting point of a Picard iteration. The metric we obtain coincides with $\widetilde{g}_m$ up to the m^{th} order. Since we can choose the degree m arbitrarily high, we can prescribe all of the initial conditions we have found. Therefore, we obtain for any choice of the initial conditions a convergent solution. The above considerations are the main ideas for the proof of the following theorem:

Theorem 3.2.24. *(See Eschenburg and Wang [32].) Let M be a manifold equipped with a cohomogeneity-one action by a compact Lie group G. We assume that the principal orbits of this action are G-equivariantly diffeomorphic to G/H and that it has a singular orbit G/K, where $H \subseteq K \subseteq G$ are closed subgroups of each other.*

Let $\mathfrak{p}$ be the tangent space of the singular orbit. Any tubular neighborhood of the singular orbit is a disc bundle. The tangent space of its fiber at a point of the singular orbit we denote by $\mathfrak{p}^\perp$. The K-module structures of $\mathfrak{p}$ and $\mathfrak{p}^\perp$ are independent of the point we consider. We assume that $\mathfrak{p}$ and $\mathfrak{p}^\perp$ have no H-submodule $\neq \{0\}$ in common.

Let g_0 be a G-invariant metric on the singular orbit. Furthermore, let g_0' be an arbitrary linear, K-equivariant map $g_0' : \mathfrak{p}^\perp \to S^2(\mathfrak{p})$, which we extend by the G-action to all of G/K. We finally choose an arbitrary $\lambda \in \mathbb{R}$. In this situation, there exists a G-invariant Einstein metric g on a sufficiently small tubular neighborhood of the singular orbit which has the following properties:

1. *g has λ as Einstein constant.*
2. *The restriction of g to the singular orbit is g_0.*
3. *The first derivation of g at the singular orbit in the directions normal to the singular orbit is g_0'.*

Furthermore, the set of all those Einstein metrics depends on additional initial conditions of higher order. The number of those initial conditions of m^{th} order is given by:

$$\begin{aligned} \dim \widetilde{W}^h_m - \dim \widetilde{W}^h_{m-2} &\quad \textit{in the directions horizontal to } G/K \\ \dim W^v_2 - 1 &\quad \textit{in the vertical directions if } m = 2 \\ 0 &\quad \textit{in the vertical directions if } m \neq 2 \end{aligned}$$

where the notations of these formulas are explained on the previous pages.

Remark 3.2.25.

1. In the cases we consider in Chapter 5, the number of possible initial conditions for the cohomogeneity-one Einstein equation will seem to be smaller than the theorem predicts. The reason for this is that metrics which are isometric to each other in the sense of Remark 3.2.21 will be identified.

2. If we insert $m = 0$ or $m = 1$ in the above formulas, we see that g_0 and g_0' can be chosen arbitrarily.

3. The geometric meaning of the freedom in the vertical direction is that we can choose the trace-free part of g_0'' freely.

4. In contrast to the rest of this section, we do not assume in the above theorem that G, H, and K are connected.

5. Our representation theoretical considerations do not depend on the value of the Einstein constant. Therefore, we can construct solutions with an arbitrary sign of the Einstein constant. Since the above theorem gives only local solutions, we do not have to be concerned with the topological restrictions for the sign of the Einstein constant.

On the next pages, we will mention a few general results on cohomogeneity-one Spin(7)-structures. We require that the action of G leaves not only the metric but also the Spin(7)-structure invariant:

Definition 3.2.26. Let (M, Ω) be a Spin(7)-manifold equipped with an isometric cohomogeneity-one action of a Lie group G. (M, Ω) is called a *Spin*(7)*-manifold of cohomogeneity one* if the action of any $g \in G$ leaves Ω invariant.

On any principal orbit of a cohomogeneity-one Spin(7)-manifold, there exists a canonical G_2-structure:

Lemma 3.2.27. *Let (M, Ω) be a Spin*(7)*-manifold with a cohomogeneity-one action by a Lie group G preserving the four-form Ω. Any principal orbit is G-equivariantly diffeomorphic to G/H, where H is a Lie subgroup of G. Furthermore, let $i : G/H \to M$ be the inclusion map of a fixed principal orbit. The pull-back $i^*(\Omega)$ is a G-invariant four-form ω^* on G/H. Its Hodge dual ω determines a G-invariant G_2-structure on G/H.*

Proof : The only thing we have to show is that ω is stabilized by a group isomorphic to G_2. Since ω is homogeneous, it suffices to show this for an arbitrary $p \in G/H$. There exists an orthonormal basis $(e_0, \ldots, e_7)$ of $T_{i(p)}M$ such that

$$\Omega_{i(p)} = \widetilde{\omega^*}_{i(p)} + e^0 \wedge \widetilde{\omega}_{i(p)}$$

and the coefficients of $\widetilde{\omega^*}_{i(p)}$ and $\widetilde{\omega}_{i(p)}$ with respect to $(e_1, \ldots, e_7)$ coincide with the coefficients of the forms $\omega, *\omega \in \bigwedge^*(\mathrm{Im}(\mathbb{O}))^*$. Since Spin(7) acts transitively on the unit sphere, there is an element $g \in \mathrm{Spin}(7)$ which maps e_0 into the tangent vector $\frac{\partial}{\partial t}$ which is orthogonal to the principal orbit. We denote the pull-back of the action of g^{-1} on $T_{i(p)}M$ by ${g^{-1}}^*$ and obtain:

$$\Omega_{i(p)} = {g^{-1}}^* \widetilde{\omega^*}_{i(p)} + dt \wedge {g^{-1}}^* \widetilde{\omega}_{i(p)} \,.$$

Since the action of Spin(7) is orthogonal, $e_1, \ldots, e_7$ are mapped to tangent vectors of the principal orbit. We apply the pull-back of i to the above equation:

$$\omega_p^* = i^*(\Omega_{i(p)}) = i^*({g^{-1}}^* \widetilde{\omega^*}_{i(p)}) .$$

ω_p^* can be obtained from $*\omega \in \bigwedge^4(\mathrm{Im}(\mathbb{O}))^*$ by the pull-back of an endomorphism $T_pG/H \to \mathrm{Im}(\mathbb{O})$. Therefore, $\omega \in \bigwedge^3(\mathrm{Im}(\mathbb{O}))^*$ and $*\omega_p^* = \omega_p \in \bigwedge^3 T_p^*G/H$ are stabilized by the same group, which is G_2.

□

The converse of the above lemma is also true:

Lemma 3.2.28. *Let ω be a G-invariant G_2-structure on a homogeneous space G/H and let $I \subseteq \mathbb{R}$ be an interval. Then, there exists a G-invariant Spin(7)-structure Ω on $G/H \times I$. Furthermore, let $t \in I$ be arbitrary and*

$$\begin{aligned} i : G/H &\to G/H \times I \\ i(p) &:= (p,t) . \end{aligned}$$

The above Ω can be chosen in such a way that $i^*(\Omega) = *\omega$.

Proof : Let t be the coordinate directed along I. We simply choose:

$$\Omega = *\omega + dt \wedge \omega .$$

□

We are first of all interested in parallel Spin(7)-structures of cohomogeneity one. The following necessary condition for $d\Omega = 0$ can easily be seen:

Lemma 3.2.29. *In the situation of Lemma 3.2.27, let $d\Omega = 0$. On any principal orbit, the G-invariant G_2-structure ω is cosymplectic, i.e., $d_{G/H} * \omega = 0$.*

Convention 3.2.30. In the above lemma, $d_{G/H}$ denotes the exterior differential operator on the principal orbit G/H. In the course of this thesis, we will often have to distinguish between the exterior differential on a manifold M and on submanifolds of M. We will therefore index d by the submanifold we consider in order to avoid confusion. Throughout this section, $*$ denotes the Hodge star operator on G/H.

Proof : We choose an arbitrary principal orbit and denote the inclusion map of this orbit into M by i. Since $i^*(\Omega) = *\omega$, we have:

$$d\Omega = 0 \quad \Rightarrow \quad i^*(d\Omega) = 0 \quad \Rightarrow \quad d_{G/H}(i^*\Omega) = 0 \quad \Rightarrow \quad d_{G/H} * \omega = 0$$

□

Conversely, any cosymplectic G-invariant G_2-structure on G/H can be uniquely extended to a parallel Spin(7)-structure:

Theorem 3.2.31. *Let G/H be a seven-dimensional homogeneous space carrying a G-invariant G_2-structure $\widetilde{\omega}$ with $d_{G/H} * \widetilde{\omega} = 0$. Then there exists an $\epsilon > 0$ and a one-parameter family $(\omega_t)_{t\in(-\epsilon,\epsilon)}$ of G-invariant three-forms on G/H such that the initial value problem*

$$\frac{\partial}{\partial t} * \omega_t = d_{G/H}\omega_t \tag{3.17}$$

$$\omega_0 = \widetilde{\omega} \tag{3.18}$$

has a unique solution on $G/H \times (-\epsilon, \epsilon)$ with the following properties:

1. *ω_t is a G_2-structure on G/H.*
2. *$d_{G/H} * \omega_t = 0$.*
3. *$*\omega_t$ is in the same cohomology class in $H^4(M, \mathbb{R})$ as $*\widetilde{\omega}$.*

In the above formula, $\frac{\partial}{\partial t}$ denotes the Lie derivative in t-direction. Furthermore, the four-form $\Omega := *\omega + dt \wedge \omega$ is a G-invariant parallel Spin(7)-structure on $G/H \times (-\epsilon, \epsilon)$.

Conversely, let Ω be a parallel Spin(7)-structure preserved by a cohomogeneity-one action of a Lie group G. The isotropy group of the G-action on the principal orbit we denote by H. We identify the union of all principal orbits G-equivariantly with $G/H \times I$, where the interval I is parameterized by arclength. In this situation, the G_2-structure on the principal orbit satisfies equation (3.17).

Proof : It is easy to see that $\Omega = *\omega + dt \wedge \omega$ is a closed form on $G/H \times I$ if and only if:

$$\begin{aligned} \tfrac{\partial}{\partial t} * \omega &= d_{G/H}\omega \\ d_{G/H} * \omega &= 0 \end{aligned}$$

Therefore, we only have to show that ω_t exists and that it satisfies the three properties stated in the theorem. The short-time existence and uniqueness of ω_t follows directly from Picard-Lindelöf's theorem. Since $d_{G/H}$ commutes with the action of G, ω_t has to be G-invariant. We now prove the three properties separately:

1. The canonical action of $GL(7)$ on $\mathrm{Im}(\mathbb{O})$ induces an action of $GL(7)$ on $\bigwedge^3(\mathrm{Im}(\mathbb{O}))^*$. The orbit of $\omega \in \bigwedge^3(\mathrm{Im}(\mathbb{O}))^*$ is an open set. Therefore, there exists an $\epsilon' \in (0, \epsilon]$ such that for all $t \in (-\epsilon', \epsilon')$ ω_t is the pull-back of ω with respect to a linear map $T_pG/H \to \mathrm{Im}(\mathbb{O})$.

2. Since we have $d_{G/H} * \omega_0 = 0$, it suffices to show the equation

$$\frac{\partial}{\partial t} d_{G/H} * \omega_t = 0$$

in order to prove $d_{G/H} * \omega_t = 0$. $d_{G/H}$ is a t-independent operator on G/H. Therefore, we have:

$$\frac{\partial}{\partial t} d_{G/H} * \omega = d_{G/H} \frac{\partial}{\partial t} * \omega = d^2_{G/H} \omega = 0\,,$$

which proves the second property of ω_t.

3. Finally, we have to prove that $*\omega_t - *\omega_0$ is an exact form. This statement can be shown with help of the fundamental theorem of calculus:

$$*\omega_t - *\omega_0 = \int_0^t \frac{\partial}{\partial \tau} * \omega_\tau d\tau = \int_0^t d_{G/H} \omega_\tau d\tau = d_{G/H} \int_0^t \omega_\tau d\tau\,.$$

□

Convention 3.2.32. Let α be a tensor field on $G/H \times I$. For the rest of this thesis, we will always denote the Lie derivative of α in the t-direction by $\frac{\partial}{\partial t}\alpha$.

Remark 3.2.33.

1. In [42], Hitchin introduced the notion of a stable form. A form on a vector space V is called *stable* if its $GL(V)$-orbit is an open set. The three- and the four-form associated to a G_2-structure are special examples of stable forms. Hitchin considered the gradient flow of a certain functional defined on the space of closed stable four-forms in a fixed cohomology class. He obtained a theorem similar to the above one for the more general case that G/H is replaced by a coclosed but not necessarily homogeneous G_2-manifold.

2. Although there always exists a short-time solution of (3.17) near a principal orbit, we will see that sometimes there are invariant metrics on the singular orbit which cannot be extended to a cohomogeneity-one metric of holonomy $\subseteq \mathrm{Spin}(7)$.

There is one special solution of the equation (3.17) which can be easily obtained:

Corollary 3.2.34. *Let G/H be a homogeneous space and let ω_0 be a G-invariant G_2-structure on G/H which is nearly parallel but not parallel. Then, there exists a cone (M, g) over G/H such that*

1. *ω_0 can be extended to a parallel G-invariant Spin(7)-structure Ω on M.*

2. *The metric associated to Ω is the cone metric g which has holonomy $\subseteq$ Spin(7).*

Furthermore, the extension of ω_0 to a parallel G-invariant Spin(7)-structure is unique.

Proof : The uniqueness of Ω follows directly from the uniqueness of the solutions of the initial value problem (3.17), (3.18). We therefore only have to solve (3.17) for this special situation. Since ω_0 is nearly parallel, we have

$$d_{G/H}\omega_0 = \lambda * \omega_0$$

for a constant $\lambda \neq 0$. We equip the cone over G/H with the Spin(7)-structure

$$\Omega := \frac{\lambda^4 t^4}{256} *_{g_0} \omega_0 - \frac{\lambda^3 t^3}{64}\omega_0 \wedge dt$$

and the metric

$$g := \frac{\lambda^2 t^2}{16} g_0 + dt^2 \, .$$

It is easy to see that g is the associated metric to Ω. Ω satisfies equation (3.17). Furthermore, the restriction of the Spin(7)-structure to the principal orbit at $t = \frac{4}{\lambda}$ equals ω_0.

$\square$

Remark 3.2.35. If G/H is not a quotient of a sphere by a discrete group, the cone has a singularity at the tip which is not an orbifold singularity. In Chapter 5, we therefore will not consider the cone metrics in much detail.

Later on, we will construct concrete examples of parallel Spin(7)-manifolds (M, Ω) of cohomogeneity one. As usual, we denote the metric associated to Ω by g. We assume that we have shown the smoothness conditions for g from Theorem 3.2.18. Of course, we want the four-form Ω to have a smooth extension to the singular orbit, too. Furthermore, we want to prove this fact without manually checking any further smoothness conditions. Since g can be smoothly extended to all of M and its holonomy is a subgroup of Spin(7), we know that there exists a parallel smooth four-form on M. We first consider the case where the holonomy equals Spin(7). In this case, the four-form is up to a constant factor unique. Therefore, it has to equal a multiple of Ω and Ω therefore can be smoothly extended. If the holonomy is a proper subgroup of Spin(7), we have a bigger family of smooth four-forms. One of those four-forms has to coincide with Ω on the union of all principal orbits. Since that four-form has a smooth extension to the singular orbit, Ω has to have a smooth extension to the singular orbit, too. All in all, we have shown that in the above situation Ω can be smoothly extended to the singular orbit if g has that property.

At the end of this section, we compare the solutions of equation (3.17) in the case where the metric is diagonal with the non-diagonal case. Before we start, we motivate why this comparison is interesting for us.

When we construct parallel Spin(7)-structures of cohomogeneity one, we will often make the following ansatz: First, we will choose a seven-dimensional homogeneous space G/H admitting a cosymplectic G_2-structure as principal orbit. As usual, G and H are chosen as compact and the action of G shall leave the G_2-structure invariant. The union of all principal orbits we identify G-equivariantly with $G/H \times I$. Next, we fix a basis $(e_i)_{1\leqslant i\leqslant 7}$ of the tangent space

$\mathfrak{m}$. This basis will often be a special one, for example, a basis adapted to a splitting of $\mathfrak{m}$ into H-submodules or an orthogonal basis with respect to a normal metric q.

The calculations needed for the construction of metrics with holonomy $\subseteq \mathrm{Spin}(7)$ will become easier if we assume that the restriction of the metric to any principal orbit is diagonal. This assumption is in many cases justified, for example if $\mathfrak{m}$ splits into pairwise inequivalent irreducible H-submodules. We consider the initial value problem (3.17), (3.18). The metric associated to the G_2-structure ω_t we denote by g_t. In Chapter 5, where we consider concrete principal orbits, we will rewrite our initial value problem in terms of a family of functions $I \to \mathbb{R}$. By considering the resulting systems of ordinary differential equations, we will often see that if g_0 is diagonal, any g_t is diagonal, too. This is a further indication that our assumption is natural.

We now compare the non-diagonal with the diagonal case. We assume that $(e_i)_{1\leqslant i\leqslant 7}$ is q-orthogonal. Let g be a not necessarily diagonal metric on G/H. From linear algebra we know that there exists a q-orthogonal endomorphism $\psi : \mathfrak{m} \to \mathfrak{m}$ such that g is diagonal with respect to $(\psi^{-1}(e_i))_{i=1,\dots,7}$. It is possible to show that we can choose ψ even as an H-equivariant map. We remark that ψ only is defined on the fixed tangent space T_pG/H we identify with $\mathfrak{m}$. Since $\psi \in \mathrm{End}(\mathfrak{m})$ is H-equivariant, it is possible to extend it to a G-invariant endomorphism field on all of G/H, which we denote by ψ, too.

Let Ω be a Spin(7)-structure which was found as an solution of (3.17). We consider the principal orbit at $t = 0$ and choose $\psi : \mathfrak{m} \to \mathfrak{m}$ as above. We can extend ψ to a t-independent endomorphism field on all of $G/H \times I$. The best case would be if $\psi^*\Omega$ was parallel, too. If this was true, we only would have to solve the equation (3.17) for the diagonal case. The non-diagonal solutions could be obtained by applying the pull-back ψ^*. We will therefore compute $d(\psi^*\Omega)$ under the assumption that $d\Omega = 0$.

First, we consider the special case that ψ is the differential of an isometry. This case will be important for our examples with holonomy $SU(4)$. In this situation, we obviously have $d(\psi^*\Omega) = 0$ if $d\Omega = 0$.

We now consider the general case. Let $X_1, \dots, X_5 \in \mathfrak{m}$ be arbitrary. According to equation (3.4), we have:

$$\begin{aligned} & d(\psi^*\Omega)(X_1,\dots,X_5) \\ = & \sum_{1\leqslant i<j\leqslant 5} (-1)^{i+j} (\psi^*\Omega)([X_i,X_j], X_1,\dots,\widehat{X_i},\dots,\widehat{X_j},\dots,X_5) \\ = & \sum_{1\leqslant i<j\leqslant 5} (-1)^{i+j} \Omega(\psi([X_i,X_j]), \psi(X_1),\dots,\widehat{\psi(X_i)},\dots,\widehat{\psi(X_j)},\dots,\psi(X_5)) \,. \end{aligned}$$

If ψ is the restriction of a Lie algebra homomorphism of $\mathfrak{g}$ to $\mathfrak{m}$, we therefore obtain:

$$d(\psi^*\Omega)(X_1,\dots,X_5) = \psi^* d\Omega(X_1,\dots,X_5) \,.$$

Under this assumption, we further have analogously to above:

$$
\begin{aligned}
&d(\psi^*\Omega)(X_1^*,\dots,X_4^*,\tfrac{\partial}{\partial t})\\
=&\sum_{i=1}^{4}(-1)^{i+1}X_i^*\left((\psi^*\Omega)(X_1^*,\dots,\widehat{X_i^*},\dots,X_4^*,\tfrac{\partial}{\partial t})\right)\\
&+(-1)^{5+1}\tfrac{\partial}{\partial t}(\psi^*\Omega)(X_1^*,\dots,X_4^*)\\
&+\sum_{1\leqslant i<j\leqslant 4}(-1)^{i+j}(\psi^*\Omega)([X_i^*,X_j^*],X_1^*,\dots,\widehat{X_i^*},\dots,\widehat{X_j^*},\dots,X_4^*,\tfrac{\partial}{\partial t})\\
&+\sum_{i=1}^{4}(-1)^{i+5}(\psi^*\Omega)([X_i^*,\tfrac{\partial}{\partial t}],X_1^*,\dots,\widehat{X_i^*},\dots,X_4^*)\\
=&\sum_{i=1}^{4}(-1)^{i}X_i^*\left((\psi^*\omega)(X_1^*,\dots,\widehat{X_i^*},\dots,X_4^*)\right)\\
&+\tfrac{\partial}{\partial t}*\omega(\psi(X_1^*),\dots,\psi(X_4^*))\\
&+\sum_{1\leqslant i<j\leqslant 4}(-1)^{i+j+1}(\psi^*\omega)([X_i^*,X_j^*],X_1^*,\dots,\widehat{X_i^*},\dots,\widehat{X_j^*},\dots,X_4^*)\\
=&-d_{G/H}(\psi^*\omega)(X_1^*,\dots,X_4^*)+\psi^*\left(\tfrac{\partial}{\partial t}*\omega(X_1^*,\dots,X_4^*)\right)\\
=&-\psi^*d_{G/H}\omega(X_1^*,\dots,X_4^*)+\psi^*\left(\tfrac{\partial}{\partial t}*\omega(X_1^*,\dots,X_4^*)\right)\\
=&\psi^*d\Omega(X_1^*,\dots,X_4^*,\tfrac{\partial}{\partial t})
\end{aligned}
$$

All in all, we have shown that in this situation $d(\psi^*\Omega)=\psi^*d\Omega$. If $d\Omega=0$, $d(\psi^*\Omega)$ therefore has to vanish, too. We sum up our results:

Lemma 3.2.36. *Let (M,Ω) be a parallel Spin(7)-manifold with a cohomogeneity-one G-action preserving Ω. The isotropy group of the G-action on the principal orbit we denote by $H\subseteq G$. As usual, the tangent space of G/H is identified with $\mathfrak{m}$. Furthermore, let $\psi:\mathfrak{g}\to\mathfrak{g}$ be a Lie algebra homomorphism which is H-equivariant and satisfies $\psi(\mathfrak{m})\subseteq\mathfrak{m}$. In this situation, the four-form $\psi^*\Omega$ defined above satisfies $d(\psi^*\Omega)=0$.*

Remark 3.2.37.

1. The Lie group G from the above lemma does not necessarily has to be compact or connected. Nevertheless, we will apply the lemma only in the case where G is compact and connected.
2. If G is semisimple, the automorphism group of $\mathfrak{g}$ is up to a discrete factor G itself acting on $\mathfrak{g}$ by the adjoint action. We assume that $\psi=\mathrm{Ad}_g$ for a $g\in G$. Since ψ has to be H-equivariant, g has to commute with H. Therefore, the action of ψ^* on Ω is in many cases induced by the action of $\mathrm{Norm}_G H$ on G/H. Nevertheless, there are further possibilities for ψ. For example, ψ could be chosen as the differential of the geodesic symmetry of a symmetric space.
3. It is easy to see that the action of ψ^* not only preserves the equation $d\Omega=0$, but also the Einstein equation $\mathrm{Ric}=\lambda g$. The reason for this is that the Ricci-tensor of a cohomogeneity-one metric can be expressed in terms of the Lie bracket. Therefore, we could state a lemma analogous to Lemma 3.2.36 on cohomogeneity-one Einstein metrics.

Chapter 4

Classification of the principal orbits

In this chapter, we classify the possible principal orbits of parallel cohomogeneity-one Spin(7)-structures. As we have seen in the previous chapter, those orbits are exactly the seven-dimensional spaces carrying a homogeneous cosymplectic G_2-structure. We will therefore first classify all spaces admitting a homogeneous G_2-structure and then consider the question if any of those G_2-structures is cosymplectic. In Section 3.1, we have seen that a homogeneous space G/H admits a G-invariant G_2-structure if and only if $H \subseteq G_2$. In this formula, H acts on the tangent space by its isotropy representation and G_2 acts as the stabilizer of the three-form. Before we classify the orbits G/H, we therefore first have to classify the subgroups of G_2:

Lemma 4.1. *As usual, let G_2 be the simply connected Lie group whose Lie algebra is the compact real form of $\mathfrak{g}_2$. Furthermore, let H be a connected Lie subgroup of G_2. We denote the Lie algebra of H by $\mathfrak{h}$. The irreducible action of G_2 on $\mathbb{R}^7$ induces an action of H on $\mathbb{R}^7$. In this situation, $\mathfrak{h}$, H, and the action of H on $\mathbb{R}^7$ are contained in the table below. Moreover, any two connected Lie subgroups of G_2 whose action on $\mathbb{R}^7$ is equivalent are conjugate not only by an element of $GL(7)$ but by an element of G_2, too.*

$\mathfrak{h}$	H	*Splitting of $\mathbb{R}^7$ into irreducible summands*
$\{0\}$	$\{e\}$	
$\mathfrak{u}(1)$	$U(1)$	$\mathbb{V}_a^{\mathbb{C}} \oplus \mathbb{V}_b^{\mathbb{C}} \oplus \mathbb{V}_c^{\mathbb{C}} \oplus \mathbb{V}_0^{\mathbb{R}}$ *with* $a+b=c$
$2\mathfrak{u}(1)$	$U(1)^2$	$\mathbb{V}_{1,0}^{\mathbb{C}} \oplus \mathbb{V}_{0,1}^{\mathbb{C}} \oplus \mathbb{V}_{1,1}^{\mathbb{C}} \oplus \mathbb{V}_{0,0}^{\mathbb{R}}$
$\mathfrak{su}(2)$	$SU(2)$	$\mathbb{V}_1^{\mathbb{C}} \oplus 3\mathbb{V}_0^{\mathbb{R}}$
$\mathfrak{su}(2)$	$SU(2)$	$\mathbb{V}_2^{\mathbb{R}} \oplus \mathbb{V}_1^{\mathbb{C}}$
$\mathfrak{su}(2)$	$SO(3)$	$2\mathbb{V}_2^{\mathbb{R}} \oplus \mathbb{V}_0^{\mathbb{R}}$
$\mathfrak{su}(2)$	$SO(3)$	$\mathbb{V}_6^{\mathbb{R}}$
$\mathfrak{su}(2) \oplus \mathfrak{u}(1)$	$U(2)$	$\mathbb{V}_1^{\mathbb{C}} \oplus 3\mathbb{V}_0^{\mathbb{R}}$ *w.r.t.* $\mathfrak{su}(2)$
$\mathfrak{su}(2) \oplus \mathfrak{u}(1)$	$U(2)$	$\mathbb{V}_2^{\mathbb{R}} \oplus \mathbb{V}_1^{\mathbb{C}}$ *w.r.t* $\mathfrak{su}(2)$
$2\mathfrak{su}(2)$	$SO(4)$	$\mathbb{V}_1^{\mathbb{C}} \oplus 3\mathbb{V}_0^{\mathbb{R}}$ *w.r.t. the first summand of* $\mathfrak{h}$ $\mathbb{V}_2^{\mathbb{R}} \oplus \mathbb{V}_1^{\mathbb{C}}$ *w.r.t. the second summand of* $\mathfrak{h}$
$\mathfrak{su}(3)$	$SU(3)$	$\mathbb{V}_{1,0}^{\mathbb{C}} \oplus \mathbb{V}_0^{\mathbb{R}}$
$\mathfrak{g}_2$	G_2	$\mathbb{V}_{1,0}^{\mathbb{R}}$

In the above table, the subscripts of the modules denote the weights of the H-action and the superscript indicates if the module is real or complex. Further details of the embeddings, in particular of those of $U(2)$ and $SO(4)$ into G_2, will be described in the proof below.

Proof : In the following, we will often consider G_2 as the automorphism group of the octonions. Therefore, we will identify from now on $\mathbb{R}^7$ with $\mathrm{Im}(\mathbb{O})$.

Since G_2 is a compact Lie group, any closed subgroup of G_2 has to be compact, too. $\mathfrak{h}$ therefore is the direct sum of an abelian and a semisimple Lie algebra. Furthermore, we have $\operatorname{rank}\mathfrak{h} \leqslant 2$ and $\dim\mathfrak{h} \leqslant 14$. The Lie algebras satisfying these criteria are:

- Rank 0: $\{0\}$,
- Rank 1: $\mathfrak{u}(1)$, $\mathfrak{su}(2)$,
- Rank 2: $2\mathfrak{u}(1)$, $\mathfrak{su}(2) \oplus \mathfrak{u}(1)$, $2\mathfrak{su}(2)$, $\mathfrak{su}(3)$, $\mathfrak{so}(5)$, $\mathfrak{g}_2$.

The case $\mathfrak{h} = \mathfrak{so}(5)$ can be excluded as follows: We assume that $\mathfrak{so}(5) \subseteq \mathfrak{g}_2$. In this situation, $\mathfrak{so}(5)$ would act on $\mathrm{Im}(\mathbb{O})$. Since the seven-dimensional representation of G_2 is faithful, $\mathfrak{so}(5)$ has to act non-trivially on $\mathrm{Im}(\mathbb{O})$. The spinor representation of $\mathfrak{so}(5)$ is complex four-dimensional. Since there is no real spinor representation of $\mathfrak{so}(5)$, the only non-trivial irreducible real $\mathfrak{so}(5)$-module of dimension $\leqslant 7$ is the 5-dimensional vector representation. Therefore, $\mathrm{Im}(\mathbb{O})$ has to decompose into an irreducible five-dimensional module and a plane on which $\mathfrak{so}(5)$ acts trivially. We assume without loss of generality that the two-dimensional trivial module is spanned by i and j. Since $\mathfrak{so}(5)$ acts by automorphisms, k has to be invariant, too. This contradicts the fact that $\mathrm{span}(i, j)$ has to be the maximal trivial submodule of $\mathrm{Im}(\mathbb{O})$.

We will see that the remaining algebras from the above list can be embedded into $\mathfrak{g}_2$. Below, we describe the possible embeddings of those algebras into $\mathfrak{g}_2$ in detail. Furthermore, we discuss the Lie subgroups of G_2 associated to the Lie algebras. After that, we will consider the question if there are any further connected Lie subgroups of G_2. It will also be shown that the embeddings we have found are unique up to conjugation by an element of G_2. We begin with the program outlined above by considering each of the possible $\mathfrak{h}$ separately:

1. $\underline{\mathfrak{h} = \{0\}}$: The trivial algebra is clearly a subalgebra of $\mathfrak{g}_2$.
2. $\underline{\mathfrak{h} = \mathfrak{u}(1)}$: On page 22, we have described a Cartan subalgebra of $\mathfrak{g}_2$ in detail. Any one-dimensional Lie subalgebra of $\mathfrak{g}_2$ is conjugate to a one-dimensional subalgebra of that Cartan subalgebra. Since G_2 is compact, any one-dimensional Lie subgroup of G_2 is isomorphic to $U(1)$.
3. $\underline{\mathfrak{h} = 2\mathfrak{u}(1)}$: In this case, $\mathfrak{h}$ is the Cartan subalgebra described on page 22 or one of it conjugates. Any Lie subgroup of G_2 whose Lie algebra is isomorphic to $2\mathfrak{u}(1)$ is a maximal torus, which is isomorphic to $U(1)^2$.
4. $\underline{\mathfrak{h} = \mathfrak{su}(2)}$: There are four subalgebras of $\mathfrak{g}_2$ which are isomorphic to $\mathfrak{su}(2)$ but pairwise non-conjugate. We will describe each of them in detail:

(a) We consider the automorphisms of $\mathbb{O}$ which act trivially on the quaternions $\mathbb{H}$ and map ϵ to $h\epsilon$, where h is a unit quaternion. These automorphisms can explicitly be described as:

$$x + y\epsilon \mapsto x + (hy)\epsilon \quad \text{for all } x, y \in \mathbb{H}\,.$$

The proof that these maps are indeed automorphisms can be found in Cacciatori et al. [17]. The above formula defines a left action of $Sp(1)$ on $\mathrm{Im}(\mathbb{O})$ which is trivial on $\mathrm{Im}(\mathbb{H})$ and irreducible on $\mathbb{H}\epsilon$. Since the set of the union quaternions is isomorphic to $SU(2)$, we have described the first embedding $SU(2) \hookrightarrow G_2$ stated in the lemma.

(b) We define for each unit quaternion h a map

$$\Phi_h(x + y\epsilon) := hxh^{-1} + (yh^{-1})\epsilon \quad \text{for all } x, y \in \mathbb{H}\,.$$

These maps define another left action of $SU(2)$ on $\mathrm{Im}(\mathbb{O})$. In [17], it is proved that the Φ_h are automorphisms of $\mathbb{O}$, too. $SU(2)$ acts on $\mathrm{Im}(\mathbb{H})$ as $SO(3)$ on $\mathbb{R}^3$. On $\mathbb{H}\epsilon$ it acts irreducibly and faithfully. Therefore, the action of $SU(2)$ on all of $\mathrm{Im}(\mathbb{O})$ is faithful, too, and we have described the second embedding of $SU(2)$ into G_2 stated in the lemma. By identifying $\mathbb{C}$ with $\mathbb{R}^2$ we obtain a canonical embedding of $\mathfrak{su}(2)$ into $\mathfrak{so}(4)$. We equip $\epsilon\mathbb{H}$ with the basis $(\epsilon, i\epsilon, k\epsilon, j\epsilon)$. The matrices of the restriction of the above $\mathfrak{su}(2)$-action to $\epsilon\mathbb{H}$ with respect to that basis describe the orthogonal complement of $\mathfrak{su}(2) \subseteq \mathfrak{so}(4)$. Therefore, the first Lie subalgebra of type $\mathfrak{su}(2)$ and this one commute.

(c) The automorphism group of $\mathbb{H}$ is isomorphic to $SO(3)$. Let H be the set of those automorphisms of $\mathbb{O}$ which map $\mathbb{H}$ to itself and fix ϵ. H is obviously a group. Since any automorphism of $\mathbb{H}$ can be extended to a unique automorphism φ of $\mathbb{O}$ with $\varphi(\epsilon) = \epsilon$, H is isomorphic $SO(3)$, too. It is easy to see that H acts irreducibly on $\mathrm{Im}(\mathbb{H})$ and $\mathrm{Im}(\mathbb{H})\epsilon$. This proves the decomposition of $\mathrm{Im}(\mathbb{O})$ into $2\mathbb{V}_2^{\mathbb{R}} \oplus \mathbb{V}_0^{\mathbb{R}}$, which we have stated in the lemma.

(d) In a paper of Dynkin [31], it is proven that there are four subalgebras of $\mathfrak{g}_2$ which are isomorphic to $\mathfrak{su}(2)$. In fact, the semisimple subalgebras of all semisimple Lie algebras are classified. Their embeddings are described in an abstract manner, which is not convenient for our considerations. Therefore, the work we do in this lemma is nevertheless necessary. The first three of the embeddings of $\mathfrak{su}(2)$ into $\mathfrak{g}_2$ which can be found in [31] are those we have described above. The fourth subalgebra acts irreducibly on $\mathrm{Im}(\mathbb{O})$ and its associated Lie group is isomorphic to $SO(3)$. For our considerations, we only need the existence of this subgroup but not its explicit description. We therefore refer the reader to the literature for further details. Dynkin has classified the subalgebras up to inner automorphisms of the greater Lie algebra. Thus, we also know that there is up to conjugation by an element of G_2 no other subalgebra of type $\mathfrak{su}(2)$ which acts irreducibly on $\mathrm{Im}(\mathbb{O})$. For reasons of completeness, we will nevertheless explicitly prove that the other Lie algebras in the table of our lemma are uniquely determined up to conjugation by an automorphism of $\mathbb{O}$.

In order to avoid confusion, we will index the Lie subalgebras of type $\mathfrak{su}(2)$ by their non-zero weights. The above four algebras will therefore be denoted by $\mathfrak{su}(2)_1$, $\mathfrak{su}(2)_{1,2}$, $\mathfrak{su}(2)_{2,2}$, and $\mathfrak{su}(2)_6$.

5. $\underline{\mathfrak{h} = \mathfrak{su}(2) \oplus \mathfrak{u}(1)}$: We describe the two subalgebras mentioned in the lemma in detail:

   (a) We embed $\mathfrak{su}(2)$ as $\mathfrak{su}(2)_1$ into $\mathfrak{g}_2$. Furthermore, we choose a certain $\Phi \in \mathfrak{g}_2$. Φ generates a Lie subalgebra of $\mathfrak{g}_2$, which is isomorphic to $\mathfrak{u}(1)$. The matrix representation of Φ with respect to the basis $(i, j, k, \epsilon, i\epsilon, j\epsilon, k\epsilon)$ shall be:

$$\Phi := \begin{pmatrix} 0 & & & & & & \\ & 0 & 2 & & & & \\ & -2 & 0 & & & & \\ & & & 0 & -1 & & \\ & & & 1 & 0 & & \\ & & & & & 0 & 1 \\ & & & & & -1 & 0 \end{pmatrix}.$$

   It is possible to show that Φ commutes with all of $\mathfrak{su}(2)_1 \subseteq \mathfrak{g}_2$. This proves that there is indeed a subalgebra of $\mathfrak{g}_2$ which is isomorphic to $\mathfrak{su}(2) \oplus \mathfrak{u}(1)$ and acts in the same way as stated in the lemma on $\mathrm{Im}(\mathbb{O})$. The corresponding Lie group is finitely covered by $SU(2) \times U(1)$ and the kernel of the covering map is $\mathbb{Z}_2 := \{(1,1); (-1,-1)\}$. Therefore, the Lie group which is associated to the Lie algebra we have constructed above is isomorphic to $(SU(2) \times U(1))/\mathbb{Z}_2 = U(2)$.

   (b) The $\mathfrak{su}(2)$-summand of the next subalgebra we consider is chosen as $\mathfrak{su}(2)_{1,2}$. We further choose the abelian summand as the Lie algebra generated by the following matrix:

$$\Psi := \begin{pmatrix} 0 & & & & & & \\ & 0 & 0 & & & & \\ & 0 & 0 & & & & \\ & & & 0 & 1 & & \\ & & & -1 & 0 & & \\ & & & & & 0 & 1 \\ & & & & & -1 & 0 \end{pmatrix}.$$

   Since Ψ commutes with any element of $\mathfrak{su}(2)_{1,2}$, the direct sum $\mathfrak{su}(2)_{1,2} \oplus \mathfrak{u}(1)$ is closed under the Lie bracket. The Lie subgroup of G_2 whose Lie algebra is given by $\mathfrak{su}(2)_{1,2} \oplus \mathfrak{u}(1)$ is isomorphic to $U(2)$ for the same reasons as in the previous case.

6. $\underline{\mathfrak{h} = 2\mathfrak{su}(2)}$: Again, we consider $\mathbb{H}$ as a subalgebra of $\mathbb{O}$. Let $\mathfrak{h}$ be the Lie algebra of all derivations of $\mathbb{O}$ which leave $\mathbb{H}$ invariant. It is easy to see that the first, second, and third subalgebra of type $\mathfrak{su}(2)$ we have constructed above are contained in $\mathfrak{h}$. The first and the second of those algebras commute and their direct sum therefore is a Lie algebra isomorphic to $2\mathfrak{su}(2)$. The third Lie algebra $\mathfrak{su}(2)_{2,2}$ is diagonally embedded into $2\mathfrak{su}(2)$. It can be shown that there is no subalgebra $\mathfrak{k}$ of $\mathfrak{g}_2$ such that $2\mathfrak{su}(2) \subsetneq \mathfrak{k} \subsetneq \mathfrak{g}_2$, i.e., $2\mathfrak{su}(2)$

is a maximal subalgebra. From this fact, it follows that $\mathfrak{h} = 2\mathfrak{su}(2)$. We define a Lie group homomorphism φ by

$$\begin{aligned} \varphi : Sp(1) \times Sp(1) &\to G_2 \\ \varphi(h,k)(x + y\epsilon) &:= hxh^{-1} + (kyh^{-1})\epsilon\,. \end{aligned}$$

It is easy to see that the kernel of this homomorphism is $\{(1,1);(-1,-1)\}$. Therefore, the Lie group associated to $2\mathfrak{su}(2) \subseteq \mathfrak{g}_2$ is isomorphic to $SO(4)$. Furthermore, we obtain the same splitting of $\mathrm{Im}(\mathbb{O})$ with respect to the action of $2\mathfrak{su}(2)$ as stated in the lemma. The idea for the construction of this subgroup we have taken from [17], too.

7. $\underline{\mathfrak{h} = \mathfrak{su}(3)}$: Let H be the group of all automorphisms of $\mathbb{O}$ which fix i. For any h of unit length in the orthogonal complement of $\mathbb{C} \subseteq \mathbb{O}$ there is an automorphism Φ with $\Phi(i) = i$, $\Phi(j) = h$. We can further choose a unit octonion $h' \in \mathrm{span}(1,i,h,ih)^{\perp}$ and require that $\Phi(\epsilon) = h'$. The choice of $\Phi(i)$, $\Phi(j)$, and $\Phi(\epsilon)$ makes Φ unique. Since any automorphism of $\mathbb{O}$ which fixes i is of this type, H is a S^3-bundle over S^5 and therefore simply connected. Furthermore, $\dim H = \dim S^5 + \dim S^3 = 8$. H acts irreducibly and faithfully on the six-dimensional space $\mathbb{C}^{\perp}$ and trivially on the one-dimensional space $\mathrm{span}(i)$. The only possible action by a compact Lie group satisfying all these conditions is by $SU(3)$ acting on $\mathrm{Im}(\mathbb{O})$ as on $\mathbb{C}^3 \oplus \mathbb{R}$.

8. $\underline{\mathfrak{h} = \mathfrak{g}_2}$: In this case, the statement of the lemma is trivially true.

It remains to show that there are up to conjugacy no other embeddings of the above groups into G_2. The cases $\mathfrak{h} = \{0\}$ and $\mathfrak{h} = \mathfrak{g}_2$ are trivial and the case where $\mathfrak{h}$ is abelian has already been handled above. Next, we consider the case $\mathfrak{h} = \mathfrak{su}(3)$. The only real $\mathfrak{su}(3)$-representations of dimension $\leqslant 7$ are the trivial one and the standard representation on $\mathbb{C}^3 \cong \mathbb{R}^6$. Since $\mathfrak{su}(3)$ has to act faithfully on $\mathrm{Im}(\mathbb{O})$, the imaginary space has to decompose into a six- and a one-dimensional irreducible submodule. By conjugating $\mathfrak{h}$ by an element of G_2, we can assume that the trivial submodule is spanned by i. In this situation, H is a connected closed group of automorphisms which preserve i. Therefore, it has to be a subgroup of the group $SU(3) \subseteq G_2$ we have constructed above. Since $\dim \mathfrak{h} = 8$, H coincides with that group. The only remaining cases are $\mathfrak{h} = \mathfrak{su}(2)$, $\mathfrak{su}(2) \oplus \mathfrak{u}(1)$, or $2\mathfrak{su}(2)$. We start with the $\mathfrak{su}(2)$-case and first give a short introduction to the theory of real $\mathfrak{su}(2)$-representations:

Any irreducible complex representation of $\mathfrak{su}(2)$ is equivalent to a symmetric power $S^m_{\mathbb{C}}(\mathbb{C}^2)$. These spaces either contain no non-trivial real $\mathfrak{su}(2)$-submodule or they split into two submodules of the same dimension. It is easy to see that $S^m(\mathbb{C}^2)$ splits if and only if the conjugation map $\tau : S^m_{\mathbb{C}}(\mathbb{C}^2) \to S^m_{\mathbb{C}}(\mathbb{C}^2)$ commutes with the action of $\mathfrak{su}(2)$. In this situation, the two real $\mathfrak{su}(2)$-submodules are the eigenspaces of τ. By some calculations, we can prove that τ and $\mathfrak{su}(2)$ commute if and only if m is even. In the even case, the Lie group associated to the action of $\mathfrak{su}(2)$ is $SO(3)$ and in the odd case it is $SU(2)$. We will denote the irreducible complex module on which $\mathfrak{su}(2)$ acts with weight m shortly by $\mathbb{V}^{\mathbb{C}}_m$. If $\mathbb{V}^{\mathbb{C}}_m$ splits into two real modules, we will denote each of them by $\mathbb{V}^{\mathbb{R}}_m$. With help of the above considerations, we can show that the only irreducible real $\mathfrak{su}(2)$-modules of real dimension $\leqslant 7$ can be described as follows:

Action of $\mathfrak{su}(2)$	Name of the module	Real dimension
Trivial action	$\mathbb{V}_0^{\mathbb{R}}$	1
Standard action of $\mathfrak{su}(2)$ on $\mathbb{C}^2$	$\mathbb{V}_1^{\mathbb{C}}$	4
Standard action of $\mathfrak{so}(3)$ on $\mathbb{R}^3$	$\mathbb{V}_2^{\mathbb{R}}$	3
Action of $\mathfrak{so}(3)$ on the trace-free, symmetric 3×3-matrices	$\mathbb{V}_4^{\mathbb{R}}$	5
Irreducible action of $\mathfrak{so}(3)$ on $\mathrm{Im}(\mathbb{O})$ as described above	$\mathbb{V}_6^{\mathbb{R}}$	7

We now prove that there are no splittings of $\mathrm{Im}(\mathbb{O})$ with respect to a $\mathfrak{su}(2) \subseteq \mathfrak{g}_2$ except those which we already have described. The case $\mathbb{V}_4^{\mathbb{R}} \subseteq \mathrm{Im}(\mathbb{O})$ can be excluded by similar arguments as $\mathfrak{so}(5) \subseteq \mathfrak{g}_2$: We assume that $\mathfrak{su}(2)$ is embedded into $\mathfrak{g}_2$ such that $\mathrm{Im}(\mathbb{O})$ contains $\mathbb{V}_4^{\mathbb{R}}$ as a $\mathfrak{su}(2)$-submodule. Since there exists no real two-dimensional irreducible $\mathfrak{su}(2)$-module, the orthogonal complement of $\mathbb{V}_4^{\mathbb{R}}$ is trivial. Without loss of generality, we can assume that the orthogonal complement is spanned by i and j. Since i and j are fixed by the action of the Lie group $SO(3) \subseteq G_2$, which is associated to $\mathfrak{su}(2)$, and G_2 acts by automorphisms, k is fixed, too. Therefore, $\mathrm{Im}(\mathbb{O})$ contains a three-dimensional trivial submodule, which contradicts our assumption.

The only other case we have to exclude is $\mathrm{Im}(\mathbb{O}) \cong \mathbb{V}_2^{\mathbb{R}} \oplus 4\mathbb{V}_0^{\mathbb{R}}$. We assume that $\mathfrak{su}(2)$ acts irreducibly on a three-dimensional subspace V of $\mathrm{Im}(\mathbb{O})$ and trivially on its complement $V^\perp$. Without loss of generality, we furthermore can assume that i and j are contained in V. Since $i \perp j$, there exists a $\psi \in \mathfrak{su}(2)$ with

$$\begin{aligned} \psi(i) &= j \\ \psi(j) &= -i \\ \psi(\mathrm{span}(i,j)^\perp) &= \{0\} \,. \end{aligned}$$

$\psi \in \mathfrak{su}(2) \subseteq \mathfrak{g}_2$ has to be a derivation. Therefore, we have:

$$\psi(\underbrace{i\epsilon}_{\perp \mathrm{span}(i,j)}) = \psi(i)\epsilon + i\psi(\epsilon) = j\epsilon \neq 0 \,.$$

Since this is a contradiction, the case $\mathrm{Im}(\mathbb{O}) \cong \mathbb{V}_2^{\mathbb{R}} \oplus 4\mathbb{V}_0^{\mathbb{R}}$ can be excluded.

Before we can finish the $\mathfrak{su}(2)$-case, we have to prove that any two subalgebras of type $\mathfrak{su}(2)$ which act by the same weights on $\mathrm{Im}(\mathbb{O})$ are conjugate by an automorphism of $\mathbb{O}$. We start with the subcase where the subalgebra, which we denote by $\mathfrak{h}$, splits $\mathrm{Im}(\mathbb{O})$ into $\mathbb{V}_1^{\mathbb{C}} \oplus 3\mathbb{V}_0^{\mathbb{R}}$. In this situation, there exists a four-dimensional subspace $V \subseteq \mathrm{Im}(\mathbb{O})$ and an orthonormal basis (x, y, z, w) of V such that the matrix representation of $\mathfrak{h}$ is the same as of $\mathfrak{su}(2) \subseteq \mathfrak{gl}(2, \mathbb{C}) \subseteq \mathfrak{gl}(4, \mathbb{R})$. If there exists an automorphism of $\mathbb{O}$ which maps (x, y, z, w) to $(\epsilon, i\epsilon, k\epsilon, j\epsilon)$, it follows that $\mathfrak{h}$ is conjugate to $\mathfrak{su}(2)_1$ by an element of G_2. Since (x, y) is orthonormal, there is an automorphism which maps (x, y) to $(\epsilon, i\epsilon)$. We therefore assume from now on that $x = \epsilon$ and $y = i\epsilon$. Let $\mathfrak{h} \ni \varphi : V \to V$ be the linear map whose matrix representation with respect to (x, y, z, w) is

$$\begin{pmatrix} \boxed{\begin{matrix} 0 & -1 \\ 1 & 0 \end{matrix}} & \\ & \boxed{\begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix}} \end{pmatrix} .$$

It is easy to see that

$$\varphi(i) = \varphi(\epsilon \cdot i\epsilon) = \varphi(\epsilon) \cdot (i\epsilon) + \epsilon \cdot \varphi(i\epsilon) = (i\epsilon) \cdot (i\epsilon) + \epsilon \cdot (-\epsilon) = 0 \,.$$

The space of all octonions on which φ acts trivially is a φ-invariant subspace of $\mathrm{Im}(\mathbb{O})$. Since $\varphi \in \mathfrak{so}(7)$, that subspace has to be orthogonal to V. The basis element z, which is orthogonal to $\mathrm{span}(i, \epsilon, i\epsilon)$, can be mapped by an automorphism to any other element of $\mathrm{span}(i, \epsilon, i\epsilon)^\perp$. We therefore can assume that $z = k\epsilon$. Since $(\epsilon, i\epsilon, k\epsilon)$ is a basis triple, the action of any $\psi \in \mathfrak{h}$ on $\mathrm{Im}(\mathbb{O})$ is determined by the matrix representation of $\psi|_V$ with respect to (x, y, z, w). We finally have proven that there is up to conjugation by an element of G_2 at most one subalgebra, namely $\mathfrak{su}(2)_1$, with the properties we have assumed. The uniqueness of $\mathfrak{su}(2)_{1,2}$ and $\mathfrak{su}(2)_{2,2}$ can be verified by similar arguments. For the $\mathfrak{su}(2)_6$-case, we already have referred to [31].

Next, we will show that there are no embeddings of $U(2) = (SU(2) \times U(1))/\mathbb{Z}_2$ into G_2 except those described above. Let x be a generator of the Lie algebra of $U(1)$. Since $SU(2)$ and $U(1)$ commute, the action φ_x of x on $\mathrm{Im}(\mathbb{O})$ is a $\mathfrak{su}(2)$-equivariant map. We can apply Schur's lemma and exclude some of the possible $\mathfrak{su}(2) \subseteq \mathfrak{g}_2$ as the first summand of a Lie subalgebra isomorphic to $\mathfrak{su}(2) \oplus \mathfrak{u}(1)$:

- $\underline{\mathfrak{su}(2)_6}$: In this situation, φ_x has to be a multiplication by a real constant. The only multiple of the identity which is contained in $\mathfrak{g}_2$ is 0. Since x has to act non-trivially, we can exclude this case.

- $\underline{\mathfrak{su}(2)_{2,2}}$: Since $x \in \mathfrak{g}_2 \subseteq \mathfrak{so}(7)$, it has to act skew-symmetrically on $\mathrm{Im}(\mathbb{O})$. Therefore, the action of x on $\mathrm{Im}(\mathbb{H}) \oplus \epsilon\mathrm{Im}(\mathbb{H})$ has to be a multiple of

$$\left(\begin{array}{ccc|ccc} 0 & & & 1 & & \\ & 0 & & & 1 & \\ & & 0 & & & 1 \\ \hline -1 & & & 0 & & \\ & -1 & & & 0 & \\ & & -1 & & & 0 \end{array}\right)$$

  and the action on ϵ has to be trivial. If the φ_x defined above was a derivation, we had

$$\begin{aligned} \varphi_x(i) &= -i\epsilon \\ \varphi_x(j) &= -j\epsilon \\ \varphi_x(k) &= -k\epsilon \end{aligned}$$

  From the definition of the derivation property, it follows that:

$$\begin{aligned}\varphi_x(ij) &= \varphi_x(i)j + i\varphi_x(j) \\ &= (-i\epsilon)j + i(-j\epsilon) \\ &= 2k\epsilon \neq -k\epsilon\,.\end{aligned}$$

Therefore, we can exclude this case, too.

The above considerations prove that the only possibilities for the semisimple part of a Lie subalgebra of $\mathfrak{g}_2$ which is isomorphic to $\mathfrak{su}(2) \oplus \mathfrak{u}(1)$ are those we have described in the lemma. Next, we will show that for any fixed choice of the $\mathfrak{su}(2)$-summand the abelian summand is unique up to conjugation by an element of G_2. Since the subalgebras $\mathfrak{su}(2)_1$ and $\mathfrak{su}(2)_{1,2}$ are unique up to conjugation, too, the uniqueness of the subalgebras $\mathfrak{su}(2)_1 \oplus \mathfrak{u}(1)$ and $\mathfrak{su}(2)_{1,2} \oplus \mathfrak{u}(1)$ then will be proven. We first consider the $\mathfrak{su}(2)_1$-case. The abelian summand $\mathfrak{u}(1)$ of $\mathfrak{su}(2)_1 \oplus \mathfrak{u}(1)$ has to act $\mathfrak{su}(2)_1$-equivariantly on $\mathrm{Im}(\mathbb{O})$. By the real version of Schur's lemma, we see that the vector space of all $\mathfrak{su}(2)_1$-equivariant maps $\mathrm{Im}(\mathbb{O}) \to \mathrm{Im}(\mathbb{O})$ is given by $M^{3\times 3}(\mathbb{R}) \oplus K$. In the following, we will describe that space and in particular the summand K in more detail: The group $SO(4)$ acts on $\mathbb{H}\epsilon$. We consider its subgroup of type $SU(2)$ whose Lie algebra commutes with $\mathfrak{su}(2)_1$. K acts on $\mathbb{H}\epsilon$ and is generated as a vector space by the above $SU(2)$ and its real multiples. The first summand of $M^{3\times 3}(\mathbb{R}) \oplus K$ acts canonically on $\mathrm{Im}(\mathbb{H})$. Since $\mathfrak{u}(1)$ has to be a subalgebra of $\mathfrak{g}_2$, we have $\mathfrak{u}(1) \subseteq (M^{3\times 3}(\mathbb{R}) \oplus K) \cap \mathfrak{g}_2 = \mathfrak{su}(2)_{1,2}$. Conversely, we can take any one-dimensional subalgebra of $\mathfrak{su}(2)_{1,2}$ as the abelian summand of $\mathfrak{su}(2)_1 \oplus \mathfrak{u}(1)$. All in all, that summand is unique up to conjugation by an element of the Lie group $SU(2)_{1,2} \subseteq G_2$ which is associated to $\mathfrak{su}(2)_{1,2}$ if the semisimple part of $\mathfrak{su}(2)_1 \oplus \mathfrak{u}(1)$ is fixed. The $\mathfrak{su}(2)_{1,2} \oplus \mathfrak{u}(1)$-case can be handled by similar arguments. The only thing we have to take care of is that $\mathfrak{su}(2)_{1,2}$ acts on $\mathbb{H}\epsilon$ as the other summand of $\mathfrak{so}(4)$ than $\mathfrak{su}(2)_1$.

Finally, we will show that $2\mathfrak{su}(2) \subseteq \mathfrak{g}_2$ is unique up to conjugation by an automorphism. Let $\mathfrak{h}$ be an arbitrary subalgebra of $\mathfrak{g}_2$ which is isomorphic to $2\mathfrak{su}(2)$. There is a subalgebra of $\mathfrak{h}$ which is isomorphic to $\mathfrak{su}(2) \oplus \mathfrak{u}(1)$. After a conjugation, we can assume that this subalgebra is one of the two subalgebras of type $\mathfrak{su}(2) \oplus \mathfrak{u}(1)$ which we have constructed earlier in this proof. Therefore, either $\mathfrak{su}(2)_1$ or $\mathfrak{su}(2)_{1,2}$ is an ideal of $\mathfrak{h}$. We assume that $\mathfrak{su}(2)_1 \trianglelefteq \mathfrak{h}$. The second summand of $\mathfrak{h}$ has to commute with $\mathfrak{su}(2)_1$. We already have classified all derivations of $\mathbb{O}$ which commute with $\mathfrak{su}(2)_1$. It therefore is easy to see that the second summand of $\mathfrak{h}$ has to be $\mathfrak{su}(2)_{1,2}$. If $\mathfrak{su}(2)_{1,2}$ is an ideal of $\mathfrak{h}$, we can show by the same arguments that $\mathfrak{h} = \mathfrak{su}(2)_1 \oplus \mathfrak{su}(2)_{1,2}$. Therefore, the subalgebra $2\mathfrak{su}(2) \subseteq \mathfrak{g}_2$ is unique up to conjugation, too.

□

We are now able to prove our theorem on the possible principal orbits. For each of the $H \subseteq G_2$ we have found, we search for the Lie groups $G \supseteq H$ such that $\dim G - \dim H = 7$ and H acts on the tangent space of G/H in the same way as $H \subseteq G_2$ on $\mathrm{Im}(\mathbb{O})$. As we have stated in Convention 3.2.1, we require that G is compact and connected. G/H is therefore compact and connected, too. Furthermore, it is determined up to a covering map by the Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ of G and H and the embedding of $\mathfrak{h}$ into $\mathfrak{g}$. Since G is compact, $\mathfrak{g}$ is the direct sum of a semisimple and an abelian Lie algebra. This will make the classification of the possible $\mathfrak{g}$ easier. We find the following list of homogeneous spaces admitting a G_2-structure:

Theorem 4.2. *Let G/H be a compact, connected seven-dimensional homogeneous space such that G is compact and connected, too. Furthermore, we assume that G/H is the product of a simply connected space and a torus, G acts almost effectively on G/H, and G/H admits a G-invariant G_2-structure. Then, G/H is either a product of a circle and a six-dimensional homogeneous space or G/H cannot be described as a product of lower-dimensional homogeneous spaces. In the first case, G and H are up to a finite cover one of the groups in the table below, and G/H can be found in that table, too:*

G	H	G/H
$U(1)^7$	$\{e\}$	T^7
$SU(2) \times U(1)^4$	$\{e\}$	$S^3 \times T^4$
$SU(2)^2 \times U(1)$	$\{e\}$	$S^3 \times S^3 \times S^1$
$SU(2)^2 \times U(1)^2$	$U(1)$	$S^3 \times S^3 \times S^1$
$SU(2)^2 \times U(1)^2$	$U(1)$	$SU(2)^2/U(1) \times T^2$
$SU(2)^3 \times U(1)$	$SU(2)$	$S^3 \times S^3 \times S^1$
$SU(3) \times U(1)^2$	$SU(2)$	$S^5 \times T^2$
$SU(3) \times U(1)$	$U(1)^2$	$SU(3)/U(1)^2 \times S^1$
$Sp(2) \times U(1)$	$Sp(1) \times U(1)$	$\mathbb{CP}^3 \times S^1$
$G_2 \times U(1)$	$SU(3)$	$S^6 \times S^1$

The information on the topology of G/H given in the above table determines the embedding of H into G except in two cases. In the fourth (fifth) row, the embedding $U(1) \hookrightarrow SU(2)^2$ has to be special in order to make $S^3 \times S^3 \times S^1$ ($SU(2)^2/U(1) \times T^2$) a space admitting a $SU(2)^2 \times U(1)^2$-invariant G_2-structure. The detailed description of those embeddings and the space $SU(2)^2/U(1) \times T^2$ can be found in the proof below.

If G/H is not a product of lower-dimensional homogeneous spaces, we obtain the following table:

G	H	G/H
$SU(3)$	$U(1)$	$N^{k,l}$ with $k, l \in \mathbb{Z}$
$SO(5)$	$SO(3)$	$V^{5,2}$
$Sp(2)$	$Sp(1)$	S^7
$SO(5)$	$SO(3)$	B^7
$SU(2)^3$	$U(1)^2$	$Q^{1,1,1}$
$SU(3) \times U(1)$	$U(1)^2$	$N^{k,l}$ with $k, l \in \mathbb{Z}$
$SU(3) \times SU(2)$	$SU(2) \times U(1)$	$M^{1,1,0}$
$SU(3) \times SU(2)$	$SU(2) \times U(1)$	$N^{1,1}$
$Sp(2) \times U(1)$	$Sp(1) \times U(1)$	S^7
$Sp(2) \times Sp(1)$	$Sp(1) \times Sp(1)$	S^7
$SU(4)$	$SU(3)$	S^7
$Spin(7)$	G_2	S^7

As in the first case, G and H are only determined up to a finite cover. In the above table, the *indices of* $M^{1,1,0}$ *(*$Q^{1,1,1}$*,* $N^{k,l}$*) denote a special embedding of* $SU(2) \times U(1)$ *(*$U(1)^2$*,* $U(1)$*) into* $SU(3) \times SU(2)$ *(*$SU(2)^3$*,* $SU(3)$*). Those embeddings will be described explicitly in Section 5.3 (5.2, 5.4).* $V^{5,2}$ *denotes the Stiefel-manifold of all orthonormal pairs in* $\mathbb{R}^5$*.* B^7 *is the Berger space, which will be described in more detail in the proof below and in Section 5.1.*

The converse of our statement is also true: Any of the above spaces G/H *admits a* G*-invariant* G_2*-structure.*

Proof : In order to prove the theorem, we consider the different possibilities for the Lie algebra $\mathfrak{h}$ of H separately:

- $\underline{\mathfrak{h} = \{0\}}$: In this situation, G is simply a seven-dimensional Lie group with the properties we have required in the theorem. The only possibilities for G are:

    1. $U(1)^7$
    2. $SU(2) \times U(1)^4$
    3. $SU(2)^2 \times U(1)$

- $\underline{\mathfrak{h} = \mathfrak{u}(1)}$: In this case, G is an eight-dimensional compact Lie group. It therefore has to be up to a cover one the groups in the following list:

    1. $U(1)^8$
    2. $SU(2) \times U(1)^5$
    3. $SU(2)^2 \times U(1)^2$
    4. $SU(3)$

    For our considerations, the following argument will often be helpful: Let $\mathfrak{h}$ be non-trivial. A Cartan subalgebra of $\mathfrak{h}$ has to act on the tangent space of G/H as a subalgebra of the Cartan subalgebra (2.1) of $\mathfrak{g}_2$. Therefore, the subspace on which $\mathfrak{h}$ acts trivially is either one- or three-dimensional. Since $\mathfrak{h}$ acts trivially on the center of $\mathfrak{g}$, the dimension of the center is at most three. In the case $\mathfrak{h} = \mathfrak{u}(1)$, which we now consider, we can exclude the first two possibilities for G by this argument.

    Next, we investigate the case $G = SU(2)^2 \times U(1)^2$. Since $\mathfrak{h}$ has to act non-trivially on the tangent space, it cannot be a subgroup of the center. Therefore, the projection of $\mathfrak{h}$ onto $2\mathfrak{su}(2) \subseteq \mathfrak{g}$ has to be injective. We first consider the case where $H \nsubseteq SU(2)^2$. It follows from Lemma 3.1.12 that in this situation $S^3 \times S^3 \times S^1$ is a $|H \cap SU(2)^2|$-fold cover of G/H. Since we want G/H to be the product of a simply connected space and a torus, $H \cap SU(2)^2$ has to be trivial. In order to satisfy this condition, the embedding of H into $SU(2)^2 \times U(1)^2$ has to be of the following type:

    $$\left(e^{i\varphi}\right) \mapsto \left(\psi_1(e^{i\varphi}), \psi_2(e^{i\varphi})\right) ,$$

    where $\psi_1 : U(1) \to SU(2)^2$ and $\psi_2 : U(1) \to U(1)^2$ are injective group homomorphisms. We assume without loss of generality that $\psi_2(e^{i\varphi}) = (1, e^{i\varphi})$. There are infinitely many non-conjugate embeddings of $U(1)$ into $SU(2)^2$ which can all be described up to conjugation by:

$$\left(e^{i\varphi}\right) \mapsto \begin{pmatrix} \boxed{\begin{matrix} e^{ik\varphi} & 0 \\ 0 & e^{-ik\varphi} \end{matrix}} & \\ & \boxed{\begin{matrix} e^{il\varphi} & 0 \\ 0 & e^{-il\varphi} \end{matrix}} \end{pmatrix} \quad \text{with } k, l \in \mathbb{Z}\,.$$

We can prove by a short calculation that $U(1)$ has in fact to be embedded by

$$\psi_1\left(e^{i\varphi}\right) := \begin{pmatrix} \boxed{\begin{matrix} e^{i\varphi} & 0 \\ 0 & e^{-i\varphi} \end{matrix}} & \\ & \boxed{\begin{matrix} e^{-i\varphi} & 0 \\ 0 & e^{i\varphi} \end{matrix}} \end{pmatrix}$$

Only in that case, H acts on the tangent space in the same way as a subgroup of G_2 on $\mathrm{Im}(\mathbb{O})$. The quotient G/H can be considered as $SU(2) \times SU(2) \times U(1)$, where $SU(2) \times SU(2) \times U(1)$ acts by left-multiplication and $\psi_1(U(1))$ acts by right-multiplication. Next, we have to consider the case where $H \subseteq SU(2)^2$. In this case, the embedding of H into $SU(2)$ has to be the same as above and G/H is a coset space of type $SU(2)^2/U(1) \times T^2$.

$G = SU(3)$ is the only remaining group from the above list. In this case, we obtain the Aloff-Wallach spaces $N^{k,l}$ which do all admit a homogeneous G_2-structure.

- $\underline{\mathfrak{h} = \mathfrak{su}(2)}$: If this is the case, G has to be 10-dimensional and the dimension of its center is $\leqslant 3$. With help of the classification theorem for the compact Lie groups we see that the universal cover of G has to be from the following list

  1. $SU(2)^3 \times U(1)$
  2. $SU(3) \times U(1)^2$
  3. $Sp(2)$

  We consider each case separately and start with the first one: No matter how we embed $SU(2)$ into $SU(2)^3$, the tangent space T_pG/H consists only of three-dimensional irreducible and trivial submodules. The only $\mathfrak{su}(2) \subseteq \mathfrak{g}_2$ which acts in that way on $\mathrm{Im}(\mathbb{O})$ is $\mathfrak{su}(2)_{2,2}$. Therefore, T_pG/H has to contain exactly two irreducible three-dimensional submodules. The only embedding of $SU(2)$ into $SU(2)^3$ which induces this kind of action is described by

$$X \mapsto \begin{pmatrix} \boxed{X} & & \\ & \boxed{X} & \\ & & \boxed{X} \end{pmatrix}.$$

  The resulting homogeneous space is diffeomorphic to $S^3 \times S^3 \times S^1$, where $SU(2)^3$ acts by a suitable action on $S^3 \times S^3$ and trivially on S^1.

  In the second case, H has to be embedded into the $SU(3)$-factor. Any embedding of $\mathfrak{su}(2)$ into $\mathfrak{su}(3)$ induces a complex three-dimensional representation of $\mathfrak{su}(2)$. Since there is only one two- and one three-dimensional complex representation of $\mathfrak{su}(2)$, H is either $SU(2) \subseteq SU(3)$ or $SO(3) \subseteq SU(3)$. We first consider the case $SU(2) \subseteq SU(3)$. The quotient $SU(3)/SU(2)$ is a five-dimensional sphere. $\mathfrak{su}(2)$ commutes with a certain one-dimensional subalgebra of $\mathfrak{su}(3)$ and acts irreducibly on the complement of $\mathfrak{u}(2) \subseteq \mathfrak{su}(3)$. The tangent space of $SU(3)/SU(2) \times U(1)^2$ therefore splits into an irreducible four-dimensional and a trivial three-dimensional $\mathfrak{su}(2)$-module. Since this $SU(2)$-action is included in the list of Lemma 4.1, we have found another space admitting a homogeneous G_2-structure. Next, we consider the case $SO(3) \subseteq SU(3)$. The only 3×3-matrices which commute with all of $SO(3)$ are the multiples of the identity. Therefore, the orthogonal complement of $\mathfrak{so}(3) \subseteq \mathfrak{su}(3)$ contains no trivial $\mathfrak{so}(3)$-module. The maximal trivial submodule of $T_pSU(3)/SO(3) \times U(1)^2$ therefore is two-dimensional. By considering the possible embeddings of $\mathfrak{su}(2)$ into $\mathfrak{g}_2$, which we have found in Lemma 4.1, we can exclude this case.

We finally consider the case where $\mathfrak{g} = \mathfrak{sp}(2) \cong \mathfrak{so}(5)$. An embedding of $\mathfrak{su}(2)$ into $\mathfrak{so}(5)$ is the same as a real five-dimensional orthogonal representation of $\mathfrak{so}(3)$. The possible splittings of $\mathbb{R}^5$ with respect to $\mathfrak{so}(3)$ are:

1. $\mathbb{V}_1^{\mathbb{C}} \oplus \mathbb{V}_0^{\mathbb{R}}$
2. $\mathbb{V}_2^{\mathbb{R}} \oplus 2\mathbb{V}_0^{\mathbb{R}}$
3. $\mathbb{V}_4^{\mathbb{R}}$

In the first case, $\mathfrak{so}(3)$ is embedded into $\mathfrak{so}(5)$ via the standard representation. The resulting simply connected homogeneous space $SO(5)/SO(3)$ is the Stiefel-manifold $V^{5,2}$. It can easily be seen that $\mathfrak{h}$ acts on the tangent space of $V^{5,2}$ as $\mathfrak{su}(2)_{2,2}$. Therefore, we have to put this space on our list.

In the second case, $\mathfrak{su}(2)$ is embedded into $\mathfrak{so}(5)$ by $\mathfrak{su}(2) \subseteq \mathfrak{so}(4) \subseteq \mathfrak{so}(5)$. In order to make the resulting space simply connected, we choose $G = Sp(2)$ and $H = Sp(1)$. The quotient G/H is the sphere S^7. $\mathfrak{su}(2)$ acts on its tangent space as $\mathfrak{su}(2)_1$. Therefore, S^7 is a possible principal orbit, too.

We finally consider the case where $\mathfrak{so}(3)$ acts on $\mathbb{R}^5$ as on the trace-free symmetric 3×3-matrices. In this situation, $\mathfrak{h}$ acts irreducibly on the tangent space of the resulting simply connected space of type $SO(5)/SO(3)$. Since $\mathfrak{su}(2)_6$ is a subalgebra of $\mathfrak{g}_2$, we have found a further candidate for a principal orbit. The space we have constructed is the seven-dimensional Berger space B^7.

- $\underline{\mathfrak{h} = 2\mathfrak{u}(1)}$: For any $\mathfrak{h}$ of rank 2 we have the following argument at hand: The Cartan subalgebra (2.1) of $\mathfrak{g}_2$ splits $\mathrm{Im}(\mathbb{O})$ into three two-dimensional irreducible modules and one one-dimensional module. Since $\mathfrak{h}$ has to act as a subalgebra of $\mathfrak{g}_2$ on the tangent space, the subspace V on which $\mathfrak{h}$ acts trivially is at most one-dimensional. Moreover, $\dim V = 1$ is only possible if $\mathfrak{h}$ is contained in $\mathfrak{su}(3)$, which is the algebra of all derivations which vanish on $i \in \mathbb{O}$. Since $\mathfrak{h}$ acts trivially on the center of $\mathfrak{g}$, the center is in that situation at most one-dimensional. If $\dim V = 0$, we even now that the center has to be trivial. Since we now assume that $\mathfrak{h} = 2\mathfrak{u}(1)$, we can conclude that $\mathfrak{g}$ has to be one of the following algebras:

  1. $3\mathfrak{su}(2)$
  2. $\mathfrak{su}(3) \oplus \mathfrak{u}(1)$

There are several non-conjugate embeddings of $U(1)^2$ into $SU(2)^3$. Quotients of $SU(2)^3$ by $U(1)^2$ will be considered in detail in Section 5.2. There, we will show that $SU(2)^3/U(1)^2$ admits only for a special embedding an invariant G_2-structure. We denote that homogeneous space by $Q^{1,1,1}$.

Next, we consider the case $\mathfrak{g} = \mathfrak{su}(3) \oplus \mathfrak{u}(1)$. There are two subcases. The first one is $\mathfrak{h} \subseteq \mathfrak{su}(3)$. Since any Cartan subalgebra of $\mathfrak{su}(3)$ is isomorphic to $2\mathfrak{u}(1)$, any two embeddings of $\mathfrak{h}$ into $\mathfrak{su}(3)$ are conjugate. It can easily be proven that $2\mathfrak{u}(1)$ acts by the same weights on the tangent space of $SU(3)/U(1)^2 \times U(1)$ as the Cartan subalgebra (2.1) of $\mathfrak{g}_2$ on $\mathrm{Im}(\mathbb{O})$. Therefore, the space $SU(3)/U(1)^2 \times U(1)$ admits a homogeneous G_2-structure.

The other case we have to consider is $\mathfrak{h} \nsubseteq \mathfrak{su}(3)$. We can assume by similar arguments as above that $U(1)^2$ is embedded into $SU(3) \times U(1)$ by

$$\left(e^{i\varphi_1}, e^{i\varphi_2}\right) \mapsto \left(\psi(e^{i\varphi_1}), e^{i\varphi_2}\right) ,$$

where $\psi : U(1) \to SU(3)$ is an injective group homomorphism. The resulting space G/H is an Aloff-Wallach space with an action of $SU(3) \times U(1)$. It is easy to see that the action of $U(1)^2$ on the tangent space is the same as the action of the maximal torus of G_2 on $\mathrm{Im}(\mathbb{O})$.

- $\underline{\mathfrak{h} = \mathfrak{su}(2) \oplus \mathfrak{u}(1)}$: If $\mathfrak{h}$ is isomorphic to $\mathfrak{su}(2) \oplus \mathfrak{u}(1)$, $\mathfrak{g}$ has to be 11-dimensional. Since the center of $\mathfrak{g}$ is at most one-dimensional, $\mathfrak{g}$ has to be one of the following:

  1. $\mathfrak{su}(3) \oplus \mathfrak{su}(2)$
  2. $\mathfrak{so}(5) \oplus \mathfrak{u}(1)$

  The first case we consider is $\mathfrak{g} = \mathfrak{su}(3) \oplus \mathfrak{su}(2)$. In order to describe the possible embeddings of $\mathfrak{su}(2) \oplus \mathfrak{u}(1)$ into $\mathfrak{su}(3) \oplus \mathfrak{su}(2)$, we introduce some notations: Let

$$\begin{aligned} i : \mathfrak{su}(2) &\to \mathfrak{su}(3) \\ i(A) &:= \begin{pmatrix} \boxed{A} & \\ & \boxed{1} \end{pmatrix} \end{aligned}$$

  Furthermore, let $j : \mathfrak{su}(2) \to \mathfrak{su}(3)$ be the differential of the embedding $SO(3) \subseteq SU(3)$. Next, we describe the possible embeddings of $\mathfrak{su}(2) \subseteq \mathfrak{h}$ into $\mathfrak{g}$. We will deal with the embedding of the abelian summand of $\mathfrak{h}$ into $\mathfrak{g}$ later on. It is easy to see that there are only the following possibilities for $\mathfrak{su}(2) \subseteq \mathfrak{g}$:

  1. $\mathfrak{su}(2)$ is the second summand of $\mathfrak{su}(3) \oplus \mathfrak{su}(2)$.
  2. $\mathfrak{su}(2)$ is embedded by the map i into the first summand of $\mathfrak{su}(3) \oplus \mathfrak{su}(2)$.
  3. $\mathfrak{su}(2)$ is embedded by the map j into the first summand of $\mathfrak{su}(3) \oplus \mathfrak{su}(2)$.
  4. $\mathfrak{su}(2)$ is embedded by a map i' into $\mathfrak{su}(3) \oplus \mathfrak{su}(2)$, where the projection of i' onto the first summand is i and the projection onto the second summand is the identity.
  5. $\mathfrak{su}(2)$ is embedded by a map j' into $\mathfrak{su}(3) \oplus \mathfrak{su}(2)$, where the projection of j' onto the first summand is j and the projection onto the second summand is the identity.

The first of these cases can easily be excluded, since in that situation $\mathfrak{su}(2)$ would act trivially on the tangent space.

The homogeneous spaces we obtain in the second case will be considered in more detail in Section 5.3. In that section, we will see that the embedding of the center of $\mathfrak{su}(2) \oplus \mathfrak{u}(1)$ into $\mathfrak{su}(3) \oplus \mathfrak{su}(2)$ has to be special in order to make G/H a space admitting a $SU(3) \times SU(2)$-invariant G_2-structure. The homogeneous space we obtain this way we denote by $M^{1,1,0}$. $\mathfrak{su}(2) \oplus \mathfrak{u}(1)$ acts in the same way as $\mathfrak{su}(2)_1 \oplus \mathfrak{u}(1) \subseteq \mathfrak{g}_2$ on the tangent space of $M^{1,1,0}$. In the proof of Lemma 4.1, we have shown that the kernel of the associated representation of $SU(2) \times U(1)$ is isomorphic to $\mathbb{Z}_2$. Nevertheless, we will describe $M^{1,1,0}$ as a quotient of type $(SU(3) \times SU(2))/(SU(2) \times U(1))$. This is possible, since we only require that G acts almost effectively on G/H. The isotropy group $SU(2) \times U(1)$ of the $SU(3) \times SU(2)$-action is not contained in G_2. Since the group $(SU(2) \times U(1))/\mathbb{Z}_2$, which acts effectively on the tangent space, is a subgroup of G_2, this is not a contradiction to Lemma 3.1.47. The other homogeneous spaces which we obtain in the case $\mathfrak{h} = \mathfrak{su}(2) \oplus \mathfrak{u}(1)$ we will also describe as quotients of type $G/(SU(2) \times U(1))$.

In the third case, we can apply similar arguments as in the case $H = SO(3)$, $G = SU(3) \times U(1)^2$: It easily follows from Schur's lemma that $H = SO(3)$ commutes with no elements of $SU(3)$ except those in the center. Therefore, the orthogonal complement of $\mathfrak{su}(2) \subseteq \mathfrak{su}(3)$ contains no trivial $\mathfrak{su}(2)$-submodule. The only five-dimensional $\mathfrak{su}(2)$-module which contains no trivial submodule is the irreducible one. $\mathfrak{g}_2$ contains no subalgebra $\mathfrak{h}$ which is isomorphic to $\mathfrak{su}(2)$ and acts by its five-dimensional irreducible representation on a subspace of $\mathrm{Im}(\mathbb{O})$. Therefore, we can exclude this case. The fifth embedding of $\mathfrak{su}(2)$ into $\mathfrak{su}(3) \oplus \mathfrak{su}(2)$ can be excluded by the same arguments.

We now consider the fourth embedding of $\mathfrak{su}(2)$ into $\mathfrak{su}(3) \oplus \mathfrak{su}(2)$. Since both summands of $\mathfrak{su}(2) \oplus \mathfrak{u}(1)$ commute, the action of $\mathfrak{u}(1)$ on the tangent space of G/H has to be a $\mathfrak{su}(2)$-equivariant map. From this fact, we can easily conclude that the action of $\mathfrak{u}(1)$ on the second summand of $\mathfrak{su}(3) \oplus \mathfrak{su}(2)$ is trivial. Therefore, the abelian factor of H is a subgroup of $SU(3)$. Since $G = SU(3) \times SU(2)$, we have an $SU(3)$-action on the coset space G/H. In the following, we will determine the orbit of this action. An element $g \in SU(3)$ fixes the point $eH \in G/H$ if and only if $g \in H$. Since $SU(3) \cap H = U(1)$, the orbit of the $SU(3)$-action is an Aloff-Wallach space $SU(3)/U(1)$. By a dimension argument, we see that the $SU(3)$-orbit has to be an open subset of G/H. Since G/H is connected, the orbit $SU(3)/U(1)$ has to be all of G/H. Next, we will determine the embedding of $U(1)$ into $SU(3)$. The $U(1)$-factor of H, which is a subgroup of $SU(3)$, is up to conjugation of the following type:

$$\left\{ \left(\begin{array}{ccc} e^{ikt} & 0 & 0 \\ 0 & e^{ilt} & 0 \\ 0 & 0 & e^{-i(k+l)t} \end{array} \right) \middle| \, t \in \mathbb{R} \right\} \quad \text{with } k, l \in \mathbb{Z} \, .$$

Since $H = SU(2) \times U(1)$ is a direct product, the above abelian group has to commute with $SU(2) \subseteq SU(3)$. This forces k to equal l. Therefore, G/H has to be a so called *exceptional Aloff-Wallach space*, which we will denote as in Section 5.4 by $N^{1,1}$. In order to make sure that $N^{1,1}$ carries a $SU(3) \times SU(2)$-invariant rather than only a $SU(3)$-invariant G_2-structure, we have to determine the action of $SU(2) \times U(1)$ on the tangent space. By an explicit calculation, we see that the tangent space of $N^{1,1}$ decomposes into two $SU(2)$-modules, which are isomorphic to $\mathbb{V}_1^{\mathbb{C}}$ and $\mathbb{V}_2^{\mathbb{R}}$. Furthermore, it is possible to prove that the center of $\mathfrak{h}$ acts in the same way on the tangent space as the second summand of $\mathfrak{su}(2)_{1,2} \oplus \mathfrak{u}(1) \subseteq \mathfrak{g}_2$ on $\mathrm{Im}(\mathbb{O})$. Therefore, we have found another description of $N^{1,1}$ as a coset space, which we will include in our list.

We finally consider the case $\mathfrak{g} = \mathfrak{so}(5) \oplus \mathfrak{u}(1)$. The projection of the center of $\mathfrak{h}$ onto $\mathfrak{so}(5)$ has to be injective. Otherwise, the action of G on G/H would not be almost effective. First, we consider the case where the projection of the center onto both summands of $\mathfrak{so}(5) \oplus \mathfrak{u}(1)$ is injective. In that situation, it follows from Lemma 3.1.12 that G/H is covered by $Sp(2)/Sp(1)$. Since we want G/H to be simply connected, G/H has to be the seven-sphere equipped with an action of $Sp(2) \times U(1)$. Below, we will construct a transitive action of $Sp(2) \times Sp(1)$ on S^7, which preserves a G_2-structure. Since the action we have described above is a restriction of that $Sp(2) \times Sp(1)$-action, it preserves a G_2-structure, too.

Next, we consider the case where $\mathfrak{su}(2) \oplus \mathfrak{u}(1)$ is embedded into the $\mathfrak{so}(5)$-summand of $\mathfrak{g}$. As we have seen earlier in this proof, there are three possible embeddings of $\mathfrak{su}(2)$ into $\mathfrak{so}(5)$. We will consider each of those embeddings separately. If $\mathfrak{su}(2)$ was embedded by its five-dimensional real representation into $\mathfrak{so}(5)$, it would act irreducibly on the center of $\mathfrak{h}$, which has to be a subalgebra of $\mathfrak{so}(5)$, too. The induced action of $\mathfrak{h}$ on $\mathbb{R}^5$ therefore would be a $\mathfrak{su}(2)$-equivariant map. Since any of those maps has to be a multiple of the identity, we can exclude that case. Next, we assume that $\mathfrak{su}(2)$ is embedded by its three-dimensional real representation into $\mathfrak{so}(5)$. In that situation, $\mathfrak{su}(2)$ would act as $\mathfrak{su}(2)_{2,2}$ on the tangent space of G/H. Since the $\mathfrak{su}(2)$-summand of any subalgebra $\mathfrak{su}(2) \oplus \mathfrak{u}(1) \subseteq \mathfrak{g}_2$ does not act in this way on $\mathrm{Im}(\mathbb{O})$, we can exclude this case, too. The only remaining possibility is that $\mathfrak{su}(2)$ is embedded by its two-dimensional complex representation into $\mathfrak{so}(5)$. Since the rank of $\mathfrak{so}(5)$ is two, there is only one choice for the embedding of the center of $\mathfrak{h}$ into $\mathfrak{so}(5)$. Since $\mathfrak{su}(2) \subseteq \mathfrak{so}(5)$ splits its orthogonal complement into $\mathbb{V}_1^{\mathbb{C}} \oplus 3\mathbb{V}_0^{\mathbb{R}}$, $\mathfrak{h}$ has to act as $\mathfrak{su}(2)_1 \oplus \mathfrak{u}(1)$ on the tangent space. By a short calculation, it is possible to show that this is indeed the case. We obtain $G/H = Sp(2)/(Sp(1) \times U(1)) \times U(1)$. Since $Sp(2)/Sp(1)$ is S^7, $Sp(2)/(Sp(1) \times U(1)) \times U(1)$ is diffeomorphic to $\mathbb{CP}^3 \times S^1$.

- $\underline{\mathfrak{h} = 2\mathfrak{su}(2)}$: In this case, G has to be a 13-dimensional compact Lie group containing a subgroup of type $SO(4)$. The action of $\mathfrak{h}$ on the tangent space of G/H has to be the same as the action of $2\mathfrak{su}(2) \subseteq \mathfrak{g}_2$ on $\mathrm{Im}(\mathbb{O})$, which we have described in Lemma 4.1. There is no non-trivial subspace of $\mathrm{Im}(\mathbb{O})$ on which $2\mathfrak{su}(2) \subseteq \mathfrak{g}_2$ acts trivially. Since $\mathfrak{h}$ acts trivially on the center of $\mathfrak{g}$, the center has to be $\{0\}$. By these arguments, we can easily exclude all candidates for $\mathfrak{g}$ except $\mathfrak{so}(5) \oplus \mathfrak{su}(2)$. The tangent space of G/H has to split into a four- and a three-dimensional irreducible submodule with respect to the action of $2\mathfrak{su}(2)$. On the four-dimensional submodule, $2\mathfrak{su}(2)$ has to act as the standard representation of $\mathfrak{so}(4)$. On the three-dimensional submodule, one of the summands of $2\mathfrak{su}(2)$ has to act as the standard representation of $\mathfrak{so}(3)$, and the other one has to act trivially. Let $i : 2\mathfrak{su}(2) \to \mathfrak{so}(5) \oplus \mathfrak{su}(2)$ be the embedding of $\mathfrak{h}$ into $\mathfrak{g}$. It is easy to see that the projection of i onto $\mathfrak{so}(5)$ has to be the canonical embedding of $\mathfrak{so}(4)$ into $\mathfrak{so}(5)$. The projection $i' : 2\mathfrak{su}(2) \to \mathfrak{su}(2)$ of i onto the second summand has to be given by $i'(x, y) = (x, 0)$. We are able to describe the space G/H explicitly:

Let $S^7 \subseteq \mathbb{H}^2$ be the seven-sphere. $Sp(2)$ acts on S^7 by matrix multiplication. Any unit quaternion $h \in Sp(1)$ acts on S^7 by scalar multiplication by h^{-1}. Since the scalar multiplication on a quaternionic vector space acts from the right, the $Sp(1)$-action we have defined is a left action. Since both actions commute, we have constructed an action of $Sp(2) \times Sp(1)$ on S^7. By a straightforward calculation, we see that the isotropy group of this action at $(1,0)^T \in S^7$ is isomorphic to $Sp(1) \times Sp(1)$. The tangent space of S^7 at $(1,0)^T$ is $\mathrm{Im}(\mathbb{H}) \oplus \mathbb{H}$ and the action of $Sp(1) \times Sp(1)$ on the tangent space can explicitly be described by

$$(h_1, h_2) \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} h_1 x h_1^{-1} \\ h_2 y h_1^{-1} \end{pmatrix},$$

where h_1 and h_2 are unit quaternions. On the Lie algebra level, we obtain the same action as of $2\mathfrak{su}(2) \subseteq \mathfrak{g}_2$ on $\mathrm{Im}(\mathbb{O})$. We therefore include the action of $Sp(2) \times Sp(1)$ on S^7 in the list of our theorem. The group $Sp(2) \times Sp(1)$ does not act effectively but almost effectively on the sphere. Furthermore, $(Sp(1) \times Sp(1))/\mathbb{Z}_2$ is a subgroup of G_2 but not its universal cover. For the same reasons as in the case $\mathfrak{h} = \mathfrak{su}(2) \oplus \mathfrak{u}(1)$, we nevertheless can put the action of $Sp(2) \times Sp(1)$ instead of $(Sp(2) \times Sp(1))/\mathbb{Z}_2$ on our list.

- $\underline{\mathfrak{h} = \mathfrak{su}(3)}$: Since $\mathfrak{su}(3)$ acts irreducibly on a six-dimensional subspace of $\mathrm{Im}(\mathbb{O})$, we can conclude by the usual arguments that the center of $\mathfrak{g}$ is at most one-dimensional. Moreover, $\mathfrak{g}$ has to be 15-dimensional. There are the following possibilities for $\mathfrak{g}$:

  1. $\mathfrak{su}(3) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1)$
  2. $\mathfrak{g}_2 \oplus \mathfrak{u}(1)$
  3. $\mathfrak{su}(4)$ $(\cong \mathfrak{so}(6))$

  The first case can be excluded, since in that situation $\mathfrak{su}(3)$ would act trivially on the orthogonal complement of $\mathfrak{h} \subseteq \mathfrak{g}$. In the second case, we have $G/H = G_2/SU(3) \times U(1) \cong S^6 \times S^1$ and in the third case $G/H = SU(4)/SU(3) \cong S^7$. In both cases, the tangent space of G/H decomposes into a six-dimensional irreducible and a trivial one-dimensional $SU(3)$-module. Therefore, both spaces admit a G-invariant G_2-structure.

- $\underline{\mathfrak{h} = \mathfrak{g}_2}$: If this is the case, $\mathfrak{g}$ is 21-dimensional and the center of $\mathfrak{g}$ vanishes. Furthermore, $\mathfrak{g}$ contains an ideal whose dimension is $\geqslant 14$. Therefore, $\mathfrak{g}$ is one the following three algebras:

  1. $\mathfrak{so}(6) \oplus 2\mathfrak{su}(2)$
  2. $\mathfrak{so}(7)$
  3. $\mathfrak{sp}(3)$

  If $\mathfrak{g}_2$ was a subalgebra of $\mathfrak{so}(6)$ $(\mathfrak{sp}(3))$, $\mathfrak{g}_2$ would have a six-dimensional real (three-dimensional quaternionic) representation. Since this is not the case, the only remaining possibility is that G/H is the sphere $S^7 = \mathrm{Spin}(7)/G_2$. S^7 admits a Spin(7)-invariant G_2-structure, which coincides at $T_1 S^7 \cong \mathrm{Im}(\mathbb{O})$ with the three-form $\omega \in \bigwedge^3(\mathrm{Im}(\mathbb{O}))^*$ defined in Section 2.1. This invariant G_2-structure is unique up to multiplication by a constant.

□

Remark 4.3. 1. We call those principal orbits which are not a product of lower-dimensional homogeneous spaces *irreducible principal orbits*. The other principal orbits, which are contained in the first table of our theorem, we call *reducible principal orbits*.

2. Some spaces appear more than once in the lists of Theorem 4.2. Namely, this happens for the following spaces:

   (a) $S^3 \times S^3 \times S^1$ appears as $SU(2)^2 \times U(1)$, $(SU(2)^2 \times U(1)^2)/U(1)$, and $(SU(2)^3 \times U(1))/SU(2)$ in our list.

   (b) $N^{k,l}$ can be described as $SU(3)/U(1)$ and as $(SU(3) \times U(1))/U(1)^2$.

   (c) The space $N^{1,1}$ has a further description as $(SU(3) \times SU(2))/(SU(2) \times U(1))$.

   (d) On the sphere S^7, there are the following five transitive actions, which all leave a certain G_2-structure invariant: $Sp(2)/Sp(1)$, $(Sp(2) \times U(1))/(Sp(1) \times U(1))$, $(Sp(2) \times Sp(1))/(Sp(1) \times Sp(1))$, $SU(4)/SU(3)$, and $\mathrm{Spin}(7)/G_2$.

   We can conclude from the existence of the different transitive actions that in some cases G is not the whole group of diffeomorphisms of G/H which leave the G_2-structure invariant.

3. There are further examples of diffeomorphisms between the spaces which we have included in our theorem. For example, some of the Aloff-Wallach spaces are diffeomorphic, although the actions of $SU(3)$ are inequivalent and should therefore be treated separately. Further details on this phenomenon can be found in Section 5.4 or in the paper of Kreck and Stolz [51].

4. It is not a priori clear if there are parallel cohomogeneity-one $\mathrm{Spin}(7)$-manifolds with principal orbit $G'/H' \times S^1$ which are not a Riemannian product of a circle and a parallel cohomogeneity-one G_2-manifold. Therefore, we will mostly focus on the irreducible principal orbits.

Our next step is to prove which of the above spaces admit not only a homogeneous G_2-structure but also a coclosed homogeneous one. Fortunately, this work has been done in part by other authors:

Remark 4.4. 1. In a paper of Friedrich, Kath, Moroianu, and Semmelmann [37] all simply connected homogeneous spaces admitting a nearly parallel G_2-structure are classified. The list of those spaces coincides with the second list in the above theorem. Since any nearly parallel G_2-structure is coclosed, all of the spaces of those list are possible principal orbits.

2. Next, we consider those G/H which are a product $G'/H' \times S^1$ of a circle and a six-dimensional homogeneous space. G/H admits a G_2-structure if and only if G'/H' admits a $SU(3)$-structure. This fact can be shown by the same methods as Lemma 3.2.27 and 3.2.28. The basic reason behind this is that the group of all automorphisms of $\mathbb{O}$ fixing $i \in \mathrm{Im}(\mathbb{O})$ is isomorphic to $SU(3)$. Six-dimensional spaces admitting a homogeneous $SU(3)$-structure are important in another context: Those spaces are possible principal orbits for cohomogeneity-one G_2-structures. For this reason, Cleyton and Swann [20]

study that kind of spaces, too. They obtain a list of homogeneous spaces which includes all of the G'/H' from our theorem except $SU(2)^2/U(1) \times U(1)$, which is included in our theorem, but seems to be missing in [20]. For many of those six-dimensional spaces it is known that they are also possible principal orbits for parallel cohomogeneity-one G_2-structures. (See, for example, the calculations carried out in [20].) Therefore, the Riemannian products of those G_2-structures with a circle are parallel Spin(7)-manifolds and $G'/H' \times S^1$ is a possible principal orbit for our considerations. Since the reducible principal orbits are not the main subject of this thesis, we will not prove the existence of a homogeneous coclosed G_2-structure on each of those spaces.

In the following, we will choose some of the G/H from Theorem 4.2 and construct parallel cohomogeneity-one Spin(7)-structures with principal orbit G/H.

Chapter 5

Parallel Spin(7)-manifolds with irreducible principal orbits

5.1 The principal orbit B^7

In this chapter, we consider parallel Spin(7)-manifolds of cohomogeneity one whose principal orbit is irreducible. For many of the possible cases from the list of Theorem 4.2, we will deduce the equations for the holonomy reduction and discuss its possible solutions. Each of the following sections will deal with a certain principal orbit. The order of the sections will reflect the complexity of the techniques we have to apply.

We first take a look at cohomogeneity-one manifolds whose principal orbit is the Berger space $B^7 := SO(5)/SO(3)$. The embedding of $SO(3)$ into $SO(5)$ is given by the five-dimensional irreducible representation of $SO(3)$. First, we recall how this representation can be explicitly described. Let W be the space of all trace-free, symmetric 3×3-matrices. It is easy to see that W is five-dimensional and that the action of $SO(3)$ on W by conjugation is irreducible and orthogonal with respect to the inner product $(X, Y) \mapsto \mathrm{tr}(XY)$ on W. This action therefore induces an injective Lie group homomorphism $SO(3) \to SO(5)$.

The biinvariant metric on $\mathfrak{so}(5)$ is as in Section 3.1 denoted by q and $\mathfrak{m}$ shall be the q-orthogonal complement of $\mathfrak{so}(3) \subseteq \mathfrak{so}(5)$. The tangent space of B^7 we identify with $\mathfrak{m}$. Next, we will describe how $\mathfrak{so}(3)$ acts on the tangent space. The five-dimensional $\mathfrak{so}(3)$-module W has the weight 4 and $\mathfrak{so}(5)$ is $\mathfrak{so}(3)$-equivariantly isomorphic to $\bigwedge^2 W$. With help of the Clebsch-Gordan formula, we decompose this module into the following irreducible submodules: $\bigwedge^2 W = \mathbb{V}_6^{\mathbb{R}} \oplus \mathbb{V}_2^{\mathbb{R}}$, where $\mathbb{V}_n^{\mathbb{R}}$ denotes as usual the irreducible real $\mathfrak{so}(3)$-module with weight n. Since $\mathbb{V}_2^{\mathbb{R}}$ is the adjoint representation of $\mathfrak{so}(3)$, $\mathfrak{m}$ has to be isomorphic to $\mathbb{V}_6^{\mathbb{R}}$. This proves that B^7 is isotropy-irreducible. It easily follows from Schur's lemma that there is up to a constant factor only one $SO(5)$-invariant metric on B^7. This metric is given by the restriction of q to $\mathfrak{m}$ and will be denoted by q, too.

The metric q plays an important role in another context: It is one of the few examples of a homogeneous metric with positive sectional curvature. This fact has been discovered by Berger [9], which explains the name "Berger space".

Our next step is to describe the set of all $SO(5)$-invariant G_2-structures on B^7. In order to do this, we first describe the space of all invariant three-forms. $\bigwedge^3 \mathfrak{m}$ decomposes into $\mathbb{V}_{12}^{\mathbb{R}} \oplus \mathbb{V}_8^{\mathbb{R}} \oplus \mathbb{V}_6^{\mathbb{R}} \oplus \mathbb{V}_4^{\mathbb{R}} \oplus \mathbb{V}_0^{\mathbb{R}}$. Since $\bigwedge^3 \mathfrak{m}$ contains only one trivial submodule, there is up to a constant factor only one invariant three-form ω on B^7. In Chapter 4, we have already shown that B^7 admits an invariant G_2-structure. Therefore, ω has to be a three-form defining a G_2-structure.

We now deduce the equation for the holonomy reduction. The $\mathfrak{so}(3)$-modules $\bigwedge^3 \mathfrak{m}$ and $\bigwedge^4 \mathfrak{m}$ are equivalent. An isomorphism of these modules is given by the Hodge star operator $*$. The exterior differential $d : \bigwedge^3 \mathfrak{m} \to \bigwedge^4 \mathfrak{m}$ is a $\mathfrak{so}(3)$-equivariant map. $d\omega$ has to be an element of the trivial submodule of $\bigwedge^4 \mathfrak{m}$. We therefore have the equation:

$$d\omega = \lambda * \omega$$

for a constant $\lambda \in \mathbb{R}$. Since $\bigwedge^5 \mathfrak{m} \cong \bigwedge^2 \mathfrak{m} = \mathbb{V}_6^{\mathbb{R}} \oplus \mathbb{V}_2^{\mathbb{R}}$ and $d * \omega$ has to be $\mathfrak{so}(3)$-invariant, we have $d * \omega = 0$. This proves that ω is a nearly parallel G_2-structure. As we have seen in Proposition 2.2.15, B^7 is an Einstein manifold with Einstein constant $\frac{3}{8}\lambda^2$. λ has to be non-zero, since we have shown in Theorem 3.1.51 that a Ricci-flat homogeneous space is necessarily flat.

Because of Corollary 3.2.34, the only torsion-free Spin(7)-structures of cohomogeneity one with principal orbit B^7 are cones over that space. Since B^7 is not a sphere, the resulting space of cohomogeneity one has a singularity at the singular orbit.

5.2 The principal orbit $Q^{1,1,1}$

The topic of this section are parallel Spin(7)-manifolds with a cohomogeneity-one action of $SU(2)^3$ which preserves the associated four-form. From the proof of Theorem 4.2, it follows that the isotropy group of the action of $SU(2)^3$ on the principal orbits is $U(1)^2$. It will turn out that a space of type $SU(2)^3/U(1)^2$ admits a homogeneous G_2-structure for a special embedding of $U(1)^2$ into $SU(2)^3$ only. This fact already has been stated in Theorem 4.2 but was left unproven. The homogeneous space of type $SU(2)^3/U(1)^2$ we obtain will be denoted by $Q^{1,1,1}$. We will search for parallel cohomogeneity-one Spin(7)-structures with this space as principal orbit. The metrics we will find have been considered in the literature before, for example, in Cvetič et al. [21], [25]. Nevertheless, we will introduce a new, more algebraic point of view. Furthermore, we will prove a classification result which guarantees that there are no further metrics of this kind.

Our first aim is to describe the special embedding of $U(1)^2$ into $SU(2)^3$ mentioned above. We will represent the elements of $SU(2)^3$ by complex 6×6-matrices:

$$SU(2)^3 := \left\{ \left(\begin{array}{ccc} \boxed{X} & & \\ & \boxed{Y} & \\ & & \boxed{Z} \end{array} \right) \;\middle|\; X, Y, Z \in SU(2) \right\} .$$

Since $U(1)^2$ and $SU(2)^3$ are both connected, an embedding $\imath : U(1)^2 \to SU(2)^3$ is determined by its differential $(d\imath)_e : 2\mathfrak{u}(1) \to 3\mathfrak{su}(3)$. In order to describe $(d\imath)_e$, we first consider the possible Lie algebra morphisms $\mathfrak{u}(1) \to 3\mathfrak{su}(3)$. We assume that the image of $\mathfrak{u}(1)$ is the Lie algebra of a closed subgroup of $SU(2)^3$. In this situation, any of these morphisms is up to a conjugation and a multiplication by a constant given by:

$$i_{k,l,m} : \mathfrak{u}(1) \to 3\mathfrak{su}(2) \quad \text{with} \quad k, l, m \in \mathbb{Z} \quad \text{and}$$

$$i_{k,l,m}(ix) := \begin{pmatrix} ikx & 0 & & & & \\ 0 & -ikx & & & & \\ & & ilx & 0 & & \\ & & 0 & -ilx & & \\ & & & & imx & 0 \\ & & & & 0 & -imx \end{pmatrix} .$$

Without loss of generality, we can assume that (k, l, m) are coprime. Furthermore, we can even restrict ourselves to non-negative values of k, l, and m: Let $\phi_P : SU(2)^3 \to SU(2)^3$ be defined by $\phi_P(Q) := PQP^{-1}$, where $P \in SU(2)^3$ is the following matrix:

$$P := \begin{pmatrix} 0 & 1 & & & & \\ -1 & 0 & & & & \\ & & 1 & 0 & & \\ & & 0 & 1 & & \\ & & & & 1 & 0 \\ & & & & 0 & 1 \end{pmatrix} .$$

It is easy to see that ϕ_P maps $i_{k,l,m}(\mathfrak{u}(1))$ into $i_{-k,l,m}(\mathfrak{u}(1))$. By a similar choice of P, we can map $i_{k,l,m}(\mathfrak{u}(1))$ into $i_{k,-l,m}(\mathfrak{u}(1))$ or $i_{k,l,-m}(\mathfrak{u}(1))$. Therefore, our restriction to the case where $k, l, m \geqslant 0$ is justified. It is also possible to choose P as an arbitrary permutation of the three components of $(\mathbb{C}^2)^3$. Since the group which is generated by the corresponding ϕ_P acts on (k, l, m) by permutations, too, we finally can assume that $k \geqslant l \geqslant m \geqslant 0$. There is a deeper reason behind this argument: The symmetry group of the root system $A_1 \times A_1 \times A_1$ is $\mathbb{Z}_2 \rtimes S_3$. This group is also the group of outer isomorphisms of $SU(2)^3$. It can be generated by the maps ϕ_P which we have constructed above. Since the automorphism group of $SU(2)^3$ is generated by the outer and the inner automorphisms, we have found the strongest possible restriction on (k, l, m). We continue describing the embeddings $(d\imath)_e : 2\mathfrak{u}(1) \to 3\mathfrak{su}(2)$. A Cartan subalgebra of $3\mathfrak{su}(2)$ is given by:

$$\left\{ \left. \begin{pmatrix} \boxed{\begin{matrix} ix & 0 \\ 0 & -ix \end{matrix}} & & \\ & \boxed{\begin{matrix} iy & 0 \\ 0 & -iy \end{matrix}} & \\ & & \boxed{\begin{matrix} iz & 0 \\ 0 & -iz \end{matrix}} \end{pmatrix} \right| \; x, y, z \in \mathbb{R} \right\}.$$

This algebra is denoted throughout this section by $3\mathfrak{u}(1)$. We can assume without loss of generality that $(d\imath)_e(2\mathfrak{u}(1)) \subseteq 3\mathfrak{u}(1)$. The equation $q(X, Y) := -\mathrm{tr}(XY)$ defines a biinvariant metric q on $SU(2)^3$. The q-orthogonal complement of $i_{k,l,m}(\mathfrak{u}(1)) \subseteq 3\mathfrak{u}(1)$ we denote by $2\mathfrak{u}(1)_{k,l,m}$ and the Lie subgroup of $SU(2)^3$ whose Lie algebra is $2\mathfrak{u}(1)_{k,l,m}$ by $U(1)^2_{k,l,m}$. The quotient $SU(2)^3/U(1)^2_{k,l,m}$ is called $Q^{k,l,m}$. We remark that since $U(1)^2 \subseteq SU(2)^3$ is compact, $i_{k,l,m}(\mathfrak{u}(1)) \subseteq 3\mathfrak{su}(2)$ had to be chosen as a Lie subalgebra whose associated Lie subgroup is compact, too. Let $(e_1, \ldots, e_9)$ be the following basis of $3\mathfrak{su}(2)$:

$$e_1 := \begin{pmatrix} \boxed{\begin{matrix} 0 & \frac{1}{2}i \\ \frac{1}{2}i & 0 \end{matrix}} & & & & \\ & \boxed{0} & & & \\ & & \boxed{0} & & \\ & & & \boxed{0} & \\ & & & & \boxed{0} \end{pmatrix}, \quad e_2 := \begin{pmatrix} \boxed{\begin{matrix} 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 \end{matrix}} & & & & \\ & \boxed{0} & & & \\ & & \boxed{0} & & \\ & & & \boxed{0} & \\ & & & & \boxed{0} \end{pmatrix},$$

$$e_3 := \begin{pmatrix} \boxed{0} & & & & \\ & \boxed{0} & & & \\ & & \boxed{\begin{matrix} 0 & \frac{1}{2}i \\ \frac{1}{2}i & 0 \end{matrix}} & & \\ & & & \boxed{0} & \\ & & & & \boxed{0} \end{pmatrix}, \quad e_4 := \begin{pmatrix} \boxed{0} & & & & \\ & \boxed{0} & & & \\ & & \boxed{\begin{matrix} 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 \end{matrix}} & & \\ & & & \boxed{0} & \\ & & & & \boxed{0} \end{pmatrix},$$

$$e_5 := \begin{pmatrix} 0 &&&&& \\ & 0 &&&& \\ && 0 &&& \\ &&& 0 && \\ &&&& 0 & \frac{1}{2}i \\ &&&& \frac{1}{2}i & 0 \end{pmatrix}, \quad e_6 := \begin{pmatrix} 0 &&&&& \\ & 0 &&&& \\ && 0 &&& \\ &&& 0 && \\ &&&& 0 & \frac{1}{2} \\ &&&& -\frac{1}{2} & 0 \end{pmatrix},$$

$$e_7 := \begin{pmatrix} \frac{1}{2}ki & 0 &&&& \\ 0 & -\frac{1}{2}ki &&&& \\ && \frac{1}{2}li & 0 && \\ && 0 & -\frac{1}{2}li && \\ &&&& \frac{1}{2}im & 0 \\ &&&& 0 & -\frac{1}{2}im \end{pmatrix}, \quad e_8 := \begin{pmatrix} \frac{1}{2}li & 0 &&&& \\ 0 & -\frac{1}{2}li &&&& \\ && -\frac{1}{2}ki & 0 && \\ && 0 & \frac{1}{2}ki && \\ &&&& 0 & 0 \\ &&&& 0 & 0 \end{pmatrix},$$

$$e_9 := \begin{pmatrix} \frac{1}{2}mki & 0 &&&& \\ 0 & -\frac{1}{2}mki &&&& \\ && \frac{1}{2}mli & 0 && \\ && 0 & -\frac{1}{2}mli && \\ &&&& -\frac{1}{2}(k^2+l^2)i & 0 \\ &&&& 0 & \frac{1}{2}(k^2+l^2)i \end{pmatrix}.$$

The above basis is orthogonal with respect to q. As in Section 3.1, we denote the q-orthogonal complement of $2\mathfrak{u}(1)_{k,l,m} \subseteq 3\mathfrak{su}(2)$ by $\mathfrak{m}$. The $2\mathfrak{u}(1)_{k,l,m}$-module $\mathfrak{m}$ can be equivariantly identified with the tangent space of $Q^{k,l,m}$. It is easy to see that $(e_1, \dots, e_7)$ is a basis of $\mathfrak{m}$, and (e_8, e_9) is a basis of $2\mathfrak{u}(1)_{k,l,m}$. We determine the matrices of $\mathrm{ad}_{e_8}|_{\mathfrak{m}}$ and $\mathrm{ad}_{e_9}|_{\mathfrak{m}}$ with respect to the basis $(e_1, \dots, e_7)$. For $\mathrm{ad}_{e_8}|_{\mathfrak{m}}$ we obtain:

$$\begin{pmatrix} 0 & l &&&&& \\ -l & 0 &&&&& \\ && 0 & -k &&& \\ && k & 0 &&& \\ &&&& 0 & 0 & \\ &&&& 0 & 0 & \\ &&&&&& 0 \end{pmatrix},$$

and for $\mathrm{ad}_{e_9}|_{\mathfrak{m}}$ we obtain:

$$\begin{pmatrix} 0 & mk &&&&& \\ -mk & 0 &&&&& \\ && 0 & ml &&& \\ && -ml & 0 &&& \\ &&&& 0 & -k^2-l^2 & \\ &&&& k^2+l^2 & 0 & \\ &&&&&& 0 \end{pmatrix}.$$

The action of the isotropy algebra $2\mathfrak{u}(1)_{k,l,m}$ on $\mathfrak{m}$ yields the following subalgebra of $\mathfrak{gl}(\mathfrak{m})$:

$$\mathrm{ad}_{2\mathfrak{u}(1)_{k,l,m}}|_{\mathfrak{m}} = \left\{ \left(\begin{array}{ccccccc} 0 & x & & & & & \\ -x & 0 & & & & & \\ & & 0 & y & & & \\ & & -y & 0 & & & \\ & & & & 0 & z & \\ & & & & -z & 0 & \\ & & & & & & 0 \end{array} \right) \middle| \; kx + ly + mz = 0 \right\} .$$

We equip $\mathbb{R}^7$ with the action of the Cartan subalgebra

$$\left\{ \left(\begin{array}{ccccccc} 0 & & & & & & \\ & 0 & x & & & & \\ & -x & 0 & & & & \\ & & & 0 & y & & \\ & & & -y & 0 & & \\ & & & & & 0 & z \\ & & & & & -z & 0 \end{array} \right) \middle| \; x + y - z = 0 \right\} \tag{5.1}$$

of $\mathfrak{g}_2$, which we have defined on page 22. We assume that there exists a $SU(2)^3$-invariant G_2-structure on $Q^{k,l,m}$. Let $p \in Q^{k,l,m}$ be arbitrary. An element of the fiber $\pi^{-1}(p)$ of the G_2-structure can be identified with a map $\psi : \mathfrak{m} \to \mathbb{R}^7$. ψ has to be a $2\mathfrak{u}(1)$-equivariant isomorphism of the $2\mathfrak{u}(1)$-modules $\mathfrak{m}$ and $\mathbb{R}^7$. We therefore have:

$$\psi \circ \mathrm{ad}_{2\mathfrak{u}(1)_{k,l,m}}|_{\mathfrak{m}} \circ \psi^{-1} = 2\mathfrak{u}(1) \subseteq \mathfrak{g}_2 \, . \tag{5.2}$$

Since the weights of the action of $2\mathfrak{u}(1)$ are invariant with respect to ψ, we also have

$$\psi \circ \mathrm{ad}_{2\mathfrak{u}(1)_{k,l,m}}|_{\mathfrak{m}} \circ \psi^{-1} = \left\{ \left(\begin{array}{ccccccc} 0 & & & & & & \\ & 0 & x & & & & \\ & -x & 0 & & & & \\ & & & 0 & y & & \\ & & & -y & 0 & & \\ & & & & & 0 & z \\ & & & & & -z & 0 \end{array} \right) \middle| \; k'x + l'y + m'z = 0 \right\} ,$$

where (k', l', m') equals (k, l, m) up to signs and a permutation. In order to satisfy (5.2), $Q^{k,l,m}$ has to be one of the spaces $Q^{\pm1,\pm1,\pm1}$. Since all of those spaces are $SU(2)^3$-equivariantly diffeomorphic, it suffices to focus on the space $Q^{1,1,1}$. Consequently, whenever we consider a Lie subgroup $U(1)^2 \subseteq SU(2)^3$ or a Lie subalgebra $2\mathfrak{u}(1) \subseteq 3\mathfrak{su}(2)$ in this section, the embedding shall be given by $U(1)^2_{1,1,1} \subseteq SU(2)^3$ or $2\mathfrak{u}(1)_{1,1,1} \subseteq 3\mathfrak{su}(2)$. For our considerations, we have to classify the $SU(2)^3$-invariant metrics on $Q^{1,1,1}$. $\mathfrak{m}$ splits into the following irreducible $2\mathfrak{u}(1)$-submodules:

$$\begin{aligned} V_1 &:= \operatorname{span}(e_1, e_2) \\ V_2 &:= \operatorname{span}(e_3, e_4) \\ V_3 &:= \operatorname{span}(e_5, e_6) \\ V_4 &:= \operatorname{span}(e_7) \end{aligned}$$

In order to describe the possible metrics, we have to check if any pair of the above $2\mathfrak{u}(1)$-modules is equivalent. It is easy to see that V_1, V_2, and V_3 are pairwise inequivalent, since on any pair of those spaces either the one-dimensional Lie algebra generated by e_8 or the Lie algebra generated by e_9 acts with different weights. V_4 cannot be equivalent to one of the other modules, since its dimension is one.

We are now able to classify the $SU(2)^3$-invariant metrics on $Q^{1,1,1}$ by Schur's lemma. Any such metric g can be identified with a $2\mathfrak{u}(1)$-equivariant endomorphism $\varphi : \mathfrak{m} \to \mathfrak{m}$ satisfying $q(\varphi(X), Y) = g(X, Y)$ for all $X, Y \in \mathfrak{m}$. Since φ has to be symmetric with respect to q, the restriction of φ to any of the V_i has to be a multiple of the identity map. Therefore, the matrix representation of g with respect to the basis $(e_1, \ldots, e_7)$ has to be of type

$$\begin{pmatrix} \boxed{\begin{matrix} a^2 & 0 \\ 0 & a^2 \end{matrix}} & & & \\ & \boxed{\begin{matrix} b^2 & 0 \\ 0 & b^2 \end{matrix}} & & \\ & & \boxed{\begin{matrix} c^2 & 0 \\ 0 & c^2 \end{matrix}} & \\ & & & \boxed{f^2} \end{pmatrix} \tag{5.3}$$

with $a, b, c, f \in \mathbb{R}\backslash\{0\}$. Conversely, any such matrix can be identified with a $SU(2)^3$-invariant metric on $Q^{1,1,1}$. The G_2-structures on $Q^{1,1,1}$ are most conveniently described by a three-form which contains odd powers of a, b, c, and f. Therefore, we allow those parameters to take negative values although this does not change the metric. Any cohomogeneity-one metric g with principal orbit $Q^{1,1,1}$ is given by:

$$g = g_t + dt^2 \ .$$

In this formula, g_t describes the invariant metric on the orbit and is identified with a matrix of the above type. If the orbit is singular, g_t degenerates. t denotes the coordinate in the direction of the geodesics which intersect all orbits perpendicularly.

Our next step is to construct a sufficiently generic Spin(7)-structure of cohomogeneity one whose principal orbit is $Q^{1,1,1}$. Let $\mathfrak{G}_{G_2}$ be an invariant G_2-structure on $Q^{1,1,1}$ with associated metric g. Any $\mathfrak{G}_{G_2} \supseteq \pi^{-1}(p) \ni \psi : (\mathfrak{m}, g) \to (\mathrm{Im}(\mathbb{O}), \langle\cdot,\cdot\rangle)$ has to preserve the scalar product and satisfy (5.2). ψ defines a frame $(\psi^{-1}(i), \ldots,\ \psi^{-1}(k\epsilon))$. This frame can canonically be extended to a frame $(f_0, \ldots, f_7)$ of a cohomogeneity-one Spin(7)-structure. A possible $(f_0, \ldots, f_7)$ which we can construct this way is:

$$\begin{aligned} &f_0 := \tfrac{\partial}{\partial t} \quad f_1 := \tfrac{1}{f} e_7 \quad f_2 := \tfrac{1}{a} e_1 \quad f_3 := \tfrac{1}{a} e_2 \\ &f_4 := \tfrac{1}{b} e_3 \quad f_5 := \tfrac{1}{b} e_4 \quad f_6 := \tfrac{1}{c} e_6 \quad f_7 := \tfrac{1}{c} e_5 \end{aligned} \tag{5.4}$$

We obtain the following four-form Ω which is associated to this frame:

$$\begin{aligned} \Omega = {} & abcf\, e^{1357} - abcf\, e^{1467} - abcf\, e^{2367} - abcf\, e^{2457} \\ & - a^2 b^2\, e^{1234} - a^2 c^2\, e^{1256} - b^2 c^2\, e^{3456} \\ & - a^2 f\, e^{127} \wedge dt - b^2 f\, e^{347} \wedge dt - c^2 f\, e^{567} \wedge dt \\ & - abc\, e^{136} \wedge dt - abc\, e^{145} \wedge dt - abc\, e^{235} \wedge dt + abc\, e^{246} \wedge dt \end{aligned} \tag{5.5}$$

It can be easily checked that this four-form is indeed $SU(2)^3$-invariant. We want to express the equation $d\Omega = 0$ as a system of ordinary differential equations for the metric functions a, b, c, and f. In order to do this, we first compute the exterior derivatives of the one-forms $(e^{1^*}, \ldots, e^{7^*})$, which are dual to the Killing vector fields $(e_1^*, \ldots, e_7^*)$. Then, we apply Remark 3.1.46 and obtain $d\Omega$. We finally see that $d\Omega = 0$ is equivalent to:

$$\begin{aligned} a' &= -\frac{1}{6} \frac{f}{a} \\ b' &= -\frac{1}{6} \frac{f}{b} \\ c' &= -\frac{1}{6} \frac{f}{c} \\ f' &= \frac{1}{6} \frac{f^2}{a^2} + \frac{1}{6} \frac{f^2}{b^2} + \frac{1}{6} \frac{f^2}{c^2} - 3 \end{aligned} \tag{5.6}$$

The details of the calculations can be found in Appendix A.

By considering the above equations, we see that if f is non-negative and the function a, b, or c is positive, its first derivative is non-positive. By replacing $\frac{\partial}{\partial t}$ by $-\frac{\partial}{\partial t}$ we could make the derivatives of those metric functions non-negative, but we will not do this change. Next, we discuss the solutions of the system (5.6). Since we have:

$$a'a = b'b = c'c = -\frac{1}{6} f \,,$$

the functions a^2, b^2, and c^2 differ only by a constant. Let the orbit on which we fix the initial conditions be at $t = 0$ and let F be a function with $F' = f$. By requiring $F(0) = 0$, we make F unique. We conclude from the above equation that

$$a^2 - a_0^2 = b^2 - b_0^2 = c^2 - c_0^2 = -\frac{1}{3} F$$

where $a_0 := a(0)$, $b_0 := b(0)$ and $c_0 := c(0)$. Analogously, $f(0)$ will from now on be denoted by f_0. We rewrite the fourth equation of (5.6):

$$f' = -\frac{1}{6}\frac{f^2}{\frac{1}{3}F - a_0^2} - \frac{1}{6}\frac{f^2}{\frac{1}{3}F - b_0^2} - \frac{1}{6}\frac{f^2}{\frac{1}{3}F - c_0^2} - 3\,.$$

By replacing t by $\tau := F(t)$, the four equations simplify to:

$$\begin{aligned} \dot a &= -\frac{1}{6a} \quad \dot b = -\frac{1}{6b} \quad \dot c = -\frac{1}{6c} \\ f\dot f &= -\frac{f^2}{2\tau - 6a_0^2} - \frac{f^2}{2\tau - 6b_0^2} - \frac{f^2}{2\tau - 6c_0^2} - 3 \end{aligned}$$

where the dot denotes the derivation with respect to τ. The last equation is linear in f^2 and can be solved explicitly. We first consider the homogeneous problem and obtain:

$$f(\tau)^2 = C\frac{1}{(\tau - 3a_0^2)(\tau - 3b_0^2)(\tau - 3c_0^2)}\,.$$

By variation of constants, we obtain the following solution of the inhomogeneous equation:

$$f(\tau)^2 = \frac{1}{(\tau - 3a_0^2)(\tau - 3b_0^2)(\tau - 3c_0^2)}\left(C - 3\int_0^\tau (s - 3a_0^2)(s - 3b_0^2)(s - 3c_0^2)ds\right)\,.$$

The constant C is fixed by the initial condition $f(0) = f_0$ at the value of $-27a_0^2b_0^2c_0^2f_0^2$. Therefore, we finally obtain:

$$\begin{aligned} f(\tau)^2 = &\frac{f_0^2}{(1 - \frac{\tau}{3a_0^2})(1 - \frac{\tau}{3b_0^2})(1 - \frac{\tau}{3c_0^2})} \\ &- \frac{3}{(\tau - 3a_0^2)(\tau - 3b_0^2)(\tau - 3c_0^2)}\int_0^\tau (s - 3a_0^2)(s - 3b_0^2)(s - 3c_0^2)ds\,. \end{aligned} \tag{5.7}$$

The resulting cohomogeneity-one metric is given by:

$$\begin{aligned} g = &(-\tfrac{1}{3}\tau + a_0^2)(e^1 \otimes e^1 + e^2 \otimes e^2) + (-\tfrac{1}{3}\tau + b_0^2)(e^3 \otimes e^3 + e^4 \otimes e^4) \\ &+ (-\tfrac{1}{3}\tau + c_0^2)(e^5 \otimes e^5 + e^6 \otimes e^6) + f(\tau)^2(e^7 \otimes e^7 + d\tau^2)\,. \end{aligned} \tag{5.8}$$

We will check the compactness and completeness of the metrics (5.8) later on, when we have determined the possible initial values at the singular orbit.

The next question we will consider is if there are other parallel Spin(7)-structures with principal orbit $Q^{1,1,1}$ which are not described by the frame (5.4) and the equations (5.6). According to Theorem 3.2.31, a parallel Spin(7)-structure of cohomogeneity one is determined by the G_2-structure on a fixed principal orbit. We will therefore classify all $SU(2)^3$-invariant G_2-structures on $Q^{1,1,1}$ which are compatible with an arbitrary but fixed metric and orientation. Because of Lemma 3.1.50, the space of those G_2-structures is described by:

$$\mathrm{Norm}_{SO(7)} U(1)^2 / \mathrm{Norm}_{G_2} U(1)^2 \,,$$

where $U(1)^2$ can be chosen as the maximal torus of G_2 with (5.1) as Lie algebra. Whenever we consider on the following pages a Lie subgroup $U(1)^2 \subseteq SO(7)$ without specifying the embedding, $U(1)^2$ shall be given by the maximal torus of $G_2 \subseteq SO(7)$. Analogously, $2\mathfrak{u}(1) \subseteq \mathfrak{so}(7)$ will denote the Cartan subalgebra of $\mathfrak{g}_2 \subseteq \mathfrak{so}(7)$. In order to describe the space of invariant G_2-structures, we first consider the Lie algebra of $\mathrm{Norm}_{SO(7)} U(1)^2$. Since $U(1)^2$ is connected, we have:

$$\mathrm{Norm}_{SO(7)} U(1)^2 = \{h \in SO(7) | h \, 2\mathfrak{u}(1) \, h^{-1} = 2\mathfrak{u}(1)\} \,.$$

Therefore, the Lie algebra of the normalizer is

$$\mathrm{Norm}_{\mathfrak{so}(7)} 2\mathfrak{u}(1) := \{x \in \mathfrak{so}(7) | \mathrm{ad}_x(2\mathfrak{u}(1)) = 2\mathfrak{u}(1)\} \,.$$

We denote the Cartan subalgebra

$$\left\{ \left(\begin{array}{ccccccc} \boxed{0} & & & & & & \\ & 0 & a & & & & \\ & -a & 0 & & & & \\ & & & 0 & b & & \\ & & & -b & 0 & & \\ & & & & & 0 & c \\ & & & & & -c & 0 \end{array} \right) \middle| \; a, b, c \in \mathbb{R} \right\} \tag{5.9}$$

of $\mathfrak{so}(7)$ by $3\mathfrak{u}(1)$. Since we will not work with the Cartan subalgebra of $3\mathfrak{su}(2)$ on the next pages, the notation "$3\mathfrak{u}(1)$" is non-ambiguous. Obviously, $3\mathfrak{u}(1) \subseteq \mathrm{Norm}_{\mathfrak{so}(7)} 2\mathfrak{u}(1)$. We will show that "$\supseteq$" is also true. Let $x \in \mathrm{Norm}_{\mathfrak{so}(7)} 2\mathfrak{u}(1)$ be arbitrary and let κ be the Killing form of $\mathfrak{so}(7)$. x has to satisfy

$$[x, z] \in 2\mathfrak{u}(1) \quad \forall z \in 2\mathfrak{u}(1) \,.$$

Since κ is associative, we have for any $y \in 2\mathfrak{u}(1)$:

$$\kappa([x, z], y) = \kappa(x, [z, y]) = 0$$

and therefore $[x, z] = 0$. This consideration proves that the normalizer equals the centralizer

$$C_{\mathfrak{so}(7)} \, 2\mathfrak{u}(1) := \{x \in \mathfrak{so}(7) | \mathrm{ad}_x(2\mathfrak{u}(1)) = \{0\}\} \,.$$

For the following considerations, we complexify all Lie algebras and return to the real case later on. Our x can be considered as an element of $\mathrm{Norm}_{\mathfrak{so}(7,\mathbb{C})}(2\mathfrak{u}(1) \otimes \mathbb{C})$. x has a Cartan decomposition

$$x = x_h + \sum_{\alpha \in \Phi} \mu_\alpha g_\alpha \quad \text{with} \quad x_h \in 3\mathfrak{u}(1) \otimes \mathbb{C}\,, \mu_\alpha \in \mathbb{C}\,, g_\alpha \in L_\alpha\,, \tag{5.10}$$

where Φ is the root system of $\mathfrak{so}(7,\mathbb{C})$ and L_α is the eigenspace to the eigenvalue $\alpha : 3\mathfrak{u}(1) \otimes \mathbb{C} \to \mathbb{C}$ of $\mathrm{ad}_{3\mathfrak{u}(1)\otimes\mathbb{C}}$. Let $z \in 2\mathfrak{u}(1) \otimes \mathbb{C}$ be arbitrary. Applying ad_z to (5.10) yields the following equation:

$$\sum_{\alpha \in \Phi} \mu_\alpha \alpha(z) g_\alpha = 0\,.$$

We want to prove that the centralizer is $3\mathfrak{u}(1) \otimes \mathbb{C}$, or equivalently that there is no non-zero x in the orthogonal complement of $3\mathfrak{u}(1) \otimes \mathbb{C}$ which commutes with $2\mathfrak{u}(1) \otimes \mathbb{C}$. If there exists an $\alpha \in \Phi$ with $\alpha(2\mathfrak{u}(1) \otimes \mathbb{C}) = 0$, then $[z, g_\alpha] = \alpha(z) g_\alpha = 0$ for all $z \in 2\mathfrak{u}(1) \otimes \mathbb{C}$. Conversely, if there is no such α, we can easily prove for all $(3\mathfrak{u}(1) \otimes \mathbb{C})^\perp \backslash \{0\} \ni x = \sum_{\alpha \in \Phi} \mu_\alpha g_\alpha$ that $\mathrm{ad}_z(x) \neq 0$ if $z \in 2\mathfrak{u}(1) \otimes \mathbb{C}$ satisfies $\alpha(z) \neq 0$ for an α with $\mu_\alpha \neq 0$. In order to answer the question if there is an α with $\alpha(2\mathfrak{u}(1) \otimes \mathbb{C}) = 0$ we have to take a closer look at the root system of $\mathfrak{so}(7,\mathbb{C})$:

In Fulton, Harris [35], the complex Lie algebra $\mathfrak{so}(7,\mathbb{C})$ is defined as the set of all endomorphisms which are skew-symmetric with respect to a bilinear form Q. The matrix representation of Q with respect to the standard basis of $\mathbb{C}^7$ is given by:

$$\begin{pmatrix} & & & 1 & 0 & 0 & \\ & & & 0 & 1 & 0 & \\ & & & 0 & 0 & 1 & \\ 1 & 0 & 0 & & & & \\ 0 & 1 & 0 & & & & \\ 0 & 0 & 1 & & & & \\ & & & & & & 1 \end{pmatrix}.$$

It is easily possible to make a change of the basis such that the matrix representation of Q becomes the identity matrix. Fulton and Harris take the following basis of the Cartan subalgebra of $\mathfrak{so}(7,\mathbb{C})$:

$$\begin{aligned} L_1 &:= \mathrm{diag}(1,0,0,-1,0,0,0) \\ L_2 &:= \mathrm{diag}(0,1,0,0,-1,0,0) \\ L_3 &:= \mathrm{diag}(0,0,1,0,0,-1,0) \end{aligned}$$

With respect to our new basis of $\mathbb{C}^7$, the matrix representation of L_1, L_2, and L_3 changes: If we set in (5.9) $a = i$ and $b = c = 0$, we obtain the first element L_1 of the Cartan subalgebra. Analogously, the second element L_2 is determined by $b = i$, $a = c = 0$, and the third element L_3 by $c = i$ and $a = b = 0$. Unfortunately, these matrices are not contained in the real Lie algebra $\mathfrak{so}(7)$. Nevertheless, we obtain the same root system if we consider $\mathfrak{so}(7,\mathbb{C})$ together with the complex basis (L_1, L_2, L_3) of $3\mathfrak{u}(1) \otimes \mathbb{C}$. Therefore, we can work with the root system which is described in [35]. This root system is given by:

$$\{\pm\theta_j | 1 \leqslant j \leqslant 3\} \cup \{\pm\theta_j \pm \theta_k | 1 \leqslant j < k \leqslant 3\} ,$$

where θ_j is the dual of L_j with respect to the Killing form. We redefine L_j and θ_j as $-i$ times the old basis elements L_j and θ_j, since this does not change our equations but guarantees that L_j is contained in the real Lie algebra $\mathfrak{so}(7)$. The Cartan subalgebra $2\mathfrak{u}(1) \otimes \mathbb{C}$ of $\mathfrak{g}_2^{\mathbb{C}} \subseteq \mathfrak{so}(7, \mathbb{C})$ is the plane which is orthogonal to $L_1 + L_2 - L_3$ (see page 106). The $\alpha \in \Phi$ which vanish on $2\mathfrak{u}(1) \otimes \mathbb{C}$ are precisely those which are multiples of $\theta_1 + \theta_2 - \theta_3$. Since there is no root of $\mathfrak{so}(7, \mathbb{C})$ with this property, we have proven that indeed $\mathrm{Norm}_{\mathfrak{so}(7,\mathbb{C})}(2\mathfrak{u}(1) \otimes \mathbb{C}) = 3\mathfrak{u}(1) \otimes \mathbb{C}$. By passing to the compact real form of $\mathfrak{so}(7, \mathbb{C})$, we can conclude that $\mathrm{Norm}_{\mathfrak{so}(7)} 2\mathfrak{u}(1) = 3\mathfrak{u}(1)$.

Let $U(1)^3$ be the maximal torus of $SO(7)$ with Lie algebra $3\mathfrak{u}(1)$. Our next step is to describe the discrete group $\Gamma := (\mathrm{Norm}_{SO(7)} U(1)^2)/U(1)^3$. In order to do this, we first consider the group $(\mathrm{Norm}_{SO(7)} U(1)^3)/U(1)^3$, which is isomorphic to the Weyl group $\mathcal{W}_{\mathfrak{so}(7)}$ of $\mathfrak{so}(7)$. This isomorphism is given by $xU(1)^3 \mapsto \mathrm{Ad}^*_x|_{3\mathfrak{u}(1)^*} : 3\mathfrak{u}(1)^* \to 3\mathfrak{u}(1)^*$, where Ad^* is the coadjoint action of $SO(7)$. This is a general statement, which is true for any semisimple Lie algebra and its Cartan subalgebra.

We will show that $\mathrm{Norm}_{SO(7)} U(1)^2 \subseteq \mathrm{Norm}_{SO(7)} U(1)^3$. Let us assume that there is an element h of $SO(7)$ such that Ad_h leaves $2\mathfrak{u}(1)$ invariant, but does not leave $3\mathfrak{u}(1)$ invariant. Then $3\mathfrak{u}(1)$ and $\mathrm{Ad}_h(3\mathfrak{u}(1))$ are two distinct Cartan subalgebras whose intersection is $2\mathfrak{u}(1)$. In this situation, the centralizer of $2\mathfrak{u}(1)$ is at least four-dimensional which is not the case. The quotient Γ we want to determine therefore is a subgroup of the Weyl group. More precisely, it is the subgroup of the Weyl group which leaves the plane $2\mathfrak{u}(1) \subseteq 3\mathfrak{u}(1)$ invariant. In order to describe this group, we have to introduce a few facts on the Weyl group of $\mathfrak{so}(7)$. $\mathcal{W}_{\mathfrak{so}(7)}$ is isomorphic to $\mathbb{Z}_2^3 \rtimes S_3$, which is of order 48. The $\mathbb{Z}_2^3$-factor of $\mathcal{W}_{\mathfrak{so}(7)}$ acts by changing the signs of the θ_i. The S_3-factor of the Weyl group consists of the permutations of $\{\theta_1, \theta_2, \theta_3\}$. The Cartan subalgebra $2\mathfrak{u}(1)$ of $\mathfrak{g}_2$ is the plane of all $x \in 3\mathfrak{u}(1)$ satisfying:

$$\theta_1(x) + \theta_2(x) - \theta_3(x) = 0 .$$

By replacing L_3 by $-L_3$ in the basis of $3\mathfrak{u}(1)$, we can change this equation to:

$$\theta_1(x) + \theta_2(x) + \theta_3(x) = 0 .$$

The subgroup of $\mathcal{W}_{\mathfrak{so}(7)}$ which leaves this equation invariant is generated by the permutations and the global change of all signs. Therefore, Γ is a semidirect product $\mathbb{Z}_2 \rtimes S_3$. It is easy to see that Γ is in fact the Dieder group D_6. All in all, we have proven:

$$\mathrm{Norm}_{SO(7)} U(1)^2 = U(1)^3 \rtimes D_6 .$$

Next, we have to determine $\mathrm{Norm}_{G_2} U(1)^2$. Since $2\mathfrak{u}(1)$ is a Cartan subalgebra of $\mathfrak{g}_2$, $\mathrm{Norm}_{G_2} U(1)^2 / U(1)^2$ is the Weyl group $\mathcal{W}_{\mathfrak{g}_2}$ of $\mathfrak{g}_2$. It is known that $\mathcal{W}_{\mathfrak{g}_2}$ is isomorphic to D_6. The group $\mathrm{Norm}_{G_2} U(1)^2$ therefore is a semidirect product $U(1)^2 \rtimes D_6$. There is the following exact sequence:

$$\pi_0(\mathrm{Norm}_{G_2} U(1)^2) \xrightarrow{\pi_0(i)} \pi_0(\mathrm{Norm}_{SO(7)} U(1)^2) \xrightarrow{\pi_0(\pi)} \pi_0(\mathrm{Norm}_{SO(7)} U(1)^2 / \mathrm{Norm}_{G_2} U(1)^2) \to \{0\} ,$$

where $\pi_0(i)$ is the map which is induced by the inclusion of $\mathrm{Norm}_{G_2}U(1)^2$ into $\mathrm{Norm}_{SO(7)}U(1)^2$ and $\pi_0(\pi)$ is induced by the projection map. $\mathrm{Norm}_{G_2}U(1)^2$ has 12 connected components and $\mathrm{Norm}_{SO(7)}U(1)^2$ has 12 connected components, too. If we were able to prove that $\pi_0(i)$ is surjective, we could conclude that $\mathrm{Norm}_{SO(7)}U(1)^2/\mathrm{Norm}_{G_2}U(1)^2$ is connected. Since $\pi_0(\mathrm{Norm}_{SO(7)}U(1)^2)$ and $\pi_0(\mathrm{Norm}_{G_2}U(1)^2)$ are both finite, we can equivalently prove the injectivity of $\pi_0(i)$. Let $x \in \mathrm{Norm}_{G_2}U(1)^2$ with $x \notin U(1)^2$ be arbitrary. Then $x = \alpha x_0$ with $\alpha \in \mathcal{W}_{\mathfrak{g}_2}\backslash\{e\}$ and $x_0 \in U(1)^2$. Since α acts non-trivially on the dual of the Cartan subalgebra $2\mathfrak{u}(1)$, it cannot be an element of the maximal torus $U(1)^3$ of $SO(7)$. Therefore, x is not an element of the identity component of $\mathrm{Norm}_{SO(7)}U(1)^2$. This consideration proves that $\pi_0(i)$ is injective and $\mathrm{Norm}_{SO(7)}U(1)^2/\mathrm{Norm}_{G_2}U(1)^2$ therefore is connected. Since the identity component of $\mathrm{Norm}_{SO(7)}U(1)^2$ is $U(1)^3$ and the identity component of $\mathrm{Norm}_{G_2}U(1)^2$ is $U(1)^2$, their quotient is isomorphic to $U(1)$.

The Weyl group of $\mathfrak{so}(7,\mathbb{C})$ (or $\mathfrak{g}_2^{\mathbb{C}}$) leaves the Cartan subalgebra of the compact real form $\mathfrak{so}(7)$ (or $\mathfrak{g}_2$) invariant. For this reason, there was no need to carry out the above calculations first on the complex and then on the real level. Instead, we were able to work directly with the real Lie algebras and groups.

$\mathrm{Norm}_{SO(7)}U(1)^2/\mathrm{Norm}_{G_2}U(1)^2$ can be represented by the subgroup of $U(1)^3$ whose Lie algebra is the orthogonal complement of $2\mathfrak{u}(1) \subseteq 3\mathfrak{u}(1)$. This group is given by:

$$\left\{ \begin{pmatrix} \boxed{1} & & & \\ & \boxed{R_\theta} & & \\ & & \boxed{R_\theta} & \\ & & & \boxed{R_\theta} \end{pmatrix} \;\middle|\; \theta \in \mathbb{R} \right\} \subseteq GL(\mathrm{Im}(\mathbb{O}))\,,$$

where R_θ denotes the rotation in the plane around the angle θ. In order to construct the set of invariant G_2-structures which are associated to a given metric and orientation, we have to describe how the above group acts on $\mathfrak{m}$. We define:

$$T := \left\{ \begin{pmatrix} \boxed{R_\theta} & & & \\ & \boxed{R_\theta} & & \\ & & \boxed{R_\theta} & \\ & & & \boxed{1} \end{pmatrix} =: T_\theta \;\middle|\; \theta \in \mathbb{R} \right\} \subseteq GL(\mathfrak{m})\,.$$

In the above formula, the matrix representation of T_θ is with respect to the basis $(e_1, \ldots, e_7)$. T commutes with the action of $U(1)^2_{1,1,1}$ on $\mathfrak{m}$. Furthermore, T_θ is orthogonal with respect to g and orientation preserving. Therefore, T describes the action of $(\mathrm{Norm}_{SO(7)}U(1)^2)/(\mathrm{Norm}_{G_2}U(1)^2)$ on $\mathfrak{m}$ and its action on a given invariant G_2-structure determines the set of all invariant G_2-structures. We consider the following subgroup of $\mathrm{Norm}_{SU(2)^3}U(1)^2$:

$$S := \left\{ \begin{pmatrix} \boxed{\begin{matrix} e^{i\frac{\theta}{2}} & 0 \\ 0 & e^{-i\frac{\theta}{2}} \end{matrix}} & & \\ & \boxed{\begin{matrix} e^{i\frac{\theta}{2}} & 0 \\ 0 & e^{-i\frac{\theta}{2}} \end{matrix}} & \\ & & \boxed{\begin{matrix} e^{i\frac{\theta}{2}} & 0 \\ 0 & e^{-i\frac{\theta}{2}} \end{matrix}} \end{pmatrix} =: S_\theta \;\middle|\; \theta \in \mathbb{R} \right\} . \tag{5.11}$$

Conjugation by S_θ is a well-defined diffeomorphism of $Q^{1,1,1}$. It is easy to see that it is even an orientation preserving isometry. By a short calculation we see that the differential of this map acts as T_θ on $\mathfrak{m}$. We sum up our results: Let ω be a $SU(2)^3$-invariant G_2-structure on $Q^{1,1,1}$. Any other invariant G_2-structure on $Q^{1,1,1}$ with the same associated metric and orientation as ω can be obtained by the action of S on ω. Furthermore, this action is isometric.

We are now able to determine the holonomy of the metrics which we obtain as solutions of (5.6): Let M be a cohomogeneity-one manifold with principal orbit $Q^{1,1,1}$, and let Ω be a parallel Spin(7)-structure which is described by (5.5) and (5.6). The union of all principal orbits we denote by M^o. We choose a diffeomorphism such that M^o is identified with $Q^{1,1,1} \times I$ where I is an interval. Conjugation by S_θ on all $Q^{1,1,1} \times \{t\}$ is an isometry of M^o. The action of S_θ maps Ω into another Spin(7)-structure, which is parallel with respect to the Levi-Civita connection, too. We therefore have found a one-parameter family of parallel Spin(7)-structures whose extension to a $SO(8)$-structure is the same. Furthermore, this family is diffeomorphic to a circle. Because of Lemma 2.3.17, the holonomy of the metric is contained in $SU(4)$. From the classification of the possible holonomy groups of a parallel Spin(7)-manifold, it follows that the holonomy is contained in $SU(3)$ or is one of the groups $SU(4)$, $Sp(2)$, or $SU(2) \times SU(2)$. In the first case, there exists a parallel vector field on the manifold, since $SU(3)$ acts trivially on a subspace of the tangent space. The holonomy bundle of a Riemannian manifold is invariant with respect to isometries. Since the Spin(7)-structure we have found is invariant, too, the extension of the holonomy bundle to a $SU(3)$-structure has to be preserved by the action of $SU(2)^3$. Therefore, we can assume that the parallel vector field we search for is invariant. If there exists a parallel vector field X on the manifold, it has to be of type $c_1 \frac{1}{f} e_7 + c_2 \frac{\partial}{\partial t}$, where c_1 and c_2 depend on t only. We first consider the case where $c_1 \not\equiv 0$. Since $\nabla_{\frac{\partial}{\partial t}} X = 0$ and the length of X is constant, the functions c_1, c_2, and f have to be constant, too. We can conclude from the equations (5.6) that a^2, b^2, and c^2 are strictly monotonous. More precisely, either all of them are strictly increasing or all of them are strictly decreasing. In any case, the right hand side of the fourth equation of (5.6) cannot be 0 for all values of t. This is a contradiction to f being constant. Next, we assume that $c_1 = 0$ and $c_2 = 1$. If $\frac{\partial}{\partial t}$ is parallel, we have $a' = b' = c' = f' = 0$, which is impossible, too. Therefore, the holonomy has to be $SU(4)$, $Sp(2)$, or $SU(2) \times SU(2)$ and the metric thus is Kähler.

Since $GL(4, \mathbb{C}) \cap \mathrm{Spin}(7) = SU(4)$, we only have to describe the complex structure J or equivalently the Kähler form η in order to determine the $SU(4)$-structure. For the same reasons as above, the $SU(4)$-structure and η have to be invariant. The two-forms

$$e^{12}\,, \quad e^{34}\,, \quad e^{56}\,, \quad e^7 \wedge dt$$

span the space of all invariant two-forms on $Q^{1,1,1}$. Let $(f_0, \dots, f_7)$ be a frame of the $SU(4)$-structure. Since that basis has to be orthonormal and we have $\eta(f_{2i}, f_{2i+1}) = 1$ for $i = 0, \dots, 3$, the Kähler form has to satisfy:

$$\eta = \epsilon_1 a^2 e^{12} + \epsilon_2 b^2 e^{34} + \epsilon_3 c^2 e^{56} + f e^7 \wedge dt \quad \text{with} \quad \epsilon_1, \epsilon_2, \epsilon_3 \in \{-1, 1\}\,. \tag{5.12}$$

With help of the exterior derivatives of the one-forms which we have calculated in Appendix A, we obtain for the exterior derivative of η:

$$\begin{aligned}
d\eta = & -(f(-\frac{1}{3}e^{12} - \frac{1}{3}e^{34} - \frac{1}{3}e^{56}) \wedge dt) \\
& + \epsilon_1\, 2a'a\, dt \wedge e^{12} - \epsilon_1\, a^2((-e^{27}) \wedge e^2 - e^1 \wedge e^{17}) \\
& + \epsilon_2\, 2b'b\, dt \wedge e^{34} - \epsilon_2\, b^2((-e^{47}) \wedge e^4 - e^3 \wedge e^{37}) \\
& + \epsilon_3\, 2c'c\, dt \wedge e^{56} - \epsilon_3\, c^2((-e^{67}) \wedge e^6 - e^5 \wedge e^{57}) \\
= & \frac{1}{3} f\, e^{12} \wedge dt + \epsilon_1 2a'a\, e^{12} \wedge dt \\
& + \frac{1}{3} f\, e^{34} \wedge dt + \epsilon_2 2b'b\, e^{34} \wedge dt \\
& + \frac{1}{3} f\, e^{56} \wedge dt + \epsilon_3 2c'c\, e^{56} \wedge dt
\end{aligned}$$

The metric g therefore has to satisfy the following equations:

$$f = -6\epsilon_1 a'a = -6\epsilon_2 b'b = -6\epsilon_3 c'c\,. \tag{5.13}$$

By comparing (5.13) with the equations (5.6), we see that we have $\epsilon_1 = \epsilon_2 = \epsilon_3 = 1$ and obtain:

$$\eta = a^2 e^{12} + b^2 e^{34} + c^2 e^{56} + f e^7 \wedge dt\,. \tag{5.14}$$

Since we already know that the $SU(4)$-structure is parallel, we do not have to prove that J is integrable, or that $\nabla\eta = 0$.

We now prove that the holonomy of the metrics which satisfy the equations (5.6) is all of $SU(4)$. Let us assume that the holonomy is a subgroup of $Sp(2)$. If this is the case, the metric is hyperkähler and there exists another symplectic form η' with the same properties as η such that (η, η') is linearly independent. η' has to be of type (5.12), too. Therefore, the equations (5.13) have to be true for another choice of the signs ϵ_i. It easily follows that $f = 0$, which cannot be the case.

We consider the behavior of the metric near a singular orbit. The following lemma yields the possible dimensions of the collapsing spheres:

Lemma 5.2.1. *Let $U(1)^2_{1,1,1} \subseteq SU(2)^3$ be chosen as in the beginning of this section. As usual, we denote this subgroup shortly by $U(1)^2$ and its Lie algebra by $2\mathfrak{u}(1)$. Furthermore, let K be a connected, closed group with $U(1)^2 \subsetneq K \subseteq SU(2)^3$. We denote the Lie algebra of K by $\mathfrak{k}$. In this situation, $\mathfrak{k}$ and K can be found in the table below. Furthermore, $K/U(1)^2$ and $SU(2)^3/K$ satisfy the following topological conditions:*

$\mathfrak{k}$	K	$K/U(1)^2$	$SU(2)^3/K$
$3\mathfrak{u}(1)$	$U(1)^3$	$\cong S^1$	$\cong S^2 \times S^2 \times S^2$
$2\mathfrak{u}(1) \oplus \mathfrak{su}(2)$	$U(1)^2 \times SU(2)$	$\cong S^3$	$\cong S^2 \times S^2$
$\mathfrak{u}(1) \oplus 2\mathfrak{su}(2)$	$U(1) \times SU(2)^2$	$\not\cong S^5/\Gamma$	$\cong S^2$
$3\mathfrak{su}(2)$	$SU(2)^3$	$= Q^{1,1,1} \not\cong S^7/\Gamma$	

where Γ is an arbitrary discrete subgroup of the orthogonal group.

Proof : $\mathfrak{k}$ is obviously a $2\mathfrak{u}(1)$-module. Since we have decomposed $\mathfrak{m}$ into its irreducible $2\mathfrak{u}(1)$-submodules, we are able to list all possibilities for $\mathfrak{k}$. We consider each case separately:

1. $\mathfrak{k} = 2\mathfrak{u}(1) \oplus V_4$: In this case, $\mathfrak{k}$ is a Lie algebra isomorphic to $3\mathfrak{u}(1)$. The corresponding Lie group K is $U(1)^3$. $K/U(1)^2$ therefore is a circle. For the quotient $SU(2)^3/K$ we obtain $SU(2)/U(1) \times SU(2)/U(1) \times SU(2)/U(1)$, which is due to the Hopf fibration diffeomorphic to $S^2 \times S^2 \times S^2$.

2. $\mathfrak{k} = 2\mathfrak{u}(1) \oplus V_i$ with $i \in \{1, 2, 3\}$: We have $[e_{2i-1}, e_{2i}] \in 3\mathfrak{u}(1)$, but $[e_{2i-1}, e_{2i}] \notin 2\mathfrak{u}(1)$. Therefore, $\mathfrak{k}$ is not closed under the Lie bracket and we can exclude this case.

3. $\mathfrak{k} = 2\mathfrak{u}(1) \oplus V_i \oplus V_4$ with $i \in \{1, 2, 3\}$. Without loss of generality, we assume that $i = 1$. $\mathfrak{k}$ is a Lie algebra isomorphic to $2\mathfrak{u}(1) \oplus \mathfrak{su}(2)$. In order to describe $K/U(1)^2$ explicitly, we first have to describe K and $U(1)^2$ in more detail: Let S_θ be the matrix

$$\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}.$$

We have:

$$K = \left\{ \begin{pmatrix} \boxed{A} & & \\ & \boxed{S_\phi} & \\ & & \boxed{S_\psi} \end{pmatrix} \middle| \; A \in SU(2),\ \phi, \psi \in [0, 2\pi) \right\},$$

and

$$U(1)^2 = \left\{ \begin{pmatrix} \boxed{S_{-\psi-\phi}} & & \\ & \boxed{S_\phi} & \\ & & \boxed{S_\psi} \end{pmatrix} \middle| \; \phi, \psi \in [0, 2\pi) \right\}.$$

It is easy to see that an element of $U(1)^2$ can only be an element of the semisimple part of K if $\phi = \psi = 0$. By applying Lemma 3.1.12 twice, it follows that $K/U(1)^2 = (SU(2) \times U(1)^2)/U(1)^2 = (SU(2) \times U(1))/U(1) = SU(2) = S^3$. Finally, we can deduce from the above description of K that $SU(2)^3/K$ is diffeomorphic to $S^2 \times S^2$.

4. $\mathfrak{k} = 2\mathfrak{u}(1) \oplus V_i \oplus V_j$ with $i, j \in \{1,2,3\}$ and $i \neq j$: Analogously to the second case, $\mathfrak{k}$ is not closed under the Lie bracket.

5. $\mathfrak{k} = 2\mathfrak{u}(1) \oplus V_i \oplus V_j \oplus V_4$ with V_i and V_j as above: We can assume without loss of generality that $i = 1$ and $j = 2$. The group K is isomorphic to $SU(2) \times SU(2) \times U(1)$. As in the third case, we apply Lemma 3.1.12. In order to do this, we consider the following two subgroups of $U(1)^2$:

$$U(1)_1 := \left\{ \begin{pmatrix} \boxed{S_{-\phi}} & & \\ & \boxed{S_\phi} & \\ & & \boxed{S_0} \end{pmatrix} \middle| \ \phi \in [0, 2\pi) \right\}$$

$$U(1)_2 := \left\{ \begin{pmatrix} \boxed{S_{-\psi}} & & \\ & \boxed{S_0} & \\ & & \boxed{S_\psi} \end{pmatrix} \middle| \ \psi \in [0, 2\pi) \right\}$$

Since $U(1)_1 \cap U(1)_2 = \{I\}$, the covering map from $U(1)_1 \times U(1)_2$ to $U(1)^2$ which maps (ϕ, ψ) to $\phi \cdot \psi$ is a diffeomorphism. An element of $U(1)_2$ can be in $SU(2) \times SU(2) \subseteq K$ only for $\psi = 0$. We therefore can rewrite $K/U(1)^2$ as $(SU(2) \times SU(2))/U(1)$, where $U(1)$ is given by:

$$\left\{ \begin{pmatrix} \boxed{S_{-\phi}} & \\ & \boxed{S_\phi} \end{pmatrix} \middle| \ \phi \in [0, 2\pi) \right\} .$$

We assume that $K/U(1)^2$ is a quotient of the sphere S^5 by a discrete subgroup Γ of $O(6)$. $S^3 \times S^3$ is a circle bundle over $K/U(1)^2$. If our assumption was true, we would have the following exact sequence:

$$\ldots \to \pi_3(S^1) \to \pi_3(S^3 \times S^3) \to \pi_3(S^5/\Gamma) \to \ldots .$$

Since the higher homotopy groups of S^5 and S^5/Γ coincide, the above exact sequence becomes:

$$\ldots \to \{0\} \to \mathbb{Z}^2 \to \{0\} \to \ldots .$$

which is impossible. This proves that $SU(2)^3/K$ is not a possible singular orbit for a smooth cohomogeneity-one manifold with principal orbit $Q^{1,1,1}$. We nevertheless note that this space is diffeomorphic to S^2 for similar reasons as in the above cases.

6. $\mathfrak{k} = 2\mathfrak{u}(1) \oplus V_1 \oplus V_2 \oplus V_3$: For the same reasons as in the second and the fourth case, $\mathfrak{k}$ is not closed under the Lie bracket.

7. $\mathfrak{k} = 2\mathfrak{u}(1) \oplus V_1 \oplus V_2 \oplus V_3 \oplus V_4$: In this case, $K/U(1)^2 = Q^{1,1,1}$. We assume that $Q^{1,1,1}$ is homeomorphic to S^7/Γ, where $\Gamma \subseteq O(8)$ is discrete. $Q^{1,1,1}$ can be described as a circle bundle over $SU(2)^3/U(1)^3 = (S^2)^3$. Therefore, we obtain the following exact sequence:

$$\ldots \to \pi_2(S^7/\Gamma) \to \pi_2((S^2)^3) \to \pi_1(S^1) \to \pi_1(S^7/\Gamma) \to \ldots ,$$

which can be explicitly written as

$$\ldots \to \{0\} \to \mathbb{Z}^3 \to \mathbb{Z} \to \Gamma \to \ldots .$$

Since there is no injective group homomorphism from $\mathbb{Z}^3$ to $\mathbb{Z}$, $Q^{1,1,1}$ is not homeomorphic to S^7/Γ.

□

Next, we investigate the geometric properties of the metrics, which satisfy the equations (5.6) and have a singular orbit. We start with a proof that none of those metrics is compact. Without loss of generality, we can assume that we have a singular orbit at $t = 0$. The only possibilities for the singular orbit are $S^2 \times S^2$ and $S^2 \times S^2 \times S^2$. In both cases, we have $f(0) = 0$. We will see below that in the first case we have $f'(0) = -\frac{3}{2}$ and in the second one we have $f'(0) = -3$. Therefore, f is negative near the singular orbit. If the manifold was compact, there would be two singular orbits. We denote the position of the second singular orbit by T. For the same reasons as above, we have $f(T) = 0$, $f'(T) \in \{-3, -\frac{3}{2}\}$, and f is positive near the second singular orbit. Therefore, there exists a $s \in (0, T)$ such that $f(s) = 0$. Since there are only two singular orbits, we have obtained a contradiction.

In order to check the completeness of our metrics, we have to find out if the geodesics which intersect all orbits perpendicularly are of infinite length and if the metric can be smoothly extended to the singular orbit. Since f is negative near the singular orbit and we have $f \neq 0$ outside of that orbit, $\tau := F(t)$ is negative, too. The initial value f_0 equals zero and the metric therefore is determined by:

$$f(\tau)^2 = -\frac{3}{(\tau - 3a_0^2)(\tau - 3b_0^2)(\tau - 3c_0^2)} \int_0^\tau (s - 3a_0^2)(s - 3b_0^2)(s - 3c_0^2)ds$$

$$a(\tau)^2 = -\tfrac{1}{3}\tau + a_0^2 \qquad b(\tau)^2 = -\tfrac{1}{3}\tau + b_0^2 \qquad c(\tau)^2 = -\tfrac{1}{3}\tau + c_0^2 \, .$$

If the singular orbit is $S^2 \times S^2 \times S^2$, all of the values of a_0, b_0, and c_0 are non-zero. If we have $S^2 \times S^2$ as singular orbit, one of those values has to vanish. Let I be the maximal interval on which we can define f. Since the formula for the metric contains the term $f(\tau)^2 d\tau^2$, we have to prove that $\int_I |f(\tau)| d\tau = \infty$. All zeros of $(s - 3a_0^2)(s - 3b_0^2)(s - 3c_0^2)$ are positive and we therefore have $I = (-\infty, 0)$. For negative values of τ, $f^2(\tau)$ behaves like $O(\tau)$. We conclude that $\int_I |f(\tau)| d\tau = \infty$ and thus have proven the first condition for the completeness.

Next, we will check if the solutions of (5.6) we have found satisfy the smoothness conditions from Theorem 3.2.18. We consider both possible singular orbits separately and start with $S^2 \times S^2 \times S^2$. In this case, we have $\mathfrak{k} = 3\mathfrak{u}(1)$ and the tangent space of the singular orbit can be identified with $\mathfrak{p} = V_1 \oplus V_2 \oplus V_3$. For the following arguments, we need some information on $S^2(\mathfrak{p})$. We will see soon that we only need to consider those $3\mathfrak{u}(1)$-submodules of $S^2(\mathfrak{p})$ which are trivial with respect to $2\mathfrak{u}(1)_{1,1,1}$. On page 107, we have classified the invariant metrics on $Q^{1,1,1}$. Since any of those metrics corresponds to a trivial $2\mathfrak{u}(1)_{1,1,1}$-submodule of $S^2(\mathfrak{m}) = S^2(\mathfrak{p} \oplus V_4)$, the maximal trivial submodule of $S^2(\mathfrak{p})$ decomposes into:

$$\begin{aligned}\operatorname{span}(e_1 \otimes e_1 + e_2 \otimes e_2) &\subseteq S^2(V_1)\,,\\ \operatorname{span}(e_3 \otimes e_3 + e_4 \otimes e_4) &\subseteq S^2(V_2)\,,\\ \operatorname{span}(e_5 \otimes e_5 + e_6 \otimes e_6) &\subseteq S^2(V_3)\,.\end{aligned}$$

We remark that all of these modules are not only trivial with respect to $2\mathfrak{u}(1)_{1,1,1}$ but also with respect to $3\mathfrak{u}(1)$. The normal space $\mathfrak{p}^\perp$ of the singular orbit has to be described as a $3\mathfrak{u}(1)$-module, too. Since $2\mathfrak{u}(1)_{1,1,1}$ is the isotropy algebra of the cohomogeneity-one action, it has to leave $\frac{\partial}{\partial t}$ invariant. The only two-dimensional $2\mathfrak{u}(1)$-module which contains invariant elements except 0 is the trivial one. Therefore, $\mathfrak{p}^\perp$ is a trivial $2\mathfrak{u}(1)_{1,1,1}$-module, too. At the beginning of this section, we have introduced the notation $i_{1,1,1}(\mathfrak{u}(1))$ for the orthogonal complement of $2\mathfrak{u}(1)_{1,1,1} \subseteq 3\mathfrak{u}(1)$. As on page 114, we denote the connected Lie subgroup of $SU(2)^3$ with Lie algebra $i_{1,1,1}(\mathfrak{u}(1))$ by S. We will need the weight of the action of S on $\mathfrak{p}^\perp$, too. The orbit of a point in $K/U(1)^2_{1,1,1}$ with respect to the action of S is a loop in that space. Since $\pi_1(K/U(1)^2_{1,1,1}) = \pi_1(S^1) = \mathbb{Z}$, the homotopy class of that loop corresponds to an integer. For geometric reasons, it coincides with the weight of the action of S on $\mathfrak{p}^\perp$. By a short calculation, we see that

$$S \cap U(1)^2_{1,1,1} \cong \left\{ \left(\begin{array}{cccccc} e^{\frac{2\pi ik}{3}} & 0 & & & & \\ 0 & e^{-\frac{2\pi ik}{3}} & & & & \\ & & e^{\frac{2\pi ik}{3}} & 0 & & \\ & & 0 & e^{-\frac{2\pi ik}{3}} & & \\ & & & & e^{\frac{2\pi ik}{3}} & 0 \\ & & & & 0 & e^{-\frac{2\pi ik}{3}} \end{array} \right) \;\middle|\; k \in \mathbb{Z} \right\} \cong \mathbb{Z}_3\,.$$

Therefore, the winding number of the S-orbit is 3 and the weight of the S-action on $\mathfrak{p}^\perp$ equals 3, too. We denote the irreducible $3\mathfrak{u}(1)$-module on which the group $U(1)_1$ from the proof of Lemma 5.2.1 acts with weight r, $U(1)_2$ with weight s and S with weight t by $\mathbb{V}_{r,s,t}$. All of those spaces have complex dimension 1 except $\mathbb{V}_{0,0,0}$ which is real and one-dimensional. In contrast to Chapter 4, we will not index those spaces by a $\mathbb{R}$ or $\mathbb{C}$. We decompose $S^m(\mathfrak{p}^\perp)$, which can be written as $S^m(\mathbb{V}_{0,0,3})$, into irreducible submodules and obtain:

$$S^m(\mathbb{V}_{0,0,3}) = \begin{cases} \mathbb{V}_{0,0,3m} \oplus \mathbb{V}_{0,0,3(m-2)} \oplus \ldots \oplus \mathbb{V}_{0,0,0} & \text{if } m \text{ is even} \\ \mathbb{V}_{0,0,3m} \oplus \mathbb{V}_{0,0,3(m-2)} \oplus \ldots \oplus \mathbb{V}_{0,0,3} & \text{if } m \text{ is odd} \end{cases}$$

The embedding of the summands into $S^m(\mathbb{V}_{0,0,3})$ can be described as follows: We identify $\mathbb{V}_{0,0,3}$ with $\mathbb{C}$. The homogeneous polynomial $\mathrm{Re}(z)^2 + \mathrm{Im}(z)^2$ of degree 2 is K-invariant. The module $\mathbb{V}_{0,0,3k}$ with $k \geqslant 2$ can be identified with the orthogonal complement of $|z|^2 \cdot S^{k-2}(\mathbb{V}_{0,0,3})$ in $S^k(\mathbb{V}_{0,0,3})$. Therefore, the embedding of $\mathbb{V}_{0,0,3k}$ into $S^m(\mathbb{V}_{0,0,3})$ with $m \geqslant k$ can be described by $p \mapsto |z|^{2(m-k)}p$.

The dimensions of the spaces $W^v_m = \mathrm{Hom}_{3\mathfrak{u}(1)}(S^m(\mathfrak{p}^\perp), S^2(\mathfrak{p}^\perp))$ and $W^h_m = \mathrm{Hom}_{3\mathfrak{u}(1)}(S^m(\mathfrak{p}^\perp), S^2(\mathfrak{p}))$ easily follow from Schur's lemma. We have $\dim \mathrm{Hom}_{3\mathfrak{u}(1)} \mathbb{V}_{0,0,0} = 1$. If $(r,s,t) \neq$

$(0,0,0)$, we have $\dim \mathrm{Hom}_{3\mathfrak{u}(1)} \mathbb{V}_{r,s,t} = 2$, since $\mathbb{V}_{r,s,t}$ is a complex space. $S^m(\mathfrak{p}^\perp)$ is a trivial $2\mathfrak{u}(1)_{1,1,1}$-module and thus we only have to take care of the trivial summands of $S^2(\mathfrak{p})$. We put our results together and obtain:

$$\dim W_m^h = \begin{cases} 3 & \text{if } m \text{ is even,} \\ 0 & \text{if } m \text{ is odd.} \end{cases} \qquad \dim W_m^v = \begin{cases} 1 & \text{if } m = 0, \\ 0 & \text{if } m \text{ is odd,} \\ 3 & \text{if } m \geqslant 2 \text{ is even.} \end{cases} \tag{5.15}$$

These dimensions we have to interpret. The horizontal part of the metric is determined by the functions a^2, b^2, and c^2. Since $\dim W_{2k+1}^h = 0$ for all $k \in \mathbb{N}_0$, the values of the $(2k+1)^{st}$ derivatives of a^2, b^2, and c^2 at $t = 0$ have to be a fixed number. All of the conditions from Theorem 3.2.18 are linear and thus we have $(a^2)^{(2k+1)}(0) = (b^2)^{(2k+1)}(0) = (c^2)^{(2k+1)}(0) = 0$. The values of the even derivatives $(a^2)^{(2k)}(0)$, $(b^2)^{(2k)}(0)$, and $(c^2)^{(2k)}(0)$ can be chosen freely, since W_{2k}^h is a three-dimensional space.

In Remark 3.2.21, we have seen that the dimension of W_m^v is not directly linked to the properties of $(f^2)^{(m)}(0)$. One reason for this is that $g(\frac{\partial}{\partial t}, e_7)$ and $g(\frac{\partial}{\partial t}, \frac{\partial}{\partial t})$ are fixed to 0 and 1 by our choice of the coordinates. Furthermore, $\dim W_m^v$ makes a statement on $(\frac{1}{t^2} f^2)^{(m)}(0)$, since we implicitly have parameterized $\mathfrak{p}^\perp$ by polar coordinates. In order to find the smoothness conditions for f, we have to describe the metric on $\mathfrak{p}^\perp$ more explicitly. There is a unique inner product g_e on $\mathfrak{p}^\perp$ which is $U(1)^3$-invariant and satisfies $g_e(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}) = 1$. Let g_t^v be the restriction of the cohomogeneity-one metric to the circle of radius t in $\mathfrak{p}^\perp$. It follows from our formula for $\dim W_m^v$ that

$$g_t^v = \left(1 + 0 \cdot t + \kappa_2 t^2 + 0 \cdot t^3 + \kappa_4 t^4 + \ldots\right) g_e = \kappa(t) g_e \tag{5.16}$$

for some coefficients $\kappa_2, \kappa_4, \ldots \in \mathbb{R}$. Since $f(t)^2 = g_t^v(e_7, e_7) = \kappa(t) g_e(e_7, e_7)$, we have to consider e_7 in more detail. Let γ be a geodesic which intersects all orbits perpendicularly. Instead of a tangent vector of $Q^{1,1,1}$, e_7 should in this situation be considered as a vector field along γ. Since S intersects $U(1)_{1,1,1}^2$ three times, its orbit can be parameterized by $\delta : [0, \frac{2\pi}{3}] \to Q^{1,1,1}$ where

$$\delta(s) := \begin{pmatrix} \boxed{\begin{matrix} e^{is} & 0 \\ 0 & e^{-is} \end{matrix}} & & \\ & \boxed{\begin{matrix} e^{is} & 0 \\ 0 & e^{-is} \end{matrix}} & \\ & & \boxed{\begin{matrix} e^{is} & 0 \\ 0 & e^{-is} \end{matrix}} \end{pmatrix} U(1)_{1,1,1}^2$$

We see that $\delta'(0) = 2e_7$ and that the length of the circle in $\mathfrak{p}^\perp$ with radius t is with respect to g_e given by:

$$\int_0^{\frac{2\pi}{3}} \sqrt{g_e(2t \cdot e_7, 2t \cdot e_7)} ds = \frac{4\pi}{3} \sqrt{g_e(e_7, e_7)} \cdot t \,.$$

On the other hand, this length has to equal $2\pi t$. Therefore, $g_e(e_7, e_7) = \frac{9}{4} t^2$ and we have:

$$f(t)^2 = \frac{9}{4}t^2 + \frac{9}{4}\kappa_2 t^4 + \frac{9}{4}\kappa_4 t^6 + \dots .$$

Any choice of $\kappa_2, \kappa_4, \dots$ satisfies the conditions from Theorem 3.2.18. Since a, b, c, and f are analytic, we have found the following necessary and sufficient conditions for the smoothness of the metric:

- a^2, b^2, and c^2 are even functions. We conclude from $a(0), b(0), c(0) \neq 0$ that a, b, and c have to be even, too.
- f is an odd function.
- $|f'(0)| = \frac{3}{2}$.

We have to check if our metrics satisfy the above conditions. Let $(a(t), b(t), c(t), f(t))$ be a solution of (5.6). It is easy to see that $(a(-t), b(-t), c(-t), -f(-t))$ is another solution of (5.6), whose value at $t = 0$ is the same. Since (5.6) has a unique solution for any initial metric on the singular orbit $S^2 \times S^2 \times S^2$, the first two conditions are satisfied. It follows directly from our equations that $f'(0) = -3$ if $f(0) = 0$ and $a(0), b(0), c(0) \neq 0$. Therefore, the third condition is not satisfied and our metrics have a singularity at the singular orbit.

As a by-product of the above considerations, we obtain new local examples of cohomogeneity-one Einstein metrics with principal orbit $Q^{1,1,1}$ and singular orbit $S^2 \times S^2 \times S^2$. Those metrics satisfy $|f'(0)| = \frac{3}{2}$ and thus are not one of our metrics with holonomy $SU(4)$. Before we can apply Theorem 3.2.24, we have to check if Assumption 3.2.19 is satisfied. In our situation, that assumption states that $\mathfrak{p}$ and $\mathfrak{p}^\perp$ have no equivalent $U(1)^2_{1,1,1}$-submodules in common. Since $\mathfrak{p}^\perp$ is trivial with respect to the action of $U(1)^2_{1,1,1}$ and $\mathfrak{p}$ decomposes into the three non-trivial modules V_1, V_2, and V_3, 3.2.19 is satisfied. The dimensions of the spaces W^h_m and W^v_m are known and we thus can calculate the number of cohomogeneity-one Einstein metrics with principal orbit $Q^{1,1,1}$ and singular orbit $S^2 \times S^2 \times S^2$. We have stated above that $\dim W^v_m$ describes the degree of freedom of $(\frac{1}{t^2}f^2)^{(m)}(0)$ instead of $(f^2)^{(m)}(0)$. If $m = 2$, we can compute $(\frac{1}{t^2}f^2)^{(m)}(0)$ explicitly by l'Hôpital's rule. We obtain a term which contains $f(0)$, $f'(0)$, $f''(0)$, and $f'''(0)$. Since the lower derivatives of f are fixed, we can take $f'''(0)$ as the free parameter which is associated to W^v_2. All in all, there are the following parameters which are sufficient to determine a unique cohomogeneity-one Einstein metric on a tubular neighborhood of $S^2 \times S^2 \times S^2$:

1. $a(0)$, $b(0)$, and $c(0)$,
2. $f'''(0)$, and
3. the Einstein constant.

Next, we have to prove the smoothness of the metrics with singular orbit $S^2 \times S^2$. In order to keep our considerations short, we only prove a sufficient condition for the smoothness. We will see that the metrics which we have constructed satisfy that condition. The spaces W^h_m and W^v_2 we will not have to describe in detail. Therefore, we will obtain only an existence result for cohomogeneity-one Einstein metrics which satisfy certain initial conditions, but no result on their uniqueness.

In our situation, we have $f(0) = 0$ and either $a(0) = 0$, $b(0) = 0$, or $c(0) = 0$. We assume without loss of generality that $a(0) = 0$. Motivated by our results for the $S^2 \times S^2 \times S^2$-case, we suppose that any metric with the following properties is smooth:

- b and c are even analytic functions.
- a and f are odd and analytic.
- $a'(0)$ and $f'(0)$ are chosen such that the sectional curvature of the collapsing sphere is $\frac{1}{t^2} + O(\frac{1}{t})$ for small values of t.

We consider the third condition in more detail. The tangent space of the collapsing sphere can be identified with the first summand of the Lie algebra $3\mathfrak{su}(2)$. The collapsing sphere itself therefore should be considered as $SU(2)$. The metric on $SU(2)$ with constant sectional curvature 1 is given by:

$$
\begin{aligned}
h : \mathfrak{su}(2) \times \mathfrak{su}(2) &\to \mathbb{R} \\
h(X,Y) &:= -\tfrac{1}{2}\mathrm{tr}(XY)\,.
\end{aligned}
$$

Since $g(e_1, e_1) = a^2(t)$ and $h(e_1, e_1) = \frac{1}{4}$, $|a'(0)|$ has to be $\frac{1}{2}$. The orbit of a point on the collapsing sphere with respect to the group which is generated by e_7 is a great circle on that sphere. Its length has to be $2\pi t + O(t^2)$ for small t. By repeating the calculations which we have done in the previous case, we see that this is only possible if $|f'(0)| = \frac{3}{2}$.

We consider the system (5.6) under the assumption that $a(0) = f(0) = 0$. With help of l'Hôpital's rule, we obtain the following system for $a'(0)$ and $f'(0)$:

$$
\begin{aligned}
a'(0) &= -\frac{1}{6}\frac{f'(0)}{a'(0)} \\
f'(0) &= \frac{1}{6}\frac{f'(0)^2}{a'(0)^2} - 3
\end{aligned}
$$

Its solutions are given by $a'(0) = \pm\frac{1}{2}$ and $f'(0) = -\frac{3}{2}$. Since we can change $(a(t), b(t), c(t), f(t))$ into $(a(-t), b(-t), c(-t), -f(-t))$, we can assume that $a'(0) = \frac{1}{2}$. In any case, the third of our supposed conditions is satisfied.

In order to prove that our conditions are sufficient for the smoothness of the metric near the singular orbit, we again have to describe $\mathfrak{p}$ and $\mathfrak{p}^\perp$ as $\mathfrak{k}$-modules. The Lie algebra $\mathfrak{k}$ is in our situation isomorphic to $\mathfrak{su}(2) \oplus 2\mathfrak{u}(1)$. The corresponding Lie group is generated by $SU(2)$ and $U(1)^2_{1,1,1}$. In most cases, we use the $SU(2)$-action for our arguments and turn only to the action of $U(1)^2_{1,1,1}$ if necessary. The space $\mathfrak{p}$ is spanned by $(e_3, \ldots, e_6)$. Since the first summand of $3\mathfrak{su}(2)$ commutes with those e_i, $\mathfrak{p}$ is a trivial $\mathfrak{su}(2)$-module. The $SU(2)$-orbit of any $v \in \mathfrak{p}^\perp \backslash \{0\}$ is a 3-sphere. $\mathfrak{p}^\perp$ therefore is a real four-dimensional irreducible $\mathfrak{su}(2)$-module.

As in Chapter 4, we will denote this module by $\mathbb{V}_1^{\mathbb{C}}$. The action of $U(1)^2_{1,1,1}$ leaves $\frac{\partial}{\partial t}$ invariant. Thus, $\mathfrak{p}^\perp$ contains a trivial submodule. At the beginning of this section, we have seen that $U(1)^2_{1,1,1}$ acts irreducibly on the space $\operatorname{span}(e_1, e_2)$, which is tangent to the collapsing sphere. As a $U(1)^2_{1,1,1}$-module, $\mathfrak{p}^\perp$ therefore has to decompose into a trivial two-dimensional and an irreducible two-dimensional submodule, which is isomorphic to V_1. We see that $\mathfrak{p}^\perp$ and $\mathfrak{p}$, which decomposes into $V_2 \oplus V_3$, have no equivalent $U(1)^2_{1,1,1}$-submodule in common and thus will be able to apply Theorem 3.2.24 later on.

We investigate the horizontal part of the metric, which is determined by b^2 and c^2. With help of Theorem 3.2.18, we will prove that the metric is smooth if b^2 and c^2 are even. Since $b(0), c(0) \neq 0$, we can conclude that in this situation $b(t) = b(-t)$ and $c(t) = c(-t)$. The odd derivatives of any even function have to vanish at $t = 0$. Since b^2 and c^2 are analytic, we thus only have to prove that we can choose $(b^2)^{(2m)}(0)$ and $(c^2)^{(2m)}(0)$ arbitrarily. As in Section 3.2, these initial values are associated to certain $\mathfrak{k}$-equivariant maps $\phi_1, \phi_2 : S^{2m}(\mathfrak{p}^\perp) \to S^2(\mathfrak{p})$. We will describe those maps in detail. Let $(\widetilde{e}_1, \ldots, \widetilde{e}_4)$ be an orthonormal basis of $\mathfrak{p}^\perp$, such that the matrix representation of $SU(2)$ with respect to this basis is the standard one. The radial coordinate t of $\mathfrak{p}^\perp$ satisfies

$$dt \otimes dt = \widetilde{e}^1 \otimes \widetilde{e}^1 + \ldots + \widetilde{e}^4 \otimes \widetilde{e}^4 \,.$$

b^2 and c^2 are the coefficients of $e^3 \otimes e^3 + e^4 \otimes e^4$ and $e^5 \otimes e^5 + e^6 \otimes e^6$. Since $(b^2)^{(2m)}(0)$ and $(c^2)^{(2m)}(0)$ are derivatives in the t-direction, ϕ_1 and ϕ_2 are determined by

$$\begin{aligned}
\phi_1 \left(\bigotimes^m \left(\widetilde{e}^1 \otimes \widetilde{e}^1 + \ldots + \widetilde{e}^4 \otimes \widetilde{e}^4 \right) \right) &= e^3 \otimes e^3 + e^4 \otimes e^4 \\
\phi_2 \left(\bigotimes^m \left(\widetilde{e}^1 \otimes \widetilde{e}^1 + \ldots + \widetilde{e}^4 \otimes \widetilde{e}^4 \right) \right) &= e^5 \otimes e^5 + e^6 \otimes e^6
\end{aligned}$$

On the orthogonal complement of $\bigotimes^m \left(\widetilde{e}^1 \otimes \widetilde{e}^1 + \ldots + \widetilde{e}^4 \otimes \widetilde{e}^4 \right)$ in $S^{2m}(\mathfrak{p}^\perp)$, both of these maps vanish. Since $SU(2) \times U(1)^2$ acts by isometries on $\mathfrak{p}^\perp$, $\widetilde{e}^1 \otimes \widetilde{e}^1 + \ldots + \widetilde{e}^4 \otimes \widetilde{e}^4$ is left invariant by this action. ϕ_1 and ϕ_2 thus map a $\mathfrak{su}(2) \oplus 2\mathfrak{u}(1)$-invariant object to another $\mathfrak{su}(2) \oplus 2\mathfrak{u}(1)$-invariant object. Any choice of $(b^2)^{(2m)}(0)$ and $(c^2)^{(2m)}(0)$ is associated to the map $(b^2)^{(2m)}(0)\,\phi_1 + (c^2)^{(2m)}(0)\,\phi_2$, which is $\mathfrak{su}(2) \oplus 2\mathfrak{u}(1)$-equivariant. From Theorem 3.2.18, it therefore follows that these initial values can be chosen arbitrarily.

The condition on a and f can be proven by similar means as above. For the same reasons as in the $S^2 \times S^2 \times S^2$-case, the $2m^{th}$ derivatives of $\frac{1}{t^2}a^2$ and $\frac{1}{t^2}f^2$ are associated to elements of W^v_{2m}. We therefore search for the maps $S^{2m}(\mathfrak{p}^\perp) \to S^2(\mathfrak{p}^\perp)$ which represent $(\frac{1}{t^2}a^2)^{(2m)}(0)$ and $(\frac{1}{t^2}f^2)^{(2m)}(0)$ and prove that they are $\mathfrak{k}$-equivariant. Let γ be the geodesic with $\gamma(0) = 0 \in \mathfrak{p}^\perp$ and $\gamma'(0) = \widetilde{e}_1$. In the coordinates which we have chosen, $\gamma(t) = t\widetilde{e}_1$. The tangent space of the collapsing sphere coincides with the orthogonal complement of $\gamma'(t)$. This space is spanned by $(\widetilde{e}_2, \widetilde{e}_3, \widetilde{e}_4)$. The tangent space of S^3 is also spanned by (e_7, e_1, e_2). From now on, we will identify both triples with each other. This is possible, since $\operatorname{span}(\widetilde{e}_2)$ and $\operatorname{span}(e_7)$ as well as $\operatorname{span}(\widetilde{e}_3, \widetilde{e}_4)$ and $\operatorname{span}(e_1, e_2)$ are equivalent with respect to the action of $U(1)^2_{1,1,1}$. We are interested in the value of

$$\frac{\partial^{2m}}{\partial\, \widetilde{e}_1^{2m}}\, g(\widetilde{e}_i, \widetilde{e}_i) \,,$$

where $i \in \{2,3,4\}$. For $i \in \{3,4\}$, this term describes $(\frac{1}{t^2}a^2)^{(2m)}(0)$ and for $i = 2$ it is $(\frac{1}{t^2}f^2)^{(2m)}(0)$. We will extend the above derivatives to $SU(2)$-equivariant maps $\varphi : \mathfrak{p}^\perp \to \mathbb{R}$. In order to do this, we first define the following function on $SU(2)$:

$$k \mapsto \frac{\partial^{2m}}{\partial\,(k \cdot \widetilde{e}_1)^{2m}}\, g(k \cdot \widetilde{e}_i, k \cdot \widetilde{e}_i)\,.$$

Since $SU(2) \cong S^3 \subseteq \mathfrak{p}^\perp$, we can extend this function by linearity to all of $\mathfrak{p}^\perp$. We consider the case $m = 1$. In this situation, we have to construct the two K-equivariant maps $\phi_1, \phi_2 : S^2(\mathfrak{p}^\perp) \to S^2(\mathfrak{p}^\perp)$ which correspond to $(\frac{1}{t^2}a^2)''(0)$ and $(\frac{1}{t^2}f^2)''(0)$. With help of the function φ, we obtain the following relations for ϕ_1 and ϕ_2:

$$\begin{aligned}
\phi_1\left((L_k^*\widetilde{e}^1) \otimes (L_k^*\widetilde{e}^1)\right) &= (L_k^*\widetilde{e}^3) \otimes (L_k^*\widetilde{e}^3) + (L_k^*\widetilde{e}^4) \otimes (L_k^*\widetilde{e}^4)\\
\phi_2\left((L_k^*\widetilde{e}^1) \otimes (L_k^*\widetilde{e}^1)\right) &= (L_k^*\widetilde{e}^2) \otimes (L_k^*\widetilde{e}^2)
\end{aligned}$$

where L_k^* denotes the pull-back of the left-multiplication with $k \in SU(2)$. By polarization, the ϕ_i become $SU(2)$-equivariant maps on all of $S^2(\mathfrak{p}^\perp)$. Since $\widetilde{e}_1 \otimes \widetilde{e}_1$, $\widetilde{e}_2 \otimes \widetilde{e}_2$ and $\widetilde{e}_3 \otimes \widetilde{e}_3 + \widetilde{e}_4 \otimes \widetilde{e}_4$ are all $U(1)^2_{1,1,1}$-invariant, ϕ_1 and ϕ_2 are equivariant with respect to all of K. If $m \geqslant 2$, we have to describe the K-equivariant maps $\phi_1^m, \phi_2^m : S^{2m}(\mathfrak{p}^\perp) \to S^2(\mathfrak{p}^\perp)$ which are associated to $(\frac{1}{t^2}a^2)^{(2m)}(0)$ and $(\frac{1}{t^2}f^2)^{(2m)}(0)$. These derivatives can be considered as the $2m^{th}$ derivatives of $\varphi : \mathfrak{p}^\perp \to \mathbb{R}$ in the radial direction. We therefore have:

$$\begin{aligned}
\phi_1^m\left(\bigotimes^{m-1}(dt \otimes dt) \vee (L_k^*\widetilde{e}^1) \otimes (L_k^*\widetilde{e}^1)\right) &= (L_k^*\widetilde{e}^3) \otimes (L_k^*\widetilde{e}^3) + (L_k^*\widetilde{e}^4) \otimes (L_k^*\widetilde{e}^4)\\
\phi_2^m\left(\bigotimes^{m-1}(dt \otimes dt) \vee (L_k^*\widetilde{e}^1) \otimes (L_k^*\widetilde{e}^1)\right) &= (L_k^*\widetilde{e}^2) \otimes (L_k^*\widetilde{e}^2)\,,
\end{aligned}$$

where as usual

$$dt \otimes dt = \widetilde{e}^1 \otimes \widetilde{e}^1 + \widetilde{e}^2 \otimes \widetilde{e}^2 + \widetilde{e}^3 \otimes \widetilde{e}^3 + \widetilde{e}^4 \otimes \widetilde{e}^4$$

and $\vee$ is the symmetric product. On the orthogonal complement of $\bigotimes^{m-1}(dt \otimes dt) \vee S^2(\mathfrak{p}^\perp) \subseteq S^{2m}(\mathfrak{p}^\perp)$ both of these maps vanish. Since $dt \otimes dt$ is K-invariant, the K-equivariance of ϕ_1^m and ϕ_2^m directly follows from the equivariance of ϕ_1 and ϕ_2. Again, we conclude with help of Theorem 3.2.18 that we can choose $(\frac{1}{t^2}a^2)^{(2m)}(0)$ and $(\frac{1}{t^2}f^2)^{(2m)}(0)$ for all $m \geqslant 1$ arbitrarily. If $\frac{1}{t^2}a^2$ and $\frac{1}{t^2}f^2$ are even analytic functions, $|a'(0)| = \frac{1}{2}$, and $|f'(0)| = \frac{3}{2}$, all conditions from Theorem 3.2.18 on a and f are satisfied. Since $a'(0), f'(0) \neq 0$, a and f have to be odd if $\frac{1}{t^2}a^2$ and $\frac{1}{t^2}f^2$ are even. By similar arguments as in the $S^2 \times S^2 \times S^2$-case, we see that our solutions of (5.6) with singular orbit $S^2 \times S^2$ satisfy all the conditions from page 122. All in all, we have proven that all of our solutions are smooth, complete, and non-compact.

In the $S^2 \times S^2$-case, we have not calculated the dimension of the spaces W^h_m and W^v_2. Nevertheless, we have constructed some elements of those spaces and thus are able to make some statements on the cohomogeneity-one Einstein metrics with singular orbit $S^2 \times S^2$. According to Theorem 3.2.24, the invariant metric on the singular orbit can be chosen arbitrarily. Since the maps $S^2(\mathfrak{p}^\perp) \to S^2(\mathfrak{p}^\perp)$ which we have described above correspond to $a'''(0)$ and $f'''(0)$, these initial values can be chosen freely, too. We finally sum up the results of this section:

Theorem 5.2.2. *Let (M,Ω) be a parallel Spin(7)-manifold with a cohomogeneity-one action of $SU(2)^3$ which preserves Ω. In this situation, the following statements are true:*

1. *The principal orbits are $SU(2)^3$-equivariantly diffeomorphic to $Q^{1,1,1}$.*
2. *The metric g which is associated to Ω is special Kähler and its holonomy is all of $SU(4)$.*
3. *g has the matrix representation (5.3) with respect to the basis $(e_1,\ldots,e_7)$ from page 104. The metric functions satisfy the equations (5.6), whose solutions are described by (5.7) and (5.8). The Kähler form is given by (5.14).*

If M has a singular orbit, which has to be the case if (M,g) is complete, it is $S^2\times S^2$ or $S^2\times S^2\times S^2$. Any $SU(2)^3$-invariant metric on the singular orbit can be extended to a unique cohomogeneity-one metric with holonomy $SU(4)$, which is non-compact. If the singular orbit is $S^2\times S^2$, the metric is smooth and complete. If the singular orbit is $S^2\times S^2\times S^2$, the metric is not differentiable at the singular orbit. Outside of the singular orbit, the metric is smooth and any geodesic which does not intersect the singular orbit can be infinitely extended. In any of these cases, M is a vector bundle over the singular orbit.

With help of Theorem 3.2.24, we have also obtained results on cohomogeneity-one Einstein metrics with principal orbit $Q^{1,1,1}$:

Theorem 5.2.3. *Let M be a cohomogeneity-one manifold, whose principal orbit is $Q^{1,1,1}$. We assume that M has a singular orbit, which has to be either $S^2\times S^2\times S^2$ or $S^2\times S^2$. Any $SU(2)^3$-invariant metric on M is described by a matrix of type (5.3) or equivalently by the four metric functions $a,b,c,f: I\to\mathbb{R}$, where I is an interval. Without loss of generality, we assume that the singular orbit is at $0\in I$. In this situation, the following statements are true:*

1. *Let the singular orbit be $S^2\times S^2\times S^2$. For any choice of $a_0,b_0,c_0,f_3,\lambda\in\mathbb{R}$, there exists a unique $SU(2)^3$-invariant Einstein metric on a tubular neighborhood of $S^2\times S^2\times S^2$ such that:*
   (a) *$a(0)=a_0$, $b(0)=b_0$, $c(0)=c_0$,*
   (b) *$f'''(0)=f_3$, and*
   (c) *the Einstein constant is λ.*
2. *Let the singular orbit be $S^2\times S^2$. For any choice of $b_0,c_0,a_3,f_3,\lambda\in\mathbb{R}$, there exists a $SU(2)^3$-invariant Einstein metric on a tubular neighborhood of $S^2\times S^2$ such that:*
   (a) *$a(0)=0$, $b(0)=b_0$, $c(0)=c_0$,*
   (b) *$a'''(0)=a_3$, $f'''(0)=f_3$, and*
   (c) *the Einstein constant is λ.*

Remark 5.2.4. 1. It is likely that the Kähler metrics with singular orbit $S^2\times S^2\times S^2$ can be modified in such a way that they become smooth. If we replace the principal orbit by a space of type $Q^{1,1,1}/\mathbb{Z}_2$, where $\mathbb{Z}_2$ is contained in the isotropy group of the singular orbit, we obtain a new space of cohomogeneity one with singular orbit $S^2\times S^2\times S^2$.

The length of the collapsing circle gets multiplied by $\frac{1}{2}$ and the smoothness condition on $f'(0)$ is satisfied. Since the isotropy group of the principal orbit has changed, it has to be checked if the other smoothness conditions have changed, too. If $\mathbb{Z}_2$ preserves the $SU(4)$-structure, it turns into a $SU(4)$-structure on the new space. Otherwise, we obtain a $SU(4) \rtimes \mathbb{Z}_2$-structure. In Section 5.4, we will describe a similar construction in detail.

2. The metrics with holonomy $SU(4)$ which we have constructed have also been considered by Cvetič, Gibbons, Lü, and Pope in [25]. In that paper, the authors construct Ricci-flat Kähler-metrics on holomorphic vector bundles over a product of several Einstein-Kähler manifolds. As a special case, they obtain Ricci-flat Kähler metrics of cohomogeneity one with principal orbit $Q^{1,1,1}$ and a single singular orbit of type $S^2 \times S^2$ or $S^2 \times S^2 \times S^2$. The authors also prove that those metrics are non-compact and complete away from the singular orbit. In [21], the same authors consider those metrics in another context. Both papers are based on earlier works by Berard-Bergery [8], Page and Pope [58], and Stenzel [64]. The examples with singular orbit $S^2 \times S^2 \times S^2$ have also been considered by Herzog and Klebanov [41]. Although the metrics with holonomy $SU(4)$ which we have constructed are known, the proof that any parallel Spin(7)-manifold with a cohomogeneity-one action of $SU(2)^3$ is one of our examples is new. Our results prove that it is impossible to deform the Kähler examples into metrics with holonomy Spin(7) without loosing the cohomogeneity-one property. Moreover, we have proven that all (not necessarily parallel) Spin(7)-structures of cohomogeneity one with principal orbit $Q^{1,1,1}$ have to reduce to a $SU(4)$-structure. This fact has not been mentioned in the literature before, either. Finally, we remark that the Einstein metrics from the above theorem which are not included in Theorem 5.2.2 have, as far as the author knows, not been considered in the literature before. All in all, we hope to have introduced an interesting, more algebraic approach to the issue of special cohomogeneity-one metrics with principal orbit $Q^{1,1,1}$.

5.3 The principal orbit $M^{1,1,0}$

In this section, we will investigate parallel cohomogeneity-one Spin(7)-structures whose principal orbit is of type $(SU(3)\times SU(2))/(SU(2)\times U(1))$, where the semisimple part of $SU(2)\times U(1)$ is embedded into $SU(3) \subseteq SU(3) \times SU(2)$. These homogeneous spaces will be denoted by $M^{k,l,m}$. The indices k, l, and m describe the embedding of the smaller group into the larger. As in the previous section, we will see that $M^{k,l,m}$ carries only for a special choice of k, l, and m a homogeneous G_2-structure. This fact already has been stated in Theorem 4.2. We will construct three classes of metrics with holonomy $\subseteq$ Spin(7) which are distinguished by their singular orbit. The first one was found by Cvetič et al. in [21], and the second and third class have been considered by the same authors in [25]. Most of the results we find in this section are analogous to those of Section 5.2.

Our first step is to describe the possible embeddings of $SU(2) \times U(1)$ into $SU(3) \times SU(2)$. In the literature, there are two conventions how to denote these embeddings. The first one is used by Friedrich, Kath, Moroianu, and Semmelmann in [37]. Castellani et al. [19] and Fabbri [33] use the second convention. For our considerations, we will take the notation of Castellani.

In order to explain the meaning of the indices k, l, and m, we first consider spaces of type $(SU(3) \times SU(2) \times U(1))/(SU(2) \times U(1) \times U(1))$ instead of $(SU(3) \times SU(2))/(SU(2) \times U(1))$. These spaces have been important in research on Kaluza-Klein supergravity in 11 dimensions (see [19], [33]). Since we want to distinguish the different abelian factors, we will denote the newly introduced spaces by $(SU(3) \times SU(2) \times U(1))/(SU(2) \times U(1)' \times U(1)'')$. The inclusion $\imath : SU(2) \times U(1)' \times U(1)'' \to SU(3) \times SU(2) \times U(1)$ is determined by its differential $(d\imath)_e : \mathfrak{su}(2) \oplus \mathfrak{u}(1)' \oplus \mathfrak{u}(1)'' \to \mathfrak{su}(3) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1)$, since both groups are connected. We describe the Lie algebra $\mathfrak{su}(3) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1)$ by the following matrices:

$$\left\{ \begin{pmatrix} \boxed{X} & & \\ & \boxed{Y} & \\ & & \boxed{Z} \end{pmatrix} \;\middle|\; X \in \mathfrak{su}(3),\ Y \in \mathfrak{su}(2),\ Z \in \mathfrak{u}(1) \right\} .$$

$\mathfrak{su}(2) \subseteq \mathfrak{su}(2) \oplus \mathfrak{u}(1)$ is embedded into this Lie algebra by:

$$(d\imath)_e \begin{pmatrix} ti & z & & \\ -\bar{z} & -ti & & \\ & & 0 & \\ & & & 0 \end{pmatrix} := \begin{pmatrix} ti & z & 0 & & & \\ -\bar{z} & -ti & 0 & & & \\ 0 & 0 & 0 & & & \\ & & & 0 & 0 & \\ & & & 0 & 0 & \\ & & & & & 0 \end{pmatrix} .$$

We have to map $\mathfrak{u}(1)'$ and $\mathfrak{u}(1)''$ in such a way into $\mathfrak{su}(3) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1)$ that the sum of $(d\imath)_e(\mathfrak{u}(1)')$, $(d\imath)_e(\mathfrak{u}(1)'')$, and $(d\imath)_e(\mathfrak{su}(2))$ is direct, and the three summands commute with each other. We choose the following Cartan subalgebra of $\mathfrak{su}(3) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1)$:

$$\mathfrak{t} := \left\{ \left(\begin{array}{cccccc} xi & 0 & 0 & & & \\ 0 & yi & 0 & & & \\ 0 & 0 & -(x+y)i & & & \\ & & & zi & 0 & \\ & & & 0 & -zi & \\ & & & & & wi \end{array} \right) \;\middle|\; x,y,z,w \in \mathbb{R} \right\} .$$

We may assume without loss of generality that $(d\imath)_e(\mathfrak{u}(1)')$, $(d\imath)_e(\mathfrak{u}(1)'')$, and the image of the standard Cartan subalgebra of $\mathfrak{su}(2)$ are subalgebras of $\mathfrak{t}$. The orthogonal complement of $(d\imath)_e(\mathfrak{su}(2) \oplus \mathfrak{u}(1)' \oplus \mathfrak{u}(1)'') \cap \mathfrak{t}$ in $\mathfrak{t}$ with respect to the Killing form is a one-dimensional Lie algebra. This algebra we will denote by $\mathfrak{u}(1)'''$. It is easy to see that $\imath$ is determined by $\mathfrak{u}(1)'''$ up to a conjugation. Moreover, there is a one-to-one correspondence between the conjugacy classes of the possible $\imath$ and the one-dimensional subalgebras of $\mathfrak{t}$ which are orthogonal to $(d\imath)_e(\mathfrak{su}(2))$. These subalgebras can be described by three numbers k, l, and m. We define:

$$\mathfrak{u}(1)'''_{k,l,m} := \left\{ \left(\begin{array}{cccccc} \frac{k}{2}ix & 0 & 0 & & & \\ 0 & \frac{k}{2}ix & 0 & & & \\ 0 & 0 & -kix & & & \\ & & & -\frac{l}{2}ix & 0 & \\ & & & 0 & \frac{l}{2}ix & \\ & & & & & mix \end{array} \right) \;\middle|\; x \in \mathbb{R} \right\} .$$

Any choice of k, l, and m determines an embedding of $SU(2) \times U(1)' \times U(1)''$ into $SU(3) \times SU(2) \times U(1)$ which we denote by $\imath_{k,l,m}$. The coset space $(SU(3) \times SU(2) \times U(1))/\imath_{k,l,m}(SU(2) \times U(1)' \times U(1)'')$ is denoted by $M^{k,l,m}$. In order to make the exponential of $(d\imath)_e(\mathfrak{u}(1)' \oplus \mathfrak{u}(1)'')$ a closed subgroup, we have to choose k, l, and m as rational numbers. Since $\mathfrak{u}(1)'''_{nk,nl,nm}$ equals $\mathfrak{u}(1)'''_{k,l,m}$, we assume from now on that (k,l,m) consists of coprime integers. Castellani et al. have proven in [19] that

$$M^{k,l,m} = M^{k,l,0}/\mathbb{Z}_r \quad \text{with} \quad r = \frac{3k^2 + l^2}{\gcd(2mk, ml, 3k^2 + l^2)} .$$

It therefore suffices to consider only the spaces of type $M^{k,l,0}$. Since in this case the abelian factor of $SU(3) \times SU(2) \times U(1)$ is a subgroup of $U(1)' \times U(1)''$, $M^{k,l,0}$ is a space of type $(SU(3) \times SU(2))/(SU(2) \times U(1))$. In the notation of [37], the space $M^{k,l,0}$ is denoted by $M^{3k,2l}$. For the rest of this section, we will shortly denote $\imath_{k,l,0}(SU(2) \times U(1))$ by $SU(2) \times U(1)$ and $(d\imath_{k,l,0})_e(\mathfrak{su}(2) \oplus \mathfrak{u}(1))$ by $\mathfrak{su}(2) \oplus \mathfrak{u}(1)$, as long as the meaning of this abbreviated notation is clear from the context. We can restrict the values of k and l even more. Let

$$P := \left(\begin{array}{ccccc} 1 & 0 & 0 & & \\ 0 & 1 & 0 & & \\ 0 & 0 & 1 & & \\ & & & 0 & i \\ & & & i & 0 \end{array} \right) \in SU(3) \times SU(2) ,$$

and let $\phi_P : SU(3) \times SU(2) \to SU(3) \times SU(2)$ be defined by $\phi_P(Q) := PQP^{-1}$. ϕ_P is an automorphism of $SU(3) \times SU(2)$ and maps $\imath_{k,l,0}(SU(2) \times U(1))$ into $\imath_{k,-l,0}(SU(2) \times U(1))$. Therefore, the spaces $M^{k,l,0}$ and $M^{k,-l,0}$ are $SU(3) \times SU(2)$-equivariantly diffeomorphic. Since $M^{k,l,0}$ and $M^{-k,-l,0}$ are the same, too, we can assume without loss of generality that $(k,l) \in \mathbb{N}_0 \times \mathbb{N}_0 \backslash \{(0,0)\}$.

Next, we decide which $M^{k,l,0}$ admit an invariant G_2-structure. In order to answer this question, we first choose a basis $(e_1, \ldots, e_{11})$ of $\mathfrak{su}(3) \oplus \mathfrak{su}(2)$:

$$e_1 := \begin{pmatrix} 0 & 0 & 1 & & \\ 0 & 0 & 0 & & \\ -1 & 0 & 0 & & \\ & & & 0 & 0 \\ & & & 0 & 0 \end{pmatrix}, \quad e_2 := \begin{pmatrix} 0 & 0 & i & & \\ 0 & 0 & 0 & & \\ i & 0 & 0 & & \\ & & & 0 & 0 \\ & & & 0 & 0 \end{pmatrix}, \quad e_3 := \begin{pmatrix} 0 & 0 & 0 & & \\ 0 & 0 & 1 & & \\ 0 & -1 & 0 & & \\ & & & 0 & 0 \\ & & & 0 & 0 \end{pmatrix},$$

$$e_4 := \begin{pmatrix} 0 & 0 & 0 & & \\ 0 & 0 & i & & \\ 0 & i & 0 & & \\ & & & 0 & 0 \\ & & & 0 & 0 \end{pmatrix}, \quad e_5 := \begin{pmatrix} 0 & 0 & 0 & & \\ 0 & 0 & 0 & & \\ 0 & 0 & 0 & & \\ & & & 0 & 1 \\ & & & -1 & 0 \end{pmatrix}, \quad e_6 := \begin{pmatrix} 0 & 0 & 0 & & \\ 0 & 0 & 0 & & \\ 0 & 0 & 0 & & \\ & & & 0 & i \\ & & & i & 0 \end{pmatrix},$$

$$e_7 := \begin{pmatrix} \frac{k}{2}i & 0 & 0 & & \\ 0 & \frac{k}{2}i & 0 & & \\ 0 & 0 & -ki & & \\ & & & -\frac{l}{2}i & 0 \\ & & & 0 & \frac{l}{2}i \end{pmatrix}, \quad e_8 := \begin{pmatrix} 0 & 1 & 0 & & \\ -1 & 0 & 0 & & \\ 0 & 0 & 0 & & \\ & & & 0 & 0 \\ & & & 0 & 0 \end{pmatrix},$$

$$e_9 := \begin{pmatrix} 0 & i & 0 & & \\ i & 0 & 0 & & \\ 0 & 0 & 0 & & \\ & & & 0 & 0 \\ & & & 0 & 0 \end{pmatrix}, \quad e_{10} := \begin{pmatrix} i & 0 & 0 & & \\ 0 & -i & 0 & & \\ 0 & 0 & 0 & & \\ & & & 0 & 0 \\ & & & 0 & 0 \end{pmatrix},$$

$$e_{11} := \begin{pmatrix} \frac{l}{3}i & 0 & 0 & & \\ 0 & \frac{l}{3}i & 0 & & \\ 0 & 0 & -\frac{2l}{3}i & & \\ & & & ki & 0 \\ & & & 0 & -ki \end{pmatrix}.$$

The bilinear form q on $\mathfrak{su}(3) \oplus \mathfrak{su}(2)$ which is defined by $q(X,Y) := -\mathrm{tr}(XY)$ determines a biinvariant metric on $SU(3) \times SU(2)$. It is easy to see that $(e_1, \ldots, e_{11})$ is a q-orthogonal basis of $\mathfrak{su}(3) \oplus \mathfrak{su}(2)$, and that $(e_8, \ldots, e_{11})$ spans $\mathfrak{su}(2) \oplus \mathfrak{u}(1)$. The other elements of our basis span the q-orthogonal complement $\mathfrak{m}$ of $\mathfrak{su}(2) \oplus \mathfrak{u}(1)$ in $\mathfrak{su}(3) \oplus \mathfrak{su}(2)$. As usual, we identify the tangent space of $M^{k,l,0}$ at a fixed point with $\mathfrak{m}$. Our next step is to describe the isotropy action of $\mathfrak{su}(2)\oplus\mathfrak{u}(1)$ on $\mathfrak{m}$. We compute the matrix representations of $\mathrm{ad}_{e_8}|_{\mathfrak{m}},\ldots,\mathrm{ad}_{e_{11}}|_{\mathfrak{m}}$ with respect to the basis $(e_1,\ldots,e_7)$ and obtain:

$$\mathrm{ad}_{e_8}|_{\mathfrak{m}} = \begin{pmatrix} 0 & 0 & 1 & 0 & & & \\ 0 & 0 & 0 & 1 & & & \\ -1 & 0 & 0 & 0 & & & \\ 0 & -1 & 0 & 0 & & & \\ & & & & 0 & 0 & \\ & & & & 0 & 0 & \\ & & & & & & 0 \end{pmatrix}, \quad \mathrm{ad}_{e_9}|_{\mathfrak{m}} = \begin{pmatrix} 0 & 0 & 0 & -1 & & & \\ 0 & 0 & 1 & 0 & & & \\ 0 & -1 & 0 & 0 & & & \\ 1 & 0 & 0 & 0 & & & \\ & & & & 0 & 0 & \\ & & & & 0 & 0 & \\ & & & & & & 0 \end{pmatrix},$$

$$\mathrm{ad}_{e_{10}}|_{\mathfrak{m}} = \begin{pmatrix} 0 & -1 & 0 & 0 & & & \\ 1 & 0 & 0 & 0 & & & \\ 0 & 0 & 0 & 1 & & & \\ 0 & 0 & -1 & 0 & & & \\ & & & & 0 & 0 & \\ & & & & 0 & 0 & \\ & & & & & & 0 \end{pmatrix}, \quad \mathrm{ad}_{e_{11}}|_{\mathfrak{m}} = \begin{pmatrix} 0 & -l & 0 & 0 & & & \\ l & 0 & 0 & 0 & & & \\ 0 & 0 & 0 & -l & & & \\ 0 & 0 & l & 0 & & & \\ & & & & 0 & -2k & \\ & & & & 2k & 0 & \\ & & & & & & 0 \end{pmatrix}.$$

We consider the Cartan subalgebra of $\mathfrak{su}(2)\oplus\mathfrak{u}(1)$ which is generated by e_{10} and e_{11}. Its adjoint action on $\mathfrak{m}$ is given by:

$$\mathrm{ad}_{\mathrm{span}(e_{10},e_{11})}|_{\mathfrak{m}} = \left\{ \begin{pmatrix} 0 & x & & & & & \\ -x & 0 & & & & & \\ & & 0 & y & & & \\ & & -y & 0 & & & \\ & & & & 0 & z & \\ & & & & -z & 0 & \\ & & & & & & 0 \end{pmatrix} \;\middle|\; \left\langle \begin{pmatrix} k \\ k \\ -l \end{pmatrix}, \begin{pmatrix} x \\ y \\ z \end{pmatrix} \right\rangle = 0 \right\}.$$

We assume that there exists a $SU(3)\times SU(2)$-invariant G_2-structure on $M^{k,l,0}$. As in Section 5.2, we compare the action of $\mathrm{span}(e_{10},e_{11})$ on $\mathfrak{m}$ with the action of the standard Cartan subalgebra $2\mathfrak{u}(1)\subseteq\mathfrak{g}_2$ on $\mathrm{Im}(\mathbb{O})$. $\mathfrak{m}$ and $\mathbb{R}^7$ have to be equivalent $2\mathfrak{u}(1)$-modules. By considering the matrix representation of $2\mathfrak{u}(1)\subseteq\mathfrak{g}_2$ from page 106, we can easily deduce that $k=1$ and $l=1$. Therefore, we will restrict ourselves from now on to the principal orbit $M^{1,1,0}$.

By a short calculation, we see that the product of $SU(2)$ and the one-dimensional Lie group generated by e_{11} is a direct one. Thus, $M^{1,1,0}$ indeed can be written as a coset space of type $(SU(3)\times SU(2))/(SU(2)\times U(1))$. We could prove by another calculation that $SU(2)\times U(1)$ acts as $(SU(2)\times U(1))/\mathbb{Z}_2$ on $\mathfrak{m}$. The action of $SU(3)\times SU(2)$ on $M^{1,1,0}$ therefore is only almost effective. Since G_2 contains a subgroup of type $(SU(2)\times U(1))/\mathbb{Z}_2$, $M^{1,1,0}$ admits a homogeneous G_2-structure although $SU(2)\times U(1)\subsetneq G_2$.

We classify the $SU(3)\times SU(2)$-invariant metrics on $M^{1,1,0}$. Before we can do this, we have to decompose the tangent space into irreducible $\mathfrak{su}(2)\oplus\mathfrak{u}(1)$-modules. The semisimple part of

$\mathfrak{su}(2) \oplus \mathfrak{u}(1)$ obviously acts by its two-dimensional complex representation on $\mathrm{span}(e_1, \ldots, e_4)$. This space is a $\mathfrak{u}(1)$-module, too, although not an irreducible one. $\mathfrak{su}(2)$ acts trivially on $\mathrm{span}(e_5, e_6)$, while $\mathfrak{u}(1)$ acts irreducibly on it. Both Lie algebras act trivially on $\mathrm{span}(e_7)$. All in all, $\mathfrak{m}$ splits into three pairwise inequivalent $\mathfrak{su}(2) \oplus \mathfrak{u}(1)$-modules V_1, V_2, and V_3:

$$\begin{aligned} V_1 &:= \mathrm{span}(e_1, e_2, e_3, e_4) \\ V_2 &:= \mathrm{span}(e_5, e_6) \\ V_3 &:= \mathrm{span}(e_7) \end{aligned}$$

Any invariant metric g on $M^{1,1,0}$ can be identified by the formula

$$q(\varphi(X), Y) := g(X, Y) \quad \forall X, Y \in \mathfrak{m}$$

with a $\mathfrak{su}(2) \oplus \mathfrak{u}(1)$-equivariant map $\varphi : \mathfrak{m} \to \mathfrak{m}$, which has to be q-symmetric and positive definite. We classify the possible φ by Schur's lemma. V_1 can be identified with $\mathbb{C}^2$ on which $SU(2) \times U(1)$ acts as $U(2)$ by its standard representation. The only equivariant endomorphisms of $\mathbb{C}^2$ are the complex multiples of the identity map. Since φ has to be symmetric, $\varphi|_{V_1}$ has in fact to be a real multiple of the identity. The other components of φ can be determined in a similar manner. We see that the matrix representation of g with respect to the basis $(e_1, \ldots, e_7)$ has to be given by:

$$\begin{pmatrix} a^2 & 0 & 0 & 0 & & & \\ 0 & a^2 & 0 & 0 & & & \\ 0 & 0 & a^2 & 0 & & & \\ 0 & 0 & 0 & a^2 & & & \\ & & & & b^2 & 0 & \\ & & & & 0 & b^2 & \\ & & & & & & c^2 \end{pmatrix} \quad \text{with} \quad a, b, c \in \mathbb{R} \setminus \{0\} \, . \tag{5.17}$$

Conversely, any such matrix corresponds to an invariant metric on $M^{1,1,0}$. As in Section 5.2, we also allow negative values of a, b, and c, since a change of their signs changes the frame of the Spin(7)-structure, although it leaves the metric invariant. Let (M, g) be a cohomogeneity-one manifold with an isometric $SU(3) \times SU(2)$-action whose principal orbit is $M^{1,1,0}$. We can identify the union of all principal orbits by a suitable diffeomorphism with $M^{1,1,0} \times I$, where I is an open interval. With this identification, g can be described by:

$$g = g_t + dt^2 \, ,$$

where t is the coordinate directed along I and g_t is an invariant metric on the principal orbit at t. Near a singular orbit, g_t will degenerate.

Next, we construct a frame $(f_0, \ldots, f_7)$ of a $SU(3) \times SU(2)$-invariant Spin(7)-structure on M. We recall that the $SU(2)$-factor of $SU(2) \times U(1)$ acts by its two-dimensional complex representation on V_1 and trivially on $V_2 \oplus V_3$. $SU(2)$ also acts on $\mathbb{H}\epsilon \subseteq \mathrm{Im}(\mathbb{O})$ by left-multiplication by unit-quaternions. Since this is the only $SU(2)$-action by a subgroup of G_2 which splits $\mathrm{Im}(\mathbb{O})$ into $\mathbb{V}_1^{\mathbb{C}} \oplus 3\mathbb{V}_0^{\mathbb{R}}$, we assume that the last four elements of the frame are in

V_1. f_0 is as usual chosen as $\frac{\partial}{\partial t}$. f_1 will be a multiple of e_7, since $i \in \mathbb{O}$ is invariant with respect to the subalgebra $\mathfrak{su}(2)_1 \oplus \mathfrak{u}(1)$ of $\mathfrak{g}_2$ from Chapter 4. $\mathrm{span}(f_2, f_3)$ therefore will coincide with V_2. We compare the action of $\mathrm{span}(e_{10}, e_{11})$ on $\mathfrak{m}$ with the action of $2\mathfrak{u}(1) \subseteq \mathfrak{g}_2$ on $\mathrm{Im}(\mathbb{O})$. $\mathrm{span}(e_1, e_2)$ and $\mathrm{span}(e_3, e_4)$ are both irreducible with respect to the action of $\mathrm{span}(e_{10}, e_{11})$ and thus will be identified with $\mathrm{span}(f_4, f_5)$ and $\mathrm{span}(f_6, f_7)$. Since the above both actions have to be equivariant, we have to change the order of some of the pairs (e_{2i-1}, e_{2i}). A frame of a Spin(7)-structure also has to be orthonormal with respect to g. Motivated by these considerations, we choose the following frame:

$$
\begin{aligned}
& f_0 := \tfrac{\partial}{\partial t} \quad f_1 := \tfrac{1}{c} e_7 \quad f_2 := \tfrac{1}{b} e_6 \quad f_3 := \tfrac{1}{b} e_5 \\
& f_4 := \tfrac{1}{a} e_1 \quad f_5 := \tfrac{1}{a} e_2 \quad f_6 := \tfrac{1}{a} e_4 \quad f_7 := \tfrac{1}{a} e_3
\end{aligned}
\tag{5.18}
$$

By a short calculation, we see that the matrix representation of the isotropy action with respect to the above basis is the same as of $\mathfrak{su}(2)_1 \oplus \mathfrak{u}(1) \subseteq \mathfrak{g}_2$. The four-form Ω which is associated to the above frame is given by:

$$
\begin{aligned}
\Omega = & -a^4\, e^{1234} + a^2 b^2\, e^{1256} + a^2 b^2\, e^{3456} \\
& + a^2 bc\, e^{1367} - a^2 bc\, e^{1457} - a^2 bc\, e^{2357} - a^2 bc\, e^{2467} \\
& - a^2 b\, e^{135} \wedge dt - a^2 b\, e^{146} \wedge dt - a^2 b\, e^{236} \wedge dt + a^2 b\, e^{245} \wedge dt \\
& - a^2 c\, e^{127} \wedge dt - a^2 c\, e^{347} \wedge dt + b^2 c\, e^{567} \wedge dt
\end{aligned}
\tag{5.19}
$$

In order to calculate $d\Omega$, we apply the same methods as in Section 5.2. After that, we can express the condition $d\Omega = 0$ as a system of ordinary differential equations for the metric functions a, b, and c:

$$
\begin{aligned}
\frac{a'}{a} &= \frac{3}{8} \frac{c}{a^2} \\
\frac{b'}{b} &= \frac{1}{4} \frac{c}{b^2} \\
\frac{c'}{c} &= 8\frac{1}{c} - \frac{1}{4}\frac{c}{b^2} - \frac{3}{4}\frac{c}{a^2}
\end{aligned}
\tag{5.20}
$$

The details of the calculations can be found in Appendix B. Next, we discuss the solutions of the above system.

Without loss of generality, we assume that the orbit on which we choose the initial conditions is at $t = 0$. We denote the initial values $a(0)$, $b(0)$, and $c(0)$ by a_0, b_0, and c_0. Furthermore, we define $C(t) := \int_0^t c(s) ds$. The first two equations of the system (5.20) can be simplified to:

$$
\begin{aligned}
a^2 &= \tfrac{3}{4} C + a_0^2 \\
b^2 &= \tfrac{1}{2} C + b_0^2
\end{aligned}
$$

By inserting this into the last equation, we obtain:

$$c' = 8 - \frac{c^2}{2C + 4b_0^2} - \frac{3c^2}{3C + 4a_0^2} \,.$$

As in Section 5.2, we define $\tau := C(t)$ and denote the derivation with respect to τ by a dot. With this notation, the above equation becomes:

$$\dot{c}c = 8 - \frac{c^2}{2\tau + 4b_0^2} - \frac{3c^2}{3\tau + 4a_0^2} \,.$$

Since this is a linear differential equation for c^2, it can be explicitly solved. For the homogeneous problem, we obtain the following solution:

$$c(\tau)^2 = K \frac{1}{(\tau + 2b_0^2)(\tau + \frac{4}{3}a_0^2)^2} \,,$$

where K is a constant. The most general solution of the inhomogeneous equation for c is:

$$c(\tau)^2 = \frac{1}{(\tau + 2b_0^2)(\tau + \frac{4}{3}a_0^2)^2} \left(K + 8 \int_0^\tau (s + 2b_0^2)(s + \tfrac{4}{3}a_0^2)^2 ds \right) \,.$$

Next, we have to determine the value of K. By inserting $\tau = 0$ into the above equation, we obtain $K = \frac{32}{9} a_0^4 b_0^2 c_0^2$:

$$\begin{aligned} c(\tau)^2 = &\frac{32 a_0^4 b_0^2 c_0^2}{9(\tau + 2b_0^2)(\tau + \frac{4}{3}a_0^2)^2} \\ &+ \frac{8}{(\tau + 2b_0^2)(\tau + \frac{4}{3}a_0^2)^2} \int_0^\tau (s + 2b_0^2)(s + \tfrac{4}{3}a_0^2)^2 ds \,. \end{aligned} \tag{5.21}$$

We are finally able to give an explicit description of the cohomogeneity-one metric:

$$\begin{aligned} g := &(\tfrac{3}{4}\tau + a_0^2)(e^1 \otimes e^1 + e^2 \otimes e^2 + e^3 \otimes e^3 + e^4 \otimes e^4) \\ &+ (\tfrac{1}{2}\tau + b_0^2)(e^5 \otimes e^5 + e^6 \otimes e^6) + c(\tau)^2 (e^7 \otimes e^7 + d\tau^2) \,. \end{aligned} \tag{5.22}$$

As in Section 5.2, we put back the question of the compactness and completeness until we have determined the possible initial values at the singular orbit.

Our next question is if there are any other cohomogeneity-one metrics with principal orbit $M^{1,1,0}$ and a holonomy contained in Spin(7). We fix an invariant metric g and an orientation on $M^{1,1,0}$ and consider the set of all invariant G_2-structures with the same associated metric and orientation. According to Lemma 3.1.50, this set can be described by:

$$(\mathrm{Norm}_{SO(7)}(\mathfrak{su}(2)_1 \oplus \mathfrak{u}(1)))/(\mathrm{Norm}_{G_2}(\mathfrak{su}(2)_1 \oplus \mathfrak{u}(1))) \,,$$

where $\mathrm{Norm}_G(\mathfrak{su}(2)_1 \oplus \mathfrak{u}(1))$ is defined as:

$$\{g \in G \mid g(\mathfrak{su}(2)_1 \oplus \mathfrak{u}(1))g^{-1} = \mathfrak{su}(2)_1 \oplus \mathfrak{u}(1)\} .$$

We do not need the explicit form of the normalizers, since we can apply the knowledge which we have acquired in Section 5.2: Any homogeneous G_2-structure on $M^{1,1,0}$ corresponds to a $\mathfrak{su}(2)_1 \oplus \mathfrak{u}(1)$-equivariant map $\psi : \mathfrak{m} \to \mathbb{R}^7$. As usual, let $2\mathfrak{u}(1) \subseteq \mathfrak{su}(2)_1 \oplus \mathfrak{u}(1)$ be the standard Cartan subalgebra of $\mathfrak{g}_2$, which acts on $\mathfrak{m}$ and on $\mathbb{R}^7$. ψ is obviously also $2\mathfrak{u}(1)$-equivariant. The set of all $2\mathfrak{u}(1)$-invariant G_2-structures on the tangent space $\mathfrak{m}$ which can be extended to a fixed $SO(7)$-structure is generated by a $U(1)$-action on one of those G_2-structures. This fact was proven in Section 5.2. The $U(1)$-action on the G_2-structures is generated by another $U(1)$-action on $\mathfrak{m}$. This action is not unique, since we can change it by elements of G_2. We have seen in Section 5.2 that one possibility for the action on $\mathfrak{m}$ is described with respect to $(e_1, \ldots, e_7)$ by the following matrices:

$$S := \left\{ \left(\begin{array}{cccc} \boxed{R_\theta} & & & \\ & \boxed{R_\theta} & & \\ & & \boxed{R_\theta} & \\ & & & \boxed{1} \end{array} \right) \middle| \ \theta \in \mathbb{R} \right\} , \tag{5.23}$$

where R_θ denotes the rotation in the plane around an angle of θ. Conjugation by any element of S leaves not only $2\mathfrak{u}(1)$ but also $\mathfrak{su}(2)_1 \oplus \mathfrak{u}(1)$ invariant. Therefore, the action of S on a fixed $SU(3) \times SU(2)$-invariant G_2-structure ω indeed generates the set of all invariant G_2-structures which have the same extension to a $SO(7)$-structure as ω. These G_2-structures also can be obtained as the pull-back of ω by certain isometries of $M^{1,1,0}$. We consider the group:

$$T := \left\{ \left(\begin{array}{cc} \boxed{\begin{array}{ccc} e^{\frac{i\varphi}{2}} & 0 & 0 \\ 0 & e^{\frac{i\varphi}{2}} & 0 \\ 0 & 0 & e^{-i\varphi} \end{array}} & \\ & \boxed{\begin{array}{cc} e^{-\frac{i\varphi}{2}} & 0 \\ 0 & e^{\frac{i\varphi}{2}} \end{array}} \end{array} \right) =: T_\varphi \ \middle| \ \varphi \in \mathbb{R} \right\} .$$

T is a subgroup of $\mathrm{Norm}_{SU(3)\times SU(2)}(SU(2) \times U(1))$. Therefore, conjugation by T_φ induces a well-defined diffeomorphism of $M^{1,1,0}$. We prove that this diffeomorphism is even an isometry. The neutral element e of $SU(3) \times SU(2)$ is fixed by conjugation by T_φ. The differential of this map at e is determined by the adjoint action of $T_\varphi \in SU(3) \times SU(2)$ restricted to $\mathfrak{m}$. Since the tangent space of T is spanned by e_7 and T is connected, we only have to prove that $\mathrm{ad}_{e_7}|_{\mathfrak{m}}$ is skew-symmetric with respect to g. We obtain by a straightforward calculation:

$$\mathrm{ad}_{e_7}|_{\mathfrak{m}} = \begin{pmatrix} 0 & -\frac{3}{2} & & & & & \\ \frac{3}{2} & 0 & & & & & \\ & & 0 & -\frac{3}{2} & & & \\ & & \frac{3}{2} & 0 & & & \\ & & & & 0 & 1 & \\ & & & & -1 & 0 & \\ & & & & & & 0 \end{pmatrix}$$

with respect to the basis $(e_1, \ldots, e_7)$. This endomorphism is clearly skew-symmetric. It is easy to see that it is orientation-preserving, too. Unfortunately, the adjoint action of T on $\mathfrak{m}$ does not coincide with the action (5.23). Nevertheless, it generates the same family of G_2-structures as the action of S on $\mathfrak{m}$. The reason for this simply is that S and the action of T both are contained in $\mathrm{Norm}_{SO(7)}(\mathfrak{su}(2)_1 \oplus \mathfrak{u}(1))$, but not in $\mathrm{Norm}_{G_2}(\mathfrak{su}(2)_1 \oplus \mathfrak{u}(1))$. In fact, we could choose T as any connected one-dimensional subgroup of the standard maximal torus of $SU(3) \times SU(2)$ which is not contained in $SU(2) \times U(1)$.

To sum up, we have proven that the one-parameter family of all invariant G_2-structures with the same associated metric and orientation as ω can be obtained by the action of certain isometries. Furthermore, this family is diffeomorphic to a circle. We conclude with help of similar arguments as in Section 5.2 that the holonomy of our metrics is $\subseteq SU(4)$. The existence of a parallel vector field can be excluded as in 5.2, too. Therefore, the holonomy is either $SU(4)$, $Sp(2)$, or $SU(2) \times SU(2)$ and the metric thus is Kähler.

The Kähler form η has to be $SU(3) \times SU(2)$-invariant. The space of all two-forms on $\mathfrak{m}$ which are invariant with respect to the $U(1)$-factor of $SU(2) \times U(1)$ is spanned by:

$$e^{12}\,, \quad e^{34}\,, \quad e^{56}\,, \quad e^7 \wedge dt\,.$$

η therefore has to be of the following type:

$$\eta = \epsilon_1 a^2\, e^{12} + \epsilon_2 a^2\, e^{34} + \epsilon_3 b^2\, e^{56} + c\, e^7 \wedge dt \quad \text{with} \quad \epsilon_1,\, \epsilon_2,\, \epsilon_3 \in \{-1, 1\}\,. \tag{5.24}$$

The exterior derivative of η is given by:

$$\begin{aligned} d\eta = {} & \epsilon_1\, 2a'a\, dt \wedge e^{12} - \epsilon_1\, a^2\, (3e^{272} + 3e^{117}) \\ & + \epsilon_2\, 2a'a\, dt \wedge e^{34} - \epsilon_2\, a^2\, (3e^{474} + 3e^{337}) \\ & + \epsilon_3\, 2b'b\, dt \wedge e^{56} - \epsilon_3\, b^2 (-2e^{676} - 2e^{557}) \\ & - c\left(\tfrac{3}{4}e^{12} + \tfrac{3}{4}e^{34} - \tfrac{1}{2}e^{56}\right) \wedge dt \end{aligned}$$

The condition $d\eta = 0$ is thus equivalent to:

$$\frac{a'}{a} = \epsilon_1 \frac{3}{8}\frac{c}{a^2}\,, \qquad \frac{a'}{a} = \epsilon_2 \frac{3}{8}\frac{c}{a^2}\,, \qquad \frac{b'}{b} = -\epsilon_3 \frac{1}{4}\frac{c}{b^2}\,.$$

In order to make these equations consistent with (5.20), we have to choose $\epsilon_1 = \epsilon_2 = 1$ and $\epsilon_3 = -1$. We obtain:

$$\eta = a^2\, e^{12} + a^2\, e^{34} - b^2\, e^{56} + c\, e^7 \wedge dt\,. \tag{5.25}$$

If η is closed, no other two-form of type (5.24) can be closed, too. Otherwise, we had $c = 0$, which is impossible, since the metric has to be positive definite. By the same arguments as in Section 5.2, it therefore follows that the holonomy of the metrics which satisfy (5.20) has to be all of $SU(4)$.

We finally consider the behavior of our metrics near a singular orbit. Any singular orbit is a homogeneous space of type $(SU(3) \times SU(2))/K$. The possibilities for K are described by the following lemma:

Lemma 5.3.1. *Let $SU(2) \times U(1)$ be embedded into $SU(3) \times SU(2)$ by the map $\imath_{1,1,0}$, which we have defined at the beginning of this section. Furthermore, let K be a connected, closed group with $SU(2) \times U(1) \subsetneq K \subseteq SU(3) \times SU(2)$. The Lie algebra of K we denote by $\mathfrak{k}$. In this situation, $\mathfrak{k}$ and K can be found in the table below. Furthermore, $K/(SU(2) \times U(1))$ and $(SU(3) \times SU(2))/K$ satisfy the following topological conditions:*

$\mathfrak{k}$	K	$K/(SU(2) \times U(1))$	$(SU(3) \times SU(2))/K$
$\mathfrak{su}(2) \oplus 2\mathfrak{u}(1)$	$U(2) \times U(1)$	$\cong S^1$	$\cong \mathbb{CP}^2 \times S^2$
$2\mathfrak{su}(2) \oplus \mathfrak{u}(1)$	$U(2) \times SU(2)$	$\cong S^3$	$\cong \mathbb{CP}^2$
$\mathfrak{su}(3) \oplus \mathfrak{u}(1)$	$SU(3) \times U(1)$	$\cong S^5/\mathbb{Z}_3$	$\cong S^2$
$\mathfrak{su}(3) \oplus \mathfrak{su}(2)$	$SU(3) \times SU(2)$	$= M^{1,1,0} \not\cong S^7/\Gamma$	

where $\Gamma \subseteq O(8)$ is an arbitrary discrete subgroup and the group $\mathbb{Z}_3$ which we divide out is $\{\lambda\, \mathrm{Id}_{\mathbb{C}^3} | \lambda^3 = 1\}$.

Proof : $\mathfrak{su}(2) \oplus \mathfrak{u}(1)$ acts on $\mathfrak{su}(3) \oplus \mathfrak{su}(2)$ by the restriction of the adjoint action. $\mathfrak{k}$ obviously is a $\mathfrak{su}(2) \oplus \mathfrak{u}(1)$-submodule of $\mathfrak{su}(3) \oplus \mathfrak{su}(2)$. Since we have decomposed $\mathfrak{su}(3) \oplus \mathfrak{su}(2)$ into its submodules, we see that there are only few possibilities for $\mathfrak{k}$:

1. $\mathfrak{k} = \mathfrak{su}(2) \oplus \mathfrak{u}(1) \oplus V_3$: Since $[e_7, \mathfrak{su}(2) \oplus \mathfrak{u}(1)] = 0$, K is covered by $SU(2) \times U(1) \times U(1)$, where the last factor is generated by e_7. $\mathfrak{k}$ contains the standard Cartan subalgebra of $\mathfrak{su}(3) \oplus \mathfrak{su}(2)$. We therefore can rewrite $\mathfrak{k}$ as $\mathfrak{su}(2) \oplus W_1 \oplus W_2$, where W_1 is a one-dimensional Lie algebra which is generated by an element of $\mathfrak{su}(3)$ commuting with $\mathfrak{su}(2)$ and W_2 is generated by an element of the $\mathfrak{su}(2)$-summand of $\mathfrak{su}(3) \oplus \mathfrak{su}(2)$. As we will see in the third case, the intersection of $SU(2)$ and the Lie group which is generated by W_1 is isomorphic to $\mathbb{Z}_2$. The Lie group which corresponds to $\mathfrak{su}(2) \oplus W_1$ therefore is isomorphic to $(SU(2) \times U(1))/\mathbb{Z}_2 \cong U(2)$. Since the intersection of the Lie groups which are associated to $\mathfrak{su}(2) \oplus W_1$ and W_2 is trivial, K is isomorphic to $U(2) \times U(1)$. The quotient $K/(SU(2) \times U(1))$ is a one-dimensional Lie group, which is diffeomorphic to a circle. Furthermore, we can conclude that $(SU(3) \times SU(2))/K = SU(3)/S(U(2) \times U(1)) \times SU(2)/U(1) = \mathbb{CP}^2 \times S^2$.

2. $\mathfrak{k} = \mathfrak{su}(2) \oplus \mathfrak{u}(1) \oplus V_2$: We can easily find elements $x, y \in V_2$ such that $q([x,y], e_7) \neq 0$. Since $e_7 \notin \mathfrak{su}(2) \oplus \mathfrak{u}(1) \oplus V_2$, $\mathfrak{k}$ is not closed under the Lie bracket.

3. $\mathfrak{k} = \mathfrak{su}(2) \oplus \mathfrak{u}(1) \oplus V_2 \oplus V_3$: It is easy to see that $[V_2, V_2] \subseteq \mathrm{span}(e_7, e_{11})$, $[V_2, V_3] \subseteq V_2$, $[V_3, V_3] = 0$, and of course $[\mathfrak{su}(2) \oplus \mathfrak{u}(1), V_2 \oplus V_3] \subseteq V_2 \oplus V_3$. These relations prove that $\mathfrak{k}$ is closed under the Lie bracket. We obtain $\mathfrak{k} = \mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1)$. The first summand is spanned by (e_8, e_9, e_{10}). The second summand coincides with the second summand of $\mathfrak{su}(3) \oplus \mathfrak{su}(2)$ and the last summand is spanned by:

$$\left(\begin{array}{ccc|cc} i & 0 & 0 & & \\ 0 & i & 0 & & \\ 0 & 0 & -2i & & \\ \hline & & & 0 & 0 \\ & & & 0 & 0 \end{array}\right). \tag{5.26}$$

Let $i_1 : SU(2) \to SU(3) \times SU(2)$ be the map which is induced by the embedding of the first $\mathfrak{su}(2)$-summand. Analogously, we denote the map which is induced by the embedding of the second $\mathfrak{su}(2)$-summand by i_2 and define a map i_3 which is associated to the embedding of the $\mathfrak{u}(1)$-summand into $\mathfrak{su}(3) \oplus \mathfrak{su}(2)$. The map

$$\begin{aligned} \pi : SU(2) \times SU(2) \times U(1) &\to K \\ \pi(X, Y, Z) &:= i_1(X) \cdot i_2(Y) \cdot i_3(Z) \end{aligned}$$

is a covering map of K. Since its kernel is

$$\left\{ \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, 1 \right), \left(\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, -1 \right) \right\},$$

$SU(2) \times SU(2) \times U(1)$ is a two-fold cover of K. More precisely, we have $K = (SU(2) \times U(1))/\mathbb{Z}_2 \times SU(2) \cong U(2) \times SU(2)$.

For our considerations, we need some information on the topology of K/H. In order to keep our notation short, we denote the isotropy group $SU(2) \times U(1)$ of the $SU(3) \times SU(2)$-action shortly by H. Let $\pi^{-1}(H)_e$ be the identity component of $\pi^{-1}(H)$. The abelian factor of H is

$$\left\{ \left(\begin{array}{ccc|cc} e^{\frac{it}{3}} & 0 & 0 & & \\ 0 & e^{\frac{it}{3}} & 0 & & \\ 0 & 0 & e^{-\frac{2it}{3}} & & \\ \hline & & & e^{it} & 0 \\ & & & 0 & e^{-it} \end{array}\right) \middle| \; t \in \mathbb{R} \right\}.$$

Therefore,

$$\pi^{-1}(H)_e = \left\{ \left(X, \begin{pmatrix} e^{3it} & 0 \\ 0 & e^{-3it} \end{pmatrix}, e^{it} \right) \middle| \; X \in SU(2), t \in \mathbb{R} \right\}.$$

Since $\ker \pi \subseteq \pi^{-1}(H)_e$, $\pi^{-1}(H)$ is connected. We take a look at the quotient $\pi^{-1}(K)/\pi^{-1}(H) \cong (SU(2) \times SU(2) \times U(1))/(SU(2) \times U(1))$. Since we can cancel the first $SU(2)$-factor, this quotient is a space of type $(SU(2) \times U(1))/U(1)$. The group

$$\left\{ \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} e^{3it} & 0 \\ 0 & e^{-3it} \end{pmatrix}, \left(e^{it}\right) \right) \,\middle|\, t \in \mathbb{R} \right\},$$

by which we divide, intersects the second factor of $SU(2) \times SU(2) \times U(1)$ only at $t \in 2\pi\mathbb{Z}$. By Lemma 3.1.12, we see that $\pi^{-1}(K)/\pi^{-1}(H) = S^3$. The map

$$\begin{aligned} \pi' \,:\, \pi^{-1}(K)/\pi^{-1}(H) &\to K/H \\ \pi'(k \cdot \pi^{-1}(H)) &:= \pi(k) \cdot H \end{aligned}$$

is a diffeomorphism, since $\pi'^{-1}(e \cdot H) = e \cdot \pi^{-1}(H)$. Thus, $K/H \cong S^3$. From the explicit description of K, which we have found above, we obtain $(SU(3) \times SU(2))/K \cong SU(3)/S(U(2) \times U(1)) \times SU(2)/SU(2) = \mathbb{CP}^2$.

4. $\mathfrak{k} = \mathfrak{su}(2) \oplus \mathfrak{u}(1) \oplus V_1$: As in the second case, we can easily find elements $x, y \in V_1$ such that $q([x,y], e_7) \neq 0$. Therefore, $\mathfrak{k}$ is not closed under the Lie bracket.

5. $\mathfrak{k} = \mathfrak{su}(2) \oplus \mathfrak{u}(1) \oplus V_1 \oplus V_3$: In this situation, we have $[V_1, V_1] \subseteq \mathfrak{su}(2) \oplus \mathfrak{u}(1) \oplus V_1 \oplus V_3$, $[V_1, V_3] \subseteq V_1 \oplus \mathfrak{su}(2)$, $[V_3, V_3] = 0$, and $[\mathfrak{su}(2) \oplus \mathfrak{u}(1), V_1 \oplus V_3] \subseteq V_1 \oplus V_3$. These facts prove that $\mathfrak{k}$ is closed under the Lie bracket. We have $K = SU(3) \times U(1)$, where the second factor is:

$$\left\{ \left(\begin{array}{ccc|cc} 0 & 0 & 0 & & \\ 0 & 0 & 0 & & \\ 0 & 0 & 0 & & \\ \hline & & & e^{it} & 0 \\ & & & 0 & e^{-it} \end{array} \right) \,\middle|\, t \in \mathbb{R} \right\}.$$

Let $U(1)'$ be the abelian factor of the isotropy group $SU(2) \times U(1) \subseteq SU(3) \times SU(2)$. The intersection $U(1)' \cap SU(3)$ is a group which is isomorphic to $\mathbb{Z}_3$ and is generated by:

$$\begin{pmatrix} e^{\frac{2\pi i}{3}} & 0 & 0 \\ 0 & e^{\frac{2\pi i}{3}} & 0 \\ 0 & 0 & e^{-\frac{4\pi i}{3}} \end{pmatrix} = e^{\frac{2\pi i}{3}} \, \mathrm{Id}_{\mathbb{C}^3} \,. \tag{5.27}$$

We conclude with help of Lemma 3.1.12 that $K/(SU(2) \times U(1)) = (SU(3)/SU(2))/\mathbb{Z}_3 = S^5/\mathbb{Z}_3$. Furthermore, we have $(SU(3) \times SU(2))/K = SU(2)/U(1) = S^2$.

6. $\mathfrak{k} = \mathfrak{su}(2) \oplus \mathfrak{u}(1) \oplus V_1 \oplus V_2$: Analogously to the fourth case, we see that $\mathfrak{k}$ is not closed under the Lie bracket.

7. $\mathfrak{k} = \mathfrak{su}(2) \oplus \mathfrak{u}(1) \oplus V_1 \oplus V_2 \oplus V_3$: In this case, $\mathfrak{k} = \mathfrak{su}(3) \oplus \mathfrak{su}(2)$, $K = SU(3) \times SU(2)$, and $K/(SU(2) \times U(1)) = M^{1,1,0}$. We assume that $M^{1,1,0}$ is a quotient of S^7 by a discrete group Γ. For our argument, we need the second homotopy group of $\mathbb{CP}^2$. We have:

$$\ldots \to \underbrace{\pi_2(S^5)}_{=\{0\}} \to \pi_2(\mathbb{CP}^2) \to \underbrace{\pi_1(S^1)}_{=\mathbb{Z}} \to \underbrace{\pi_1(S^5)}_{=\{0\}} \to \ldots$$

and therefore $\pi_2(\mathbb{CP}^2) = \mathbb{Z}$. $M^{1,1,0}$ is a circle bundle over the space $\mathbb{CP}^2 \times S^2$. Hence, we have the following exact sequence:

$$\ldots \to \underbrace{\pi_2(S^7/\Gamma)}_{=\{0\}} \to \underbrace{\pi_2(\mathbb{CP}^2 \times S^2)}_{=\mathbb{Z}^2} \to \underbrace{\pi_1(S^1)}_{=\mathbb{Z}} \to \ldots .$$

Since there exists no injective group homomorphism from $\mathbb{Z}^2$ to $\mathbb{Z}$, the above exact sequence is impossible.

□

Remark 5.3.2. In the case $K = SU(3) \times U(1)$, the space $K/(SU(2) \times U(1))$ is a quotient of a sphere by an effective action of a non-trivial discrete group. The resulting space of cohomogeneity one therefore is an orbifold instead of a manifold.

We investigate the geometric properties of the cohomogeneity-one metrics satisfying (5.20) which have a singular orbit. Without loss of generality, we assume that the singular orbit we consider is at $t = 0$. For each of the three possible cases from the above lemma we have $c(0) = 0$. On the following pages, we will see that $c'(0)$ is either 8, 4, or $\frac{8}{3}$. If the underlying space of cohomogeneity one was compact, there would be exactly two singular orbits. Near the orbit at $t = 0$, c would have to be positive and near the second one it would be negative. Therefore, there would exist a third orbit on which c vanishes. Since this is impossible, the space is non-compact.

We will check if the metrics which we have found are complete. As in Section 5.2, we only have to investigate the smoothness conditions from Theorem 3.2.18 and if $\int_0^\infty |c(\tau)| d\tau = \infty$ in order to do this. The reason why we consider the integral is that $\int_0^\infty |c(\tau)| d\tau$ measures the length of the geodesics which intersect all orbits perpendicularly. On the previous pages, we have seen that our metrics are described by the equations (5.21) and (5.22). Since $c(0)$ vanishes, we have

$$c(\tau)^2 = \frac{8}{(\tau + 2b_0^2)(\tau + \frac{4}{3}a_0^2)^2} \int_0^\tau (s + 2b_0^2)(s + \tfrac{4}{3}a_0^2)^2 ds \,.$$

The above term behaves as $O(\tau)$. Therefore, we have proven one of the conditions for the completeness. We are going to check the smoothness conditions for each of the possible singular orbits separately and start with $\mathbb{CP}^2 \times S^2$. The tangent space $\mathfrak{p}$ of $\mathbb{CP}^2 \times S^2$ decomposes into $V_1 \oplus V_2$ with respect to the action of $SU(2) \times U(1)$. This splitting is also preserved by the action of K. For our considerations, we need some information on the K-module $S^2(\mathfrak{p})$. As in Section 5.2, it suffices to know the trivial submodules of this space. The maximal trivial $SU(2) \times U(1)$-submodule of $S^2(\mathfrak{p})$ is:

$$\mathrm{span}(e_1 \otimes e_1 + e_2 \otimes e_2 + e_3 \otimes e_3 + e_4 \otimes e_4,\ e_5 \otimes e_5 + e_6 \otimes e_6) \,.$$

It is easy to see that this space is trivial with respect to K, too. The normal space $\mathfrak{p}^\perp$ is of dimension two. Moreover, our chosen geodesic which intersects all orbits perpendicularly is fixed by the action of $SU(2) \times U(1)$. Since there is a straight line in $\mathfrak{p}^\perp$ which is left invariant, $\mathfrak{p}^\perp$ is a trivial module with respect to $SU(2) \times U(1)$. The subgroup of K which is generated by e_7 acts on $\mathfrak{p}^\perp$ with a non-zero weight r, which we need not to specify for now. We decompose $S^m(\mathfrak{p}^\perp)$ into irreducible K-modules and obtain similar formulas as for the S^1-collapse in Section 5.2:

$$S^m(\mathfrak{p}^\perp) = \begin{cases} \mathbb{V}_{rm} \oplus \mathbb{V}_{r(m-2)} \oplus \ldots \oplus \mathbb{V}_0 & \text{if } m \text{ is even} \\ \mathbb{V}_{rm} \oplus \mathbb{V}_{r(m-2)} \oplus \ldots \oplus \mathbb{V}_r & \text{if } m \text{ is odd} \end{cases}$$

In this formula, $\mathbb{V}_s$ denotes the irreducible K-module on which e_7 acts with weight s and $SU(2) \times U(1)$ acts trivially on. We take a look at the space W^h_m which consists of all K-equivariant maps from $S^m(\mathfrak{p}^\perp)$ to $S^2(\mathfrak{p})$. Since any element of $S^m(\mathfrak{p}^\perp)$ is $SU(2) \times U(1)$-invariant, its image has to be $SU(2) \times U(1)$-invariant, too. Furthermore, since the $SU(2) \times U(1)$-invariant elements of $S^2(\mathfrak{p})$ are also K-invariant, their preimages have to be K-invariant, too. The above considerations prove that W^h_m is trivial if and only if m is odd. If m is even, W^h_m is two-dimensional and the $\phi \in W^h_m$ are given by:

$$\phi\left(\bigotimes^{\frac{m}{2}} (\tilde{e}_1 \otimes \tilde{e}_1 + \tilde{e}_2 \otimes \tilde{e}_2)\right) = \lambda_1(e_1 \otimes e_1 + e_2 \otimes e_2 + e_3 \otimes e_3 + e_4 \otimes e_4) + \lambda_2(e_5 \otimes e_5 + e_6 \otimes e_6)\,.$$

$(\tilde{e}_1, \tilde{e}_2)$ denotes an orthonormal basis of $\mathfrak{p}^\perp$, λ_1 and λ_2 are in $\mathbb{R}$, and ϕ shall vanish on the orthogonal complement of $\bigotimes^m (\tilde{e}_1 \otimes \tilde{e}_1 + \tilde{e}_2 \otimes \tilde{e}_2)$. Therefore, we have:

$$\dim W^h_m = \begin{cases} 2 & \text{if } m \text{ is even} \\ 0 & \text{if } m \text{ is odd} \end{cases}$$

The interpretation of these numbers is that the odd derivatives of a^2 and b^2 have to vanish and that we can choose $(a^2)^{(2m)}(0)$ and $(b^2)^{(2m)}(0)$ arbitrarily. Therefore, a and b are either odd or even functions. Since $a(0), b(0) \neq 0$, a and b have to be even functions in order to make the metric smooth. We turn to the vertical part of the metric, which is described by $W^v_m = \operatorname{Hom}_K(S^m(\mathfrak{p}^\perp), S^2(\mathfrak{p}^\perp))$. $S^2(\mathfrak{p}^\perp)$ decomposes into an irreducible two-dimensional and a trivial one-dimensional module. As in Section 5.2, we can deduce that

$$\dim W^v_m = \begin{cases} 1 & \text{if } m = 0 \\ 0 & \text{if } m \text{ is odd} \\ 3 & \text{if } m \geqslant 2 \text{ is even} \end{cases}$$

We have to discuss the dimensions of the spaces W^v_m, too. Let g^v_t be the metric on the collapsing circle. The elements of W^v_m correspond to the possible values of $\lim_{t\to 0}\left(\frac{1}{t^2} g^v_t\right)^{(m)}$. In the case $m = 0$, we thus obtain the degree of freedom for $c'(0)$. By the choice of our coordinates, we have fixed $\|\frac{\partial}{\partial t}\|$ to be 1. Therefore, g^v_t has to coincide up to first order with the unique K-invariant metric q on $\mathfrak{p}^\perp$ which satisfies $q(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}) = 1$. It follows that $c'(0)$ is in fact fixed. We recall the arguments we have made for the S^1-collapse in Section 5.2.

Analogously to that case, $2|c'(0)|$ has to coincide with the cardinality of $U(1)' \cap (SU(2) \times U(1))$, where $U(1)'$ is generated by e_7. We have:

$$U(1)' = \left\{ \left(\begin{array}{ccc|cc} e^{\frac{it}{2}} & 0 & 0 & & \\ 0 & e^{\frac{it}{2}} & 0 & & \\ 0 & 0 & e^{-it} & & \\ \hline & & & e^{-\frac{it}{2}} & 0 \\ & & & 0 & e^{\frac{it}{2}} \end{array} \right) \,\middle|\, t \in \mathbb{R} \right\}$$

and

$$SU(2) \times U(1) = \left\{ \left(\begin{array}{ccc|cc} a_{11} & a_{12} & 0 & & \\ a_{21} & a_{22} & 0 & & \\ 0 & 0 & e^{-\frac{2it}{3}} & & \\ \hline & & & e^{it} & 0 \\ & & & 0 & e^{-it} \end{array} \right) \,\middle|\, \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & e^{-\frac{2it}{3}} \end{pmatrix} \in SU(3)\,, \quad t \in \mathbb{R} \right\}$$

By comparing the third and the fourth line of the above two matrices, we obtain the equations $e^{-is} = e^{-\frac{2it}{3}}$ and $e^{-\frac{is}{2}} = e^{it}$, which the elements of $U(1)' \cap (SU(2) \times U(1))$ have to satisfy for some $s, t \in \mathbb{R}$. These conditions are equivalent to the following system:

$$\begin{array}{rcl} -s + \frac{2}{3}t & = & 2\pi k_1 \\ -\frac{1}{2}s - t & = & 2\pi k_2 \end{array}$$

where $k_1, k_2 \in \mathbb{Z}$. Its solution is

$$\begin{array}{rcl} s & = & -\frac{3}{2}\pi k_1 - \pi k_2 \\ t & = & \frac{3}{4}\pi k_1 - \frac{3}{2}\pi k_2 \end{array}$$

The smallest possible non-zero value of s is $\frac{\pi}{2}$. Therefore

$$U(1)' \cap SU(2) \times U(1) = \left\{ \left(\begin{array}{ccc|cc} e^{\frac{ik}{4}\pi} & 0 & 0 & & \\ 0 & e^{\frac{ik}{4}\pi} & 0 & & \\ 0 & 0 & e^{-\frac{ik}{2}\pi} & & \\ \hline & & & e^{-\frac{ik}{4}\pi} & 0 \\ & & & 0 & e^{\frac{ik}{4}\pi} \end{array} \right) \,\middle|\, k \in \mathbb{Z} \right\}$$

The cardinality of this group is 8. Thus, we have $|c'(0)| = 4$. Next, we consider the spaces W^v_{2m} with $m \geqslant 1$. The three dimensions of those spaces correspond to the higher derivatives of $g_t(\frac{\partial}{\partial t}, \frac{\partial}{\partial t})$, $g_t(\frac{\partial}{\partial t}, e_7)$, and $g_t(e_7, e_7)$. By the choice of our coordinates, we have $g_t(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}) = 1$ and $g_t(\frac{\partial}{\partial t}, e_7) = 0$. Therefore, our only freedom is in the choice of $g_t(e_7, e_7)$. By the same arguments as on page 120, we see that c^2 is an even function. Since $c'(0) \neq 0$, c has to be odd. All in all, we have found the following necessary and sufficient conditions for the smoothness of a metric of type (5.17) where a, b, and c are analytic functions:

- a and b are even.
- c is odd.
- $|c'(0)| = 4$.

If $(a(t), b(t), c(t))$ is a solution of the equations (5.20), $(a(-t), b(-t), -c(-t))$ solves those equations, too. Moreover, the initial values of both solutions are the same. The metrics with singular orbit $\mathbb{CP}^2 \times S^2$ which we have found thus satisfy the first two of our conditions. It directly follows from (5.20) that for any metric with $a(0), b(0) \neq 0$ and $c(0) = 0$ we have $c'(0) = 8$. Therefore, our metrics do not satisfy the conditions from Theorem 3.2.18 and are not smooth at the singular orbit.

We will classify the cohomogeneity-one Einstein metrics with principal orbit $M^{1,1,0}$ and singular orbit $\mathbb{CP}^2 \times S^2$. Those metrics are not one of our examples with holonomy $SU(4)$, since they satisfy $|c'(0)| = 4$. $\mathfrak{p}^\perp$ is a trivial $SU(2) \times U(1)$-module and $\mathfrak{p}$ decomposes into the non-trivial modules V_1 and V_2. Assumption 3.2.19 thus is satisfied. Furthermore, we have calculated the dimension of W^h_m for all $m \in \mathbb{N}_0$ and of W^v_2. Therefore, we are able to apply Theorem 3.2.24. With help of that theorem, we see that in the horizontal direction there are two initial conditions of 0^{th} order which we can prescribe. They correspond to $a(0)$ and $b(0)$. The free parameter in the vertical direction, which corresponds to W^v_2, is for similar reasons as in the previous section given by $c'''(0)$.

Next, we take a look at the case $\mathfrak{k} = \mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1)$. Instead of describing a necessary and sufficient condition for the smoothness of a metric with singular orbit $\mathbb{CP}^2$, we only prove a sufficient condition. We claim that any metric of type (5.17) for which

- a, b, and c are analytic,
- a is even,
- b and c are odd,
- $|b'(0)| = 1$, and $|c'(0)| = 4$,

is smooth at the singular orbit. We will be able to prove that there exist cohomogeneity-one Einstein metrics with certain initial conditions. The uniqueness of those metrics could only be proven by calculating $\dim W^v_2$ and $\dim W^h_m$, which we will not do. As in the proof of Lemma 5.3.1, the first summand of $\mathfrak{k}$ denotes the subalgebra of $\mathfrak{su}(3) \subseteq \mathfrak{su}(3) \oplus \mathfrak{su}(2)$ which is contained in the isotropy algebra. The second summand coincides with the second summand of $\mathfrak{su}(3) \oplus \mathfrak{su}(2)$ and the abelian summand is generated by

$$\begin{pmatrix} \boxed{\begin{matrix} i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -2i \end{matrix}} & \\ & \boxed{\begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix}} \end{pmatrix}$$

The tangent space $\mathfrak{p}$ of the singular orbit can be identified with V_1. The first summand of $\mathfrak{k}$ acts by the complex two-dimensional representation of $\mathfrak{su}(2)$ on $\mathfrak{p}$ and the second one acts trivially. When we have classified the invariant metrics on $M^{1,1,0}$, we have seen that the only trivial $\mathfrak{su}(2) \oplus \mathfrak{u}(1)$-submodule of $S^2(\mathfrak{p})$ is

$$\mathrm{span}(e_1 \otimes e_1 + e_2 \otimes e_2 + e_3 \otimes e_3 + e_4 \otimes e_4) .$$

This space is of course invariant with respect to the second summand of $\mathfrak{su}(3) \oplus \mathfrak{su}(2)$, too, and therefore is a K-module. The normal space $\mathfrak{p}^\perp$ of the singular orbit is four-dimensional. We discuss the action of the three summands of $\mathfrak{k}$ on $\mathfrak{p}^\perp$ separately. The first $\mathfrak{su}(2)$-summand fixes one point of the collapsing sphere, since it is contained in the isotropy algebra. The tangent space of the sphere at that point can be identified with the complement of $\mathfrak{su}(2) \oplus \mathfrak{u}(1)$ in $\mathfrak{k}$. This complement is given by $\mathrm{span}(e_5, e_6, e_7)$. Since the first $\mathfrak{su}(2)$-summand commutes with e_5, e_6, and e_7, its action on the tangent space is trivial. Therefore, the action of the first summand on $\mathfrak{p}^\perp$ is trivial, too. The second $\mathfrak{su}(2)$-summand of $\mathfrak{k}$ cannot act trivially on $\mathfrak{p}^\perp$. Otherwise, the K-orbits of the non-zero elements of $\mathfrak{p}^\perp$ would not be spheres. Since the collapsing sphere is a coset space of type $(SU(2) \times U(1))/U(1)$, we moreover can deduce that the second summand acts by the two-dimensional complex representation of $\mathfrak{su}(2)$ on $\mathfrak{p}^\perp$. $\mathfrak{k}$ can be generated by $2\mathfrak{su}(2) \subseteq \mathfrak{k}$ and the abelian summand of the isotropy algebra $\mathfrak{su}(2) \oplus \mathfrak{u}(1)$. In order to describe the action of all of $\mathfrak{k}$ on $\mathfrak{p}^\perp$, it suffices to investigate how the above summand acts on $\mathfrak{p}^\perp$ instead of the abelian part of $\mathfrak{k}$. The action of the abelian summand of $\mathfrak{su}(2) \oplus \mathfrak{u}(1)$ is determined by the action of e_7 on the tangent space $V_2 \oplus V_3$ of the collapsing sphere. The weights with which e_7 act can be easily calculated. Since we will not need their value, we will not do that calculation. Instead of decomposing $S^m(\mathfrak{p}^\perp)$ into its submodules, we will explicitly describe the $\mathfrak{k}$-equivariant maps which correspond to the derivatives of the metric we are interested in. Let $(\widetilde{e}_1, \ldots, \widetilde{e}_4)$ be an orthonormal basis of $\mathfrak{p}^\perp$ such that the matrix representation of the second $\mathfrak{su}(2)$-summand with respect to $(\widetilde{e}_1, \ldots, \widetilde{e}_4)$ is the standard one. As in the previous section, we have

$$dt \otimes dt = \widetilde{e}^1 \otimes \widetilde{e}^1 + \widetilde{e}^2 \otimes \widetilde{e}^2 + \widetilde{e}^3 \otimes \widetilde{e}^3 + \widetilde{e}^4 \otimes \widetilde{e}^4 .$$

Since we want a to be even, we have to consider the value of $(a^2)^{(2m)}(0)$. The derivative of a^2 is in the t-direction and we thus obtain the following map $\phi : S^{2m}(\mathfrak{p}^\perp) \to S^2(\mathfrak{p})$ which is associated to $(a^2)^{(2m)}(0)$:

$$\phi\left(\bigotimes^m \left(\widetilde{e}^1 \otimes \widetilde{e}^1 + \widetilde{e}^2 \otimes \widetilde{e}^2 + \widetilde{e}^3 \otimes \widetilde{e}^3 + \widetilde{e}^4 \otimes \widetilde{e}^4\right)\right) = e^1 \otimes e^1 + e^2 \otimes e^2 + e^3 \otimes e^3 + e^4 \otimes e^4 .$$

As usual, we assume that ϕ vanishes on the orthogonal complement of $\bigotimes^m(\widetilde{e}^1 \otimes \widetilde{e}^1 + \widetilde{e}^2 \otimes \widetilde{e}^2 + \widetilde{e}^3 \otimes \widetilde{e}^3 + \widetilde{e}^4 \otimes \widetilde{e}^4)$. Since ϕ maps a K-invariant object to another K-invariant object, it is a

K-equivariant map. Therefore, we can choose $(a^2)^{(2m)}(0)$ arbitrarily. In particular, a can be chosen as any even analytic function.

Next, we have to consider the vertical part of the metric. Since we have a group of type $SU(2)$ acting irreducibly on the four-dimensional space $\mathfrak{p}^\perp$, we are in a similar situation as in Section 5.2, when we have considered the S^3-collapse. By the same method as in that case, we are able to construct the maps which are associated to the odd derivatives of b and c. More precisely: Let γ be our chosen geodesic which intersects all orbits perpendicularly. We assume without loss of generality that $\gamma'(0) = \widetilde{e}_1$. The tangent space of the collapsing sphere is spanned by $(\widetilde{e}_2, \widetilde{e}_3, \widetilde{e}_4)$. Since that tangent space can also be identified with the complement of $\mathfrak{su}(2) \oplus \mathfrak{u}(1)$ in $\mathfrak{k}$, we will work with the basis (e_5, e_6, e_7) instead of $(\widetilde{e}_2, \widetilde{e}_3, \widetilde{e}_4)$. $\mathrm{span}(e_5, e_6, e_7)$ is a $SU(2) \times U(1)$-module, which decomposes into $V_2 \oplus V_3$. We can extend any term of type

$$b^2(e^5 \otimes e^5 + e^6 \otimes e^6) + c^2\, e^7 \otimes e^7$$

by the action of K to a well-defined K-invariant metric on the sphere. Let $\phi_1^m, \phi_2^m : S^{2m}(\mathfrak{p}^\perp) \to S^2(\mathfrak{p}^\perp)$ be the maps which correspond to $\left(\frac{1}{t^2}b^2\right)^{(2m)}(0)$ and $\left(\frac{1}{t^2}c^2\right)^{(2m)}(0)$. Motivated by the arguments from Section 5.2 and the above considerations, we see that these maps satisfy:

$$\begin{aligned} \phi_1^m\left(\bigotimes^{m-1}(dt\otimes dt) \vee (L_k^*\widetilde{e}^1) \otimes (L_k^*\widetilde{e}^1)\right) &= (L_k^* e^5) \otimes (L_k^* e^5) + (L_k^* e^6) \otimes (L_k^* e^6) \\ \phi_2^m\left(\bigotimes^{m-1}(dt\otimes dt) \vee (L_k^*\widetilde{e}^1) \otimes (L_k^*\widetilde{e}^1)\right) &= (L_k^* e^7) \otimes (L_k^* e^7)\,, \end{aligned} \tag{5.28}$$

where L_k^* is the pull-back of the left-multiplication by $k \in SU(2)$ and the ϕ_i^m with $i \in \{1,2\}$ are extended by polarization to all of $\bigotimes^{m-1}(dt\otimes dt) \vee S^2(\mathfrak{p}^\perp)$. On the orthogonal complement of that space, both maps vanish. By their definition, ϕ_1^m and ϕ_2^m are equivariant with respect to the second $\mathfrak{su}(2)$-summand. With respect to the first $\mathfrak{su}(2)$-summand, they are equivariant, too, since that algebra acts trivially on $\mathfrak{p}^\perp$. The abelian part of the isotropy algebra $\mathfrak{su}(2)\oplus\mathfrak{u}(1)$ leaves the terms on the left and the right hand side of (5.28) invariant. The maps ϕ_1^m and ϕ_2^m thus are $\mathfrak{k}$-equivariant. We have proven that the value of $\left(\frac{1}{t^2}b^2\right)^{(2m)}(0)$ and $\left(\frac{1}{t^2}c^2\right)^{(2m)}(0)$ can be chosen arbitrarily for all $m \geqslant 1$. Therefore, the $(2m+1)^{th}$ derivatives of b and c can be chosen arbitrarily, too, if $m \geqslant 1$.

It remains to prove that $b'(0)$ and $c'(0)$ have the value which we have claimed. As we have remarked in the previous section, the metric h on $SU(2)$ with constant sectional curvature 1 is determined by $h(X,Y) = -\frac{1}{2}\mathrm{tr}(XY)$ for all $X, Y \in \mathfrak{su}(2)$. The following matrices are an orthonormal basis of $\mathfrak{su}(2)$ with respect to h:

$$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

The second and the third matrix correspond to e_5 and e_6, since it is the second factor of $SU(3) \times SU(2)$ which acts irreducibly on $\mathfrak{p}^\perp$. We want the metric on the collapsing sphere to have sectional curvature $\frac{1}{t^2} + O(\frac{1}{t})$ for small values of t. This is only possible if $|b'(0)| = 1$. The matrix e_7 whose length is measured by c generates the one-dimensional Lie group $U(1)'$. The action of the group $U(1)'$ generates a great circle on the collapsing sphere which coincides with the collapsing circle in the previous case. We thus can repeat the arguments which we have made in that case. We again obtain $|c'(0)| = 4$. All in all, we have proven that the criteria which we have proposed are indeed sufficient for the smoothness of the metric.

In order to complete our considerations, we have to prove that the solutions of (5.20) satisfy these sufficient conditions. By a short calculation, we see that any solution of (5.20) with $b(0) = c(0) = 0$ satisfies $b'(0) \in \{-1, 1\}$ and $c'(0) = 4$. We remark that it suffices to restrict ourselves to the case $b'(0) = 1$, since we can replace b by $-b$ without changing our system of differential equations. Let $(a(t), b(t), c(t))$ be a solution of (5.20). Then $(a(-t), -b(-t), -c(-t))$ is a solution of those equations, too. Furthermore, its values at $t = 0$ coincide with the initial values of $(a(t), b(t), c(t))$. Since the metric we obtain from the initial value problem is in both cases the same, we have proven that a is even and b as well as c is odd.

As a by-product of our considerations, we can prove the existence of certain cohomogeneity-one Einstein metrics with singular orbit $\mathbb{CP}^2$. The semisimple part of the isotropy group $SU(2) \times U(1)$ acts irreducibly on $\mathfrak{p}$, which coincides with V_1, and trivially on $\mathfrak{p}^\perp$. Therefore, Assumption 3.2.19 is satisfied and we can work with Theorem 3.2.24. Since we always can prescribe the metric on the singular orbit in the setting of Theorem 3.2.24, the initial value $a(0)$ can be chosen freely. The maps (5.28) are two linearly independent elements of W_2^v. Since those maps correspond to the functions b and c, we also can prescribe $b'''(0)$ and $c'''(0)$ arbitrarily.

We finally consider the case where $\mathfrak{k} = \mathfrak{su}(3) \oplus \mathfrak{u}(1)$ and we have S^2 as singular orbit. As in the previous case, we only prove a sufficient condition for the smoothness of the metric. Our claim is that any metric of type (5.17), where

- a, b, and c are analytic,
- b is even,
- a and c are odd,
- $|a'(0)| = 1$, and $|c'(0)| = 4$,

is a smooth metric on the underlying orbifold. Again, we will prove the existence of cohomogeneity-one Einstein metrics with singular orbit S^2 and certain initial values. In our situation, the tangent space $\mathfrak{p}$ of the singular orbit can be identified with V_2. The Lie algebra $\mathfrak{su}(3)$ clearly acts trivially on that space. It is easy to see that the submodule

$$\operatorname{span}(e_5 \otimes e_5 + e_6 \otimes e_6)$$

of $S^2(\mathfrak{p})$ is trivial with respect to all of $\mathfrak{su}(3) \oplus \mathfrak{u}(1)$ and that it is the only trivial submodule of $S^2(\mathfrak{p})$. The normal space $\mathfrak{p}^\perp$ of the singular orbit is a quotient of a six-dimensional vector space by a discrete group which is isomorphic to $\mathbb{Z}_3$. Since that group acts by complex multiples of the identity map, our considerations are essentially the same as if $\mathfrak{p}^\perp$ was a vector space and the underlying space was a manifold. In particular, we can apply the results of Eschenburg and Wang [32]. Since $\mathfrak{p}^\perp$ is six-dimensional and the orbit of any non-zero element of $\mathfrak{p}^\perp$ has to be of type $S^5/\mathbb{Z}_3$, $\mathfrak{su}(3)$ acts by its six-dimensional irreducible representation on $\mathfrak{p}^\perp$.

In order to prove our claim, we have to describe the K-equivariant maps which correspond to the derivatives of the horizontal and vertical part of the metric. Let $(\widetilde{e}_1, \ldots, \widetilde{e}_6)$ be an orthonormal basis of $\mathfrak{p}^\perp$ such that the matrix representation of $\mathfrak{su}(3)$ with respect to this basis is the same as of $\mathfrak{su}(3) \subseteq \mathfrak{gl}(6,\mathbb{C})$. The map $\phi : S^{2m}(\mathfrak{p}^\perp) \to S^2(\mathfrak{p})$ which corresponds to $(b^2)^{(2m)}(0)$ is analogously to the previous case determined by:

$$\phi\left(\bigotimes^m \left(\widetilde{e}^1 \otimes \widetilde{e}^1 + \ldots + \widetilde{e}^6 \otimes \widetilde{e}^6\right)\right) = e^5 \otimes e^5 + e^6 \otimes e^6 .$$

On the rest of $S^{2m}(\mathfrak{p}^\perp)$, ϕ vanishes. Since this map is $\mathfrak{k}$-equivariant, we can choose the even derivatives of b^2 arbitrarily. Thus, b can be chosen as any analytic even function. In order to prove that we can choose the odd derivatives of a and c freely, we make a similar argument as in the case $\mathfrak{k} = \mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1)$. Again, we assume that the geodesic γ which we have fixed satisfies $\gamma'(0) = \widetilde{e}_1$. The normal space of $\widetilde{e}_1$ can be identified with the tangent space $V_1 \oplus V_3$ of $S^5/\mathbb{Z}_3$. Any invariant metric on $S^5/\mathbb{Z}_3$ is of the following type:

$$a^2(e^1 \otimes e^1 + e^2 \otimes e^2 + e^3 \otimes e^3 + e^4 \otimes e^4) + c^2\, e^7 \otimes e^7 .$$

We define the following map $\widetilde{\phi} : S^{2m}(\mathfrak{p}^\perp) \to S^2(\mathfrak{p}^\perp)$:

$$\begin{aligned}\widetilde{\phi}\left(\bigotimes^{m-1}(dt \otimes dt) \vee (L_k^*\widetilde{e}^1) \otimes (L_k^*\widetilde{e}^1)\right) := {}& a^2((L_k^* e^1) \otimes (L_k^* e^1) + (L_k^* e^2) \otimes (L_k^* e^2) \\ & + (L_k^* e^3) \otimes (L_k^* e^3) + (L_k^* e^4) \otimes (L_k^* e^4)) \\ & + c^2 (L_k^* e^7) \otimes (L_k^* e^7)\end{aligned}$$

L_k^* denotes the pull-back of the left-multiplication by $k \in SU(3)$. The above map is as usually extended by polarization and vanishes on the complement of $\bigotimes^{m-1}(dt \otimes dt) \vee S^2(\mathfrak{p}^\perp)$. If k is the neutral element, the terms on both sides of the above equation are invariant with respect to the isotropy group $SU(2) \times U(1)$ of the K-action. Therefore, the above map is well-defined. $\widetilde{\phi}$ is $SU(3)$-equivariant by its definition. We have to prove if it is also equivariant with respect to the abelian summand of $\mathfrak{k}$. Since $\mathfrak{k}$ can be generated by $\mathfrak{su}(3)$ and the abelian part of the isotropy algebra $\mathfrak{su}(2) \oplus \mathfrak{u}(1)$, we can prove instead the equivariance with respect to the abelian summand. We have already remarked that $\widetilde{\phi}$ is equivariant with respect to $SU(2) \times U(1)$ and thus have proven the K-equivariance. $\widetilde{\phi}$ is the map which is associated to $\left(\frac{1}{t^2} g_t^v\right)^{(2m)}(0)$, where g_t^v is the metric on the collapsing sphere. We therefore can conclude by the usual arguments that $a^{(2m+1)}(0)$ and $c^{(2m+1)}(0)$ can be chosen freely for $m \geqslant 1$.

The values of $a'(0)$ and $c'(0)$ which would make our metric smooth up to first order can be obtained as in the previous case. The metric h on $SU(3)/SU(2)$ with constant sectional curvature 1 is induced by $h(X,Y) = -\frac{1}{2}\mathrm{tr}(XY)$ for all X and Y in the orthogonal complement of $\mathfrak{su}(2) \subseteq \mathfrak{su}(3)$. Since V_1 is contained in $\mathfrak{su}(3)$, we can conclude that $|a'(0)| = h(e_1, e_1) = 1$. For the same reasons as in the other cases, we again obtain $|c'(0)| = 4$.

We check if our solutions of (5.20) satisfy the conditions for the smoothness which we have found. It is easy to see that indeed b is even and a as well as c are odd. By a short calculation, we obtain $a'(0) = \pm 1$ and $c'(0) = \frac{8}{3}$ as initial values. Since $|c'(0)| \neq 4$, the metrics with singular orbit S^2 have are not smooth at the singular orbit.

As in the case $\mathfrak{k} = \mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1)$, we can prove with help of Theorem 3.2.24 the existence of certain cohomogeneity-one Einstein metrics. $\mathfrak{p}$ can be identified with V_2 and

$\mathfrak{p}^\perp$ decomposes with respect to $SU(2) \times U(1)$ into V_1 and a two-dimensional trivial module. Therefore, $\mathfrak{p}$ and $\mathfrak{p}^\perp$ have no equivalent $SU(2) \times U(1)$-submodules in common and Assumption 3.2.19 is satisfied. It follows by similar arguments as in the previous case that for any choice of $b(0)$, $a'''(0)$, and $c'''(0)$ there exists a cohomogeneity-one Einstein metric with principal orbit $M^{1,1,0}$ and singular orbit S^2. Since those metrics satisfy $|c'(0)| = 4$, our solutions of (5.20) are not one of them. Finally, we sum up the results of this section:

Theorem 5.3.3. *Let (M, Ω) be a parallel Spin(7)-orbifold with a cohomogeneity-one action of $SU(3) \times SU(2)$ which preserves Ω. We assume that the principal orbit is a coset space of type $M^{k,l,0}$. In this situation, the following statements are true:*

1. *The principal orbit is $SU(3) \times SU(2)$-equivariantly diffeomorphic to $M^{1,1,0}$.*
2. *The metric g which is associated to Ω is special Kähler and its holonomy is all of $SU(4)$.*
3. *g has the matrix representation (5.17) with respect to the basis $(e_1, \ldots, e_7)$ from page 129. The metric functions satisfy the equations (5.20), whose solutions are described by (5.21) and (5.22). The Kähler form is given by (5.25).*

If M has a singular orbit, which has to be the case if (M, g) is complete, it is $\mathbb{CP}^2 \times S^2$, $\mathbb{CP}^2$, or S^2. Any $SU(3) \times SU(2)$-invariant metric on the singular orbit can be extended to a unique cohomogeneity-one metric with holonomy $SU(4)$, which is non-compact.

1. *If the singular orbit is $\mathbb{CP}^2 \times S^2$, M is a manifold, but the metric g is not differentiable at the singular orbit. Outside of the singular orbit, g is smooth and any geodesic which does not intersect the singular orbit can be infinitely extended.*
2. *If the singular orbit is $\mathbb{CP}^2$, M is a manifold and the metric is smooth and complete.*
3. *If the singular orbit is S^2, M is an orbifold but not a manifold. At the singular orbit, the metric is not differentiable, i.e. it is not a smooth orbifold metric. Outside of the singular orbit, g is smooth and any geodesic which does not intersect the singular orbit can be infinitely extended.*

In the first two cases M is a vector bundle over the singular orbit and in the third case it is a $\mathbb{C}^3/\mathbb{Z}_3$-bundle over S^2.

With help of Theorem 3.2.24, we have also obtained results on cohomogeneity-one Einstein metrics with principal orbit $M^{1,1,0}$:

Theorem 5.3.4. *Let M be a cohomogeneity-one orbifold whose principal orbit is $M^{1,1,0}$. We assume that M has a singular orbit, which has to be $\mathbb{CP}^2 \times S^2$, $\mathbb{CP}^2$, or S^2. Any $SU(3) \times SU(2)$-invariant metric on M is described by a matrix of type (5.17) or equivalently by the three metric functions $a, b, c : I \to \mathbb{R}$, where I is an interval. Without loss of generality, we assume that the singular orbit is at $0 \in I$. In this situation, the following statements are true:*

1. *Let the singular orbit be $\mathbb{CP}^2 \times S^2$. For any choice of $a_0, b_0, c_3, \lambda \in \mathbb{R}$, there exists a unique $SU(3) \times SU(2)$-invariant Einstein metric on a tubular neighborhood of $\mathbb{CP}^2 \times S^2$ such that:*

   (a) $a(0) = a_0$, $b(0) = b_0$,
   (b) $c'''(0) = c_3$, *and*
   (c) *the Einstein constant is* λ.

2. *Let the singular orbit be* $\mathbb{CP}^2$. *For any choice of* $a_0, b_3, c_3, \lambda \in \mathbb{R}$, *there exists a* $SU(3) \times SU(2)$*-invariant Einstein metric on a tubular neighborhood of* $\mathbb{CP}^2$ *such that:*
   (a) $a(0) = a_0$,
   (b) $b'''(0) = b_3$, $c'''(0) = c_3$, *and*
   (c) *the Einstein constant is* λ.

3. *Let the singular orbit be* S^2. *For any choice of* $a_3, b_0, c_3, \lambda \in \mathbb{R}$, *there exists a* $SU(3) \times SU(2)$*-invariant Einstein metric on a tubular neighborhood of* S^2 *such that:*
   (a) $b(0) = b_0$,
   (b) $a'''(0) = a_3$, $c'''(0) = c_3$, *and*
   (c) *the Einstein constant is* λ.

Remark 5.3.5. 1. It is likely that the Ricci-flat Kähler metrics with singular orbit $\mathbb{CP}^2 \times S^2$ can be modified by a similar construction as we have described at the end of Section 5.2 in such a way that they become smooth.

2. The Kähler metrics with singular orbit $\mathbb{CP}^2 \times S^2$ have been considered by Herzog and Klebanov in [41] and by Cvetič et al. in [21]. The examples with singular orbit $\mathbb{CP}^2$ or S^2 are mentioned by the same authors in [21] and [25]. Our considerations prove that any complete parallel Spin(7)-orbifold of cohomogeneity-one with principal orbit $M^{1,1,0}$ is one of our examples. This result and those of the above Einstein metrics, which are not contained in Theorem 5.3.3, are, as far as the author knows, new.

5.4 The Aloff-Wallach spaces as principal orbits

The subject of this section are parallel cohomogeneity-one Spin(7)-structures whose principal orbit is of type $SU(3)/U(1)$. A coset space of that kind is called an *Aloff-Wallach space*. There are infinitely many distinct Aloff-Wallach spaces depending on the embedding of $U(1)$ into $SU(3)$. For two special kinds of embeddings, we obtain coset spaces which have different features than the other ones. They are called *exceptional Aloff-Wallach spaces*. Any Aloff-Wallach space admits a cosymplectic G_2-structure. Therefore, there exist (not necessarily complete) parallel cohomogeneity-one Spin(7)-structures Ω whose principal orbit can be chosen as any Aloff-Wallach space. By considering the equation $d\Omega = 0$ near a singular orbit, we obtain local examples of parallel Spin(7)-structures. They include those which have been found by Cvetič, Gibbons, Lü, and Pope [24], by Gukov and Sparks [39], and by Kanno and Yasui [48], [49]. We also will construct new examples with an exceptional Aloff-Wallach space as principal orbit. In the setting of this section, it is possible to prove classification results for the cohomogeneity-one metrics with reduced holonomy and the Einstein metrics, which are similar to the Theorems 5.2.2, 5.2.3, 5.3.3, and 5.3.4. In the course of our considerations, it will turn out that the calculations needed for the proof of such theorems would be more difficult than those in Section 5.2 and 5.3. The main reason for this is the existence of non-diagonal metrics on the exceptional Aloff-Wallach spaces. This makes the space of homogeneous metrics and G_2-structures larger than in the previous sections. We therefore will omit some cases, but still cover the most important aspects. Another difference to the previous sections is that the equations for the holonomy reduction have no explicit solutions except in some special cases. Therefore, it is not a priori clear if our examples can be extended to complete metrics. This question has been considered by Cvetič et al. in [24] and in the recent paper of Bazaikin [5], but here we mostly focus on the classification of the local metrics.

Before we construct any examples of parallel Spin(7)-structures or Einstein metrics, we have to describe the Aloff-Wallach spaces themselves in more detail. The possible embeddings of $U(1)$ into $SU(3)$ are up to conjugacy given by:

$$\begin{aligned} i_{k,l} &: U(1) \to SU(3) \\ i_{k,l}\left(e^{i\varphi}\right) &:= \begin{pmatrix} e^{ik\varphi} & 0 & 0 \\ 0 & e^{il\varphi} & 0 \\ 0 & 0 & e^{-i(k+l)\varphi} \end{pmatrix} \quad \text{with } k,l \in \mathbb{Z}\,. \end{aligned}$$

We denote the image of $U(1)$ with respect to $i_{k,l}$ by $U(1)_{k,l}$ and the quotient $SU(3)/U(1)_{k,l}$ by $N^{k,l}$. For different pairs (k,l) we sometimes obtain the same space $N^{k,l}$.

Let, for example, $\lambda \in \gcd(k,l)^{-1}\mathbb{Z}$ be arbitrary. The spaces $N^{k,l}$ and $N^{\lambda k,\lambda l}$ coincide. Therefore, we will assume from now on that k and l are coprime. Let σ be a permutation of the triple $(k,l,-k-l)$. It is easy to see that the spaces $N^{k,l}$ and $N^{\sigma(k),\sigma(l)}$ are $SU(3)$-equivariantly diffeomorphic. This diffeomorphism could be explicitly described by conjugation by the permutation matrix of σ. The S_3-action on the indices $(k,l,-k-l)$ also has a deeper meaning: It is well-known that the root system of the Lie algebra $\mathfrak{su}(3)$ is of type A_2. Its Weyl group is the permutation group S_3, which acts on the dual of any Cartan subalgebra of $\mathfrak{su}(3)$. In this section, we fix the following Cartan subalgebra:

$$\mathfrak{t} := \left\{ \left(\begin{array}{ccc} ix & 0 & 0 \\ 0 & iy & 0 \\ 0 & 0 & -i(x+y) \end{array} \right) \middle| \ x, y \in \mathbb{R} \right\}$$

and identify $\mathfrak{t}$ with its dual space by the Killing form. The action on $\mathfrak{t}$ by a σ in the Weyl group maps the Lie algebra $\mathfrak{u}(1)_{k,l}$ of $U(1)_{k,l}$ into $\mathfrak{u}(1)_{\sigma(k),\sigma(l)}$. Therefore, the action of S_3 on $(k, l, -k-l)$ is induced by the action of the Weyl group. Our considerations motivate the following convention:

Convention 5.4.1. Let $k, l, k', l' \in \mathbb{Z}$ and $(k, l, -k-l) = \lambda(\sigma(k'), \sigma(l'), \sigma(-k'-l'))$ for a $\lambda \in \gcd(\sigma(k'), \sigma(l'))^{-1}\mathbb{Z}$ and a $\sigma \in S_3$. Since the Aloff-Wallach spaces $N^{k,l}$ and $N^{k',l'}$ are $SU(3)$-equivariantly diffeomorphic, we will write in this situation $N^{k,l} \cong N^{k',l'}$. "$\cong$" is obviously an equivalence relation. The Aloff-Wallach spaces in the equivalence class of $N^{1,0}$ or $N^{1,1}$ are called *exceptional* and the other ones are called *generic Aloff-Wallach spaces*. Since we have to consider only one representant in the equivalence class of $N^{k,l}$, we may assume that k and l are coprime and $k \geqslant l \geqslant -k-l$. If $l \leqslant 0$, we replace $N_{k,l}$ by $N_{k+l,-l}$. Therefore, it is permitted to assume without loss of generality that k and l are both non-negative. On the next pages, we will always use the convention $k \geqslant l \geqslant 0$. Later on, we will turn to another convention which will be introduced at that point.

We briefly review the history of mathematical research on the Aloff-Wallach spaces: The spaces $N^{k,l}$ have been first considered by Simon Aloff and Nolan Wallach. They have proven in their paper [2] that on any $N^{k,l}$ except on $N^{1,0}$ there exists a homogeneous metric with strictly positive sectional curvature.

There are infinitely many homotopy types of Aloff-Wallach spaces. This follows from the fact that

$$H^4(N^{k,l}, \mathbb{Z}) = \mathbb{Z}_{k^2+lk+l^2} \, .$$

Another interesting feature of the Aloff-Wallach spaces is that some of them are homeomorphic but not diffeomorphic. Examples of this fact are given in a paper of Kreck and Stolz [51].

On any Aloff-Wallach space there are up to homothety exactly two $SU(3)$-invariant Einstein metrics. First examples of those metrics have been constructed in the papers of Kobayashi [52], Jensen [46], and Wang [65]. It turned out that on the spaces $N^{k,l}$ there exist two Einstein metrics, which depend on k and l. Those metrics all have weak holonomy G_2 (see Cvetič et al. [24]). Therefore, the cones over those Riemannian manifolds have a holonomy, which is contained in Spin(7). Since we will see that the Aloff-Wallach spaces are not diffeomorphic to any quotient of a sphere by a discrete group, the cones have a singularity at the tip, which is not an orbifold singularity. Page and Pope [57] have proven that there are no further homogeneous Einstein metrics on the generic Aloff-Wallach spaces. In a paper of Nikonorov [56], the exceptional case is treated. The author proves that on $N^{1,1}$ the above two Einstein metrics are the only homogeneous Einstein metrics, too. On the space $N^{1,0}$ the two Einstein metrics are homothetic. Surprisingly, there exists another non-diagonal homogeneous Einstein metric on $N^{1,0}$, which is not homothetic to the other ones.

Our next step is to describe all $SU(3)$-invariant metrics on the Aloff-Wallach spaces. We first choose a basis $(e_1, \ldots, e_8)$ of $\mathfrak{su}(3)$:

$$e_1 := \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_2 := \begin{pmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$e_3 := \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad e_4 := \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix},$$

$$e_5 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad e_6 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{pmatrix},$$

$$e_7 := \begin{pmatrix} (2l+k)i & 0 & 0 \\ 0 & (-2k-l)i & 0 \\ 0 & 0 & (k-l)i \end{pmatrix}, \quad e_8 := \begin{pmatrix} ki & 0 & 0 \\ 0 & li & 0 \\ 0 & 0 & -(k+l)i \end{pmatrix}$$

We fix the biinvariant background metric q on $\mathfrak{su}(3)$ which is defined by $q(X,Y) := -tr(XY)$ for all $X, Y \in \mathfrak{su}(3)$. It is easy to see that $(e_1, \ldots, e_8)$ is orthogonal with respect to q. Let $\mathfrak{m}$ be the q-orthogonal complement of $\mathfrak{u}(1)_{k,l}$ in $\mathfrak{su}(3)$. $\mathfrak{m}$ is spanned by $(e_1, \ldots, e_7)$, and $\mathfrak{u}(1)_{k,l}$ is spanned by e_8. As usual, we fix a $p \in N^{k,l}$ and identify $T_pN^{k,l}$ by a $U(1)_{k,l}$-equivariant map with $\mathfrak{m}$. The matrix representation of $\mathrm{ad}_{e_8}|_{\mathfrak{m}}$ with respect to the basis $(e_1, \ldots, e_7)$ of $\mathfrak{m}$ is:

$$\begin{pmatrix} 0 & -k+l & & & & & \\ k-l & 0 & & & & & \\ & & 0 & -2k-l & & & \\ & & 2k+l & 0 & & & \\ & & & & 0 & -k-2l & \\ & & & & k+2l & 0 & \\ & & & & & & 0 \end{pmatrix} \tag{5.29}$$

It is easy to see that $\mathfrak{m}$ decomposes into the following irreducible $\mathfrak{u}(1)_{k,l}$-submodules:

$$\begin{aligned} V_1 &:= \mathrm{span}(e_1, e_2) \\ V_2 &:= \mathrm{span}(e_3, e_4) \\ V_3 &:= \mathrm{span}(e_5, e_6) \\ V_4 &:= \mathrm{span}(e_7) \end{aligned} \tag{5.30}$$

Later on, we will need the weight of the isotropy action on the above modules. Let $z \in U(1) \subseteq \mathbb{C}$ and $w \in \mathbb{C}$ be arbitrary. The $U(1)$-action on $\mathbb{C}$ with weight n is defined by $z.w := z^n w$. By identifying the complex number $a + bi := z^n$ with the matrix

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

we obtain an action of weight n on $\mathbb{R}^2$. The weight of a $U(1)$-action is determined only up to the sign, since we can replace n by $-n$ by a change of the basis of $\mathbb{R}^2$. In this section, we will make the following convention: If an irreducible $U(1)$-module comes with a basis such that the action of $U(1)$ has the above matrix representation, we denote the weight by $+n$. With respect to the bases which we have chosen in (5.30), the action of $U(1)_{k,l}$ on V_1, V_2, and V_3 has the following weights:

$$k-l,\; 2k+l,\; k+2l\,.$$

If we choose $k=l=1$, the module V_1 is trivial. Therefore, it decomposes into two one-dimensional submodules. Furthermore, the modules V_2 and V_3 are equivalent in this case. In the case $k=1$, $l=0$, the modules V_1 and V_3 have the same weight. In the other cases, all of the three modules are irreducible and pairwise inequivalent. This is the reason why $N^{1,1}$ and $N^{1,0}$ are called the exceptional Aloff-Wallach spaces. In the following, we consider the generic Aloff-Wallach spaces, $N^{1,0}$, and $N^{1,1}$ separately.

We now start describing the possible $SU(3)$-invariant metrics on the Aloff-Wallach spaces. As in the previous two sections, we can identify any such metric g on $N^{k,l}$ with a $\mathfrak{u}(1)_{k,l}$-equivariant, q-symmetric, positive definite endomorphism $\varphi : \mathfrak{m} \to \mathfrak{m}$, which is defined by $q(\varphi(X),Y) := g(X,Y)$. Conversely, any such φ defines an invariant metric on $N^{k,l}$. The possible φ can be classified by Schur's lemma. We first consider the generic Aloff-Wallach spaces. Since the submodules in which $\mathfrak{m}$ decomposes are pairwise inequivalent, the possible metrics are precisely those whose matrix representation with respect to the basis $(e_1,\ldots,e_7)$ is of type:

$$\begin{pmatrix} a^2 & 0 & & & & & \\ 0 & a^2 & & & & & \\ & & b^2 & 0 & & & \\ & & 0 & b^2 & & & \\ & & & & c^2 & 0 & \\ & & & & 0 & c^2 & \\ & & & & & & f^2 \end{pmatrix} \quad \text{with } a,b,c,f \in \mathbb{R}\backslash\{0\}\,. \tag{5.31}$$

As in Section 5.2 and 5.3, we allow both signs for a, b, c, and f, since the four-forms, which we will consider, contain odd powers of these parameters. Next, we consider $N^{1,0}$. The invariant metrics on this space can be described with respect to the basis $(e_1,e_2,e_5,e_6,e_3,e_4,e_7)$ by the following matrices:

$$\begin{pmatrix} a^2 & 0 & \beta_{1,5} & \beta_{1,6} & & & \\ 0 & a^2 & -\beta_{1,6} & \beta_{1,5} & & & \\ \beta_{1,5} & -\beta_{1,6} & c^2 & 0 & & & \\ \beta_{1,6} & \beta_{1,5} & 0 & c^2 & & & \\ & & & & b^2 & 0 & \\ & & & & 0 & b^2 & \\ & & & & & & f^2 \end{pmatrix} \quad \begin{array}{ll} \text{with} & a,b,c,f,\beta_{1,5},\beta_{1,6} \in \mathbb{R}\,, \\ & a^2c^2 \geqslant \beta_{1,5}^2+\beta_{1,6}^2\,, b \neq 0\,, f \neq 0\,. \end{array} \tag{5.32}$$

The last three conditions are necessary, since the determinant of the above matrix is $(a^2c^2 - \beta_{1,5}^2 - \beta_{1,6}^2)^2 b^4 f^2$. It should be noted that the upper left part of this matrix can be considered as the Hermitian matrix

$$\begin{pmatrix} a^2 & \beta_{1,5} - \beta_{1,6} i \\ \beta_{1,5} + \beta_{1,6} i & c^2 \end{pmatrix} .$$

Finally, we describe the $SU(3)$-invariant metrics on $N^{1,1}$. These metrics are precisely those whose matrix representation with respect to $(e_1, e_2, e_7, e_3, e_4, e_5, e_6)$ is a positive definite matrix of the following type:

$$\begin{pmatrix} a_1^2 & \beta_{1,2} & \beta_{1,7} & & & & \\ \beta_{1,2} & a_2^2 & \beta_{2,7} & & & & \\ \beta_{1,7} & \beta_{2,7} & f^2 & & & & \\ & & & b^2 & 0 & \beta_{3,5} & \beta_{3,6} \\ & & & 0 & b^2 & -\beta_{3,6} & \beta_{3,5} \\ & & & \beta_{3,5} & -\beta_{3,6} & c^2 & 0 \\ & & & \beta_{3,6} & \beta_{3,5} & 0 & c^2 \end{pmatrix} \tag{5.33}$$

In Lemma 3.2.36 and Remark 3.2.37 we have proven that the action of the normalizer on each principal orbit preserves the equation $d\Omega = 0$ and the cohomogeneity-one Einstein equation. It therefore may be possible to reduce the number of metrics which we have to consider by some parameters. In order to check if this is possible we have to determine the normalizer $\mathrm{Norm}_{SU(3)} U(1)_{k,l}$ for all k and l.

For our result, we need the following facts: Let $D := \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)$ with $\lambda_i \neq \lambda_j$ for $i \neq j$ be a diagonal matrix and let $A \in GL(3, \mathbb{C})$. If $\mathrm{Ad}_A(D)$ is a diagonal matrix, too, it has to be $\mathrm{diag}(\lambda_{\sigma(1)}, \lambda_{\sigma(2)}, \lambda_{\sigma(3)})$ for a permutation $\sigma \in S_3$. Furthermore, we can prove by a short calculation that $\mathrm{Ad}_A(D) = D$ is equivalent to A being a diagonal matrix. From these facts we can conclude that if $N^{k,l}$ is not $\cong N^{1,1}$, the normalizer acts trivially on $U(1)_{k,l}$ and we have:

$$\mathrm{Norm}_{SU(3)} U(1)_{k,l} = \left\{ \begin{pmatrix} e^{i\mu_1} & 0 & 0 \\ 0 & e^{i\mu_2} & 0 \\ 0 & 0 & e^{i\mu_3} \end{pmatrix} \middle| \; e^{i\mu_1} e^{i\mu_2} e^{i\mu_3} = 1 \right\} \cong U(1)^2 .$$

Next, we take a look at the $N^{1,1}$-case. Again, we first consider a diagonal matrix $D = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)$, but now assume that $\lambda_1 = \lambda_2 \neq \lambda_3$. By similar arguments as above, we can prove that $\mathrm{Ad}_A(D) = D$ if and only if

$$A = \begin{pmatrix} a_{11} & a_{12} & \\ a_{21} & a_{22} & \\ & & a_{33} \end{pmatrix} \quad \text{for some } a_{ij} \in \mathbb{C} .$$

Therefore,

$$\mathrm{Norm}_{SU(3)} U(1)_{1,1} = \left\{ \begin{pmatrix} a_{11} & a_{12} & \\ a_{21} & a_{22} & \\ & & a_{33} \end{pmatrix} \middle| \; a_{11}, a_{12}, a_{21}, a_{22}, a_{33} \in \mathbb{C} \right\} \cap SU(3) .$$

In fact, we have:

$$\mathrm{Norm}_{SU(3)}U(1)_{1,1} = \left\{ \begin{pmatrix} \boxed{A} & \\ & \boxed{\det A^{-1}} \end{pmatrix} \,\middle|\, A \in U(2) \right\} \cong U(2)\,.$$

We call the subgroup of all $A \in \mathrm{Norm}_{SU(3)}U(1)_{1,1}$ with $\det A = 1$ shortly $SU(2) \subseteq \mathrm{Norm}_{SU(3)}$ $U(1)_{1,1}$ or the $SU(2)$-factor of $\mathrm{Norm}_{SU(3)}U(1)_{1,1}$, although $U(2)$ is not isomorphic to $SU(2) \times U(1)$.

Let g be a $SU(3)$-invariant metric on $N^{k,l}$ and let φ be the corresponding endomorphism of $\mathfrak{m}$. The adjoint action of any $A \in \mathrm{Norm}_{SU(3)}U(1)_{k,l}$ on $\mathfrak{su}(3)$ leaves $\mathfrak{m}$ invariant. We identify A with its action on $\mathfrak{m}$. In the generic case, we have $A^T\varphi A = \varphi$ for all A in the normalizer. Therefore, the action of A does not change the metric and we cannot reduce the number of metrics which we have to consider in the generic case.

Next, we consider the $N^{1,0}$-case. If we choose A as an element of $U(1)_{1,0}$, its action of course leaves the metric invariant. If A is chosen as

$$A = \begin{pmatrix} e^{it} & 0 & 0 \\ 0 & e^{-2it} & 0 \\ 0 & 0 & e^{it} \end{pmatrix},$$

which is contained in $\mathrm{Norm}_{SU(3)}U(1)_{1,0}$, too, $\beta_{1,5} + \beta_{1,6}i$ is replaced by $e^{6it}(\beta_{1,5} + \beta_{1,6}i)$. Since the action of A on each principal orbit induces an isometry, we can identify all the metrics which we obtain by this action. Our calculations therefore prove that it suffices to consider a family of metrics which depends on 5 instead of 6 parameters.

We finally consider the $N^{1,1}$-case. The subgroup $SU(2)$ of $\mathrm{Norm}_{SU(3)}U(1)_{1,1}$ acts as $SO(3)$ on $\mathrm{span}(e_1, e_2, e_7)$. Since we can diagonalize the upper left part of the matrix (5.33) by this action, we can reduce the set of metrics which we have to consider to a 7-parameter family.

The isotropy group K of the $SU(3)$-action on the singular orbit is greater than $U(1)_{k,l}$. We will see that the $SU(3)$-invariance of the metric on the singular orbit makes that metric diagonal in all cases but one. The only exception is if the principal orbit is $N^{1,1}$ and K is generated by e_8 and a $x \in V_1 \oplus V_4$ which is not a multiple of e_1, e_2, or e_7. The fact that the metric on $SU(3)/K$ is diagonal will often force the metric on the whole manifold to be diagonal, too. For this reason, we will mostly consider the differential equations for the diagonal case and only make some short remarks on the non-diagonal case. Another reason, why we focus on the diagonal case, is of course that our calculations are much more easier in that case.

Our next step is to deduce a sufficient condition for the holonomy reduction. Before we can express the torsion-freeness of a $SU(3)$-invariant $\mathrm{Spin}(7)$-structure by a system of ordinary differential equations, we have to choose a frame for it. We start with the generic Aloff-Wallach spaces. Any invariant G_2-structure on $N^{k,l}$ is determined by a single element $\psi : \mathfrak{m} \to \mathrm{Im}(\mathbb{O})$ of the fiber. We consider $\mathfrak{u}(1)_{k,l}$ as a subalgebra of $\mathfrak{gl}(\mathfrak{m})$. ψ can be chosen in such a way that $\psi \circ \mathfrak{u}(1)_{k,l} \circ \psi^{-1}$ is contained in the Cartan subalgebra of $\mathfrak{g}_2$ which we have defined on page 22. Conversely, any ψ with this property determines an invariant G_2-structure. If we want the G_2-structure to be extendable to a chosen $SO(7)$-structure, we also have to require that ψ is orthogonal and orientation-preserving.

We will now construct an explicit ψ. The only imaginary octonions which are invariant with respect to the Cartan subalgebra of $\mathfrak{g}_2$ are the multiples of i. Since the action of $\mathfrak{u}(1)_{k,l}$ on $\mathfrak{m}$ leaves only e_7 invariant, ψ has to map e_7 to a multiple of i. The weights with which $\mathfrak{u}(1)_{k,l}$ acts on $\mathfrak{m}$ satisfy the relation:

$$(k-l) + (-1)(2k+l) = (-1)(k+2l)\,.$$

By reversing the order of (e_3, e_4) and (e_5, e_6), we also change the sign of the weights of V_2 and V_3. After that change, we are able to take $\mathfrak{u}(1)_{k,l}$ as a subalgebra of $\mathfrak{g}_2$. Since we can extend any G_2-structure on the principal orbit in a canonical way to a Spin(7)-structure on the whole manifold, the following $(f_0, \ldots, f_7)$ is a frame of an invariant Spin(7)-structure:

$$\begin{aligned} &f_0 := \tfrac{\partial}{\partial t} \quad && f_1 := \tfrac{1}{f} e_7 \quad && f_2 := \tfrac{1}{a} e_1 \quad && f_3 := \tfrac{1}{a} e_2 \\ &f_4 := \tfrac{1}{b} e_4 && f_5 := \tfrac{1}{b} e_3 && f_6 := \tfrac{1}{c} e_6 && f_7 := \tfrac{1}{c} e_5 \end{aligned} \tag{5.34}$$

The $SU(3)$-invariant four-form Ω, which is associated to this frame, is:

$$\begin{aligned} \Omega = & -abcf\, e^{1367} + abcf\, e^{1457} - abcf\, e^{2357} - abcf\, e^{2467} \\ & + a^2 b^2\, e^{1234} - a^2 c^2\, e^{1256} + b^2 c^2\, e^{3456} \\ & - a^2 f\, e^{127} \wedge dt + b^2 f\, e^{347} \wedge dt - c^2 f\, e^{567} \wedge dt \\ & - abc\, e^{135} \wedge dt - abc\, e^{146} \wedge dt + abc\, e^{236} \wedge dt - abc\, e^{245} \wedge dt \end{aligned} \tag{5.35}$$

Next, we express the condition $d\Omega = 0$ as a system of ordinary differential equations for the metric functions a, b, c, and f. By the usual methods we obtain:

$$\begin{aligned} \frac{a'}{a} &= \frac{b^2 + c^2 - a^2}{abc} + \frac{-k-l}{2\Delta} \frac{f}{a^2} \\ \frac{b'}{b} &= \frac{c^2 + a^2 - b^2}{abc} + \frac{l}{2\Delta} \frac{f}{b^2} \\ \frac{c'}{c} &= \frac{a^2 + b^2 - c^2}{abc} + \frac{k}{2\Delta} \frac{f}{c^2} \\ \frac{f'}{f} &= -\frac{-k-l}{2\Delta} \frac{f}{a^2} - \frac{l}{2\Delta} \frac{f}{b^2} - \frac{k}{2\Delta} \frac{f}{c^2} \end{aligned} \tag{5.36}$$

In the above system, we denote $k^2 + lk + l^2$ by Δ. Furthermore, we have replaced t by $-t$ for cosmetic reasons. This convention will be maintained throughout this section. The calculations we have made in order to derive the four-form (5.35) and the system (5.36) can be found in Appendix C.1.

Next, we consider the exceptional Aloff-Wallach space $N^{1,0}$. If we make the assumption that the restriction of the metric to any principal orbit is diagonal, we can choose the same frame and the same four-form Ω as above. Consequently, the equation $d\Omega = 0$ is equivalent to the system (5.36) with $k = 1$, $l = 0$, and $\Delta = 1$.

Finally, we consider $N^{1,1}$ as principal orbit. We again assume that the restriction of the metric to any principal orbit is diagonal. For the same reasons as in the other cases, there is a natural choice of the frame $(f_0, \ldots, f_7)$:

$$\begin{aligned} &f_0 := \tfrac{\partial}{\partial t} \quad && f_1 := \tfrac{1}{f} e_7 \quad && f_2 := \tfrac{1}{a_1} e_1 \quad && f_3 := \tfrac{1}{a_2} e_2 \\ &f_4 := \tfrac{1}{b} e_4 \quad && f_5 := \tfrac{1}{b} e_3 \quad && f_6 := \tfrac{1}{c} e_6 \quad && f_7 := \tfrac{1}{c} e_5 \end{aligned} \tag{5.37}$$

For the four-form Ω, which is associated to this frame, we obtain:

$$\begin{aligned} \Omega = &-a_1bcf\, e^{1367} + a_1bcf\, e^{1457} - a_2bcf\, e^{2357} - a_2bcf\, e^{2467} \\ &+ a_1a_2b^2\, e^{1234} - a_1a_2c^2\, e^{1256} + b^2c^2\, e^{3456} \\ &- a_1a_2f\, e^{127} \wedge dt + b^2f\, e^{347} \wedge dt - c^2f\, e^{567} \wedge dt \\ &- a_1bc\, e^{135} \wedge dt - a_1bc\, e^{146} \wedge dt + a_2bc\, e^{236} \wedge dt - a_2bc\, e^{245} \wedge dt \end{aligned} \tag{5.38}$$

The equation $d\Omega = 0$ is equivalent to:

$$\begin{aligned} \frac{a_1'}{a_1} &= \frac{b^2+c^2-a_1^2}{a_1bc} + 3\frac{a_1^2-a_2^2}{a_1a_2f} - \frac{1}{3}\frac{f}{a_1a_2} \\ \frac{a_2'}{a_2} &= \frac{b^2+c^2-a_2^2}{a_2bc} + 3\frac{a_2^2-a_1^2}{a_1a_2f} - \frac{1}{3}\frac{f}{a_1a_2} \\ \frac{b'}{b} &= \frac{1}{2}\frac{a_1^2+c^2-b^2}{a_1bc} + \frac{1}{2}\frac{a_2^2+c^2-b^2}{a_2bc} + \frac{1}{6}\frac{f}{b^2} \\ \frac{c'}{c} &= \frac{1}{2}\frac{a_1^2+b^2-c^2}{a_1bc} + \frac{1}{2}\frac{a_2^2+b^2-c^2}{a_2bc} + \frac{1}{6}\frac{f}{c^2} \\ \frac{f'}{f} &= -3\frac{(a_1-a_2)^2}{a_1a_2f} + \frac{1}{3}\frac{f}{a_1a_2} - \frac{1}{6}\frac{f}{b^2} - \frac{1}{6}\frac{f}{c^2} \end{aligned} \tag{5.39}$$

In the above system, we again have replaced t by $-t$. The detailed calculations which were necessary to obtain the four-form (5.38) and the system (5.39) can be found in Appendix C.2.

We consider the question if there are any $SU(3)$-invariant G_2-structures on the Aloff-Wallach spaces except those which we have constructed on the previous pages. Let g be an arbitrary invariant metric on an Aloff-Wallach space $N^{k,l}$. In Lemma 3.1.50 we have proven that the space of all invariant G_2-structures on $N^{k,l}$ whose associated metric is g and whose orientation is fixed can be described by:

$$\mathrm{Norm}_{SO(7)} U(1)_{k,l} / \mathrm{Norm}_{G_2} U(1)_{k,l}\,.$$

In this situation, $U(1)_{k,l}$ is identified with its representation on $\mathfrak{m}$ or $\mathrm{Im}(\mathbb{O})$ respectively. We first investigate the problem on the Lie algebra level. The tangent space of the above coset space can be identified with:

$$\mathrm{Norm}_{\mathfrak{so}(7)} \mathfrak{u}(1)_{k,l} / \mathrm{Norm}_{\mathfrak{g}_2} \mathfrak{u}(1)_{k,l}\,.$$

The Lie algebra $\mathfrak{u}(1)_{k,l}$ is generated by a $z \in \mathfrak{gl}(\mathrm{Im}(\mathbb{O}))$ whose matrix representation with respect to the standard basis of $\mathrm{Im}(\mathbb{O})$ is:

$$\begin{pmatrix} 0 & & & & & & \\ & 0 & -k+l & & & & \\ & k-l & 0 & & & & \\ & & & 0 & 2k+l & & \\ & & & -2k-l & 0 & & \\ & & & & & 0 & k+2l \\ & & & & & -k-2l & 0 \end{pmatrix}$$

By similar arguments as on page 110, we see that both of the above normalizers coincide with the corresponding centralizers. Let $\mathfrak{g}$ be either $\mathfrak{g}_2$ or $\mathfrak{so}(7)$ and let κ be the Killing form of $\mathfrak{g}$. A $x \in \mathfrak{g}$ is contained in the normalizer $\mathrm{Norm}_{\mathfrak{g}}\mathfrak{u}(1)_{k,l}$ if and only if

$$[x, z] = \lambda z \quad \text{for a} \quad \lambda \in \mathbb{R}\,.$$

From this relation, it follows that

$$0 = \kappa(x, [z, z]) = \kappa([x, z], z) = \lambda\, \kappa(z, z)\,.$$

The above equation can be satisfied for $\lambda = 0$ only and thus we have

$$\mathrm{Norm}_{\mathfrak{g}}\mathfrak{u}(1)_{k,l} = \{x \in \mathfrak{g} | [x, z] = 0\} = C_{\mathfrak{g}}\, \mathfrak{u}(1)_{k,l}\,.$$

We are going to determine $C_{\mathfrak{g}}\, \mathfrak{u}(1)_{k,l}$. Our considerations first will be done for the complexification $\mathfrak{g} \otimes \mathbb{C}$, since this will simplify some of our arguments. Any $x \in \mathfrak{g} \otimes \mathbb{C}$ has a Cartan decomposition

$$x = h + \sum_{\alpha \in \Phi} \mu_\alpha x_\alpha \quad \text{with } \mu_\alpha \in \mathbb{C}\,.$$

In the above formula, h is an element of the standard Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g} \otimes \mathbb{C}$, which is the complexification of the subalgebra from page 22 or 110 respectively. Furthermore, Φ is the root system of $\mathfrak{g} \otimes \mathbb{C}$ and x_α is a suitable generator of the root space L_α of α. With this notation, the centralizer can be described as follows:

$$C_{\mathfrak{g} \otimes \mathbb{C}}\left(\mathfrak{u}(1)_{k,l} \otimes \mathbb{C}\right) = \left\{ x \in \mathfrak{g} \otimes \mathbb{C} \,\middle|\, [z, x] = \sum_{\alpha \in \Phi} \alpha(z) \mu_\alpha x_\alpha = 0 \right\}.$$

Let $\Phi' := \{\alpha \in \Phi | \alpha(z) = 0\}$. The above formula can be simplified to:

$$C_{\mathfrak{g} \otimes \mathbb{C}}\left(\mathfrak{u}(1)_{k,l} \otimes \mathbb{C}\right) = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi'} L_\alpha\,.$$

We specialize to the case $\mathfrak{g} = \mathfrak{so}(7)$. As usual, we will identify the Cartan subalgebra of $\mathfrak{so}(7,\mathbb{C})$ and its dual space by the Killing form. In Section 5.2, we have chosen a basis $(\theta_1,\theta_2,\theta_3)$ of the Cartan subalgebra of $\mathfrak{so}(7,\mathbb{C})$. With respect to this basis, z becomes:

$$z = (-k+l)\theta_1 + (2k+l)\theta_2 + (k+2l)\theta_3 \,.$$

In the above formula, we had to multiply the weights by which z acts on $\mathrm{Im}(\mathbb{O})$ by -1. The reason for this are the signs we had chosen in Section 5.2 when we fixed a basis of the Cartan subalgebra of $\mathfrak{so}(7)$. Let $\alpha = \sum_{i=1}^{3}\alpha_i\theta_i$ be a weight of $\mathfrak{so}(7,\mathbb{C})$. The equation $\alpha(z) = 0$ is equivalent to:

$$(-k+l)\alpha_1 + (2k+l)\alpha_2 + (k+2l)\alpha_3 = 0 \,.$$

We recall that the root system of $\mathfrak{so}(7,\mathbb{C})$ is

$$\{\pm\theta_i | 1 \leqslant i \leqslant 3\} \cup \{\pm\theta_i \pm \theta_j | 1 \leqslant i < j \leqslant 3\} \,.$$

A root of the first kind can be contained in Φ' only if one of the coefficients $-k+l$, $2k+l$, $k+2l$ vanishes. This is the case if and only if $N^{k,l} \cong N^{1,1}$. In that situation, it is exactly one of those terms which vanishes. A root of the second kind can be in Φ' only if two of the above terms coincide up to the sign. In both of the exceptional cases, this happens exactly once. If $N^{k,l}$ is generic, none of these phenomena do happen and Φ' is empty. We are now able to describe the normalizer in all of the three cases:

- Let $N^{k,l}$ be a generic Aloff-Wallach space. Since the set Φ' is empty, $\mathrm{Norm}_{\mathfrak{so}(7,\mathbb{C})}(\mathfrak{u}(1)_{k,l} \otimes \mathbb{C})$ is the Cartan subalgebra of $\mathfrak{so}(7,\mathbb{C})$. The normalizer $\mathrm{Norm}_{\mathfrak{so}(7)}\mathfrak{u}(1)_{k,l}$ therefore has to be isomorphic to $3\mathfrak{u}(1)$.
- If $N^{k,l} \cong N^{1,0}$, the set Φ' consists of two roots $\pm\alpha$. $\mathrm{Norm}_{\mathfrak{so}(7,\mathbb{C})}(\mathfrak{u}(1)_{k,l} \otimes \mathbb{C})$ therefore has to be isomorphic to $\mathfrak{sl}(2,\mathbb{C}) \oplus 2\,\mathfrak{gl}(1,\mathbb{C})$. Since $\mathrm{Norm}_{\mathfrak{so}(7)}\mathfrak{u}(1)_{k,l}$ is the compact real form of $\mathrm{Norm}_{\mathfrak{so}(7,\mathbb{C})}(\mathfrak{u}(1)_{k,l} \otimes \mathbb{C})$, it is isomorphic to $\mathfrak{su}(2) \oplus 2\mathfrak{u}(1)$. We are able to describe the embedding of the semisimple summand into $\mathfrak{so}(7)$ explicitly: If $k = 1$ and $l = 0$, the matrix representation of z with respect to the basis $(i, j, k, k\epsilon, j\epsilon, \epsilon, i\epsilon)$ is:

$$\begin{pmatrix} 0 & & & & & & \\ & 0 & -1 & & & & \\ & 1 & 0 & & & & \\ & & & 0 & -1 & & \\ & & & 1 & 0 & & \\ & & & & & 0 & 2 \\ & & & & & -2 & 0 \end{pmatrix}$$

  The Lie algebra $\mathfrak{su}(2)$ acts irreducibly on $\mathrm{span}(j, k, k\epsilon, j\epsilon)$ by its complex two-dimensional representation. This action commutes with the action of z on $\mathrm{Im}(\mathbb{O})$. Therefore, the subalgebra $\mathfrak{su}(2) \subseteq \mathfrak{gl}(\mathrm{span}(j, k, k\epsilon, j\epsilon))$ which we have constructed has to be contained in the normalizer. This subalgebra and the standard Cartan subalgebra of $\mathfrak{so}(7)$ together generate all of $\mathrm{Norm}_{\mathfrak{so}(7)}\mathfrak{u}(1)_{1,0}$.

- If $N^{k,l} \cong N^{1,1}$, the set Φ' consists of two pairs of roots $\{\pm\alpha\} \cup \{\pm\beta\}$. $\mathrm{Norm}_{\mathfrak{so}(7,\mathbb{C})}(\mathfrak{u}(1)_{k,l} \otimes \mathbb{C})$ therefore is the direct sum of a Lie algebra with Dynkin diagram $A_1 \times A_1$ and a one-dimensional Lie algebra. Since $A_1 \times A_1$ is the Dynkin diagram of $\mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C})$ and $\mathrm{Norm}_{\mathfrak{so}(7)}\mathfrak{u}(1)_{k,l}$ is the compact real form of $\mathrm{Norm}_{\mathfrak{so}(7,\mathbb{C})}(\mathfrak{u}(1)_{k,l} \otimes \mathbb{C})$, $\mathrm{Norm}_{\mathfrak{so}(7)}\mathfrak{u}(1)_{k,l}$ is isomorphic to $2\mathfrak{su}(2) \oplus \mathfrak{u}(1)$. This normalizer can be described explicitly, too: If $k = l = 1$, the matrix representation of z with respect to the standard basis of $\mathrm{Im}(\mathbb{O})$ is:

$$\begin{pmatrix} 0 & 0 & 0 & & & & \\ 0 & 0 & 0 & & & & \\ 0 & 0 & 0 & & & & \\ & & & 0 & 3 & & \\ & & & -3 & 0 & & \\ & & & & & 0 & 3 \\ & & & & & -3 & 0 \end{pmatrix}$$

$\mathfrak{su}(2)$ acts irreducibly on $\mathrm{Im}(\mathbb{H})$ and on $\mathbb{H}\epsilon$. Furthermore, the subalgebra of type $2\mathfrak{su}(2) \subseteq \mathfrak{gl}(\mathrm{Im}(\mathbb{O}))$ which is defined by this action is contained in $\mathfrak{so}(7)$ and commutes with z. Therefore, the algebra has to be contained in $\mathrm{Norm}_{\mathfrak{so}(7)}\mathfrak{u}(1)_{1,1}$, too. Analogously to the previous case, $\mathrm{Norm}_{\mathfrak{so}(7)}\mathfrak{u}(1)_{1,1}$ is generated by the Lie algebra of type $2\mathfrak{su}(2)$ which we have constructed and the standard Cartan subalgebra of $\mathfrak{so}(7)$.

We denote the Cartan subalgebra of $\mathfrak{g}_2$ which we have defined on page 22 shortly by $2\mathfrak{u}(1)$. Since we have:

$$2\mathfrak{u}(1) \subseteq \mathrm{Norm}_{\mathfrak{g}_2}\mathfrak{u}(1)_{k,l} \subseteq \mathfrak{g}_2 \,,$$

the rank of $\mathrm{Norm}_{\mathfrak{g}_2}\mathfrak{u}(1)_{k,l}$ has to equal 2. By checking the subalgebras of $\mathrm{Norm}_{\mathfrak{so}(7)}\mathfrak{u}(1)_{k,l}$ of rank two, we thus can determine $\mathrm{Norm}_{\mathfrak{g}_2}\mathfrak{u}(1)_{k,l}$. Again, we consider the three cases separately:

- Let $N^{k,l}$ be a generic Aloff-Wallach space. Under this assumption, we have

$$\mathrm{Norm}_{\mathfrak{g}_2}\mathfrak{u}(1)_{k,l} = (\mathrm{Norm}_{\mathfrak{so}(7)}\mathfrak{u}(1)_{k,l}) \cap \mathfrak{g}_2 = 2\mathfrak{u}(1) \,.$$

- Let $N^{k,l} \cong N^{1,0}$. Without loss of generality, we assume that $k = 1$, $l = 0$. We have to check if the semisimple part $\mathfrak{k}$ of $\mathrm{Norm}_{\mathfrak{so}(7)}\mathfrak{u}(1)_{1,0}$ is contained in $\mathfrak{g}_2$. If this is the case, it follows from $\mathrm{rank}(\mathrm{Norm}_{\mathfrak{g}_2}\mathfrak{u}(1)_{1,0}) = 2$ that $\mathrm{Norm}_{\mathfrak{g}_2}\mathfrak{u}(1)_{1,0} \cong \mathfrak{su}(2) \oplus \mathfrak{u}(1)$.

  The intersection of $\mathfrak{k}$ and $2\mathfrak{u}(1)$ is obviously one-dimensional and contained in $\mathfrak{g}_2$. It suffices to proof that there is another $x \in \mathfrak{k} \setminus (\mathfrak{k} \cap 2\mathfrak{u}(1))$ which is contained in $\mathfrak{g}_2$. The reason for this is that in this situation $\mathfrak{k} \cap 2\mathfrak{u}(1)$, x, and $[\mathfrak{k} \cap 2\mathfrak{u}(1), x]$ span all of $\mathfrak{k} \cong \mathfrak{su}(2)$ and thus $\mathfrak{k} \subseteq \mathrm{Norm}_{\mathfrak{g}_2}\mathfrak{u}(1)_{1,1}$. The matrix

$$\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \in \mathfrak{su}(2)$$

  corresponds to a $x \in \mathfrak{k} \subseteq \mathfrak{so}(7)$, whose matrix representation with respect to the standard basis of $\mathrm{Im}(\mathbb{O})$ is:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

As we have seen in Section 2.1, x is contained in $\mathfrak{g}_2$ if and only if the three-form ω is annihilated by the pull-back of x. We have:

$$\begin{aligned} -x^*\omega &= -x^*(dx^{123} + dx^{145} - dx^{167} + dx^{246} + dx^{257} + dx^{347} - dx^{356}) \\ &= (0 + dx^{163} - dx^{127}) \\ &\quad + (0 + 0 + 0) \\ &\quad - (0 - dx^{127} + dx^{163}) \\ &\quad + (dx^{646} + 0 - dx^{242}) \\ &\quad + (dx^{657} + 0 + dx^{253}) \\ &\quad + (-dx^{747} + 0 + dx^{343}) \\ &\quad - (-dx^{756} + 0 - dx^{352}) = 0\,. \end{aligned}$$

Therefore, $x \in \mathfrak{g}_2$ and we can conclude that $\mathrm{Norm}_{\mathfrak{g}_2}\mathfrak{u}(1)_{1,0} \cong \mathfrak{su}(2) \oplus \mathfrak{u}(1)$.

- Finally, let $N^{k,l} \cong N^{1,1}$. We take a look at the proof of Lemma 4.1, where we explicitly have described all subalgebras of $\mathfrak{g}_2$. It is easy to see that the subalgebra $\mathfrak{su}(2)_{1,2} \oplus \mathfrak{u}(1)$ of $\mathfrak{g}_2$ is also contained in $\mathrm{Norm}_{\mathfrak{so}(7)}\mathfrak{u}(1)_{1,1}$. Since $\mathrm{rank}(\mathrm{Norm}_{\mathfrak{g}_2}\mathfrak{u}(1)_{1,1}) = 2$, the only possibilities for $\mathrm{Norm}_{\mathfrak{g}_2}\mathfrak{u}(1)_{1,1}$ are $\mathfrak{su}(2)_{1,2} \oplus \mathfrak{u}(1)$ and $2\mathfrak{su}(2)$. By a short calculation, we can exclude the second possibility. Thus, we have $\mathrm{Norm}_{\mathfrak{g}_2}\mathfrak{u}(1)_{1,1} \cong \mathfrak{su}(2) \oplus \mathfrak{u}(1)$.

The connected components of the coset space $\mathrm{Norm}_{SO(7)}U(1)_{k,l}/\mathrm{Norm}_{G_2}U(1)_{k,l}$ are all diffeomorphic and the diffeomorphisms are given by the action of an element of $\mathrm{Norm}_{SO(7)}U(1)_{k,l}$. By passing from the Lie algebras to the corresponding connected Lie groups we are able to describe the connected components of that space. We will denote the identity component of a Lie group by the index "e". The quotient $(\mathrm{Norm}_{SO(7)}U(1)_{k,l})_e/(\mathrm{Norm}_{G_2}U(1)_{k,l})_e$ covers any connected component of $\mathrm{Norm}_{SO(7)}U(1)_{k,l}/\mathrm{Norm}_{G_2}U(1)_{k,l}$. This fact will help us to describe a transitive, almost free group action on the connected components of $\mathrm{Norm}_{SO(7)}U(1)_{k,l}/\mathrm{Norm}_{G_2}U(1)_{k,l}$:

- If $N^{k,l}$ is generic, we have:

$$(\mathrm{Norm}_{SO(7)}U(1)_{k,l})_e/(\mathrm{Norm}_{G_2}U(1)_{k,l})_e \cong U(1)^3/U(1)^2 \cong U(1)\,.$$

Let T be a one-dimensional connected Lie subgroup of the standard maximal torus of $SO(7)$. We choose T in such a way that $G_2 \cap T$ is trivial. The action of T on a frame of a fixed homogeneous G_2-structure ω generates a family of G_2-structures, which is diffeomorphic to the circle. This family is a connected component of the space of all $SU(3)$-invariant G_2-structures on $N^{k,l}$ which have the same associated metric and orientation as ω. One possible choice of T is:

$$T := \left\{ \left(\begin{array}{c|cc|cc|cc} 1 & & & & & & \\ \hline & 1 & 0 & & & & \\ & 0 & 1 & & & & \\ \hline & & & \cos\theta & -\sin\theta & & \\ & & & \sin\theta & \cos\theta & & \\ \hline & & & & & 1 & 0 \\ & & & & & 0 & 1 \end{array} \right) =: T_\theta \;\middle|\; \theta \in \mathbb{R} \right\} \tag{5.40}$$

- If $N^{k,l} \cong N^{1,0}$, we have:

$$(\mathrm{Norm}_{SO(7)} U(1)_{k,l})_e / (\mathrm{Norm}_{G_2} U(1)_{k,l})_e \cong (U(2) \times U(1))/U(2) \cong U(1)\,.$$

It is easy to see that $T \subseteq \mathrm{Norm}_{SO(7)} U(1)_{k,l}$ and that $T \cap G_2$ is trivial. Therefore, the space of all $SU(3)$-invariant G_2-structures which have the same associated metric and orientation is in this case generated by the action of T, too.

- The third case which we have to consider is $N^{k,l} \cong N^{1,1}$. First, we take a look at the situation on the level of Lie algebras. We recall that:

  - $\mathrm{Norm}_{\mathfrak{so}(7)} \mathfrak{u}(1)_{1,1} \cong 2\mathfrak{su}(2) \oplus \mathfrak{u}(1)$
  - $\mathrm{Norm}_{\mathfrak{g}_2} \mathfrak{u}(1)_{1,1} \cong \mathfrak{su}(2) \oplus \mathfrak{u}(1)$

We compare the abelian part of both algebras. The abelian summand of $\mathrm{Norm}_{\mathfrak{so}(7)} \mathfrak{u}(1)_{1,1}$ has to be a subalgebra of the standard Cartan subalgebra of $\mathfrak{so}(7)$. Furthermore, it has to commute with $2\mathfrak{su}(2) \subseteq \mathrm{Norm}_{\mathfrak{so}(7)} \mathfrak{u}(1)_{1,1}$. These conditions leave only one choice for the abelian summand open. It has to be generated by:

$$\left(\begin{array}{c|cc|cc|cc} 0 & & & & & & \\ \hline & 0 & 0 & & & & \\ & 0 & 0 & & & & \\ \hline & & & 0 & -1 & & \\ & & & 1 & 0 & & \\ \hline & & & & & 0 & -1 \\ & & & & & 1 & 0 \end{array} \right)$$

The abelian part of $\mathrm{Norm}_{\mathfrak{g}_2} \mathfrak{u}(1)_{1,1} = \mathfrak{su}(2)_{1,2} \oplus \mathfrak{u}(1)$ is generated by the same matrix. We therefore can cancel that summand and obtain:

$$(\mathrm{Norm}_{SO(7)} U(1)_{1,1})_e / (\mathrm{Norm}_{G_2} U(1)_{1,1})_e \cong (SO(3) \times SU(2))/\triangle SU(2)\,.$$

The group $\triangle SU(2)$ by which we divide is not the second factor of $SO(3)\times SU(2)$ but the Lie subgroup of G_2 which is associated to $\mathfrak{su}(2)_{1,2}$. $\triangle SU(2)$ is "diagonally" embedded into $SO(3)\times SU(2)$. More precisely, it acts as $SU(2)/\mathbb{Z}_2$ on the first and canonically on the second factor. We take a look at the orbit of $SO(3)$ in $(SO(3)\times SU(2))/\triangle SU(2)$. Since the intersection of $SO(3)$ and $\triangle SU(2)$ is discrete, the orbit is a three-dimensional submanifold of $(SO(3)\times SU(2))/\triangle SU(2)$. In particular, it is an open set. Since $SO(3)$ is a closed subgroup of $SO(3)\times SU(2)$, the orbit is closed, too. $(SO(3)\times SU(2))/\triangle SU(2)$ is a connected space and we can conclude that $SO(3)$ acts transitively on that space. All in all, we have proven that the action of $SO(3)$ on $\mathrm{Im}(\mathbb{H})$ generates any connected component of the space which we consider.

The $SO(3)$-action on $\mathrm{span}(f_1,f_2,f_3)$, which we have considered above, is in general not the same as the action of $SU(2)\subseteq \mathrm{Norm}_{SU(3)}U(1)_{1,1}$ on $\mathrm{span}(e_1,e_2,e_7)$. The reason for this simply is that the first action preserves the $SU(3)$-invariant metric which we have fixed, while the action of the normalizer preserves the normal metric q. Nevertheless, if some of the elements of $\{f_1,f_2,f_3\}$ have the same length with respect to q, a subset of the G_2-structures which we consider can be generated by the action of a subgroup of $\mathrm{Norm}_{SU(3)}U(1)_{1,1}$. In that case, that subgroup even acts by isometries. Any element of $SO(3)$ is conjugate to a matrix of type:

$$\begin{pmatrix} \boxed{1} & & \\ & \cos\theta & -\sin\theta \\ & \sin\theta & \cos\theta \end{pmatrix}.$$

Therefore, we will later on consider the G_2-structures which are generated by the action of

$$S := \left\{ \begin{pmatrix} 1 & & & & & & \\ & \cos\theta & -\sin\theta & & & & \\ & \sin\theta & \cos\theta & & & & \\ & & & 1 & 0 & & \\ & & & 0 & 1 & & \\ & & & & & 1 & 0 \\ & & & & & 0 & 1 \end{pmatrix} \middle| \ \theta\in\mathbb{R} \right\}$$

on a fixed frame. We hope that the other invariant G_2-structures which have the same extension to a $SO(7)$-structure behave similar as these special ones. Since $S\mathrm{Norm}_{G_2}U(1)_{1,1} = T\,\mathrm{Norm}_{G_2}U(1)_{1,1}$, we can replace the action of S by the action of T.

Until now, we have only determined the type of the connected components of $\mathrm{Norm}_{SO(7)}U(1)_{k,l}/\mathrm{Norm}_{G_2}U(1)_{k,l}$, but not their number. In the generic case, this number could be determined by similar methods as in Section 5.2. Unfortunately, those methods do not work in the two exceptional cases. We will briefly explain why our arguments would fail: In Section 5.2, we successfully have determined the number of connected components of the space of G_2-structures. In order to do this, we had to prove that $\mathrm{Norm}_{SO(7)}U(1)^2\subseteq \mathrm{Norm}_{SO(7)}U(1)^3$, where $U(1)^3$ is the maximal torus of $SO(7)$. This allowed us to work with the Weyl group $\mathrm{Norm}_{SO(7)}U(1)^3/U(1)^3$.

If $N^{k,l}$ is exceptional, $\mathrm{Norm}_{SO(7)}U(1)_{k,l}$ is not a subgroup of $\mathrm{Norm}_{SO(7)}U(1)^3$, since the identity component of the second group is $U(1)^3$ and the identity component $\mathrm{Norm}_{SO(7)}U(1)_{k,l}$ is a larger group. Therefore, we cannot rule out by the above methods that there are further homogeneous G_2-structures, which cannot be obtained by the group actions from the previous pages. It is possible that the additional G_2-structures could be extended to new examples of parallel cohomogeneity-one Spin(7)-structures. We will nevertheless not further discuss this issue. There are three reasons for our decision: First, the study of the Spin(7)-structures whose restriction to a principal orbit is one of the G_2-structures which we have constructed so far is already a rewarding project even if further examples existed. Second, it may be possible that we switch to another connected component of $\mathrm{Norm}_{SO(7)}U(1)_{k,l}/\mathrm{Norm}_{G_2}U(1)_{k,l}$ if we change the sign of some of the functions a, a_1, a_2, b, c, and f. Therefore, we may already describe more than one or even all connected components by our ansatz for Ω. A third reason is that the results which we have found are sufficient to make statements on the holonomy of our metrics.

On the following pages, we take a look at the new cohomogeneity-one Spin(7)-structures whose existence follows from the considerations we have made. We first consider the case where $N^{k,l}$ is generic or $\cong N^{1,0}$. The action of the matrix T_θ from page 161 on the frame (5.34) yields the following more general frame:

$$\begin{aligned} &f_0 := \tfrac{\partial}{\partial t} && f_1 := \tfrac{1}{f} e_7 && f_2 := \tfrac{1}{a} e_1 \quad f_3 := \tfrac{1}{a} e_2 \\ &f_4 := -\tfrac{1}{b}\sin\theta\, e_3 + \tfrac{1}{b}\cos\theta\, e_4 && f_5 := \tfrac{1}{b}\cos\theta\, e_3 + \tfrac{1}{b}\sin\theta\, e_4 && f_6 := \tfrac{1}{c} e_6 \quad f_7 := \tfrac{1}{c} e_5 \end{aligned} \tag{5.41}$$

With help of a short MAPLE program, we can determine the four-form Ω which is associated to this frame. We can calculate $d\Omega$, too. Since we have made that calculations for $\theta = 0$ by hand, we can compare our results with the four-form (5.35) in order to check our program. The exterior derivative $d\Omega$ which we have calculated in Appendix C.1 can also be used for checking our results. We obtain:

$$\begin{aligned} \Omega = & -abcf\ \cos\theta\, e^{1367} + abcf\ \cos\theta\, e^{1457} - abcf\ \cos\theta\, e^{2357} - abcf\ \cos\theta\, e^{2467} \\ & + abcf\ \sin\theta\, e^{1357} + abcf\ \sin\theta\, e^{1467} - abcf\ \sin\theta\, e^{2367} + abcf\ \sin\theta\, e^{2457} \\ & + a^2b^2\, e^{1234} - a^2c^2\, e^{1256} + b^2c^2\, e^{3456} \\ & - a^2 f\, e^{127} \wedge dt + b^2 f\, e^{347} \wedge dt - c^2 f\, e^{567} \wedge dt \\ & - abc\ \cos\theta\, e^{135} \wedge dt - abc\ \cos\theta\, e^{146} \wedge dt + abc\ \cos\theta\, e^{236} \wedge dt - abc\ \cos\theta\, e^{245} \wedge dt \\ & - abc\ \sin\theta\, e^{136} \wedge dt + abc\ \sin\theta\, e^{145} \wedge dt - abc\ \sin\theta\, e^{235} \wedge dt - abc \sin\theta\, e^{246} \wedge dt \end{aligned}$$

The exterior derivative of $*\omega$ is given by:

$$d * \omega = 4abcf\ \sin\theta (e^{12347} - e^{12567} + e^{34567}) .$$

This term has to vanish in order to make Ω a closed form. We have $d * \omega = 0$ if and only if $\theta \in \pi\mathbb{Z}$. If $\theta \in 2\pi\mathbb{Z}$, we again obtain the special form (5.35) which we have considered earlier in this section. Therefore, we only have to treat the case $\theta = \pi$. We insert $\theta = \pi$ in the frame

(5.41). The result coincides with (5.34), except that f_4 and f_5 have the opposite sign. This change of signs is equivalent to changing the sign of the function b. Since we have allowed positive and negative values for the function b, we do not have to consider the case $\theta = \pi$ separately.

Let g be a cohomogeneity-one metric which has a singular orbit, satisfies (5.36), and whose principal orbit is a generic Aloff-Wallach space or $\cong N^{1,0}$. Since any complete cohomogeneity-one metric with holonomy $\subseteq \mathrm{Spin}(7)$ which is not a Riemannian product has a singular orbit, we can assume that our first condition on g is justified. We will prove that the holonomy of g is all of Spin(7). If the holonomy was not all of Spin(7), it would be contained in G_2 or $SU(4)$. In the first case, there would exist a parallel vector field X on the manifold. Let η be the dual of X. Since η has to be $SU(3)$-invariant, $\eta = \alpha(t)e^7 + \beta(t)dt$ for some functions α and β. If X is parallel, η is closed. de^7 and $dt \wedge e^7$ are linearly independent and we can easily conclude from $d\eta = 0$ that $\alpha \equiv 0$. Since X has to be of constant length, it has to be a multiple of dt. In that situation, we have $a' = b' = c' = f' = 0$ and the manifold is a Riemannian product. Since g is Ricci-flat, its restriction to the principal orbit is Ricci-flat, too, which is impossible. In the second case, it follows from Lemma 2.3.17 that in this situation there exists a one-parameter family of parallel Spin(7)-structures. Since we have proven that the set of all invariant coclosed G_2-structures on the principal orbit is discrete, this cannot be the case. Therefore, the holonomy of g has to be all of Spin(7).

We proceed to the $N^{1,1}$-case. We assume that $k = l = 1$ and let T_θ act on the frame (5.37):

$$\begin{aligned}
&f_0 := \tfrac{\partial}{\partial t} && f_1 := \tfrac{1}{f}e_7 && f_2 := \tfrac{1}{a_1}e_1 \quad f_3 := \tfrac{1}{a_2}e_2 \\
&f_4 := -\tfrac{1}{b}\sin\theta\, e_3 + \tfrac{1}{b}\cos\theta\, e_4 && f_5 := \tfrac{1}{b}\cos\theta\, e_3 + \tfrac{1}{b}\sin\theta\, e_4 && f_6 := \tfrac{1}{c}e_6 \quad f_7 := \tfrac{1}{c}e_5
\end{aligned} \tag{5.42}$$

Again, we write a MAPLE program and obtain for the four-form Ω which corresponds to (5.42):

$$\begin{aligned}
\Omega = &-a_1bcf\ \cos\theta\, e^{1367} + a_1bcf\ \cos\theta\, e^{1457} - a_2bcf\ \cos\theta\, e^{2357} - a_2bcf\ \cos\theta\, e^{2467} \\
&+ a_1bcf\ \sin\theta\, e^{1357} + a_1bcf\ \sin\theta\, e^{1467} - a_2bcf\ \sin\theta\, e^{2367} + a_2bcf\ \sin\theta\, e^{2457} \\
&+ a_1a_2b^2\, e^{1234} - a_1a_2c^2\, e^{1256} + b^2c^2\, e^{3456} \\
&- a_1a_2f\, e^{127} \wedge dt + b^2f\, e^{347} \wedge dt - c^2f\, e^{567} \wedge dt \\
&- a_1bc\ \cos\theta\, e^{135} \wedge dt - a_1bc\ \cos\theta\, e^{146} \wedge dt \\
&+ a_2bc\ \cos\theta\, e^{236} \wedge dt - a_2bc\ \cos\theta\, e^{245} \wedge dt \\
&- a_1bc\ \sin\theta\, e^{136} \wedge dt + a_1bc\ \sin\theta\, e^{145} \wedge dt \\
&- a_2bc\ \sin\theta\, e^{235} \wedge dt - a_2bc\ \sin\theta\, e^{246} \wedge dt
\end{aligned} \tag{5.43}$$

The exterior derivative of this four-form is:

$$
\begin{aligned}
d\Omega = {} & (a_1'bcf\ \sin\theta + a_1b'cf\ \sin\theta + a_1bc'f\ \sin\theta + a_1bcf'\ \sin\theta + a_1bcf\theta'\ \cos\theta \\
& - 6a_1bc\ \sin\theta + 6a_2bc\ \sin\theta)e^{1357} \wedge dt \\
& + (-a_1'bcf\ \cos\theta - a_1b'cf\ \cos\theta - a_1bc'f\ \cos\theta - a_1bcf'\ \cos\theta + a_1bcf\theta'\ \sin\theta \\
& + 6a_1bc\ \cos\theta - 6a_2bc\ \cos\theta + a_1a_2f + b^2f + c^2f)e^{1367} \wedge dt \\
& + (a_1'bcf\ \cos\theta + a_1b'cf\ \cos\theta + a_1bc'f\ \cos\theta + a_1bcf'\ \cos\theta - a_1bcf\theta'\ \sin\theta \\
& - 6a_1bc\ \cos\theta + 6a_2bc\ \cos\theta - a_1a_2f - b^2f - c^2f)e^{1457} \wedge dt \\
& + (a_1'bcf\ \sin\theta + a_1b'cf\ \sin\theta + a_1bc'f\ \sin\theta + a_1bcf'\ \sin\theta + a_1bcf\theta'\ \cos\theta \\
& - 6a_1bc\ \sin\theta + 6a_2bc\ \sin\theta)e^{1467} \wedge dt \\
& + (-a_2'bcf\ \cos\theta - a_2b'cf\cos\theta - a_2bc'f\cos\theta - a_2bcf'\ \cos\theta + a_2bcf\theta'\ \sin\theta \\
& - 6a_1bc\ \cos\theta + 6a_2bc\ \cos\theta + a_1a_2f + b^2f + c^2f)e^{2357} \wedge dt \\
& + (-a_2'bcf\ \sin\theta - a_2b'cf\ \sin\theta - a_2bc'f\ \sin\theta - a_2bcf'\ \sin\theta - a_2bcf\theta'\ \cos\theta \\
& - 6a_1bc\ \sin\theta + 6a_2bc\ \sin\theta)e^{2367} \wedge dt \\
& + (a_2'bcf\ \sin\theta + a_2b'cf\ \sin\theta + a_2bc'f\ \sin\theta + a_2bcf'\ \sin\theta + a_2bcf\theta'\ \cos\theta \\
& + 6a_1bc\ \sin\theta - 6a_2bc\ \sin\theta)e^{2457} \wedge dt \\
& + (-a_2'bcf\ \cos\theta - a_2b'cf\ \cos\theta - a_2bc'f\ \cos\theta - a_2bcf'\ \cos\theta + a_2bcf\theta'\ \sin\theta \\
& - 6a_1bc\ \cos\theta + 6a_2bc\ \cos\theta + a_1a_2f + b^2f + c^2f)e^{2467} \wedge dt \\
& + (a_1'a_2b^2 + a_1a_2'b^2 + 2a_1a_2bb' - \tfrac{1}{3}a_1a_2f + \tfrac{2}{3}b^2f - 2a_1bc\ \cos\theta - 2a_2bc\ \cos\theta)e^{1234} \wedge dt \\
& + (-a_1'a_2c^2 - a_1a_2'c^2 - 2a_1a_2cc' + \tfrac{1}{3}a_1a_2f - \tfrac{2}{3}c^2f + 2a_1bc\ \cos\theta + 2a_2bc\ \cos\theta)e^{1256} \wedge dt \\
& + (2bb'c^2 + 2b^2cc' - \tfrac{1}{3}b^2f - \tfrac{1}{3}c^2f - 2a_1bc\cos\theta - 2a_2bc\cos\theta)e^{3456} \wedge dt \\
& + (2a_1bcf\ \sin\theta + 2a_2bcf\ \sin\theta)e^{12347} \\
& + (-2a_1bcf\ \sin\theta - 2a_2bcf\ \sin\theta)e^{12567} \\
& + (2a_1bcf\ \sin\theta + 2a_2bcf\ \sin\theta)e^{34567}
\end{aligned}
$$

We want $d * \omega$ to vanish and thus either have to require that $\sin\theta = 0$ or that $a_2 = -a_1$. In the first case, we can make the same arguments as before and obtain no other examples as in the special case $\theta = 0$, which we have already considered in this section. We therefore analyze the equation $d\Omega = 0$ under the assumption that $a_2 = -a_1$. In that situation, the condition that the coefficients of $e^{1367} \wedge dt$ and $e^{2357} \wedge dt$ vanish is equivalent to:

$$
\begin{aligned}
-a_1'bcf\ \cos\theta - a_1b'cf\ \cos\theta - a_1bc'f\ \cos\theta - a_1bcf'\ \cos\theta + a_1bcf\theta'\ \sin\theta & \\
+12a_1bc\ \cos\theta - a_1^2f + b^2f + c^2f &= 0 \\
a_1'bcf\ \cos\theta + a_1b'cf\ \cos\theta + a_1bc'f\ \cos\theta + a_1bcf'\ \cos\theta - a_1bcf\theta'\ \sin\theta & \\
-12a_1bc\ \cos\theta - a_1^2f + b^2f + c^2f &= 0
\end{aligned}
$$

Since f has to be non-zero, we obtain $a_1^2 = b^2 + c^2$ from the above two equations. With this additional relation, it is easy to see that $d\Omega = 0$ is equivalent to:

$$
\begin{aligned}
a_1'bcf\ \sin\theta + a_1b'cf\ \sin\theta + a_1bc'f\ \sin\theta + a_1bcf'\ \sin\theta + a_1bcf\theta'\ \cos\theta - 12a_1bc\ \sin\theta &= 0\\
-a_1'bcf\ \cos\theta - a_1b'cf\ \cos\theta - a_1bc'f\ \cos\theta - a_1bcf'\ \cos\theta + a_1bcf\theta'\ \sin\theta + 12a_1bc\ \cos\theta &= 0\\
-2a_1'a_1b^2 - 2a_1^2b'b + \tfrac{1}{3}a_1^2f + \tfrac{2}{3}b^2f &= 0\\
2a_1'a_1c^2 + 2a_1^2c'c - \tfrac{1}{3}a_1^2f - \tfrac{2}{3}c^2f &= 0\\
2b'bc^2 + 2b^2c'c - \tfrac{1}{3}b^2f - \tfrac{1}{3}c^2f &= 0
\end{aligned}
$$

By dividing the above equations by suitable products of the metric functions, we obtain:

$$
\begin{aligned}
\left(\frac{a_1'}{a_1} + \frac{b'}{b} + \frac{c'}{c} + \frac{f'}{f}\right)\sin\theta + \theta'\ \cos\theta &= 12\frac{1}{f}\sin\theta\\
\left(\frac{a_1'}{a_1} + \frac{b'}{b} + \frac{c'}{c} + \frac{f'}{f}\right)\cos\theta - \theta'\ \sin\theta &= 12\frac{1}{f}\cos\theta\\
\frac{a_1'}{a_1} + \frac{b'}{b} &= \frac{1}{3}\frac{f}{a_1^2} + \frac{1}{6}\frac{f}{b^2}\\
\frac{a_1'}{a_1} + \frac{c'}{c} &= \frac{1}{3}\frac{f}{a_1^2} + \frac{1}{6}\frac{f}{c^2}\\
\frac{b'}{b} + \frac{c'}{c} &= \frac{1}{6}\frac{f}{b^2} + \frac{1}{6}\frac{f}{c^2}
\end{aligned}
$$

From the first two equations, we conclude that θ can be chosen as an arbitrary constant. Our system finally simplifies to:

$$
\frac{a_1'}{a_1} = \frac{1}{3}\frac{f}{a_1^2} \qquad \frac{b'}{b} = \frac{1}{6}\frac{f}{b^2} \qquad \frac{c'}{c} = \frac{1}{6}\frac{f}{c^2} \qquad \frac{f'}{f} = 12\frac{1}{f} - \frac{1}{3}\frac{f}{a_1^2} - \frac{1}{6}\frac{f}{b^2} - \frac{1}{6}\frac{f}{c^2}\,.
$$

If we consider the system (5.39) under the assumption that $a_2 = -a_1$ and $a_1^2 = b^2 + c^2$, we obtain the same four equations as above. There are two conclusions we can draw from this fact: First, we have proven that we obtain no new metrics by considering the more general Spin(7)-structures which are described by (5.42) and (5.43). Second, it follows that any solution of (5.39) which satisfies $a_2 = -a_1$ and $a_1^2 = b^2 + c^2$ yields a metric whose holonomy is contained in $SU(4)$. The reason for this is that if the two relations hold, we can choose θ arbitrarily and have found a one-parameter family of parallel Spin(7)-structures with the same associated metric. We apply Lemma 2.3.17 and conclude that the holonomy is a subgroup of $SU(4)$. The existence of a parallel vector field can be excluded by similar arguments as in the previous case. The dual η of such a vector field has to coincide with $\alpha(t)e^1 + \beta(t)e^2 + \gamma(t)e^7 + \delta(t)dt$ for some functions α, β, γ, and δ. Since $d\eta = 0$ and de^1, de^2, de^7, $dt \wedge e^1$, $dt \wedge e^2$, $dt \wedge e^7$ are all linearly independent, it follows that $\alpha, \beta, \gamma \equiv 0$. Therefore, we have $\eta = dt$, which can be excluded by the same arguments as in the previous case. We have proven that there exists no non-zero tangent vector which is left invariant by the holonomy group. The holonomy thus is either $SU(4)$, $Sp(2)$, or $SU(2) \times SU(2)$. It is possible to prove that there actually are solutions of (5.39) which obey $a_2 = -a_1$ and $a_1^2 = b^2 + c^2$. Metrics of that kind have been considered by Kanno and Yasui in [49].

We will make some short remarks on the question if the above case is the only one where the holonomy is $\subsetneq$ Spin(7). For the same reasons as above, we only have to decide if the holonomy is contained in $SU(4)$. We recall that the four-forms of type (5.43) are not the only possible invariant Spin(7)-structures, since there is a bigger family of invariant G_2-structures on $N^{1,1}$ which is generated by a $SO(3)$- rather than a $U(1)$-action. If we had considered a more general frame which was obtained by the action of $SO(3) \subseteq \mathrm{Norm}_{SO(7)}U(1)_{1,1}$ on (5.37), we could have found other sufficient conditions for the holonomy to be contained in $SU(4)$. It even might be possible that another choice of the Spin(7)-structure yielded a diagonal cohomogeneity-one metric with holonomy $\subseteq$ Spin(7) which did not obey the system (5.39). Since the calculations for the more general frame would be much more extensive than the above ones, we have restricted ourselves to a special case. The great number of possible parallel Spin(7)-structures is also the reason why we will not determine the holonomy but only characterize it as a subgroup of a bigger group.

The results which we have obtained so far motivate why we only consider metrics which satisfy the systems (5.36) and (5.39). Unlike in Section 5.2 and 5.3, there is no easy method to solve those systems of differential equations except in special cases. Since the equations are highly non-linear, it is even hard to decide if the resulting metrics are complete.

In order to prove at least local existence results, we make a power series ansatz for (5.36) and (5.39) near a singular orbit. At least one of the metric functions has to converge to zero at the singular orbit. The fact that we divide by this function in the differential equations will cause some difficulties, which we have to solve. We will investigate which invariant metrics on the singular orbit can be extended to cohomogeneity-one metrics with holonomy $\subseteq$ Spin(7) and if there are any initial conditions of higher order which we can prescribe. Before we start this program, we first have to classify the possible singular orbits:

Lemma 5.4.2. *Let $U(1)$ be embedded into $SU(3)$ as $U(1)_{k,l}$. Furthermore, let K be a connected, closed group with $U(1)_{k,l} \subsetneq K \subseteq SU(3)$. The Lie algebra of K we denote by $\mathfrak{k}$. In this situation, $\mathfrak{k}$ and K can be found in the table below. Moreover, $K/U(1)_{k,l}$ and $SU(3)/K$ satisfy the following topological conditions:*

$\mathfrak{k}$	K	$K/U(1)_{k,l}$	$SU(3)/K$	*Condition on k and l*
$2\mathfrak{u}(1)$	$U(1)^2$	$\cong S^1$	$= SU(3)/U(1)^2$	
$\mathfrak{su}(2)$	$SU(2)$	$\cong S^2$	$\cong S^5$	$k \cdot l \cdot (-k-l) = 0$
$\mathfrak{su}(2)$	$SO(3)$	$\cong S^2$	$= SU(3)/SO(3)$	$k \cdot l \cdot (-k-l) = 0$
$\mathfrak{su}(2) \oplus \mathfrak{u}(1)$	$U(2)$	$\cong S^3/\mathbb{Z}_{\lvert k+l \rvert}$	$\cong \mathbb{CP}^2$	$k \cdot l \cdot (-k-l) \neq 0$
$\mathfrak{su}(2) \oplus \mathfrak{u}(1)$	$U(2)$	$\cong S^3$	$\cong \mathbb{CP}^2$	$k \cdot l \cdot (-k-l) = 0$
$\mathfrak{su}(2) \oplus \mathfrak{u}(1)$	$U(2)$	$\cong S^2 \times S^1$	$\cong \mathbb{CP}^2$	$k \cdot l \cdot (-k-l) = 0$
$\mathfrak{su}(3)$	$SU(3)$	$= N^{k,l} \not\cong S^7/\Gamma$		

In the above table, Γ denotes an arbitrary discrete subgroup of $O(8)$ and the group $\mathbb{Z}_{\lvert k+l \rvert}$, by which we divide S^3, is explicitly described by (5.46).

Proof : The fact that $\mathfrak{k}$ has to be an $\mathfrak{u}(1)_{k,l}$-module, reduces the number of subspaces of $\mathfrak{su}(3)$ which we have to consider. We will check for all $\mathfrak{u}(1)_{k,l}$-modules $\mathfrak{k}$ with $\mathfrak{u}(1)_{k,l} \subsetneq \mathfrak{k} \subseteq \mathfrak{su}(3)$ if they are closed under the Lie bracket and if $K/U(1)_{k,l}$ is covered by a sphere. If $N^{k,l}$ is an exceptional Aloff-Wallach space, $\mathfrak{m}$ contains pairs (U, U') of equivalent submodules. This will make our considerations more complicated than in the previous two sections, since there may be submodules of $\mathfrak{k}$ which are transversely embedded into $U \oplus U'$. Therefore, we often have to distinguish the different types of Aloff-Wallach spaces in the course of this proof. In order to find all the possibilities for $\mathfrak{k}$, we consider each of the possible values of $\dim \mathfrak{k}$ separately:

1. $\mathfrak{k} = \mathfrak{u}(1)_{k,l} \oplus W$, W one-dimensional: If $N^{k,l} \not\cong N^{1,1}$, W has to be V_4, since this is the only one-dimensional submodule of $\mathfrak{m}$. In this case, $\mathfrak{k}$ is obviously a Lie algebra and isomorphic to $2\mathfrak{u}(1)$. Thus, we have $K = U(1)^2$, $K/U(1)_{k,l} \cong S^1$, and $SU(3)/K = SU(3)/U(1)^2$.

   If $N^{k,l} \cong N^{1,1}$, either V_1, V_2, or V_3 is trivial. Without loss of generality, we assume that $k = l = 1$, which makes V_1 trivial. In this situation, we can choose W as $\mathrm{span}(x)$, where $x \in (V_1 \oplus V_4)\backslash\{0\}$ is arbitrary. Since x commutes with $\mathfrak{u}(1)_{1,1}$, $\mathfrak{k}$ is closed under the Lie bracket and abelian. Therefore, we again have: $K = U(1)^2$, $K/U(1)_{1,1} = S^1$, and $SU(3)/K = SU(3)/U(1)^2$.

2. $\mathfrak{k} = \mathfrak{u}(1)_{k,l} \oplus W$, W two-dimensional: If $N^{k,l}$ is a generic Aloff-Wallach space, the only two-dimensional submodules of $\mathfrak{m}$ are V_1, V_2, and V_3. We consider the case $W = V_1$. V_1 is spanned by e_1 and e_2 and we have:

$$[e_1, e_2] = \begin{pmatrix} 2i & 0 & 0 \\ 0 & -2i & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{5.44}$$

   The commutator $[e_1, e_2]$ is contained in $\mathfrak{u}(1)_{k,l} \oplus V_1$ if and only if $k = \pm 1$ and $l = \mp 1$. Analogously, we can prove that the case $W = V_2$ is only possible for $k = \pm 1$ and $l = 0$, and that the case $W = V_3$ is only possible for $k = 0$ and $l = \pm 1$. Therefore, $\mathfrak{k}$ cannot be closed in the generic case.

   Next, we consider the case $N^{k,l} \cong N^{1,0}$. We assume without loss of generality that $k = 1$ and $l = -1$. Formula (5.44) together with the fact that V_1 is a non-trivial $\mathfrak{u}(1)_{1,-1}$-module proves that $\mathfrak{k}$ is a Lie algebra, which is isomorphic to $\mathfrak{su}(2)$. In this situation, we have $K = SU(2)$ and $K/U(1)_{1,-1} \cong S^2$ due to the Hopf fibration. Moreover, $SU(3)/K \cong S^5$.

   Since the modules V_2 and V_3 are $\mathfrak{u}(1)_{1,-1}$-equivariantly isomorphic, there may exist a two-dimensional space W which is transversely embedded into $V_2 \oplus V_3$ such that $\mathfrak{u}(1)_{1,-1} \oplus W$ is closed under the Lie bracket. The first of the exceptional cases therefore is not yet fully handled. We assume that such a W exists. Since $\mathfrak{k}$ is the Lie algebra of a compact Lie group, it is either isomorphic to $3\mathfrak{u}(1)$ or to $\mathfrak{su}(2)$. The first case can be excluded, since the rank of $\mathfrak{su}(3)$ is two. Therefore, $\mathfrak{k}$ has to be isomorphic to $\mathfrak{su}(2)$. There are up to conjugation two embeddings of $\mathfrak{su}(2)$ into $\mathfrak{su}(3)$. The first embedding is induced by the inclusion of $SO(3)$ into $SU(3)$. The second subalgebra of type $\mathfrak{su}(2)$ is given by $\mathfrak{u}(1)_{1,-1} \oplus V_1$ and can be identified with a subalgebra of $\mathfrak{gl}(2, \mathbb{C})$. In the first case, $\mathfrak{k}$ is a conjugate of $\mathfrak{so}(3)$ and has to contain $\mathfrak{u}(1)_{1,-1}$. One possible choice of $\mathfrak{k}$ is:

$$\mathfrak{k} = A^{-1}\,\mathrm{span}(e_1, e_3, e_5)\, A$$

with

$$A := \begin{pmatrix} \frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} & 0 \\ \frac{1}{2}\sqrt{2}\, i & \frac{1}{2}\sqrt{2}\, i & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We obtain:

$$\mathfrak{k} = \mathrm{span}(e_8,\ \tfrac{1}{2}\sqrt{2}\, e_3 - \tfrac{1}{2}\sqrt{2}\, e_5,\ -\tfrac{1}{2}\sqrt{2}\, e_4 - \tfrac{1}{2}\sqrt{2}\, e_6)\,.$$

If there is another $\mathfrak{k}$ which is also tangent to a Lie group isomorphic to $SO(3)$, it is conjugate to the above $\mathfrak{k}$ and also contains $\mathfrak{u}(1)_{1,-1}$ as a Cartan subalgebra. We can assume that the matrix B by which we conjugate preserves $\mathfrak{u}(1)_{1,-1}$. In other words: $B \in \mathrm{Norm}_{SU(3)} U(1)_{1,-1}$. We already have proven that this normalizer is the standard maximal torus of $SU(3)$. The action of $U(1)_{1,-1}$ by conjugation obviously preserves the algebra $\mathfrak{k}$ which we have chosen. By a short calculation, we see that the action of $U(1)_{1,1}$ does not preserve $\mathfrak{k}$. We therefore have found a one-parameter family of Lie algebras which are conjugate to each other. The associated Lie group K of any Lie algebra $\mathfrak{k}$ in this family is isomorphic to $SO(3)$. By an inner automorphism of $SU(3)$, it can be mapped to the standard embedding of $SO(3)$ into $SU(3)$. The group $U(1)_{1,-1}$ is conjugate to the following subgroup of $SO(3)$:

$$\left\{ \begin{pmatrix} \boxed{A} & \\ & \boxed{1} \end{pmatrix} \middle|\ A \in SO(2) \right\}.$$

Therefore, we have $K/U(1)_{1,-1} \cong S^2$ and the singular orbit is the symmetric space $SU(3)/SO(3)$.

Next, we assume that the embedding of $\mathfrak{k}$ into $\mathfrak{su}(3)$ is of the second type. In this case, $\mathfrak{k}$ is conjugate to the Lie algebra

$$\left\{ \begin{pmatrix} \boxed{x} & \\ & \boxed{0} \end{pmatrix} \middle|\ x \in \mathfrak{su}(2) \right\}$$

and contains $\mathfrak{u}(1)_{1,-1}$ as a Cartan subalgebra. We can prove by the same arguments as above that $\mathfrak{k}$ is conjugate by an element of $\mathrm{Norm}_{SU(3)} U(1)_{1,-1}$ to the above Lie algebra. Since that algebra is preserved by the normalizer, $\mathfrak{k}$ is in this situation unique. Moreover, it coincides with a Lie subalgebra of $\mathfrak{su}(3)$ which we have already considered.

We finally have to investigate the case $N^{k,l} \cong N^{1,1}$. If $k = l = 1$, V_1 is trivial and V_2, V_3 are equivalent modules. There are two possibilities for W: Either $W = V_1$ or $W \subseteq V_2 \oplus V_3$. We first assume that $W = V_1$. Since $[e_1, e_2] \in \mathfrak{u}(1)_{1,-1} \backslash \{0\}$, $\mathfrak{k}$ is not closed under the Lie bracket. If $W \subseteq V_2 \oplus V_3$, $\mathfrak{k}$ would have to be a Lie algebra which is isomorphic to $\mathfrak{su}(2)$. Furthermore, $\mathfrak{u}(1)_{1,1}$ would be a Cartan subalgebra of $\mathfrak{k}$. No matter how we embed $\mathfrak{su}(2)$ into $\mathfrak{su}(3)$, its Cartan subalgebras are all conjugate to $\mathfrak{u}(1)_{1,-1}$. Therefore, we can exclude this case.

3. $\mathfrak{k} = \mathfrak{u}(1)_{k,l} \oplus W$, W three-dimensional: We again consider the three different types of Aloff-Wallach spaces separately. First, we assume that $N^{k,l}$ is a generic Aloff-Wallach space. In this situation, we have $W = V_i \oplus V_4$ for an $i \in \{1,2,3\}$. Since the direct sum $\mathfrak{u}(1)_{k,l} \oplus V_4$ is a two-dimensional abelian Lie algebra, we will denote that space by $2\mathfrak{u}(1)$ in the course of this proof. It is easy to see that $[V_i, V_i] \subseteq 2\mathfrak{u}(1)$ and $[2\mathfrak{u}(1), V_i] \subseteq V_i$. Therefore, $\mathfrak{k}$ is closed under the Lie bracket. As a Lie algebra it is isomorphic to $\mathfrak{su}(2) \oplus \mathfrak{u}(1) = \mathfrak{u}(2)$. We assume without loss of generality that $i = 1$. The Lie group K is given by:

$$\left\{ \left(\begin{array}{cc} \boxed{A} & \\ & \boxed{\det A^{-1}} \end{array} \right) \middle| \; A \in U(2) \right\} .$$

K is isomorphic to $U(2)$. The group $U(2)$ is not a direct product $SU(2) \times U(1)$, but it is covered by that product. We can therefore analyze the topology of $K/U(1)_{k,l}$ by the same method as in the proof of Lemma 3.1.12: We define a map $\pi : SU(2) \to K/U(1)_{k,l}$ by $\pi(h) := hU(1)_{k,l}$, where $SU(2)$ is the Lie subgroup of $SU(3)$ with Lie algebra $\mathfrak{u}(1)_{1,-1} \oplus V_1$. It is easy to see that π is a covering map. Its kernel is $SU(2) \cap U(1)_{k,l}$. This intersection is, except for $k = 1$, $l = -1$,

$$\left\{ \left(\begin{array}{cc} e^{2\pi i \frac{mk}{k+l}} & 0 \\ 0 & e^{2\pi i \frac{ml}{k+l}} \end{array} \right) \middle| \; m \in \mathbb{Z} \right\} . \tag{5.45}$$

Since k and l are coprime, the above group is in fact:

$$\left\{ \left(\begin{array}{cc} e^{2\pi i \frac{m}{k+l}} & 0 \\ 0 & e^{-2\pi i \frac{m}{k+l}} \end{array} \right) \middle| \; m \in \mathbb{Z} \right\} . \tag{5.46}$$

$SU(2)$ is diffeomorphic to S^3 and the quotient $K/U(1)_{k,l}$ thus is a lens space which is usually denoted by $L(k+l, 1)$. We will nevertheless denote $K/U(1)_{k,l}$ by $S^3/\mathbb{Z}_{|k+l|}$, since in this thesis $\mathbb{Z}_{|k+l|}$ will always be the above discrete group. The singular orbit which we obtain for our choice of K is $SU(3)/S(U(2) \times U(1)) \cong \mathbb{CP}^2$ for all k and l.

Next, we consider the exceptional case $k = 1$, $l = -1$. If we choose W as $V_i \oplus V_4$ with $i \in \{2,3\}$, K again is isomorphic to $U(2)$. By conjugating $SU(3)$ by a permutation matrix, we can replace K by the Lie subgroup of $SU(3)$ whose Lie algebra is $2\mathfrak{u}(1) \oplus V_1$ and $U(1)_{1,-1}$ by $U(1)_{0,1}$ or $U(1)_{-1,0}$. Therefore, we obtain $K/U(1)_{1,-1} = S^3$ for the original K. For $W = V_1 \oplus V_4$, we can prove that the kernel of the covering map from $SU(2)/U(1)_{1,-1} \times U(1)_{1,1}$ to $K/U(1)_{1,-1}$ which is defined by $(h_1 U(1)_{1,-1}, h_2) \mapsto h_1 h_2 U(1)_{1,-1}$ is contained in $U(1)_{1,1}$. Therefore, we obtain $K/U(1)_{1,-1} = SU(2)/U(1)_{1,-1} \times U(1)_{1,1}/\mathbb{Z}_2 = S^2 \times S^1$. In both cases, we have as before $SU(3)/K = \mathbb{CP}^2$. In order to finish the first exceptional case, we have to check if we can choose $W = W' \oplus V_4$ with $W' \subseteq V_2 \oplus V_3$ but $W' \neq V_2, V_3$. Since $2\mathfrak{u}(1)$ is contained in $\mathfrak{k}$, W' has to be a $2\mathfrak{u}(1)$-module. The q-orthogonal complement of $2\mathfrak{u}(1)$ in $\mathfrak{su}(3)$ decomposes with respect to $2\mathfrak{u}(1)$ into $V_1 \oplus V_2 \oplus V_3$ and the three modules are pairwise inequivalent. Therefore, $\mathfrak{k} = 2\mathfrak{u}(1) \oplus V_i$ with $i \in \{1,2,3\}$ and the above possibilities for $\mathfrak{k}$ are the only ones.

We finally consider the case $k = l = 1$. In this case, $\mathfrak{k}$ necessarily has to be a Lie algebra which is isomorphic to $\mathfrak{su}(2) \oplus \mathfrak{u}(1)$, since K is compact and $\operatorname{rank} \mathfrak{k} \leqslant 2$. We decompose $\mathfrak{k}$ into the direct sum $\mathfrak{h} \oplus W'$ of $\mathfrak{u}(1)_{1,1}$-modules, where $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{k}$ and W' is the q-orthogonal complement of $\mathfrak{h}$. Since $\operatorname{span}(e_1, e_2, e_7)$ is the centralizer of $\mathfrak{u}(1)_{1,1}$, $\mathfrak{h}$ can be any of the following spaces:

$$\mathfrak{h} = \mathfrak{u}(1)_{1,1} \oplus \operatorname{span}(x) \quad \text{with} \quad x \in \operatorname{span}(e_1, e_2, e_7) \backslash \{0\} \, .$$

The $\mathfrak{u}(1)_{1,1}$-module W' is either a subspace of $\operatorname{span}(e_1, e_2, e_7)$ or of $V_2 \oplus V_3$. In the first case, we have to choose W' as the complement of $\operatorname{span}(x)$ in $\operatorname{span}(e_1, e_2, e_7)$. $\mathfrak{k}$ therefore is $2\mathfrak{u}(1) \oplus V_1$, which is isomorphic to $\mathfrak{su}(2) \oplus \mathfrak{u}(1)$. As in the other cases, we have $K/U(1)_{1,1} = S^3/\mathbb{Z}_2 = \mathbb{RP}^3$ and $SU(3)/K = \mathbb{CP}^2$. We turn to the case where $W' \subseteq V_2 \oplus V_3$. It is easy to see that there exists an

$$A \in \left\{ \left(\begin{array}{cc} \boxed{A'} & \\ & \boxed{1} \end{array} \right) \middle| \; A' \in SU(2) \right\} \quad \text{with} \quad A \, \mathfrak{h} \, A^{-1} = \mathfrak{u}(1)_{1,1} \oplus \mathfrak{u}(1)_{1,-1} = 2\mathfrak{u}(1) \, .$$

We conjugate our decomposition of $\mathfrak{k}$ by A and obtain:

$$A \, \mathfrak{k} \, A^{-1} = 2\mathfrak{u}(1) \oplus AW'A^{-1} \, .$$

Since $AW'A^{-1}$ is a $2\mathfrak{u}(1)$-module, we can prove by the same arguments as in the case $k = 1$, $l = -1$ that $AW'A^{-1}$ is either V_2 or V_3. If $A = \mathrm{Id}_{\mathbb{C}^3}$, we obtain by the same arguments as in the other exceptional case that $K/U(1)_{1,1} = S^3$ and $SU(3)/K = \mathbb{CP}^2$. If $A \neq \mathrm{Id}_{\mathbb{C}^3}$, $\mathfrak{k}$ still is a Lie algebra with $\mathfrak{u}(1)_{1,1} \subseteq \mathfrak{k}$ and our results on $K/U(1)_{1,1}$ and $SU(3)/K$ remain true. We therefore have found a continuous family of subalgebras which yield the same singular orbit and the same collapsing sphere.

4. $\mathfrak{k} = \mathfrak{u}(1)_{k,l} \oplus W$, W four-dimensional: In this case, $\mathfrak{k}$ has to be a five-dimensional Lie algebra. Since for any root α of $\mathfrak{k}$, $-\alpha$ is a root of $\mathfrak{k}$, too, we have $\dim \mathfrak{k} \equiv \operatorname{rank} \mathfrak{k} \pmod 2$. Therefore, the rank of $\mathfrak{k}$ has to be odd. Since $\operatorname{rank} \mathfrak{su}(3) = 2$, the only possibility for $\operatorname{rank} \mathfrak{k}$ is in fact 1. For that reason, $\mathfrak{k}$ has to be either $\mathfrak{u}(1)$ or $\mathfrak{su}(2)$. Since both of these Lie algebras are not five-dimensional, we can exclude this case.

5. $\mathfrak{k} = \mathfrak{u}(1)_{k,l} \oplus W$, W five-dimensional: $\mathfrak{k}$ has to be six-dimensional and closed under the Lie bracket. The only six-dimensional Lie algebra of rank $\leqslant 2$ which belongs to a compact Lie group is $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$. No matter how we embed the first summand into $\mathfrak{su}(3)$, we cannot construct another three-dimensional subspace of $\mathfrak{su}(3)$ on which the first summand acts trivially. Therefore, this case can be excluded, too.

6. $\mathfrak{k} = \mathfrak{u}(1)_{k,l} \oplus W$, W six-dimensional: We can prove by the same arguments as we have made for $\dim W = 4$ that this case is impossible.

7. $\mathfrak{k} = \mathfrak{u}(1)_{k,l} \oplus W$, W seven-dimensional: In this case, $\mathfrak{k}$ is all of $\mathfrak{su}(3)$. The only thing left to be done in this proof is to exclude that $N^{k,l}$ is a quotient of S^7 by a discrete subgroup $\Gamma \subseteq O(8)$. We have proven that for $(k, l) \neq (1, -1)$ the space $N^{k,l}$ is a $S^3/\mathbb{Z}_{|k+l|}$-bundle over $\mathbb{CP}^2$. Under the assumption that $N^{k,l} \cong S^7/\Gamma$, we have the following exact sequence:

$$\ldots \to \pi_2(S^7/\Gamma) \to \pi_2(\mathbb{CP}^2) \to \pi_1(S^3/\mathbb{Z}_{|k+l|}) \to \pi_1(S^7/\Gamma) \to \pi_1(\mathbb{CP}^2) \to \ldots .$$

If we make this exact sequence explicit, we obtain:

$$\ldots \to \{0\} \to \mathbb{Z} \to \mathbb{Z}_{|k+l|} \to \Gamma \to \{0\} \to \ldots .$$

Since there is no injective group homomorphism from $\mathbb{Z}$ to $\mathbb{Z}_{|k+l|}$, the Aloff-Wallach space $N^{k,l}$ cannot be covered by a sphere.

In the case $(k,l) = (1,-1)$, we would have to replace $S^3/\mathbb{Z}_{|k+l|}$ by $S^2 \times S^1$ and our argument would fail. Since $N^{1,-1}$ is diffeomorphic to $N^{1,0}$, that space nevertheless can be described as a S^3-bundle over $\mathbb{CP}^2$. Therefore, our argument holds for all Aloff-Wallach spaces.

□

Remark 5.4.3. Let K and K' be Lie subgroups of $SU(3)$ which satisfy $U(1)_{k,l} \subseteq K, K'$ and $K' = AKA^{-1}$ for an $A \in \mathrm{Norm}_{SU(3)}U(1)_{k,l}$. In that situation, $SU(3)/K$ and $SU(3)/K'$ as well as $K/U(1)_{k,l}$ and $K'/U(1)_{k,l}$ are $SU(3)$-equivariantly diffeomorphic. We have already mentioned all of those cases in the above proof. We will nevertheless explicitly state them again, since different choices of K yield different initial conditions for our differential equations.

1. If $N^{k,l} \cong N^{1,1}$, there are several groups K which satisfy $U(1)_{k,l} \subseteq K \subseteq SU(3)$ and are isomorphic to $U(1)^2$. These groups can be obtained by conjugation of the standard maximal torus of $SU(3)$ by an arbitrary element of $SU(2) \subseteq \mathrm{Norm}_{SU(3)}U(1)_{k,l}$.

2. If $N^{k,l} \cong N^{1,-1}$, there is a one-parameter family of groups which are isomorphic to $SO(3)$ and can be obtained by a $U(1)$-action on a fixed group. That $U(1)$-action coincides with the action of a subgroup of the normalizer $\mathrm{Norm}_{SU(3)}U(1)_{k,l}$.

3. If $N^{k,l} \cong N^{1,1}$, we obtain by an action of $SU(2)$ a further family of Lie algebras, which are all isomorphic to $\mathfrak{u}(2)$. That $SU(2)$-action coincides with the action of $SU(2) \subseteq \mathrm{Norm}_{SU(3)}U(1)_{k,l}$.

For any $N^{k,l}$, there are three different Lie algebras $\mathfrak{k}$ which satisfy $\mathfrak{u}(1)_{k,l} \subseteq \mathfrak{k} \subseteq \mathfrak{su}(3)$ and are isomorphic to $\mathfrak{u}(2)$, namely $2\mathfrak{u}(1) \oplus V_i$ with $i \in \{1,2,3\}$. These algebras can be obtained from each other by conjugation by a permutation matrix. If we drop the condition $k \geqslant l \geqslant 0$ and consider all possible Aloff-Wallach spaces, we can restrict ourselves to the case $\mathfrak{k} = 2\mathfrak{u}(1) \oplus V_1$. Therefore, it was sufficient to include only that case, in which we have $K/U(1)_{k,l} \cong S^3/\mathbb{Z}_{|k+l|}$ in the table of the lemma.

Convention 5.4.4. We want to treat the three Lie algebras from Remark 5.4.3 in a uniform manner. Therefore, we will drop the convention $k \geqslant l \geqslant 0$ whenever we consider the case $\mathfrak{k} \cong \mathfrak{u}(2)$. This will allow us to fix $\mathfrak{k}$ as $2\mathfrak{u}(1) \oplus V_1$. Since we can replace (k,l) by $(-k,-l)$, we can still assume that $k \geqslant l$.

Our next aim is to classify the cohomogeneity-one metrics with at least one singular orbit which have an Aloff-Wallach space as principal orbit and whose holonomy is contained in Spin(7). We will restrict ourselves to diagonal metrics. In the $N^{1,1}$-case, we will furthermore consider Spin(7)-structures with a frame which is given by (5.37) only. Therefore, we can work with the systems (5.36) and (5.39). There are several differences between those systems of ordinary differential equations and the systems (5.6) and (5.20) from Section 5.2 and 5.3. First, there are $SU(3)$-invariant metrics on the singular orbit which cannot be extended to a short-time solution of (5.36) or (5.39). Second, in some cases the solutions of those systems depend on additional free parameters of higher order. Third, our systems have, except in special cases, no explicit solution.

In order to classify the solutions of (5.36) and (5.39), we make an explicit power series ansatz. By this method, we find the possible initial metrics on the singular orbit and the free parameters of higher order. Furthermore, we can prove that the free parameters which we have found are the only ones and that the power series expansion exists up to infinite order. We want to know if the power series converges and satisfies the smoothness conditions from Theorem 3.2.18 for any choice of our parameters.

After having deduced the smoothness conditions, we can easily check them by symmetry arguments. Since the equations for the holonomy reduction are a special case of the cohomogeneity-one Einstein condition, we can apply Theorem 3.2.24. It follows that our power series converges on a tubular neighborhood of the singular orbit. Moreover, we also obtain an existence result for local examples of cohomogeneity-one Einstein metrics. Since the free parameters for the metrics with special holonomy are a subset of the free parameters from Theorem 3.2.24, we have found a further, more conceptual explanation for the parameters which we have found. Theorem 3.2.24 also holds in the non-diagonal case. Therefore, some of our Einstein metrics are non-diagonal, too. In the cases where non-diagonal Einstein metrics exist, the existence of non-diagonal metrics whose holonomy is contained in Spin(7) cannot be excluded. Since our program is already an extensive one, we will not deal further with this issue.

Remark 5.4.5. In some of the cases which we will consider, Assumption 3.2.19 is not satisfied. Later in this section, we will give arguments why we can nevertheless apply Theorem 3.2.24. For those arguments, we need to know if the differential equations for the cohomogeneity-one Einstein condition respect the splitting $S^2(\mathfrak{p}) \oplus S^2(\mathfrak{p}^\perp)$. In order to show this and for reasons of completeness, we deduce the equations for diagonal metrics. We include the system for the generic Aloff-Wallach spaces and for $N^{1,0}$ in the Appendix C.3. The system for $N^{1,1}$ can be found in Appendix C.4. We also include the derivation of the equations in the appendix.

On the following pages, we carry out the program which we have outlined above for each of the possible singular orbits. We will have to distinguish between the three different types of Aloff-Wallach spaces. First, we consider the most simple case:

Principal orbit: A generic $\mathbf{N^{k,l}}$, singular orbit: $\mathbf{SU(3)/U(1)^2}$

As in the previous two sections, we assume that the singular orbit is at $t = 0$. In Lemma 5.4.2, we have proven that the Lie algebra $2\mathfrak{u}(1)$ of the isotropy group $U(1)^2$ is given by $\mathfrak{u}(1)_{k,l} \oplus V_4$. We therefore have $f(0) = 0$ as an initial condition. The functions a, b, and c are all non-zero at the singular orbit. Since f has to satisfy the equation

$$f' = -\frac{-k-l}{2\Delta}\frac{f^2}{a^2} - \frac{l}{2\Delta}\frac{f^2}{b^2} - \frac{k}{2\Delta}\frac{f^2}{c^2}$$

from the system (5.36), we also have $f'(0) = 0$. With help of a short induction proof, we see that the higher derivatives of f vanish at $t = 0$, too. Any solution of the system (5.36) has to be analytic. Therefore, f vanishes for any value of t. The system (5.36) hence becomes:

$$\begin{aligned} \frac{a'}{a} &= \frac{b^2+c^2-a^2}{abc} \\ \frac{b'}{b} &= \frac{c^2+a^2-b^2}{abc} \\ \frac{c'}{c} &= \frac{a^2+b^2-c^2}{abc} \end{aligned} \tag{5.47}$$

These equations also appear in the paper of Cleyton and Swann [20] and in the next chapter. They describe the reduction of the holonomy to a subgroup of G_2 in the case of a cohomogeneity-one metric with principal orbit $SU(3)/U(1)^2$. Our considerations prove that the eight-dimensional metric (5.31) degenerates into a seven-dimensional one. Since we are mainly interested in eight-dimensional metrics, we will not further investigate the equations (5.47).

Our next aim is to classify the cohomogeneity-one Einstein metrics with a generic Aloff-Wallach space as principal orbit and $SU(3)/U(1)^2$ as singular orbit. This classification is of interest of its own, but also has a further aim, since the modules which we have to determine for our classification will be needed for other purposes, too. Before we start our calculations, we have to describe the tangent and the normal space of the singular orbit as $2\mathfrak{u}(1)$-modules. As usual, we identify the tangent space of the singular orbit with the q-orthogonal complement $\mathfrak{p}$ of $2\mathfrak{u}(1) \subseteq \mathfrak{su}(3)$. $\mathfrak{p}$ decomposes into the direct sum $V_1 \oplus V_2 \oplus V_3$ of $\mathfrak{u}(1)_{k,l}$-modules. It is easy to see that the summands V_1, V_2, and V_3 are invariant with respect to $2\mathfrak{u}(1)$, too. We will describe these $2\mathfrak{u}(1)$-modules in more detail. $2\mathfrak{u}(1)$ is spanned by e_7 and $\mathfrak{u}(1)_{k,l}$. The basis element e_7 is given by the matrix $diag((2l+k)i, (-2k-l)i, (k-l)i)$. The two-dimensional real $U(1)^2$-module on which $diag(\exp((2l+k)i\varphi), \exp((-2k-l)i\varphi), \exp((k-l)i\varphi))$ acts as $diag(\exp((2l+k)ir\varphi), \exp((-2k-l)ir\varphi), \exp((k-l)ir\varphi))$ and $diag(\exp(ki\varphi), \exp(li\varphi), \exp((-k-l)i\varphi))$ acts as $diag(\exp(kis\varphi), \exp(lis\varphi), \exp((-k-l)is\varphi))$ we denote by $\mathbb{V}_{r,s}$. Since $2l+k$ and $-2k-l$ are not necessarily coprime if k and l are coprime, the weights of the $U(1)^2$-action on $\mathbb{V}_{r,s}$ are r and $s \cdot gcd(2l+k, -2k-l)$ instead of r and s. We will nevertheless call (r,s) the weight of the $U(1)^2$-action and we are able to calculate with (r,s) in the same way as we are used to. Unlike in Section 5.2 and 5.3, $\mathbb{V}_{0,0}$ does not denote the irreducible trivial module, which we will shortly denote by $\mathbb{R}$, but a two-dimensional reducible one. Since we have $\mathrm{span}(e_7) = \mathfrak{u}(1)_{2l+k,-2k-l}$, we can determine the weights of the action of $\mathrm{span}(e_7)$ on the V_i by the same method as we have used for the action of $\mathfrak{u}(1)_{k,l}$. We obtain:

$$\begin{aligned} V_1 &= \mathbb{V}_{(2l+k)-(-2k-l),k-l} = \mathbb{V}_{3k+3l,k-l} \\ V_2 &= \mathbb{V}_{2(2l+k)-2k-l,2k+l} = \mathbb{V}_{3l,2k+l} \\ V_3 &= \mathbb{V}_{(2l+k)+2(-2k-l),k+2l} = \mathbb{V}_{-3k,k+2l} \end{aligned}$$

The following two relations will be helpful to decompose $S^2(\mathfrak{p})$ into irreducible $2\mathfrak{u}(1)$-submodules:

$$\begin{aligned} S^2(\mathbb{V}_{r,s}) &= \mathbb{V}_{2r,2s} \oplus \mathbb{R} \\ \mathbb{V}_{r_1,s_1} \otimes \mathbb{V}_{r_2,s_2} &= \mathbb{V}_{r_1+r_2,s_1+s_2} \oplus \mathbb{V}_{r_1-r_2,s_1-s_2} \end{aligned}$$

The two-dimensional space $\mathbb{V}_{r,s}$ can be $2\mathfrak{u}(1)$-equivariantly identified with $\mathbb{C}$. With this notation, the embedding of the trivial summand into $S^2(\mathbb{V}_{r,s})$ is given by:

$$\lambda \mapsto \lambda |z|^2 = \lambda \left(\mathrm{Re}(z) \otimes \mathrm{Re}(z) + \mathrm{Im}(z) \otimes \mathrm{Im}(z) \right) \ .$$

The projections onto the two summands of the decomposition of $\mathbb{V}_{r_1,s_1} \otimes \mathbb{V}_{r_2,s_2}$ can be described by:

$$\begin{aligned} z_1 \otimes z_2 &\mapsto z_1 \cdot z_2 \\ z_1 \otimes z_2 &\mapsto z_1 \cdot \overline{z_2} \end{aligned}$$

The isomorphism

$$\mathbb{V}_{r,s} \cong \mathbb{V}_{-r,-s} \ ,$$

which can be identified with the conjugation map, will be needed later on, too. With help of the above relations, we obtain:

$$\begin{aligned} S^2(\mathfrak{p}) &= S^2(\mathbb{V}_{3k+3l,k-l} \oplus \mathbb{V}_{3l,2k+l} \oplus \mathbb{V}_{-3k,k+2l}) \\ &= S^2(\mathbb{V}_{3k+3l,k-l}) \oplus S^2(\mathbb{V}_{3l,2k+l}) \oplus S^2(\mathbb{V}_{-3k,k+2l}) \\ &\quad \oplus (\mathbb{V}_{3k+3l,k-l} \otimes \mathbb{V}_{3l,2k+l}) \oplus (\mathbb{V}_{3k+3l,k-l} \otimes \mathbb{V}_{-3k,k+2l}) \oplus (\mathbb{V}_{3l,2k+l} \otimes \mathbb{V}_{-3k,k+2l}) \\ &= \mathbb{V}_{6k+6l,2k-2l} \oplus \mathbb{V}_{6l,4k+2l} \oplus \mathbb{V}_{-6k,2k+4l} \oplus 3\mathbb{R} \\ &\quad \oplus \mathbb{V}_{3k+6l,3k} \oplus \mathbb{V}_{3k,-k-2l} \oplus \mathbb{V}_{3l,2k+l} \oplus \mathbb{V}_{6k+3l,-3l} \\ &\quad \oplus \mathbb{V}_{-3k+3l,3k+3l} \oplus \mathbb{V}_{3k+3l,k-l} \end{aligned}$$

For our considerations, we have to determine the weight of the $2\mathfrak{u}(1)$-action on the normal space $\mathfrak{p}^\perp$ of the singular orbit. $\mathfrak{p}^\perp$ is a real two-dimensional $2\mathfrak{u}(1)$-module. The tangent vector $\frac{\partial}{\partial t}$ of the geodesic γ which we have fixed is contained in $\mathfrak{p}^\perp$, since γ intersects the singular orbit perpendicularly. $U(1)_{k,l}$ is generated by $e_8 \in 2\mathfrak{u}(1)$. Since $U(1)_{k,l}$ is the isotropy group of the $SU(3)$-action on the principal orbits, its action fixes $\frac{\partial}{\partial t}$. e_8 therefore acts trivially on $\mathfrak{p}^\perp$. $e_7 \in 2\mathfrak{u}(1)$ generates the group $U(1)_{2l+k,-2k-l} \subseteq SU(3)$, which we will shortly denote by $U(1)'$. The weight of the action of $U(1)'$ on $\mathfrak{p}^\perp$ can be determined by an argument which we have used in Section 5.2. The orbit of the $U(1)'$-action on a point $p \in U(1)^2/U(1)_{k,l}$ is a loop in that space. Since $\pi_1(U(1)^2/U(1)_{k,l}) = \mathbb{Z}$, the homotopy class of the orbit corresponds to an integer. For geometric reasons, that integer coincides with the weight of the $U(1)'$-action on $\mathfrak{p}^\perp$. The group $U(1)'$ consists of the following 3×3-matrices:

$$\left\{ \left(\begin{array}{ccc} e^{(2l+k)i\varphi} & 0 & 0 \\ 0 & e^{(-2k-l)i\varphi} & 0 \\ 0 & 0 & e^{(k-l)i\varphi} \end{array} \right) =: A_\varphi \,\middle|\, \varphi \in \mathbb{R} \right\} .$$

Analogously, we have:

$$U(1)_{k,l} = \left\{ \left(\begin{array}{ccc} e^{ki\psi} & 0 & 0 \\ 0 & e^{li\psi} & 0 \\ 0 & 0 & e^{-(k+l)i\psi} \end{array} \right) =: B_\psi \,\middle|\, \psi \in \mathbb{R} \right\} .$$

We want to describe the intersection of $U(1)'$ and $U(1)_{k,l}$, since its cardinality equals the winding number of the $U(1)'$-orbit around $U(1)^2/U(1)_{k,l}$. Therefore, we search for the smallest positive φ such that there exists a $\psi \in \mathbb{R}$ with $A_\varphi = B_\psi$. It is easy to see that $U(1)' \cap U(1)_{k,l} = \{A_{j\varphi} | j \in \mathbb{Z}\}$. The equation $A_\varphi = B_\psi$ is equivalent to:

$$\begin{array}{rrcl} (-2k-l)\varphi & -l\psi & = & 2\pi j_1 \\ (k-l)\varphi & +(k+l)\psi & = & 2\pi j_2 \end{array} \tag{5.48}$$

for a pair $(j_1, j_2) \in \mathbb{Z}^2$. We solve the above system for all $(j_1, j_2) \in \mathbb{Z}^2$ and obtain:

$$\left(\begin{array}{c} \varphi \\ \psi \end{array} \right) = \frac{-1}{2(k^2+lk+l^2)} \left(\begin{array}{cc} (k+l)(2\pi j_1) & +l(2\pi j_2) \\ (-k+l)(2\pi j_1) & +(-2k-l)(2\pi j_2) \end{array} \right) .$$

Since $k+l$ and l are coprime, we can choose j_1 and j_2 in such a way that φ becomes $\frac{\pi}{k^2+lk+l^2}$. This number obviously is the smallest possible value of $|\varphi|$. We let φ run from 0 to 2π no matter if $2l+k$ and $-2k-l$ are coprime or not. Loosely speaking, $U(1)'$ thus intersects $U(1)_{k,l}$ $2(k^2+lk+l^2)$ times. Since the number of intersections coincides with the weight of the $U(1)'$-action on $\mathfrak{p}^\perp$, we have proven that in our notation:

$$\mathfrak{p}^\perp = \mathbb{V}_{2(k^2+lk+l^2),0} \,.$$

We are now able to decompose $S^m(\mathfrak{p}^\perp)$ into irreducible $2\mathfrak{u}(1)$-submodules and obtain:

$$S^m(\mathfrak{p}^\perp) = \left\{ \begin{array}{ll} \mathbb{V}_{2m(k^2+lk+l^2),0} \oplus \mathbb{V}_{2(m-2)(k^2+lk+l^2),0} \oplus \ldots \oplus \mathbb{V}_{4(k^2+lk+l^2),0} \oplus \mathbb{R} & \text{if } m \text{ is even} \\ \mathbb{V}_{2m(k^2+lk+l^2),0} \oplus \mathbb{V}_{2(m-2)(k^2+lk+l^2),0} \oplus \ldots \oplus \mathbb{V}_{2(k^2+lk+l^2),0} & \text{if } m \text{ is odd} \end{array} \right.$$

The elements of $S^m(\mathfrak{p}^\perp)$ can be identified with polynomials of m^{th} order on the real two-dimensional vector space $\mathbb{C}$. After this identification, the embedding of $\mathbb{V}_{2(m-2p)(k^2+lk+l^2),0}$ (or $\mathbb{R}$) into $S^m(\mathfrak{p}^\perp)$ simply becomes $w \mapsto w \cdot |z|^{2p}$ (or $\lambda \mapsto \lambda \cdot |z|^m$).

Before we can apply Theorem 3.2.24, we have to check if Assumption 3.2.19 is satisfied. This assumption states that $\mathfrak{p}$ and $\mathfrak{p}^\perp$ have no equivalent $U(1)_{k,l}$-submodules in common. Since $U(1)_{k,l}$ acts trivially on $\mathfrak{p}^\perp$, our assumption is violated if and only if any of the modules V_1, V_2, or V_3 is trivial. As we have seen at the beginning of this section, this can only be the case if $N^{k,l} \cong N^{1,1}$. Since we are now considering the generic case, Theorem 3.2.24 can be applied.

In order to determine the number of cohomogeneity-one Einstein metrics, we have to describe the spaces $W_2^v := \mathrm{Hom}_{U(1)^2}(S^2(\mathfrak{p}^\perp), S^2(\mathfrak{p}^\perp))$ and $W_m^h := \mathrm{Hom}_{U(1)^2}(S^m(\mathfrak{p}^\perp), S^2(\mathfrak{p}))$. We start with the vertical case. $S^2(\mathfrak{p}^\perp)$ decomposes into two inequivalent irreducible $2\mathfrak{u}(1)$-submodules:

$$S^2(\mathfrak{p}^\perp) = \mathbb{V}_{4(k^2+lk+l^2),0} \oplus \mathbb{R}\,.$$

The module $\mathbb{V}_{4(k^2+lk+l^2),0}$ can be identified with $\mathbb{C}$ which is equipped with a certain $U(1)^2$-action. The $U(1)^2$-equivariant maps from $\mathbb{C}$ to itself are described by the multiplication by an arbitrary complex number. Since $\dim_{\mathbb{R}} \mathbb{C} = 2$, we have $\dim \mathrm{Hom}_{U(1)^2}(\mathbb{V}_{4(k^2+lk+l^2),0}, \mathbb{V}_{4(k^2+lk+l^2),0}) = 2$. Therefore, $\dim W_2^v = 3$. Since the vertical part of the metric is determined by f, there is only one initial condition for the vertical part which we can prescribe. For the same reasons as in Section 5.2 and 5.3, this initial condition describes the value of $f'''(0)$.

Next, we calculate the number of free parameters in the horizontal direction. Since the principal orbit is a generic Aloff-Wallach space, the weight of the $U(1)_{k,l}$-action on any of the submodules of $S^2(\mathfrak{p})$, except on $3\mathbb{R}$, is non-zero. Therefore, the image of any $U(1)^2$-equivariant homomorphism from $S^m(\mathfrak{p}^\perp)$ to $S^2(\mathfrak{p})$ is contained in $3\mathbb{R}$ and we obtain:

$$\dim W_m^h = \begin{cases} 3 & \text{if } m \text{ is even} \\ 0 & \text{if } m \text{ is odd} \end{cases}$$

We recall that the number of free parameters of order $m \geqslant 2$ in the horizontal direction is $\dim W_m^h - \dim W_{m-2}^h$. The number of free parameters of 0^{th} (1^{st}) order is $\dim W_0^h$ ($\dim W_1^h$). Therefore, we only have 3 parameters of 0^{th} order. This means that we can choose $a(0)$, $b(0)$, and $c(0)$ freely. Since we have no choice for the first derivative of the metric, the initial shape operator has to vanish and the singular orbit has to be totally geodesic. All in all, we have proven that for any choice of $f'''(0)$ and the Einstein constant, there is a unique extension of any initial metric on the singular orbit to a cohomogeneity-one Einstein metric. We turn to the next case:

Principal orbit: A $\mathbf{N^{k,l}}$ which is $\cong \mathbf{N^{1,0}}$, singular orbit: $\mathbf{SU(3)/U(1)^2}$

In this situation, the only trivial $U(1)_{k,l}$-submodule of $\mathfrak{m}$ again is V_4 and we have $f(0) = 0$ as initial condition. Since we restrict ourselves to diagonal metrics, we can work as before with the system (5.36). Therefore, we can prove by the same arguments as in the previous case that f vanishes and the metric degenerates.

We apply the methods of Eschenburg and Wang [32] in order to describe the cohomogeneity-one Einstein metrics with the orbit structure which we have chosen. Since none of the $U(1)_{k,l}$-modules V_1, V_2, and V_3 is trivial, Assumption 3.2.19 is satisfied and we are in the situation of Theorem 3.2.24. If $k = 1$ and $l = 0$, the decompositions of $S^2(\mathfrak{p})$ and $S^m(\mathfrak{p}^\perp)$ which we have found specialize to:

$$\begin{aligned} S^2(\mathfrak{p}) = {} & \mathbb{V}_{6,2} \oplus \mathbb{V}_{0,4} \oplus \mathbb{V}_{-6,2} \oplus 3\mathbb{R} \\ & \oplus \mathbb{V}_{3,3} \oplus \mathbb{V}_{3,-1} \oplus \mathbb{V}_{0,2} \oplus \mathbb{V}_{6,0} \\ & \oplus \mathbb{V}_{-3,3} \oplus \mathbb{V}_{3,1} \end{aligned}$$

and

$$S^m(\mathfrak{p}^\perp) = \begin{cases} \mathbb{V}_{2m,0} \oplus \mathbb{V}_{2(m-2),0} \oplus \ldots \oplus \mathbb{V}_{4,0} \oplus \mathbb{R} & \text{if } m \text{ is even} \\ \mathbb{V}_{2m,0} \oplus \mathbb{V}_{2(m-2),0} \oplus \ldots \oplus \mathbb{V}_{2,0} & \text{if } m \text{ is odd} \end{cases}$$

$S^2(\mathfrak{p})$ contains a summand which is isomorphic to $\mathbb{V}_{6,0}$. Therefore, we have $\dim W^h_m = 2$ if m is odd and $\geqslant 3$. We calculate $\dim W^h_m$ for all $m \in \mathbb{N}_0$ and obtain:

$$\dim W^h_m = \begin{cases} 3 & \text{if } m \text{ is even} \\ 0 & \text{if } m = 1 \\ 2 & \text{if } m \geqslant 3 \text{ is odd} \end{cases}$$

We conclude that there are three initial conditions of 0^{th} and two of 3^{rd} order which we can choose. Of course, there again is a further free initial condition for the vertical part of the metric, which describes the freedom in $f'''(0)$. Since the $\mathbb{V}_{6,0}$-summand is contained in $V_1 \otimes V_3$, the two parameters of 3^{rd} order describe how the initial metric, which is diagonal, changes into a non-diagonal one. If we had considered a system of differential equations which also described the non-diagonal metrics, we therefore might have obtained new examples of metrics with holonomy Spin(7). We will nevertheless not carry out the calculations which were needed to check the existence of such examples.

Principal orbit: A $\mathbf{N^{k,l}}$ which is $\cong \mathbf{N^{1,1}}$, singular orbit: $\mathbf{SU(3)/U(1)^2}$

Analogously to the previous case, we assume that the metric on any principal orbit is a diagonal metric of type (5.33). Furthermore, we assume that the Spin(7)-structure is given by the frame (5.37) and the four-form (5.38). We thus have to work with the system (5.39), which is more intricate than (5.36). Since the module V_1 is in this situation trivial, we can choose the Lie algebra of the isotropy group $U(1)^2$ as $\mathfrak{u}(1)_{k,l} \oplus \mathrm{span}(x)$ with an arbitrary non-zero $x \in V_1 \oplus V_4$. Therefore, we have further possibilities for the component of the metric which vanishes at $t = 0$. Since we assume that the metric is diagonal, we can reduce the number of the possible x. The metric on the principal orbit $N^{1,1}$ depends on a parameter t and will thus be denoted by g_t. We have $\lim_{t\to 0} g_t(x,x) = 0$. x is a linear combination $\alpha e_1 + \beta e_2 + \gamma e_3$ for some coefficients $\alpha, \beta, \gamma \in \mathbb{R}$. With the notation of (5.33), we have:

$$g_t(x,x) = \alpha^2 a_1(t)^2 + \beta^2 a_2(t)^2 + \gamma^2 f(t)^2 \,.$$

This term converges to zero if and only if one of the three metric functions a_1, a_2, f converges to zero and the coefficients of the other two functions vanish. Therefore, we have $x \in \{e_1, e_2, e_7\}$ and the only cases which we have to consider are $a_1(0) = 0$, $a_2(0) = 0$, and $f(0) = 0$. We will replace e_7 by $\frac{1}{3}e_7$, since this will simplify some of our calculations. We will prove that the first two cases are equivalent to the third one. Let $x \neq \frac{1}{3}e_7$. At the beginning of this section, we have seen that the $SU(2)$-factor of the normalizer $\mathrm{Norm}_{SU(3)}U(1)_{1,1}$ acts on $V_1 \oplus V_4$ as $SO(3)$ on $\mathbb{R}^3$ and on $V_2 \oplus V_3$ as $SU(2)$ on $\mathbb{C}^2$. Therefore, there exists a $\sigma \in \mathrm{Norm}_{SU(3)}U(1)_{1,1}$ which interchanges x and $\frac{1}{3}e_7$. We denote the third element of $\{e_1, e_2, \frac{1}{3}e_7\}$ by y. Since σ acts as an element of $SO(3)$ on $V_1 \oplus V_4$ and preserves the normal metric q, its matrix representation with respect to the basis $(x, \frac{1}{3}e_7, y)$ necessarily is:

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

We will prove that there exists a $\tau \in \mathrm{Norm}_{SU(3)}U(1)_{1,1}$ which acts as σ on $\mathrm{span}(e_1, e_2, \frac{1}{3}e_7)$ and maps the metric g_t to another diagonal metric. Let $\rho : SU(2) \to SO(3)$ be the canonical covering homomorphism. $\rho(h)$, where h is a unit quaternion, can be described by the linear map $k \mapsto hkh^{-1}$, which acts on $\mathrm{Im}(\mathbb{H}) \cong \mathbb{R}^3$. In order to find the $\tau \in \mathrm{Norm}_{SU(3)}U(1)_{1,1}$ which we are looking at, we first construct a $\tau' \in SU(2)$ such that $\rho(\tau')$ is the above 3×3-matrix. That matrix is a rotation around the axis $\mathrm{span}((1,1,0)^T)$ by an angle of π. It is easy to see that τ' has to be $\pm\frac{1}{\sqrt{2}}(i + j) \subseteq SU(2)$. Motivated by this observation, we define τ as $\pm\frac{1}{\sqrt{2}}(x + \frac{1}{3}e_7) \subseteq SU(2)$. As usual, the group $SU(2)$ is embedded into $SU(3)$ by:

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \mapsto \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

By a short calculation, we see that τ acts indeed as σ on $\mathrm{span}(e_1, e_2, \frac{1}{3}e_7)$. Moreover, we can easily determine the action of τ on $S^2(\mathrm{span}(e_3, e_4, e_5, e_6))$. Since the sign of $\frac{1}{\sqrt{2}}(x + \frac{1}{3}e_7)$ gets canceled, we obtain the same action on $S^2(\mathrm{span}(e_3, e_4, e_5, e_6))$ for both values of τ. For the following considerations, we assume that $x = e_1$. For $x = e_2$, we would obtain an analogous result. We examine the bilinear form whose matrix representation with respect to (e_3, e_4, e_5, e_6) is

$$\begin{pmatrix} b^2 & 0 & & \\ 0 & b^2 & & \\ & & c^2 & 0 \\ & & 0 & c^2 \end{pmatrix}$$

This form is mapped by the action of τ to:

$$\begin{pmatrix} \frac{1}{2}(b^2 + c^2) & 0 & 0 & \frac{1}{2}(b^2 - c^2) \\ 0 & \frac{1}{2}(b^2 + c^2) & \frac{1}{2}(-b^2 + c^2) & 0 \\ 0 & \frac{1}{2}(-b^2 + c^2) & \frac{1}{2}(b^2 + c^2) & 0 \\ \frac{1}{2}(b^2 - c^2) & 0 & 0 & \frac{1}{2}(b^2 + c^2) \end{pmatrix}$$

Therefore, the metric which we obtain by the action of τ is diagonal if and only if $b^2 = c^2$. We consider the system (5.39) with the initial conditions $f(0) = 0$ and $b(0)^2 = c(0)^2$. Furthermore, we assume that $b^2(t) = c^2(t)$ for all values of t. Although the system (5.39) is preserved after we interchange b and c, this assumption might not be automatically satisfied. The reason for this is that there may exist initial conditions of higher order which we can prescribe in such a way that $b^2(t) - c^2(t)$ becomes $\neq 0$. The action of τ turns our initial value problem into a new one. Since $\tau \in \mathrm{Norm}_{SU(3)}U(1)_{1,1}$ transforms the metric into another diagonal metric, the new metric obeys according to Lemma 3.2.36 again the system (5.39) with the new initial conditions $a_1(0) = 0$ and $b(0)^2 = c(0)^2$. We consider the third equation of the system (5.39) under the assumption that we still have $a_1(0) = 0$ but the initial values of the other metric functions are arbitrary non-zero numbers. Since $b(0) \neq 0$, the right hand side of that equation has to converge for $t \to 0$ to a real number. In particular, the term $\frac{1}{2}\frac{a_1^2+c^2-b^2}{a_1bc}$ from the right hand side has to converge to a real number, too. From this fact, it easily follows that $b(0)^2$ has to equal $c(0)^2$. By the action of τ^{-1}, this initial metric becomes another metric which satisfies $f(0) = 0$ and $b(0)^2 = c(0)^2$. Therefore, the diagonal metrics with the initial condition $a_1(0) = 0$ are isometric to those with $f(0) = 0$ and $b^2(0) = c^2(0)$ provided that there are no free parameters of higher order for b and c. Conversely, the metrics with $f(0) = 0$ and $b(0)^2 = c(0)^2$ correspond in our situation to diagonal metrics with $a_1(0) = 0$ and those with $b(0)^2 \neq c(0)^2$ correspond to non-diagonal metrics.

For the above reasons, we can restrict ourselves to the case $f(0) = 0$. This case we will now consider in detail. We take a look at the following three equations, which we have taken from (5.39):

$$\begin{aligned} a_1' &= \frac{b^2+c^2-a_1^2}{bc} + 3\frac{a_1^2-a_2^2}{a_2 f} - \frac{1}{3}\frac{f}{a_2} \\ a_2' &= \frac{b^2+c^2-a_2^2}{bc} + 3\frac{a_2^2-a_1^2}{a_1 f} - \frac{1}{3}\frac{f}{a_1} \\ f' &= -3\frac{(a_1-a_2)^2}{a_1a_2} + \frac{1}{3}\frac{f^2}{a_1a_2} - \frac{1}{6}\frac{f^2}{b^2} - \frac{1}{6}\frac{f^2}{c^2} \end{aligned}$$

Since we need $f'(0) \neq 0$ in order to make the metric smooth at the singular orbit, we have $a_1(0) - a_2(0) \neq 0$. The numerator of the term $\frac{a_1^2-a_2^2}{a_2 f}$ has to vanish, since we do not want $\lim_{t\to 0} a_i'(t) = \pm\infty$. Therefore, we have $a_1(0) = a_0 = -a_2(0)$ for a $a_0 \in \mathbb{R}\backslash\{0\}$. We choose $b(0)$ and $c(0)$ as arbitrary real constants b_0 and c_0. With these initial values, we make a power series ansatz for the system (5.39) and obtain:

$$\begin{aligned} a_1(t) &= a_0 - \frac{1}{2}\frac{a_0^2-b_0^2-c_0^2}{b_0c_0}t + \frac{1}{8}\frac{3a_0^4-2a_0^2b_0^2-2a_0^2c_0^2-b_0^4+14b_0^2c_0^2-c_0^4}{a_0b_0^2c_0^2}t^2 + \ldots \\ a_2(t) &= -a_0 - \frac{1}{2}\frac{a_0^2-b_0^2-c_0^2}{b_0c_0}t - \frac{1}{8}\frac{3a_0^4-2a_0^2b_0^2-2a_0^2c_0^2-b_0^4+14b_0^2c_0^2-c_0^4}{a_0b_0^2c_0^2}t^2 + \ldots \\ b(t) &= b_0 + 0\cdot t - \frac{1}{4}\frac{a_0^4-6a_0^2c_0^2-b_0^4+c_0^4}{a_0^2b_0c_0^2}t^2 + \ldots \\ c(t) &= c_0 + 0\cdot t - \frac{1}{4}\frac{a_0^4-6a_0^2b_0^2+b_0^4-c_0^4}{a_0^2b_0^2c_0}t^2 + \ldots \\ f(t) &= 0 + 12t + 0\cdot t^2 + \ldots \end{aligned} \tag{5.49}$$

The next issue which we will discuss is how the smoothness conditions from Theorem 3.2.18 translate into conditions on the coefficients of the above power series. In order to answer this question, we have to consider the modules

$$W_m = \mathrm{Hom}_{U(1)^2}(S^m(\mathfrak{p}^\perp), S^2(\mathfrak{p}\oplus\mathfrak{p}^\perp))\,.$$

Since $\mathfrak{p} = \mathbb{V}_{6,0} \oplus \mathbb{V}_{3,3} \oplus \mathbb{V}_{-3,3}$ and $\mathfrak{p}^\perp = \mathbb{V}_{6,0}$ have one submodule in common, Assumption 3.2.19 is not satisfied and we do not have the splitting $W_m = W_m^h \oplus W_m^v$. Since the metrics which we investigate are diagonal, they can be considered as elements of $S^2(\mathfrak{p}) \oplus S^2(\mathfrak{p}^\perp)$ at a point of the singular orbit. Therefore, the smaller modules

$$W'_m := \mathrm{Hom}_{U(1)^2}(S^m(\mathfrak{p}^\perp), S^2(\mathfrak{p}) \oplus S^2(\mathfrak{p}^\perp)) = W^h_m \oplus W^v_m$$

are sufficient for our considerations. We describe W^h_m and W^v_m in detail. In our situation, we have:

$$\begin{aligned} S^2(\mathfrak{p}) = {} & \mathbb{V}_{12,0} \oplus \mathbb{V}_{6,6} \oplus \mathbb{V}_{-6,6} \oplus 3\mathbb{R} \\ & \oplus \mathbb{V}_{9,3} \oplus \mathbb{V}_{3,-3} \oplus \mathbb{V}_{3,3} \oplus \mathbb{V}_{9,-3} \\ & \oplus \mathbb{V}_{0,6} \oplus \mathbb{V}_{6,0} \end{aligned}$$

For $S^m(\mathfrak{p}^\perp)$, we obtain:

$$S^m(\mathfrak{p}^\perp) = \begin{cases} \mathbb{V}_{6m,0} \oplus \mathbb{V}_{6(m-2),0} \oplus \ldots \oplus \mathbb{V}_{12,0} \oplus \mathbb{R} & \text{if } m \text{ is even} \\ \mathbb{V}_{6m,0} \oplus \mathbb{V}_{6(m-2),0} \oplus \ldots \oplus \mathbb{V}_{6,0} & \text{if } m \text{ is odd} \end{cases}$$

The dimensions of W^h_m and W^v_m can be calculated with help of Schur's lemma:

$$\dim W^h_m = \begin{cases} 3 & \text{if } m = 0 \\ 5 & \text{if } m \geqslant 2 \text{ and even} \\ 2 & \text{if } m \text{ odd} \end{cases} \qquad \dim W^v_m = \begin{cases} 1 & \text{if } m = 0 \\ 3 & \text{if } m \geqslant 2 \text{ and even} \\ 0 & \text{if } m \text{ odd} \end{cases}$$

We have to discuss the geometric meaning of these numbers. First, we interpret the dimensions of the spaces W^h_m. W^h_0 describes the $SU(3)$-invariant metrics on the singular orbit. By a short calculation, we see that any diagonal metric on the singular orbit which satisfies $a_1(0)^2 = a_2(0)^2$ is $SU(3)$-invariant. Since $\dim W^h_0 = 3$, there are no further invariant metrics on $SU(3)/U(1)^2$. If m is even and $\geqslant 2$, there are two additional parameters. Since the $\mathbb{V}_{12,0}$-summand, which is responsible for these parameters, is contained in $S^2(V_1)$, the m^{th} derivative of the restriction of the metric to V_1 may depend on three parameters. One of the two additional parameters describes the m^{th} derivative of the off-diagonal function $\beta_{1,2}$ and the other one describes the change of $a_1^2 - a_2^2$ away from zero. Since we only consider diagonal metrics, we can ignore the degree of freedom for $\beta^{(m)}_{1,2}(0)$. The three parameters which were already there in the case $m = 0$ are responsible for $(b^2)^{(m)}(0)$, $(c^2)^{(m)}(0)$, and the m^{th} derivative of the coefficient of $e^1 \otimes e^1 + e^2 \otimes e^2$. Our considerations prove that we can choose $(a_1^2)^{(m)}(0)$, $(a_2^2)^{(m)}(0)$, $(b^2)^{(m)}(0)$, and $(c^2)^{(m)}(0)$ freely if m is even and $\geqslant 2$. Next, we consider the case where m is odd. In that case, the m^{th} derivative of the metric depends on two parameters. The $\mathbb{V}_{6,0}$-summand of $S^2(\mathfrak{p})$, which is responsible for these parameters, is contained in $V_2 \otimes V_3$. Therefore, the two parameters describe how the off-diagonal functions $\beta_{3,5}$ and $\beta_{3,6}$ behave and we have no degree of freedom for the other coefficients. Since the restrictions on the coefficients are linear, $(a_1^2)^{(m)}(0)$, $(a_2^2)^{(m)}(0)$, $(b^2)^{(m)}(0)$, and $(c^2)^{(m)}(0)$ have to vanish if m is odd. All in all, we have proven that a_1^2, a_2^2, b^2, and c^2 are even functions. Since these functions are $\neq 0$ at the singular orbit, the smoothness conditions on a_1, a_2, b, and c are that they all have to be even and that $a_1(0)^2 = a_2(0)^2$. The power series (5.49) does not seem to satisfy these conditions, since a_1 and a_2 are not even. We will deal with this problem later on.

Next, we explicitly describe the smoothness conditions for the vertical part of the metric. We can make use of the same arguments as in Section 5.2 and 5.3 and conclude that f has to be an odd function. The fact that the length of the collapsing circle $U(1)^2/U(1)_{k,l}$ has to be $2\pi t + O(t^2)$ for small values of t determines the value of $|f'(0)|$. As in the previous two sections, these two conditions on f are necessary and sufficient for the smoothness of the vertical part. We calculate the value of $|f'(0)|$ which makes the metric smooth. The following map γ parameterizes $U(1)^2/U(1)_{k,l}$:

$$\gamma(t) := \begin{pmatrix} e^{(2l+k)it} & 0 & 0 \\ 0 & e^{(-2k-l)it} & 0 \\ 0 & 0 & e^{(k-l)it} \end{pmatrix} U(1)_{k,l}$$

We have already proven that the Lie group which is generated by e_7 intersects $U(1)_{k,l}$ for the first time at $\frac{\pi}{k^2+kl+l^2}$. Since we assume that $k = l = 1$, we have to define γ on $[0, \frac{\pi}{3}]$ in order to parameterize all of the circle. The length of the collapsing circle with radius t therefore is:

$$\int_0^{\frac{\pi}{3}} \sqrt{g(e_7, e_7)} ds = \frac{\pi}{3} |f(t)| \,.$$

Since this length should equal $2\pi t + O(t^2)$, we have

$$\frac{|f(t)|}{t} = 6 + O(t) \,.$$

and conclude that $|f'(0)|$ has to be 6. From the power series ansatz (5.49), it follows that in our situation we actually have $|f'(0)| = 12$. Therefore, the metrics with principal orbit $N^{1,1}$ and singular orbit $SU(3)/U(1)^2$ which solve (5.39) cannot be smoothly extended to the singular orbit. Nevertheless, there is a method to obtain smooth cohomogeneity-one metrics whose holonomy is contained in Spin(7) from the solutions of that system. Let

$$h := \begin{pmatrix} i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & 1 \end{pmatrix} \in SU(3) \,.$$

Furthermore, let $K \subseteq SU(3)$ be the group which is generated by $U(1)_{1,1}$ and h. It is easy to see that K is a direct product of type $U(1) \times \mathbb{Z}_2$. The action of h on $\mathfrak{m}$ we denote by a dot. We have:

$$\begin{aligned} &h.e_1 = -e_1 \quad h.e_2 = -e_2 \quad h.e_3 = e_4 \quad h.e_4 = -e_3 \\ &h.e_5 = -e_6 \quad h.e_6 = e_5 \qquad h.e_7 = e_7 \end{aligned} \tag{5.50}$$

These equations prove that the action of h preserves any diagonal metric of type (5.33). Therefore, the map $\varphi : N^{1,1} \to N^{1,1}$ with $\varphi(gU(1)_{1,1}) := hgh^{-1}U(1)_{1,1}$ is an isometry. From this fact, it follows that $N^{1,1}$ is a double cover of $SU(3)/K$ and that the covering map can be chosen as a local isometry. K is contained in the maximal torus of $SU(3)$ which acts as a subgroup of G_2 on $\mathfrak{m}$. h therefore preserves the G_2-structure on $N^{1,1}$ and $SU(3)/K$ admits a $SU(3)$-invariant G_2-structure which is induced by the covering map. We divide the eight-dimensional cohomogeneity-one manifold which we consider by the group which is generated by the action of h on all orbits. Since h preserves the G_2-structure, the Spin(7)-structure on our manifold turns into another Spin(7)-structure on the new space. The holonomy of the metric which we obtain this way therefore remains the same. The principal orbits of the old manifold turn into spaces of type $SU(3)/K$. The singular orbit $SU(3)/U(1)^2$ remains the same, since h is an element of $U(1)^2 \subseteq SU(3)$. The action of h preserves the orientation. Therefore, the new space has a conical orbifold singularity at the singular orbit, which we will straighten out. It is easy to see that any circle which was wrapped r times around the origin of $\mathfrak{p}^\perp$ is wrapped $2r$ times around the origin of the new normal space. We reconsider the arguments which we have made and see that in this new situation we have to require $|f'(0)| = 12$ instead of $|f'(0)| = 6$ in order to make the metric smooth at the singular orbit $SU(3)/U(1)^2$. Since $U(1)^2$ now acts on $\mathfrak{p}^\perp$ as $\mathbb{V}_{12,0}$ rather than $\mathbb{V}_{6,0}$, we have

$$\dim W_m^h = \begin{cases} 3 & \text{if } m \text{ even} \\ 2 & \text{if } m \text{ odd} \end{cases}$$

Since the values of $\dim W_m^h$ have changed, the smoothness conditions on the functions a_1, a_2, b, and c have changed, too. By similar arguments as for the principal orbit $N^{1,1}$, we can prove that for even m we still can choose $(b^2)^{(m)}(0)$ and $(c^2)^{(m)}(0)$ arbitrarily. The value of $(a_1^2)^{(m)}(0)$ has to equal $(a_2^2)^{(m)}(0)$ and can be chosen arbitrarily, too. If m is odd, there are two parameters on which the m^{th} derivative of the horizontal part of the metric may depend. These parameters correspond to the equivariant maps from $\mathbb{V}_{12,0} \subseteq S^2(V_1)$ to $\mathfrak{p}^\perp$. $\mathbb{V}_{12,0}$ is the trace-free part of $S^2(V_1)$. Therefore, $(b^2)^{(m)}(0)$ as well as $(c^2)^{(m)}(0)$ have to vanish and the two parameters describe the m^{th} derivatives of the coefficients of $e^1 \otimes e^1 - e^2 \otimes e^2$ and $e^1 \otimes e^2 + e^2 \otimes e^1$. Since we assume that the metric is diagonal, we only have to consider the coefficient of $e^1 \otimes e^1 - e^2 \otimes e^2$. It follows that $(a_1^2)^{(m)}(0)$ has to coincide with $-(a_2^2)^{(m)}(0)$ and can be chosen arbitrarily. Since a_1 and a_2 are analytic functions, we obtain $a_1^2(t) = a_2^2(-t)$ as necessary smoothness condition. With help of $a_1(0) = -a_2(0)$, we conclude that $a_1(t) = -a_2(-t)$. As in the previous case, b and c have to be even. The dimensions of W_m^v stay the same and we have the same conditions on f as before, except $|f'(0)| = 12$.

On page 180, we have calculated the first summands of a formal power series solution of the system (5.39). Those summands satisfy the smoothness conditions which we have deduced above. Nevertheless, it is not a priori clear if there really exists a power series solution of (5.39) which is of infinite order and satisfies the conditions from Theorem 3.2.18. On the following pages, we will prove that there is indeed a unique power series solution of (5.39) for any choice of $a_1(0)$, $b(0)$, and $c(0)$. Since the system (5.39) is preserved if we replace $(a_1(t), a_2(t), b(t), c(t), f(t))$ by $(-a_2(-t), -a_1(-t), b(-t), c(-t), -f(-t))$, that power series has to satisfy the smoothness conditions. Any formal power series which satisfies (5.39) and the smoothness conditions also satisfies the cohomogeneity-one Einstein condition. According to Eschenburg and Wang [32], the power series converges near the singular orbit if the cohomogeneity-one action obeys the condition which we have stated in 3.2.19. The result of Eschenburg and Wang on the convergence is also true in our situation, although we have not yet checked if a manifold with principal orbit $SU(3)/K$ and singular orbit $SU(3)/U(1)^2$ satisfies Assumption 3.2.19. The reason for this is that the convergence result is proven with help of a Picard-iteration. Assumption 3.2.19 is only needed to secure that the metric is contained in $S^2(\mathfrak{p}) \oplus S^2(\mathfrak{p}^\perp)$ and that the Picard iteration takes its values in the same space. We take a look at the cohomogeneity-one Einstein condition, which we have explicitly described in Appendix C.4. The differential equations which we have deduced in that appendix preserve the space $S^2(\mathfrak{p}) \oplus S^2(\mathfrak{p}^\perp)$. Therefore, the Picard-iteration cannot leave that space, too, and we can show by the same arguments as Eschenburg and Wang [32] that the power series converges. We thus will have proven the existence of local cohomogeneity-one metrics whose holonomy is contained in Spin(7) as soon as it is clear that a unique formal power series solution exists. The system (5.39) can be rewritten as:

$$\begin{aligned}
a_1'a_2bcf &= (b^2+c^2-a_1^2)a_2f + 3(a_1^2-a_2^2)bc - \frac{1}{3}bcf^2 \\
a_1a_2'bcf &= (b^2+c^2-a_2^2)a_1f + 3(a_2^2-a_1^2)bc - \frac{1}{3}bcf^2 \\
b' &= \frac{1}{2}\frac{a_1^2+c^2-b^2}{a_1c} + \frac{1}{2}\frac{a_2^2+c^2-b^2}{a_2c} + \frac{1}{6}\frac{f}{b} \\
c' &= \frac{1}{2}\frac{a_1^2+b^2-c^2}{a_1b} + \frac{1}{2}\frac{a_2^2+b^2-c^2}{a_2b} + \frac{1}{6}\frac{f}{c} \\
f' &= -3\frac{(a_1-a_2)^2}{a_1a_2} + \frac{1}{3}\frac{f^2}{a_1a_2} - \frac{1}{6}\frac{f^2}{b^2} - \frac{1}{6}\frac{f^2}{c^2}
\end{aligned}$$

By considering the $(m-1)^{st}$ derivatives of the last three equations at $t=0$, we see that $b^{(m)}(0)$, $c^{(m)}(0)$, and $f^{(m)}(0)$ are uniquely determined by the lower derivatives of a_1, a_2, b, c, and f. It remains to prove that $a_1^{(m)}(0)$ and $a_2^{(m)}(0)$ are uniquely determined, too. We consider the m^{th} derivatives of the first two equations at $t=0$. We will put all terms which contain any m^{th} derivative of a function on the left hand side. The other terms will be placed on the right hand side. Since any product which contains $f(0)$ vanishes, only few terms survive and we obtain:

$$\begin{aligned}
ma_1^{(m)}(0)a_2(0)b(0)c(0)f'(0) + a_1'(0)a_2(0)b(0)c(0)f^{(m)}(0) \\
-(b(0)^2+c(0)^2-a_1(0)^2)a_2(0)f^{(m)}(0) \\
-3(2a_1^{(m)}(0)a_1(0) - 2a_2^{(m)}(0)a_2(0))b(0)c(0) \\
-3(a_1^2(0)-a_2^2(0))(b^{(m)}(0)c(0)+b(0)c^{(m)}(0)) &= R_1
\end{aligned}$$

$$\begin{aligned}
ma_1(0)a_2^{(m)}(0)b(0)c(0)f'(0) + a_1(0)a_2'(0)b(0)c(0)f^{(m)}(0) \\
-(b(0)^2+c(0)^2-a_2(0)^2)a_1(0)f^{(m)}(0) \\
-3(2a_2^{(m)}(0)a_2(0) - 2a_1^{(m)}(0)a_1(0))b(0)c(0) \\
-3(a_2^2(0)-a_1^2(0))(b^{(m)}(0)c(0)+b(0)c^{(m)}(0)) &= R_2
\end{aligned}$$

where R_1 and R_2 are polynomials which depend on the lower derivatives of our functions only. The reason for the factor m in the above equations is that the term $a_1^{(m)}(0)f'(0)$ appears m

times if we apply the product rule to $(a_1' f)^{(m)}(0)$. Since we have already calculated a power series expansion up to second order, we can assume that $m \geqslant 2$. The above two equations are thus linear with respect to the m^{th} derivatives of our functions. $b^{(m)}(0)$, $c^{(m)}(0)$, and $f^{(m)}(0)$ can be expressed in terms of the lower derivatives. Therefore, we can also put them on the right hand side, too. We define $a_0 := a_1(0) = -a_2(0)$, $b_0 := b(0)$, and $c_0 := c(0)$. Since $f'(0) = 12$, the above equations simplify to:

$$\begin{aligned} -(12m+6)\, a_0 b_0 c_0\, a_1^{(m)}(0) - 6\, a_0 b_0 c_0\, a_2^{(m)}(0) &= \widetilde{R}_1 \\ 6\, a_0 b_0 c_0\, a_1^{(m)}(0) + (12m+6)\, a_0 b_0 c_0\, a_2^{(m)}(0) &= \widetilde{R}_2 \end{aligned}$$

where $\widetilde{R}_1$ and $\widetilde{R}_2$ also are polynomials which depend on the lower derivatives of a_1, a_2, b, c, and f only. Since the above system has always a unique solution $(a_1^{(m)}(0), a_2^{(m)}(0))$, we have proven the existence of a unique power series which satisfies (5.39).

Before we turn to the next case, we will discuss how many cohomogeneity-one Einstein metrics exist in our situation. If the principal orbit is $N^{1,1}$, Assumption 3.2.19 is not satisfied and we cannot make a definitive statement on the number of Einstein metrics. The main problem is that we do not automatically know that if the metric g is in $S^2(\mathfrak{p}) \oplus S^2(\mathfrak{p}^\perp)$, the differential equations for the cohomogeneity-one Einstein condition respect that splitting, too. We will nevertheless apply Theorem 3.2.24 to our situation under the assumption that $g \in S^2(\mathfrak{p}) \oplus S^2(\mathfrak{p}^\perp)$, which is equivalent to $\beta_{1,7} = \beta_{2,7} = 0$, and state our result as a conjecture. If the principal orbit is $N^{1,1}$ and the singular orbit is $SU(3)/U(1)^2$, we suppose that there are

- 3 parameters of 0^{th} order in the horizontal direction, which correspond to the initial values $a_1(0)$, $b(0)$, and $c(0)$,
- 2 parameters of 1^{st} order in the horizontal direction, which correspond to $\beta_{3,5}'(0)$ and $\beta_{3,6}'(0)$, and
- 2 parameters of 2^{nd} order in the horizontal direction, which correspond to $\beta_{1,2}''(0)$ and $(a_1+a_2)''(0)$.

As usual, $f'''(0)$ and the Einstein constant can be chosen arbitrarily, too. Since those metrics satisfy $f'(0) = \pm 6$, none of them is described by a power series of type (5.49).

If we restrict ourselves to diagonal metrics, the metric and the differential equations both live inside the subspace $S^2(\mathfrak{p}) \oplus S^2(\mathfrak{p}^\perp) \subseteq S^2(\mathfrak{p} \oplus \mathfrak{p}^\perp)$. Moreover, the differential equations leave any diagonal metric diagonal. We thus can go through the proof of Eschenburg and Wang [32] and see that all of their arguments hold true in our situation, too. If we omit the non-diagonal free parameters from the above list, we therefore have found all the free parameters for the diagonal case.

We will check if Assumption 3.2.19 is satisfied for $SU(3)/K$ as principal orbit. $U(1)_{1,1}$ acts trivially on the normal space of the old cohomogeneity-one manifold. The normal space $\mathfrak{p}^\perp$ of our new space with principal orbit $SU(3)/K$ is obtained by taking the quotient with respect to the action of h. Therefore, h and K both act trivially on $\mathfrak{p}^\perp$. We take a look at (5.50) and see that h acts non-trivially on V_1, V_2, and V_3. Assumption 3.2.19 thus is satisfied. If the principal orbit is $SU(3)/K$, we can therefore freely choose

- 3 parameters of 0^{th} order in the horizontal direction, which correspond to $a_1(0)$, $b(0)$, and $c(0)$, and
- 2 parameters of 1^{st} order in the horizontal direction, which correspond to $\beta'_{1,2}(0)$ and $(a_1 + a_2)'(0)$.

Again, we can choose $f'''(0)$ and the Einstein constant freely. Since there is a free parameter of first order which describes how the diagonal metric on the singular orbit changes into a non-diagonal one, it might be possible to construct further metrics with special holonomy if we allow $\beta_{1,2}$ to be $\neq 0$.

Principal orbit: A $\mathbf{N^{k,l}}$ which is $\cong \mathbf{N^{1,0}}$, singular orbit: $\mathbf{S^5}$

For reasons of simplicity, we assume that $k = 1$ and $l = -1$. Any $SU(3)$-invariant metric on $N^{1,-1}$ is described by a matrix which is similar to (5.32). The only difference is that the off-diagonal coefficients are contained in $V_2 \otimes V_3 \oplus V_3 \otimes V_2$ instead of $V_1 \otimes V_3 \oplus V_3 \otimes V_1$ and hence will be denoted by $\beta_{3,5}$ and $\beta_{3,6}$ instead of $\beta_{1,5}$ and $\beta_{1,6}$. Therefore, we still speak of "metrics of type (5.32)". In our situation, we have $a(0) = 0$ as initial condition, since $V_1 \subseteq \mathfrak{k}$. We only consider diagonal metrics and thus can work with the system (5.36). For $k = 1$ and $l = -1$, this system becomes:

$$\begin{aligned} a' &= \frac{b^2 + c^2 - a^2}{bc} \\ \frac{b'}{b} &= \frac{c^2 + a^2 - b^2}{abc} - \frac{1}{2}\frac{f}{b^2} \\ \frac{c'}{c} &= \frac{a^2 + b^2 - c^2}{abc} + \frac{1}{2}\frac{f}{c^2} \\ \frac{f'}{f} &= \frac{1}{2}\frac{f}{b^2} - \frac{1}{2}\frac{f}{c^2} \end{aligned} \tag{5.51}$$

We want that the left hand sides of the second and the third equation converge to a real number for $t \to 0$. On the right hand side of these equations, we divide by a, which converges to zero. Therefore, we need to have $b(0) = \pm c(0)$ as initial condition. If we replace c by $-c$ and f by $-f$ in the above system, the whole right hand side changes its sign. This change is the same as if we had replaced t by $-t$. Since this replacement is nothing more than a change of the orientation, we do not need to consider the case $b(0) = -c(0)$ separately. Therefore, we assume from now on that $b(0) = c(0)$. We shortly denote $b(0)$ and $c(0)$ by b_0 and $f(0)$ by f_0. By a power series ansatz, we obtain:

$$\begin{array}{rclclclclcl} a(t) & = & 0 & + & 2t & + & 0 \cdot t^2 & - & \frac{1}{27}\frac{36b_0^2 - f_0^2}{b_0^4} t^3 & + & \dots \\ b(t) & = & b_0 & - & \frac{1}{6}\frac{f_0}{b_0} t & + & \frac{1}{72}\frac{72b_0^2 - 5f_0^2}{b_0^3} t^2 & + & \frac{1}{6480}\frac{f_0(504b_0^2 - 167f_0^2)}{b_0^5} t^3 & + & \dots \\ c(t) & = & b_0 & + & \frac{1}{6}\frac{f_0}{b_0} t & + & \frac{1}{72}\frac{72b_0^2 - 5f_0^2}{b_0^3} t^2 & - & \frac{1}{6480}\frac{f_0(504b_0^2 - 167f_0^2)}{b_0^5} t^3 & + & \dots \\ f(t) & = & f_0 & + & 0 \cdot t & + & \frac{1}{6}\frac{f_0^3}{b_0^4} t^2 & + & 0 \cdot t^3 & + & \dots \end{array} \tag{5.52}$$

Before we discuss if our initial value problem has smooth solutions, we have to describe the smoothness conditions for a cohomogeneity-one metric with principal orbit $N^{1,-1}$ and singular orbit S^5 explicitly. The isotropy group of the $SU(3)$-action on the singular orbit is isomorphic to $SU(2)$. In order to describe the smoothness conditions, we therefore have to investigate several real $SU(2)$-modules.

$SU(2)$ acts irreducibly on the three-dimensional space $\mathfrak{p}^\perp$, since its orbits are spheres. $\mathfrak{p}^\perp$ therefore can be identified with $\mathbb{R}^3$, on which $SU(2)$ acts as $SO(3)$ by matrix multiplication. We will give a short introduction to the representation theory of $SU(2)$, before we decompose the symmetric powers of $\mathfrak{p}^\perp$ into irreducible $SU(2)$-submodules. The complex two-dimensional representation of $SU(2)$ we denote by $\mathbb{C}^2$. On the following pages, we will consider $\mathbb{C}^2$ not as a complex two-dimensional but as a real four-dimensional vector space. We denote the space of all homogeneous polynomials of m^{th} order on $\mathbb{C}^2$ by $S^m_{\mathbb{R}}(\mathbb{C}^2)$, where the index "$\mathbb{R}$" indicates that $\mathbb{C}^2$ is considered as real. Our first aim is to decompose the complex vector space $S^m_{\mathbb{R}}(\mathbb{C}^2)\otimes\mathbb{C}$ of all homogeneous polynomials of m^{th} order with complex coefficients. Let $s \in \{0,\dots,m\}$ be arbitrary. $S^m_{\mathbb{R}}(\mathbb{C}^2)\otimes\mathbb{C}$ obviously contains the space of all homogeneous polynomials which depend on s complex and $m-s$ conjugate complex variables as a submodule. Moreover, it is the direct sum of all those modules. The submodules of $S^m_{\mathbb{R}}(\mathbb{C}^2)\otimes\mathbb{C}$ which we have introduced are not necessarily irreducible. We can split each of them into the space of those polynomials which are a multiple of $z_1 \otimes \overline{z_1} + z_2 \otimes \overline{z_2}$ and its complement. If m and $m-s$ are both $\geqslant 1$, both of these summands are non-trivial. If m and $m-s$ are both $\geqslant 2$, we can repeat this process. A polynomial P with $P(z_1,\overline{z_1},\dots) + P(z_2,\overline{z}_2,\dots) = 0$ we call *trace-free*. The $SU(2)$-module which consists of all homogeneous trace-free polynomials which depend on s complex and $m-s$ conjugate complex variables we denote by $\mathbb{V}'_{s,\overline{m-s}}$. It is possible to prove that all $\mathbb{V}'_{s,\overline{m-s}}$ are irreducible. The above considerations prove the following decomposition formula:

$$S^m_{\mathbb{R}}(\mathbb{C}^2)\otimes\mathbb{C} = \bigoplus_{p=0}^{\lfloor\frac{m}{2}\rfloor}\bigoplus_{s=0}^{m-2p}\mathbb{V}'_{s,\overline{m-2p-s}}\,.$$

Any complex irreducible $SU(2)$-module is equivalent to $S^p_{\mathbb{C}}(\mathbb{C}^2)$, where $p \in \mathbb{N}_0$ and $\mathbb{C}^2$ now is considered as a complex vector space. In Chapter 4, we have denoted that module by $\mathbb{V}^{\mathbb{C}}_p$. By a short calculation, we see that $\dim\mathbb{V}'_{s,\overline{m-2p-s}} = m-2p+1$. The module $\mathbb{V}'_{s,\overline{m-2p-s}}$ therefore has to be equivalent to $\mathbb{V}^{\mathbb{C}}_{m-2p}$ and our decomposition formula can be rewritten as:

$$S^m_{\mathbb{R}}(\mathbb{C}^2)\otimes\mathbb{C} = \bigoplus_{p=0}^{\lfloor\frac{m}{2}\rfloor}(m-2p+1)\mathbb{V}^{\mathbb{C}}_{m-2p}\,,$$

where $(m-2p+1)\mathbb{V}^{\mathbb{C}}_{m-2p}$ denotes the direct sum of $m-2p+1$ copies of $\mathbb{V}^{\mathbb{C}}_{m-2p}$. For our considerations, the space $S^m_{\mathbb{R}}(\mathbb{C}^2)$ of all homogeneous polynomials of m^{th} order with real coefficients is more important than $S^m_{\mathbb{R}}(\mathbb{C}^2)\otimes\mathbb{C}$. It is easy to see that $S^m_{\mathbb{R}}(\mathbb{C}^2)$ consists of exactly those elements of $S^m_{\mathbb{R}}(\mathbb{C}^2)\otimes\mathbb{C}$ which are invariant with respect to the conjugation map. The conjugation map maps $\mathbb{V}'_{s,\overline{m-2p-s}}$ into $\mathbb{V}'_{m-2p-s,\overline{s}}$ and vice versa. If $m-2p$ is even and $s \neq \frac{m-2p}{2}$, the subspace of $\mathbb{V}'_{s,\overline{m-2p-s}} \oplus \mathbb{V}'_{m-2p-s,\overline{s}}$ which is invariant under conjugation decomposes into two real submodules of the same dimension. As in Chapter 4, we denote the real irreducible $SU(2)$-module with weight $m-2p$ by $\mathbb{V}^{\mathbb{R}}_{m-2p}$. Since both of the above real submodules are irreducible, they have to be equivalent to $\mathbb{V}^{\mathbb{R}}_{m-2p}$. In the case $s=\frac{m-2p}{2}$, we obtain only one real module which is of the same type. If $m-2p$ is odd, the subspace of $\mathbb{V}'_{\overline{s,m-2p-s}} \oplus \mathbb{V}'_{m-2p-s,\overline{s}}$ which is invariant under conjugation is irreducible. Since its real dimension is $2(m-2p+1)$, it has to be equivalent to $\mathbb{V}^{\mathbb{C}}_{m-2p}$. We therefore have the following decomposition formula:

$$S^m_{\mathbb{R}}(\mathbb{C}^2) = \begin{cases} \displaystyle\bigoplus_{p=0}^{\frac{m}{2}} (2p+1)\mathbb{V}^{\mathbb{R}}_{2p} & \text{if } m \text{ is even} \\ \displaystyle\bigoplus_{p=0}^{\frac{m-1}{2}} (p+1)\mathbb{V}^{\mathbb{C}}_{2p+1} & \text{if } m \text{ is odd} \end{cases}$$

We are now able to deduce a decomposition formula for $S^m(\mathfrak{p}^\perp)$. $\mathfrak{p}^\perp$ is equivalent to the $SU(2)$-module $\mathbb{V}^{\mathbb{R}}_2$. That module can also be considered as the submodule of $S^2_{\mathbb{R}}(\mathbb{C}^2)$ which consists of all homogeneous polynomials which depend on one complex and one conjugate complex variable. $S^m(\mathbb{V}^{\mathbb{R}}_2)$ therefore consists of certain polynomials of $2m^{th}$ order which depend on m complex and m conjugate complex variables. The only way of obtaining submodules of $S^m(\mathbb{V}^{\mathbb{R}}_2)$ therefore is to take the trace and we have:

$$S^m(\mathbb{V}^{\mathbb{R}}_2) = \bigoplus_{p=0}^{\lfloor \frac{m}{2} \rfloor} \mathbb{V}'_{m-2p,m-2p} = \bigoplus_{p=0}^{\lfloor \frac{m}{2} \rfloor} \mathbb{V}^{\mathbb{R}}_{2m-4p}\,.$$

Next, we consider $S^2(\mathfrak{p})$ as a $SU(2)$-module. $SU(2)$ acts irreducibly on $\operatorname{span}(e_3,e_4,e_5,e_6)$ and trivially on $\operatorname{span}(e_7)$. We therefore have

$$\mathfrak{p} = \mathbb{V}^{\mathbb{C}}_1 \oplus \mathbb{V}^{\mathbb{R}}_0$$

and

$$\begin{aligned} S^2(\mathfrak{p}) &= S^2(\mathbb{V}^{\mathbb{C}}_1) \oplus \mathbb{V}^{\mathbb{C}}_1 \otimes \mathbb{V}^{\mathbb{R}}_0 \oplus S^2(\mathbb{V}^{\mathbb{R}}_0) \\ &= 3\mathbb{V}^{\mathbb{R}}_2 \oplus \mathbb{V}^{\mathbb{C}}_1 \oplus 2\mathbb{V}^{\mathbb{R}}_0\,. \end{aligned}$$

We will describe the spaces W^h_m and W^v_m in detail. First, we consider the horizontal part of the metric. It easily follows from Schur's lemma that

$$\dim W^h_m = \begin{cases} 2 & \text{if } m \text{ is even} \\ 3 & \text{if } m \text{ is odd} \end{cases}$$

These dimensions we have to interpret. We start with the case $m=0$. It is easy to see that any $SU(2)$-invariant metric on the singular orbit has to be diagonal and satisfy $b(0)^2=c(0)^2$. Therefore, the equation $\dim W^h_0 = 2$ simply means that we can choose b_0 and f_0 freely and that there are no further restrictions on the initial values of the equations (5.51). We search for a geometric interpretation of the dimensions of the other W_m^h. For the following considerations, we denote the coefficients of the power series for a, b, c, and f by a_m, b_m, c_m, and f_m. We take a look at the power series (5.52) and see that the coefficients of low order obey the following relations:

$$\begin{array}{llll} b_m = c_m & & & \text{if } m \text{ is even} \\ b_m = -c_m & \text{and} & f_m = 0 & \text{if } m \text{ is odd} \end{array}$$

Since b, c, and f are analytic, these relations are equivalent to:

$$\begin{array}{rcl} b(t) & = & c(-t) \\ f(t) & = & f(-t) \end{array} \tag{5.53}$$

We will prove that the horizontal part of an analytic diagonal metric of type (5.32) has a smooth extension to the singular orbit if and only if it satisfies (5.53). First, we prove the "only if"-part of this statement. We identify the isotropy group $SU(2)$ of the $SU(3)$-action on the singular orbit with $Sp(1)$. Let R_j be the matrix in $Sp(1)$ which corresponds to the right-multiplication by $j \in \mathbb{H}$. This matrix acts on $\mathrm{span}(e_3, e_4, e_5, e_6, e_7)$ as

$$\begin{pmatrix} 0 & 0 & -1 & 0 & \\ 0 & 0 & 0 & -1 & \\ 1 & 0 & 0 & 0 & \\ 0 & 1 & 0 & 0 & \\ & & & & 1 \end{pmatrix}$$

On $\mathfrak{p}^\perp$, R_j acts as an element of $SO(3)$. Since R_j is a rotation around an angle of $\frac{\pi}{2}$, it has to act on $\mathfrak{p}^\perp$ as a rotation around an angle of π. $R_i \in U(1)_{1,-1}$ leaves $\frac{\partial}{\partial t}$ invariant and thus acts on $\mathfrak{p}^\perp$ as a rotation around the axis $\mathrm{span}(\frac{\partial}{\partial t})$. R_j therefore has to act as a rotation around an axis which is perpendicular to $\mathrm{span}(\frac{\partial}{\partial t})$. This observation proves that R_j maps $\frac{\partial}{\partial t}$ into its negative. Since the action of R_j has to leave the metric invariant, we obtain the following two relations:

$$\begin{array}{rcl} g_t(e_3, e_3) & = & g_{-t}(e_5, e_5) \\ g_t(e_7, e_7) & = & g_{-t}(e_7, e_7) \end{array}$$

These equations can be rewritten as:

$$\begin{array}{rcl} b^2(t) & = & c^2(-t) \\ f^2(t) & = & f^2(-t) \end{array}$$

Since we have $b(0) = c(0) \neq 0$ and $f(0) \neq 0$, (5.53) has to be satisfied and we have proven the "only if"-part of our statement. In order to prove the "if"-part of our statement, we assume that g is an analytic diagonal metric which satisfies (5.53). We choose an arbitrary unit quaternion h and investigate the action of R_h on our metric. After we have done a calculation, which is similar to the above calculation, we see that R_h preserves the metric. We consider the action of R_h on the power series for b, c, and f and see that the invariance with respect to $SU(2)$ which we have proven is the same as the $SU(2)$-equivariance in the statement of Theorem 3.2.18. Our metric thus is smooth and we have proven the "if"-part, too.

If m is odd, there is only one degree of freedom for the m^{th} derivative of a metric which satisfies (5.53). Since in this case $\dim W_m^h = 3$, the other two parameters have to correspond to the off-diagonal coefficients of the metric.

We now consider the vertical part of the metric. With help of the decomposition formula for $S^m(\mathbb{V}_2^{\mathbb{R}})$, we obtain:

$$S^2(\mathfrak{p}^\perp) = \mathbb{V}_4^{\mathbb{R}} \oplus \mathbb{V}_0^{\mathbb{R}} \,.$$

Since we have $\dim \mathrm{Hom}_{SU(2)}(\mathbb{V}_4^{\mathbb{R}}, \mathbb{V}_4^{\mathbb{R}}) = 1$, it follows that

$$\dim W_m^v = \begin{cases} 1 & \text{if } m = 0 \\ 0 & \text{if } m \text{ is odd} \\ 2 & \text{if } m \geqslant 2 \text{ and even} \end{cases}$$

Since the action of $U(1)_{1,-1}$ on the tangent space V_1 of the collapsing sphere is irreducible, any invariant metric on the sphere has to be a multiple of the standard metric with sectional curvature 1. The vertical part of the metric therefore is solely described by the function a. By the usual arguments, we can prove that the vertical part is smooth if and only if a is odd and the length of any great circle of the collapsing sphere is $2\pi t + O(t^2)$ for small t. The Lie group which is generated by e_1 intersects $U(1)_{1,-1}$ twice. Therefore, the length of the great circle is $\sqrt{g_t(e_1, e_1)}\,\pi$, which equals $|a(t)|\pi$. By the same argument as for the S^1-collapse, it follows that $|a'(0)| = 2$. The first coefficients of the power series (5.52) thus satisfy all of the smoothness conditions. Moreover, it is possible to prove that the solutions of (5.51) satisfy those conditions, too. We again choose b_0 and f_0 arbitrarily. From the first equation of (5.51), we immediately obtain $a'(0) = 2$. If we replace a by $-a$, b by c, and c by b, the right hand side of (5.51) changes its sign. If we further replace t by $-t$, we again obtain the original sign. Therefore, $(a(t), b(t), c(t), f(t))$ and $(-a(-t), c(-t), b(-t), f(-t))$ are solutions of the same initial value problem. We will prove that the system (5.51) has for any choice of b_0 and f_0 a unique power series solution. From that uniqueness result, it follows that the power series satisfies the equations $a(t) = -a(-t)$, $b(t) = c(-t)$, $c(t) = b(-t)$, and $f(t) = f(-t)$ which are the remaining smoothness conditions. Any power series which solves (5.51) and is smooth at the singular orbit also satisfies the cohomogeneity-one Einstein condition. We can conclude with help of Theorem 3.2.24 that the power series converges. Since $U(1)_{1,-1}$ acts trivially on e_7 and on a one-dimensional subspace of $\mathfrak{p}^\perp$, Assumption 3.2.19 is, as in the previous case, not satisfied. The metric g is nevertheless contained in $S^2(\mathfrak{p}) \oplus S^2(\mathfrak{p}^\perp)$, since we always have $g(e_7, \frac{\partial}{\partial t}) = 0$. We thus can work around this problem by the same argument as before. After we have shown the existence and uniqueness of our power series solution, we finally have proven the local existence and smoothness of our metrics. In order to do begin with our proof, we rewrite the system (5.51) as:

$$\begin{aligned} a'bc &= b^2 + c^2 - a^2 \\ ab'bc &= (c^2 + a^2 - b^2)b - \tfrac{1}{2}acf \\ abc'c &= (a^2 + b^2 - c^2)c + \tfrac{1}{2}abf \\ b^2c^2f' &= \tfrac{1}{2}c^2f^2 - \tfrac{1}{2}b^2f^2 \end{aligned} \tag{5.54}$$

We consider the m^{th} derivatives of the above equations at $t = 0$ and put the terms which contain the $(m+1)^{st}$ derivative of a, b, c, or f on the left hand side. The other terms we put on the right hand side of our equations. After dividing by $(m+1)!$, we obtain:

$$\begin{aligned} a_{m+1}b_0c_0 &= R_1 \\ a_0b_{m+1}b_0c_0 &= R_2 \\ a_0b_0c_{m+1}c_0 &= R_3 \\ f_{m+1}b_0^2c_0^2 &= R_4 \end{aligned}$$

where $R_1, \ldots, R_4$ are polynomials which depend only on $a_0, \ldots, f_m$. Since $a_0 = 0$, the second and the third equation contain no information on b_{m+1} or c_{m+1}. We therefore have to consider the $(m+1)^{st}$ derivatives of the second and third equation of (5.54) and obtain:

$$\begin{aligned} & a_{m+1}b_1b_0c_0 + (m+1)a_1b_{m+1}b_0c_0 + (m+1)a_0b_{m+1}b_1c_0 + (m+1)a_0b_{m+1}b_0c_1 + a_0b_1b_{m+1}c_0 \\ & \quad + a_0b_1b_0c_{m+1} \\ =& (2c_0c_{m+1} + 2a_0a_{m+1} - 2b_0b_{m+1})b_0 + (c_0^2 + a_0^2 - b_0^2)b_{m+1} \\ & \quad - \tfrac{1}{2}a_{m+1}c_0f_0 - \tfrac{1}{2}a_0c_{m+1}f_0 - \tfrac{1}{2}a_0c_0f_{m+1} + \widetilde{R}_2 \end{aligned}$$

$$\begin{aligned} & a_{m+1}b_0c_1c_0 + a_0b_{m+1}c_1c_0 + (m+1)a_1b_0c_{m+1}c_0 + (m+1)a_0b_1c_{m+1}c_0 + (m+1)a_0b_0c_{m+1}c_1 \\ & \quad + a_0b_0c_1c_{m+1} \\ =& (2a_0a_{m+1} + 2b_0b_{m+1} - 2c_0c_{m+1})c_0 + (a_0^2 + b_0^2 - c_0^2)c_{m+1} \\ & \quad + \tfrac{1}{2}a_{m+1}b_0f_0 + \tfrac{1}{2}a_0b_{m+1}f_0 + \tfrac{1}{2}a_0b_0f_{m+1} + \widetilde{R}_3 \end{aligned}$$

where $\widetilde{R}_2$ and $\widetilde{R}_3$ also are polynomials which depend only on $a_0, \ldots, f_m$. As in the previous case, we assume that $m \geqslant 2$ and the above system becomes linear with respect to $a_{m+1}, \ldots, f_{m+1}$. By setting $a_0 = 0$, $c_0 = b_0$, $a_1 = 2$, $b_1 = -\frac{1}{6}\frac{f_0}{b_0}$, and $c_1 = \frac{1}{6}\frac{f_0}{b_0}$, we obtain the following linear system for $a_{m+1}, \ldots, f_{m+1}$:

$$\begin{aligned} b_0^2a_{m+1} &= R_1 \\ -\tfrac{1}{6}b_0f_0a_{m+1} + 2(m+1)b_0^2b_{m+1} &= 2b_0^2c_{m+1} - 2b_0^2b_{m+1} - \tfrac{1}{2}b_0f_0a_{m+1} + \widetilde{R}_2 \\ \tfrac{1}{6}b_0f_0a_{m+1} + 2(m+1)b_0^2c_{m+1} &= 2b_0^2b_{m+1} - 2b_0^2c_{m+1} + \tfrac{1}{2}b_0f_0a_{m+1} + \widetilde{R}_3 \\ b_0^4f_{m+1} &= R_4 \end{aligned}$$

which is equivalent to

$$\begin{pmatrix} b_0^2 & 0 & 0 & 0 \\ \frac{1}{3}b_0 f_0 & 2(m+2)b_0^2 & -2b_0^2 & 0 \\ -\frac{1}{3}b_0 f_0 & -2b_0^2 & 2(m+2)b_0^2 & 0 \\ 0 & 0 & 0 & b_0^4 \end{pmatrix} \begin{pmatrix} a_{m+1} \\ b_{m+1} \\ c_{m+1} \\ f_{m+1} \end{pmatrix} = \begin{pmatrix} R_1 \\ \widetilde{R}_2 \\ \widetilde{R}_3 \\ R_4 \end{pmatrix}$$

The above matrix obviously has always rank 4. Therefore, there exists a unique formal power series which solves our initial value problem. All in all, we have proven the existence of a family of non-homothetic metrics with holonomy Spin(7) which depend on one parameter. Since there are non-diagonal smooth metrics with principal orbit $N^{1,-1}$ and singular orbit S^5, there also may exist further non-diagonal examples of metrics with holonomy Spin(7).

We finally classify the cohomogeneity-one Einstein metrics with principal orbit $N^{1,-1}$ and singular orbit S^5. Although Assumption 3.2.19 is not satisfied, we can apply Theorem 3.2.24 to our situation. The reason for this is as above that since $g(e_7, \frac{\partial}{\partial t}) = 0$, the metric is always contained in $S^2(\mathfrak{p}) \oplus S^2(\mathfrak{p}^\perp)$. Moreover, we as usual have $\mathrm{Ric}(e_7, \frac{\partial}{\partial t}) = 0$ and therefore $\mathrm{Ric} \in S^2(\mathfrak{p}) \oplus S^2(\mathfrak{p}^\perp)$. We thus obtain the following numbers of initial conditions which we can prescribe:

1. 2 initial conditions of 0^{th} order in the horizontal direction, which describe the values of $b(0)$ $(= c(0))$ and $f(0)$,
2. 3 initial conditions of 1^{st} order in the horizontal direction, which describe the values of $(b-c)'(0)$, $\beta'_{3,5}(0)$, and $\beta'_{3,6}(0)$.
3. 1 initial condition in the vertical direction, which describes the value of $a'''(0)$.

As usual, the Einstein constant can also be fixed at an arbitrary value.

Principal orbit: A $\mathbf{N^{k,l}}$ which is $\cong \mathbf{N^{1,0}}$, singular orbit: $\mathbf{SU(3)/SO(3)}$

As in the previous case, we assume that $k = 1$, $l = -1$ and denote the isotropy group of the $SU(3)$-action on the singular orbit by K and its Lie algebra by $\mathfrak{k}$. We have seen that there exists a one-parameter family of Lie groups K with $U(1)_{1,-1} \subseteq K \subseteq SU(3)$ which are all isomorphic to $SO(3)$. That family is generated by the action of $U(1)_{1,1}$, which is a subset of $\mathrm{Norm}_{SU(3)}U(1)_{1,-1}$. The action of the normalizer preserves any diagonal metric. Therefore, we have to consider one special choice of K only. In Lemma 5.4.2, we have chosen $\mathfrak{k}$ as:

$$\mathfrak{k} = \mathrm{span}(e_8, \tfrac{1}{2}\sqrt{2}\, e_3 - \tfrac{1}{2}\sqrt{2}\, e_5, -\tfrac{1}{2}\sqrt{2}\, e_4 - \tfrac{1}{2}\sqrt{2}\, e_6)\,.$$

The restriction of the metric to $\mathfrak{k}$ has to vanish at the singular orbit. Since we only consider diagonal metrics, we have $\|e_3 - e_5\|^2 = b^2(0) + c^2(0) = 0$. This equation forces $b(0)$ and $c(0)$ to be 0. In this situation, the metric vanishes on all of $\mathrm{span}(e_3, e_4, e_5, e_6)$ and the singular orbit is a space which is different from $SU(3)/SO(3)$. In order to construct cohomogeneity-one metrics with singular orbit $SU(3)/SO(3)$, we have to consider non-diagonal metrics or change K by an element $k \in SU(3)$ which is not contained in $U(1)_{1,-1}$. We choose the second ansatz. It is easy to see that if $k \in \mathrm{Norm}_{SU(3)}U(1)_{1,-1}$ and the metric is diagonal, we are again in the same situation as above. Therefore, we will work with a $k \in SU(3)$ which does not preserve $U(1)_{1,-1}$. Nevertheless, the space $SU(3)/kU(1)_{1,-1}k^{-1}$ is still $SU(3)$-equivariantly diffeomorphic to $N^{1,-1}$. By conjugation by a suitable $k \in SU(3)$, we can map $U(1)_{1,-1}$ to the group

$$\left\{ \left(\begin{array}{cc|c} \cos\varphi & -\sin\varphi & \\ \sin\varphi & \cos\varphi & \\ \hline & & 1 \end{array} \right) \middle| \varphi \in \mathbb{R} \right\},$$

which we shortly denote by $SO(2)$. K can be mapped by the same element to the standard embedding of $SO(3)$ into $SU(3)$. In this situation, the isotropy group of the $SU(3)$-action on the principal orbit is generated by e_1. The tangent space of the principal orbit $SU(3)/SO(2)$ can be identified with $\mathfrak{m} = \mathrm{span}(e_2, \ldots, e_8)$. It is easily possible to calculate the action of $SO(2)$ on $\mathfrak{m}$. As a $SO(2)$-module, $\mathfrak{m}$ splits into the following irreducible submodules:

$$\begin{aligned} V_0' &:= \mathrm{span}(e_7) \\ V_1' &:= \mathrm{span}(e_3, e_5) \\ V_2' &:= \mathrm{span}(e_4, e_6) \\ V_3' &:= \mathrm{span}(e_2, e_8) \end{aligned}$$

Since V_1' and V_2' are equivalent modules, there exist metrics on $SU(3)/SO(2)$ which are non-diagonal with respect to any basis of $\mathfrak{m}$ which is adapted to the above splitting. Since $SU(3)/SO(2)$ is $SU(3)$-equivariantly diffeomorphic to the exceptional Aloff-Wallach space $N^{1,-1}$, this is not surprising. In order to simplify our calculations, we fix the basis $(e_7, e_3, e_5, e_4, e_6, e_2, e_8)$ of $\mathfrak{m}$ and consider only metrics which are diagonal with respect to this basis. The normalizer $\mathrm{Norm}_{SU(3)}SO(2)$ is generated by e_1 and e_7. V_1' is not invariant under the action of e_7. Therefore, $\mathrm{Norm}_{SU(3)}SO(2)$ maps diagonal into non-diagonal metrics and vice versa. If we consider neither the conjugates of K with respect to $\mathrm{Norm}_{SU(3)}SO(2)$ nor non-diagonal metrics, there will be metrics with singular orbit $SU(3)/SO(3)$ which will not be explored. For reasons of brevity, we nevertheless will not investigate those cases. The matrix representation of any of the diagonal metrics has the following form:

$$\begin{pmatrix} f^2 & & & & & & \\ & a^2 & 0 & & & & \\ & 0 & a^2 & & & & \\ & & & b^2 & 0 & & \\ & & & 0 & b^2 & & \\ & & & & & c^2 & 0 \\ & & & & & 0 & c^2 \end{pmatrix}$$

where $a, b, c, f \in \mathbb{R}\backslash\{0\}$. The Lie algebra $\mathfrak{k}$ is spanned by (e_1, e_3, e_5). Near the singular orbit, a therefore has to converge to zero. Conversely, any cohomogeneity-one metric of the above type with $a = 0$ and $b, c, f \neq 0$ on some orbit has $SU(3)/SO(3)$ as a singular orbit. With help of our new ansatz, we thus can construct a big class of diagonal cohomogeneity-one metrics with the required orbit structure.

By interchanging e_2 and e_8, we obtain a new basis of $\mathfrak{m}$. The matrix representation of $\mathrm{ad}_{e_1}|_{\mathfrak{m}}$ with respect to this new basis is:

$$\begin{pmatrix} \boxed{0} & & & & & & \\ & 0 & 1 & & & & \\ & -1 & 0 & & & & \\ & & & 0 & 1 & & \\ & & & -1 & 0 & & \\ & & & & & 0 & 2 \\ & & & & & -2 & 0 \end{pmatrix}$$

The above matrix is contained in $\mathfrak{g}_2$. Therefore, we have found the following frame of a cohomogeneity-one Spin(7)-structure with principal orbit $SU(3)/SO(2)$:

$$f_0 := \tfrac{\partial}{\partial t} \quad f_1 := \tfrac{1}{f} e_7 \quad f_2 := \tfrac{1}{a} e_3 \quad f_3 := \tfrac{1}{a} e_5$$

$$f_4 := \tfrac{1}{b} e_4 \quad f_5 := \tfrac{1}{b} e_6 \quad f_6 := \tfrac{1}{c} e_8 \quad f_7 := \tfrac{1}{c} e_2$$

There exists a $U(1)$-action on the tangent space which generates a one-parameter family of $SU(3)$-invariant Spin(7)-structures with the same associated metric. A possible frame of those structures is:

$$\begin{aligned} f_0 &:= \tfrac{\partial}{\partial t} & f_1 &:= \tfrac{1}{f} e_7 \\ f_2 &:= \tfrac{1}{a}\cos\theta\, e_3 + \tfrac{1}{a}\sin\theta\, e_5 & f_3 &:= -\tfrac{1}{a}\sin\theta\, e_3 + \tfrac{1}{a}\cos\theta\, e_5 \\ f_4 &:= \tfrac{1}{b}\cos\theta\, e_4 + \tfrac{1}{b}\sin\theta\, e_6 & f_5 &:= -\tfrac{1}{b}\sin\theta\, e_4 + \tfrac{1}{b}\cos\theta\, e_6 \\ f_6 &:= \tfrac{1}{c}\cos\theta\, e_8 + \tfrac{1}{c}\sin\theta\, e_2 & f_7 &:= -\tfrac{1}{c}\sin\theta\, e_8 + \tfrac{1}{c}\cos\theta\, e_2 \end{aligned} \tag{5.55}$$

Analogously to the case $N^{1,-1}$, the $U(1)$-action generates any connected component of the space of all $SU(3)$-invariant G_2-structures which have the same associated metric. We calculate the three-form ω and the four-form $*\omega$ which are associated to the above frame. By a straight-forward but lengthy calculation, we obtain:

$$\begin{aligned} \omega = {} & a^2 f\, e^{357} + b^2 f\, e^{467} - c^2 f\, e^{278} \\ & + abc\,\sin\theta\, e^{234} + abc\,\cos\theta\, e^{236} - abc\,\cos\theta\, e^{245} - abc\,\sin\theta\, e^{256} \\ & + abc\,\cos\theta\, e^{348} - abc\,\sin\theta\, e^{368} + abc\,\sin\theta\, e^{458} - abc\,\cos\theta\, e^{568} \end{aligned} \tag{5.56}$$

and

$$\begin{aligned} *\omega = {} & abcf\,\cos\theta\, e^{2347} - abcf\,\sin\theta\, e^{2367} + abcf\,\sin\theta\, e^{2457} - abcf\,\cos\theta\, e^{2567} \\ & + abcf\,\sin\theta\, e^{3478} + abcf\,\cos\theta\, e^{3678} - abcf\,\cos\theta\, e^{4578} - abcf\,\sin\theta\, e^{5678} \\ & + a^2 b^2\, e^{3456} - a^2 c^2 e^{2358} - b^2 c^2 e^{2468} \end{aligned} \tag{5.57}$$

The Spin(7)-structure is parallel only if $d_{SU(3)/SO(2)} * \omega = 0$, where $d_{SU(3)/SO(2)}$ denotes the exterior differential on the principal orbit. With help of equation (3.4), we calculate $(d_{SU(3)/SO(2)} * \omega)_{eSO(2)}(e_3^*, e_4^*, e_5^*, e_6^*, e_7^*)$ and obtain:

$$\begin{aligned}
&(d_{SU(3)/SO(2)} * \omega)_{eSO(2)}(e_3^*, e_4^*, e_5^*, e_6^*, e_7^*)\\
&= - * \omega_{eSO(2)}([e_3, e_4]^*, e_5^*, e_6^*, e_7^*) + *\omega_{eSO(2)}([e_3, e_5]^*, e_4^*, e_6^*, e_7^*)\\
&\quad - *\omega_{eSO(2)}([e_3, e_6]^*, e_4^*, e_5^*, e_7^*) + *\omega_{eSO(2)}([e_3, e_7]^*, e_4^*, e_5^*, e_6^*)\\
&\quad - *\omega_{eSO(2)}([e_4, e_5]^*, e_3^*, e_6^*, e_7^*) + *\omega_{eSO(2)}([e_4, e_6]^*, e_3^*, e_5^*, e_7^*)\\
&\quad - *\omega_{eSO(2)}([e_4, e_7]^*, e_3^*, e_5^*, e_6^*) - *\omega_{eSO(2)}([e_5, e_6]^*, e_3^*, e_4^*, e_7^*)\\
&\quad + *\omega_{eSO(2)}([e_5, e_7]^*, e_3^*, e_4^*, e_6^*) - *\omega_{eSO(2)}([e_6, e_7]^*, e_3^*, e_4^*, e_5^*)\\
&= *\omega_{eSO(2)}(e_7^* - e_8^*, e_5^*, e_6^*, e_7^*) - *\omega_{eSO(2)}(e_1^*, e_4^*, e_6^*, e_7^*)\\
&\quad - *\omega_{eSO(2)}(e_2^*, e_4^*, e_5^*, e_7^*) + 3 * \omega_{eSO(2)}(e_4^*, e_4^*, e_5^*, e_6^*)\\
&\quad + *\omega_{eSO(2)}(e_2^*, e_3^*, e_6^*, e_7^*) - *\omega_{eSO(2)}(e_1^*, e_3^*, e_5^*, e_7^*)\\
&\quad + 3 * \omega_{eSO(2)}(e_3^*, e_3^*, e_5^*, e_6^*) + *\omega_{eSO(2)}(e_7^* + e_8^*, e_3^*, e_4^*, e_7^*)\\
&\quad + 3 * \omega_{eSO(2)}(e_6^*, e_3^*, e_4^*, e_6^*) + 3 * \omega_{eSO(2)}(e_5^*, e_3^*, e_4^*, e_5^*)\\
&= -4abcf \sin\theta
\end{aligned}$$

θ therefore has to be either 0 or π. The frames for $\theta = 0$ and $\theta = \pi$ coincide up to a change of the signs of a, b, and c. We thus can assume that $\theta = 0$. In that situation, some of the coefficients of $*\omega$ vanish and $d_{SU(3)/SO(2)} * \omega$ can more easily be calculated:

$$\begin{aligned}
d_{SU(3)/SO(2)} * \omega &= d_{SU(3)/SO(2)}(abcf\, e^{2347} - abcf\, e^{2567} + abcf\, e^{3678} - abcf\, e^{4578}\\
&\quad + a^2b^2\, e^{3456} - a^2c^2 e^{2358} - b^2c^2 e^{2468})\\
&= abcf((-e^{36} + e^{45}) \wedge e^{347} - e^2 \wedge (e^{26} + 3e^{47} - e^{48}) \wedge e^{47}\\
&\qquad + e^{23} \wedge (-e^{25} - 3e^{37} + e^{38}) \wedge e^7 - e^{234} \wedge (e^{34} + e^{56}))\\
&\quad - abcf((-e^{36} + e^{45}) \wedge e^{567} - e^2 \wedge (e^{24} + 3e^{67} + e^{68}) \wedge e^{67}\\
&\qquad + e^{25} \wedge (-e^{23} - 3e^{57} - e^{58}) \wedge e^7 - e^{256} \wedge (e^{34} + e^{56}))\\
&\quad + abcf((e^{26} + 3e^{47} - e^{48}) \wedge e^{678} - e^3 \wedge (-e^{23} - 3e^{57} - e^{58}) \wedge e^{78}\\
&\qquad + e^{36} \wedge (e^{34} + e^{56}) \wedge e^8 - e^{367} \wedge (-e^{34} + e^{56}))\\
&\quad - abcf((-e^{25} - 3e^{37} + e^{38}) \wedge e^{578} - e^4 \wedge (e^{24} + 3e^{67} + e^{68}) \wedge e^{78}\\
&\qquad + e^{45} \wedge (e^{34} + e^{56}) \wedge e^8 - e^{457} \wedge (-e^{34} + e^{56}))\\
&\quad + a^2b^2((e^{26} + 3e^{47} - e^{48}) \wedge e^{456} - e^3 \wedge (-e^{25} - 3e^{37} + e^{38}) \wedge e^{56}\\
&\qquad + e^{34} \wedge (e^{24} + 3e^{67} + e^{68}) \wedge e^6 - e^{345} \wedge (-e^{23} - 3e^{57} - e^{58}))\\
&\quad - a^2c^2((-e^{36} + e^{45}) \wedge e^{358} - e^2 \wedge (e^{26} + 3e^{47} - e^{48}) \wedge e^{58}\\
&\qquad + e^{23} \wedge (e^{24} + 3e^{67} + e^{68}) \wedge e^8 - e^{235} \wedge (-e^{34} + e^{56}))\\
&\quad - b^2c^2((-e^{36} + e^{45}) \wedge e^{468} - e^2 \wedge (-e^{25} - 3e^{37} + e^{38}) \wedge e^{68}\\
&\qquad + e^{24} \wedge (-e^{23} - 3e^{57} - e^{58}) \wedge e^8 - e^{246} \wedge (-e^{34} + e^{56}))
\end{aligned}$$

$$= (3b^2c^2 - 3a^2c^2)e^{23678} + (3b^2c^2 - 3a^2c^2)e^{24578}$$

Since the metric on the principal orbit has to be positive definite, we have $c \neq 0$. Therefore, the equation $d_{SU(3)/SO(2)} * \omega = 0$ is satisfied if and only if $a = \pm b$. Let d be the exterior differential on the eight-dimensional cohomogeneity-one manifold which we consider. We calculate $(d\Omega)_{eSO(2)}(e_2^*, e_3^*, e_4^*, e_8^*, \frac{\partial}{\partial t})$ under the assumption that $\theta = 0$ and $a = \pm b$:

$$\begin{aligned}
&(d\Omega)_{eSO(2)}(e_2^*, e_3^*, e_4^*, e_8^*, \tfrac{\partial}{\partial t}) \\
&= \tfrac{\partial}{\partial t} * \omega_{eSO(2)}(e_2^*, e_3^*, e_4^*, e_8^*) - (d_{SU(3)/SO(2)}\omega)_{eSO(2)}(e_2^*, e_3^*, e_4^*, e_8^*) \\
&= \omega_{eSO(2)}([e_2, e_3]^*, e_4^*, e_8^*) - \omega_{eSO(2)}([e_2, e_4]^*, e_3^*, e_8^*) + \omega_{eSO(2)}([e_2, e_8]^*, e_3^*, e_4^*) \\
&\quad + \omega_{eSO(2)}([e_3, e_4]^*, e_2^*, e_8^*) - \omega_{eSO(2)}([e_3, e_8]^*, e_2^*, e_4^*) + \omega_{eSO(2)}([e_4, e_8]^*, e_2^*, e_3^*) \\
&= \omega_{eSO(2)}(e_6^*, e_4^*, e_8^*) - \omega_{eSO(2)}(-e_5^*, e_3^*, e_8^*) + \omega_{eSO(2)}(2e_1^*, e_3^*, e_4^*) \\
&\quad + \omega_{eSO(2)}(e_8^* - e_7^*, e_2^*, e_8^*) - \omega_{eSO(2)}(-e_4^*, e_2^*, e_4^*) + \omega_{eSO(2)}(e_3^*, e_2^*, e_3^*) \\
&= -c^2 f
\end{aligned}$$

The above calculation proves that a Spin(7)-structure which has a frame of type (5.55) can be parallel only if $c = 0$ or $f = 0$. Since the metric on the principal orbit has to be positive definite, none of these conditions can be satisfied. Nevertheless, the G_2-structure ω on the principal orbit is cosymplectic if $\theta = 0$ and $a = \pm b$. In Theorem 3.2.31, we have proven that $*\omega$ can be extended to a parallel Spin(7)-structure on a neighborhood of the principal orbit if it is cosymplectic. Therefore, there have to exist other G_2-structures on $SU(3)/SO(2)$ which cannot be described by a frame of type $(f_1, \ldots, f_7)$. The G_2-structures on the principal orbits have to remain in the same connected component of the space of all invariant G_2-structures on $SU(3)/SO(2)$ as ω. The only G_2-structures which satisfy these conditions are those which are connected to ω and whose associated metric is non-diagonal with respect to our basis. All in all, we have proven that any diagonal metric which is equipped with a G_2-structure of type (5.56) is immediately changed by the equation $d\Omega = 0$ into a non-diagonal one.

The tangent space of the singular orbit $SU(3)/SO(3)$ splits with respect to the action of $SO(2)$ into $V_0' \oplus V_2' \oplus V_3'$. Since V_1' and V_2' are equivalent $SO(2)$-modules, Assumption 3.2.19 is not satisfied. The modules V_0', V_2', and V_3' are pairwise inequivalent and we thus can prove with help of Schur's lemma that any $SU(3)$-invariant metric on the singular orbit is diagonal. We will investigate how a diagonal metric on the singular orbit can smoothly be extended into a non-diagonal cohomogeneity-one metric. In order to answer this question, we work with Theorem 3.2.18 and thus have to decompose certain $SO(3)$-modules. Since 3.2.19 is not satisfied, we have to consider the space $W_m = \mathrm{Hom}_{SO(3)}(S^m(\mathfrak{p}^\perp), S^2(\mathfrak{p} \oplus \mathfrak{p}^\perp))$ instead of the smaller space $\mathrm{Hom}_{SO(3)}(S^m(\mathfrak{p}^\perp), S^2(\mathfrak{p}) \oplus S^2(\mathfrak{p}^\perp))$. The normal space $\mathfrak{p}^\perp$ of the singular orbit is a three-dimensional irreducible $\mathfrak{su}(2)$-module and therefore equivalent to $\mathbb{V}_2^\mathbb{R}$. Since the action of $SO(3)$ on $\mathfrak{gl}(3, \mathbb{C})$ by conjugation leaves only the multiples of the identity matrix invariant, the subspace $\mathfrak{p}$ of $\mathfrak{su}(3)$ contains no trivial submodule. By considering the low-dimensional $\mathfrak{su}(2)$-modules, we see that $\mathfrak{p}$ in fact has to be irreducible. The only five-dimensional irreducible $\mathfrak{su}(2)$-module is $\mathbb{V}_4^\mathbb{R}$ and we thus have proven that $\mathfrak{p} \cong \mathbb{V}_4^\mathbb{R}$. It is possible to prove with help of the Clebsch-Gordan formula that the space $S^2(\mathfrak{p})$ decomposes as $\mathbb{V}_8^\mathbb{R} \oplus \mathbb{V}_4^\mathbb{R} \oplus \mathbb{V}_0^\mathbb{R}$. The tensor product $\mathfrak{p} \otimes \mathfrak{p}^\perp$ splits into $\mathbb{V}_6^\mathbb{R} \oplus \mathbb{V}_4^\mathbb{R} \oplus \mathbb{V}_2^\mathbb{R}$ and we have:

$$\begin{aligned} S^2(\mathfrak{p}\oplus\mathfrak{p}^\perp) &= S^2(\mathfrak{p})\oplus(\mathfrak{p}\otimes\mathfrak{p}^\perp)\oplus S^2(\mathfrak{p}^\perp) \\ &= (\mathbb{V}_8^{\mathbb{R}}\oplus\mathbb{V}_4^{\mathbb{R}}\oplus\mathbb{V}_0^{\mathbb{R}})\oplus(\mathbb{V}_6^{\mathbb{R}}\oplus\mathbb{V}_4^{\mathbb{R}}\oplus\mathbb{V}_2^{\mathbb{R}})\oplus(\mathbb{V}_4^{\mathbb{R}}\oplus\mathbb{V}_0^{\mathbb{R}}) \\ &= \mathbb{V}_8^{\mathbb{R}}\oplus\mathbb{V}_6^{\mathbb{R}}\oplus 3\mathbb{V}_4^{\mathbb{R}}\oplus\mathbb{V}_2^{\mathbb{R}}\oplus 2\mathbb{V}_0^{\mathbb{R}} \end{aligned}$$

The module $S^m(\mathfrak{p}^\perp)$ decomposes in the same way as in the case where the singular orbit is S^5. We therefore obtain for the dimensions of W_m:

$$\dim W_m = \begin{cases} 2 & \text{if } m=0 \\ 1 & \text{if } m=1 \\ 5 & \text{if } m=2 \\ 2 & \text{if } m\geqslant 3 \text{ and odd} \\ 6 & \text{if } m\geqslant 4 \text{ and even} \end{cases} \tag{5.58}$$

The space of all diagonal $SU(3)$-invariant metrics on $N^{1,-1}$ has only four dimensions. The metric on the singular orbit therefore can smoothly change into an invariant, non-diagonal metric on $N^{1,-1}$. Our observation is even true if we take into account that the vertical part of the metric in fact depends on one parameter instead of $\dim \mathrm{Hom}_{SO(3)}(S^{2m+2}(\mathfrak{p}^\perp), S^2(\mathfrak{p}^\perp)) = 2$ parameters. This is a hint that parallel cohomogeneity-one Spin(7)-structures with principal orbit $N^{1,-1}$ and singular orbit $SU(3)/SO(3)$ may indeed exist. In order to construct those parallel Spin(7)-structures, we have to consider a more general equation for the holonomy reduction which also describes non-diagonal metrics. Although we will not carry out that calculation, it nevertheless seems to be a promising endeavor.

As we have already remarked, Assumption 3.2.19 is in our situation not satisfied. Furthermore, we have not discussed the geometric meaning of the dimensions of W_m. Therefore, we will not further consider the issue of cohomogeneity-one Einstein metrics with principal orbit $N^{1,-1}$ and singular orbit $SU(3)/SO(3)$.

Principal orbit: An arbitrary Aloff-Wallach space $\mathbf{N^{k,l}}$, singular orbit: $\mathbb{CP}^2$

We will consider the three different types of Aloff-Wallach spaces simultaneously. The reason for this is that many arguments which work for one of these cases will also work for the other ones. The Lie algebra $\mathfrak{k}$ of the isotropy group of the $SU(3)$-action on $\mathbb{CP}^2$ is isomorphic to $\mathfrak{u}(2)$. If the principal orbit is a generic Aloff-Wallach space or $\cong N^{1,0}$, $\mathfrak{k}$ is either $\mathfrak{u}(1)_{k,l}\oplus V_1\oplus V_4$, $\mathfrak{u}(1)_{k,l}\oplus V_2\oplus V_4$, or $\mathfrak{u}(1)_{k,l}\oplus V_3\oplus V_4$. As we have remarked in 5.4.4, it suffices to work with the case $\mathfrak{k}=\mathfrak{u}(1)_{k,l}\oplus V_1\oplus V_4$ if we consider all pairs (k,l) of coprime integers with $k\geqslant l$. The pair $(1,-1)$ has to be excluded from our considerations, since in that situation the space $K/U(1)_{k,l}$ is diffeomorphic to $S^2\times S^1$ and not to a quotient of S^3 by a discrete group. If $k=l=1$, the action of $SU(2)\subseteq \mathrm{Norm}_{SU(3)}U(1)_{1,1}$ on $\mathfrak{u}(1)_{1,1}\oplus V_2\oplus V_4$, generates a family of Lie algebras which are all isomorphic to $\mathfrak{u}(2)$. It is easy to see that the action of the normalizer on $\mathfrak{u}(1)_{1,1}\oplus V_3\oplus V_4$ generates the same family of Lie algebras. The algebra $\mathfrak{u}(1)_{1,1}\oplus V_1\oplus V_4$ is the Lie algebra of $\mathrm{Norm}_{SU(3)}U(1)_{1,1}$ and is therefore left invariant by the action of the normalizer. According to Lemma 5.4.2, the above $\mathfrak{k}$ are the only Lie algebras which are isomorphic to $\mathfrak{u}(2)$ and satisfy $\mathfrak{u}(1)_{k,l}\subseteq\mathfrak{k}\subseteq\mathfrak{su}(3)$. For our considerations, we will always assume that the metric is diagonal. In this situation, we can easily show that even if $k=l=1$, $\mathfrak{k}$ has to be one of the three subalgebras which are also there in the generic case.

Since we can obtain $\mathfrak{u}(1)_{1,1} \oplus V_3 \oplus V_4$ from $\mathfrak{u}(1)_{1,1} \oplus V_2 \oplus V_4$ by the action of the normalizer, we will restrict ourselves to the case $\mathfrak{k} = \mathfrak{u}(1)_{k,l} \oplus V_i \oplus V_4$ with $i \in \{1, 2\}$ if $k = l = 1$. All in all, we have to consider the following three initial values problems:

1. The system (5.36) with $a(0) = f(0) = 0$,

2. The system (5.39) with $a_1(0) = a_2(0) = f(0) = 0$,

3. The system (5.39) with $b(0) = f(0) = 0$.

In the last of these cases, we have $K/U(1)_{k,l} = S^3/\mathbb{Z}_{|1+(-2)|} = S^3$ and the space on which $SU(3)$ acts with cohomogeneity one is a manifold. We make a power series ansatz for all of these three initial value problems and start with the first one. It follows from the second and third equation of (5.36) that we necessarily have $b(0)^2 = c(0)^2$ if $a(0) = f(0) = 0$. If $(a(t), b(t), c(t), f(t))$ is a solution of (5.36), $(a(-t), b(-t), -c(-t), -f(-t))$ is a solution of (5.36), too. For the same reasons as in the case where the singular orbit is S^5, we can therefore assume that $b(0) = c(0) =: b_0$ and do not have to consider the case $b(0) = -c(0)$ separately. With help of a MAPLE program, we see that the Taylor expansion of any solution of our initial value problem has to begin with

$$
\begin{aligned}
a(t) &= 0 + t + 0 \cdot t^2 - \frac{1}{24b_0^2}\frac{12\Delta + q(k+l)}{\Delta}t^3 + 0 \cdot t^4 + \ldots \\
b(t) &= b_0 + 0 \cdot t + \frac{1}{6b_0}\frac{4k+5l}{k+l}t^2 + 0 \cdot t^3 \\
&\quad + \frac{1}{288b_0^3}\frac{(-104k^2 - 224kl - 140l^2)\Delta + q(-k^3 - k^2 l + kl^2 + l^3)}{\Delta^2}t^4 + \ldots \\
c(t) &= b_0 + 0 \cdot t + \frac{1}{6b_0}\frac{5k+4l}{k+l}t^2 + 0 \cdot t^3 \\
&\quad + \frac{1}{288b_0^3}\frac{(-140k^2 - 224kl - 104l^2)\Delta + q(k^3 + k^2 l - kl^2 - l^3)}{\Delta^2}t^4 + \ldots \\
f(t) &= 0 + \frac{2\Delta}{k+l}t + 0 \cdot t^2 + \frac{q}{6b_0^2}t^3 + 0 \cdot t^4 + \ldots
\end{aligned}
\tag{5.59}
$$

The parameter q of third order which appears in the above power series can be chosen arbitrarily. Before we discuss the meaning of q, we take a look at the other two initial value problems. Analogously to the previous case, any solution of (5.39) with $a_1(0) = a_2(0) = f(0)$ has to satisfy $b(0)^2 = c(0)^2$. Since the system (5.39) is preserved if we replace c by $-c$, f by $-f$, and t by $-t$ we again can assume that $b(0) = c(0) =: b_0$. We obtain the following Taylor expansion for our functions:

$$\begin{array}{rcrcrcrcrcrcrc}
a_1(t) & = & 0 & + & t & + & 0\cdot t^2 & + & \frac{q_1}{6b_0^2}t^3 & + & 0\cdot t^4 & + & \frac{2q_2^1-3q_1q_2-3q_2^2-3q_1-18q_2}{60b_0^4}t^5 & +\ldots\\
a_2(t) & = & 0 & + & t & + & 0\cdot t^2 & + & \frac{q_2}{6b_0^2}t^3 & + & 0\cdot t^4 & + & \frac{-3q_1^2-3q_1q_2+2q_2^2-18q_1-3q_2}{60b_0^4}t^5 & +\ldots\\
b(t) & = & b_0 & + & 0\cdot t & + & \frac{3}{4b_0}t^2 & + & 0\cdot t^3 & - & \frac{39}{96b_0^3}t^4 & + & 0\cdot t^5 & +\ldots\\
c(t) & = & b_0 & + & 0\cdot t & + & \frac{3}{4b_0}t^2 & + & 0\cdot t^3 & - & \frac{39}{96b_0^3}t^4 & + & 0\cdot t^5 & +\ldots\\
f(t) & = & 0 & + & 3t & + & 0\cdot t^2 & - & \frac{6+q_1+q_2}{2b_0^2}t^3 & + & 0\cdot t^4 & + & \frac{2q_1^2+7q_1q_2+2q_2^2+27q_1+27q_2+90}{20b_0^4}t^5 & +\ldots
\end{array} \tag{5.60}$$

The functions b and c coincide up to fifth order. We therefore suppose that $b(t) = c(t)$ for all values of t. Analogously to the previous case, there are two parameters q_1 and q_2 which can be chosen freely. We will prove that $b \equiv c$ and that q_1 and q_2 are the only free parameters later on. Now, we consider the equations (5.39) under the assumption that $b(0) = f(0) = 0$. It is easy to see that in this situation we necessarily have $c(0)^2 = a_1(0)^2 = a_2(0)^2$. Let $a_0 := a_1(0)$. Since there are two possibilities for the sign of $a_2(0)$ and $c(0)$, there are four kinds of initial value problems which we have to consider. Because of the symmetry of (5.39) which we have mentioned in the previous case, we can assume that the sign of $a_1(0)$ and $c(0)$ is the same. Therefore, the only two subcases which we have to consider are $a_1(0) = a_2(0)$ and $a_1(0) = -a_2(0)$. The initial condition $a_1(0) = a_2(0)$ yields the following power series:

$$\begin{array}{rcrcrcrcrcrcrc}
a_1(t) & = & a_0 & + & 0\cdot t & + & \frac{1}{a_0}t^2 & + & 0\cdot t^3 & - & \frac{q+21}{24a_0^3}t^4 & + & 0\cdot t^5 & +\ldots\\
a_2(t) & = & a_0 & + & 0\cdot t & + & \frac{1}{a_0}t^2 & + & 0\cdot t^3 & - & \frac{q+21}{24a_0^3}t^4 & + & 0\cdot t^5 & +\ldots\\
b(t) & = & 0 & + & t & + & 0\cdot t^2 & + & \frac{q}{6a_0^2}t^3 & + & 0\cdot t^4 & - & \frac{8q^2+42q+9}{120a_0^4}t^5 & +\ldots\\
c(t) & = & a_0 & + & 0\cdot t & + & \frac{1}{2a_0}t^2 & + & 0\cdot t^3 & + & \frac{q-6}{24a_0^3}t^4 & + & 0\cdot t^5 & +\ldots\\
f(t) & = & 0 & - & 6t & + & 0\cdot t^2 & + & \frac{2(q+3)}{a_0^2}t^3 & + & 0\cdot t^4 & - & \frac{11q^2+54q+123}{10a_0^4}t^5 & +\ldots
\end{array} \tag{5.61}$$

where q is a free parameter. Next, we consider the equations (5.39) under the assumption that $b(0) = f(0) = 0$ and $a_1(0) = -a_2(0)$. By a short calculation, we obtain a system of quadratic equations for the first derivatives of the metric functions at $t = 0$. $b'(0)$ and $f'(0)$ have to be in $\mathbb{R}\backslash\{0\}$ for geometric reasons. Otherwise, the length of a great circle on the universal cover S^3 of $K/U(1)_{1,1}$ would be $O(t^2)$ for small t and the metric thus would not be smooth. Therefore, we can exclude some solutions of the quadratic system. Nevertheless, the two possible solutions $(a_1'(0), a_2'(0), b'(0), c'(0), f'(0)) = (0, 0, 1, 0, 6)$ and $(a_1'(0), a_2'(0), b'(0), c'(0), f'(0)) = (0, 0, -1, 0, 6)$ remain. It is easy to see that if $(a_1(t), a_2(t), b(t), c(t), f(t))$ is a solution of (5.39), $(-a_2(-t), -a_1(-t), b(-t), c(-t), -f(-t))$ is a solution of (5.39), too. Since this symmetry maps a solution with $b'(0) = 1$ into a solution with $b'(0) = -1$, we only need to consider the case where $b'(0) = 1$. In this case, we obtain:

$$
\begin{array}{rclcrcccccccl}
a_1(t) & = & a_0 & + & 0\cdot t & + & \frac{1}{a_0}t^2 & + & 0\cdot t^3 & + & \frac{3q-23}{24a_0^3}t^4 & + & 0\cdot t^5 + \ldots \\
a_2(t) & = & -a_0 & + & 0\cdot t & + & \frac{q-2}{a_0}t^2 & + & 0\cdot t^3 & - & \frac{12q^2-25q-7}{24a_0^3}t^4 & + & 0\cdot t^5 + \ldots \\
b(t) & = & 0 & + & t & + & 0\cdot t^2 & - & \frac{1}{6a_0^2}t^3 & + & 0\cdot t^4 & - & \frac{39q^2-114q+25}{240a_0^4}t^5 + \ldots \\
c(t) & = & a_0 & + & 0\cdot t & + & \frac{q}{2a_0}t^2 & + & 0\cdot t^3 & + & \frac{3q^2-13q+3}{24a_0^3}t^4 & + & 0\cdot t^5 + \ldots \\
f(t) & = & 0 & + & 6t & + & 0\cdot t^2 & - & \frac{4}{a_0^2}t^3 & + & 0\cdot t^4 & + & \frac{3q^2-18q+175}{20a_0^4}t^5 + \ldots
\end{array}
\tag{5.62}
$$

As usual, q can be chosen freely. Our aim is to prove that for any of the three initial value problems and any choice of the parameters a_0, b_0, q, q_1, q_2 there exists a unique smooth metric which solves the initial value problem. In order to do this, we need a condition which guarantees the smoothness of the metric. We take a look at the power series which we have found and see that the Taylor polynomials of low order which correspond to the horizontal part of the metric are even functions and those which correspond to the vertical part are odd. We therefore propose the following sufficient conditions for the smoothness:

1. Let (M, g) be a cohomogeneity-one orbifold whose principal orbit is an Aloff-Wallach space which is $\not\cong N^{1,1}$. Furthermore, let the matrix representation of g with respect to the basis $(e_1, \ldots, e_7)$ from page 150 be of type (5.31). If

   (a) a, b, c, and f are analytic,

   (b) $a(0) = f(0) = 0$,

   (c) $b(0)^2 = c(0)^2$,

   (d) the metric on the collapsing space $K/U(1)_{k,l}$ has sectional curvature $\frac{1}{t^2} + O(\frac{1}{t})$ for small t,

   (e) a, f are odd, and

   (f) b, c are even,

   g can be smoothly extended to the singular orbit $\mathbb{CP}^2$.

2. Let (M, g) be a cohomogeneity-one orbifold whose principal orbit is $N^{1,1}$. Furthermore, let the matrix representation of g with respect to the basis $(e_1, \ldots, e_7)$ from page 150 be a diagonal matrix of type (5.33). If

   (a) a_1, a_2, b, c, and f are analytic,

   (b) $a_1(0) = a_2(0) = f(0) = 0$,

   (c) $b(0)^2 = c(0)^2$,

   (d) the metric on the collapsing space $K/U(1)_{1,1} \cong S^3/\mathbb{Z}_2$ has sectional curvature $\frac{1}{t^2} + O(\frac{1}{t})$ for small t,

(e) a_1, a_2, f are odd, and

(f) b, c are even,

g can be smoothly extended to the singular orbit $\mathbb{CP}^2$.

3. Let (M, g) be a cohomogeneity-one manifold whose principal orbit is $N^{1,1}$. Furthermore, let the matrix representation of g with respect to the basis $(e_1, \ldots, e_7)$ from page 150 be a diagonal matrix of type (5.33). If

   (a) a_1, a_2, b, c, and f are analytic,

   (b) $b(0) = f(0) = 0$,

   (c) $a_1^2(0) = a_2^2(0) = c^2(0)$,

   (d) the metric on the collapsing sphere $K/U(1)_{1,1} \cong S(U(2) \times U(1))/U(1)_{1,-2} \cong S^3$ has sectional curvature $\frac{1}{t^2} + O(\frac{1}{t})$ for small t,

   (e) b, f are odd, and

   (f) a_1, a_2, c are even,

   g can be extended smoothly to the singular orbit $\mathbb{CP}^2$.

We will work with Theorem 3.2.18. If we are able to show that the maps from $S^m(\mathfrak{p}^\perp)$ to $S^2(\mathfrak{p})$ (or $S^2(\mathfrak{p}^\perp)$) which correspond to the even derivatives of the horizontal (or vertical) part of the metric are K-equivariant, our statement follows by the same arguments as in Section 5.2 and 5.3. We therefore will describe those maps in detail. It is possible to consider all of the three cases simultaneously. We start with the vertical part g^v of the metric. Let $(f_i)_{i\in I}$ be the family of the metric functions which correspond to g^v. We want to prove that we can choose the f_i as arbitrary odd functions. Analogously to the previous sections, it suffices to prove that the even derivatives of g^v can be chosen freely. Let $m \geqslant 1$. As we have explained in Section 3.2, there exists a certain map $\phi : S^{2m}(\mathfrak{p}^\perp) \to S^2(\mathfrak{p}^\perp)$ which is associated to the $2m^{th}$ derivative of g^v. ϕ satisfies:

$$\phi\left(\bigotimes^m (dt \otimes dt)\right) = \frac{\partial^{2m}}{\partial t^{2m}} g^v$$

where t is the radial direction in the normal space $\mathfrak{p}^\perp$. On the orthogonal complement of $\bigotimes^m (dt \otimes dt)$, ϕ vanishes. $dt \otimes dt$ is invariant with respect to the action of K. The orbit of the K-action on any $v \in \mathfrak{p}^\perp \backslash \{0\}$ is K-equivariantly diffeomorphic to $K/U(1)_{k,l}$. Since the f_i define a K-invariant metric on $K/U(1)_{k,l}$, the right hand side of the above equation is K-invariant, too. We thus can conclude that ϕ is a K-equivariant map. Next, we prove the smoothness condition for the horizontal part of the metric. This can be done by a geometric argument. Since we have to prove the smoothness of the metric at the singular orbit only, we can restrict our considerations to a tubular neighborhood of the singular orbit. That neighborhood has the structure of a fiber bundle over $\mathbb{CP}^2$. We define a bilinear form g_h on that bundle which coincides in the horizontal direction with g and vanishes in the vertical direction. Since $SU(3)$ acts transitively on the set of the fibers of the bundle, we only need to consider the restriction of g_h to a single fiber. We describe a construction which also yields g_h: Let γ be a geodesic which intersects all orbits perpendicularly. Without loss of generality, we assume that $\gamma(0) \in \mathbb{CP}^2$. We first define g_h on γ only. By the action of K, g_h is extended to a tensor field on a fiber of the tubular neighborhood. Let $k \in K$ be an element which maps $\gamma(t)$ to $\gamma(-t)$. Since we assume that $g_h \circ \gamma$ is even, $k.(g_h(\gamma(t))) = g_h(\gamma(-t))$. Since we furthermore assume that $g_h(\gamma(0))$ is K-invariant, the extension of g_h by the action of K is well-defined. Since $g_h \circ \gamma$ is a smooth function and $k^{2m} \in U(1)_{k,l} \subseteq K$, the derivative $\frac{\partial^{2m}}{\partial t^{2m}}|_{t=0}\, g_h \circ \gamma$ is well-defined, too. We conclude that the map $\phi : S^{2m}(\mathfrak{p}^\perp) \to S^2(\mathfrak{p})$ which corresponds to the $2m^{th}$ derivative of g_h is also well-defined. Its K-equivariance follows from the construction which we have made and we have proven our statement.

In order to check if our power series satisfy the smoothness conditions, we have to calculate the values which the $f_i'(0)$ have to take. Let q be the metric with constant sectional curvature 1 on $K/U(1)_{k,l}$. As in Section 5.2 and 5.3, we need to have:

$$\frac{\partial}{\partial t}\bigg|_{t=0} \|e_i\|_g = \|e_i\|_q$$

for all e_i whose length vanishes at the singular orbit. Before we can check if the above equation is satisfied, we have to describe q explicitly. The metric $\widetilde{q}$ on $SU(2)$ with constant sectional curvature 1 is given by $\widetilde{q}(X,Y) = -\frac{1}{2}\mathrm{tr}(XY)$ for all $X, Y \in \mathfrak{su}(2)$. The map $\pi : SU(2) \to K/U(1)_{k,l}$ with $\pi(k) := kU(1)_{k,l}$ where $SU(2)$ is canonically embedded into $K \subseteq SU(3)$ is a covering map. q therefore has to satisfy:

$$\|d\pi(e_i)\|_q = \|e_i\|_{\widetilde{q}}\,.$$

With help of this formula, we see that

1. in the case where $a(0) = f(0) = 0$, $\|e_1\|_q = \|e_2\|_q = 1$ and $a'(0)$ thus has to equal ± 1,
2. in the case where $a_1(0) = a_2(0) = f(0)$, $\|e_1\|_q = \|e_2\|_q = 1$ and $a_1'(0)$ and $a_2'(0)$ thus have to equal ± 1,
3. in the case where $b(0) = f(0) = 0$, $\|e_3\|_q = \|e_4\|_q = 1$ and $b'(0)$ thus has to equal ± 1.

In all three cases, these conditions are indeed satisfied. We next consider the condition on $f'(0)$. The matrix $\mathrm{diag}(i, -i) \in \mathfrak{su}(2)$ which is of unit length is mapped by $d\pi$ in the first two cases to

$$\begin{pmatrix} i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & 0 \end{pmatrix} \mathfrak{u}(1)_{k,l} \in T_{eU(1)_{k,l}} N^{k,l}$$

and in the third case to

$$\begin{pmatrix} i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -i \end{pmatrix} \mathfrak{u}(1)_{1,1} \in T_{eU(1)_{1,1}} N^{1,1}\,.$$

We project the above two matrices onto $\mathfrak{m}$ and see that the two tangent vectors can be identified with

$$\frac{k+l}{2\Delta}e_7 \in \mathfrak{m} \quad \text{and} \quad \frac{1}{6}e_7 \in \mathfrak{m}\,.$$

These vectors are of unit length with respect to q. It therefore follows that

1. in the case where $a(0) = f(0) = 0$, $\|e_7\|_q = |\frac{2\Delta}{k+l}|$ and $f'(0)$ thus has to equal $\pm\frac{2\Delta}{k+l}$,
2. in the case where $a_1(0) = a_2(0) = f(0)$, $\|e_7\|_q = 3$ and $f'(0)$ thus has to equal ± 3,
3. in the case where $b(0) = f(0) = 0$, $\|e_7\|_q = 6$ and $f'(0)$ thus has to equal ± 6.

Since $f'(0)$ indeed takes these values, the first summands of our power series satisfy all of the smoothness conditions. We will prove that for any choice of the initial metric and the free parameters q, q_1, and q_2 there exists a unique formal power series solution of our equations which satisfies the smoothness conditions. The cohomogeneity-one metric $g = g_t + dt^2$, which has to be analytic, is determined by the power series

$$g_t = \sum_{m=0}^{\infty} g_m t^m\,.$$

The coefficient g_m satisfies an equation of type

$$L_m(g_m) = P_m(g_0, \ldots, g_{m-1})\,,$$

where L_m is a linear map and P_m is a polynomial. As in the other cases which we have considered in this section, we will show our result by calculating the rank of L_m. The smoothness conditions we will prove with help of symmetry arguments. All of these steps can be carried out no matter if Assumption 3.2.19 is satisfied. In the cases where that assumption holds, the convergence of the power series follows directly from the results of Eschenburg and Wang. In the other cases, the convergence follows by the same argument as we have made earlier in this section. The reason again is that the cohomogeneity-one Einstein condition is an equation which operates on the space $S^2(\mathfrak{p}) \oplus S^2(\mathfrak{p}^{\perp})$ if the metric is diagonal.

We will carry out the program which we have outlined above for each of our cases separately and start with the case where $N^{k,l} \not\cong N^{1,1}$ and $a(0) = f(0) = 0$. The system (5.36) can be rewritten as:

$$\begin{aligned}
a'abc &= (b^2+c^2-a^2)a - \tfrac{k+l}{2\Delta}bcf \\
ab'bc &= (a^2+c^2-b^2)b + \tfrac{l}{2\Delta}acf \\
abc'c &= (a^2+b^2-c^2)c + \tfrac{k}{2\Delta}abf \\
a^2b^2c^2f' &= \tfrac{k+l}{2\Delta}b^2c^2f^2 - \tfrac{l}{2\Delta}a^2c^2f^2 - \tfrac{k}{2\Delta}a^2b^2f^2
\end{aligned} \tag{5.63}$$

We assume that $m \geqslant 2$ and take the m^{th} derivatives of the first three equations and the $(m+1)^{st}$ derivative of the fourth equation. Analogously to the cases where the singular orbit is $SU(3)/U(1)^2$ or S^5, we obtain:

$$\begin{aligned}
(m+1)b_0^2a^{(m)}(0) &= 2b_0^2a^{(m)}(0) - \tfrac{k+l}{2\Delta}b_0^2f^{(m)}(0) + R_1 \\
mb_0^2b^{(m)}(0) &= (2b_0c^{(m)}(0) - 2b_0b^{(m)}(0))b_0 + R_2 \\
mb_0^2c^{(m)}(0) &= (2b_0b^{(m)}(0) - 2b_0c^{(m)}(0))b_0 + R_3 \\
2(m+1)\tfrac{2\Delta}{k+l}b_0^4a^{(m)}(0) + m(m+1)b_0^4f^{(m)}(0) &= 2(m+1)\tfrac{k+l}{2\Delta}\tfrac{2\Delta}{k+l}b_0^4f^{(m)}(0) + R_4
\end{aligned}$$

In the above system, $R_1, \ldots, R_4$ are polynomials which depend on the lower derivatives of a, b, c, and f only. We have already inserted $b(0) = c(0) = b_0$, $a'(0) = 1$, and $f'(0) = \frac{2\Delta}{k+l}$. The reason for the factor $m(m+1)$ in the fourth equation is that the term $a'^2b^2c^2f^{(m)}(0)$ has the multiplicity $m(m+1)$ if we apply the product rule $m+1$ times to $a^2b^2c^2f'$. We obtain the following linear system for the m^{th} derivatives of the functions a, b, c, and f:

$$\begin{pmatrix} (m-1)b_0^2 & 0 & 0 & \frac{k+l}{2\Delta}b_0^2 \\ 0 & (m+2)b_0^2 & -2b_0^2 & 0 \\ 0 & -2b_0^2 & (m+2)b_0^2 & 0 \\ (m+1)\frac{4\Delta}{k+l}b_0^4 & 0 & 0 & (m-2)(m+1)b_0^4 \end{pmatrix} \begin{pmatrix} a^{(m)}(0) \\ b^{(m)}(0) \\ c^{(m)}(0) \\ f^{(m)}(0) \end{pmatrix} = \begin{pmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \end{pmatrix}$$

By a short calculation, we see that the matrix on the left hand side has rank 4, except for $m = 3$, where the rank is 3. Since we have already calculated the power series (5.59), we know that the solution space of the above linear system is one-dimensional if $m = 3$. We therefore have proven that there exists a formal power series solution of (5.36) for any choice of b_0 and q. Moreover, that solution is unique. q thus is the only free parameter even in the cases where $\mathfrak{p}$ and $\mathfrak{p}^\perp$ have an equivalent $U(1)_{k,l}$-submodule in common. If we replace $(a(t), b(t), c(t), f(t))$ by $(-a(-t), b(-t), c(-t), -f(-t))$, we again obtain a solution of (5.36) with the same values of $b(0)$ and $f'''(0)$. Therefore, we have $a(t) = -a(-t), b(t) = b(-t), c(t) = c(-t), f(t) = -f(-t)$ and the power series satisfies all of the smoothness conditions. Since the power series converges, we finally have proven the existence of a family of local cohomogeneity-one metrics with holonomy Spin(7).

Next, we turn to the case where $k = l = 1$ and $a_1(0) = a_2(0) = f(0) = 0$. For our calculations, we rewrite the system (5.39) as:

$$\begin{aligned}
a_1'a_2bcf &= (b^2+c^2-a_1^2)a_2f + 3(a_1^2-a_2^2)bc - \tfrac{1}{3}bcf^2 \\
a_1a_2'bcf &= (b^2+c^2-a_2^2)a_1f + 3(a_2^2-a_1^2)bc - \tfrac{1}{3}bcf^2 \\
a_1a_2b'bc &= \tfrac{1}{2}(a_1^2+c^2-b^2)a_2b + \tfrac{1}{2}(a_2^2+c^2-b^2)a_1b + \tfrac{1}{6}a_1a_2cf \\
a_1a_2bc'c &= \tfrac{1}{2}(a_1^2+b^2-c^2)a_2c + \tfrac{1}{2}(a_2^2+b^2-c^2)a_1c + \tfrac{1}{6}a_1a_2bf \\
a_1a_2b^2c^2f' &= -3(a_1-a_2)^2b^2c^2 + \tfrac{1}{3}b^2c^2f^2 - \tfrac{1}{6}a_1a_2c^2f^2 - \tfrac{1}{6}a_1a_2b^2f^2
\end{aligned} \tag{5.64}$$

We assume that $m \geqslant 2$ and consider the $(m+1)^{st}$ derivatives of the above five equations at $t = 0$. As in the previous case, we directly insert the values of $b(0), c(0), a_1'(0), a_2'(0)$, and $f'(0)$, which are determined by the power series (5.60), and obtain the following equations:

$$\begin{aligned}
3m(m+1)b_0^2 a_1^{(m)}(0) + 3(m+1)b_0^2 a_2^{(m)}(0) + (m+1)b_0^2 f^{(m)}(0) &= 2(m+1)b_0^2(3a_2^{(m)}(0) + f^{(m)}(0)) \\
&\quad +6(m+1)b_0^2(a_1^{(m)}(0) - a_2^{(m)}(0)) \\
&\quad -2(m+1)b_0^2 f^{(m)}(0) + R_1 \\
3(m+1)b_0^2 a_1^{(m)}(0) + 3m(m+1)b_0^2 a_2^{(m)}(0) + (m+1)b_0^2 f^{(m)}(0) &= 2(m+1)b_0^2(3a_1^{(m)}(0) + f^{(m)}(0)) \\
&\quad +6(m+1)b_0^2(a_2^{(m)}(0) - a_1^{(m)}(0)) \\
&\quad -2(m+1)b_0^2 f^{(m)}(0) + R_2 \\
m(m+1)b_0^2 b^{(m)}(0) &= 2(m+1)(b_0^2 c^{(m)}(0) - b_0^2 b^{(m)}(0)) \\
&\quad +R_3 \\
m(m+1)b_0^2 c^{(m)}(0) &= 2(m+1)(b_0^2 b^{(m)}(0) - b_0^2 c^{(m)}(0)) \\
&\quad +R_4 \\
m(m+1)b_0^4 f^{(m)}(0) + 3(m+1)b_0^4 a_1^{(m)}(0) + 3(m+1)b_0^4 a_2^{(m)}(0) &= 2(m+1)b_0^4 f^{(m)}(0) + R_5
\end{aligned}$$

In the above system, $R_1, \ldots, R_5$ are polynomials which depend on the lower derivatives of the coefficients of the metric only. The factors of type $m(m+1)$ appear for the same reason as in the previous case. Our equations are equivalent to the following linear system:

$$L_m \begin{pmatrix} a_1^{(m)}(0) \\ a_2^{(m)}(0) \\ b^{(m)}(0) \\ c^{(m)}(0) \\ f^{(m)}(0) \end{pmatrix} = \begin{pmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \\ R_5 \end{pmatrix} \tag{5.65}$$

where

$$L_m = \begin{pmatrix} (3m-6)(m+1)b_0^2 & 3(m+1)b_0^2 & 0 & 0 & (m+1)b_0^2 \\ 3(m+1)b_0^2 & (3m-6)(m+1)b_0^2 & 0 & 0 & (m+1)b_0^2 \\ 0 & 0 & (m+1)(m+2)b_0^2 & -2(m+1)b_0^2 & 0 \\ 0 & 0 & -2(m+1)b_0^2 & (m+1)(m+2)b_0^2 & 0 \\ 3(m+1)b_0^4 & 3(m+1)b_0^4 & 0 & 0 & (m-2)(m+1)b_0^4 \end{pmatrix}$$

The rank of L_3 is 3 and for all other $m \geqslant 2$ it is 5. Since we have been able to calculate the power series expansion (5.60) up to fifth order, the system (5.65) has a solution for $m = 3$. Our initial value problem thus has a formal power series solution for any choice of b_0, q_1, and q_2. Moreover, we have proven that the two free parameters of third order are indeed the only ones. If we replace $(a_1(t), a_2(t), b(t), c(t), f(t))$ by $(-a_1(-t), -a_2(-t), b(-t), c(-t), -f(-t))$, we obtain a new solution of (5.39) with the same values of $b(0)$, $a_1'''(0)$, and $a_2'''(0)$. We conclude that b and c are even and that a_1, a_2, and f are odd. All of the smoothness conditions thus are satisfied. The system (5.39) is also preserved if we interchange b and c. By the same arguments as above, we can conclude that any solution of our initial value problem satisfies $b(t) = c(t)$. With help of the convergence result of Eschenburg and Wang [32], it follows that the power series converges to a metric whose holonomy is contained in Spin(7).

Finally, we consider the case where $k = l = 1$ and $b(0) = f(0) = 0$. There are two subcases, namely $a_2(0) = a_1(0)$ and $a_2(0) = -a_1(0)$. The subcase $a_2(0) = a_1(0)$ we investigate first.

We take the $(m+1)^{st}$ derivatives of the first, second, and fifth equation of (5.64) and the m^{th} derivatives of the third and fourth equation. As usual, we assume that $m \geqslant 2$ and obtain the following equations:

$$\begin{aligned}
-6m(m+1)a_0^2a_1^{(m)}(0) &= -6(m+1)a_0(2a_0c^{(m)}(0) - 2a_0a_1^{(m)}(0)) \\
&\quad +3(m+1)a_0(2a_0a_1^{(m)}(0) - 2a_0a_2^{(m)}(0)) \\
&\quad +R_1 \\
-6m(m+1)a_0^2a_2^{(m)}(0) &= -6(m+1)a_0(2a_0c^{(m)}(0) - 2a_0a_2^{(m)}(0)) \\
&\quad +3(m+1)a_0(2a_0a_2^{(m)}(0) - 2a_0a_1^{(m)}(0)) \\
&\quad +R_2 \\
(m+1)a_0^3b^{(m)}(0) &= 2a_0^3b^{(m)}(0) + \tfrac{1}{6}a_0^3f^{(m)}(0) + R_3 \\
ma_0^3c^{(m)}(0) &= a_0^2(a_0a_1^{(m)}(0) - a_0c^{(m)}(0)) \\
&\quad +a_0^2(a_0a_2^{(m)}(0) - a_0c^{(m)}(0)) + R_4 \\
-12(m+1)a_0^4b^{(m)}(0) + m(m+1)a_0^4f^{(m)}(0) &= 2(m+1)a_0^4f^{(m)}(0) + R_5
\end{aligned}$$

where $R_1, \ldots, R_5$ are polynomials which depend on the lower derivatives of a_1, a_2, b, c, and f only. The above equations are equivalent to a linear system of type (5.65) where

$$L_m = \begin{pmatrix} -6(m+1)(m+3)a_0^2 & 6(m+1)a_0^2 & 0 & 12(m+1)a_0^2 & 0 \\ 6(m+1)a_0^2 & -6(m+1)(m+3)a_0^2 & 0 & 12(m+1)a_0^2 & 0 \\ 0 & 0 & (m-1)a_0^3 & 0 & -\frac{1}{6}a_0^3 \\ -a_0^3 & -a_0^3 & 0 & (m+2)a_0^3 & 0 \\ 0 & 0 & -12(m+1)a_0^4 & 0 & (m-2)(m+1)a_0^4 \end{pmatrix}$$

For $m = 3$ the rank of L_m is 4. Since (5.61) is a power series expansion up to fifth order, the system (5.65) has in that case a solution. If $m = 2$ or $m \geqslant 4$, L_m is of full rank. Therefore, there exists for any choice of a_0 and q a unique formal power series solution of our initial value problem. Since we can replace $(a_1(t), a_2(t), b(t), c(t), f(t))$ by $(a_1(-t), a_2(-t), -b(-t), c(-t), -f(-t))$ without changing the system (5.39), we can prove by the same methods as in the previous cases that a_1, a_2, and c are even and that b and f are odd. The power series (5.61) converges and we thus have proven the existence of a family of local cohomogeneity-one metrics whose holonomy is contained in Spin(7). From the fact that the system (5.39) is preserved if we interchange a_1 and a_2, we can conclude that all of those metrics satisfy $a_1(t) = a_2(t)$.

Next, we investigate the subcase $a_2(0) = -a_1(0)$. We again take the $(m+1)^{st}$ derivatives of the first, second, and fifth equation of (5.64) and the m^{th} derivatives of the other two equations. As before, we assume without loss of generality that $b'(0) = 1$. We obtain the following five equations:

$$
\begin{aligned}
-6m(m+1)a_0^2 a_1^{(m)}(0) &= -6(m+1)a_0(2a_0c^{(m)}(0) - 2a_0a_1^{(m)}(0)) \\
&\quad +3(m+1)a_0(2a_0a_1^{(m)}(0) + 2a_0a_2^{(m)}(0)) \\
&\quad +R_1 \\
6m(m+1)a_0^2 a_2^{(m)}(0) &= 6(m+1)a_0(2a_0c^{(m)}(0) + 2a_0a_2^{(m)}(0)) \\
&\quad +3(m+1)a_0(-2a_0a_2^{(m)}(0) - 2a_0a_1^{(m)}(0)) \\
&\quad +R_2 \\
-(m+1)a_0^3 b^{(m)}(0) &= -\tfrac{1}{6}a_0^3 f^{(m)}(0) + R_3 \\
-ma_0^3 c^{(m)}(0) &= -a_0^2(a_0a_1^{(m)}(0) - a_0c^{(m)}(0)) \\
&\quad +a_0^2(-a_0a_2^{(m)}(0) - a_0c^{(m)}(0)) + R_4 \\
-12(m+1)a_0^4 b^{(m)}(0) - m(m+1)a_0^4 f^{(m)}(0) &= -24(m+1)a_0^4 b^{(m)}(0) \\
&\quad +2(m+1)a_0^4 f^{(m)}(0) + R_5
\end{aligned}
$$

where $R_1, \ldots, R_5$ are polynomials which depend on the lower derivatives of a_1, a_2, b, c, and f only. These equations are equivalent to a linear system of type (5.65) where

$$
L_m = \begin{pmatrix}
-6(m+1)(m+3)a_0^2 & -6(m+1)a_0^2 & 0 & 12(m+1)a_0^2 & 0 \\
6(m+1)a_0^2 & 6(m-1)(m+1)a_0^2 & 0 & -12(m+1)a_0^2 & 0 \\
0 & 0 & -(m+1)a_0^3 & 0 & \frac{1}{6}a_0^3 \\
a_0^3 & a_0^3 & 0 & -ma_0^3 & 0 \\
0 & 0 & 12(m+1)a_0^4 & 0 & -(m+1)(m+2)a_0^4
\end{pmatrix}
$$

The rank of L_m is 5 for all $m \geqslant 3$ and for $m = 2$ it is 4. Since we have calculated the power series (5.62) up to fifth order, the system (5.65) has indeed a solution for $m = 2$. We thus have proven that there exists a unique power series solution of our initial value problem for any choice of a_0 and q. By the symmetry of (5.39) which we have used in the previous subcase, we can show that a_1, a_2, and c are even and that b and f are odd. Therefore, the power series satisfies all of the smoothness conditions. Since the series converges, we have proven the existence of a further family of local cohomogeneity-one metrics whose holonomy is contained in Spin(7).

We will not calculate the dimensions of the spaces W_m^h and W_2^v. Nevertheless, we are able to make some statements on the number of cohomogeneity-one Einstein metrics whose principal orbit is an Aloff-Wallach space and whose singular orbit is $\mathbb{CP}^2$. There are choices of k and l such that Assumption 3.2.19 is not satisfied. The action of the group $U(1)_{k,l}$ splits $\mathfrak{m}$ into

$$\mathbb{V}_{k-l} \oplus \mathbb{V}_{2k+l} \oplus \mathbb{V}_{k+2l} \oplus \mathbb{R}\,,$$

where $\mathbb{V}_r$ denotes the two-dimensional module on which $U(1)_{k,l}$ acts with weight r. The action of $U(1)_{k,l}$ on $\mathbb{R}$ of course is trivial. We assume that $\mathfrak{k} = \mathfrak{u}(1)_{k,l} \oplus V_1 \oplus V_4$. In this situation, we have the following splittings:

$$\mathfrak{p}^\perp = \mathbb{V}_0 \oplus \mathbb{V}_{k-l} \quad \text{and} \quad \mathfrak{p} = \mathbb{V}_{2k+l} \oplus \mathbb{V}_{k+2l}\,.$$

The normal space $\mathfrak{p}^\perp$ is not a vector space but a quotient of a four-dimensional space by a group of type $\mathbb{Z}_{|k+l|}$. The above formula for the splitting of $\mathfrak{p}^\perp$ thus should be understood as follows: The action of $U(1)_{k,l}$ on $\mathfrak{p}^\perp$ can be lifted to an action on a four-dimensional vector space which splits into $\mathbb{V}_0 \oplus \mathbb{V}_{k-l}$. Since the action on $\mathfrak{p}^\perp$ can be lifted, Theorem 3.2.24 also holds with the obvious replacements, although the underlying space is an orbifold instead of a manifold. Assumption 3.2.19 is violated if and only if one of the modules $\mathbb{V}_{2k+l}$ or $\mathbb{V}_{k+2l}$ is trivial or equivalent to $\mathbb{V}_{k-l}$. By a short calculation, we see that this is equivalent to

$$(k,l) \in \{(0,-1),(1,-2),(1,0),(2,-1)\}\ .$$

The case, where $k = l = 1$ and $b(0) = f(0) = 0$, is equivalent to $\mathfrak{k} = \mathfrak{u}(1)_{1,-2} \oplus V_1 \oplus V_4$. Therefore, our assumption is in that situation not satisfied. If $k \cdot l = 0$ and $a(0) = f(0) = 0$, Assumption 3.2.19 is not satisfied, too. We have justified why Theorem 3.2.24 also holds if 3.2.19 is not true but the metric and its Ricci-tensor still take their values in $S^2(\mathfrak{p}) \oplus S^2(\mathfrak{p}^\perp)$. Moreover, we show in Appendix C.3 and C.4 that this weaker assumption is satisfied if the metric is diagonal. For this reason, we notwithstanding apply the theorem to all of the cases which we have considered. By taking a look at the sufficient smoothness conditions which we have found, we find lower bounds on $\dim W^h_m$ and $\dim W^v_2$. In all of our cases, the metric on the singular orbit is described by a single parameter. We therefore have $\dim W^h_0 = 1$. The second derivatives of the functions which correspond to the horizontal part of the metric can be chosen arbitrarily without violating the smoothness conditions. The number of those functions thus yields a lower bound on $\dim W^h_2$. Since the third derivatives of the functions which correspond to the vertical part of the metric can be chosen arbitrarily, too, we also have a lower bound on $\dim W^v_2$. All in all, we can at least prescribe the following initial values:

1. If $N^{k,l} \not\cong N^{1,1}$ and $a(0) = f(0) = 0$, we can choose
   (a) $b(0)$ $(= c(0))$,
   (b) $(b-c)''(0)$,
   (c) $a'''(0)$, and $f'''(0)$

   arbitrarily.
2. If $k = l = 1$ and $a_1(0) = a_2(0) = f(0) = 0$, we can choose
   (a) $b(0)$ $(= c(0))$,
   (b) $(b-c)''(0)$,
   (c) $a_1'''(0)$, $a_2'''(0)$, and $f'''(0)$

   arbitrarily.
3. If $k = l = 1$ and $b(0) = f(0) = 0$, we can choose
   (a) $a_1(0)$ $(= a_2(0) = c(0))$,
   (b) two elements of $(a_1''(0), a_2''(0), f''(0))$,
   (c) $b'''(0)$, and $f'''(0)$

   arbitrarily.

As usual, the Einstein constant can be fixed at an arbitrary value, too. The above considerations put the free parameters q, q_1, q_2 of our metrics in the context of the research of Eschenburg and Wang [32].

In the following, we consider $SU(2)$ as a subgroup of K. Moreover, we consider $\mathrm{Hom}_{SU(2)}(S^{m+2}(\mathfrak{p}^\perp), S^2(\mathfrak{p}) \oplus S^2(\mathfrak{p}^\perp))$ as a subset of $\mathrm{Hom}_{SU(2)}(S^m(\mathfrak{p}^\perp), S^2(\mathfrak{p}) \oplus S^2(\mathfrak{p}^\perp))$. $S^2(\mathfrak{p})$ and $S^2(\mathfrak{p}^\perp)$ both split with respect to the action of $SU(2)$ into $3\mathbb{V}_2^{\mathbb{R}} \oplus \mathbb{V}_0^{\mathbb{R}}$. If m is odd, $S^m(\mathfrak{p}^\perp)$ decomposes into $SU(2)$-submodules with odd weights only. If $m \geqslant 2$ is even, the complement of $S^m(\mathfrak{p}^\perp) \subseteq S^{m+2}(\mathfrak{p}^\perp)$ decomposes into $SU(2)$-submodules whose weights are greater than 2. With help of these facts, we can easily show that

$$\mathrm{Hom}_{SU(2)}(S^{m+2}(\mathfrak{p}^\perp), S^2(\mathfrak{p}) \oplus S^2(\mathfrak{p}^\perp)) \cong \mathrm{Hom}_{SU(2)}(S^m(\mathfrak{p}^\perp), S^2(\mathfrak{p}) \oplus S^2(\mathfrak{p}^\perp))$$

for all $m \geqslant 1$. Let $\varphi : S^{m+2}(\mathfrak{p}^\perp) \to S^2(\mathfrak{p}) \oplus S^2(\mathfrak{p}^\perp)$ be a linear, K-equivariant map. φ has to be $SU(2)$-equivariant, too. From the above formula it follows that φ has to vanish on the complement of $S^m(\mathfrak{p}^\perp)$ in $S^{m+2}(\mathfrak{p}^\perp)$. This argument proves that

$$\mathrm{Hom}_K(S^{m+2}(\mathfrak{p}^\perp), S^2(\mathfrak{p}) \oplus S^2(\mathfrak{p}^\perp)) \cong \mathrm{Hom}_K(S^m(\mathfrak{p}^\perp), S^2(\mathfrak{p}) \oplus S^2(\mathfrak{p}^\perp))$$

for all $m \geqslant 1$. Therefore, we have $\dim W^h_{m+2} = \dim W^v_{m+2} = 0$. We conclude with help of Theorem 3.2.24 that in the diagonal case where we can apply that theorem there are no free initial conditions of higher order than those which we have found in the previous paragraph. The focus of this thesis is on metrics whose holonomy is contained in Spin(7). Therefore, we will not further discuss the issue of cohomogeneity-one Einstein metrics in this context.

Since we have not explicitly solved the systems (5.36) and (5.39), we cannot prove under which conditions the metrics from this section are complete. Moreover, the arguments which we have made in Section 5.2 and 5.3 do not exclude the existence of compact solutions in our situation. We will illustrate this by an example: Let the principal orbit of the cohomogeneity-one orbifold which we consider be a generic Aloff-Wallach space $N^{k,l}$ with $k \geqslant l \geqslant 0$. Moreover, we assume that there are two singular orbits which are located at $t = 0$ and $t = T$ for a $T > 0$. If $a(0) = f(0) = 0$, we have $f'(0) = \frac{2\Delta}{k+l} > 0$. If in addition $b(T) = f(T) = 0$, it follows that $f'(T) = -\frac{2\Delta}{l} < 0$. In this situation, we cannot conclude that any of the functions a, b, c, or f has an additional zero in $(0, T)$. Therefore, we can neither prove nor exclude the existence of compact examples by our methods. We finally sum up the results of this section:

Theorem 5.4.6. *Let (M, Ω) be a Spin(7)-orbifold with a cohomogeneity-one action of $SU(3)$ which preserves Ω. The metric associated to Ω we denote by g. In this situation, the following statements are true:*

1. *All principal orbits are $SU(3)$-equivariantly diffeomorphic to an Aloff-Wallach space $N^{k,l}$. We assume from now on that (k, l) is coprime.*
2. (a) *If $N^{k,l}$ is generic, the matrix representation of g with respect to the basis $(e_1, \dots, e_7)$ from page 150 is of type (5.31).*
   (b) *If $k = 1$ and $l = 0$, the matrix representation of g with respect to the basis $(e_1, e_2, e_5, e_6, e_3, e_4, e_7)$ is of type (5.32).*

(c) *If $k = 1$ and $l = 1$, the matrix representation of g with respect to the basis $(e_1, e_2, e_7, e_3, e_4, e_5, e_6)$ is of type (5.33).*

3. (a) *We assume that $N^{k,l}$ is generic or $N^{1,0}$ and that g is diagonal with respect to the basis from page 150. Furthermore, we assume that there is a continuous map $\gamma : [0,1] \to \Gamma(\bigwedge^4 T^*M)$ such that $\gamma(0) = \Omega$, $\gamma(1)$ is a four-form of type (5.35), and $\gamma(t)$ is for all $t \in [0,1]$ a $SU(3)$-invariant four-form with the same associated metric as Ω. In this situation, Ω already is a four-form of type (5.35) or Ω is not parallel.*

   (b) *We assume that $k = 1$, $l = 1$, and that g is diagonal. If Ω is of type (5.43), there exists a parallel four-form of type (5.38) on M which has the same associated metric and orientation as Ω or Ω is not parallel.*

4. (a) *If $N^{k,l}$ is generic or $N^{1,0}$ and Ω is of type (5.35), Ω is parallel if and only if the system (5.36) is satisfied. In that case, the holonomy of g is all of Spin(7).*

   (b) *If $k = l = 1$ and Ω is of type (5.38), Ω is parallel if and only if the system (5.39) is satisfied. In that case, the holonomy of g is contained in Spin(7). If $a_1 = -a_2$ and $a_1^2 = b^2 + c^2$, the holonomy is contained in $SU(4)$.*

If M has a singular orbit, which has to be the case if (M,Ω) is parallel and complete, it is $SU(3)/U(1)^2$, S^5, $SU(3)/SO(3)$, or $\mathbb{CP}^2$. If the singular orbit is S^5 or $SU(3)/SO(3)$, $N^{k,l}$ has to be $\cong N^{1,0}$. If the singular orbit is $SU(3)/U(1)^2$, there are no restrictions on k and l. If the singular orbit is $\mathbb{CP}^2$, we can assume without loss of generality that the function a vanishes at the singular orbit (or if $k = l = 1$, a_1 and a_2 vanish at the singular orbit). In this situation, the only restriction on k and l is that $k + l \neq 0$. For any choice of the principal and the singular orbit, the orbifold M is a manifold, except in the case where the singular orbit is $\mathbb{CP}^2$ and $|k + l| \neq 1$. In that case, M is a $D^4/\mathbb{Z}_{|k+l|}$-bundle over $\mathbb{CP}^2$, where D^4 is a disc in $\mathbb{R}^4$ and $\mathbb{Z}_{|k+l|}$ is generated by:

$$\begin{pmatrix} e^{\frac{2\pi i}{k+l}} & 0 \\ 0 & e^{-\frac{2\pi i}{k+l}} \end{pmatrix}.$$

From now on, we assume that the singular orbit is at $t = 0$.

1. *If the singular orbit is $SU(3)/U(1)^2$ and $N^{k,l}$ is generic or $N^{1,0}$, the four-form (5.35) cannot be parallel.*

2. *We assume that the singular orbit is $SU(3)/U(1)^2$, the principal orbit is $N^{1,1}$, and $f(0) = 0$. For any choice of the initial values $a_1(0), a_2(0), b(0), c(0) \in \mathbb{R}\backslash\{0\}$ with $a_1(0) + a_2(0) = 0$, there exists a unique parallel four-form Ω of type (5.38) for all $t \in (0,\epsilon)$ and an $\epsilon > 0$. Ω cannot be smoothly extended to the singular orbit. If we replace the principal orbit by $N^{1,1}/\mathbb{Z}_2$ where $\mathbb{Z}_2$ is generated by diag$(i,-i,1)$, there also exists a unique parallel four-form Ω for any choice of the initial values. In this new situation, it is possible to extend Ω smoothly to the singular orbit.*

3. *We assume that the singular orbit is S^5 and the principal orbit is $N^{1,-1}$. The space of all $SU(3)$-invariant metrics on the singular orbit is two-dimensional. For any of those*

metrics and any choice of the signs of $b(0)$ *and* $c(0)$*, there exists a unique smooth and parallel four-form* Ω *of type (5.38) on a tubular neighborhood of the singular orbit.*

4. *We assume that the singular orbit is* $SU(3)/SO(3)$ *and the principal orbit is* $N^{1,-1}$*.* $N^{1,-1}$ *is* $SU(3)$*-equivariantly diffeomorphic to* $SU(3)/SO(2)$ *where the group* $SO(2)$ *is generated by* e_1*. The embedding of* $SO(3)$ *into* $SU(3)$ *shall be the standard one. We fix the basis* $(e_7, e_3, e_5, e_4, e_6, e_2, e_8)$ *of the tangent space of* $SU(3)/SO(2)$*. Let* Ω *be a* $SU(3)$*-invariant Spin*(7)*-structure whose associated metric is diagonal and which can be connected by a path of such structures to a Spin*(7)*-structure with a frame of type (5.55). In this situation,* Ω *already is a Spin*(7)*-structure with a frame of type (5.55). There exists no parallel four-form on a tubular neighborhood of the singular orbit which belongs to the above class of Spin*(7)*-structures. Moreover, there does not even exist such a parallel four-form on a neighborhood of a principal orbit.*

5. *We assume that the singular orbit is* $\mathbb{CP}^2$ *and that the principal orbit is a generic Aloff-Wallach space,* $N^{0,-1}$*, or* $N^{1,0}$*. Furthermore, let* $a(0) = f(0) = 0$*. The space of all* $SU(3)$*-invariant metrics on the singular orbit is one-dimensional. For any choice of the invariant metric on* $\mathbb{CP}^2$*, of the signs of* $b(0)$ *and* $c(0)$*, and of* $f'''(0)$*, there exists a unique smooth and parallel four-form* Ω *of type (5.35) on a tubular neighborhood of the singular orbit.*

6. *We assume that the singular orbit is* $\mathbb{CP}^2$ *and that the principal orbit is* $N^{1,1}$*. Furthermore, let* $a_1(0) = a_2(0) = f(0) = 0$*. The space of all* $SU(3)$*-invariant metrics on the singular orbit is one-dimensional. For any choice of the invariant metric on* $\mathbb{CP}^2$*, of the signs of* $b(0)$ *and* $c(0)$*, of* $a_1'''(0)$*, and of* $a_2'''(0)$*, there exists a unique smooth and parallel four-form* Ω *of type (5.38) on a tubular neighborhood of the singular orbit. Moreover, we have* $b(t) = c(t)$ *for all values of* t*.*

7. *We assume that the singular orbit is* $\mathbb{CP}^2$ *and that the principal orbit is* $N^{1,1}$*. Furthermore, let* $b(0) = f(0) = 0$ *and* $a_1(0) = a_2(0)$*. The space of all* $SU(3)$*-invariant metrics on the singular orbit is one-dimensional. For any choice of the invariant metric on* $\mathbb{CP}^2$*, of the signs of* $a_1(0)$ *and* $c(0)$*, and of* $b'''(0)$*, there exists a unique smooth and parallel four-form* Ω *of type (5.38) on a tubular neighborhood of the singular orbit. Moreover, we have* $a_1(t) = a_2(t)$ *for all values of* t*.*

8. *We assume that the singular orbit is* $\mathbb{CP}^2$ *and that the principal orbit is* $N^{1,1}$*. Furthermore, let* $b(0) = f(0) = 0$*,* $a_1(0) = -a_2(0)$*, and* $b'(0) = 1$*. The space of all* $SU(3)$*-invariant metrics on the singular orbit is one-dimensional. For any choice of the invariant metric on* $\mathbb{CP}^2$*, of the signs of* $a_1(0)$ *and* $c(0)$*, and of* $c''(0)$*, there exists a unique smooth and parallel four-form* Ω *of type (5.38) on a tubular neighborhood of the singular orbit.*

Remark 5.4.7. 1. For any of the above results which holds for a principal orbit $N^{k,l}$, there is an analogous result which holds for the other Aloff-Wallach spaces which are $\cong N^{k,l}$. We have already discussed in the course of this section that if the principal orbit is covered by $N^{1,1}$, the singular orbit is $SU(3)/U(1)^2$, and the metric is diagonal, it suffices to consider the initial condition $f(0) = 0$. If the isotropy group of the singular orbit is conjugate to $SO(3)$ and the isotropy group of the principal orbit is as usual generated by $\mathrm{diag}(i, -i, 0)$ and not by e_1, the metric has to be non-diagonal. The cases where

$(k,l) \in \{(1,-2),(2,-1)\}$ and $a(0) = f(0) = 0$ are both equivalent to the initial value problem with $k = l = 1$ and $b(0) = f(0) = 0$ which we have included in our theorem. In the last case we have mentioned, we necessarily have $|b'(0)| = 1$. We did not need to consider the case $b'(0) = -1$, since there is a symmetry of the system (5.39) which changes $b(t)$ into $b(-t)$. For the above reasons, we have investigated all possible initial conditions for the systems (5.36) and (5.39).

2. The metrics which are associated to any of the Spin(7)-structures from the above theorem are diagonal. At least if $k = l = 1$, there are further cohomogeneity-one Spin(7)-structures whose associated metric is diagonal. We suppose that those structures have a similar behavior.

With help of Theorem 3.2.24, we also have obtained results on cohomogeneity-one Einstein metrics with an Aloff-Wallach space as principal orbit:

Theorem 5.4.8. *Let M be a cohomogeneity-one orbifold whose principal orbit is an Aloff-Wallach space $N^{k,l}$. We assume that M has a singular orbit, which has to be one of those from Theorem 5.4.6. Let $I \subseteq \mathbb{R}$ be an interval which is isometric to $M/SU(3)$. Any diagonal $SU(3)$-invariant metric on M is described by the four functions $a,b,c,f : I \to \mathbb{R}$ if $N^{k,l}$ is generic or $\cong N^{1,-1}$. If $k = l = 1$, the metric is described by the functions $a_1, a_2, b, c, f : I \to \mathbb{R}$. We denote the off-diagonal coefficients of the metric g by $\beta_{i,j} := g(e_i, e_j)$. Without loss of generality, we assume that the singular orbit is at $0 \in I$. In this situation, the following statements are true:*

1. *Let $N^{k,l}$ be generic and the singular orbit be $SU(3)/U(1)^2$. For any choice of $a_0, b_0, c_0, f_3, \lambda \in \mathbb{R}$, there exists a unique $SU(3)$-invariant Einstein metric on a tubular neighborhood of $SU(3)/U(1)^2$ such that:*
   (a) *$a(0) = a_0$, $b(0) = b_0$, $c(0) = c_0$,*
   (b) *$f'''(0) = f_3$, and*
   (c) *the Einstein constant is λ.*
2. *Let $k = 1$, $l = 0$, and the singular orbit be $SU(3)/U(1)^2$. For any choice of $a_0, b_0, c_0, \widetilde{\beta}_{1,5,3}, \widetilde{\beta}_{1,6,3}, f_3, \lambda \in \mathbb{R}$, there exists a unique $SU(3)$-invariant Einstein metric on a tubular neighborhood of $SU(3)/U(1)^2$ such that:*
   (a) *$a(0) = a_0$, $b(0) = b_0$, $c(0) = c_0$,*
   (b) *$\beta'''_{1,5}(0) = \widetilde{\beta}_{1,5,3}$, $\beta'''_{1,6}(0) = \widetilde{\beta}_{1,6,3}$,*
   (c) *$f'''(0) = f_3$, and*
   (d) *the Einstein constant is λ.*
3. *Let $N^{k,l}$ be generic, the singular orbit be $\mathbb{CP}^2$, and $a(0) = f(0) = 0$. For any choice of $b_0, b_2, a_3, f_3, \lambda \in \mathbb{R}$, there exists a $SU(3)$-invariant Einstein metric on a tubular neighborhood of $\mathbb{CP}^2$ such that:*
   (a) *$b(0) = c(0) = b_0$,*
   (b) *$(b-c)''(0) = b_2$,*

(c) $a'''(0) = a_3$, $f'''(0) = f_3$, *and*

(d) *the Einstein constant is* λ.

4. *Let* $k = l = 1$, *the singular orbit be* $\mathbb{CP}^2$, *and* $a_1(0) = a_2(0) = f(0) = 0$. *For any choice of* $b_0, b_2, a_{1,3}, a_{2,3}, f_3, \lambda \in \mathbb{R}$, *there exists a* $SU(3)$*-invariant Einstein metric on a tubular neighborhood of* $\mathbb{CP}^2$ *such that:*

   (a) $b(0) = c(0) = b_0$,

   (b) $(b - c)''(0) = b_2$,

   (c) $a_1'''(0) = a_{1,3}$, $a_2'''(0) = a_{2,3}$, $f'''(0) = f_3$, *and*

   (d) *the Einstein constant is* λ.

For the case, where the principal orbit is $N^{1,1}/\mathbb{Z}_2$, we have a similar result:

Theorem 5.4.9. *Let* M *be a cohomogeneity-one manifold with principal orbit* $N^{1,1}/\mathbb{Z}_2$ *where* $\mathbb{Z}_2$ *is generated by diag*$(i, -i, 1)$. *Since the principal orbit is covered by* $N^{1,1}$, *we can use the same notation as in Theorem 5.4.8. We assume that* M *has* $SU(3)/U(1)^2$ *as a singular orbit, that the singular orbit is located at* $t = 0$, *and that* $f(0) = 0$. *In this situation, the following statement is true:*

For any choice of $a_0, b_0, c_0, a_1, \widetilde{\beta}_{1,2,1}, f_3, \lambda \in \mathbb{R}$, *there exists a unique* $SU(3)$*-invariant Einstein metric on a tubular neighborhood of* $SU(3)/U(1)^2$ *such that:*

1. $a_1(0) = a_2(0) = a_0$, $b(0) = b_0$, $c(0) = c_0$,

2. $(a_1 - a_2)'(0) = a_1$, $\beta'_{1,2}(0) = \widetilde{\beta}_{1,2,1}$,

3. $f'''(0) = f_3$, *and*

4. *the Einstein constant is* λ.

There are certain cases where Assumption 3.2.19 is not satisfied, but Theorem 3.2.24 still holds true. In those cases, we obtain further results on cohomogeneity-one Einstein metrics with an Aloff-Wallach space as principal orbit:

Theorem 5.4.10. *Let* M *be a cohomogeneity-one manifold whose principal orbit is an Aloff-Wallach space* $N^{k,l}$. *We assume that* M *has a singular orbit at* $t = 0$ *and use the same notation as in Theorem 5.4.8. In this situation, the following statements are true:*

1. *Let* $k = 1$, $l = -1$, *and the singular orbit be* S^5. *For any choice of* $b_0, f_0, b_1, \widetilde{\beta}_{3,5,1}, \widetilde{\beta}_{3,6,1}, a_3, \lambda \in \mathbb{R}$, *there exists a unique* $SU(3)$*-invariant Einstein metric on a tubular neighborhood of* S^5 *such that:*

   (a) $b(0) = c(0) = b_0$, $f(0) = f_0$,

   (b) $(b - c)'(0) = b_1$, $\beta'_{3,5}(0) = \widetilde{\beta}_{3,5,1}$, $\beta'_{3,6}(0) = \widetilde{\beta}_{3,6,1}$,

   (c) $a'''(0) = a_3$, *and*

   (d) *the Einstein constant is* λ.

2. *Let* $(k, l) \in \{(0, -1), (1, 0)\}$*, the singular orbit be* $\mathbb{CP}^2$*, and* $a(0) = f(0) = 0$*. For any choice of* $b_0, b_2, a_3, f_3, \lambda \in \mathbb{R}$*, there exists a* $SU(3)$*-invariant Einstein metric on a tubular neighborhood of* $\mathbb{CP}^2$ *such that:*

   (a) $b(0) = c(0) = b_0$,
   (b) $(b - c)''(0) = b_2$,
   (c) $a'''(0) = a_3$, $f'''(0) = f_3$, *and*
   (d) *the Einstein constant is* λ.

3. *Let* $k = l = 1$*, the singular orbit be* $\mathbb{CP}^2$*, and* $b(0) = f(0) = 0$*. For any choice of* $a_0, a_{1,2}, a_{2,2}, b_3, f_3, \lambda \in \mathbb{R}$*, there exists a* $SU(3)$*-invariant Einstein metric on a tubular neighborhood of* $\mathbb{CP}^2$ *such that:*

   (a) $a_1(0) = a_2(0) = c(0) = a_0$,
   (b) $(a_1 - a_2)''(0) = a_{1,2}$, $(a_1 + a_2 - 2c)''(0) = a_{2,2}$,
   (c) $b'''(0) = b_3$, $f'''(0) = f_3$, *and*
   (d) *the Einstein constant is* λ.

There is one case left open for which we have not proven that it is possible to apply Theorem 3.2.24. We thus state the result of our calculations as a conjecture:

Conjecture 5.4.11. *In the situation of Theorem 5.4.10, let* $k = l = 1$*, the singular orbit be* $SU(3)/U(1)^2$*,* $f(0) = 0$*, and* $\beta_{1,7}(t) = \beta_{2,7}(t) = 0$ *for all* t*. For any choice of* $a_0, b_0, c_0, \widetilde{\beta}_{3,5,1}, \widetilde{\beta}_{3,6,1}, a_2, \widetilde{\beta}_{1,2,2}, f_3, \lambda \in \mathbb{R}$*, there exists a unique* $SU(3)$*-invariant Einstein metric with* $\beta_{1,7}(t) = \beta_{2,7}(t) = 0$ *on a tubular neighborhood of* $SU(3)/U(1)^2$ *such that:*

1. $a_1(0) = a_2(0) = a_0$, $b(0) = b_0$, $c(0) = c_0$,
2. $\beta'_{3,5}(0) = \widetilde{\beta}_{3,5,1}$, $\beta'_{3,6}(0) = \widetilde{\beta}_{3,6,1}$,
3. $(a_1 - a_2)''(0) = a_2$, $\beta''_{1,2}(0) = \widetilde{\beta}_{1,2,2}$,
4. $f'''(0) = f_3$, *and*
5. *the Einstein constant is* λ.

If we specialize the above conjecture to the case of diagonal metrics, we obtain the following result:

Theorem 5.4.12. *In the situation of Theorem 5.4.10, let* $k = l = 1$*, the singular orbit be* $SU(3)/U(1)^2$*,* $f(0) = 0$*, and the metric be diagonal for all* t*. For any choice of* $a_0, b_0, c_0, a_2, f_3, \lambda \in \mathbb{R}$*, there exists a unique diagonal* $SU(3)$*-invariant Einstein metric on a tubular neighborhood of* $SU(3)/U(1)^2$ *such that:*

1. $a_1(0) = a_2(0) = a_0$, $b(0) = b_0$, $c(0) = c_0$,
2. $(a_1 - a_2)''(0) = a_2$,

3. *$f'''(0) = f_3$, and*

4. *the Einstein constant is λ.*

Remark 5.4.13.

1. If we could show that $\mathrm{Ric}(e_1, e_7) = \mathrm{Ric}(e_2, e_7) = 0$ for any cohomogeneity-one metric with principal orbit $N^{1,1}$ and $\beta_{1,7}(t) = \beta_{2,7}(t) = 0$, Conjecture 5.4.11 would be proven. Alternatively, it may be possible to generalize the Theorem of Eschenburg and Wang [32] to arbitrary cohomogeneity-one Einstein manifolds which do not necessarily satisfy Assumption 3.2.19. The generalized theorem likely predicts free parameters which are associated to $\mathrm{Hom}_K(S^m(\mathfrak{p}^\perp), \mathfrak{p} \otimes \mathfrak{p}^\perp)$. If we could prove such a theorem, we could also prove the conjecture and even drop the restriction $\beta_{1,7}(t) = \beta_{2,7}(t) = 0$.

2. As in Theorem 5.4.6, any of the above results which holds for an Aloff-Wallach space $N^{k,l}$ can be carried over to all Aloff-Wallach spaces which are related to $N^{k,l}$ by $\cong$. Let K be the isotropy group of the $SU(3)$-action on the singular orbit. In some cases, we can replace K by a conjugate K' of K such that we still have $U(1)_{k,l} \subseteq K' \subseteq SU(3)$. In those cases, we can change the whole metric by an isometry to a metric with singular orbit $SU(3)/K$. For the same reasons as in Remark 5.4.7, our results cover up to a conjugation of K almost all cohomogeneity-one Einstein metrics with an Aloff-Wallach space as principal orbit. The only exception is the case where the singular is orbit $SU(3)/SO(3)$, which we have considered briefly only.

At the end of this section, we compare our results with those of other authors:

Remark 5.4.14.

1. The four-form (5.35) has been described in the literature before, for example in Kanno, Yasui [48]. In [49], the same authors also present the form (5.38) which is defined on a product of $N^{1,1}$ and an interval. With help of these forms, the authors derive two systems of ordinary differential equations for the reduction of the holonomy which are equivalent to our systems. There is a further method to find our sufficient condition for the holonomy reduction: The condition for the Ricci-flatness of a diagonal cohomogeneity-one metric can be described with help of a Lagrangian. By constructing a superpotential, we obtain the metrics which satisfy the equations (5.36) as a subclass of the Ricci-flat metrics. This ansatz has been made by Cvetič et al. [22] and by Kanno, Yasui [48]. The contribution of this thesis is the discussion on the existence of further homogeneous G_2-structures or non-diagonal metrics on the Aloff-Wallach spaces which also yield cohomogeneity-one metrics with special holonomy.

2. Some of our metrics can also be obtained by the construction of Bazaikin [5]. The starting point of this construction is a cone over a seven-dimensional compact 3-Sasakian manifold M. It should be noted that there are only few compact regular 3-Sasakian manifolds of dimension 7, namely S^7, $\mathbb{RP}^7$, and $N^{1,1}$ (see Boyer, Galicki [13] and Friedrich, Kath [36]). We are mainly interested in the last case, although the construction is the same for any M. The holonomy of the cone metric is contained in $Sp(2)$. It is possible to deform the structure on the cone in such a way that the holonomy group Hol becomes Spin(7). Since the 3-Sasakian structure on $N^{1,1}$ is homogeneous, the deformed cones over that space are cohomogeneity-one manifolds. The condition Hol = Spin(7) can be written as a system of ordinary differential equations. These equations are equivalent to our system (5.39) under the additional restriction that $b = c$. The conical singularity can be resolved into a six- or a four-dimensional space, which is determined by the 3-Sasakian structure. If $M = N^{1,1}$, these spaces become $SU(3)/U(1)^2$ and $\mathbb{CP}^2$. For both

cases, Bazaikin investigated the smoothness near the singular orbit. It is also proven that in the $SU(3)/U(1)^2$-case the principal orbit has to be replaced by $N^{1,1}/\mathbb{Z}_2$ in order to make the metric smooth. The main result of Bazaikin is the following: Let the conical singularity be resolved into a six-dimensional space. For a special choice of the initial values, the differential equations have a solution which is *asymptotically conical (AC)*, i.e. it approaches a cone metric for $t \to \infty$. The holonomy of the AC-metric is $SU(4)$. Metrics of that kind have first been considered by Berard-Bergery [7] and by Page, Pope [58]. For a more general choice of the initial values, there is a family of non-homothetic metrics with holonomy Spin(7), which depends on one parameter. These metrics are *asymptotically locally conical (ALC)*, i.e. they approach a circle bundle over a cone for $t \to \infty$. If $M = N^{1,1}/\mathbb{Z}_2$, the ALC-metrics coincide with those of our metrics which have $SU(3)/U(1)^2$ as a singular orbit and satisfy $b(t) = c(t)$ and $b(0) > a_1(0) > 0$. The global behavior of the metrics with singular orbit $\mathbb{CP}^2$ and the free parameters of third order are not the subject of [5]. In a forthcoming paper [6], Bazaikin and Malkovich will deform a cone metric whose holonomy is $SU(4)$. By this deformation, they will obtain further examples of cohomogeneity-one metrics with holonomy Spin(7). Since we have allowed b to be $\neq c$, our examples with singular orbit $SU(3)/U(1)^2$ complement the analysis of [5].

3. In the paper of Cvetič et al. [24], cohomogeneity-one metrics with singular orbit S^5 and principal orbit $N^{1,-1}$ are investigated. Those metrics correspond to solutions of (5.36) with $k = 1$, $l = -1$, $a(0) = 0$, and $b(0) = c(0)$. For small values of $\left|\frac{f(0)}{b(0)}\right|$ the solution is ALC and for great values of $\left|\frac{f(0)}{b(0)}\right|$ it is incomplete. In the limiting case, we obtain an AC metric. All these observations are made by numerical methods. Our discussion of the smoothness conditions and our proof that for any choice of $b(0)$ and $f(0)$ there actually are unique local metrics which solve (5.36) is new.

4. The possible singular orbit $SU(3)/SO(3)$ of a cohomogeneity-one manifold with principal orbit $N^{1,-1}$ is investigated in this thesis for the first time. Therefore, our discussion of special metrics with this orbit structure is new, too.

5. The metrics with an Aloff-Wallach space $N^{k,l}$ which is $\not\cong N^{1,1}$ as principal and $\mathbb{CP}^2$ as singular orbit are intensively studied in the literature. In this context, the papers of Cvetič et al. [24], of Gukov, Sparks [39], and of Kanno, Yasui [48] should be mentioned. In [24], the first summands of the power series expansion of the corresponding initial value problem are calculated and the free parameter q of third order is discovered. The construction of the authors can also be carried over to the case $k = l = 1$ and $a_1(t) = a_2(t)$. Numerical simulations for different values of q are made. It turns out that there is a critical value $q_0 > 0$, such that the metric is AC if $q = q_0$ and ALC if $q > q_0$. The authors give further heuristic arguments which affirm the numerical results. Cvetič et al. also construct for any choice of (k,l) for which $S(U(2) \times U(1))/U(1)_{k,l}$ is not $S^2 \times S^1$ an explicit solution of the differential equations which corresponds to a metric with singular orbit $\mathbb{CP}^2$. This metric is asymptotically locally conical. It is possible to characterize the metric by a choice $\widetilde{q}$ of the parameter q. $\widetilde{q}$ depends in a non-trivial way on k and l and is calculated for special orbits $N^{k,l}$. The authors also discuss the topology of the coset spaces $S(U(2) \times U(1))/U(1)_{k,l} \cong S^3/\mathbb{Z}_{|k+l|}$ and the set of all values of (k,l) which have to be considered. The weak G_2-structures on $N^{k,l}$,

which are important for the description of the AC metrics, are explicitly given, too.

Kanno and Yasui [48] worked on this issue at the same time as Cvetič et al. Therefore, the papers [48] and [24] partly overlap. In particular, the free parameter q, the explicit solution with singular orbit $\mathbb{CP}^2$, and the structure of $S(U(2) \times U(1))/U(1)_{k,l}$ are discussed in [48], too.

In the paper of Gukov and Sparks [39], the connections between metrics with holonomy Spin(7) and M-theory are investigated. One chapter of this work is devoted to cohomogeneity-one orbifolds with an Aloff-Wallach space as principal orbit. The type of the orbifold singularities is important from a physical perspective. For this reason, the topology of the spaces $S(U(2) \times U(1))/U(1)_{k,l}$ is studied in detail. Moreover, a special case of the explicit metric with singular orbit $\mathbb{CP}^2$ is first discovered in [39]. In our notation, that metric satisfies $k = 0$, $l = -1$, and $a(0) = f(0) = 0$.

Although much research has been done on metrics of the above kind, our interpretation of the free parameter q with help of the results of Eschenburg and Wang [32] is new. Furthermore, we have rigorously proven that q is the only free parameter and that the power series indeed converges against a smooth metric which is defined on a tubular neighborhood of $\mathbb{CP}^2$.

6. In [49], Kanno and Yasui investigate cohomogeneity-one metrics with special holonomy and $N^{1,1}$ as principal orbit. They also obtain the system (5.39), which we have used throughout this thesis, as a sufficient condition for the holonomy reduction. By a natural ansatz for the Kähler form, they show that in the case $a_2 = -a_1$ and $a_1^2 = b^2 + c^2$ the holonomy reduces to a subgroup of $SU(4)$. This result we have proven independently. Kanno and Yasui moreover prove that in this special situation the system (5.39) has an explicit solution which depends solely on the two parameters $b(0)$ and $c(0)$. All of the resulting metrics are asymptotically conical. For most choices of the two parameters the metric has holonomy $SU(4)$. If we further assume that $b(0) = 0$, we obtain a metric with singular orbit $\mathbb{CP}^2$, which has holonomy $Sp(2)$. This metric coincides with the hyper-Kähler metric on $T^*\mathbb{CP}^2$ which has been discovered by Calabi [18]. Other references for this metric are Dancer, Swann [28] and Cvetič et al. [22]. Kanno and Yasui [49] also study the possible AC- and ALC-solutions of (5.39) which are defined for a sufficiently large t. This analysis is done with help of a power series in $\frac{1}{t}$. The authors also calculate the first terms of the power series expansion of (5.39) for the initial conditions $a_1(0) = -a_2(0) \neq 0$, $b(0), c(0) \neq 0$, and $f(0) = 0$. The solutions of (5.39) with these initial conditions correspond to metrics whose principal orbit is covered by $N^{1,1}$ and whose singular orbit is $SU(3)/U(1)^2$. For the initial conditions $b(0) = f(0) = 0$ and $c(0) = a_1(0) = \pm a_2(0)$, the same calculations are made and the authors also observe the free parameter of third (second) order.

The case where $a_1(0) = a_2(0) = f(0) = 0$ and the two parameters q_1 and q_2 of third order seem to be discussed in this thesis for the first time. For all of the four cases with singular orbit $\mathbb{CP}^2$, we have interpreted the free parameter(s) in the context of Eschenburg's and Wang's work [32] and have proven that there are no further free parameters. Moreover, we have shown an existence and smoothness result for these metrics. For the case $SU(3)/U(1)^2$, we have made a similar analysis. All of these results are original work of the author.

7. Cohomogeneity-one Einstein metrics with an Aloff-Wallach space as principal orbit are also studied by Dancer and Wang [29]. In their paper, they consider the cohomogeneity-one Einstein condition as a Hamiltonian system. They assume that its solution has a Painlevé expansion and study the behavior of the cohomogeneity-one Einstein metric for great values of t. If the solution of the system exists for arbitrarily large t, the metric is AC or ALC. If the metric is non-compact and exists for $t < t_0$ only, at least one of the metric functions has a pole of type $O((t_0 - t)^\alpha)$ with $\alpha \in \mathbb{Q}$. The AC- and ALC-cases and the possible poles are classified by the authors. All of their results describe the metric for $t \to \infty$ (or $t \to t_0$) only. We consider the behavior of the Einstein metric near a singular orbit. Therefore it is, except for the global examples which we describe in this remark, unclear if our metrics evolve into those of Dancer and Wang or vice versa.

8. On the quaternionic projective space $\mathbb{HP}^2$, there is a certain isometric action of $U(3)$ which is of cohomogeneity one. Its principal orbit is the Aloff-Wallach space $N^{1,0} \cong (SU(3) \times U(1))/U(1)^2$, where the abelian factor of $SU(3) \times U(1)$ acts by right-multiplication with elements of $\mathrm{Norm}_{SU(3)} U(1)_{1,0}$. The cohomogeneity-one action has two singular orbits, namely $\mathbb{CP}^2$ and S^5. An explicit description of this action can be found in a paper of Püttmann and Rigas [59]. The cohomogeneity-one Einstein metrics in this thesis which have an Aloff-Wallach space as principal orbit and are neither $\mathbb{HP}^2$ nor one of the examples with holonomy $\subseteq \mathrm{Spin}(7)$ are new contributions of the author.

Chapter 6

Parallel Spin(7)-manifolds with reducible principal orbits

6.1 General considerations

Again, let (M, Ω) be a parallel Spin(7)-manifold admitting an action of cohomogeneity one which preserves Ω. In this chapter, we will consider the case where the principal orbit is of type $G/H \times U(1)$, where $G \times U(1)$ is the group acting with cohomogeneity one and $H \subseteq G$ is the isotropy group. Unlike in Chapter 5, we will not try to classify all $G \times U(1)$-invariant parallel Spin(7)-structures on M. Instead, we will focus on the question if any interesting examples exist. In the first section of this chapter, we will consider the equations for the holonomy reduction without specifying G or H. In the second section, we will consider a special reducible principal orbit.

There is one class of examples which is easy to obtain. Let (M', ω) be a parallel G_2-manifold with principal orbit G/H. Then, the Riemannian product $M := M' \times U(1)$ carries a canonical Spin(7)-structure Ω such that (M, Ω) is a parallel Spin(7)-manifold with principal orbit $G/H \times U(1)$. The holonomy of this space is contained in G_2. Our main concern is if there exists any parallel cohomogeneity-one Spin(7)-structure on M which is not of this type. A possible example would be a warped product of M' and $U(1)$. In order to decide this question, we have to take a closer look at the different structures on M.

Since $G/H \times U(1)$ admits a G_2-structure, we can show by the same arguments as in Lemma 3.2.27 that G/H admits a $SU(3)$-structure: Let $u : T_{(p,1)}G/H \times U(1) \to \mathbb{R}^7$ be an element of the G_2-structure which maps tangent vectors in the $U(1)$-direction into the space $\{(x_1, 0, \ldots, 0) | x_1 \in \mathbb{R}\} \subseteq \mathbb{R}^7$. The $SU(3)$-structure on G/H is given by the restrictions $T_pG/H \to \mathbb{R}^6 \cong \{(0, x_2, \ldots, x_7) | x_2, \ldots, x_7 \in \mathbb{R}\}$ of those u. We will describe the relation between the G_2- and the $SU(3)$-structure on the level of tensor fields. First, we choose a frame $(e_1, \ldots, e_7)$ of the G_2-structure on $G/H \times U(1)$. The three-form ω associated to the G_2-structure is as usual given by:

$$\omega = e^{123} + e^{145} - e^{167} + e^{246} + e^{257} + e^{347} - e^{356} .$$

We assume without loss of generality that the direction of the circle $U(1)$ is e_1. Our claim is that $(e_2, e_3, e_4, e_5, e_6, -e_7)$ is a frame of the $SU(3)$-structure on G/H. We therefore replace from now on e_7 by $-e_7$. After this replacement, ω becomes:

$$\omega = e^1 \wedge (e^{23} + e^{45} + e^{67}) + e^{246} - e^{257} - e^{347} - e^{356} .$$

On any manifold equipped with a $SU(3)$-structure, there is a two-form α which becomes symplectic if the $SU(3)$-structure is torsion-free. Furthermore, we have a complex volume form. We will denote its real part by θ. With this notation, we obtain:

$$\omega = e^1 \wedge \alpha + \theta . \tag{6.1}$$

Since we can identify α and the complex volume form by our basis with the standard symplectic and complex volume form on $\mathbb{C}^3$, $(e_2, \ldots, e_7)$ is indeed a frame of the $SU(3)$-structure. By a short calculation, we can compute the following Hodge duals:

$$\begin{aligned} *_{G/H\times U(1)}\theta &= -e^1 \wedge (J^*\theta) \\ *_{G/H\times U(1)}(e^1 \wedge \alpha) &= -\alpha \wedge \alpha \end{aligned}$$

where J is the almost complex structure associated to the $SU(3)$-structure. Since we will consider geometric structures on several different spaces, we will always index the Hodge star and the exterior differential with the space we consider. For the understanding of the above formulas it is important to note that the replacement of e_7 by $-e_7$ has changed the orientation. The Hodge star operator $*_{G/H\times U(1)}$ is defined by the former orientation induced by the G_2-structure. For the four-form associated to the G_2-structure we therefore have:

$$*_{G/H\times U(1)}\omega = -\alpha \wedge \alpha - e^1 \wedge (J^*\theta) . \tag{6.2}$$

We now consider another G_2-structure on a certain space. Let M^o be the union of all principal orbits of M. Furthermore, let $I^o := M^o/(G \times U(1))$. We define $M^{o'}$ as $G/H \times I^o$. Let t be the coordinate of I^o, which we parameterize by arclength. There is a G-invariant G_2-structure on $M^{o'}$ with a frame which is given by $(\frac{\partial}{\partial t}, e_2, e_3, e_4, e_5, e_6, e_7)$. Analogously to formulas (6.1) and (6.2), we obtain for the three-form $\widetilde{\omega}$ and the four-form $*_{M^{o'}}\widetilde{\omega}$ on $M^{o'}$:

$$\begin{aligned} \widetilde{\omega} &= dt \wedge \alpha + \theta \\ *_{M^{o'}}\widetilde{\omega} &= -\alpha \wedge \alpha - dt \wedge (J^*\theta) \end{aligned}$$

By applying $d_{M^{o'}}$ to these equations, we obtain the condition for the holonomy reduction to a subgroup of G_2:

$$\begin{aligned} d_{M^{o\prime}}(dt) \wedge \alpha - dt \wedge \left(d_{G/H}\alpha + dt \wedge \left(\tfrac{\partial}{\partial t}\alpha\right)\right) + d_{G/H}\theta + dt \wedge \left(\tfrac{\partial}{\partial t}\theta\right) &= 0 \\ -2\left(d_{G/H}\alpha\right) \wedge \alpha - 2dt \wedge \left(\tfrac{\partial}{\partial t}\alpha\right) \wedge \alpha - d_{M^{o\prime}}(dt) \wedge (J^*\theta) + dt \wedge d_{G/H}(J^*\theta) & \\ + dt \wedge dt \wedge \left(\tfrac{\partial}{\partial t}J^*\theta\right) &= 0 \end{aligned}$$

As in Convention 3.2.32, we denote the Lie derivative $\mathcal{L}_{\frac{\partial}{\partial t}}$ shortly by $\frac{\partial}{\partial t}$. The above system can be simplified to:

$$\begin{aligned} d_{G/H}\alpha \wedge dt + d_{G/H}\theta - \left(\tfrac{\partial}{\partial t}\theta\right) \wedge dt &= 0 \\ 2\left(d_{G/H}\alpha\right) \wedge \alpha + 2\left(\tfrac{\partial}{\partial t}\alpha\right) \wedge \alpha \wedge dt - d_{G/H}(J^*\theta) \wedge dt &= 0 \end{aligned}$$

We split the above equations into those terms which contain an dt and those which do not contain a dt:

$$\begin{aligned} d_{G/H}\theta &= 0 \\ \tfrac{\partial}{\partial t}\theta &= d_{G/H}\alpha \\ d_{G/H}\alpha \wedge \alpha &= 0 \\ 2\left(\tfrac{\partial}{\partial t}\alpha\right) \wedge \alpha &= d_{G/H}(J^*\theta) \end{aligned}$$

Our aim is to compare these equations with the equation $d_{M^o}\Omega = 0$, which is equivalent to:

$$\begin{aligned} \tfrac{\partial}{\partial t} *_{G/H\times U(1)} \omega &= d_{G/H\times U(1)}\omega \\ d_{G/H\times U(1)} *_{G/H\times U(1)} \omega &= 0 \end{aligned}$$

There is a canonical G-equivariant diffeomorphism $M^{o'} \times U(1) \to M^o$. We will therefore identify those spaces. By inserting the equations (6.1) and (6.2), we obtain:

$$\begin{aligned} -2\left(\tfrac{\partial}{\partial t}\alpha\right) \wedge \alpha - \left(\tfrac{\partial}{\partial t}e^1\right) \wedge (J^*\theta) - e^1 \wedge \left(\tfrac{\partial}{\partial t}(J^*\theta)\right) &= \\ d_{G/H\times U(1)}e^1 \wedge \alpha - e^1 \wedge d_{G/H\times U(1)}\alpha + d_{G/H\times U(1)}\theta & \end{aligned} \tag{6.3}$$

$$-2\left(d_{G/H\times U(1)}\alpha\right) \wedge \alpha - d_{G/H\times U(1)}e^1 \wedge (J^*\theta) + e^1 \wedge d_{G/H\times U(1)}(J^*\theta) = 0 \tag{6.4}$$

There exists a homogeneous vector field on $G/H \times U(1)$ which is directed along the circle $U(1)$ and coincides with e_1 at the point we consider. In the above formulas, we identify e_1 with this vector field. The bundle of all k-forms has the following splitting:

$$\bigwedge^k T^*(G/H \times U(1)) = \left(\mathrm{span}(e^1) \wedge \bigwedge^{k-1} T^*G/H\right) \oplus \bigwedge^k T^*G/H .$$

We denote the projection onto the second summand by π_2. Analogously, we denote the projection onto the first summand by p_1 and define π_1 by

$$e^1 \wedge \pi_1(\beta) := p_1(\beta) \quad \forall \beta \in \bigwedge^* T^*(G/H \times U(1)) .$$

Our next step is to decompose the equations (6.3) and (6.4) with respect to the above splitting. Let β be an arbitrary k-form on G/H which we extend by the $U(1)$-action to $G/H \times U(1)$. We have:

$$d_{G/H\times U(1)}\beta = d_{G/H}\beta + e^1 \wedge (\mathcal{L}_{e_1}\beta) .$$

Furthermore, we have:

$$\mathcal{L}_{e_1}(\beta(X_1,\ldots,X_k)) = (\mathcal{L}_{e_1}\beta)(X_1,\ldots,X_k) + \sum_{i=1}^{k} \beta(X_1,\ldots,[e_1,X_i],\ldots,X_k) .$$

We choose the X_i as vector fields on G/H which we extend by the action of $U(1)$ to vector fields on all of $G/H \times U(1)$. Since $G/H \times U(1)$ is a Cartesian product, the commutator $[e_1, X_i]$ vanishes. Furthermore, $\beta(X_1,\ldots,X_k)$ is constant in the e_1-direction. These considerations show that $\mathcal{L}_{e_1}\beta = 0$ and therefore

$$d_{G/H\times U(1)}\beta = d_{G/H}\beta ,$$

or equivalently $\pi_1(d_{G/H\times U(1)}\beta) = 0$. Let q be a normal metric on $G/H \times U(1)$. In the next two formulas, $\widetilde{e}_1$ denotes the vector which is with respect to q dual to e^1. We have:

$$d_{G/H\times U(1)}e^1(\widetilde{e}_1, e_j^*) = -q(\widetilde{e}_1, [\widetilde{e}_1, e_j^*]) = q([\widetilde{e}_1, \widetilde{e}_1], e_j^*) = 0$$

and therefore $\pi_1(d_{G/H\times U(1)}e^1) = 0$. Now let $j, k > 1$. We have:

$$d_{G/H\times U(1)}e^1(e_j^*, e_k^*) = q(\widetilde{e}_1, [e_j, e_k]^*) = 0 .$$

All in all, we have shown that $d_{G/H\times U(1)}e^1 = 0$. It is easy to see that $\left(\frac{\partial}{\partial t}e^1\right)(e_j) = \left(\frac{\partial}{\partial t}g\right)(e_1, e_j)$. Since we have made no assumptions on the Spin(7)-structure or the metric, we do not assume that $\pi_1(\frac{\partial}{\partial t}e^1) = 0$ or $\pi_2(\frac{\partial}{\partial t}e^1) = 0$. Now we are able to split the equations (6.3) and (6.4):

$$\begin{aligned}
-\pi_1\left(\tfrac{\partial}{\partial t}e^1\right)e^1\wedge(J^*\theta)-e^1\wedge\left(\tfrac{\partial}{\partial t}(J^*\theta)\right) &= -e^1\wedge d_{G/H}\alpha\\
-2\left(\tfrac{\partial}{\partial t}\alpha\right)\wedge\alpha-\pi_2\left(\tfrac{\partial}{\partial t}e^1\right)\wedge(J^*\theta) &= d_{G/H}\theta\\
e^1\wedge d_{G/H}(J^*\theta) &= 0\\
-2\left(d_{G/H}\alpha\right)\wedge\alpha &= 0
\end{aligned}$$

We can simplify this to:

$$\begin{aligned}
\pi_1\left(\tfrac{\partial}{\partial t}e^1\right)(J^*\theta)+\left(\tfrac{\partial}{\partial t}(J^*\theta)\right) &= d_{G/H}\alpha\\
-2\left(\tfrac{\partial}{\partial t}\alpha\right)\wedge\alpha-\pi_2\left(\tfrac{\partial}{\partial t}e^1\right)\wedge(J^*\theta) &= d_{G/H}\theta\\
d_{G/H}(J^*\theta) &= 0\\
(d_{G/H}\alpha)\wedge\alpha &= 0
\end{aligned}\tag{6.5}$$

We now consider the case where M is a Riemannian product of a circle and a G_2-manifold. In this situation, we have $\frac{\partial}{\partial t}e^1=0$. Therefore, the above equations become:

$$\begin{aligned}
\tfrac{\partial}{\partial t}\left(J^*\theta\right) &= d_{G/H}\alpha\\
-2\left(\tfrac{\partial}{\partial t}\alpha\right)\wedge\alpha &= d_{G/H}\theta\\
d_{G/H}\left(J^*\theta\right) &= 0\\
(d_{G/H}\alpha)\wedge\alpha &= 0
\end{aligned}$$

Next, we consider the $SU(3)$-structure which is determined by the metric g, the two-form α, and the complex volume-form whose real part is $\widetilde{\theta}:=J^*\theta$. The subgroup of $U(3)$ which leaves the complex volume-form invariant is in both cases the same. Therefore, the new $SU(3)$-structure coincides with the original $SU(3)$-structure. Since $J^*\widetilde{\theta}=-\theta$, the above equations become:

$$\begin{aligned}
\tfrac{\partial}{\partial t}\widetilde{\theta} &= d_{G/H}\alpha\\
2\left(\tfrac{\partial}{\partial t}\alpha\right)\wedge\alpha &= d_{G/H}\left(J^*\widetilde{\theta}\right)\\
d_{G/H}\widetilde{\theta} &= 0\\
(d_{G/H}\alpha)\wedge\alpha &= 0
\end{aligned}$$

These are exactly the conditions for the torsion-freeness of the G_2-structure on $M^{o'}$. We have proven the following theorem:

Theorem 6.1.1. *Let G/H be a six-dimensional homogeneous space admitting a G-invariant $SU(3)$-structure. We assume that there is an interval $I\subseteq\mathbb{R}$ and a parallel G-invariant*

G_2-structure on $N := G/H \times I$. We define $M := N \times U(1)$ where "$\times$" denotes the Riemannian product. In this situation, there exists a canonical parallel $G \times U(1)$-invariant Spin(7)-structure on M.

Conversely, let $N \times S^1$ be a Riemannian product equipped with a parallel Spin(7)-structure which is preserved by a cohomogeneity-one action by a compact, connected Lie group $G \times U(1)$. We assume that the $U(1)$-factor acts canonically on S^1. Then there exists a G-invariant G_2-structure on N, which is parallel.

Remark 6.1.2. The above theorem and the equations (6.5) do not forbid the existence of parallel cohomogeneity-one Spin(7)-structures which are not Riemannian products with a circle. In that case, the G_2-structure on $M^{o'}$ has to have non-vanishing intrinsic torsion. Our hope is that if we choose a G_2-structure on $M^{o'}$ with a certain intrinsic torsion, we can extend it to a parallel Spin(7)-structure on M. Because of Theorem 6.1.1, this Spin(7)-structure will not be a Riemannian product. Unfortunately, we are not able to easily deduce a theorem of the following type:

"If the G_2-structure on $M^{o'} = G/H \times I^o$ has the following intrinsic torsion, then there exists a parallel Spin(7)-structure on the Cartesian product $M^o = M^{o'} \times U(1)$: ..."

The following considerations illustrate the difficulties we face: Let the G_2-structure $\widetilde{\omega}$ on $M^{o'}$ be nearly parallel. The equations $d\widetilde{\omega} = \lambda * \widetilde{\omega}$ and $d * \widetilde{\omega} = 0$ can be written as:

$$\begin{array}{rcl} d_{G/H}\theta &=& -\lambda\alpha \wedge \alpha \\ d_{G/H}\alpha &=& \frac{\partial}{\partial t}\theta + \lambda J^*\theta \\ (d_{G/H}\alpha) \wedge \alpha &=& 0 \\ 2\left(\frac{\partial}{\partial t}\alpha\right) \wedge \alpha &=& d_{G/H}J^*\theta \end{array} \tag{6.6}$$

Now, we consider the equation $d\Omega = 0$ on M. We assume that $\pi_2\left(\frac{\partial}{\partial t}e^1\right)$ vanishes and that $\pi_1\left(\frac{\partial}{\partial t}e^1\right)$ is a constant function which we denote by λ, too. Then, we have:

$$\begin{array}{rcl} \lambda\widetilde{\theta} + \frac{\partial}{\partial t}\widetilde{\theta} &=& d_{G/H}\alpha \\ 2\left(\frac{\partial}{\partial t}\alpha\right) \wedge \alpha &=& d_{G/H}J^*\widetilde{\theta} \\ d_{G/H}\widetilde{\theta} &=& 0 \\ (d_{G/H}\alpha) \wedge \alpha &=& 0 \end{array} \tag{6.7}$$

If $\lambda \neq 0$, (6.6) and (6.7) are not equivalent. If we make other natural assumptions on $\frac{\partial}{\partial t}e^1$, we do not get a system of type (6.6), either. For other choices of the intrinsic torsion of $\widetilde{\omega}$, we face similar difficulties.

In order to decide if there are examples of parallel cohomogeneity-one Spin(7)-structures with a principal orbit of type $G/H \times U(1)$, whose holonomy is not contained in G_2, we will consider a concrete example.

6.2 The principal orbit $SU(3)/U(1)^2 \times U(1)$

For our considerations, we choose $SU(3)/U(1)^2 \times U(1)$ as principal orbit. In the paper of Cleyton and Swann [20], cohomogeneity-one G_2-structures with different principal orbits have been treated in great detail. Since the principal orbit $SU(3)/U(1)^2$ is one of the most interesting ones, our choice of the principal orbit seems promising, too. We assume without loss of generality that $U(1)^2 \subseteq SU(3)$ is the maximal torus whose Lie algebra is given by:

$$2\mathfrak{u}(1) := \left\{ \left(\begin{array}{ccc} ix & 0 & 0 \\ 0 & iy & 0 \\ 0 & 0 & iz \end{array} \right) \middle| \, x, y, z \in \mathbb{R}, \; x + y + z = 0 \right\} .$$

$SU(3)/U(1)^2 \times U(1)$ and the Aloff-Wallach spaces are both of type $(SU(3) \times U(1))/U(1)^2$. Since $\pi_1(SU(3)/U(1)^2 \times U(1)) = \mathbb{Z}$ and $\pi_1(N^{k,l}) = \{0\}$, the principal orbit we consider is not an Aloff-Wallach space. The aim of this section is to study the parallel Spin(7)-structures of cohomogeneity one with $SU(3)/U(1)^2 \times U(1)$ as principal orbit and to compare them with the parallel cohomogeneity-one G_2-structures with principal orbit $SU(3)/U(1)^2$. In order to do this, we will briefly review the results of [20] on the $SU(3)/U(1)^2$-case. First, we choose an appropriate basis of $\mathfrak{su}(3)$. We define $e_1, \ldots, e_6 \in \mathfrak{su}(3)$ as in Section 5.4 and define

$$e_7 := \left(\begin{array}{ccc} i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & 0 \end{array} \right) \quad e_8 := \left(\begin{array}{ccc} i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -2i \end{array} \right) .$$

As usual, we define a biinvariant background metric q on $\mathfrak{su}(3)$ by $q(X,Y) := -\mathrm{tr}(XY)$ and denote the q-orthogonal complement of $2\mathfrak{u}(1) \subseteq \mathfrak{su}(3)$ by $\mathfrak{m}$. Since the basis $(e_1, \ldots, e_8)$ is orthogonal with respect to q, $(e_1, \ldots, e_6)$ spans $\mathfrak{m}$ and (e_7, e_8) spans $2\mathfrak{u}(1)$. The matrix representation of $\mathrm{ad}_{e_7}|_{\mathfrak{m}}$ and $\mathrm{ad}_{e_8}|_{\mathfrak{m}}$ with respect to the basis $(e_1, \ldots, e_6)$ is, as in Section 5.4, given by:

$$\mathrm{ad}_{e_7}|_{\mathfrak{m}} = \left(\begin{array}{cccccc} 0 & -2 & & & & \\ 2 & 0 & & & & \\ & & 0 & -1 & & \\ & & 1 & 0 & & \\ & & & & 0 & 1 \\ & & & & -1 & 0 \end{array} \right) \quad \mathrm{ad}_{e_8}|_{\mathfrak{m}} = \left(\begin{array}{cccccc} 0 & 0 & & & & \\ 0 & 0 & & & & \\ & & 0 & -3 & & \\ & & 3 & 0 & & \\ & & & & 0 & -3 \\ & & & & 3 & 0 \end{array} \right) .$$

$\mathfrak{m}$ decomposes therefore into the following $2\mathfrak{u}(1)$-modules:

$$\begin{aligned} V_1 &:= \mathrm{span}(e_1, e_2) \\ V_2 &:= \mathrm{span}(e_3, e_4) \\ V_3 &:= \mathrm{span}(e_5, e_6) \end{aligned}$$

e_7 and e_8 act with different weights on the modules V_i. Therefore, V_1, V_2, and V_3 are pairwise inequivalent. By the same arguments as in Chapter 5 we can show that the matrix representation of g with respect to the basis $(e_1, \ldots, e_6)$ has to be of the following type:

$$\begin{pmatrix} \boxed{\begin{matrix} a^2 & 0 \\ 0 & a^2 \end{matrix}} & & \\ & \boxed{\begin{matrix} b^2 & 0 \\ 0 & b^2 \end{matrix}} & \\ & & \boxed{\begin{matrix} c^2 & 0 \\ 0 & c^2 \end{matrix}} \end{pmatrix} \quad \text{with } a, b, c \in \mathbb{R}\backslash\{0\} .$$

Conversely, any such matrix defines a $SU(3)$-invariant metric on $SU(3)/U(1)^2$. Next, we construct a frame of an invariant $SU(3)$-structure on $SU(3)/U(1)^2$. If we first interchange (e_1, e_2) and then (e_5, e_6), and furthermore identify $\mathbb{R}^6$ with $\mathbb{C}^3$, the matrix representation of $\mathrm{ad}_{e_7}|_{\mathfrak{m}}$ and $\mathrm{ad}_{e_8}|_{\mathfrak{m}}$ becomes:

$$\mathrm{ad}_{e_7}|_{\mathfrak{m}} = \begin{pmatrix} -2i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & i \end{pmatrix} \quad \text{and} \quad \mathrm{ad}_{e_8}|_{\mathfrak{m}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3i & 0 \\ 0 & 0 & -3i \end{pmatrix} .$$

Since both of these matrices are contained in $\mathfrak{su}(3)$, a frame $(h_1, \ldots, h_6)$ of an invariant $SU(3)$-structure on $SU(3)/U(1)^2$ whose associated metric is g is therefore given by:

$$\begin{aligned} &h_1 := \tfrac{1}{a} e_2 \quad h_2 := \tfrac{1}{a} e_1 \quad h_3 := \tfrac{1}{b} e_3 \\ &h_4 := \tfrac{1}{b} e_4 \quad h_5 := \tfrac{1}{c} e_6 \quad h_6 := \tfrac{1}{c} e_5 \end{aligned} \tag{6.8}$$

Our next question is if this $SU(3)$-structure is unique. More concretely, we want to determine the set of all invariant $SU(3)$-structures whose extension to a $SO(6)$-structure is the same. We can show by the same arguments as in the proof of Lemma 3.1.50 that this set can be generated by the action of

$$\{\phi \in SO(6) | \phi U(1)^2 \phi^{-1} \subseteq SU(3)\}$$

on the frame $(h_1, \ldots, h_6)$. Any element of $SU(3)$ is conjugate to an element of the maximal torus $U(1)^2$. Since the action of $SU(3)$ leaves the $SU(3)$-structure invariant, the above set can be replaced by

$$\{\phi \in SO(6) | \phi U(1)^2 \phi^{-1} \subseteq U(1)^2\}$$

or equivalently by $\mathrm{Norm}_{SO(6)} U(1)^2$. Let

$$U(1)^3 := \left\{ \begin{pmatrix} \boxed{\begin{matrix} \cos\theta_1 & \sin\theta_1 \\ -\sin\theta_1 & \cos\theta_1 \end{matrix}} & & \\ & \boxed{\begin{matrix} \cos\theta_2 & \sin\theta_2 \\ -\sin\theta_2 & \cos\theta_2 \end{matrix}} & \\ & & \boxed{\begin{matrix} \cos\theta_3 & \sin\theta_3 \\ -\sin\theta_3 & \cos\theta_3 \end{matrix}} \end{pmatrix} \middle| \, \theta_1, \theta_2, \theta_3 \in \mathbb{R} \right\}$$

be the standard maximal torus of $SO(6)$. $U(1)^2 \subseteq SU(3) \subseteq SO(6)$ is the set of all elements of $U(1)^3$ satisfying $\theta_1+\theta_2+\theta_3 = 0$. By a short calculation, we can see that the only 6×6-matrices ϕ with $\phi\, U(1)^2 \subseteq U(1)^3\phi$ are of type

$$\begin{pmatrix} \boxed{\begin{matrix} x_1 & y_1 \\ -y_1 & x_1 \end{matrix}} & & \\ & \boxed{\begin{matrix} x_2 & y_2 \\ -y_2 & x_2 \end{matrix}} & \\ & & \boxed{\begin{matrix} x_3 & y_3 \\ -y_3 & x_3 \end{matrix}} \end{pmatrix}$$

All those ϕ also satisfy $\phi\, U(1)^2 = U(1)^2\phi$. Since we only search for matrices in $SO(6)$, we obtain $\mathrm{Norm}_{SO(6)}U(1)^2 = U(1)^3$. The action of $U(1)^2$ on $(h_1,\ldots,h_6)$ leaves the $SU(3)$-structure invariant. Therefore, the set of all $SU(3)$-structures is generated by a $U(1)$-action. We can choose the group acting on $(h_1,\ldots,h_6)$ as the set of all matrices in $U(1)^3$ with $\theta_1 = \theta_2 = \theta_3 =: \theta$. A frame of the most general invariant $SU(3)$-structure on $SU(3)/U(1)^2$ therefore is:

$$\begin{aligned} h_1 &:= \tfrac{1}{a}\sin\theta\, e_1 + \tfrac{1}{a}\cos\theta\, e_2 & h_2 &:= \tfrac{1}{a}\cos\theta\, e_1 - \tfrac{1}{a}\sin\theta\, e_2 \\ h_3 &:= \tfrac{1}{b}\cos\theta\, e_3 + \tfrac{1}{b}\sin\theta\, e_4 & h_4 &:= -\tfrac{1}{b}\sin\theta\, e_3 + \tfrac{1}{b}\cos\theta\, e_4 \\ h_5 &:= \tfrac{1}{c}\sin\theta\, e_5 + \tfrac{1}{c}\cos\theta\, e_6 & h_6 &:= \tfrac{1}{c}\cos\theta\, e_5 - \tfrac{1}{c}\sin\theta\, e_6 \end{aligned} \tag{6.9}$$

with $\theta \in \mathbb{R}$. Let M be a cohomogeneity-one manifold with principal orbit $SU(3)/U(1)^2$. As we have seen in the first section of this chapter, we can obtain a frame of an invariant G_2-structure on M by changing the sign of h_6 and supplementing our basis with $\frac{\partial}{\partial t}$. We obtain:

$$\begin{aligned} f_1 &:= \tfrac{\partial}{\partial t} & f_2 &:= \tfrac{1}{a}\sin\theta\, e_1 + \tfrac{1}{a}\cos\theta\, e_2 \\ f_3 &:= \tfrac{1}{a}\cos\theta\, e_1 - \tfrac{1}{a}\sin\theta\, e_2 & f_4 &:= \tfrac{1}{b}\cos\theta\, e_3 + \tfrac{1}{b}\sin\theta\, e_4 \\ f_5 &:= -\tfrac{1}{b}\sin\theta\, e_3 + \tfrac{1}{b}\cos\theta\, e_4 & f_6 &:= \tfrac{1}{c}\sin\theta\, e_5 + \tfrac{1}{c}\cos\theta\, e_6 \\ f_7 &:= -\tfrac{1}{c}\cos\theta\, e_5 + \tfrac{1}{c}\sin\theta\, e_6 & & \end{aligned} \tag{6.10}$$

The three-form which is associated to this G_2-structure we denote by ω_θ. We obtain:

$$\begin{aligned} \omega_\theta = {} & abc\,\cos 3\theta(-e^{135} - e^{146} + e^{236} - e^{245}) - abc\,\sin 3\theta(e^{136} - e^{145} + e^{235} + e^{246}) \\ & - a^2\, e^{12} \wedge dt + b^2\, e^{34} \wedge dt - c^2\, e^{56} \wedge dt\,. \end{aligned} \tag{6.11}$$

Furthermore, we have:

$$\begin{aligned} *\omega_\theta &= a^2b^2\, e^{1234} - a^2c^2\, e^{1256} + b^2c^2\, e^{3456} \\ &\quad - abc\, \cos 3\theta (e^{136} - e^{145} + e^{235} + e^{246}) \wedge dt \\ &\quad - abc\, \sin 3\theta (-e^{135} - e^{146} + e^{236} - e^{245}) \wedge dt \end{aligned} \tag{6.12}$$

Next, we determine $d\omega_\theta$ and $d * \omega_\theta$. Since $d\omega_\theta$ contains terms of type $4abc\, \cos 3\theta\, e^{ijkl}$, $\cos 3\theta$ has to vanish if we want the G_2-structure to be parallel. As we will see in Appendix D, it is possible to assume without loss of generality that $\theta = \frac{\pi}{6}$. Therefore, the equations for the reduction of the holonomy to a subgroup of G_2 do not contain θ anymore. These equations are equivalent to:

$$\begin{aligned} \frac{a'}{a} &= \frac{b^2 + c^2 - a^2}{abc} \\ \frac{b'}{b} &= \frac{a^2 + c^2 - b^2}{abc} \\ \frac{c'}{c} &= \frac{a^2 + b^2 - c^2}{abc} \end{aligned}$$

The detailed calculations and arguments needed for the derivation of these equations can be found in Appendix D. We now show that the metrics we obtain as solutions of the above system have holonomy G_2. In order to do this, we assume that the holonomy is smaller. In that case, the holonomy is a subgroup of $SU(3)$ and there exists a one-dimensional subspace of the tangent space on which the holonomy acts trivially. The holonomy bundle of $SU(3)/U(1)^2$ is $SU(3)$-invariant, since the metric is invariant. Furthermore, the subbundle of the tangent bundle on which the holonomy acts trivially, is invariant, too. Its fibers have to be of positive dimension. We first consider the case where the fibers are one-dimensional. Since any submodule of $\mathfrak{m}$ is at least two-dimensional, the fiber of the bundle is span$(\frac{\partial}{\partial t})$. In this situation, the manifold has to be a Riemannian product of $SU(3)/U(1)^2$ and an interval. Therefore, we have $a' = b' = c' = 0$. By adding the above three equations up we obtain $a = b = c = 0$, which is impossible. Next, we assume that the space on which the holonomy acts trivially is at least two-dimensional. That space has to contain V_1, V_2, or V_3 as a subspace. We can conclude with help of the de Rham decomposition theorem that $SU(3)/U(1)^2$ is a product, which has a two-dimensional torus as factor. Since $\pi_1(SU(3)/U(1)^2) = \{0\}$, we can exclude this case, too.

Next, we construct parallel cohomogeneity-one Spin(7)-structures with principal orbit $SU(3)/U(1)^2 \times U(1)$. In order to do this, we choose a basis of $\mathfrak{su}(3) \oplus \mathfrak{u}(1)$. Let $i : \mathfrak{su}(3) \to \mathfrak{su}(3) \oplus \mathfrak{u}(1)$ be the canonical embedding. The family

$$i(e_1), \ldots, i(e_6), \begin{pmatrix} 0 & 0 & 0 & \\ 0 & 0 & 0 & \\ 0 & 0 & 0 & \\ & & & i \end{pmatrix}, i(e_7), i(e_8)$$

is a basis of $\mathfrak{su}(3) \oplus \mathfrak{u}(1)$. For reasons of simplicity, we denote this new basis by $(e_1, \ldots, e_9)$. span$(e_1, \ldots, e_7)$ can be identified with the tangent space of $SU(3)/U(1)^2 \times U(1)$ at a point

p and (e_8, e_9) spans $2\mathfrak{u}(1) \subseteq \mathfrak{su}(3)$. The tangent space splits into the following irreducible $2\mathfrak{u}(1)$-modules:

$$\begin{aligned} V_1 &:= \mathrm{span}(e_1, e_2) \\ V_2 &:= \mathrm{span}(e_3, e_4) \\ V_3 &:= \mathrm{span}(e_5, e_6) \\ V_4 &:= \mathrm{span}(e_7) \end{aligned}$$

Since $V_1, \dots, V_4$ are pairwise inequivalent, any $SU(3) \times U(1)$-invariant metric g on the principal orbit has the following matrix representation with respect to $(e_1, \dots, e_7)$:

$$\begin{pmatrix} a^2 & 0 & & & & & \\ 0 & a^2 & & & & & \\ & & b^2 & 0 & & & \\ & & 0 & b^2 & & & \\ & & & & c^2 & 0 & \\ & & & & 0 & c^2 & \\ & & & & & & f^2 \end{pmatrix} \quad \text{with } a, b, c, f \in \mathbb{R}\backslash\{0\} .$$

As in Section 6.1, we obtain a three- and a four-form associated to an invariant G_2-structure by replacing dt by fe^7 in (6.11) and (6.12). Therefore, we have found a family of G_2-structures depending on a parameter θ. Since we do not want to introduce a new notation, we denote the three- and the four-form associated to these G_2-structures by the same symbols as before. We obtain:

$$\begin{aligned} \omega_\theta &= abc\, \cos 3\theta(-e^{135} - e^{146} + e^{236} - e^{245}) - abc\, \sin 3\theta(e^{136} - e^{145} + e^{235} + e^{246}) \\ &\quad - a^2 f\, e^{127} + b^2 f\, e^{347} - c^2 f\, e^{567} \\ *\omega_\theta &= a^2 b^2\, e^{1234} - a^2 c^2\, e^{1256} + b^2 c^2\, e^{3456} \\ &\quad - abcf\, \cos 3\theta(e^{1367} - e^{1457} + e^{2357} + e^{2467}) \\ &\quad - abcf\, \sin 3\theta(-e^{1357} - e^{1467} + e^{2367} - e^{2457}) \end{aligned}$$

with $\theta \in \mathbb{R}$. We can see by the same arguments as for the principal orbit $SU(3)/U(1)^2$ that these are all possible invariant G_2-structures. The evolution equation for the holonomy reduction is:

$$\frac{\partial}{\partial t} * \omega_\theta = d\omega_\theta .$$

Furthermore, the G_2-structure on the principal orbit has to satisfy:

$$d * \omega_\theta = 0 .$$

Let e^{7*} be the one-form defined by $e^{7*}(e_i^*) := \delta_i^7$. Since e_7 commutes with $\mathfrak{su}(3)$, we have $de_p^{7*} = 0$. Therefore, we obtain by a calculation, which is analogous to the one on page 268:

$$d * \omega_\theta = -4abcf\ \sin 3\theta (e^{12347} - e^{12567} + e^{34567})\ .$$

Since we want $d * \omega_\theta$ to vanish, θ has to be in $\frac{\pi}{3}\mathbb{Z}$. Next, we determine the system of ordinary differential equations for the holonomy reduction. The exterior derivative of the three-form ω_θ is:

$$\begin{aligned} d\omega_\theta &= 4abc\ \cos 3\theta (e^{1234} - e^{1256} + e^{3456}) \\ &\quad - (a^2 f + b^2 f + c^2 f)(e^{1367} - e^{1457} + e^{2357} + e^{2467})\ . \end{aligned}$$

Furthermore, we have:

$$\begin{aligned} \frac{\partial}{\partial t} * \omega_\theta = &\left(2\frac{a'}{a} + 2\frac{b'}{b}\right) a^2 b^2\, e^{1234} - \left(2\frac{a'}{a} + 2\frac{c'}{c}\right) a^2 c^2\, e^{1256} + \left(2\frac{b'}{b} + 2\frac{c'}{c}\right) b^2 c^2\, e^{3456} \\ &- \left(\left(\frac{a'}{a} + \frac{b'}{b} + \frac{c'}{c} + \frac{f'}{f}\right) \cos 3\theta - 3\theta' \sin 3\theta\right) abcf\, (e^{1367} - e^{1457} + e^{2357} + e^{2467}) \\ &- \left(\left(\frac{a'}{a} + \frac{b'}{b} + \frac{c'}{c} + \frac{f'}{f}\right) \sin 3\theta + 3\theta' \cos 3\theta\right) abcf\, (-e^{1357} - e^{1467} + e^{2367} - e^{2457}) \end{aligned}$$

Therefore, the equations for the holonomy reduction are given by:

$$\begin{aligned} \frac{a'}{a} + \frac{b'}{b} &= 2\frac{c}{ab} \cos 3\theta \\ \frac{a'}{a} + \frac{c'}{c} &= 2\frac{b}{ac} \cos 3\theta \\ \frac{b'}{b} + \frac{c'}{c} &= 2\frac{a}{bc} \cos 3\theta \\ \left(\frac{a'}{a} + \frac{b'}{b} + \frac{c'}{c} + \frac{f'}{f}\right) \cos 3\theta - 3\theta' \sin 3\theta &= \frac{a^2 + b^2 + c^2}{abc} \\ \left(\frac{a'}{a} + \frac{b'}{b} + \frac{c'}{c} + \frac{f'}{f}\right) \sin 3\theta + 3\theta' \cos 3\theta &= 0 \end{aligned}$$

For analogous reasons as in the case where the principal orbit is $SU(3)/U(1)^2$, we can assume without loss of generality that $\theta = 0$. By adding the first three equations up, we obtain:

$$\frac{a'}{a} + \frac{b'}{b} + \frac{c'}{c} = \frac{a^2 + b^2 + c^2}{abc}\ .$$

With the help of this relation, the above equations reduce to:

$$\begin{aligned}
\frac{a'}{a} &= \frac{b^2+c^2-a^2}{abc}\\
\frac{b'}{b} &= \frac{a^2+c^2-b^2}{abc}\\
\frac{c'}{c} &= \frac{a^2+b^2-c^2}{abc}\\
\frac{f'}{f} &= 0
\end{aligned}$$

The first three of these equations are exactly the equations we have obtained in the $SU(3)/U(1)^2$-case. Since $f' = 0$, the only examples we obtain for the principal orbit $SU(3)/U(1)^2 \times U(1)$ are Riemannian products of a circle and a parallel cohomogeneity-one G_2-manifold:

Theorem 6.2.1. *Let (M, Ω) be a parallel Spin(7)-manifold with a $SU(3) \times U(1)$-action of cohomogeneity one which preserves Ω. We assume that the stabilizer group of the action on the principal orbit is $U(1)^2 \subseteq SU(3)$. In this situation, M is the Riemannian product of a circle and a parallel G_2-manifold of cohomogeneity one.*

Although we have obtained only negative results in this chapter, we still hope that there are new examples which are not of this simple type. One reason for our optimism is found in a paper of Gukov, Yau, and Zaslow [40]. In that paper, the authors construct parallel cohomogeneity-one G_2-manifolds whose principal orbit is $SU(2) \times U(1)^3$. The metrics they obtain are not Riemannian products of a circle and a manifold with holonomy contained in $SU(3)$. Therefore, we hope that in the Spin(7)-case there also exist examples with the desired properties.

Chapter 7

Conclusion and outlook

We shortly recapitulate what we have done. One aim of this thesis was to find a partial classification result on parallel Spin(7)-manifolds of cohomogeneity one and many of our results indeed include some kind of classification.

In Theorem 4.2, we have listed all possible principal orbits of cohomogeneity-one Spin(7)-manifolds under some mild restrictions. Some of those orbits we have considered in detail and have found further classification results.

We first have taken a look at the Berger space B^7 as principal orbit. In this case, the only parallel cohomogeneity-one Spin(7)-manifold is the cone over B^7. The reason for this is that B^7 is an isotropy-irreducible homogeneous space.

In Section 5.2 and 5.3, we also have described all parallel cohomogeneity-one Spin(7)-manifolds (and -orbifolds) whose principal orbit is $Q^{1,1,1}$ or $M^{1,1,0}$ and which have a singular orbit. The following facts have simplified our arguments: First, all homogeneous metrics on those orbits are diagonal. This makes the space of all metrics which we had to consider sufficiently small. Second, the space of all G_2-structures on the principal orbit which have the same associated metric and orientation is diffeomorphic to a circle. We therefore did not have to make any simplifying assumptions on the cohomogeneity-one Spin(7)-structure. Third, the equations for the holonomy reduction can be solved explicitly, which was helpful, too. All of our examples have been considered in the literature before (see Berard-Bergery [8]; Stenzel [64]; Page, Pope [58]; Herzog, Klebanov [41]; Cvetič et al. [21], [25]). Nevertheless, our classification result has until now not been stated explicitly and we have introduced a new point of view on those metrics.

The case where the principal orbit is a product of a circle and another homogeneous space has been shortly investigated, too. The only principal orbit which we have explored in detail is $SU(3)/U(1)^2 \times U(1)$. In this case, we also have been able to classify all homogeneous G_2-structures on the principal orbit. Furthermore, we have seen that in our situation the only cohomogeneity-one metrics whose holonomy is contained in Spin(7) are Riemannian products of a circle and a parallel cohomogeneity-one G_2-manifold. Since that kind of manifolds is classified by Cleyton and Swann [20], an analogous result for our case follows.

In the case where the principal orbit is an Aloff-Wallach space $N^{k,l}$, we had to make some simplifications. On the two exceptional Aloff-Wallach spaces $N^{1,0}$ and $N^{1,1}$, there exist non-diagonal $SU(3)$-invariant metrics. We nevertheless have always assumed that the metric on the principal orbit is diagonal. For each type of Aloff-Wallach space, we have described the space of all homogeneous G_2-structures whose associated metric and orientation is an arbitrary but fixed one. We have been able to determine the type of the connected components of that space but not their number. Therefore, we have restricted our attention to G_2-structures of a special kind or equivalently to a subset of the connected components of that space. In the case $k = l = 1$, we moreover have considered a one-parameter family of G_2-structures in that space only. Under these restrictions, we have been able to classify all parallel cohomogeneity-one Spin(7)-structures with a singular orbit. In our situation, there is no explicit solution of the equations for the holonomy reduction, except in special cases. Therefore, we could prove the existence of our metrics on a tubular neighborhood of the singular orbit only. Another difference to the other principal orbits is that sometimes our metrics depend on additional initial conditions of higher order. The number and order of those initial conditions we have described, too. We have reproduced the metrics which can be found in the literature (see Cvetič et al. [24]; Gukov, Sparks [39]; Kanno, Yasui [48], [49]; Bazaikin [5]), but also have found new examples.

Although we have proven many results on parallel cohomogeneity-one Spin(7)-manifolds with an Aloff-Wallach space $N^{k,l}$ as principal orbit, there are still many questions left open. One of these questions is if there are any further Spin(7)-manifolds of this kind. In order to find further examples, we either have to consider non-diagonal metrics on $N^{k,l}$ or if the metric is diagonal, at least other G_2-structures which do not satisfy our simplifying assumptions. On the singular orbit, the $SU(3)$-invariance forces the metric in most cases to be diagonal. The only exception is the case where the principal orbit is covered by $N^{1,1}$ and the isotropy algebra of the singular orbit is generated by $\mathrm{diag}(i, i, -2i)$ and an element of $\mathrm{span}(e_1, e_2, e_7)$ which is not an e_i. Nevertheless, it may possible that the metric on the singular orbit becomes non-diagonal as soon as we move away from that orbit. Therefore, the existence of non-diagonal examples cannot be a priori excluded. Calculations which have been made by the author seem to indicate that the G_2-structure on the principal orbit is often not coclosed if the metric is non-diagonal or if the G_2-structure is different from the G_2-structure which we have chosen. If it is possible to make a more general ansatz for the coclosed G_2-structure on $N^{k,l}$, our differential equations contain more functions and are more difficult to handle. Nevertheless, it is possible to prove local existence results by the same methods which we have developed in this thesis.

Another interesting question is if our examples can be extended to complete or compact metrics. The holonomy of a compact parallel Spin(7)-manifold which is not covered by a Riemannian product is determined by its $\hat{A}$-genus. It therefore may be possible to find restrictions on the orbit structure of compact examples by calculating the $\hat{A}$-genus of cohomogeneity-one manifolds with two singular orbits and an Aloff-Wallach space as principal orbit. If we could make a statement on the long term behavior of the solutions of our differential equations, the compactness and completeness of our metrics could be verified. In the paper of Bazaikin [5], methods were introduced which may be helpful to answer these question in further cases, beside those which are treated in [5] or in his forthcoming paper [6]. All of the above questions can be the subject of future research.

In this thesis, we have investigated all irreducible principal orbits, except the Stiefel-manifold $V^{5,2} = SO(5)/SO(3)$ and the sphere $S^7 = Sp(2)/Sp(1)$. Cohomogeneity-one metrics with special holonomy whose principal orbit is covered by one of those spaces have been studied in the literature. On the cotangent bundle T^*S^4 of the four-dimensional sphere, there is a canonical $SO(5)$-action of cohomogeneity one. The principal orbit of this action is $V^{5,2}$ and the zero section is a singular orbit with isotropy group $SO(4)$. On this space, a complete $SO(5)$-invariant metric with holonomy $SU(4)$ is known. That metric is contained in the construction of Stenzel [64] and is explicitly described by Cvetič et al. [25]. The complete metric with holonomy Spin(7) which has been found by Bryant and Salamon [15] is a cohomogeneity-one metric with S^7 as principal and S^4 as singular orbit. Further metrics of a similar kind have been discovered in a series of papers by Cvetič et al. [23], [24], [26], [27]. Among them are cohomogeneity-one manifolds with principal orbit $S^7/\mathbb{Z}_4$ and singular orbit $\mathbb{CP}^3$ as well as manifolds with principal orbit S^7 and singular orbit S^4. The methods of Bazaikin [5] yield further examples of the first kind. All of those metrics are diagonal with respect to a natural choice of the basis of the tangent space. Furthermore, they are all non-compact. For both orbit structures, AC- and ALC-examples do exist. Since much research has been done on the principal orbits $V^{5,2}$ and S^7 (or $S^7/\mathbb{Z}_4$ respectively), we have not investigated them in this thesis.

We will not further review the above results, but will shortly discuss if we can obtain any new examples by considering non-diagonal metrics. For this discussion, we denote the irreducible representations of $\mathfrak{su}(2)$ in the same way as in the rest of this thesis. The tangent space of $V^{5,2}$ splits into $\mathbb{V}_2^{\mathbb{R}} \oplus \mathbb{V}_2^{\mathbb{R}} \oplus \mathbb{V}_0^{\mathbb{R}}$ with respect to the isotropy group $SO(3)$. In the same way, the tangent space of the sphere splits into $\mathbb{V}_1^{\mathbb{C}} \oplus 3\mathbb{V}_0^{\mathbb{R}}$ with respect to $Sp(1)$. Since both splittings contain equivalent summands, there exist non-diagonal homogeneous metrics on both spaces. By a short calculation, we see that $SO(3) \times SO(2)$ and $Sp(1) \times Sp(1)$ are the identity components of the normalizers $\mathrm{Norm}_{SO(5)}SO(3)$ and $\mathrm{Norm}_{Sp(2)}Sp(1)$. The second factor of $SO(3) \times SO(2)$ acts on the second factor of $\mathbb{V}_2^{\mathbb{R}} \otimes \mathbb{R}^2$ by matrix multiplication. The second factor of $Sp(1) \times Sp(1)$ acts as $SO(3)$ on $3\mathbb{V}_0^{\mathbb{R}}$. Any homogeneous metric on $V^{5,2}$ or S^7 can be made diagonal by the action of a suitable element of the normalizer. Let (M, Ω) be a parallel cohomogeneity-one Spin(7)-manifold with $V^{5,2}$ or S^7 as principal orbit. The associated metric on M we denote by g. We consider the restriction of the metric to a fixed principal orbit $\mathcal{O}$. Let φ be a diffeomorphism of $\mathcal{O}$ which is induced by an element of the normalizer and makes the metric diagonal. The simultaneous action of φ on all principal orbits transforms g into a new metric g', which is diagonal on $\mathcal{O}$. If g' satisfies the equations for the holonomy reduction from the literature [24], [25], g' as a whole is diagonal, too. Let ω be the G_2-structure on the principal orbit $\mathcal{O}$ and let $\omega' := \varphi^*\omega$. The only possibility to obtain a non-diagonal example which is not isometric to one of the known ones is if ω' satisfies one of the following two conditions:

1. The metric g' is always diagonal, but satisfies a system of differential equations which is different from the known ones. This can be the case only if ω' is an unknown coclosed G_2-structure on the principal orbit which yields a diagonal metric.

2. The restriction of g' to $\mathcal{O}$ is diagonal, but the equation $\frac{\partial}{\partial t} * \omega' = d\omega'$ changes g' into a non-diagonal metric. In this situation, ω' has to be an unknown coclosed G_2-structure, too.

Although it seems unlikely that such an ω' exists, only an analysis which is similar to that in Section 5.4 can fully answer the question if there are any non-obvious non-diagonal examples.

We finally remark that although there are still interesting questions open, the number of parallel cohomogeneity-one Spin(7)-manifolds is limited. The reason for this is that all possible principal orbits are contained in the table of Theorem 4.2 and that any parallel cohomogeneity-one Spin(7)-structure is determined by the initial values at the singular orbit. Despite the fact that it may be possible to construct compact manifolds with exceptional holonomy by our methods, in the long run one has to turn to other techniques in order to construct such manifolds.

Appendix A

Calculations for the principal orbit $Q^{1,1,1}$

Let (M, Ω) be a parallel Spin(7)-manifold admitting a cohomogeneity-one action of $SU(2)^3$ which preserves Ω. From Theorem 4.2, it follows that the isotropy group of the action on the principal orbit is isomorphic to $U(1)^2$. Furthermore, we have shown in Section 5.2 that the principal orbit is $SU(2)^3$-equivariantly diffeomorphic to $Q^{1,1,1}$. The aim of this chapter is to express the equation $d\Omega = 0$ for the Ω which we have chosen in 5.2 as a set of equations for the metric functions.

Our Ω is defined by the frame (5.4) from page 108. We denote the linear map associated to this basis by $\psi : \mathfrak{m} \oplus \operatorname{span}\left(\frac{\partial}{\partial t}\right) \to \mathbb{R}^8$ and obtain the following $SU(2)^3$-invariant four-form:

$$
\begin{aligned}
\Omega &= \psi^*(dx^{0123} + dx^{0145} - dx^{0167} + dx^{0246} + dx^{0257} + dx^{0347} - dx^{0356} \\
&\quad - dx^{1247} + dx^{1256} + dx^{1346} + dx^{1357} - dx^{2345} + dx^{2367} + dx^{4567}) \\
&= -a^2 f\, e^{712} \wedge dt - b^2 f\, e^{734} \wedge dt + c^2 f\, e^{765} \wedge dt \\
&\quad - abc\, e^{136} \wedge dt - abc\, e^{145} \wedge dt - abc\, e^{235} \wedge dt + abc\, e^{246} \wedge dt \\
&\quad - abcf\, e^{7135} + abcf\, e^{7146} + abcf\, e^{7236} + abcf\, e^{7245} \\
&\quad - a^2b^2\, e^{1234} + a^2c^2\, e^{1265} + b^2c^2\, e^{3465} \\
&= abcf\, e^{1357} - abcf\, e^{1467} - abcf\, e^{2367} - abcf\, e^{2457} \\
&\quad - a^2b^2\, e^{1234} - a^2c^2\, e^{1256} - b^2c^2\, e^{3456} \\
&\quad - a^2 f\, e^{127} \wedge dt - b^2 f\, e^{347} \wedge dt - c^2 f\, e^{567} \wedge dt \\
&\quad - abc\, e^{136} \wedge dt - abc\, e^{145} \wedge dt - abc\, e^{235} \wedge dt + abc\, e^{246} \wedge dt
\end{aligned}
$$

By a straightforward but lengthy calculation we could show that this form is indeed invariant with respect to the action of $SU(2)^3$. In order to obtain the equations for the holonomy reduction we have to determine $d\Omega$. Let $p \in Q^{1,1,1}$ be a point whose tangent space we identify with $\mathfrak{m}$. First, we calculate the exterior derivatives of the one-forms e^{i^*} which satisfy $e^{i^*}(e_j^*) = \delta^i_j$. By Lemma 3.1.44 we obtain:

$$
\begin{aligned}
de_p^{1*} &= -e_p^{27*}\\
de_p^{2*} &= e_p^{17*}\\
de_p^{3*} &= -e_p^{47*}\\
de_p^{4*} &= e_p^{37*}\\
de_p^{5*} &= -e_p^{67*}\\
de_p^{6*} &= e_p^{57*}\\
de_p^{7*} &= -\frac{1}{3}e_p^{12*} - \frac{1}{3}e_p^{34*} - \frac{1}{3}e_p^{56*}
\end{aligned}
$$

We are now able to compute $d\Omega$:

$$
\begin{aligned}
d\Omega = &\left(\frac{a'}{a}+\frac{b'}{b}+\frac{c'}{c}+\frac{f'}{f}\right) abcf\, e^{1357*}\wedge dt - (-abcf\, e^{27357*} + abcf\, e^{14757*} - abcf\, e^{13677*}\\
&+\frac{1}{3}abcf(e^{13512*}+e^{13534*}+e^{13556*}))\\
&-\left(\frac{a'}{a}+\frac{b'}{b}+\frac{c'}{c}+\frac{f'}{f}\right) abcf\, e^{1467*}\wedge dt + (-abcf\, e^{27467*} - abcf\, e^{13767*} + abcf\, e^{14577*}\\
&+\frac{1}{3}abcf(e^{14612*}+e^{14634*}+e^{14656*}))\\
&-\left(\frac{a'}{a}+\frac{b'}{b}+\frac{c'}{c}+\frac{f'}{f}\right) abcf\, e^{2367*}\wedge dt + (abcf\, e^{17367*} + abcf\, e^{24767*} + abcf\, e^{23577*}\\
&+\frac{1}{3}abcf(e^{23612*}+e^{23634*}+e^{23656*}))\\
&-\left(\frac{a'}{a}+\frac{b'}{b}+\frac{c'}{c}+\frac{f'}{f}\right) abcf\, e^{2457*}\wedge dt + (abcf\, e^{17457*} - abcf\, e^{23757*} - abcf\, e^{24677*}\\
&+\frac{1}{3}abcf(e^{24512*}+e^{24534*}+e^{24556*}))\\
&-\left(2\frac{a'}{a}+2\frac{b'}{b}\right) a^2b^2\, e^{1234*}\wedge dt + (-a^2b^2\, e^{27234*} - a^2b^2\, e^{11734*} - a^2b^2\, e^{12474*} - a^2b^2\, e^{12337*})\\
&-\left(2\frac{a'}{a}+2\frac{c'}{c}\right) a^2c^2\, e^{1256*}\wedge dt + (-a^2c^2\, e^{27256*} - a^2c^2\, e^{11756*} - a^2c^2\, e^{12676*} - a^2c^2\, e^{12557*})\\
&-\left(2\frac{b'}{b}+2\frac{c'}{c}\right) b^2c^2\, e^{3456*}\wedge dt + (-b^2c^2\, e^{47456*} - b^2c^2\, e^{33756*} - b^2c^2\, e^{34676*} - b^2c^2\, e^{34557*})\\
&+(-a^2f\, e^{2727*}\wedge dt - a^2f\, e^{1177*}\wedge dt - \frac{1}{3}a^2f(e^{1212*}\wedge dt + e^{1234*}\wedge dt + e^{1256*}\wedge dt))\\
&+(-b^2f\, e^{4747*}\wedge dt - b^2f\, e^{3377*}\wedge dt - \frac{1}{3}b^2f(e^{3412*}\wedge dt + e^{3434*}\wedge dt + e^{3456*}\wedge dt))\\
&+(-c^2f\, e^{6767*}\wedge dt - c^2f\, e^{5577*}\wedge dt - \frac{1}{3}c^2f(e^{5612*}\wedge dt + e^{5634*}\wedge dt + e^{5656*}\wedge dt))
\end{aligned}
$$

$$
\begin{aligned}
&+ (-abc\, e^{2736*} \wedge dt + abc\, e^{1476*} \wedge dt + abc\, e^{1357*} \wedge dt) \\
&+ (-abc\, e^{2745*} \wedge dt - abc\, e^{1375*} \wedge dt - abc\, e^{1467*} \wedge dt) \\
&+ (abc\, e^{1735*} \wedge dt + abc\, e^{2475*} \wedge dt - abc\, e^{2367*} \wedge dt) \\
&- (abc\, e^{1746*} \wedge dt - abc\, e^{2376*} \wedge dt + abc\, e^{2457*} \wedge dt) \\
= &\left(\frac{a'}{a} + \frac{b'}{b} + \frac{c'}{c} + \frac{f'}{f}\right) abcf\; e^{1357*} \wedge dt \\
&- \left(\frac{a'}{a} + \frac{b'}{b} + \frac{c'}{c} + \frac{f'}{f}\right) abcf\; e^{1467*} \wedge dt \\
&- \left(\frac{a'}{a} + \frac{b'}{b} + \frac{c'}{c} + \frac{f'}{f}\right) abcf\; e^{2367*} \wedge dt \\
&- \left(\frac{a'}{a} + \frac{b'}{b} + \frac{c'}{c} + \frac{f'}{f}\right) abcf\; e^{2457*} \wedge dt \\
&- \left(2\frac{a'}{a} + 2\frac{b'}{b}\right) a^2 b^2\; e^{1234*} \wedge dt \\
&- \left(2\frac{a'}{a} + 2\frac{c'}{c}\right) a^2 c^2\; e^{1256*} \wedge dt \\
&- \left(2\frac{b'}{b} + 2\frac{c'}{c}\right) b^2 c^2\; e^{3456*} \wedge dt \\
&+ \frac{1}{3} a^2 f (-e^{1234*} \wedge dt - e^{1256*} \wedge dt) \\
&+ \frac{1}{3} b^2 f (-e^{1234*} \wedge dt - e^{3456*} \wedge dt) \\
&+ \frac{1}{3} c^2 f (-e^{1256*} \wedge dt - e^{3456*} \wedge dt) \\
&- abc\, e^{2367*} \wedge dt - abc\, e^{1467*} \wedge dt + abc\, e^{1357*} \wedge dt \\
&- abc\, e^{2457*} \wedge dt + abc\, e^{1357*} \wedge dt - abc\, e^{1467*} \wedge dt \\
&+ abc\, e^{1357*} \wedge dt - abc\, e^{2457*} \wedge dt - abc\, e^{2367*} \wedge dt \\
&- abc\, e^{1467*} \wedge dt - abc\, e^{2367*} \wedge dt - abc\, e^{2457*} \wedge dt \\
= &\left(-\left(2\frac{a'}{a} + 2\frac{b'}{b}\right) - \frac{1}{3} a^2 f - \frac{1}{3} b^2 f\right) e^{1234*} \wedge dt \\
&+ \left(-\left(2\frac{a'}{a} + 2\frac{c'}{c}\right) - \frac{1}{3} a^2 f - \frac{1}{3} c^2 f\right) e^{1256*} \wedge dt \\
&+ \left(-\left(2\frac{b'}{b} + 2\frac{c'}{c}\right) - \frac{1}{3} b^2 f - \frac{1}{3} c^2 f\right) e^{3456*} \wedge dt
\end{aligned}
$$

$$
\begin{aligned}
&+\left(\left(\frac{a'}{a}+\frac{b'}{b}+\frac{c'}{c}+\frac{f'}{f}\right)abcf+3abc\right)e^{1357*}\wedge dt\\
&+\left(-\left(\frac{a'}{a}+\frac{b'}{b}+\frac{c'}{c}+\frac{f'}{f}\right)abcf-3abc\right)e^{1467*}\wedge dt\\
&+\left(-\left(\frac{a'}{a}+\frac{b'}{b}+\frac{c'}{c}+\frac{f'}{f}\right)abcf-3abc\right)e^{2367*}\wedge dt\\
&+\left(-\left(\frac{a'}{a}+\frac{b'}{b}+\frac{c'}{c}+\frac{f'}{f}\right)abcf-3abc\right)e^{2457*}\wedge dt
\end{aligned}
$$

From this calculation, it follows that the necessary condition $d*\omega=0$ for the existence of a parallel Spin(7)-structure is automatically satisfied. Furthermore, we see that the condition for the holonomy reduction is equivalent to:

$$
\begin{aligned}
2\frac{a'}{a}+2\frac{b'}{b} &= -\frac{1}{3}\frac{f}{a^2}-\frac{1}{3}\frac{f}{b^2}\\
2\frac{a'}{a}+2\frac{c'}{c} &= -\frac{1}{3}\frac{f}{a^2}-\frac{1}{3}\frac{f}{c^2}\\
2\frac{b'}{b}+2\frac{c'}{c} &= -\frac{1}{3}\frac{f}{b^2}-\frac{1}{3}\frac{f}{c^2}\\
\frac{a'}{a}+\frac{b'}{b}+\frac{c'}{c}+\frac{f'}{f} &= -3\frac{1}{f}
\end{aligned}
$$

We can simplify this system to:

$$
\begin{aligned}
a' &= -\frac{1}{6}\frac{f}{a}\\
b' &= -\frac{1}{6}\frac{f}{b}\\
c' &= -\frac{1}{6}\frac{f}{c}\\
f' &= \frac{1}{6}\frac{f^2}{a^2}+\frac{1}{6}\frac{f^2}{b^2}+\frac{1}{6}\frac{f^2}{c^2}-3
\end{aligned}
$$

These are the equations which we use throughout Section 5.2.

Appendix B

Calculations for the principal orbit $M^{1,1,0}$

Let M be a cohomogeneity-one manifold with principal orbit $M^{1,1,0}$. The frame (5.18) from page 132 defines a $SU(3) \times SU(2)$-invariant Spin(7)-structure on M. In this chapter, we will calculate the four-form Ω which is associated to this Spin(7)-structure. Furthermore, we express the condition $d\Omega = 0$ as a system of ordinary differential equations for the metric functions a, b, and c, which are defined by (5.17).

Let $\psi : T_pM \cong \mathfrak{m} \oplus \mathrm{span}\left(\frac{\partial}{\partial t}\right) \to \mathbb{R}^8$ be the linear map which corresponds to the frame (5.18). In our situation, Ω is the following four-form:

$$
\begin{aligned}
\Omega &= \psi^*(dx^{0123} + dx^{0145} - dx^{0167} + dx^{0246} + dx^{0257} + dx^{0347} - dx^{0356} \\
&\quad - dx^{1247} + dx^{1256} + dx^{1346} + dx^{1357} - dx^{2345} + dx^{2367} + dx^{4567}) \\
&= b^2c\, dt \wedge e^{765} + a^2c\, dt \wedge e^{712} - a^2c\, dt \wedge e^{743} + a^2b\, dt \wedge e^{614} \\
&\quad + a^2b\, dt \wedge e^{623} + a^2b\, dt \wedge e^{513} - a^2b\, dt \wedge e^{524} \\
&\quad - a^2bc\, e^{7613} + a^2bc\, e^{7624} + a^2bc\, e^{7514} + a^2bc\, e^{7523} \\
&\quad - a^2b^2\, e^{6512} + a^2b^2\, e^{6543} + a^4\, e^{1243} \\
&= -a^4\, e^{1234} + a^2b^2\, e^{1256} + a^2b^2\, e^{3456} \\
&\quad + a^2bc\, e^{1367} - a^2bc\, e^{1457} - a^2bc\, e^{2357} - a^2bc\, e^{2467} \\
&\quad - a^2b\, e^{135} \wedge dt - a^2b\, e^{146} \wedge dt - a^2b\, e^{236} \wedge dt + a^2b\, e^{245} \wedge dt \\
&\quad - a^2c\, e^{127} \wedge dt - a^2c\, e^{347} \wedge dt + b^2c\, e^{567} \wedge dt
\end{aligned}
$$

By the considerations we have made on page 131, we see that Ω has to be $SU(3) \times SU(2)$-invariant. This fact could also be proven by explicitly considering the action of ad_{e_8}, ad_{e_9}, and $\mathrm{ad}_{e_{11}}$ on Ω. The Spin(7)-structure which we obtain by the $SU(3) \times SU(2)$-invariant extension of the frame is thus well-defined.

Before we turn our attention to $d\Omega$, we have to know the exterior derivative at $p \in M^{1,1,0}$ of the one-forms e^{i^*} which are defined by $e^{i^*}(e_j^*) = \delta^i_j$:

$$
\begin{aligned}
de_p^{1*} &= 3e_p^{27*}\\
de_p^{2*} &= -3e_p^{17*}\\
de_p^{3*} &= 3e_p^{47*}\\
de_p^{4*} &= -3e_p^{37*}\\
de_p^{5*} &= -2e_p^{67*}\\
de_p^{6*} &= 2e_p^{57*}\\
de_p^{7*} &= \frac{3}{4}e_p^{12*} + \frac{3}{4}e_p^{34*} - \frac{1}{2}e_p^{56*}
\end{aligned}
$$

By the usual methods, we are now able to calculate $d\Omega$:

$$
\begin{aligned}
d\Omega = {} & -4a'a^3\, e^{1234} \wedge dt + a^4(3e^{27234} + 3e^{11734} + 3e^{12474} + 3e^{12337})\\
& + \left(2\frac{a'}{a} + 2\frac{b'}{b}\right) a^2b^2\, e^{1256} \wedge dt - a^2b^2(3e^{27256} + 3e^{11756} - 2e^{12676} - 2e^{12557})\\
& + \left(2\frac{a'}{a} + 2\frac{b'}{b}\right) a^2b^2\, e^{3456} \wedge dt - a^2b^2(3e^{47456} + 3e^{33756} - 2e^{34676} - 2e^{34557})\\
& + \left(2\frac{a'}{a} + \frac{b'}{b} + \frac{c'}{c}\right) a^2bc\, e^{1367} \wedge dt\\
& - a^2bc\left(3e^{27367} - 3e^{14767} + 2e^{13577} - \left(\frac{3}{4}e^{13612} + \frac{3}{4}e^{13634} - \frac{1}{2}e^{13656}\right)\right)\\
& - \left(2\frac{a'}{a} + \frac{b'}{b} + \frac{c'}{c}\right) a^2bc\, e^{1457} \wedge dt\\
& + a^2bc\left(3e^{27457} + 3e^{13757} - 2e^{14677} - \left(\frac{3}{4}e^{14512} + \frac{3}{4}e^{14534} - \frac{1}{2}e^{14556}\right)\right)\\
& - \left(2\frac{a'}{a} + \frac{b'}{b} + \frac{c'}{c}\right) a^2bc\, e^{2357} \wedge dt\\
& + a^2bc\left(-3e^{17357} - 3e^{24757} - 2e^{23677} - \left(\frac{3}{4}e^{23512} + \frac{3}{4}e^{23534} - \frac{1}{2}e^{23556}\right)\right)\\
& - \left(2\frac{a'}{a} + \frac{b'}{b} + \frac{c'}{c}\right) a^2bc\, e^{2467} \wedge dt\\
& + a^2bc\left(-3e^{17467} + 3e^{23767} + 2e^{24577} - \left(\frac{3}{4}e^{24612} + \frac{3}{4}e^{24634} - \frac{1}{2}e^{24656}\right)\right)\\
& + a^2b(3e^{2735} \wedge dt - 3e^{1475} \wedge dt - 2e^{1367} \wedge dt)\\
& + a^2b(3e^{2746} \wedge dt + 3e^{1376} \wedge dt + 2e^{1457} \wedge dt)\\
& + a^2b(-3e^{1736} \wedge dt - 3e^{2476} \wedge dt + 2e^{2357} \wedge dt)\\
& - a^2b(-3e^{1745} \wedge dt + 3e^{2375} \wedge dt - 2e^{2467} \wedge dt)
\end{aligned}
$$

$$
\begin{aligned}
&+ a^2c\left(3e^{2727}\wedge dt + 3e^{1177}\wedge dt + \frac{3}{4}e^{1212}\wedge dt + \frac{3}{4}e^{1234}\wedge dt - \frac{1}{2}e^{1256}\wedge dt\right)\\
&+ a^2c\left(3e^{4747}\wedge dt - 3e^{3377}\wedge dt + \frac{3}{4}e^{3412}\wedge dt + \frac{3}{4}e^{3434}\wedge dt - \frac{1}{2}e^{3456}\wedge dt\right)\\
&- b^2c\left(-2e^{6767}\wedge dt - 2e^{5577}\wedge dt + \frac{3}{4}e^{5612}\wedge dt + \frac{3}{4}e^{5634}\wedge dt - \frac{1}{2}e^{5656}\wedge dt\right)\\
= &-4a'a^3\, e^{1234}\wedge dt\\
&+ \left(2\frac{a'}{a} + 2\frac{b'}{b}\right) a^2b^2\, e^{1256}\wedge dt\\
&+ \left(2\frac{a'}{a} + 2\frac{b'}{b}\right) a^2b^2\, e^{3456}\wedge dt\\
&+ \left(2\frac{a'}{a} + \frac{b'}{b} + \frac{c'}{c}\right) a^2bc\, e^{1367}\wedge dt\\
&- \left(2\frac{a'}{a} + \frac{b'}{b} + \frac{c'}{c}\right) a^2bc\, e^{1457}\wedge dt\\
&- \left(2\frac{a'}{a} + \frac{b'}{b} + \frac{c'}{c}\right) a^2bc\, e^{2357}\wedge dt\\
&- \left(2\frac{a'}{a} + \frac{b'}{b} + \frac{c'}{c}\right) a^2bc\, e^{2467}\wedge dt\\
&+ a^2b(3e^{2357}\wedge dt + 3e^{1457}\wedge dt - 2e^{1367}\wedge dt)\\
&+ a^2b(3e^{2467}\wedge dt - 3e^{1367}\wedge dt + 2e^{1457}\wedge dt)\\
&+ a^2b(-3e^{1367}\wedge dt + 3e^{2467}\wedge dt + 2e^{2357}\wedge dt)\\
&- a^2b(-3e^{1457}\wedge dt - 3e^{2357}\wedge dt - 2e^{2467}\wedge dt)\\
&+ a^2c\left(\frac{3}{4}e^{1234}\wedge dt - \frac{1}{2}e^{1256}\wedge dt\right)\\
&+ a^2c\left(\frac{3}{4}e^{1234}\wedge dt - \frac{1}{2}e^{3456}\wedge dt\right)\\
&- b^2c\left(\frac{3}{4}e^{1256}\wedge dt + \frac{3}{4}e^{3456}\wedge dt\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \left(-4a'a^3 + \frac{3}{2}a^2c\right) e^{1234} \wedge dt \\
&\quad + \left(\left(2\frac{a'}{a} + 2\frac{b'}{b}\right) a^2b^2 - \frac{1}{2}a^2c - \frac{3}{4}b^2c\right) e^{1256} \wedge dt \\
&\quad + \left(\left(2\frac{a'}{a} + \frac{b'}{b} + \frac{c'}{c}\right) a^2bc - 8a^2b\right) e^{1367} \wedge dt \\
&\quad + \left(-\left(2\frac{a'}{a} + \frac{b'}{b} + \frac{c'}{c}\right) a^2bc + 8a^2b\right) e^{1457} \wedge dt \\
&\quad + \left(-\left(2\frac{a'}{a} + \frac{b'}{b} + \frac{c'}{c}\right) a^2bc + 8a^2b\right) e^{2357} \wedge dt \\
&\quad + \left(-\left(2\frac{a'}{a} + \frac{b'}{b} + \frac{c'}{c}\right) a^2bc + 8a^2b\right) e^{2467} \wedge dt \\
&\quad + \left(\left(2\frac{a'}{a} + 2\frac{b'}{b}\right) a^2b^2 - \frac{1}{2}a^2c - \frac{3}{4}b^2c\right) e^{3456} \wedge dt
\end{aligned}
$$

The equation $d\Omega = 0$ therefore is equivalent to:

$$
\begin{aligned}
4\frac{a'}{a} &= \frac{3}{2}\frac{c}{a^2} \\
2\frac{a'}{a} + 2\frac{b'}{b} &= \frac{1}{2}\frac{c}{b^2} + \frac{3}{4}\frac{c}{a^2} \\
2\frac{a'}{a} + \frac{b'}{b} + \frac{c'}{c} &= 8\frac{1}{c}
\end{aligned}
$$

We can simplify this system and obtain:

$$
\begin{aligned}
\frac{a'}{a} &= \frac{3}{8}\frac{c}{a^2} \\
\frac{b'}{b} &= \frac{1}{4}\frac{c}{b^2} \\
\frac{c'}{c} &= 8\frac{1}{c} - \frac{1}{4}\frac{c}{b^2} - \frac{3}{4}\frac{c}{a^2}
\end{aligned}
$$

These are exactly the equations (5.20) with which we work in Section 5.3.

Appendix C

Calculations for the Aloff-Wallach spaces as principal orbits

C.1 The Spin(7)-structure for the Aloff-Wallach spaces except $N^{1,1}$ as principal orbits and the condition for its torsion-freeness

Let M be a cohomogeneity-one manifold whose principal orbit is an Aloff-Wallach space. In this section, we assume that the principal orbit is not the exceptional Aloff-Wallach space $N^{1,1}$. Our first aim is to calculate the $SU(3)$-invariant four-form Ω which is associated to the frame (5.34) from page 155. Furthermore, we will deduce a system of ordinary differential equations for the metric functions a, b, c, and f which is equivalent to $d\Omega = 0$.

Let $p \in M$ be a point whose tangent space we identify with $\mathfrak{m} \oplus \operatorname{span}\left(\frac{\partial}{\partial t}\right)$, and let $\psi : T_pM \to \mathbb{R}^8$ be the linear map which maps the frame to the standard basis of $\mathbb{R}^8$. For Ω we obtain:

$$
\begin{aligned}
\Omega = \psi^*(&dx^{0123} + dx^{0145} - dx^{0167} + dx^{0246} + dx^{0257} + dx^{0347} - dx^{0356} \\
&- dx^{1247} + dx^{1256} + dx^{1346} + dx^{1357} - dx^{2345} + dx^{2367} + dx^{4567}) \\
= &-a^2 f\, e^{712} \wedge dt - b^2 f\, e^{743} \wedge dt + c^2 f\, e^{765} \wedge dt \\
&- abc\, e^{146} \wedge dt - abc\, e^{135} \wedge dt - abc\, e^{245} \wedge dt + abc\, e^{236} \wedge dt \\
&- abcf\, e^{7145} + abcf\, e^{7136} + abcf\, e^{7246} + abcf\, e^{7235} \\
&- a^2 b^2\, e^{1243} + a^2 c^2\, e^{1265} + b^2 c^2\, e^{4365} \\
= &-abcf\, e^{1367} + abcf\, e^{1457} - abcf\, e^{2357} - abcf\, e^{2467} \\
&+ a^2 b^2\, e^{1234} - a^2 c^2\, e^{1256} + b^2 c^2\, e^{3456} \\
&- a^2 f\, e^{127} \wedge dt + b^2 f\, e^{347} \wedge dt - c^2 f\, e^{567} \wedge dt \\
&- abc\, e^{135} \wedge dt - abc\, e^{146} \wedge dt + abc\, e^{236} \wedge dt - abc\, e^{245} \wedge dt
\end{aligned}
$$

In order to calculate $d\Omega$, we need the exterior derivatives of the one-forms e^{i^*} which are defined by $e^{i^*}(e_j^*) = \delta^i_j$. These derivatives are given by:

$$
\begin{aligned}
de_p^{1*} &= (3k+3l)e_p^{27*} - e_p^{35*} - e_p^{46*}\\
de_p^{2*} &= -(3k+3l)e_p^{17*} + e_p^{36*} - e_p^{45*}\\
de_p^{3*} &= e_p^{15*} - e_p^{26*} + 3l\, e_p^{47*}\\
de_p^{4*} &= e_p^{16*} + e_p^{25*} - 3l\, e_p^{37*}\\
de_p^{5*} &= -e_p^{13*} - e_p^{24*} - 3k\, e_p^{67*}\\
de_p^{6*} &= -e_p^{14*} + e_p^{23*} + 3k\, e_p^{57*}\\
de_p^{7*} &= \frac{k+l}{k^2+lk+l^2}e_p^{12*} + \frac{l}{k^2+lk+l^2}e_p^{34*} - \frac{k}{k^2+lk+l^2}e_p^{56*}
\end{aligned}
$$

where $p := eU(1)_{k,l} \in N^{k,l}$. We are now able to express $d\Omega$ in terms of our basis of T_pM:

$$
\begin{aligned}
d\Omega = &-\left(\frac{a'}{a}+\frac{b'}{b}+\frac{c'}{c}+\frac{f'}{f}\right) abcf\, e^{1367} \wedge dt\\
&- abcf(-(3k+3l)e^{27} + e^{35} + e^{46}) \wedge e^{367}\\
&+ abcf\, e^{1} \wedge (-e^{15} + e^{26} - 3l\, e^{47}) \wedge e^{67}\\
&- abcf\, e^{13} \wedge (e^{14} - e^{23} - 3k\, e^{57}) \wedge e^{7}\\
&+ abcf\, e^{136} \wedge \left(-\frac{l+k}{k^2+lk+l^2}e^{12} - \frac{l}{k^2+lk+l^2}e^{34} + \frac{k}{k^2+lk+l^2}e^{56}\right)\\
&+ \left(\frac{a'}{a}+\frac{b'}{b}+\frac{c'}{c}+\frac{f'}{f}\right) abcf\, e^{1457} \wedge dt\\
&+ abcf(-(3k+3l)e^{27} + e^{35} + e^{46}) \wedge e^{457}\\
&- abcf\, e^{1} \wedge (-e^{16} - e^{25} + 3l\, e^{37}) \wedge e^{57}\\
&+ abcf\, e^{14} \wedge (e^{13} + e^{24} + 3k\, e^{67}) \wedge e^{7}\\
&- abcf\, e^{145} \wedge \left(-\frac{l+k}{k^2+lk+l^2}e^{12} - \frac{l}{k^2+lk+l^2}e^{34} + \frac{k}{k^2+lk+l^2}e^{56}\right)\\
&- \left(\frac{a'}{a}+\frac{b'}{b}+\frac{c'}{c}+\frac{f'}{f}\right) abcf\, e^{2357} \wedge dt\\
&- abcf((3k+3l)e^{17} - e^{36} + e^{45}) \wedge e^{357}\\
&+ abcf\, e^{2} \wedge (-e^{15} + e^{26} - 3l\, e^{47}) \wedge e^{57}\\
&- abcf\, e^{23} \wedge (e^{13} + e^{24} + 3k\, e^{67}) \wedge e^{7}\\
&+ abcf\, e^{235} \wedge \left(-\frac{l+k}{k^2+lk+l^2}e^{12} - \frac{l}{k^2+lk+l^2}e^{34} + \frac{k}{k^2+lk+l^2}e^{56}\right)
\end{aligned}
$$

$$
\begin{aligned}
&- \left(\frac{a'}{a} + \frac{b'}{b} + \frac{c'}{c} + \frac{f'}{f}\right) abcf\, e^{2467} \wedge dt \\
&- abcf((3k+3l)e^{17} - e^{36} + e^{45}) \wedge e^{467} \\
&+ abcf\, e^{2} \wedge (-e^{16} - e^{25} + 3l\, e^{37}) \wedge e^{67} \\
&- abcf\, e^{24} \wedge (e^{14} - e^{23} - 3k\, e^{57}) \wedge e^{7} \\
&+ abcf\, e^{246} \wedge \left(-\frac{l+k}{k^2+lk+l^2} e^{12} - \frac{l}{k^2+lk+l^2} e^{34} + \frac{k}{k^2+lk+l^2} e^{56}\right) \\
&+ \left(2\frac{a'}{a} + 2\frac{b'}{b}\right) a^2 b^2\, e^{1234} \wedge dt \\
&+ a^2 b^2 (-(3k+3l)e^{27} + e^{35} + e^{46}) \wedge e^{234} \\
&- a^2 b^2\, e^{1} \wedge ((3k+3l)e^{17} - e^{36} + e^{45}) \wedge e^{34} \\
&+ a^2 b^2\, e^{12} \wedge (-e^{15} + e^{26} - 3l\, e^{47}) \wedge e^{4} \\
&- a^2 b^2\, e^{123} \wedge (-e^{16} - e^{25} + 3l\, e^{37}) \\
&- \left(2\frac{a'}{a} + 2\frac{c'}{c}\right) a^2 c^2\, e^{1256} \wedge dt \\
&- a^2 c^2 (-(3k+3l)e^{27} + e^{35} + e^{46}) \wedge e^{256} \\
&+ a^2 c^2\, e^{1} \wedge ((3k+3l)e^{17} - e^{36} + e^{45}) \wedge e^{56} \\
&- a^2 c^2\, e^{12} \wedge (e^{13} + e^{24} + 3k\, e^{67}) \wedge e^{6} \\
&+ a^2 c^2\, e^{125} \wedge (e^{14} - e^{23} - 3k\, e^{57}) \\
&+ \left(2\frac{b'}{b} + 2\frac{c'}{c}\right) b^2 c^2\, e^{3456} \wedge dt \\
&+ b^2 c^2 (-e^{15} + e^{26} - 3l\, e^{47}) \wedge e^{456} \\
&- b^2 c^2\, e^{3} \wedge (-e^{16} - e^{25} + 3l\, e^{37}) \wedge e^{56} \\
&+ b^2 c^2\, e^{34} \wedge (e^{13} + e^{24} + 3k\, e^{67}) \wedge e^{6} \\
&- b^2 c^2\, de^{345} \wedge (e^{14} - e^{23} - 3k\, e^{57}) \\
&- a^2 f(-(3k+3l)e^{27} + e^{35} + e^{46}) \wedge e^{27} \wedge dt \\
&+ a^2 f\, e^{1} \wedge ((3k+3l)e^{17} - e^{36} + e^{45}) \wedge e^{7} \wedge dt \\
&- a^2 f\, e^{12} \wedge \left(-\frac{l+k}{k^2+lk+l^2} e^{12} - \frac{l}{k^2+lk+l^2} e^{34} + \frac{k}{k^2+lk+l^2} e^{56}\right) \wedge dt \\
&+ b^2 f(-e^{15} + e^{26} - 3l\, e^{47}) \wedge e^{47} \wedge dt \\
&- b^2 f\, e^{3} \wedge (-e^{16} - e^{25} + 3l\, e^{37}) \wedge e^{7} \wedge dt \\
&+ b^2 f\, e^{34} \wedge \left(-\frac{l+k}{k^2+lk+l^2} e^{12} - \frac{l}{k^2+lk+l^2} e^{34} + \frac{k}{k^2+lk+l^2} e^{56}\right) \wedge dt \\
&- c^2 f(e^{13} + e^{24} + 3k\, e^{67}) \wedge e^{67} \wedge dt \\
&+ c^2 f\, e^{5} \wedge (e^{14} - e^{23} - 3k\, e^{57}) \wedge e^{7} \wedge dt \\
&- c^2 f\, e^{56} \wedge \left(-\frac{l+k}{k^2+lk+l^2} e^{12} - \frac{l}{k^2+lk+l^2} e^{34} + \frac{k}{k^2+lk+l^2} e^{56}\right) \wedge dt
\end{aligned}
$$

$$
\begin{aligned}
&- abc(-(3k+3l)e^{27} + e^{35} + e^{46}) \wedge e^{35} \wedge dt \\
&+ abc\, e^{1} \wedge (-e^{15} + e^{26} - 3l\, e^{47}) \wedge e^{5} \wedge dt \\
&- abc\, e^{13} \wedge (e^{13} + e^{24} + 3k\, e^{67}) \wedge dt \\
&- abc(-(3k+3l)e^{27} + e^{35} + e^{46}) \wedge e^{46} \wedge dt \\
&+ abc\, e^{1} \wedge (-e^{16} - e^{25} + 3l\, e^{37}) \wedge e^{6} \wedge dt \\
&- abc\, e^{14} \wedge (e^{14} - e^{23} - 3k\, e^{57}) \wedge dt \\
&+ abc((3k+3l)e^{17} - e^{36} + e^{45}) \wedge e^{36} \wedge dt \\
&- abc\, e^{2} \wedge (-e^{15} + e^{26} - 3l\, e^{47}) \wedge e^{6} \wedge dt \\
&+ abc\, e^{23} \wedge (e^{14} - e^{23} - 3k\, e^{57}) \wedge dt \\
&- abc((3k+3l)e^{17} - e^{36} + e^{45}) \wedge e^{45} \wedge dt \\
&+ abc\, e^{2} \wedge (-e^{16} - e^{25} + 3l\, e^{37}) \wedge e^{5} \wedge dt \\
&- abc\, e^{24} \wedge (e^{13} + e^{24} + 3k\, e^{67}) \wedge dt \\
= &- \left(\frac{a'}{a} + \frac{b'}{b} + \frac{c'}{c} + \frac{f'}{f}\right) abcf\, e^{1367} \wedge dt \\
&- 0 + 0 - 0 + 0 \\
&+ \left(\frac{a'}{a} + \frac{b'}{b} + \frac{c'}{c} + \frac{f'}{f}\right) abcf\, e^{1457} \wedge dt \\
&+ 0 - 0 + 0 - 0 \\
&- \left(\frac{a'}{a} + \frac{b'}{b} + \frac{c'}{c} + \frac{f'}{f}\right) abcf\, e^{2357} \wedge dt \\
&- 0 + 0 - 0 + 0 \\
&- \left(\frac{a'}{a} + \frac{b'}{b} + \frac{c'}{c} + \frac{f'}{f}\right) abcf\, e^{2467} \wedge dt \\
&- 0 + 0 - 0 + 0 \\
&+ \left(2\frac{a'}{a} + 2\frac{b'}{b}\right) a^2 b^2\, e^{1234} \wedge dt \\
&+ 0 - 0 + 0 - 0 \\
&- \left(2\frac{a'}{a} + 2\frac{c'}{c}\right) a^2 c^2\, e^{1256} \wedge dt \\
&- 0 + 0 - 0 + 0 \\
&+ \left(2\frac{b'}{b} + 2\frac{c'}{c}\right) b^2 c^2\, e^{3456} \wedge dt \\
&+ 0 - 0 + 0 - 0 \\
&+ a^2 f(-e^{2357} \wedge dt - e^{2467} \wedge dt) \\
&+ a^2 f(-e^{1367} \wedge dt + e^{1457} \wedge dt) \\
&+ a^2 f\left(\frac{l}{k^2 + lk + l^2} e^{1234} \wedge dt - \frac{k}{k^2 + lk + l^2} e^{1256} \wedge dt\right)
\end{aligned}
$$

$$
\begin{aligned}
&+ b^2 f(e^{1457} \wedge dt - e^{2467} \wedge dt)\\
&+ b^2 f(-e^{1367} \wedge dt - e^{2357} \wedge dt)\\
&+ b^2 f\left(-\frac{l+k}{k^2+lk+l^2} e^{1234} \wedge dt + \frac{k}{k^2+lk+l^2} e^{3456} \wedge dt\right)\\
&+ c^2 f(-e^{1367} \wedge dt - e^{2467} \wedge dt)\\
&+ c^2 f(e^{1457} \wedge dt - e^{2357} \wedge dt)\\
&+ c^2 f\left(\frac{l+k}{k^2+lk+l^2} e^{1256} \wedge dt + \frac{l}{k^2+lk+l^2} e^{3456} \wedge dt\right)\\
&+ abc((3k+3l)e^{2357} \wedge dt + e^{3456} \wedge dt)\\
&+ abc(-e^{1256} \wedge dt + 3l\, e^{1457} \wedge dt)\\
&+ abc(e^{1234} \wedge dt - 3k\, e^{1367} \wedge dt)\\
&+ abc((3k+3l)e^{2467} \wedge dt + e^{3456} \wedge dt)\\
&+ abc(-e^{1256} \wedge dt - 3l\, e^{1367} \wedge dt)\\
&+ abc(e^{1234} \wedge dt + 3k\, e^{1457} \wedge dt)\\
&+ abc((3k+3l)e^{1367} \wedge dt + e^{3456} \wedge dt)\\
&+ abc(-e^{1256} \wedge dt - 3l\, e^{2467} \wedge dt)\\
&+ abc(e^{1234} \wedge dt - 3k\, e^{2357} \wedge dt)\\
&+ abc(-(3k+3l)e^{1457} \wedge dt + e^{3456} \wedge dt)\\
&+ abc(-e^{1256} \wedge dt - 3l\, e^{2357} \wedge dt)\\
&+ abc(e^{1234} \wedge dt - 3k\, e^{2467} \wedge dt)\\
= &\left(\left(-\frac{a'}{a} - \frac{b'}{b} - \frac{c'}{c} - \frac{f'}{f}\right) abcf - a^2 f - b^2 f - c^2 f - 3l\, abc - 3k\, abc + (3k+3l)\, abc\right) e^{1367} \wedge dt\\
&+ \left(\left(\frac{a'}{a} + \frac{b'}{b} + \frac{c'}{c} + \frac{f'}{f}\right) abcf + a^2 f + b^2 f + c^2 f + 3k\, abc + 3l\, abc - (3k+3l)\, abc\right) e^{1457} \wedge dt\\
&+ \left(\left(-\frac{a'}{a} - \frac{b'}{b} - \frac{c'}{c} - \frac{f'}{f}\right) abcf - a^2 f - b^2 f - c^2 f + (3k+3l)abc - 3l\, abc - 3k\, abc\right) e^{2357} \wedge dt\\
&+ \left(\left(-\frac{a'}{a} - \frac{b'}{b} - \frac{c'}{c} - \frac{f'}{f}\right) abcf - a^2 f - b^2 f - c^2 f + (3k+3l)abc - 3k\, abc - 3l\, abc\right) e^{2467} \wedge dt\\
&+ \left(\left(2\frac{a'}{a} + 2\frac{b'}{b}\right) a^2 b^2 + \frac{l}{k^2+lk+l^2}\, a^2 f - \frac{l+k}{k^2+lk+l^2} b^2 f + 4abc\right) e^{1234} \wedge dt\\
&+ \left(\left(-2\frac{a'}{a} - 2\frac{c'}{c}\right) a^2 c^2 - \frac{k}{k^2+lk+l^2}\, a^2 f + \frac{l+k}{k^2+lk+l^2}\, c^2 f - 4abc\right) e^{1256} \wedge dt\\
&+ \left(\left(2\frac{b'}{b} + 2\frac{c'}{c}\right) b^2 c^2 + \frac{k}{k^2+lk+l^2}\, b^2 f + \frac{l}{k^2+lk+l^2}\, c^2 f + 4abc\right) e^{3456} \wedge dt
\end{aligned}
$$

The above calculations prove that $d\Omega = 0$ is equivalent to:

$$\begin{aligned}
\frac{a'}{a}+\frac{b'}{b}+\frac{c'}{c}+\frac{f'}{f} &= -\frac{a^2+b^2+c^2}{abc}\\
2\left(\frac{a'}{a}+\frac{b'}{b}\right) &= -\frac{l}{k^2+lk+l^2}\frac{f}{b^2}-\frac{-k-l}{k^2+lk+l^2}\frac{f}{a^2}-4\frac{c}{ab}\\
2\left(\frac{a'}{a}+\frac{c'}{c}\right) &= -\frac{k}{k^2+lk+l^2}\frac{f}{c^2}-\frac{-k-l}{k^2+lk+l^2}\frac{f}{a^2}-4\frac{b}{ac}\\
2\left(\frac{b'}{b}+\frac{c'}{c}\right) &= -\frac{k}{k^2+lk+l^2}\frac{f}{c^2}-\frac{l}{k^2+lk+l^2}\frac{f}{b^2}-4\frac{a}{bc}
\end{aligned}$$

The above system can be simplified to:

$$\begin{aligned}
\frac{a'}{a} &= \frac{a^2-b^2-c^2}{abc}-\frac{-k-l}{2(k^2+lk+l^2)}\frac{f}{a^2}\\
\frac{b'}{b} &= \frac{b^2-c^2-a^2}{abc}-\frac{l}{2(k^2+lk+l^2)}\frac{f}{b^2}\\
\frac{c'}{c} &= \frac{c^2-a^2-b^2}{abc}-\frac{k}{2(k^2+lk+l^2)}\frac{f}{c^2}\\
\frac{f'}{f} &= \frac{-k-l}{2(k^2+lk+l^2)}\frac{f}{a^2}+\frac{l}{2(k^2+lk+l^2)}\frac{f}{b^2}+\frac{k}{2(k^2+lk+l^2)}\frac{f}{c^2}
\end{aligned}$$

These equations are equivalent to those in the literature. If we reverse for example the direction of t, rescale f by a factor of $-2(k^2+lk+l^2)$, and rename the coefficients $(k,l,-k-l)$, we obtain the equations for the holonomy reduction which have been found by Kanno and Yasui in [48].

In order to keep our notation short, we denote the factor k^2+lk+l^2 by Δ. Furthermore, we choose the same direction of t as Kanno and Yasui, since we want to reduce the number of terms with a minus. We finally obtain the equations which we use throughout Section 5.4:

$$\begin{aligned}
\frac{a'}{a} &= \frac{b^2+c^2-a^2}{abc}+\frac{-k-l}{2\Delta}\frac{f}{a^2}\\
\frac{b'}{b} &= \frac{c^2+a^2-b^2}{abc}+\frac{l}{2\Delta}\frac{f}{b^2}\\
\frac{c'}{c} &= \frac{a^2+b^2-c^2}{abc}+\frac{k}{2\Delta}\frac{f}{c^2}\\
\frac{f'}{f} &= -\frac{-k-l}{2\Delta}\frac{f}{a^2}-\frac{l}{2\Delta}\frac{f}{b^2}-\frac{k}{2\Delta}\frac{f}{c^2}
\end{aligned}$$

C.2 The Spin(7)-structure for the principal orbit $N^{1,1}$ and the condition for its torsion-freeness

In this section, we assume that M is a cohomogeneity-one manifold whose principal orbit is the exceptional Aloff-Wallach space $N^{1,1}$. We consider the frame (5.37), which is defined on page 156. This frame defines a Spin(7)-structure of cohomogeneity one on M, whose associated metric is a diagonal metric of type (5.33). The aim of this section is to calculate the $SU(3)$-invariant four-form Ω which is associated to this Spin(7)-structure and to find a system of ordinary differential equations for the metric functions a_1, a_2, b, c, and f which is equivalent to $d\Omega = 0$.

First, we calculate the four-form. As usual, let $p \in M$ be a point whose tangent space we identify with $\mathfrak{m} \oplus \mathrm{span}\left(\frac{\partial}{\partial t}\right)$. Furthermore, let $\psi : T_pM \to \mathbb{R}^8$ be the linear map which maps the frame to the standard basis of $\mathbb{R}^8$. We obtain for Ω:

$$
\begin{aligned}
\Omega = \psi^*(& dx^{0123} + dx^{0145} - dx^{0167} + dx^{0246} + dx^{0257} + dx^{0347} - dx^{0356} \\
& - dx^{1247} + dx^{1256} + dx^{1346} + dx^{1357} - dx^{2345} + dx^{2367} + dx^{4567}) \\
= & -a_1a_2f\, e^{712} \wedge dt - b^2f\, e^{743} \wedge dt + c^2f\, e^{765} \wedge dt \\
& - a_1bc\, e^{146} \wedge dt - a_1bc\, e^{135} \wedge dt - a_2bc\, e^{245} \wedge dt + a_2bc\, e^{236} \wedge dt \\
& - a_1bcf\, e^{7145} + a_1bcf\, e^{7136} + a_2bcf\, e^{7246} + a_2bcf\, e^{7235} \\
& - a_1a_2b^2\, e^{1243} + a_1a_2c^2\, e^{1265} + b^2c^2\, e^{4365} \\
= & -a_1bcf\, e^{1367} + a_1bcf\, e^{1457} - a_2bcf\, e^{2357} - a_2bcf\, e^{2467} \\
& + a_1a_2b^2\, e^{1234} - a_1a_2c^2\, e^{1256} + b^2c^2\, e^{3456} \\
& - a_1a_2f\, e^{127} \wedge dt + b^2f\, e^{347} \wedge dt - c^2f\, e^{567} \wedge dt \\
& - a_1bc\, e^{135} \wedge dt - a_1bc\, e^{146} \wedge dt + a_2bc\, e^{236} \wedge dt - a_2bc\, e^{245} \wedge dt
\end{aligned}
$$

The exterior derivatives of the one-forms which are dual to the Killing vector fields e_i^* can be obtained by Lemma 3.1.44 from page 54:

$$
\begin{aligned}
de_p^{1*} &= 6e_p^{27*} - e_p^{35*} - e_p^{46*} \\
de_p^{2*} &= -6e_p^{17*} + e_p^{36*} - e_p^{45*} \\
de_p^{3*} &= e_p^{15*} - e_p^{26*} + 3\, e_p^{47*} \\
de_p^{4*} &= e_p^{16*} + e_p^{25*} - 3\, e_p^{37*} \\
de_p^{5*} &= -e_p^{13*} - e_p^{24*} - 3\, e_p^{67*} \\
de_p^{6*} &= -e_p^{14*} + e_p^{23*} + 3\, e_p^{57*} \\
de_p^{7*} &= \frac{2}{3}e_p^{12*} + \frac{1}{3}e_p^{34*} - \frac{1}{3}e_p^{56*}
\end{aligned}
$$

where $p := eU(1)_{1,1} \in N^{1,1}$. We are now able to express $d\Omega$ in terms of our basis of T_pM:

$$
\begin{aligned}
d\Omega = &-\left(\frac{a_1'}{a_1}+\frac{b'}{b}+\frac{c'}{c}+\frac{f'}{f}\right)a_1bcf\,e^{1367}\wedge dt\\
&+a_1bcf\Bigg((6e^{27}-e^{35}-e^{46})\wedge e^{367}-e^1\wedge(e^{15}-e^{26}+3e^{47})\wedge e^{67}\\
&+e^{13}\wedge(-e^{14}+e^{23}+3e^{57})\wedge e^7-e^{136}\wedge\left(\frac{2}{3}e^{12}+\frac{1}{3}e^{34}-\frac{1}{3}e^{56}\right)\Bigg)\\
&+\left(\frac{a_1'}{a_1}+\frac{b'}{b}+\frac{c'}{c}+\frac{f'}{f}\right)a_1bcf\,e^{1457}\wedge dt\\
&-a_1bcf\Bigg((6e^{27}-e^{35}-e^{46})\wedge e^{457}-e^1\wedge(e^{16}+e^{25}-3e^{37})\wedge e^{57}\\
&+e^{14}\wedge(-e^{13}-e^{24}-3e^{67})\wedge e^7-e^{145}\wedge\left(\frac{2}{3}e^{12}+\frac{1}{3}e^{34}-\frac{1}{3}e^{56}\right)\Bigg)\\
&-\left(\frac{a_2'}{a_2}+\frac{b'}{b}+\frac{c'}{c}+\frac{f'}{f}\right)a_2bcf\,e^{2357}\wedge dt\\
&+a_2bcf\Bigg((-6e^{17}+e^{36}-e^{45})\wedge e^{357}-e^2\wedge(e^{15}-e^{26}+3e^{47})\wedge e^{57}\\
&+e^{23}\wedge(-e^{13}-e^{24}-3e^{67})\wedge e^7-e^{235}\wedge\left(\frac{2}{3}e^{12}+\frac{1}{3}e^{34}-\frac{1}{3}e^{56}\right)\Bigg)\\
&-\left(\frac{a_2'}{a_2}+\frac{b'}{b}+\frac{c'}{c}+\frac{f'}{f}\right)a_2bcf\,e^{2467}\wedge dt\\
&+a_2bcf\Bigg((-6e^{17}+e^{36}-e^{45})\wedge e^{467}-e^2\wedge(e^{16}+e^{25}-3e^{37})\wedge e^{67}\\
&+e^{24}\wedge(-e^{14}+e^{23}+3e^{57})\wedge e^7-e^{246}\wedge\left(\frac{2}{3}e^{12}+\frac{1}{3}e^{34}-\frac{1}{3}e^{56}\right)\Bigg)\\
&+\left(\frac{a_1'}{a_1}+\frac{a_2'}{a_2}+2\frac{b'}{b}\right)a_1a_2b^2\,e^{1234}\wedge dt\\
&-a_1a_2b^2((6e^{27}-e^{35}-e^{46})\wedge e^{234}-e^1\wedge(-6e^{17}+e^{36}-e^{45})\wedge e^{34}\\
&+e^{12}\wedge(e^{15}-e^{26}+3e^{47})\wedge e^4-e^{123}\wedge(e^{16}+e^{25}-3e^{37}))\\
&-\left(\frac{a_1'}{a_1}+\frac{a_2'}{a_2}+2\frac{c'}{c}\right)a_1a_2c^2\,e^{1256}\wedge dt\\
&+a_1a_2c^2((6e^{27}-e^{35}-e^{46})\wedge e^{256}-e^1\wedge(-6e^{17}+e^{36}-e^{45})\wedge e^{56}\\
&+e^{12}\wedge(-e^{13}-e^{24}-3e^{67})\wedge e^6-e^{125}\wedge(-e^{14}+e^{23}+3e^{57}))\\
&+\left(2\frac{b'}{b}+2\frac{c'}{c}\right)b^2c^2\,e^{3456}\wedge dt\\
&-b^2c^2((e^{15}-e^{26}+3e^{47})\wedge e^{456}-e^3\wedge(e^{16}+e^{25}-3e^{37})\wedge e^{56}\\
&+e^{34}\wedge(-e^{13}-e^{24}-3e^{67})\wedge e^6-e^{345}\wedge(-e^{14}+e^{23}+3e^{57}))
\end{aligned}
$$

$$
\begin{aligned}
&+ a_1a_2f\Bigg((6e^{27} - e^{35} - e^{46}) \wedge e^{27} \wedge dt - e^1 \wedge (-6e^{17} + e^{36} - e^{45}) \wedge e^7 \wedge dt \\
&+e^{12} \wedge \left(\frac{2}{3}e^{12} + \frac{1}{3}e^{34} - \frac{1}{3}e^{56}\right) \wedge dt\Bigg) \\
&- b^2f\Bigg((e^{15} - e^{26} + 3e^{47}) \wedge e^{47} \wedge dt - e^3 \wedge (e^{16} + e^{25} - 3e^{37}) \wedge e^7 \wedge dt \\
&+e^{34} \wedge \left(\frac{2}{3}e^{12} + \frac{1}{3}e^{34} - \frac{1}{3}e^{56}\right) \wedge dt\Bigg) \\
&+ c^2f\Bigg((-e^{13} - e^{24} - 3e^{67}) \wedge e^{67} \wedge dt - e^5 \wedge (-e^{14} + e^{23} + 3e^{57}) \wedge e^7 \wedge dt \\
&+e^{56} \wedge \left(\frac{2}{3}e^{12} + \frac{1}{3}e^{34} - \frac{1}{3}e^{56}\right) \wedge dt\Bigg) \\
&+ a_1bc((6e^{27} - e^{35} - e^{46}) \wedge e^{35} \wedge dt - e^1 \wedge (e^{15} - e^{26} + 3e^{47}) \wedge e^5 \wedge dt \\
&+ e^{13} \wedge (-e^{13} - e^{24} - 3e^{67}) \wedge dt) \\
&+ a_1bc((6e^{27} - e^{35} - e^{46}) \wedge e^{46} \wedge dt - e^1 \wedge (e^{16} + e^{25} - 3e^{37}) \wedge e^6 \wedge dt \\
&+ e^{14} \wedge (-e^{14} + e^{23} + 3e^{57}) \wedge dt) \\
&- a_2bc((-6e^{17} + e^{36} - e^{45}) \wedge e^{36} \wedge dt - e^2 \wedge (e^{15} - e^{26} + 3e^{47}) \wedge e^6 \wedge dt \\
&+ e^{23} \wedge (-e^{14} + e^{23} + 3e^{57}) \wedge dt) \\
&+ a_2bc((-6e^{17} + e^{36} - e^{45}) \wedge e^{45} \wedge dt - e^2 \wedge (e^{16} + e^{25} - 3e^{37}) \wedge e^5 \wedge dt \\
&+ e^{24} \wedge (-e^{13} - e^{24} - 3e^{67}) \wedge dt) \\
=& - \left(\frac{a_1'}{a_1} + \frac{b'}{b} + \frac{c'}{c} + \frac{f'}{f}\right) a_1bcf\, e^{1367} \wedge dt \\
&+ \left(\frac{a_1'}{a_1} + \frac{b'}{b} + \frac{c'}{c} + \frac{f'}{f}\right) a_1bcf\, e^{1457} \wedge dt \\
&- \left(\frac{a_2'}{a_2} + \frac{b'}{b} + \frac{c'}{c} + \frac{f'}{f}\right) a_2bcf\, e^{2357} \wedge dt \\
&- \left(\frac{a_2'}{a_2} + \frac{b'}{b} + \frac{c'}{c} + \frac{f'}{f}\right) a_2bcf\, e^{2467} \wedge dt \\
&+ \left(\frac{a_1'}{a_1} + \frac{a_2'}{a_2} + 2\frac{b'}{b}\right) a_1a_2b^2\, e^{1234} \wedge dt \\
&- \left(\frac{a_1'}{a_1} + \frac{a_2'}{a_2} + 2\frac{c'}{c}\right) a_1a_2c^2\, e^{1256} \wedge dt \\
&+ \left(2\frac{b'}{b} + 2\frac{c'}{c}\right) b^2c^2\, e^{3456} \wedge dt
\end{aligned}
$$

$$
\begin{aligned}
&- a_1a_2f\, e^{2357} \wedge dt - a_1a_2f\, e^{2467} \wedge dt - a_1a_2f\, e^{1367} \wedge dt + a_1a_2f\, e^{1457} \wedge dt \\
&+ \frac{1}{3}a_1a_2f\, e^{1234} \wedge dt - \frac{1}{3}a_1a_2f\, e^{1256} \wedge dt \\
&+ b^2f\, e^{1457} \wedge dt - b^2f\, e^{2467} \wedge dt - b^2f\, e^{1367} \wedge dt - b^2f\, e^{2357} \wedge dt \\
&- \frac{2}{3}b^2f\, e^{1234} \wedge dt + \frac{1}{3}b^2f\, e^{3456} \wedge dt \\
&- c^2f\, e^{1367} \wedge dt - c^2f\, e^{2467} \wedge dt + c^2f\, e^{1457} \wedge dt - c^2f\, e^{2357} \wedge dt \\
&+ \frac{2}{3}c^2f\, e^{1256} \wedge dt + \frac{1}{3}c^2f\, e^{3456} \wedge dt \\
&+ 6a_1bc\, e^{2357} \wedge dt + a_1bc\, e^{3456} \wedge dt - a_1bc\, e^{1256} \wedge dt + 3a_1bc\, e^{1457} \wedge dt \\
&+ a_1bc\, e^{1234} \wedge dt - 3a_1bc\, e^{1367} \wedge dt \\
&+ 6a_1bc\, e^{2467} \wedge dt + a_1bc\, e^{3456} \wedge dt - a_1bc\, e^{1256} \wedge dt - 3a_1bc\, e^{1367} \wedge dt \\
&+ a_1bc\, e^{1234} \wedge dt + 3a_1bc\, e^{1457} \wedge dt \\
&+ 6a_2bc\, e^{1367} \wedge dt + a_2bc\, e^{3456} \wedge dt - a_2bc\, e^{1256} \wedge dt - 3a_2bc\, e^{2467} \wedge dt \\
&+ a_2bc\, e^{1234} \wedge dt - 3a_2bc\, e^{2357} \wedge dt \\
&- 6a_2bc\, e^{1457} \wedge dt + a_2bc\, e^{3456} \wedge dt - a_2bc\, e^{1256} \wedge dt - 3a_2bc\, e^{2357} \wedge dt \\
&+ a_2bc\, e^{1234} \wedge dt - 3a_2bc\, e^{2467} \wedge dt \\
= &\left(\left(\frac{a_1'}{a_1} + \frac{a_2'}{a_2} + 2\frac{b'}{b}\right) a_1a_2b^2 + \frac{1}{3}a_1a_2f - \frac{2}{3}b^2f + 2a_1bc + 2a_2bc\right) e^{1234} \wedge dt \\
&+ \left(-\left(\frac{a_1'}{a_1} + \frac{a_2'}{a_2} + 2\frac{c'}{c}\right) a_1a_2c^2 - \frac{1}{3}a_1a_2f + \frac{2}{3}c^2f - 2a_1bc - 2a_2bc\right) e^{1256} \wedge dt \\
&+ \left(-\left(\frac{a_1'}{a_1} + \frac{b'}{b} + \frac{c'}{c} + \frac{f'}{f}\right) a_1bcf - a_1a_2f - b^2f - c^2f - 6a_1bc + 6a_2bc\right) e^{1367} \wedge dt \\
&+ \left(\left(\frac{a_1'}{a_1} + \frac{b'}{b} + \frac{c'}{c} + \frac{f'}{f}\right) a_1bcf + a_1a_2f + b^2f + c^2f + 6a_1bc - 6a_2bc\right) e^{1457} \wedge dt \\
&+ \left(-\left(\frac{a_2'}{a_2} + \frac{b'}{b} + \frac{c'}{c} + \frac{f'}{f}\right) a_2bcf - a_1a_2f - b^2f - c^2f + 6a_1bc - 6a_2bc\right) e^{2357} \wedge dt \\
&+ \left(-\left(\frac{a_2'}{a_2} + \frac{b'}{b} + \frac{c'}{c} + \frac{f'}{f}\right) a_2bcf - a_1a_2f - b^2f - c^2f + 6a_1bc - 6a_2bc\right) e^{2467} \wedge dt \\
&+ \left(\left(2\frac{b'}{b} + 2\frac{c'}{c}\right) b^2c^2 + \frac{1}{3}b^2f + \frac{1}{3}c^2f + 2a_1bc + 2a_2bc\right) e^{3456} \wedge dt
\end{aligned}
$$

The equation $d\Omega = 0$ therefore is equivalent to the following system of ordinary differential equations:

$$\begin{aligned}
\frac{a_1'}{a_1} + \frac{a_2'}{a_2} + 2\frac{b'}{b} &= -\frac{1}{3}\frac{f}{b^2} + \frac{2}{3}\frac{f}{a_1 a_2} - 2\frac{c}{a_2 b} - 2\frac{c}{a_1 b} \\
\frac{a_1'}{a_1} + \frac{a_2'}{a_2} + 2\frac{c'}{c} &= -\frac{1}{3}\frac{f}{c^2} + \frac{2}{3}\frac{f}{a_1 a_2} - 2\frac{b}{a_2 c} - 2\frac{b}{a_1 c} \\
\frac{a_1'}{a_1} + \frac{b'}{b} + \frac{c'}{c} + \frac{f'}{f} &= -\frac{a_2}{bc} - \frac{b}{a_1 c} - \frac{c}{a_1 b} - 6\frac{1}{f} + 6\frac{a_2}{a_1 f} \\
\frac{a_2'}{a_2} + \frac{b'}{b} + \frac{c'}{c} + \frac{f'}{f} &= -\frac{a_1}{bc} - \frac{b}{a_2 c} - \frac{c}{a_2 b} + 6\frac{a_1}{a_2 f} - 6\frac{1}{f} \\
2\frac{b'}{b} + 2\frac{c'}{c} &= -\frac{1}{3}\frac{f}{c^2} - \frac{1}{3}\frac{f}{b^2} - 2\frac{a_1}{bc} - 2\frac{a_2}{bc}
\end{aligned}$$

This system can be simplified to:

$$\begin{aligned}
\frac{a_1'}{a_1} &= \frac{a_1}{bc} - 3\frac{a_1}{a_2 f} + 3\frac{a_2}{a_1 f} - \frac{b}{a_1 c} - \frac{c}{a_1 b} + \frac{1}{3}\frac{f}{a_1 a_2} \\
\frac{a_2'}{a_2} &= \frac{a_2}{bc} + 3\frac{a_1}{a_2 f} - 3\frac{a_2}{a_1 f} - \frac{b}{a_2 c} - \frac{c}{a_2 b} + \frac{1}{3}\frac{f}{a_1 a_2} \\
\frac{b'}{b} &= -\frac{1}{2}\frac{a_1}{bc} - \frac{1}{2}\frac{a_2}{bc} + \frac{1}{2}\frac{b}{a_1 c} + \frac{1}{2}\frac{b}{a_2 c} - \frac{1}{2}\frac{c}{a_1 b} - \frac{1}{2}\frac{c}{a_2 b} - \frac{1}{6}\frac{f}{b^2} \\
\frac{c'}{c} &= -\frac{1}{2}\frac{a_1}{bc} - \frac{1}{2}\frac{a_2}{bc} - \frac{1}{2}\frac{b}{a_1 c} - \frac{1}{2}\frac{b}{a_2 c} + \frac{1}{2}\frac{c}{a_1 b} + \frac{1}{2}\frac{c}{a_2 b} - \frac{1}{6}\frac{f}{c^2} \\
\frac{f'}{f} &= 3\frac{a_1}{a_2 f} + 3\frac{a_2}{a_1 f} - \frac{1}{3}\frac{f}{a_1 a_2} + \frac{1}{6}\frac{f}{b^2} + \frac{1}{6}\frac{f}{c^2} - 6\frac{1}{f}
\end{aligned}$$

For cosmetic reasons, we rearrange the summands on the right hand side:

$$\begin{aligned}
\frac{a_1'}{a_1} &= \frac{a_1^2 - b^2 - c^2}{a_1 bc} + 3\frac{a_2^2 - a_1^2}{a_1 a_2 f} + \frac{1}{3}\frac{f}{a_1 a_2} \\
\frac{a_2'}{a_2} &= \frac{a_2^2 - b^2 - c^2}{a_2 bc} + 3\frac{a_1^2 - a_2^2}{a_1 a_2 f} + \frac{1}{3}\frac{f}{a_1 a_2} \\
\frac{b'}{b} &= \frac{1}{2}\frac{b^2 - a_1^2 - c^2}{a_1 bc} + \frac{1}{2}\frac{b^2 - a_2^2 - c^2}{a_2 bc} - \frac{1}{6}\frac{f}{b^2} \\
\frac{c'}{c} &= \frac{1}{2}\frac{c^2 - a_1^2 - b^2}{a_1 bc} + \frac{1}{2}\frac{c^2 - a_2^2 - b^2}{a_2 bc} - \frac{1}{6}\frac{f}{c^2} \\
\frac{f'}{f} &= 3\frac{(a_1 - a_2)^2}{a_1 a_2 f} - \frac{1}{3}\frac{f}{a_1 a_2} + \frac{1}{6}\frac{f}{b^2} + \frac{1}{6}\frac{f}{c^2}
\end{aligned}$$

The above system is equivalent to those equations which can be found in the literature. By changing t into $-t$ and f into $-6f$, we obtain the equations which have been considered by Kanno and Yasui in [49].

As in the previous section, we also reverse the direction of t. We finally obtain the system which we will use throughout Section 5.4:

$$
\begin{aligned}
\frac{a_1'}{a_1} &= \frac{b^2+c^2-a_1^2}{a_1bc} + 3\frac{a_1^2-a_2^2}{a_1a_2f} - \frac{1}{3}\frac{f}{a_1a_2}\\
\frac{a_2'}{a_2} &= \frac{b^2+c^2-a_2^2}{a_2bc} + 3\frac{a_2^2-a_1^2}{a_1a_2f} - \frac{1}{3}\frac{f}{a_1a_2}\\
\frac{b'}{b} &= \frac{1}{2}\frac{a_1^2+c^2-b^2}{a_1bc} + \frac{1}{2}\frac{a_2^2+c^2-b^2}{a_2bc} + \frac{1}{6}\frac{f}{b^2}\\
\frac{c'}{c} &= \frac{1}{2}\frac{a_1^2+b^2-c^2}{a_1bc} + \frac{1}{2}\frac{a_2^2+b^2-c^2}{a_2bc} + \frac{1}{6}\frac{f}{c^2}\\
\frac{f'}{f} &= -3\frac{(a_1-a_2)^2}{a_1a_2f} + \frac{1}{3}\frac{f}{a_1a_2} - \frac{1}{6}\frac{f}{b^2} - \frac{1}{6}\frac{f}{c^2}
\end{aligned}
$$

C.3 The Einstein condition for the generic Aloff-Wallach spaces and $N^{1,-1}$ as principal orbits

Let (M, g) be a Riemannian cohomogeneity-one manifold whose principal orbit is an Aloff-Wallach space $N^{k,l}$. In this section, we assume that $N^{k,l}$ is generic or $\cong N^{1,-1}$. The restriction of the metric to a principal orbit we will denote by g, too. In the $N^{1,-1}$-case, we require that g is diagonal with respect to the basis $(e_i)_{1\leqslant i\leqslant 7}$ of $\mathfrak{m}$. If $N^{k,l}$ is generic, this condition is automatically satisfied. In both cases, we are able to describe g by a matrix of type (5.31). The aim of this section is to express the Einstein condition as a system of ordinary differential equations for the metric functions a, b, c, and f. In order to do this, we have to calculate the Ricci-curvature of $N^{k,l}$. We will work with the Corollary 3.1.37 from page 51. Before we can make use of the formula from that Corollary, we first have to extend $g \in S^2(\mathfrak{m})$ to an inner product on all of $\mathfrak{su}(3)$. This will be done by setting $g(e_8, e_8) := 1$. We choose the following g-orthonormal basis $(X_i)_{1\leqslant i\leqslant 8}$ of $\mathfrak{su}(3)$:

$$X_1 := \tfrac{1}{a}e_1 \quad X_2 := \tfrac{1}{a}e_2 \quad X_3 := \tfrac{1}{b}e_3 \quad X_4 := \tfrac{1}{b}e_4$$

$$X_5 := \tfrac{1}{c}e_5 \quad X_6 := \tfrac{1}{c}e_6 \quad X_7 := \tfrac{1}{f}e_7 \quad X_8 := e_8$$

The explicit formulas for the de^{i*}_p contain most of the structure constants of the Lie algebra $\mathfrak{su}(3)$ as coefficients. Therefore, the $c_{ij}^k := g([X_i, X_j], X_k)$ can be obtained from the formulas on page 246 and the commutator relations for e_8. We have:

$c_{27}^1 = (3k+3l)\frac{1}{f}$	$c_{28}^1 = k - l$	$c_{35}^1 = -\frac{a}{bc}$	$c_{46}^1 = -\frac{a}{bc}$
$c_{17}^2 = (-3k-3l)\frac{1}{f}$	$c_{18}^2 = -k + l$	$c_{36}^2 = \frac{a}{bc}$	$c_{45}^2 = -\frac{a}{bc}$
$c_{15}^3 = \frac{b}{ac}$	$c_{26}^3 = -\frac{b}{ac}$	$c_{47}^3 = 3l\frac{1}{f}$	$c_{48}^3 = 2k + l$
$c_{16}^4 = \frac{b}{ac}$	$c_{25}^4 = \frac{b}{ac}$	$c_{37}^4 = -3l\frac{1}{f}$	$c_{38}^4 = -2k - l$
$c_{13}^5 = -\frac{c}{ab}$	$c_{24}^5 = -\frac{c}{ab}$	$c_{67}^5 = -3k\frac{1}{f}$	$c_{68}^5 = 2l + k$
$c_{14}^6 = -\frac{c}{ab}$	$c_{23}^6 = \frac{c}{ab}$	$c_{57}^6 = 3k\frac{1}{f}$	$c_{58}^6 = -2l - k$
$c_{12}^7 = \frac{k+l}{k^2+lk+l^2}\frac{f}{a^2}$	$c_{34}^7 = \frac{l}{k^2+lk+l^2}\frac{f}{b^2}$	$c_{56}^7 = -\frac{k}{k^2+lk+l^2}\frac{f}{c^2}$	
$c_{12}^8 = \frac{k-l}{k^2+lk+l^2}\frac{1}{a^2}$	$c_{34}^8 = \frac{2k+l}{k^2+lk+l^2}\frac{1}{b^2}$	$c_{56}^8 = \frac{2l+k}{k^2+lk+l^2}\frac{1}{c^2}$	

The c_{ij}^k which are not contained in the above table and are not determined by the skew-symmetry vanish. We calculate the coefficients of the Killing form κ. The biinvariant metric q on $\mathfrak{su}(3)$ which is defined by $q(X, Y) := -\mathrm{tr}(XY)$ has to be a multiple of κ. We apply formula (3.2) and obtain:

$$\begin{aligned}
&\kappa(X_1,X_1)\\
&= 2\left(c_{12}^7c_{17}^2 + c_{12}^8c_{18}^2 + c_{13}^5c_{15}^3 + c_{14}^6c_{16}^4\right)\\
&= 2\left(\frac{k+l}{k^2+lk+l^2}\frac{f}{a^2}(-3k-3l)\frac{1}{f} + \frac{k-l}{k^2+lk+l^2}\frac{1}{a^2}(-k+l) + \left(-\frac{c}{ab}\right)\frac{b}{ac} + \left(-\frac{c}{ab}\right)\frac{b}{ac}\right)\\
&= -\frac{12}{a^2}
\end{aligned}$$

Since $q(e_1, e_1) = 2$, we have $\kappa = -6q$. Therefore:

$$\begin{aligned}
\kappa(X_2,X_2) &= -\frac{12}{a^2}\\
\kappa(X_3,X_3) = \kappa(X_4,X_4) &= -\frac{12}{b^2}\\
\kappa(X_5,X_5) = \kappa(X_6,X_6) &= -\frac{12}{c^2}\\
\kappa(X_7,X_7) &= -\frac{36(k^2+lk+l^2)}{f^2}
\end{aligned}$$

The off-diagonal coefficients of κ vanish, since $(X_i)_{1\leqslant i\leqslant 8}$ is a q-orthogonal basis. We are now able to apply Corollary 3.1.37 and calculate the Ricci-curvature Ric of $N^{k,l}$. Since Ric $\in S^2(\mathfrak{m})$ has to be $U(1)_{k,l}$-invariant, we can easily prove by Schur's lemma that Ric is diagonal if $N^{k,l}$ is generic. It is possible to prove that if $N^{k,l} \cong N^{1,-1}$ and g is diagonal, Ric is diagonal, too. We assume without loss of generality that $k = 1$, $l = -1$. Ric $\in S^2(\mathfrak{m}) \cong S^2(T_{eU(1)_{1,-1}}N^{1,-1})$ is invariant under all isometries which fix the point $eU(1)_{1,-1} \in N^{1,-1}$. The following maps are well-defined and fix $eU(1)_{1,-1}$:

$$gU(1)_{1,-1} \mapsto hgh^{-1}U(1)_{1,-1} \quad \text{where } h \in \mathrm{Norm}_{SU(3)}U(1)_{1,-1} \cong U(1)^2$$

If the metric on $N^{1,-1}$ is diagonal, all of these maps are isometries. Ric therefore has to be invariant under the action of the normalizer. By decomposing $S^2(\mathfrak{m})$ into irreducible $U(1)^2$-modules, we can deduce that Ric has to be diagonal, too. Therefore, the Ricci-tensor is in any case determined by the values of the four coefficients below:

$$\begin{aligned}
\mathrm{Ric}(X_1,X_1) &= -\frac{1}{2}\left((c_{17}^2)^2 + (c_{15}^3)^2 + (c_{16}^4)^2 + (c_{13}^5)^2 + (c_{14}^6)^2 + (c_{12}^7)^2\right)\\
&\quad + \frac{1}{4}\left((c_{27}^1)^2 + (c_{35}^1)^2 + (c_{46}^1)^2 + (c_{53}^1)^2 + (c_{64}^1)^2 + (c_{72}^1)^2\right) - \frac{1}{2}\kappa(X_1,X_1)\\
&= -\frac{1}{2}\left((3k+3l)^2\frac{1}{f^2} + \frac{b^2}{a^2c^2} + \frac{b^2}{a^2c^2} + \frac{c^2}{a^2b^2} + \frac{c^2}{a^2b^2} + \frac{(k+l)^2}{(k^2+lk+l^2)^2}\frac{f^2}{a^4}\right)\\
&\quad + \frac{1}{4}\left((3k+3l)^2\frac{1}{f^2} + \frac{a^2}{b^2c^2} + \frac{a^2}{b^2c^2} + \frac{a^2}{b^2c^2} + \frac{a^2}{b^2c^2} + (3k+3l)^2\frac{1}{f^2}\right) + \frac{6}{a^2}\\
&= \frac{6}{a^2} - \frac{1}{2}\frac{(k+l)^2}{(k^2+lk+l^2)^2}\frac{f^2}{a^4} + \frac{a^4-b^4-c^4}{a^2b^2c^2}
\end{aligned}$$

$$\begin{aligned}
\mathrm{Ric}(X_3,X_3) &= -\frac{1}{2}\left((c_{35}^1)^2+(c_{36}^2)^2+(c_{37}^4)^2+(c_{31}^5)^2+(c_{32}^6)^2+(c_{34}^7)^2\right)\\
&\quad+\frac{1}{4}\left((c_{15}^3)^2+(c_{26}^3)^2+(c_{47}^3)^2+(c_{51}^3)^2+(c_{62}^3)^2+(c_{74}^3)^2\right)-\frac{1}{2}\kappa(X_3,X_3)\\
&= -\frac{1}{2}\left(\frac{a^2}{b^2c^2}+\frac{a^2}{b^2c^2}+(3l)^2\frac{1}{f^2}+\frac{c^2}{a^2b^2}+\frac{c^2}{a^2b^2}+\frac{l^2}{(k^2+lk+l^2)^2}\frac{f^2}{b^4}\right)\\
&\quad+\frac{1}{4}\left(\frac{b^2}{a^2c^2}+\frac{b^2}{a^2c^2}+(3l)^2\frac{1}{f^2}+\frac{b^2}{a^2c^2}+\frac{b^2}{a^2c^2}+(3l)^2\frac{1}{f^2}\right)+\frac{6}{b^2}\\
&= \frac{6}{b^2}-\frac{1}{2}\frac{l^2}{(k^2+lk+l^2)^2}\frac{f^2}{b^4}+\frac{b^4-a^4-c^4}{a^2b^2c^2}
\end{aligned}$$

$$\begin{aligned}
\mathrm{Ric}(X_5,X_5) &= -\frac{1}{2}\left((c_{53}^1)^2+(c_{54}^2)^2+(c_{51}^3)^2+(c_{52}^4)^2+(c_{57}^6)^2+(c_{56}^7)^2\right)\\
&\quad+\frac{1}{4}\left((c_{13}^5)^2+(c_{24}^5)^2+(c_{31}^5)^2+(c_{42}^5)^2+(c_{67}^5)^2+(c_{76}^5)^2\right)-\frac{1}{2}\kappa(X_5,X_5)\\
&= -\frac{1}{2}\left(\frac{a^2}{b^2c^2}+\frac{a^2}{b^2c^2}+\frac{b^2}{a^2c^2}+\frac{b^2}{a^2c^2}+(3k)^2\frac{1}{f^2}+\frac{k^2}{(k^2+lk+l^2)^2}\frac{f^2}{c^4}\right)\\
&\quad+\frac{1}{4}\left(\frac{c^2}{a^2b^2}+\frac{c^2}{a^2b^2}+\frac{c^2}{a^2b^2}+\frac{c^2}{a^2b^2}+(3k)^2\frac{1}{f^2}+(3k)^2\frac{1}{f^2}\right)+\frac{6}{c^2}\\
&= \frac{6}{c^2}-\frac{1}{2}\frac{k^2}{(k^2+lk+l^2)^2}\frac{f^2}{c^4}+\frac{c^4-a^4-b^4}{a^2b^2c^2}
\end{aligned}$$

$$\begin{aligned}
\mathrm{Ric}(X_7,X_7) &= -\frac{1}{2}\left((c_{72}^1)^2+(c_{71}^2)^2+(c_{74}^3)^2+(c_{73}^4)^2+(c_{76}^5)^2+(c_{75}^6)^2\right)\\
&\quad+\frac{1}{4}\left((c_{12}^7)^2+(c_{21}^7)^2+(c_{34}^7)^2+(c_{43}^7)^2+(c_{56}^7)^2+(c_{65}^7)^2\right)-\frac{1}{2}\kappa(X_7,X_7)\\
&= -\frac{1}{2}\left((3k+3l)^2\frac{1}{f^2}+(3k+3l)^2\frac{1}{f^2}+(3l)^2\frac{1}{f^2}+(3l)^2\frac{1}{f^2}+(3k)^2\frac{1}{f^2}+(3k)^2\frac{1}{f^2}\right)\\
&\quad+\frac{1}{4}\left(\frac{(k+l)^2}{(k^2+lk+l^2)^2}\frac{f^2}{a^4}+\frac{(k+l)^2}{(k^2+lk+l^2)^2}\frac{f^2}{a^4}+\frac{l^2}{(k^2+lk+l^2)^2}\frac{f^2}{b^4}\right.\\
&\quad\left.+\frac{l^2}{(k^2+lk+l^2)^2}\frac{f^2}{b^4}+\frac{k^2}{(k^2+lk+l^2)^2}\frac{f^2}{c^4}+\frac{k^2}{(k^2+lk+l^2)^2}\frac{f^2}{c^4}\right)+\frac{18(k^2+lk+l^2)}{f^2}\\
&= \frac{1}{2}\frac{(k+l)^2}{(k^2+lk+l^2)^2}\frac{f^2}{a^4}+\frac{1}{2}\frac{l^2}{(k^2+lk+l^2)^2}\frac{f^2}{b^4}+\frac{1}{2}\frac{k^2}{(k^2+lk+l^2)^2}\frac{f^2}{c^4}
\end{aligned}$$

With help of the formulas (3.13), we see that the metric g is Einstein with Einstein constant λ if and only if the following equations hold:

$$\begin{aligned}
-\frac{a''}{a}+\frac{a'^2}{a^2}-\frac{a'}{a}\left(2\frac{a'}{a}+2\frac{b'}{b}+2\frac{c'}{c}+\frac{f'}{f}\right)+\frac{6}{a^2}-\frac{1}{2}\frac{(k+l)^2}{(k^2+lk+l^2)^2}\frac{f^2}{a^4}+\frac{a^4-b^4-c^4}{a^2b^2c^2} &= \lambda\\
-\frac{b''}{b}+\frac{b'^2}{b^2}-\frac{b'}{b}\left(2\frac{a'}{a}+2\frac{b'}{b}+2\frac{c'}{c}+\frac{f'}{f}\right)+\frac{6}{b^2}-\frac{1}{2}\frac{l^2}{(k^2+lk+l^2)^2}\frac{f^2}{b^4}+\frac{b^4-a^4-c^4}{a^2b^2c^2} &= \lambda\\
-\frac{c''}{c}+\frac{c'^2}{c^2}-\frac{c'}{c}\left(2\frac{a'}{a}+2\frac{b'}{b}+2\frac{c'}{c}+\frac{f'}{f}\right)+\frac{6}{c^2}-\frac{1}{2}\frac{k^2}{(k^2+lk+l^2)^2}\frac{f^2}{c^4}+\frac{c^4-a^4-b^4}{a^2b^2c^2} &= \lambda\\
-\frac{f''}{f}+\frac{f'^2}{f^2}-\frac{f'}{f}\left(2\frac{a'}{a}+2\frac{b'}{b}+2\frac{c'}{c}+\frac{f'}{f}\right)+\frac{1}{2}\frac{(k+l)^2}{(k^2+lk+l^2)^2}\frac{f^2}{a^4}+\frac{1}{2}\frac{l^2}{(k^2+lk+l^2)^2}\frac{f^2}{b^4} &\\
+\frac{1}{2}\frac{k^2}{(k^2+lk+l^2)^2}\frac{f^2}{c^4} &= \lambda\\
-2\frac{a''}{a}-2\frac{b''}{b}-2\frac{c''}{c}-\frac{f''}{f} &= \lambda
\end{aligned}$$

As we have remarked in 3.2.23, the above system should not be considered as over-determined, since it has unique short-time solutions for any choice of the metric and its first derivative on a principal orbit. The above equations are invariant under the change of the sign of t. Therefore, we may assume that t takes the same direction as in Section C.1. By a straightforward but lengthy calculation, it is possible to prove that the above system is automatically satisfied with $\lambda = 0$ if a, b, c, and f satisfy the equations for the holonomy reduction from Section C.1. This is not surprising, since any metric whose holonomy is contained in Spin(7) has to be Ricci-flat. Nevertheless, that calculation is an additional test that our equations are correct.

C.4 The Einstein condition for the exceptional Aloff-Wallach space $N^{1,1}$ as principal orbit

Again, let (M,g) be a Riemannian cohomogeneity-one manifold whose principal orbit is an Aloff-Wallach space. In this section, we assume that the principal orbit is the exceptional Aloff-Wallach space $N^{1,1}$ and that the restriction of the metric to any principal orbit is a diagonal metric of type (5.33). The aim of this section is to express the Einstein condition on g as a system of ordinary differential equations for the metric functions a_1, a_2, b, c, and f. As in the previous section, we will first calculate the Ricci-curvature of $N^{1,1}$ with help of Corollary 3.1.37. By defining $g(e_8,e_8) := 1$, g is as before extended to an inner product on all of $\mathfrak{su}(3)$. We choose the following g-orthonormal basis $(X_i)_{1\leqslant i\leqslant 8}$ of $\mathfrak{su}(3)$:

$$X_1 := \tfrac{1}{a_1}e_1 \quad X_2 := \tfrac{1}{a_2}e_2 \quad X_3 := \tfrac{1}{b}e_3 \quad X_4 := \tfrac{1}{b}e_4$$

$$X_5 := \tfrac{1}{c}e_5 \quad X_6 := \tfrac{1}{c}e_6 \quad X_7 := \tfrac{1}{f}e_7 \quad X_8 := e_8$$

If $\{i,j,k\}\cap\{1,2\} = \varnothing$, the coefficients c_{ij}^k, which describe the commutator relations of $\mathfrak{su}(3)$, remain the same as in the previous section. If exactly one of the indices is 1 (or 2), we have to replace the metric function a by a_1 (or a_2). The remaining coefficients are:

$$c_{27}^1 = 6\tfrac{a_1}{a_2 f} \quad c_{17}^2 = -6\tfrac{a_2}{a_1 f} \quad c_{12}^7 = \tfrac{2}{3}\tfrac{f}{a_1 a_2}$$

$$c_{28}^1 = 0 \quad c_{18}^2 = 0 \quad c_{12}^8 = 0$$

For the Killing-form κ, we obtain:

$$\kappa(X_1,X_1) = -\tfrac{12}{a_1^2} \quad \kappa(X_2,X_2) = -\tfrac{12}{a_2^2} \quad \kappa(X_3,X_3) = -\tfrac{12}{b^2} \quad \kappa(X_4,X_4) = -\tfrac{12}{b^2}$$

$$\kappa(X_5,X_5) = -\tfrac{12}{c^2} \quad \kappa(X_6,X_6) = -\tfrac{12}{c^2} \quad \kappa(X_7,X_7) = -\tfrac{108}{f^2}$$

The off-diagonal coefficients of κ vanish for the same reason as in the previous section. Since the Ricci-tensor has to be $U(1)_{1,1}$-invariant, we have $\mathrm{Ric}(X_3,X_3) = \mathrm{Ric}(X_4,X_4)$ and $\mathrm{Ric}(X_5,X_5) = \mathrm{Ric}(X_6,X_6)$. Unlike in the generic case, $\mathrm{span}(X_1,X_2)$ is not an irreducible $U(1)_{1,1}$-module. Therefore, $\mathrm{Ric}(X_1,X_1)$ does not necessarily equal $\mathrm{Ric}(X_2,X_2)$. By some calculations, which are similar to those in the previous section, we obtain:

$$\begin{aligned}\mathrm{Ric}(X_1,X_1) &= -\frac{1}{2}\left(36\frac{a_2^2}{a_1^2 f^2} + \frac{b^2}{a_1^2 c^2} + \frac{b^2}{a_1^2 c^2} + \frac{c^2}{a_1^2 b^2} + \frac{c^2}{a_1^2 b^2} + \frac{4}{9}\frac{f^2}{a_1^2 a_2^2}\right) \\ &\quad + \frac{1}{4}\left(36\frac{a_1^2}{a_2^2 f^2} + \frac{a_1^2}{b^2c^2} + \frac{a_1^2}{b^2c^2} + \frac{a_1^2}{b^2c^2} + \frac{a_1^2}{b^2c^2} + 36\frac{a_1^2}{a_2^2 f^2}\right) + \frac{6}{a_1^2} \\ &= \frac{6}{a_1^2} - \frac{2}{9}\frac{f^2}{a_1^2 a_2^2} + 18\frac{a_1^4 - a_2^4}{a_1^2 a_2^2 f^2} + \frac{a_1^4 - b^4 - c^4}{a_1^2 b^2 c^2}\end{aligned}$$

$$
\begin{aligned}
\operatorname{Ric}(X_2, X_2) &= -\frac{1}{2}\left(36\frac{a_1^2}{a_2^2 f^2} + \frac{b^2}{a_2^2 c^2} + \frac{b^2}{a_2^2 c^2} + \frac{c^2}{a_2^2 b^2} + \frac{c^2}{a_2^2 b^2} + \frac{4}{9}\frac{f^2}{a_1^2 a_2^2}\right) \\
&\quad + \frac{1}{4}\left(36\frac{a_2^2}{a_1^2 f^2} + \frac{a_2^2}{b^2 c^2} + \frac{a_2^2}{b^2 c^2} + \frac{a_2^2}{b^2 c^2} + \frac{a_2^2}{b^2 c^2} + 36\frac{a_2^2}{a_1^2 f^2}\right) + \frac{6}{a_2^2} \\
&= \frac{6}{a_2^2} - \frac{2}{9}\frac{f^2}{a_1^2 a_2^2} + 18\frac{a_2^4 - a_1^4}{a_1^2 a_2^2 f^2} + \frac{a_2^4 - b^4 - c^4}{a_2^2 b^2 c^2}
\end{aligned}
$$

$$
\begin{aligned}
\operatorname{Ric}(X_3, X_3) &= -\frac{1}{2}\left(\frac{a_1^2}{b^2 c^2} + \frac{a_2^2}{b^2 c^2} + 9\frac{1}{f^2} + \frac{c^2}{a_1^2 b^2} + \frac{c^2}{a_2^2 b^2} + \frac{1}{9}\frac{f^2}{b^4}\right) \\
&\quad + \frac{1}{4}\left(\frac{b^2}{a_1^2 c^2} + \frac{b^2}{a_2^2 c^2} + 9\frac{1}{f^2} + \frac{b^2}{a_1^2 c^2} + \frac{b^2}{a_2^2 c^2} + 9\frac{1}{f^2}\right) + \frac{6}{b^2} \\
&= \frac{6}{b^2} - \frac{1}{18}\frac{f^2}{b^4} + \frac{b^4 - a_1^4 - c^4}{2a_1^2 b^2 c^2} + \frac{b^4 - a_2^4 - c^4}{2a_2^2 b^2 c^2}
\end{aligned}
$$

$$
\begin{aligned}
\operatorname{Ric}(X_5, X_5) &= -\frac{1}{2}\left(\frac{a_1^2}{b^2 c^2} + \frac{a_2^2}{b^2 c^2} + \frac{b^2}{a_1^2 c^2} + \frac{b^2}{a_2^2 c^2} + 9\frac{1}{f^2} + \frac{1}{9}\frac{f^2}{c^4}\right) \\
&\quad + \frac{1}{4}\left(\frac{c^2}{a_1^2 b^2} + \frac{c^2}{a_2^2 b^2} + \frac{c^2}{a_1^2 b^2} + \frac{c^2}{a_2^2 b^2} + 9\frac{1}{f^2} + 9\frac{1}{f^2}\right) + \frac{6}{c^2} \\
&= \frac{6}{c^2} - \frac{1}{18}\frac{f^2}{c^4} + \frac{c^4 - a_1^4 - b^4}{2a_1^2 b^2 c^2} + \frac{c^4 - a_2^4 - b^4}{2a_2^2 b^2 c^2}
\end{aligned}
$$

$$
\begin{aligned}
\operatorname{Ric}(X_7, X_7) &= -\frac{1}{2}\left(36\frac{a_1^2}{a_2^2 f^2} + 36\frac{a_2^2}{a_1^2 f^2} + 9\frac{1}{f^2} + 9\frac{1}{f^2} + 9\frac{1}{f^2} + 9\frac{1}{f^2}\right) \\
&\quad + \frac{1}{4}\left(\frac{4}{9}\frac{f^2}{a_1^2 a_2^2} + \frac{4}{9}\frac{f^2}{a_1^2 a_2^2} + \frac{1}{9}\frac{f^2}{b^4} + \frac{1}{9}\frac{f^2}{b^4} + \frac{1}{9}\frac{f^2}{c^4} + \frac{1}{9}\frac{f^2}{c^4}\right) + \frac{54}{f^2} \\
&= \frac{36}{f^2} - 18\frac{a_1^2}{a_2^2 f^2} - 18\frac{a_2^2}{a_1^2 f^2} + \frac{2}{9}\frac{f^2}{a_1^2 a_2^2} + \frac{1}{18}\frac{f^2}{b^4} + \frac{1}{18}\frac{f^2}{c^4}
\end{aligned}
$$

Unfortunately, we cannot prove by the same arguments as in Section C.3 that the Ricci-tensor is diagonal. The reason for this is that those arguments can be applied only if g is invariant under the standard maximal torus of $SU(3)$. In that situation, we necessarily have $a_1^2 = a_2^2$. Since we want to choose a_1 and a_2 freely, we have to explicitly calculate the possible non-diagonal coefficients of the Ricci-tensor. Since the basis $(X_i)_{1\leqslant i\leqslant 8}$ is q-orthogonal, the summand $-\frac{1}{2}\kappa$ in the formula for the Ricci-curvature vanishes. We obtain:

$$
\begin{aligned}
\operatorname{Ric}(X_1, X_2) &= -\frac{1}{2}\left(c_{17}^2 c_{27}^2 + c_{15}^3 c_{25}^3 + c_{16}^4 c_{26}^4 + c_{13}^5 c_{23}^5 + c_{14}^6 c_{24}^6 + c_{12}^7 c_{22}^7\right) \\
&\quad + \frac{1}{4}\left(c_{27}^1 c_{27}^2 + c_{35}^1 c_{35}^2 + c_{46}^1 c_{46}^2 + c_{53}^1 c_{53}^2 + c_{64}^1 c_{64}^2 + c_{72}^1 c_{72}^2\right) \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Ric}(X_1, X_7) &= -\frac{1}{2}\left(c_{17}^2 c_{77}^2 + c_{15}^3 c_{75}^3 + c_{16}^4 c_{76}^4 + c_{13}^5 c_{73}^5 + c_{14}^6 c_{74}^6 + c_{12}^7 c_{72}^7\right) \\
&\quad + \frac{1}{4}\left(c_{27}^1 c_{27}^7 + c_{35}^1 c_{35}^7 + c_{46}^1 c_{46}^7 + c_{53}^1 c_{53}^7 + c_{64}^1 c_{64}^7 + c_{72}^1 c_{72}^7\right) \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Ric}(X_2, X_7) &= -\frac{1}{2}\left(c_{27}^1 c_{77}^1 + c_{26}^3 c_{76}^3 + c_{25}^4 c_{75}^4 + c_{24}^5 c_{74}^5 + c_{23}^6 c_{73}^6 + c_{21}^7 c_{71}^7\right) \\
&\quad + \frac{1}{4}\left(c_{17}^2 c_{17}^7 + c_{36}^2 c_{36}^7 + c_{45}^2 c_{45}^7 + c_{54}^2 c_{54}^7 + c_{63}^2 c_{63}^7 + c_{71}^2 c_{71}^7\right) \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Ric}(X_3, X_5) &= -\frac{1}{2}\left(c_{35}^1 c_{55}^1 + c_{36}^2 c_{56}^2 + c_{37}^4 c_{57}^4 + c_{31}^5 c_{51}^5 + c_{32}^6 c_{52}^6 + c_{34}^7 c_{54}^7\right) \\
&\quad + \frac{1}{4}\left(c_{15}^3 c_{15}^5 + c_{26}^3 c_{26}^5 + c_{47}^3 c_{47}^5 + c_{51}^3 c_{51}^5 + c_{62}^3 c_{62}^5 + c_{74}^3 c_{74}^5\right) \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Ric}(X_3, X_6) &= -\frac{1}{2}\left(c_{35}^1 c_{65}^1 + c_{36}^2 c_{66}^2 + c_{37}^4 c_{67}^4 + c_{31}^5 c_{61}^5 + c_{32}^6 c_{62}^6 + c_{34}^7 c_{64}^7\right) \\
&\quad + \frac{1}{4}\left(c_{15}^3 c_{15}^6 + c_{26}^3 c_{26}^6 + c_{47}^3 c_{47}^6 + c_{51}^3 c_{51}^6 + c_{62}^3 c_{62}^6 + c_{74}^3 c_{74}^6\right) \\
&= 0
\end{aligned}
$$

The above calculations prove that the Ricci-tensor is indeed diagonal. By inserting the coefficients of the Ricci-tensor into the formulas (3.13), we obtain the following system of ordinary differential equations which is equivalent to the Einstein condition:

$$
\begin{aligned}
-\frac{a_1''}{a_1} + \frac{a_1'^2}{a_1^2} - \frac{a_1'}{a_1}\left(\frac{a_1'}{a_1} + \frac{a_2'}{a_2} + 2\frac{b'}{b} + 2\frac{c'}{c} + \frac{f'}{f}\right) + \frac{6}{a_1^2} - \frac{2}{9}\frac{f^2}{a_1^2 a_2^2} + 18\frac{a_1^4 - a_2^4}{a_1^2 a_2^2 f^2} + \frac{a_1^4 - b^4 - c^4}{a_1^2 b^2 c^2} &= \lambda \\
-\frac{a_2''}{a_2} + \frac{a_2'^2}{a_2^2} - \frac{a_2'}{a_2}\left(\frac{a_1'}{a_1} + \frac{a_2'}{a_2} + 2\frac{b'}{b} + 2\frac{c'}{c} + \frac{f'}{f}\right) + \frac{6}{a_2^2} - \frac{2}{9}\frac{f^2}{a_1^2 a_2^2} + 18\frac{a_2^4 - a_1^4}{a_1^2 a_2^2 f^2} + \frac{a_2^4 - b^4 - c^4}{a_2^2 b^2 c^2} &= \lambda \\
-\frac{b''}{b} + \frac{b'^2}{b^2} - \frac{b'}{b}\left(\frac{a_1'}{a_1} + \frac{a_2'}{a_2} + 2\frac{b'}{b} + 2\frac{c'}{c} + \frac{f'}{f}\right) + \frac{6}{b^2} - \frac{1}{18}\frac{f^2}{b^4} + \frac{b^4 - a_1^4 - c^4}{2a_1^2 b^2 c^2} + \frac{b^4 - a_2^4 - c^4}{2a_2^2 b^2 c^2} &= \lambda \\
-\frac{c''}{c} + \frac{c'^2}{c^2} - \frac{c'}{c}\left(\frac{a_1'}{a_1} + \frac{a_2'}{a_2} + 2\frac{b'}{b} + 2\frac{c'}{c} + \frac{f'}{f}\right) + \frac{6}{c^2} - \frac{1}{18}\frac{f^2}{c^4} + \frac{c^4 - a_1^4 - b^4}{2a_1^2 b^2 c^2} + \frac{c^4 - a_2^4 - b^4}{2a_2^2 b^2 c^2} &= \lambda \\
-\frac{f''}{f} + \frac{f'^2}{f^2} - \frac{f'}{f}\left(\frac{a_1'}{a_1} + \frac{a_2'}{a_2} + 2\frac{b'}{b} + 2\frac{c'}{c} + \frac{f'}{f}\right) + \frac{36}{f^2} - 18\frac{a_1^2}{a_2^2 f^2} - 18\frac{a_2^2}{a_1^2 f^2} + \frac{2}{9}\frac{f^2}{a_1^2 a_2^2} + \frac{1}{18}\frac{f^2}{b^4} + \frac{1}{18}\frac{f^2}{c^4} &= \lambda \\
-\frac{a_1''}{a_1} - \frac{a_2''}{a_2} - 2\frac{b''}{b} - 2\frac{c''}{c} - \frac{f''}{f} &= \lambda
\end{aligned}
$$

As in the previous section, λ denotes the Einstein constant. The remarks which we have made on the system in Section C.3 are also true for the above equations: It is possible to change the sign of t without changing the equations. Therefore, we can assume that this sign is the same as in Section C.2. Furthermore, the above system is not overdetermined and we can prove that it is automatically satisfied with $\lambda = 0$ if a_1, a_2, b, c, and f satisfy the first-order system which we have obtained in C.2.

Appendix D

Calculations for the principal orbit $SU(3)/U(1)^2$

In this chapter, we will carry out the calculations which we have omitted in Section 6.2. In particular, we will deduce the equations for the reduction of the holonomy to a subgroup of G_2.

Let (M, g) be a Riemannian manifold equipped with an isometric $SU(3)$-action of cohomogeneity one with principal orbits of type $SU(3)/U(1)^2$. We have proven that any $SU(3)$-invariant G_2-structure on M has a frame which is described by (6.10). This frame depends on a parameter θ. As in Section 6.2, we denote the three-form which is associated to this frame by ω_θ. First, we compute ω_0. We obtain:

$$\begin{aligned}
\omega_0 &= f^{123} + f^{145} - f^{167} + f^{246} + f^{257} + f^{347} - f^{356} \\
&= -a^2\, e^{12} \wedge dt + b^2\, e^{34} \wedge dt - c^2\, e^{56} \wedge dt + abc\, e^{236} - abc\, e^{245} - abc\, e^{135} - abc\, e^{146} \\
&= -abc\, e^{135} - abc\, e^{146} + abc\, e^{236} - abc\, e^{245} - a^2\, e^{12} \wedge dt + b^2\, e^{34} \wedge dt - c^2\, e^{56} \wedge dt
\end{aligned}$$

Let R_θ be the matrix

$$\begin{pmatrix}
\cos\theta & \sin\theta & & & & \\
-\sin\theta & \cos\theta & & & & \\
& & \cos\theta & \sin\theta & & \\
& & -\sin\theta & \cos\theta & & \\
& & & & \cos\theta & \sin\theta \\
& & & & -\sin\theta & \cos\theta
\end{pmatrix}$$

acting on the basis (6.8). It is easily possible to show the following relations:

$$\begin{aligned}
R_\theta^* e^{12} &= e^{12}\\
R_\theta^* e^{34} &= e^{34}\\
R_\theta^* e^{56} &= e^{56}\\
R_\theta^*(-e^{135}-e^{146}+e^{236}-e^{245}) &= \cos 3\theta(-e^{135}-e^{146}+e^{236}-e^{245})\\
&\quad -\sin 3\theta(e^{136}-e^{145}+e^{235}+e^{246})\\
R_\theta^*(e^{136}-e^{145}+e^{235}+e^{246}) &= \sin 3\theta(-e^{135}-e^{146}+e^{236}-e^{245})\\
&\quad +\cos 3\theta(e^{136}-e^{145}+e^{235}+e^{246})
\end{aligned} \tag{D.1}$$

where the asterisk denotes the canonical action of $GL(6)$ on $\bigwedge^* T^*M$. Therefore, we obtain for the three-form ω_θ:

$$\begin{aligned}
\omega_\theta = {}& abc\ \cos 3\theta(-e^{135}-e^{146}+e^{236}-e^{245}) - abc\ \sin 3\theta(e^{136}-e^{145}+e^{235}+e^{246})\\
&- a^2\, e^{12}\wedge dt + b^2\, e^{34}\wedge dt - c^2\, e^{56}\wedge dt\,.
\end{aligned}$$

By a short calculation, we can determine the four-form $*\omega_0$:

$$\begin{aligned}
*\omega_0 &= -f^{1247}+f^{1256}+f^{1346}+f^{1357}-f^{2345}+f^{2367}+f^{4567}\\
&= dt\wedge abc\, e^{235} + dt\wedge abc\, e^{246} + dt\wedge abc\, e^{136} - dt\wedge abc\, e^{145}\\
&\quad - a^2b^2\, e^{2134} - a^2c^2\, e^{2165} - b^2c^2\, e^{3465}\\
&= -abc\, e^{235}\wedge dt - abc\, e^{246}\wedge dt - abc\, e^{136}\wedge dt + abc\, e^{145}\wedge dt\\
&\quad + a^2b^2\, e^{1234} - a^2c^2\, e^{1256} + b^2c^2\, e^{3456}\\
&= a^2b^2\, e^{1234} - a^2c^2\, e^{1256} + b^2c^2\, e^{3456}\\
&\quad - abc\, e^{136}\wedge dt + abc\, e^{145}\wedge dt - abc\, e^{235}\wedge dt - abc\, e^{246}\wedge dt
\end{aligned}$$

With help of the formulas (D.1), we obtain for $*\omega_\theta$:

$$\begin{aligned}
*\omega_\theta = {}& a^2b^2\, e^{1234} - a^2c^2\, e^{1256} + b^2c^2\, e^{3456}\\
&- abc\ \cos 3\theta(e^{136}-e^{145}+e^{235}+e^{246})\wedge dt\\
&- abc\ \sin 3\theta(-e^{135}-e^{146}+e^{236}-e^{245})\wedge dt
\end{aligned}$$

Analogously to Appendix C, we obtain for the exterior derivatives of the one-forms e^{i^*} which are dual to the Killing vector fields e_i^*:

$$\begin{aligned}
de_p^{1*} &= -e_p^{35*} - e_p^{46*}\\
de_p^{2*} &= e_p^{36*} - e_p^{45*}\\
de_p^{3*} &= e_p^{15*} - e_p^{26*}\\
de_p^{4*} &= e_p^{16*} + e_p^{25*}\\
de_p^{5*} &= -e_p^{13*} - e_p^{24*}\\
de_p^{6*} &= -e_p^{14*} + e_p^{23*}
\end{aligned}$$

In the above equations, $p \in SU(3)/U(1)^2$ denotes the point whose tangent space we identify with $\mathfrak{m}$. We are now able to determine exterior derivative of ω_θ:

$$
\begin{aligned}
d\omega_\theta = &-\left(\left(\frac{a'}{a}+\frac{b'}{b}+\frac{c'}{c}\right)\cos 3\theta - 3\theta'\ \sin 3\theta\right) abc\,(-e^{135}-e^{146}+e^{236}-e^{245})\wedge dt\\
&+ abc\ \cos 3\theta(-e^{3535}-e^{4635}-e^{1155}+e^{1265}-e^{1313}-e^{1324})\\
&+ abc\ \cos 3\theta(-e^{3546}-e^{4646}-e^{1166}-e^{1256}-e^{1414}+e^{1423})\\
&- abc\ \cos 3\theta(e^{3636}-e^{4536}-e^{2156}+e^{2266}-e^{2314}+e^{2323})\\
&+ abc\ \cos 3\theta(e^{3645}-e^{4545}-e^{2165}-e^{2255}-e^{2413}-e^{2424})\\
&+\left(\left(\frac{a'}{a}+\frac{b'}{b}+\frac{c'}{c}\right)\sin 3\theta + 3\theta'\ \cos 3\theta\right) abc\,(e^{136}-e^{145}+e^{235}+e^{246})\wedge dt\\
&+ abc\ \sin 3\theta(-e^{3536}-e^{4636}-e^{1156}+e^{1266}-e^{1314}+e^{1323})\\
&- abc\ \sin 3\theta(-e^{3545}-e^{4645}-e^{1165}-e^{1255}-e^{1413}-e^{1424})\\
&+ abc\ \sin 3\theta(e^{3635}-e^{4535}-e^{2155}+e^{2265}-e^{2313}-e^{2324})\\
&+ abc\ \sin 3\theta(e^{3646}-e^{4546}-e^{2166}-e^{2256}-e^{2414}+e^{2423})\\
&+ a^2(-e^{35}-e^{46})\wedge e^2\wedge dt - a^2\, e^1\wedge(e^{36}-e^{45})\wedge dt\\
&- b^2(e^{15}-e^{26})\wedge e^4\wedge dt + b^2\, e^3\wedge(e^{16}+e^{25})\wedge dt\\
&+ c^2(-e^{13}-e^{24})\wedge e^6\wedge dt - c^2\, e^5\wedge(-e^{14}+e^{23})\wedge dt\\
= &-\left(\left(\frac{a'}{a}+\frac{b'}{b}+\frac{c'}{c}\right)\cos 3\theta - 3\theta'\ \sin 3\theta\right) abc\,(-e^{135}-e^{146}+e^{236}-e^{245})\wedge dt\\
&+ abc\ \cos 3\theta(e^{3456}-e^{1256}+e^{1234}) + abc\ \cos 3\theta(e^{3456}-e^{1256}+e^{1234})\\
&+ abc\ \cos 3\theta(e^{3456}-e^{1256}+e^{1234}) + abc\ \cos 3\theta(e^{3456}-e^{1256}+e^{1234})\\
&+\left(\left(\frac{a'}{a}+\frac{b'}{b}+\frac{c'}{c}\right)\sin 3\theta + 3\theta'\ \cos 3\theta\right) abc\,(e^{136}-e^{145}+e^{235}+e^{246})\wedge dt\\
&+ a^2(-e^{235}\wedge dt - e^{246}\wedge dt) + a^2(-e^{136}\wedge dt + e^{145}\wedge dt)\\
&+ b^2(e^{145}\wedge dt - e^{246}\wedge dt) + b^2(-e^{136}\wedge dt - e^{235}\wedge dt)\\
&+ c^2(-e^{136}\wedge dt - e^{246}\wedge dt) + c^2(e^{145}\wedge dt - e^{235}\wedge dt)\\
= &-\left(\left(\frac{a'}{a}+\frac{b'}{b}+\frac{c'}{c}\right)\cos 3\theta - 3\theta'\ \sin 3\theta\right) abc\,(-e^{135}-e^{146}+e^{236}-e^{245})\wedge dt\\
&+\left(\left(\frac{a'}{a}+\frac{b'}{b}+\frac{c'}{c}\right)\sin 3\theta + 3\theta'\ \cos 3\theta\right) abc\,(e^{136}-e^{145}+e^{235}+e^{246})\wedge dt\\
&+ (-a^2-b^2-c^2)(e^{136}-e^{145}+e^{235}+e^{246})\wedge dt\\
&+ 4abc\ \cos 3\theta(e^{1234}-e^{1256}+e^{3456})
\end{aligned}
$$

The equation $d\omega_\theta = 0$ therefore is equivalent to:

$$\begin{aligned}
\left(\frac{a'}{a}+\frac{b'}{b}+\frac{c'}{c}\right) abc\cos 3\theta - 3abc\,\theta'\sin 3\theta &= 0\\
\left(\frac{a'}{a}+\frac{b'}{b}+\frac{c'}{c}\right) abc\sin 3\theta + 3abc\,\theta'\cos 3\theta &= a^2+b^2+c^2\\
4abc\cos 3\theta &= 0
\end{aligned}$$

Since $a,b,c>0$, $\cos 3\theta$ has to be zero and the above system becomes:

$$\pm\left(\frac{a'}{a}+\frac{b'}{b}+\frac{c'}{c}\right) = \frac{a^2+b^2+c^2}{abc}\,.$$

We assume that $\theta=\frac{\pi}{6}$. In this situation, the sign on the left hand side is positive. By changing the orientation, we could replace $\frac{\partial}{\partial t}$ by $-\frac{\partial}{\partial t}$ in the frame. That change would make the sign negative again. Therefore, our assumption is without loss of generality justified. Next, we compute $d*\omega_\theta$:

$$\begin{aligned}
d*\omega_\theta = &\left(2\frac{a'}{a}+2\frac{b'}{b}\right) a^2b^2\, e^{1234}\wedge dt\\
&- a^2b^2(-e^{35234}-e^{46234}-e^{13634}+e^{14534}+e^{12154}-e^{12264}-e^{12316}-e^{12325})\\
&-\left(2\frac{a'}{a}+2\frac{c'}{c}\right) a^2c^2\, e^{1256}\wedge dt\\
&+ a^2c^2(-e^{35256}-e^{46256}-e^{13656}+e^{14556}-e^{12136}-e^{12246}+e^{12514}-e^{12523})\\
&+\left(2\frac{b'}{b}+2\frac{c'}{c}\right) b^2c^2\, e^{3456}\wedge dt\\
&- b^2c^2(e^{15456}-e^{26456}-e^{31656}-e^{32556}-e^{34136}-e^{34246}+e^{34514}-e^{34523})\\
&+ abc\cos 3\theta((-e^{35}-e^{46})\wedge e^{36}-e^1\wedge(e^{15}-e^{26})\wedge e^6+e^{13}\wedge(-e^{14}+e^{23}))\wedge dt\\
&- abc\cos 3\theta((-e^{35}-e^{46})\wedge e^{45}-e^1\wedge(e^{16}+e^{25})\wedge e^5+e^{14}\wedge(-e^{13}-e^{24}))\wedge dt\\
&+ abc\cos 3\theta((e^{36}-e^{45})\wedge e^{35}-e^2\wedge(e^{15}-e^{26})\wedge e^5+e^{23}\wedge(-e^{13}-e^{24}))\wedge dt\\
&+ abc\cos 3\theta((e^{36}-e^{45})\wedge e^{46}-e^2\wedge(e^{16}+e^{25})\wedge e^6+e^{24}\wedge(-e^{14}+e^{23}))\wedge dt\\
&- abc\sin 3\theta((-e^{35}-e^{46})\wedge e^{35}-e^1\wedge(e^{15}-e^{26})\wedge e^5+e^{13}\wedge(-e^{13}-e^{24}))\wedge dt\\
&- abc\sin 3\theta((-e^{35}-e^{46})\wedge e^{46}-e^1\wedge(e^{16}+e^{25})\wedge e^6+e^{14}\wedge(-e^{14}+e^{23}))\wedge dt\\
&+ abc\sin 3\theta((e^{36}-e^{45})\wedge e^{36}-e^2\wedge(e^{15}-e^{26})\wedge e^6+e^{23}\wedge(-e^{14}+e^{23}))\wedge dt\\
&- abc\sin 3\theta((e^{36}-e^{45})\wedge e^{45}-e^2\wedge(e^{16}+e^{25})\wedge e^5+e^{24}\wedge(-e^{13}-e^{24}))\wedge dt\\
= &\left(2\frac{a'}{a}+2\frac{b'}{b}\right) a^2b^2\, e^{1234}\wedge dt - \left(2\frac{a'}{a}+2\frac{c'}{c}\right) a^2c^2\, e^{1256}\wedge dt + \left(2\frac{b'}{b}+2\frac{c'}{c}\right) b^2c^2\, e^{3456}\wedge dt\\
&- 4abc\sin 3\theta(e^{1234}-e^{1256}+e^{3456})\wedge dt
\end{aligned}$$

Since we assume that $\theta = \frac{\pi}{6}$, the equation $d * \omega_\theta = 0$ is equivalent to:

$$
\begin{aligned}
\frac{a'}{a} + \frac{b'}{b} &= 2\frac{c}{ab} \\
\frac{a'}{a} + \frac{c'}{c} &= 2\frac{b}{ac} \\
\frac{b'}{b} + \frac{c'}{c} &= 2\frac{a}{bc}
\end{aligned}
$$

If we add these equations up, we obtain the equation for $d\omega_\theta = 0$. Therefore, the above system describes the condition for the holonomy reduction to a subgroup of G_2. This systems finally simplifies to:

$$
\begin{aligned}
\frac{a'}{a} &= \frac{b^2 + c^2 - a^2}{abc} \\
\frac{b'}{b} &= \frac{a^2 + c^2 - b^2}{abc} \\
\frac{c'}{c} &= \frac{a^2 + b^2 - c^2}{abc}
\end{aligned}
\tag{D.2}
$$

The system (D.2) coincides with the system which has been found by Cleyton and Swann [20].

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