Optimal control of rate-independent systems with non-convex energies via viscous approximation

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Dissertation
Optimal control of rate-independent systems with non-convex energies via viscous approximation

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Zusammenfassung

In dieser Dissertation befassen wir uns mit ratenunabhängigen Systemen im Kontext der optimalen Steuerung. Genauer gesagt untersuchen wir ein Optimalsteuerungsproblem, dessen Steuerungsvariable durch eine äußere Last gegeben ist und der entsprechende Zustand ergibt sich als Lösung des zu dieser Last gehörenden ratenunabhängigen Systems. Wir benutzen hier den Lösungsbegriff der sogenannten parametrized balanced viscosity Lösungen und zeigen die Existenz solcher Lösungen unter geeigneten Annahmen mittels verschwindender viskoser Regularisierung. Des Weiteren approximieren wir das Optimalsteuerungsproblem durch eine Folge von einfacheren Problemen, indem wir jeweils die zulässige Menge durch Lösungen des regularisierten Systems ersetzen und den Regularisierungsparameter gegen Null gehen lassen. Außerdem schauen wir uns noch den Fall an, dass die äußeren Kräfte unstetig sind. Dies führt im Kontext der optimalen Steuerung auf die relaxierten Lösungen, deren Vor- und Nachteile wir hier aufzeigen werden.

Abstract

In this dissertation we deal with rate-independent systems in the context of optimal control. To be more precise, we investigate an optimal control problem, whose control variable is given by an external load and the corresponding state results as a solution of the rate-independent system driven by this load. Here we use the so-called parametrized balanced viscosity solutions as the underlying solution concept and show their existence under suitable assumptions by means of a vanishing viscosity approach. Furthermore, we approximate this optimal control problem by a sequence of simpler problems, where we replace the respective feasible set by solutions of regularized systems and pass the regularization parameter to zero. Moreover, we consider the case that the external forces are discontinuous. In the context of optimal control, this leads to the relaxed solutions, whose advantages and drawbacks we will point out here.

Contents

List of Symbols

$\mathcal{Z}, \mathcal{V}$	Hilbert spaces 14
$X, \mathcal{X}$	Banach spaces 14
$\mathrm{AC}^p([0,T];\mathcal{R})$	space of $\mathcal{R}$-absolutely continuous functions 59
$BV([0,T];X)$	space of functions $[0,T] \to X$ of pointwise bounded variation 98
$\mathrm{BV}(0,T;X)$	space of equivalence classes of functions $[0,T] \to X$ of bounded variation 98
$C([0,T];X)$	Bochner space of continuous functions $[0,T] \to X$ 13
$C_{weak}([0,T];X)$	Bochner space of weakly continuous functions $[0,T] \to X$ 13
$C^{0,\alpha}([0,T];X)$	Bochner space of Hölder continuous functions with exponent $\alpha \in (0,1]$ 13
$L^p(0,T;X)$	Bochner-Lebesgue space of functions $[0,T] \to X$ 13
$W^{k,p}(0,T;X)$	Bochner-Sobolev space of functions $[0,T] \to X$ 13
$\mathcal{I} : \mathcal{V}^* \times \mathcal{Z} \to \mathbb{R}$	energy functional 14
$\mathcal{E} : \mathcal{Z} \to \mathbb{R}$	reduced time independent energy functional 14
$\mathcal{F} : \mathcal{Z} \to \mathbb{R}$	non-linear part of the energy functional . . . 14
$\ell : [0,T] \to \mathcal{V}^*$	external load 16
$\mathcal{R}, \mathcal{R}^*$	dissipation potential and its convex conjugate 17, 18
$\mathcal{R}_\epsilon, \mathcal{R}_\epsilon^*$	viscous regularized dissipation potential and its convex conjugate 48, 54
$\mathcal{R}_{\epsilon,\delta}, \mathcal{R}_{\epsilon,\delta}^*$	twice viscous regularized dissipation potential and its convex conjugate 38, 38
$\partial\mathcal{R}, \partial\mathcal{R}_\epsilon, \partial\mathcal{R}_{\epsilon,\delta}$	convex subdifferentials 17, 48, 38
$\mathrm{Var}(z;[0,T])$	pointwise variation of a function z on $[0,T]$ 98
$\mathrm{Var}_{\mathcal{R}}(z;[0,T])$	pointwise variation of a function z on $[0,T]$ w.r.t. to $\mathcal{R}$ 33
(RIS)	rate-independent system 11
$(\mathrm{RIS})_\epsilon$	viscous regularized rate-independent system 48

Chapter 1

Introduction

This dissertation is concerned with the analysis and optimal control of rate-independent systems in an infinite dimensional setting. The property of rate-independence means that certain time scaling of the input of the system leads to an equally scaled output. Thus, if we consider a process driven by an external force, the behavior of the state does not depend on the rate at which the force is applied. To be more precise, it may depend on the direction of the rate but not on the magnitude of the rate. This notion of rate-independence goes back to [TN65] and [PR65], where a first mathematical formulation can be found. It then took several years for Moreau to put this field of research on a mathematical foundation with the help of convex analysis techniques, see [Mor70, Mor73, Mor11]. Building on this, Halphen and Nguyen introduced the concept of generalized standard materials, cf. [HSN75, GSNS83], which provides a general framework for modelling materials. The concept of rate-independent systems became more and more popular in the years that followed and was studied by various authors in different fields of applications in mechanics and physics that are elasto-plasticity, phase transformations in shape-memory alloys, damage modelling, ferroelectricity, magnetization and many more. Exemplarily, we refer here to [MM09, MR16, AMS08, Mai04, BS96, KRZ13, KN17] and the references therein for a more detailed insight into the respective application areas.
In this thesis, we consider rate-independent systems of the form

$$0 \in \partial\mathcal{R}(z'(t)) + \mathrm{D}_z\mathcal{I}(\ell(t), z(t)) \quad \text{f.a.a. } t \in (0,T), \quad z(0) = z_0, \tag{RIS}$$

where $\mathcal{R} : \mathcal{Z} \to [0,\infty]$ is a convex and positive 1-homogeneous dissipation and $\mathcal{I} : \mathcal{V}^* \times \mathcal{Z} \to \mathbb{R}$ denotes a potentially non-convex energy functional. Moreover, $T > 0$ describes the final time and is fixed throughout the entire thesis, $z_0 \in \mathcal{Z}$ is a given initial value and $\ell : [0,T] \to \mathcal{V}^*$ denotes the external driving force. For an associated solution z and a strictly monoton increasing function α we have that

$$-\mathrm{D}_z\mathcal{I}(\ell(\alpha t), z(\alpha t)) \in \partial\mathcal{R}(\alpha'(t)z'(\alpha(t))) = \partial\mathcal{R}(z'(\alpha(t))) \tag{1.0.1}$$

for almost all $t \in (0,T)$ due to the 0-homogeneity of the subdifferential, see (1.2.23). Therefore, $z \circ \alpha$ is a solution associated with the rescaled external load $\ell \circ \alpha$ and the system has indeed the rate-independent structure. The precise assumptions on the data are specified in Section 1.2 below. A variety of solution concepts related to (RIS) have been developed in recent years with the aim of ensuring solvability with correspondingly weaker assumptions for the data. In Chapter 2 we will present a small selection of these. In particular, we will derive the existence of the so-called parametrized balanced viscosity (BV) solutions in detail as this is the solution concept we are dealing with in this work. To this end, we consider the viscous regularized system

$$0 \in \partial\mathcal{R}(z'(t)) + \epsilon z'(t) + \mathrm{D}_z\mathcal{I}(\ell(t), z(t)) \quad \text{f.a.a. } t \in (0,T) \quad z(0) = z_0, \tag{1.0.2}$$

with a fixed parameter $\epsilon > 0$. We shall see that, in contrast to (RIS), the unique solvability of (1.0.2) by a weakly differentiable function z_ϵ is always ensured under our preconditions to the data. After a suitable parametrization, we can then pass the viscosity parameter ϵ to zero and obtain a parametrized balanced viscosity solution as the limit value. Following this, we will consider optimal control problems that are governed by rate-independent systems in Chapter 4. There, the state is described by a parametrized balanced viscosity solution and the external load ℓ plays the role of the control variable. The aim is to show the approximability of this optimal control problem by simpler solvable problems, where we replace the feasible set consisting of parametrized balanced viscosity solutions associated with the corresponding external load ℓ by solutions of the viscous regularized system (1.0.2). Lastly, in Chapter 5 we extend the set of admissible control variables to possibly discontinuous loads $\ell \in BV(0,T)$, which then leads to a more comprehensive solution concept that we call here *relaxed solutions*. We present some properties of them that are essential in the context of optimal control, namely a stability result with regard to converging external loads or, in other words, a compactness result of the solution set for those relaxed solutions. Furthermore, we will compare them with other solution concepts and point out possible disadvantages and underline it by various explicit examples.

The results of Chapter 5 are also content of [AM24]. A further paper that was produced in the course of this doctoral project is [AM23]. However, its contents are not discussed here.

1.1 Notation

For a given Banach space X, we denote the norm by $\|\cdot\|_X : X \to \mathbb{R}$ and $\langle\cdot,\cdot\rangle_{X^*,X} : X^* \times X \to \mathbb{R}$ is the dual pairing with its dual space X^*. If, especially, X is a Hilbert space, then $\langle\cdot,\cdot\rangle_X : X \times X \to \mathbb{R}$ denotes the scalar product. With $B_X(x,r)$ we label the open ball in X with radius $r > 0$ centered in $x \in X$. If the underlying space is $X = \mathbb{R}^d$, $d \in \mathbb{N}$, then we suppress the subscript. Moreover, $\mathcal{L}(X,Y)$ is the set of all linear and continuous mappings from X to a further Banach space Y. If X is continuously embedded in Y, we write $X \hookrightarrow Y$ and $X \hookrightarrow^{c,d} Y$ if the embedding is additionally compact and dense.

With $L^p(0,T;X)$, $1 \leq p \leq \infty$ and $T > 0$, we denote the space of equivalence classes of Bochner p-integrable functions $v : [0,T] \to X$ and equip it with the norm

$$\|v\|_{L^p(0,T;X)} := \left(\int_0^T \|v(t)\|_X^p \, \mathrm{d}t \right)^{\frac{1}{p}}, \quad p \in [1,\infty)$$

and

$$\|v\|_{L^\infty(0,T;X)} := \operatorname*{ess\,sup}_{t\in[0,T]} \|v(t)\|_X, \quad p = \infty.$$

For functions $v \in L^p(0,T;X)$ that possesses a weak derivative $v' \in L^p(0,T;X)$, we denote the Bochner-Sobolev space by

$$W^{1,p}(0,T;X) := \Big\{ v \in L^p(0,T;X) : v' \in L^p(0,T;X) \Big\}$$

with its norm

$$\|v\|_{W^{1,p}(0,T;X)} := \|v\|_{L^p(0,T;X)} + \|v'\|_{L^p(0,T;X)}.$$

Accordingly, we define the spaces for higher derivatives and write $H^k(0,T;X) = W^{k,p}(0,T;X)$, $k \in \mathbb{N}$, in the case $p = 2$. If v is only Bochner integrable on every compact subset $K \subset (0,T)$, we say $v \in L^p_{loc}(0,T;X)$ and accordingly we define $W^{k,p}_{loc}(0,T;X)$. Furthermore, $C([0,T];X)$ is the space of all continuous functions $v : [0,T] \to X$, $C_{weak}([0,T];X)$ denotes the set of weakly continuous functions and $C^{0,\alpha}([0,T];X)$ consists of all Hölder continuous functions with exponent $\alpha \in (0,1]$. Lastly, $BV([0,T];X)$ consists of all functions $v : [0,T] \to X$ with bounded variation, i.e.,

$$\mathrm{Var}(v;[0,T]) := \sup \left\{ \sum_{i=1}^n \|v(t_i) - v(t_{i-1})\|_X \ : \ 0 = t_0 < \cdots < t_n = T, \ n \in \mathbb{N} \right\} < \infty.$$

For a given convex function $f : X \to \mathbb{R} \cup \{\infty\}$, its subdifferential is denoted by $\partial f(x) \subset X^*$ and $f^* : X^* \to \mathbb{R}$, $x^* \mapsto \sup_{x\in X} \langle x^*, x\rangle_{X^*,X} - f(x)$ is the conjugate functional. In addition, if $f : \mathbb{R} \times X \to Y$, $(t,z) \mapsto f(t,z)$, is partially Fréchet differentiable, then $\partial_t f(t,z)$ denotes the partial derivative w.r.t. t and $\mathrm{D}_z f(t,z)$ is the Fréchet derivative w.r.t z. For the one sided limits of a function $f : [0,T] \to X$, we write $f(t+) := \lim_{s\searrow t} f(s)$ and $f(t-) := \lim_{s\nearrow t} f(s)$, respectively.
Finally, with $\lambda(M)$ we denote the Lebesgue measure of measurable set $M \subset \mathbb{R}^d$ and $C, \tilde{C} > 0$ are generic constants, whose value may change from line to line. If the dependency on a specific variable is important, we indicate this with a corresponding subscript.

1.2 Standing Assumptions

Now we introduce the required assumptions. They are valid throughout the entire thesis.

Spaces

Let $\mathcal{Z}, \mathcal{V}$ be Hilbert spaces and $\mathcal{X}$ a Banach space such that $\mathcal{Z} \hookrightarrow^{c,d} \mathcal{V} \hookrightarrow \mathcal{X}$.

Remark 1.2.1. At some points we identify the Hilbert space $\mathcal{V}$ with its dual space $\mathcal{V}^*$ and for an element $v \in \mathcal{V}$ embedded by the Riesz isomorphism $\mathcal{J}_\mathcal{V} : \mathcal{V} \to \mathcal{V}^*$ we simply write v instead of $\mathcal{J}_\mathcal{V}(v)$.

Energy

We assume that the energy $\mathcal{I} : \mathcal{V}^* \times \mathcal{Z} \to \mathbb{R}$ has the following structure:

$$\mathcal{I}(\ell, z) := \frac{1}{2} \langle Az, z\rangle_{\mathcal{Z}^*,\mathcal{Z}} + \mathcal{F}(z) - \langle \ell, z\rangle_{\mathcal{V}^*,\mathcal{V}}. \tag{1.2.1}$$

We denote the first two terms that are independent of the external load ℓ by

$$\mathcal{E}(z) := \frac{1}{2} \langle Az, z\rangle_{\mathcal{Z}^*,\mathcal{Z}} + \mathcal{F}(z), \tag{1.2.2}$$

so that the energy can be written as $\mathcal{I}(\ell, z) = \mathcal{E}(z) - \langle \ell, z\rangle_{\mathcal{V}^*,\mathcal{V}}$. Herein, $A \in \mathcal{L}(\mathcal{Z}, \mathcal{Z}^*)$ is supposed to be symmetric and coercive, i.e., it exists $\alpha > 0$ with

$$\langle Az, z\rangle_{\mathcal{Z}^*,\mathcal{Z}} \geq \alpha \|z\|^2_\mathcal{Z} \quad \text{and} \quad \langle Az, w\rangle_{\mathcal{Z}^*,\mathcal{Z}} = \langle Aw, z\rangle_{\mathcal{Z}^*,\mathcal{Z}} \tag{1.2.3}$$

for all $z, w \in \mathcal{Z}$. Since $A \in \mathcal{L}(\mathcal{Z}, \mathcal{Z}^*)$ is coercive and continuous, $|\cdot|_\mathcal{Z} : \mathcal{Z} \to \mathbb{R}$ given by

$$|z|_\mathcal{Z} := \left(\langle Az, z\rangle_{\mathcal{Z}^*,\mathcal{Z}}\right)^{\frac{1}{2}}, \quad z \in \mathcal{Z}.$$

defines a norm on $\mathcal{Z}$ that is equivalent to the canonical norm $\|\cdot\|_\mathcal{Z}$. Furthermore, we assume that $\mathcal{F} : \mathcal{Z} \to \mathbb{R}$ satisfies

$$\mathcal{F} \in C^2(\mathcal{Z}; \mathbb{R}),\ \mathcal{F} \geq 0, \tag{1.2.4}$$

$$\mathrm{D}_z\mathcal{F} \in C^1(\mathcal{Z}; \mathcal{V}^*), \quad \|\mathrm{D}^2_z\mathcal{F}(z)v\|_{\mathcal{V}^*} \leq C(1 + \|z\|^q_\mathcal{Z})\|v\|_\mathcal{Z}\ \forall v, z \in \mathcal{Z} \tag{1.2.5}$$

for some $q \geq 1$ and $\mathrm{D}_z\mathcal{F} : \mathcal{Z} \to \mathcal{Z}^*$ is supposed to be weak-weak continuous, i.e.,

$$z_k \rightharpoonup z \text{ in } \mathcal{Z} \implies \mathrm{D}_z\mathcal{F}(z_k) \rightharpoonup \mathrm{D}_z\mathcal{F}(z) \text{ in } \mathcal{Z}^*. \tag{1.2.6}$$

Additionally, we require that $\mathrm{D}^2_z\mathcal{F} : \mathcal{Z} \to \mathcal{L}(\mathcal{Z}, \mathcal{Z}^*)$ is Lipschitz continuous on bounded sets, i.e., for all $r > 0$ there exists $L_r > 0$ such that

$$\|(\mathrm{D}^2_z\mathcal{F}(z_1) - \mathrm{D}^2_z\mathcal{F}(z_2))v\|_{\mathcal{Z}^*} \leq L_r\|z_1 - z_2\|_\mathcal{Z}\|v\|_\mathcal{Z} \tag{1.2.7}$$

for all $z_1, z_2 \in B_\mathcal{Z}(0, r)$ and $v \in \mathcal{Z}$.
In comparison to other literature, instead of defining the energy with respect to the time $t \in [0, T]$ and the state $z \in \mathcal{Z}$, here we want to emphasize the dependence of the external load ℓ since this will be the control variable describing the costs in the

later investigated optimal control problems. But for a given load $\ell : [0,T] \to \mathcal{V}^*$ the energy can also be written in dependence of time and state, i.e.,

$$\tilde{\mathcal{I}}(t,z) = \frac{1}{2}\langle Az, z\rangle_{\mathcal{Z}^*,\mathcal{Z}} + \mathcal{F}(z) - \langle \ell(t), z\rangle_{\mathcal{V}^*,\mathcal{V}}, \quad t \in [0,T], z \in \mathcal{Z}.$$

Let us now collect some properties of the energy functional in terms of continuity and growth conditions. First of all, we have that $\mathcal{F}$ is weakly continuous in $\mathcal{Z}$, i.e.,

$$z_k \rightharpoonup z \text{ in } \mathcal{Z} \implies \mathcal{F}(z_k) \to \mathcal{F}(z) \text{ in } \mathbb{R}, \tag{1.2.8}$$

which can be seen as follows. Applying the mean value theorem yields the existence of $\tau_k \in (0,1)$ with

$$\begin{aligned}
&\|\mathcal{F}(z_k) - \mathcal{F}(z)\|_{\mathcal{Z}} \\
&= |\langle \mathrm{D}_z\mathcal{F}(z), z_k - z\rangle_{\mathcal{V}^*,\mathcal{V}}| + \frac{1}{2}\Big|\langle \mathrm{D}_z^2\mathcal{F}(z + \tau_k(z_k - z))(z_k - z), z_k - z\rangle_{\mathcal{V}^*,\mathcal{V}}\Big| \\
&\leq \|\mathrm{D}_z\mathcal{F}(z)\|_{\mathcal{V}^*}\|z_k - z\|_{\mathcal{V}} + \frac{1}{2}C\big(1 + \|z + \tau_k(z_k - z)\|_{\mathcal{Z}}^q\big)\|z_k - z\|_{\mathcal{Z}}\|z_k - z\|_{\mathcal{V}} \\
&\to 0, \text{ as } k \to \infty,
\end{aligned}$$

where we used (1.2.5), the boundedness of $(z_k)_k \subset \mathcal{Z}$ by its weak convergence and the strong convergence $z_k \to z$ in $\mathcal{V}$ by the compact embedding $\mathcal{Z} \hookrightarrow^c \mathcal{V}$.

Furthermore, the energy complies with the following estimate:

Lemma 1.2.2. *There exists constants $\lambda, \mu > 0$ such that*

$$\|z\|_{\mathcal{Z}}^2 \leq \lambda\mathcal{I}(\ell, z) + \mu\|\ell\|_{\mathcal{V}^*}^2 \tag{1.2.9}$$

holds for all $z \in \mathcal{Z}, \ell \in \mathcal{V}^*$.

Proof. We exploit the coercivity of $A \in \mathcal{L}(\mathcal{Z}, \mathcal{Z}^*)$ and non-negativity of $\mathcal{F}$ by (1.2.3) and (1.2.4), respectively, in order to obtain for arbitrary $z \in \mathcal{Z}$ and $\ell \in \mathcal{V}^*$

$$\begin{aligned}
\frac{\alpha}{2}\|z\|_{\mathcal{Z}}^2 &\leq \frac{1}{2}\langle Az, z\rangle_{\mathcal{Z}^*,\mathcal{Z}} \\
&\leq \frac{1}{2}\langle Az, z\rangle_{\mathcal{Z}^*,\mathcal{Z}} + \mathcal{F}(z) - \langle \ell, z\rangle_{\mathcal{V}^*,\mathcal{V}} + \langle \ell, z\rangle_{\mathcal{V}^*,\mathcal{V}} \\
&\leq \mathcal{I}(\ell, z) + \|\ell\|_{\mathcal{V}^*}\|z\|_{\mathcal{V}},
\end{aligned}$$

where we used the Cauchy-Schwarz inequality in the last step. Then applying Young's inequality with arbitrary $\epsilon > 0$ and making use of $\|z\|_{\mathcal{V}} \leq C\|z\|_{\mathcal{Z}}$ by the assumed continuous embedding $\mathcal{Z} \hookrightarrow \mathcal{V}$ leads to

$$\frac{\alpha}{2}\|z\|_{\mathcal{Z}}^2 \leq \mathcal{I}(\ell, z) + \frac{1}{2\epsilon}\|\ell\|_{\mathcal{V}^*}^2 + \frac{\epsilon}{2}\|z\|_{\mathcal{V}}^2$$

$$\leq \mathcal{I}(\ell, z) + \frac{1}{2\epsilon}\|\ell\|_{\mathcal{V}^*}^2 + \frac{C\epsilon}{2}\|z\|_{\mathcal{Z}}^2$$

Inserting $\epsilon := \frac{\alpha}{2C}$ in the above estimate and rearranging the terms results in

$$\frac{\alpha}{4}\|z\|_{\mathcal{Z}}^2 \leq \mathcal{I}(\ell, z) + \frac{C}{\alpha}\|\ell\|_{\mathcal{V}^*}^2$$

such that dividing by $\frac{\alpha}{4}$ yields the assertion with $\lambda = \frac{4}{\alpha}$ and $\mu = \frac{4C}{\alpha^2}$. □

For the further analysis, we assume from now on that we have given an external load $\ell \in H^1(0,T;\mathcal{V}^*)$ that satisfies

$$\|\ell\|_{H^1(0,T;\mathcal{V}^*)} \leq \tilde{C} \tag{1.2.10}$$

with a constant $\tilde{C} > 0$ sufficiently large. Surely, for a fixed load, this assumption seems superfluous since it is trivially fulfilled for any $\tilde{C} \geq \|\ell\|_{H^1(0,T;\mathcal{V}^*)}$. However, since the loads are our control variables, we will study varying loads from Section 3.3 on. These loads will be bounded in $H^1(0,T;\mathcal{V}^*)$ and thus comply with (1.2.10). The upcoming a priori estimates concerned with the solutions of the viscous regularized systems in the next chapter that actually depends on the norm of the associated load will instead depend on the above constant.

The next lemma deals with a further useful estimate of the energy functional.

Lemma 1.2.3. *Let $\ell \in H^1(0,T;\mathcal{V}^*)$ be a given external load that complies with* (1.2.10). *Then there exists a constant $c_0 > 0$ such that*

$$\|z\|_{\mathcal{Z}} - c_0 \leq \mathcal{I}(\ell(t), z) \tag{1.2.11}$$

holds for all $z \in \mathcal{Z}$ and $t \in [0,T]$.

Proof. First of all, the continuous embedding $H^1(0,T;\mathcal{V}^*) \hookrightarrow C([0,T];\mathcal{V}^*)$ yields that ℓ is pointwise bounded by a constant depending on $\tilde{C}$ from (1.2.10). Then by exploiting the coercivity of A, the non-negativity of $\mathcal{F}$ and the compact embedding $\mathcal{Z} \hookrightarrow^c \mathcal{V}$, we obtain

$$\mathcal{I}(\ell(t), z) \geq \frac{\alpha}{2}\|z\|_{\mathcal{Z}}^2 - C\|z\|_{\mathcal{Z}} \tag{1.2.12}$$

for all $z \in \mathcal{Z}$ and $t \in [0,T]$. The further calculation

$$\begin{aligned}
&\frac{\alpha}{2}\|z\|^2 - C\|z\|_{\mathcal{Z}} \geq \|z\|_{\mathcal{Z}} - c_0 \\
\Leftrightarrow\ &\|z\|^2 - \frac{2(C+1)}{\alpha}\|z\|_{\mathcal{Z}} \geq -\frac{2c_0}{\alpha} \\
\Leftrightarrow\ &\Big(\|z\|_{\mathcal{Z}} - \frac{C+1}{\alpha}\Big)^2 - \Big(\frac{C+1}{\alpha}\Big)^2 \geq -\frac{2c_0}{\alpha}
\end{aligned}$$

shows that the claim is fulfilled with $c_0 := \frac{(C+1)^2}{2\alpha}$. □

Now that we have established the requirements for the energy, let us turn our attention to the dissipation potential.

Dissipation

For the dissipation $\mathcal{R} : \mathcal{Z} \to [0, \infty)$, we assume that

$$\mathcal{R} \text{ is proper, convex, and lower semicontinuous,} \tag{1.2.13}$$
$$\mathcal{R} \text{ is positive 1-homogeneous, i.e., } \mathcal{R}(\lambda z) = \lambda \mathcal{R}(z)\ \forall\, z \in \mathcal{Z},\ \lambda > 0, \tag{1.2.14}$$
$$\exists\, c, C > 0, \text{ such that } c\, \|z\|_{\mathcal{X}} \leq \mathcal{R}(z) \leq C\, \|z\|_{\mathcal{V}}\ \forall\, z \in \mathcal{Z}. \tag{1.2.15}$$

From the convexity and the upper bound in (1.2.15) it follows that $\mathcal{R}$ is even continuous with respect to strong topology on $\mathcal{V}$, which can be seen as follows. The convexity of $\mathcal{R}$ yields that for all $v, w \in \mathcal{Z}$ it holds

$$\frac{1}{2}\mathcal{R}(w) = \mathcal{R}\left(\frac{1}{2}w\right) = \mathcal{R}\left(\frac{1}{2}(w-v) + \frac{1}{2}v\right) \leq \frac{1}{2}\mathcal{R}(w-v) + \frac{1}{2}\mathcal{R}(v), \tag{1.2.16}$$

which directly implies $\mathcal{R}(w) - \mathcal{R}(v) \leq \mathcal{R}(w-v)$. Now let a sequence with $z_n \to z$ in $\mathcal{V}$ be given. From (1.2.15) we infer that

$$\mathcal{R}(z_n) - \mathcal{R}(z) \leq \mathcal{R}(z_n - z) \leq C\|z_n - z\|_{\mathcal{V}} \to 0, \text{ as } n \to \infty. \tag{1.2.17}$$

Since this argumentation also holds with reversed roles of z_n and z, the continuity of $\mathcal{R}$ is shown. Moreover, from (1.2.17) we deduce that for a sequence $(z_n)_n \subset \mathcal{Z}$

$$|\mathcal{R}(z_n) - \mathcal{R}(z_m)| \leq C\|z_n - z_m\|_{\mathcal{V}} \text{ for all } n, m \in \mathbb{N}.$$

Therefore, by exploiting the density of $\mathcal{Z} \subset \mathcal{V}$, the dissipation can be extended to a mapping from $\mathcal{V}$ to $\mathbb{R}$. This means that we could have assumed $\mathcal{R}$ directly as a continuous function living on $\mathcal{V}$. However, for our purpose it is sufficient to consider $\mathcal{R}$ as a functional on $\mathcal{Z}$. For some parts of the analysis we can also allow unbounded dissipation potentials $\mathcal{R} : \mathcal{Z} \to [0, \infty]$, which makes sense from an application point of view, e.g., in damage modelling, cf. exemplarily [KRZ13], [KRZ15], [KRZ19], where such an unbounded dissipation occurs. For the sake of simplicity, however, we always assume in the prerequisites of the upcoming theorems that $\mathcal{R}$ satisfies (1.2.13)–(1.2.15), although the upper bound in (1.2.15) is actually not required in this step.

Since the subdifferential of $\mathcal{R}$ plays a crucial role in (RIS), let us now collect some properties thereof. In the following we consider the subdifferential $\partial\mathcal{R}(\cdot)$ with respect to the $\mathcal{Z}$-$\mathcal{Z}^*$-duality because for now only $\mathrm{D}_z\mathcal{I}(\ell, z) \in \mathcal{Z}^*$ due to $A \in \mathcal{L}(\mathcal{Z}, \mathcal{Z}^*)$. But it will turn out later that even $\mathrm{D}_z\mathcal{I}(\ell(t), z_\epsilon(t)) \in \mathcal{V}^*$ for a solution z_ϵ of the viscous regularized version of (RIS), which will be shown under exploitation of the upper bound in (1.2.15). Further, in Lemma A.2 and Lemma A.3 it is proven that it actually does not matter with respect to which duality the subdifferential is considered.

Lemma 1.2.4. *Assume that the dissipation $\mathcal{R}$ complies with* (1.2.13)-(1.2.15)*. Then the following properties are fulfilled for all $v \in \mathcal{Z}$ and $\eta \in \mathcal{Z}^*$.*

$$\partial\mathcal{R}(0) = \left\{\xi \in \mathcal{Z}^* : \mathcal{R}(w) \geq \langle \xi, w\rangle_{\mathcal{Z}^*,\mathcal{Z}} \ \forall w \in \mathcal{Z}\right\} \tag{1.2.18}$$

$$\partial\mathcal{R}(v) = \left\{\xi \in \partial\mathcal{R}(0) : \mathcal{R}(v) = \langle \xi, v\rangle_{\mathcal{Z}^*,\mathcal{Z}}\right\} \tag{1.2.19}$$

$$\partial\mathcal{R}(v) \subset \partial\mathcal{R}(0) \tag{1.2.20}$$

$$\mathcal{R}^*(\eta) = I_{\partial\mathcal{R}(0)}(\eta) \tag{1.2.21}$$

Proof. The definition of the subdifferential yields for $\xi \in \mathcal{R}(v) \subset \mathcal{Z}^*$

$$\mathcal{R}(w) - \mathcal{R}(v) \geq \langle \xi, w - v\rangle_{\mathcal{Z}^*,\mathcal{Z}} \text{ for all } w \in \mathcal{Z}. \tag{1.2.22}$$

By inserting $v = 0$ and exploiting the fact that $\mathcal{R}(0) = 0$, we directly obtain (1.2.18). To show (1.2.19), we test (1.2.22) with $w = 0$ and $w = 2v$, respectively, and using the 1-homogeneity of $\mathcal{R}$ by (1.2.14) yields

$$\mathcal{R}(v) = \langle \xi, v\rangle_{\mathcal{Z}^*,\mathcal{Z}}$$

such that (1.2.19) is shown. Now (1.2.20) is a direct consequence. Another use of the 1-homogeneity of $\mathcal{R}$ implies for the conjugated functional

$$\begin{aligned}
\mathcal{R}^*(\eta) &= \sup_{w\in\mathcal{Z}}(\langle \eta, w\rangle_{\mathcal{Z}^*,\mathcal{Z}} - \mathcal{R}(w)) \\
&= \sup_{\lambda>0}\sup_{w\in\mathcal{Z}}(\langle \eta, \lambda w\rangle_{\mathcal{Z}^*,\mathcal{Z}} - \mathcal{R}(\lambda w)) \\
&= \sup_{\lambda>0}\lambda\sup_{w\in\mathcal{Z}}(\langle \eta, w\rangle_{\mathcal{Z}^*,\mathcal{Z}} - \mathcal{R}(w)) \\
&= \begin{cases} 0, & \text{if } \mathcal{R}(w) \geq \langle \eta, w\rangle_{\mathcal{Z}^*,\mathcal{Z}} \ \forall w \in \mathcal{Z}, \\ \infty, & \text{else,} \end{cases}
\end{aligned}$$

which proves (1.2.21). □

With the second property (1.2.19) at hand, one directly obtains by exploiting the 1-homogeneity of $\mathcal{R}$ that the subdifferential is positive 0-homogeneous, i.e.,

$$\partial\mathcal{R}(\lambda v) = \partial\mathcal{R}(v) \text{ for all } v \in \mathcal{Z}, \lambda > 0. \tag{1.2.23}$$

Thus the system is indeed rate independent, see (1.0.1).
Utilizing the additional upper bound in (1.2.15), we have the following boundedness property of the subdifferential.

Lemma 1.2.5. *Let the dissipation $\mathcal{R} : \mathcal{Z} \to [0, \infty)$ comply with* (1.2.13)–(1.2.15) *and $z \in \mathcal{Z}$ be arbitrarily given. Then the subdifferential $\partial\mathcal{R}(z)$ is a bounded subset of $\mathcal{V}^*$.*

Proof. Let $\xi \in \partial\mathcal{R}(z) \subset \mathcal{Z}^*$ be given. From (1.2.16) we infer that for all $w \in \mathcal{Z}$ it holds $\mathcal{R}(w) - \mathcal{R}(z) \leq \mathcal{R}(w-z)$ such that the definition of the subdifferential along with (1.2.15) yields

$$\langle \xi, w - z\rangle_{\mathcal{Z}^*,\mathcal{Z}} \leq \mathcal{R}(w) - \mathcal{R}(v) \leq \mathcal{R}(w-z) \leq C\|w-z\|_{\mathcal{V}}$$

Now choosing $w := z + v$ with arbitrary $v \in \mathcal{Z}$ results in

$$\langle \xi, v\rangle_{\mathcal{Z}^*,\mathcal{Z}} \leq C\|v\|_{\mathcal{V}}$$

Thus, ξ can be extended to an element of $\mathcal{V}^*$ with $\|\xi\|_{\mathcal{V}^*} \leq C$, where the constant $C > 0$ from (1.2.15) does not depend on ξ, which gives the claim. □

The next lemma is concerned with the so called uniform subdifferentiability of the energy, cf. [MRS16, Section 2.1].

Lemma 1.2.6. *For all $\rho > 0$ there exists a constant $C_\rho > 0$ such that for all $z_1, z_2 \in \mathcal{Z}$ with $\|z_i\|_{\mathcal{Z}} \leq \rho$, $i = 1, 2$, and all $t \in [0, T]$*

$$\begin{aligned}\mathcal{I}(\ell(t), z_2) - \mathcal{I}(\ell(t), z_1) \geq \langle \mathrm{D}_z\mathcal{I}(\ell(t), z_1), z_2 - z_1\rangle_{\mathcal{Z}^*,\mathcal{Z}}& \\ + \frac{\alpha}{4}\|z_2 - z_1\|^2_{\mathcal{Z}} - C_\rho\mathcal{R}(z_2 - z_1)\|z_2 - z_1\|_{\mathcal{V}}&.\end{aligned} \tag{1.2.24}$$

Proof. Let $z_1, z_2 \in \mathcal{Z}$ and $t \in [0, T]$ be given as in the assumption. Then the mean value theorem implies

$$\begin{aligned}\mathcal{I}(\ell(t), z_2) - \mathcal{I}(\ell(t), z_1) &= \langle \mathrm{D}_z\mathcal{I}(\ell(t), z_1), z_2 - z_1\rangle_{\mathcal{Z}^*,\mathcal{Z}} \\ &\quad + \frac{1}{2}\Big\langle \mathrm{D}_z^2\mathcal{I}\big(\ell(t), z_1 + \tau(z_2 - z_1)\big)z_2 - z_1, z_2 - z_1\Big\rangle_{\mathcal{Z}^*,\mathcal{Z}}\end{aligned}$$

with $\tau \in [0, 1]$. Now by exploiting the concrete structure of the energy and (1.2.3), the second term can be estimated by

$$\begin{aligned}&\frac{1}{2}\Big\langle \mathrm{D}_z^2\mathcal{I}(\ell(t), z_1 + \tau(z_2 - z_1))z_2 - z_1, z_2 - z_1\Big\rangle_{\mathcal{Z}^*,\mathcal{Z}} \\ &\qquad\geq \frac{\alpha}{2}\|z_2 - z_1\|^2_{\mathcal{Z}} + \frac{1}{2}\Big\langle \mathrm{D}_z^2\mathcal{F}(z_1 + \tau(z_2 - z_1))z_2 - z_1, z_2 - z_1\Big\rangle_{\mathcal{Z}^*,\mathcal{Z}}.\end{aligned}$$

Next, from (1.2.5) and Ehrling's lemma with $\epsilon = \frac{\alpha}{4C(1+\rho^q)}$ we infer the existence of $C_\epsilon > 0$ such that

$$\begin{aligned}&\frac{1}{2}\Big|\langle \mathrm{D}_z^2\mathcal{F}(z_1 + \tau(z_2 - z_1))z_2 - z_1, z_2 - z_1\rangle_{\mathcal{Z}^*,\mathcal{Z}}\Big| \\ &\leq \frac{1}{2}C(1 + \rho^q)\|z_2 - z_1\|_{\mathcal{Z}}\|z_2 - z_1\|_{\mathcal{V}} \\ &\leq \frac{1}{2}C(1 + \rho^q)\|z_2 - z_1\|_{\mathcal{Z}}\big(\epsilon\|z_2 - z_1\|_{\mathcal{Z}} + C_\epsilon\|z_2 - z_1\|_{\mathcal{X}}\big) \\ &\leq \frac{\alpha}{8}\|z_2 - z_1\|^2_{\mathcal{Z}} + \frac{1}{2}C_\epsilon C(1 + \rho^q)\|z_2 - z_1\|_{\mathcal{Z}}\|z_2 - z_1\|_{\mathcal{X}}.\end{aligned}$$

Regarding the last term, we apply Young's inequality with $\delta = \frac{4C_\epsilon C(1+\rho^q)}{\alpha}$ in order to obtain

$$\begin{aligned}
&\frac{1}{2}\Big|\langle \mathrm{D}_z^2\mathcal{F}(z_1+\tau(z_2-z_1))z_2-z_1, z_2-z_1\rangle_{\mathcal{Z}^*,\mathcal{Z}}\Big| \\
&\leq \frac{\alpha}{8}\|z_2-z_1\|_{\mathcal{Z}}^2 + \frac{C_\epsilon C(1+\rho^q)}{2\delta}\|z_2-z_1\|_{\mathcal{Z}}^2 + \frac{C_\epsilon C(1+\rho^q)\delta}{2}\|z_2-z_1\|_{\mathcal{X}}^2 \\
&\leq \frac{\alpha}{4}\|z_2-z_1\|_{\mathcal{Z}}^2 + C_\rho \mathcal{R}(z_2-z_1)\|z_2-z_1\|_{\mathcal{V}},
\end{aligned}$$

where we exploited (1.2.15) and the continuity of the embedding $\mathcal{V} \hookrightarrow \mathcal{X}$. In total, this yields the assertion. □

In order to ensure the existence of solutions of (RIS), we still have to make assumptions about the initial data.

Initial data

For a given initial value $z_0 \in \mathcal{Z}$ we assume that the external load $\ell \in H^1(0,T;\mathcal{V}^*)$ complies with

$$-\mathrm{D}_z\mathcal{I}(\ell(0), z_0) = -Az_0 - \mathrm{D}_z\mathcal{F}(z_0) + \ell(0) \in \partial\mathcal{R}(0). \tag{1.2.25}$$

Remark 1.2.7. This condition is necessary to guarantee the existence of a differential solution of (RIS) under additional assumptions on the energy, cf. Theorem 2.2.4. It is also needed in Lemma 3.1.3 and thus in the following a priori estimates concerned with the solutions of the viscous regularized versions of (RIS). Furthermore, for physical reasons, (1.2.25) is used as an additional constraint in the later considered optimal control problems. Regarding this optimal control problems, if we choose $\ell_0 = \ell(0)$, we directly obtain the existence of $\ell_0 \in \mathcal{V}^*$ with

$$-\mathrm{D}_z\mathcal{I}(\ell_0, z_0) = -Az_0 - \mathrm{D}_z\mathcal{F}(z_0) + \ell_0 \in \partial\mathcal{R}(0). \tag{1.2.26}$$

This will guarantee the existence of a feasible point, cf. Chapter 4.

Remark 1.2.8. In order to derive the concept of parametrized BV solutions by the vanishing viscosity approach, (1.2.25) can be relaxed to $-\mathrm{D}_z\mathcal{I}(\ell(0), z_0) \in \mathcal{V}^*$. In [KZ21], for example, the authors only require this less restricted precondition to prove the existence of parametrized balanced viscosity solutions. Nevertheless, since we are also interested in constructing a recovery sequence, which is feasible for the viscous regularized optimal control problems and converges to a given solution of the origin optimal control problem, we need that the initial state of this solution complies with (1.2.25). Moreover, it simplifies the analysis regarding the a priori estimates as we will see in Section 3.1. We therefore assume the less general condition (1.2.25) from the outset.

Now that we have clarified all the prerequisites, we would like to give an explicit example at the end, which fits into this setting.

Example 1.2.9. Let $\Omega \subset \mathbb{R}^3$ be an open bounded domain with Lipschitz boundary. We set

$$\mathcal{Z} = H_0^1(\Omega), \quad \mathcal{V} = L^2(\Omega), \quad \mathcal{X} = L^1(\Omega)$$

such that $\mathcal{Z} \hookrightarrow^c \mathcal{V} \hookrightarrow \mathcal{X}$. The dissipation is given by

$$\mathcal{R} : L^1(\Omega) \to \mathbb{R}, \quad z \mapsto \int_\Omega |z(x)| \,\mathrm{d}x$$

and the energy reads

$$\mathcal{I}(\ell, z) = \frac{1}{2}\langle Az, z\rangle_{\mathcal{Z}^*,\mathcal{Z}} + \mathcal{F}(z) - \langle \ell, z\rangle_{\mathcal{V}^*,\mathcal{V}}$$

with $A = -\Delta : H_0^1(\Omega) \to H^{-1}(\Omega)$, i.e., $\langle -\Delta z, v\rangle_{\mathcal{Z}^*,\mathcal{Z}} = \int_\Omega \nabla z(x)\nabla v(x)\,\mathrm{d}x$ for all $v \in \mathcal{Z}$ and the non-linearity is

$$\mathcal{F} : \mathcal{Z} \to \mathbb{R}, \quad z \mapsto \int_\Omega f(z(x))\,\mathrm{d}x$$

with a non-negative function $f \in C^2(\mathbb{R};\mathbb{R})$. In following we will see what properties this f has to satisfy such that (1.2.4)–(1.2.7) holds. After that we give a concrete example for f. First, $\mathcal{F} : \mathcal{Z} \to \mathbb{R}$ inherits its continuity from $f \in C^2(\mathbb{R};\mathbb{R})$ such that (1.2.4) is fulfilled. Moreover, $\mathrm{D}_z\mathcal{F}(z)v = \int_\Omega f'(z(x))v(x)\,\mathrm{d}x$ is well defined for $v \in L^2(\Omega)$ if

$$f'(z) \in L^2(\Omega) \quad \text{for all } z \in H^1(\Omega) \tag{1.2.27}$$

holds. Then $\mathrm{D}_z\mathcal{F}(z) \in \mathcal{L}(\mathcal{Z}, \mathbb{R})$ can be extended to an element of $\mathcal{L}(\mathcal{V}, \mathbb{R})$, which we denote by the same symbol. The second derivative $\mathrm{D}_z^2\mathcal{F} : \mathcal{Z} \to \mathcal{L}(\mathcal{Z}, \mathcal{V}^*)$, given by

$$\mathrm{D}_z^2\mathcal{F}(z)[h_1, h_2] = \int_\Omega f''(z(x))h_1(x)h_2(x)\,\mathrm{d}x, \quad h_1 \in H^1(\Omega),\ h_2 \in L^2(\Omega),$$

is also well defined if the second derivative of f satisfies

$$f''(z) \in L^3(\Omega) \quad \text{for all } z \in H^1(\Omega) \tag{1.2.28}$$

because then $f''(z)h_1 \in L^2(\Omega)$ as $h_1 \in L^6(\Omega)$ by Sobolev embeddings. Moreover, by using Hölder's inequality, we obtain for all $v \in H^1(\Omega)$

$$\begin{aligned}
\|\mathrm{D}_z^2\mathcal{F}(z)v\|_{\mathcal{V}^*} &= \sup_{w\in L^2(\Omega), \|w\|_{L^2(\Omega)}\le 1} \int_\Omega f''(z(x))v(x)w(x)\,\mathrm{d}x \\
&\le \sup_{w\in L^2(\Omega), \|w\|_{L^2(\Omega)}\le 1} \|w\|_{L^2(\Omega)}\|f''(z)v\|_{L^2(\Omega)} \\
&\le \|f''(z)\|_{L^3(\Omega)}\|v\|_{L^6(\Omega)}
\end{aligned}$$

$$\leq C\|f''(z)\|_{L^3(\Omega)}\|v\|_{H^1(\Omega)}$$

such that (1.2.5) is fulfilled if

$$\|f''(z)\|_{L^3(\Omega)} \leq C\Big(1 + \|z\|^q_{H^1(\Omega)}\Big) \quad \text{for all } z \in H^1(\Omega) \tag{1.2.29}$$

with some $q \geq 1$. The weak-weak continuity of $\mathrm{D}_z\mathcal{F}$, i.e.,

$$\int_\Omega f'(z_k(x))v(x)\,\mathrm{d}x \to \int_\Omega f'(z(x))v(x)\,\mathrm{d}x \quad \text{for all } v \in H^1(\Omega),$$

is satisfied if f' complies with

$$z_k \rightharpoonup z \text{ in } H^1(\Omega) \implies f'(z_k) \rightharpoonup f'(z) \text{ in } L^{\frac{6}{5}}(\Omega) \tag{1.2.30}$$

since $H^1(\Omega) \hookrightarrow L^6(\Omega)$ continuously. Finally, we dedicate ourselves to (1.2.7), which holds true if f'' fulfills

$$|f''(v_1) - f''(v_2)| \leq C|v_1 - v_2|^s \quad \text{for all } v_1, v_2 \in \mathbb{R} \tag{1.2.31}$$

with some $1 \leq s \leq 4$, because then by exploiting Hölder's inequality and the continuous Sobolev embedding $H^1(\Omega) \hookrightarrow L^6(\Omega)$, we have for all $z_1, z_2 \in B_{H^1(\Omega)}(0,r)$, $r > 0$, and all $v \in H^1(\Omega)$

$$\begin{aligned}
&\|(\mathrm{D}_z^2\mathcal{F}(z_1) - \mathrm{D}_z^2\mathcal{F}(z_2))v\|_{\mathcal{Z}^*} \\
&= \sup_{w \in H^1_0(\Omega), \|w\|_{H^1_0(\Omega)} \leq 1} \int_\Omega (f''(z_1(x)) - f''(z_2(x)))v(x)w(x)\,\mathrm{d}x \\
&\leq C\|v\|_{H^1(\Omega)}\bigg(\int_\Omega |z_1(x) - z_2(x)|^{\frac{3s}{2}}\,\mathrm{d}x\bigg)^{\frac{2}{3}} \\
&\leq C\|v\|_{H^1(\Omega)}\|z_1 - z_2\|^s_{L^6(\Omega)} \\
&\leq C\|v\|_{H^1(\Omega)}\|z_1 - z_2\|_{H^1(\Omega)}\big(\|z_1\|_{H^1(\Omega)} + \|z_2\|_{H^1(\Omega)}\big)^{s-1} \\
&\leq L_r\|v\|_{H^1(\Omega)}\|z_1 - z_2\|_{H^1(\Omega)},
\end{aligned}$$

which is (1.2.7). A similar calculation shows that if additionally

$$\begin{aligned} f''(z) \in L^3(\Omega) \quad &\text{for all } z \in L^q(\Omega), \\ z_k \to z \text{ in } L^q(\Omega) \implies &f''(z_k) \to f''(z) \text{ in } L^3(\Omega) \end{aligned} \tag{1.2.32}$$

with $2 < q < 6$ such that $H^1(\Omega) \hookrightarrow^c L^q(\Omega)$ compactly, then the second derivative $\mathrm{D}_z^2\mathcal{F}$ can be extended on $\mathcal{W} := L^q(\Omega)$ and is continuous, i.e., $\mathrm{D}_z^2\mathcal{F} \in \mathcal{L}(\mathcal{W}, \mathcal{L}(\mathcal{Z}, \mathcal{V}^*))$, cf. the later introduced Assumption 4.2.5.
As a concrete example we choose the non-convex double well potential

$$\mathcal{F} : \mathcal{Z} \to \mathbb{R}, \quad z \mapsto \int_\Omega |z(x)|^3 - z(x)^2 + 1\,\mathrm{d}x$$

such that $f \in C^2(\mathbb{R};\mathbb{R})$ is given by $f(x) = |x|^3 - x^2 + 1$. Its first and second derivative is

$$f'(x) = \begin{cases} 3x^2 - 2x, & \text{if } x \geq 0, \\ -3x^2 - 2x, & \text{if } x < 0, \end{cases} \qquad f''(x) = \begin{cases} 6x - 2, & \text{if } x \geq 0, \\ -6x - 2, & \text{if } x < 0, \end{cases}$$

such that one easily sees by exploiting the continuous embedding of $H^1(\Omega) \hookrightarrow L^6(\Omega)$ that (1.2.27)–(1.2.29) holds. Obviously, the second derivative of f is globally Lipschitz continuous such that (1.2.31) holds with $s = 1$ and $C = 6$. In order to prove (1.2.30) and (1.2.32), we use

$$z_n \to z \text{ in } L^p(\Omega) \implies |z_n| \to |z| \text{ in } L^p(\Omega),\ p \in [1, \infty), \tag{1.2.33}$$

which follows by exploiting the triangle inequality and the monotonicity of the power function, i.e.,

$$\begin{aligned} \big\| |z_n| - |z| \big\|_{L^p(\Omega)}^p &= \int_\Omega \big| |z_n(x)| - |z(x)| \big|^p \,\mathrm{d}x \\ &\leq \int_\Omega |z_n(x) - z(x)|^p \,\mathrm{d}x = \|z_n - z\|_{L^p(\Omega)}^p \to 0. \end{aligned}$$

In view of (1.2.33) and $f''(x) = 6|x| - 2$, (1.2.32) is satisfied for every $3 \leq p < 6$ by means of the compact Sobolev embedding $H^1(\Omega) \hookrightarrow^c L^p(\Omega)$. Lastly, this embedding along with (1.2.33) yields for a weakly converging sequence $z_k \rightharpoonup z$ in $H^1(\Omega)$ that

$$\begin{aligned} \int_\Omega f'(z_k(x))v(x) \,\mathrm{d}x &= \int_\Omega (3|z_k(x)|z_k(x) - 2z_k(x))v(x) \,\mathrm{d}x \\ &\to \int_\Omega (3|z(x)|z(x) - 2z(x))v(x) \,\mathrm{d}x = \int_\Omega f'(z(x))v(x) \,\mathrm{d}x \end{aligned}$$

for all $v \in L^6(\Omega)$, which is (1.2.30).
Another example is given by $f : \mathbb{R} \to \mathbb{R},\ x \mapsto \sin(x) + 1$. Surely, this function is twice continuously differentiable and non-negative. From the boundedness of f and its derivatives, we conclude that the conditions (1.2.27)–(1.2.29) are satisfied and the Lipschitz continuity of the derivatives directly implies the other conditions (1.2.30)–(1.2.32).

Chapter 2

Solution concepts for rate-independent systems

In this chapter we will give a brief overview of different types of solutions of (RIS). We will see that the existence of weakly differentiable function satisfying (RIS) almost everywhere (also called a differential solution) is in general not ensured, which gives rise to relaxation of the solution concept. We start with deriving alternative formulations of the subdifferential inclusion in (RIS). Based on these, we can formulate the more general solution concepts that allow solvability under less restrictive assumptions.

2.1 Equivalent formulations for (RIS)

Before we define the different notions of solutions of the rate-independent system

$$0 \in \partial\mathcal{R}(z'(t)) + \mathrm{D}_z\mathcal{I}(\ell(t), z(t)) \quad \text{f.a.a. } t \in (0,T), \quad z(0) = z_0, \tag{2.1.1}$$

we reformulate this subdifferential inclusion equivalently, which then gives rise for a variety of generalizations. We assume that the solution z of (2.1.1) has the required regularity such that all reformulations are well defined. First of all, by using the definition of the subdifferential, (2.1.1) can be written as an evolutionary variational inequality, i.e.,

$$\mathcal{R}(v) - \mathcal{R}(z'(t)) + \langle \mathrm{D}_z\mathcal{I}(\ell(t), z(t)), v - z'(t)\rangle_{\mathcal{Z}^*,\mathcal{Z}} \geq 0 \quad \text{for all } v \in \mathcal{Z}$$

almost everywhere in $(0,T)$. This or, more precisely, the evolutionary variational inequality formulation of the viscous regularized system, will be the base, when we prove the existence of a solution of this viscous regularized system, cf. Theorem 3.2.3. Moreover, it is also used in Chapter 4, where we deal with optimal control problems whose state function is given as a solution of such a system. Another reformulation is the following. Since the dissipation is proper, convex and lower semicontinuous, see (1.2.13), by using standard arguments from convex analysis, we can rewrite (2.1.1) by

$$z'(t) \in \partial \mathcal{R}^*(-\mathrm{D}_z\mathcal{I}(\ell(t), z(t))) \quad \text{for almost all } t \in (0, T).$$

In the case of the twice regularization of (2.1.1), we will see in Theorem 3.1.1 that the right hand side will be single-valued and Lipschitz continuous such that we obtain the existence of a solution by the Picard-Lindelöf theorem. In the next step, we want to derive an alternative formulation of (2.1.1), which forms the basis for the relaxation of the concept of solvability. To this end, we start with using the Fenchel-Young equality, note that $\mathcal{R}$ is proper and convex by assumption (1.2.13), in order to obtain

$$\mathcal{R}(z'(t)) + \mathcal{R}^*(-\mathrm{D}_z\mathcal{I}(\ell(t), z(t))) = \langle -\mathrm{D}_z\mathcal{I}(\ell(t), z(t)), z'(t)\rangle_{\mathcal{Z}^*,\mathcal{Z}} \tag{2.1.2}$$

for almost all $t \in (0, T)$. Applying the chain rule from Lemma C.3, i.e.,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{I}(\ell(t), z(t)) &= \langle Az(t), z'(t)\rangle_{\mathcal{Z}^*,\mathcal{Z}} + \langle \mathrm{D}_z\mathcal{F}(z(t)), z'(t)\rangle_{\mathcal{Z}^*,\mathcal{Z}} \\ &\qquad - \langle \ell(t), z'(t)\rangle_{\mathcal{V}^*,\mathcal{V}} - \langle \ell'(t), z(t)\rangle_{\mathcal{V}^*,\mathcal{V}} \\ &= \langle \mathrm{D}_z\mathcal{I}(\ell(t), z(t)), z'(t)\rangle_{\mathcal{Z}^*,\mathcal{Z}} - \langle \ell'(t), z(t)\rangle_{\mathcal{V}^*,\mathcal{V}}\end{aligned}$$

to the right hand side results in

$$\mathcal{R}(z'(t)) + \mathcal{R}^*(-\mathrm{D}_z\mathcal{I}(\ell(t), z(t))) = -\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{I}(\ell(t), z(t)) - \langle \ell'(t), z(t)\rangle_{\mathcal{V}^*,\mathcal{V}}.$$

Eventually, let $t \in [0, T]$ be arbitrary and integrating over $(0, t)$ leads to

$$\begin{aligned}\mathcal{I}(\ell(t), z(t)) &+ \int_0^t \mathcal{R}(z'(r)) + \mathcal{R}^*(-\mathrm{D}_z\mathcal{I}(\ell(r), z(r)))\,\mathrm{d}r \\ &= \mathcal{I}(0, z_0) - \int_0^t \langle \ell'(r), z(r)\rangle_{\mathcal{V}^*,\mathcal{V}}\,\mathrm{d}r.\end{aligned} \tag{2.1.3}$$

In view of (1.2.21), the above energy equality, and thus (2.1.1), can equivalently written via the two conditions

$$-\mathrm{D}_z\mathcal{I}(\ell(t), z(t)) \in \partial\mathcal{R}(0) \quad \text{f.a.a. } t \in (0, T) \tag{2.1.4}$$

and

$$\begin{aligned}\mathcal{I}(\ell(t), z(t)) &+ \int_0^t \mathcal{R}(z'(r))\,\mathrm{d}r \\ &= \mathcal{I}(0, z_0) - \int_0^t \langle \ell'(r), z(r)\rangle_{\mathcal{V}^*,\mathcal{V}}\,\mathrm{d}r \quad \forall t \in [0, T].\end{aligned} \tag{2.1.5}$$

We say that a state $z \in \mathcal{Z}$ satisfying (2.1.4), i.e., $-\mathrm{D}_z\mathcal{I}(\ell(t), z) \in \partial\mathcal{R}(0)$ is locally stable at the time $t \in [0, T]$ and define the set of all locally stable states by

$$S_{\mathrm{loc}}(t) := \Big\{z \in \mathcal{Z} : -\mathrm{D}_z\mathcal{I}(\ell(t), z) \in \partial\mathcal{R}(0)\Big\}. \tag{2.1.6}$$

If the energy is additionally convex, then the local stability condition (2.1.4) is equivalent to the global stability condition

$$\mathcal{I}(\ell(t), z(t)) \leq \mathcal{I}(\ell(t), v) + \mathcal{R}(v - z(t)) \text{ for all } v \in \mathcal{Z}. \tag{2.1.7}$$

Accordingly, we define the set of global stability by

$$S_{\mathrm{glob}}(t) := \Big\{z \in \mathcal{Z} : \mathcal{I}(\ell(t), z) \leq \mathcal{I}(\ell(t), v) + \mathcal{R}(v - z) \text{ for all } v \in \mathcal{Z}\Big\}. \tag{2.1.8}$$

Indeed, we have the following relation between $S_{\mathrm{loc}}(t)$ and $S_{\mathrm{glob}}(t)$.

Lemma 2.1.1. *It always holds $S_{\mathrm{glob}}(t) \subset S_{\mathrm{loc}}(t)$. If $\mathcal{I}(\ell(t), \cdot)$ is additionally convex, then we have $S_{\mathrm{glob}}(t) = S_{\mathrm{loc}}(t)$.*

Proof. Let $z \in S_{\mathrm{glob}}(t)$ be given, i.e., $\mathcal{I}(\ell(t), z) \leq \mathcal{I}(\ell(t), v) + \mathcal{R}(v - z)$ for all $v \in \mathcal{Z}$. Testing this inequality with $v := z + \epsilon w$, $\epsilon > 0$, $w \in \mathcal{Z}$ and rearranging the terms yields

$$\mathcal{I}(\ell(t), z) - \mathcal{I}(\ell(t), z + \epsilon w) \leq \epsilon\mathcal{R}(w),$$

where we exploited the 1-homogeneity of the dissipation, cf. (1.2.14). Eventually, dividing by $\epsilon > 0$ and passing $\epsilon \searrow 0$ results in

$$\langle -\mathrm{D}_z\mathcal{I}(\ell(t), z), w\rangle_{\mathcal{Z}^*,\mathcal{Z}} = \lim_{\epsilon \searrow 0} \frac{1}{\epsilon}\big(\mathcal{I}(\ell(t), z) - \mathcal{I}(\ell(t), z + \epsilon w)\big) \leq \mathcal{R}(w).$$

Since this holds true for all $w \in \mathcal{Z}$, in view of (1.2.18), we have $-\mathrm{D}_z\mathcal{I}(\ell(t), z) \in \partial\mathcal{R}(0)$ such that z is locally stable.
In order to verify the second assertion, assume that $\mathcal{I}(\ell(t), \cdot)$ is additionally convex, which immediately gives

$$\langle \mathrm{D}_z\mathcal{I}(\ell(t), z), v - z\rangle_{\mathcal{Z}^*,\mathcal{Z}} \leq \mathcal{I}(\ell(t), v) - \mathcal{I}(\ell(t), z) \text{ for all } v \in \mathcal{Z}. \tag{2.1.9}$$

For $z \in S_{\mathrm{loc}}(t)$ we obtain from (1.2.18) that

$$\langle -\mathrm{D}_z\mathcal{I}(\ell(t), z), w\rangle_{\mathcal{Z}^*,\mathcal{Z}} \leq \mathcal{R}(w) \text{ for all } w \in \mathcal{Z}$$

Thus, inserting $w = v - z$ gives in view of (2.1.9)

$$\mathcal{I}(\ell(t), z) - \mathcal{I}(\ell(t), v) \leq \langle -\mathrm{D}_z\mathcal{I}(\ell(t), z), v - z\rangle_{\mathcal{Z}^*,\mathcal{Z}} \leq \mathcal{R}(v - z)$$

such that $z \in S_{\mathrm{glob}}(t)$. □

2.2 Solution concepts for (RIS)

Various authors have developed a variety of solution concepts for (RIS) in recent years. In the following, we present only a small selection of these solution concepts, and only those that we need for our purposes namely *differential, energetic, local* and *parametrized solutions*. We note here that these solution concepts can also be formulated for more general settings such that the energy does not need to have the structure from (1.2.1) but instead the energy has to satisfy appropriate continuity and growth conditions such that the existence of solutions is ensured. But for the sake of simplicity and better readability we use the setting from Section 1.2. With regard to a general energy framework and for a comprehensive overview of further concepts we refer to [MR15]. We start with introducing the most natural concept, namely a weakly differentiable function z that satisfies the subdifferential inclusion (RIS) almost everywhere in $(0, T)$.

Definition 2.2.1 (Differential solution)**.** We call $z \in W^{1,1}(0, T; \mathcal{Z})$ a *differential solution* of (RIS) if

$$0 \in \partial\mathcal{R}(z'(t)) + \mathrm{D}_z\mathcal{I}(\ell(t), z(t)) \tag{2.2.1}$$

for almost all $t \in (0, T)$ and $z(0) = z_0$.

The first existence result of such differential solutions in the case of a quadratic energy goes back to [Bré73]. A more general result for smooth and uniformly convex energies can be found in [CV90]. In order to ensure the existence of such a differential solution with regard to our energy and dissipation framework from Section 1.2, we also require an additional property of the energy, namely the uniform convexity, which is:

Definition 2.2.2. We say that the energy $\mathcal{I}$ is uniformly convex if there exists a constant $\gamma > 0$ such that

$$\gamma\|v\|_{\mathcal{Z}}^2 \leq \langle \mathrm{D}_z^2\mathcal{I}(\ell(t), z)v, v\rangle_{\mathcal{Z}^*,\mathcal{Z}} \tag{2.2.2}$$

for all $z, v \in \mathcal{Z}$, $t \in [0, T]$.

Remark 2.2.3. Due to the concrete structure of the energy by (1.2.1), the condition (2.2.2) is equivalent to

$$\gamma\|v\|_{\mathcal{Z}}^2 \leq \langle \mathrm{D}_z^2\mathcal{E}(z)v, v\rangle_{\mathcal{Z}^*,\mathcal{Z}} = \langle (A + \mathrm{D}_z^2\mathcal{F}(z))v, v\rangle_{\mathcal{Z}^*,\mathcal{Z}} \tag{2.2.3}$$

for all $z \in \mathcal{Z}$, $v \in \mathcal{Z}$.

Assuming the additional uniform convexity of the energy, we can now prove the following existence and uniqueness result for differential solutions.

Theorem 2.2.4. *Let the energy $\mathcal{I}$ be uniformly convex. Then for every initial value $z_0 \in \mathcal{Z}$ with $-\mathrm{D}_z\mathcal{I}(\ell(t), z_0) \in \partial\mathcal{R}(0)$ there exists a unique differential solution $z \in H^1(0, T; \mathcal{Z})$ of* (RIS).

Proof. Since the concept of differential solutions is not in the focus of this thesis and is only needed for the approximation results in Section 4.2, we only give a brief overview of the steps to prove the existence of a differential solution $z \in H^1(0,T;\mathcal{Z})$. The proof is based on results that we will only prove later, as they are also needed to derive the concept of parametrized balanced viscosity solutions. The relevant details can be found in the following chapters. In return, we will prove the uniqueness of a solution in detail.
First, we consider the viscous regularized system

$$0 \in \partial\mathcal{R}_\epsilon(z'(t)) + \mathrm{D}_z\mathcal{I}(\ell(t), z(t)) \quad \text{f.a.a. } t \in (0,T), \quad z(0) = z_0, \tag{2.2.4}$$

where $\mathcal{R}_\epsilon(z) = \mathcal{R}(z) + \frac{\epsilon}{2}\|z\|_{\mathcal{V}}^2$ with $\epsilon > 0$. The existence of a solution $z_\epsilon \in H^1(0,T;\mathcal{Z})$ of (2.2.4) is guaranteed by Theorem 3.2.3. Under the assumption of a uniformly convex energy we can derive a priori estimates that does not depend on the viscosity parameter, i.e., $\|z_\epsilon\|_{H^1(0,T;\mathcal{Z})} \leq C$, cf. Section 4.2 and (4.2.20). Therefore, we can extract a weakly converging subsequence and in the proof of Lemma 4.2.3 it is shown that the weak limit solves the subdifferential inclusion. Thus, it is a differential solution. A different approach is taken, for example, in [MT04, Sec. 7]. There the authors first prove the existence a global energetic solution by using a time discretization approach. For the precise definition of this solution concept see Definition 2.2.6. Under suitable continuity assumptions on the energy, especially the uniform convexity, it is shown that this global energetic solution has higher regularity and is even a differential solution.
In the next step, we show the uniqueness of the solution. For this purpose, we assume that there exists two differential solutions $z_1, z_2 \in H^1(0,T;\mathcal{Z})$ and follow the lines of [MR15, Sec. 3.4.4]. From (2.2.1) and the definition of the subdifferential we infer that

$$\begin{aligned} \mathcal{R}(v) &\geq \mathcal{R}(z_1'(t)) + \langle -\mathrm{D}_z\mathcal{I}(\ell(t), z_1(t)), v - z_1'(t)\rangle_{\mathcal{Z}^*,\mathcal{Z}} \\ \mathcal{R}(v) &\geq \mathcal{R}(z_2'(t)) + \langle -\mathrm{D}_z\mathcal{I}(\ell(t), z_2(t)), v - z_2'(t)\rangle_{\mathcal{Z}^*,\mathcal{Z}} \end{aligned}$$

for all $v \in \mathcal{Z}$ and almost all $t \in (0,T)$. Now inserting $v = z_2'(t)$ in the first inequality and $v = z_1'(t)$ in the second one and adding up both inequalities results in

$$\begin{aligned} 0 &\geq \langle \mathrm{D}_z\mathcal{I}(\ell(t), z_1(t)) - \mathrm{D}_z\mathcal{I}(\ell(t), z_2(t)), z_1'(t) - z_2'(t)\rangle_{\mathcal{Z}^*,\mathcal{Z}} \\ &= \langle \mathrm{D}_z\mathcal{E}(z_1(t)) - \mathrm{D}_z\mathcal{E}(z_2(t)), z_1'(t) - z_2'(t)\rangle_{\mathcal{Z}^*,\mathcal{Z}} \end{aligned} \tag{2.2.5}$$

for almost all $t \in (0,T)$, where we exploited the concrete structure of the energy for the last equality. Let $\mu(t) := \langle \mathrm{D}_z\mathcal{E}(z_1(t)) - \mathrm{D}_z\mathcal{E}(z_2(t)), z_1(t) - z_2(t)\rangle_{\mathcal{Z}^*,\mathcal{Z}}$. Then the mean value theorem yields

$$\begin{aligned} \mu(t) &= \int_0^1 \langle \mathrm{D}_z^2\mathcal{E}(z_1(t) + \tau(z_2(t) - z_1(t)))(z_1(t) - z_2(t)), z_1(t) - z_2(t)\rangle_{\mathcal{Z}^*,\mathcal{Z}}\,\mathrm{d}\tau \\ &\geq \gamma\|z_1(t) - z_2(t)\|_{\mathcal{Z}}^2, \end{aligned} \tag{2.2.6}$$

where we exploited the uniform convexity (2.2.3) in the last estimate. Moreover, the symmetry of $\mathrm{D}_z^2\mathcal{E}$ yields for the derivative of μ

$$
\begin{aligned}
\mu'(t) &= \langle \mathrm{D}_z^2\mathcal{E}(z_1(t))z_1'(t) - \mathrm{D}_z^2\mathcal{E}(z_2(t))z_2'(t), z_1(t) - z_2(t)\rangle_{\mathcal{Z}^*,\mathcal{Z}} \\
&\quad + \langle \mathrm{D}_z\mathcal{E}(z_1(t)) - \mathrm{D}_z\mathcal{E}(z_2(t)), z_1'(t) - z_2'(t)\rangle_{\mathcal{Z}^*,\mathcal{Z}} \\
&= \langle \mathrm{D}_z^2\mathcal{E}(z_1(t))(z_1(t) - z_2(t)), z_1'(t)\rangle_{\mathcal{Z}^*,\mathcal{Z}} - \langle \mathrm{D}_z^2\mathcal{E}(z_2(t))(z_1(t) - z_2(t)), z_2'(t)\rangle_{\mathcal{Z}^*,\mathcal{Z}} \\
&\quad + \langle \mathrm{D}_z\mathcal{E}(z_1(t)) - \mathrm{D}_z\mathcal{E}(z_2(t)), z_1'(t) - z_2'(t)\rangle_{\mathcal{Z}^*,\mathcal{Z}} \\
&= \langle \mathrm{D}_z^2\mathcal{E}(z_1(t))(z_1(t) - z_2(t)), z_1'(t)\rangle_{\mathcal{Z}^*,\mathcal{Z}} + \langle \mathrm{D}_z\mathcal{E}(z_2(t)) - \mathrm{D}_z\mathcal{E}(z_1(t)), z_1'(t)\rangle_{\mathcal{Z}^*,\mathcal{Z}} \\
&\quad - \langle \mathrm{D}_z^2\mathcal{E}(z_2(t))(z_1(t) - z_2(t)), z_2'(t)\rangle_{\mathcal{Z}^*,\mathcal{Z}} + \langle \mathrm{D}_z\mathcal{E}(z_1(t)) - \mathrm{D}_z\mathcal{E}(z_2(t)), z_2'(t)\rangle_{\mathcal{Z}^*,\mathcal{Z}} \\
&\quad + 2\langle \mathrm{D}_z\mathcal{E}(z_1(t)) - \mathrm{D}_z\mathcal{E}(z_2(t)), z_1'(t) - z_2'(t)\rangle_{\mathcal{Z}^*,\mathcal{Z}} \\
&= \langle \mathrm{D}_z^2\mathcal{F}(z_1(t))(z_1(t) - z_2(t)) + \mathrm{D}_z\mathcal{F}(z_2(t)) - \mathrm{D}_z\mathcal{F}(z_1(t)), z_1'(t)\rangle_{\mathcal{Z}^*,\mathcal{Z}} \\
&\quad - \langle \mathrm{D}_z^2\mathcal{F}(z_2(t))(z_1(t) - z_2(t)) - \mathrm{D}_z\mathcal{F}(z_1(t)) - \mathrm{D}_z\mathcal{F}(z_2(t)), z_2'(t)\rangle_{\mathcal{Z}^*,\mathcal{Z}} \\
&\quad + 2\langle \mathrm{D}_z\mathcal{E}(z_1(t)) - \mathrm{D}_z\mathcal{E}(z_2(t)), z_1'(t) - z_2'(t)\rangle_{\mathcal{Z}^*,\mathcal{Z}}
\end{aligned}
$$

almost everywhere in $(0, T)$, where we used the linearity of A in the last step such that the respective terms have been cancelled. Another use of the mean value theorem yields along with (1.2.7)

$$
\begin{aligned}
&\big|\langle \mathrm{D}_z^2\mathcal{F}(z_1(t))(z_1(t) - z_2(t)) + \mathrm{D}_z\mathcal{F}(z_2(t)) - \mathrm{D}_z\mathcal{F}(z_1(t)), z_1'(t)\rangle_{\mathcal{Z}^*,\mathcal{Z}}\big| \\
&\le \int_0^1 \big\|[\mathrm{D}_z^2\mathcal{F}(z_1(t)) - \mathrm{D}_z^2\mathcal{F}(z_1(t) + \tau(z_2(t) - z_1(t)))](z_1(t) - z_2(t))\big\|_{\mathcal{Z}^*}\,\mathrm{d}\tau \|z_1'(t)\|_{\mathcal{Z}} \\
&\le \int_0^1 L_r\tau\|z_1(t) - z_2(t)\|_{\mathcal{Z}}^2\,\mathrm{d}\tau\|z_1'(t)\|_{\mathcal{Z}} \\
&\le C\|z_1(t) - z_2(t)\|_{\mathcal{Z}}^2\|z_1'(t)\|_{\mathcal{Z}},
\end{aligned}
$$

where $r = \max\{\|z_1\|_{L^\infty(0,T;\mathcal{Z})}, \|z_2\|_{L^\infty(0,T;\mathcal{Z})}\}$. Note that $r < \infty$ due to the continuous embedding $H^1(0, T; \mathcal{Z}) \hookrightarrow C([0, T]; \mathcal{Z})$. In the same way, it can be shown that

$$
\begin{aligned}
&|\langle \mathrm{D}_z^2\mathcal{F}(z_2(t))(z_1(t) - z_2(t)) - \mathrm{D}_z\mathcal{F}(z_1(t)) - \mathrm{D}_z\mathcal{F}(z_2(t)), z_2'(t)\rangle_{\mathcal{Z}^*,\mathcal{Z}}| \\
&\qquad\qquad \le C\|z_1(t) - z_2(t)\|_{\mathcal{Z}}^2\|z_2'(t)\|_{\mathcal{Z}}
\end{aligned}
$$

such that together

$$
\begin{aligned}
\mu'(t) &\le C\|z_1(t) - z_2(t)\|_{\mathcal{Z}}^2(\|z_1'(t)\|_{\mathcal{Z}} + \|z_2'(t)\|_{\mathcal{Z}}) \\
&\qquad + 2\langle \mathrm{D}_z\mathcal{E}(z_1(t)) - \mathrm{D}_z\mathcal{E}(z_2(t)), z_1'(t) - z_2'(t)\rangle_{\mathcal{Z}^*,\mathcal{Z}} \\
&\le C\|z_1(t) - z_2(t)\|_{\mathcal{Z}}^2(\|z_1'(t)\|_{\mathcal{Z}} + \|z_2'(t)\|_{\mathcal{Z}}) \\
&\le \frac{C}{\gamma}\big(\|z_1'(t)\|_{\mathcal{Z}} + \|z_2'(t)\|_{\mathcal{Z}}\big)\mu(t),
\end{aligned}
$$

where we also exploited (2.2.5) and (2.2.6). Now integrating over $(0, t)$ with $t \in [0, T]$ gives

$$
\mu(t) \le \int_0^t \frac{C}{\gamma}\big(\|z_1'(r)\|_{\mathcal{Z}} + \|z_2'(r)\|_{\mathcal{Z}}\big)\mu(r)\,\mathrm{d}r
$$

since $\mu(0) = 0$ as both differential solutions comply with the initial condition, i.e., $z_1(0) = z_0 = z_2(0)$. Eventually, applying Gronwall's lemma yields

$$\gamma \|z_1(t) - z_2(t)\|_{\mathcal{Z}}^2 \leq \mu(t) \leq 0$$

such that $z_1 = z_2$ as claimed. □

The following one-dimensional example shows that the additional uniform convexity is indeed needed to ensure the existence of a differential solution.

Example 2.2.5. Let dissipation and energy be given by

$$\mathcal{R}(z) = |z|, \qquad \mathcal{I}(\ell(t), z) = \frac{1}{3}|z|^3 - \frac{1}{2}z^2 - \ell(t)z.$$

with $\ell(t) = t$. The final time is $T = 2$ and the initial value is set to $z_0 = 0$. By direct calculations we see that

$$\partial\mathcal{R}(z') = \begin{cases} \{1\}, & \text{if } z' > 0, \\ [-1,1], & \text{if } z' = 0, \\ \{-1\}, & \text{if } z' < 0, \end{cases} \qquad \mathrm{D}_z\mathcal{I}(\ell(t), z) = \begin{cases} z^2 - z - t, & \text{if } z \geq 0, \\ -z^2 - z - t, & \text{if } z < 0 \end{cases}$$

such that the initial value satisfies

$$-\mathrm{D}_z\mathcal{I}(\ell(t), z_0) = -z_0^2 + z_0 + t = t \in [-1,1] \quad \text{for all } t \in [0,1]$$

and is thus locally stable. Furthermore, we see that $z(t) \equiv z_0 = 0$ describes a differential solution in the interval $[0,1]$. For $t > 1$ we have $-\mathrm{D}_z\mathcal{I}(\ell(t), z_0) > 1 \notin \partial\mathcal{R}(0)$ such that the constant function would leave the set of local stability, which in turn contradicts the notion of differential solutions. Therefore, on $[1,2]$ the solution has to be given by another function. In order to determine this function, assume first that $z'(t) > 0$. Then (2.2.1) becomes

$$-z(t)^2 + z(t) + t = 1,$$

which provides the solutions

$$z_{\pm}(t) = \frac{1}{2} \pm \sqrt{\frac{1}{4} + t - 1}.$$

Since we consider the case that $z'(t) > 0$, z_- is eliminated as a possible solution. Moreover, z_+ is only defined for $t \geq \frac{3}{4}$. Thus, it only describes a differential solution on $[\frac{3}{4}, 2]$ with initial value $z_{\frac{3}{4}} = \frac{1}{2}$. Otherwise, if $z'(t) < 0$, then (2.2.1) can be written as

$$z(t)^2 + z(t) + t = -1,$$

whose solutions are

$$z_\pm(t) = -\frac{1}{2} \pm \sqrt{\frac{1}{4} - t - 1}.$$

Because $t \geq 0$, we have $z_\pm(t) \notin \mathbb{R}$ such that these functions are excluded as a possible solution, too. The only possible solutions are visualized in Figure 2.1. To sum up, the only differential solution that complies with the initial state $z_0 = 0$ is the constant function $z \equiv 0$ but only on the first half of the time horizon. At the latest at time $t = 1$ the solution has to jump to $z(t) = \frac{1}{2} + \sqrt{\frac{1}{4} + t - 1}$, which in turn results in a discontinuous solution. Hence, there exists no differential solution $z \in W^{1,1}(0, 2; \mathbb{R})$ in this setting.

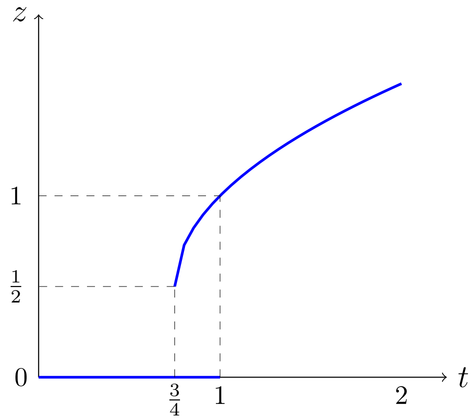

Figure 2.1: Graphs of both possible differential solutions in the setting of Example 2.2.5 with non-convex energy on their respective time intervals with (suitable) associated initial value such that a solution living on the entire interval $[0, 2]$ has to jump between $t = \frac{3}{4}$ and $t = 1$.

We have seen that additional requirements on the energy in terms of uniform convexity are necessary in order to ensure the solvability of (RIS). Thus, if we aim for existence of solutions under milder assumptions, we have to weaken the properties of a potential solution. This leads to the concept of the so called *global energetic solutions*. These energetic solutions are already investigated by various authors in recent years in different context of application. We refer here exemplarily to [MT99], [MT04], [MTL02], [MTL98], where the authors developed this notion of solution in the context of phase transformations in elastic materials with hysteresis. The main advantage of these energetic solutions compared to the differential solutions is that they do not require differentiability and convexity of the energy functional with respect to the state variable and differentiability of the solution itself. In view of the equivalent formulations for (RIS) from the previous section, in particular (2.1.5) and (2.1.7), we define:

Definition 2.2.6 (Global energetic solution)**.** We call $z \in BV(0,T;\mathcal{V}) \cap L^\infty(0,T;\mathcal{Z})$ a *global energetic solution* of (RIS) if $z(t)$ is globally stable for all $t \in [0,T]$, i.e.,

$$\mathcal{I}(\ell(t), z(t)) \leq \mathcal{I}(\ell(t), v) + \mathcal{R}(v - z(t)) \text{ for all } v \in \mathcal{Z} \tag{2.2.7}$$

and for all $t \in [0,T]$ the energy balance

$$\mathcal{I}(\ell(t), z(t)) + \mathrm{Var}_{\mathcal{R}}(z, [0,t]) = \mathcal{I}(\ell(0), z_0) - \int_0^t \langle \ell'(r), z(r) \rangle_{\mathcal{V}*,\mathcal{V}} \,\mathrm{d}r \tag{2.2.8}$$

holds, where

$$\begin{aligned} &\mathrm{Var}_{\mathcal{R}}(z; [t_1, t_2]) := \\ &\quad \sup \left\{ \sum_{i=1}^n \mathcal{R}(z(\xi_i) - z(\xi_{i-1})) \;\middle|\; t_1 = \xi_0 < \xi_1 < \cdots < \xi_n = t_2,\ n \in \mathbb{N} \right\}. \end{aligned} \tag{2.2.9}$$

Remark 2.2.7. The integral in (2.1.5) containing the derivative of z is replaced by the $\mathcal{R}$-variation $\mathrm{Var}_{\mathcal{R}}(z, [0,t])$. If z is sufficiently regular, Lemma B.3 states that both coincide such that this is indeed a generalization. The assumed regularity of the energetic solution, i.e., $z \in BV(0,T;\mathcal{Z})$, in combination with the upper bound in (1.2.15) implies that $\mathrm{Var}_{\mathcal{R}}(z, [0,t]) < \infty$.

The existence of such global energetic solutions is guaranteed under less restrictive assumptions than for differential solutions. Because this notion of solution does not play a crucial role in this thesis, we refer here to [MR15, Sec. 2.1, Thm: 2.16] for a detailed proof and only present the ideas of the proof briefly. In [MR15, Sec. 2.1] the authors show the existence via a time discretization approach. To be more precise, for a given partition

$$P_n = \Big\{ 0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = T \Big\}$$

of $[0,T]$ and an initial value $z_0 \in S_{glob}(0)$, one first solves the following incremental minimization problem

$$z_k \in \arg\min\{\mathcal{I}(\ell(t_k), z) + \mathcal{R}(z - z_{k-1}) : z \in \mathcal{Z}\},\ k = 1, \ldots, n. \tag{IMP}$$

To motivate this, the stationary condition of (IMP) is given by

$$0 \in \mathrm{D}_z\mathcal{I}(\ell(t_k), z_k) + \partial\mathcal{R}(z_k - z_{k-1}) = \mathrm{D}_z\mathcal{I}(\ell(t_k), z_k) + \partial\mathcal{R}\Big(\frac{z_k - z_{k-1}}{\tau_k}\Big),$$

where $\tau_k := t_k - t_{k-1} > 0$ and the positive 0-homogeneity of $\partial\mathcal{R}(\cdot)$ by (1.2.23) is used in the last step. This equals a discrete version of (RIS). The solvability of (IMP) can be shown by the direct method in the calculus of variations. Eventually, with these solutions at hand, one defines the piecewise constant right and left continuous interpolants

$$\underline{z}^n(t) := z_{k-1}, \quad t \in [t_{k-1}, t_k),\ k = 1, \ldots, n, \quad \text{and} \quad \underline{z}^n(T) = z_n$$

and

$$\underline{z}^n(t) := z_{k-1}, \quad t \in (t_{k-1}, t_k], \ k = 1, \dots, n, \quad \text{and} \quad \underline{z}^n(0) = z_0,$$

respectively. They will satisfy discrete versions of the stability condition (2.2.7) and the energy identity (2.2.8) and suitable a priori estimates. Then by passing the fineness of the partition

$$h_n := \max\Big\{\tau_k = t_k - t_{k_1} : k = 1, \dots, n\Big\}$$

to zero, one can extract a converging subsequence and can show that the limit fulfills (2.2.7) and (2.2.8), which in turn yields the existence of a global energetic solution.

We have to mention here that not every differential solution is automatically a global energetic solution, too. The reason for this is that we require global stability of the solution, cf. (2.2.7), and not only local stability. If the energy is not convex, one cannot guarantee that the set of local stability coincide with the global one, i.e., $S_{\mathrm{loc}}(t) \neq S_{\mathrm{glob}}(t)$. The following example from [MR15, Ex. 1.8.2] illustrate this.

Example 2.2.8 ([MR15, Ex. 1.8.2])**.** Let dissipation and energy be given by

$$\mathcal{R}(z) = |z|, \quad \mathcal{I}(\ell(t), z) = \mathcal{E}(z) - \ell(t)z$$

with

$$\mathcal{E}(z) = \begin{cases} \frac{1}{2}(z+4)^2, & \text{if } z \leq -2, \\ 4 - \frac{1}{2}z^2, & \text{if } -2 < z < 2, \\ \frac{1}{2}(z-4)^2, & \text{if } z \geq 2, \end{cases} \quad \ell(t) = \min\{t, 4-t\}.$$

The initial value is set to $z_0 = -5$. By direct calculations one verifies that a differential solution z_{DS} and a global energetic solution z_{ES} are given by

$$z_{DS}(t) = \begin{cases} t-5, & 0 \leq t \leq 2, \\ -3, & 2 \leq t \leq 4, \\ 1-t, & 4 \leq t, \end{cases} \qquad z_{ES}(t) = \begin{cases} t-5, & 0 \leq t \leq 1, \\ t+3, & 1 < t < 2, \\ 5, & 2 \leq t \leq 4, \\ 9-t, & 4 < t < 5, \\ 1-t, & 5 \leq t. \end{cases}$$

After the time $t = 1$, the two solutions differ since the differential solution is not globally stable anymore, which can seen as follows. On the interval $(1, 2]$ the differential solution reads $z_{DS}(t) = t - 5$ such that

$$\mathcal{I}(\ell(t), z_{DS}(t)) = \frac{1}{2}((t-5)+4)^2 - t(t-5) = -\frac{1}{2}t^2 + 4t + \frac{1}{2}.$$

If we choose $v = 4$, we see that the global stability condition (2.2.7) is not satisfies because for all $t \in (1, 2]$ we have

$$\mathcal{I}(\ell(t), v) + \mathcal{R}(v - z_{DS}(t)) = -4t + |4 - (t - 5)| = -5t + 9$$

such that

$$\mathcal{I}(\ell(t), z_{DS}(t)) > \mathcal{I}(\ell(t), v) + \mathcal{R}(v - z_{DS}(t))$$

for all $t \in (1, 2]$ with $v = 4$.

A solution concept that includes both, differential and global energetic solution, is the notion of *local solutions*. Here we replace the global stability condition (2.2.7) by its local version (2.1.4) and only require an energy inequality instead of equality as in (2.2.8). That means a possible solution may jump to a lower energy level and the loss of energy has not to be compensated by the dissipation. The precise definition follows now.

Definition 2.2.9 (Local solution)**.** We call $z \in BV(0, T; \mathcal{V}) \cap L^\infty(0, T; \mathcal{Z})$ a *local solution* of (RIS), if z is locally stable almost everywhere, i.e.,

$$0 \in \partial\mathcal{R}(0) + \mathrm{D}_z\mathcal{I}(t, z(t)) \quad \text{f.a.a. } t \in [0, T], \tag{2.2.10}$$

and the following energy inequality holds

$$\begin{aligned} &\mathcal{I}(t_2, z(t_2)) + \mathrm{Var}_{\mathcal{R}}(z; [t_1, t_2]) \\ &\qquad \le \mathcal{I}(t_1, z(t_1)) - \int_{t_1}^{t_2} \langle \ell'(r), z(r) \rangle_{\mathcal{V}^*, \mathcal{V}} \, \mathrm{d}r \end{aligned} \tag{2.2.11}$$

for all $0 \le t_1 \le t_2 \le T$.

Another approach to handle discontinuities in the solution is given by the so-called *parametrized balanced viscosity (BV) solutions*. This concept goes back to [EM06] and has been further developed for different settings. Exemplarily, we refer here to [MRS09, MRS12, MZ14, KRZ13, MRS16]. The main idea behind these parametrized solutions is to introduce an additional artificial time that extends the time horizon in order to resolve the arising discontinuities in the solution. This is the solution concept that we will mainly use throughout this thesis. Thus we discuss it in detail in the next chapter. The precise definition can be found in Definition 3.3.4 and in Section 3.3 we show the existence of such parametrized balanced viscosity solutions under the assumptions from Section 1.2 by using a vanishing viscosity approach.

For the sake of completeness, it should be mentioned that there exists a variety of further solution concepts for rate-independent systems, for example *semidifferential solutions*, *CD (Continuous Dissipation) solutions*, *weak solutions* and many more. For the interested reader, we refer here to [MR15] for an overview of the various solution concepts.

Chapter 3

Existence of $\mathfrak{p}$-parametrized balanced viscosity solutions

In this chapter we derive the notion parametrized balanced viscosity solutions via the vanishing viscosity approach. To do this, we consider the viscous regularized system

$$0 \in \partial \mathcal{R}_\epsilon(z'(t)) + \mathrm{D}_z \mathcal{I}(\ell(t), z(t)) \quad \text{f.a.a. } t \in (0,T), \quad z(0) = z_0 \tag{3.0.1}$$

with $\mathcal{R}_\epsilon(z) := \mathcal{R}(z) + \frac{\epsilon}{2}\|z\|_{\mathcal{V}}^2$ and $\epsilon > 0$. Although the vanishing viscosity analysis is meanwhile well understood and has been analyzed by various authors, e.g., [EM06, MRS09, MRS12, MRS16], we present the underlying in detail, since in our setting, the external loads are not fixed, but we consider a sequence of loads converging weakly in $H^1(0,T;\mathcal{V}^*)$, see Theorem 3.3.6. This has not been investigated in the literature so far, but is essential in our optimal control context in Chapter 4. Assuming that (3.0.1) is solvable, we introduce a suitable parametrization of the solution and want to drive $\epsilon \searrow 0$. It will turn out that the sequence of these parametrized functions is uniformly bounded in appropriate spaces such that we can extract a weakly converging subsequence. The limit will satisfy certain properties that will form the basis of the definition of parametrized balanced viscosity solutions. To this end, we have to verify the solvability of the viscous regularized system and have to derive the required a priori bounds. One common method for this is a time discretization approach. This has been investigated, for example, in [Col92] and [RMS08], or in a setting more closely related to ours in [KZ21] and [Tho22]. But we take a different approach and follow the idea used in [KRZ11, KT23], which is considering a further regularization of (3.0.1), for which the Picard-Lindelöf theorem ensures the existence and uniqueness of solutions. Using the assumptions about energy, we will derive a priori estimates for these solutions that are independent of the second regularization parameter. Hence, these estimates transfer to the limit by passing the second parameter to zero and the limit solves the once regualrized system. Furthermore, we infer additional properties of the solutions of the twice regularized system, which will be later useful in the context of approximability of differential solutions by means of solutions of a penalized viscous regularized system, cf. Section 4.2. This is the reason why we pursue the twice regularization approach instead of time discretization.

3.1 The twice viscous regularized system $(\mathrm{RIS})_{\epsilon,\delta}$

In this section we investigate the twice viscous regularized system, which reads as

$$0 \in \partial\mathcal{R}_{\epsilon,\delta}(z'(t)) + \mathrm{D}_z\mathcal{I}(\ell(t), z(t)) \quad \text{f.a.a. } t \in (0,T), \quad z(0) = z_0, \tag{$\mathrm{RIS}_{\epsilon,\delta}$}$$

where $\mathcal{R}_{\epsilon,\delta}(z) := \mathcal{R}(z) + \frac{\epsilon}{2}\|z\|_{\mathcal{V}}^2 + \frac{\delta}{2}|z|_{\mathcal{Z}}^2$, $z \in \mathcal{Z}$, $\epsilon, \delta > 0$ and $\ell \in H^1(0,T;\mathcal{V}^*)$. First of all, we deal with the question of whether $(\mathrm{RIS}_{\epsilon,\delta})$ can be solved at all. This will be part of the following theorem along with additional a priori bounds of the solution.

Theorem 3.1.1. *Let $\epsilon, \delta > 0$ and $z_0 \in \mathcal{Z}$ be given. Then there exists a unique solution $z_{\epsilon,\delta} \in H^2(0,T;\mathcal{Z})$ of $(\mathrm{RIS}_{\epsilon,\delta})$. Moreover, $z_{\epsilon,\delta}$ satisfies the following energy equality*

$$\begin{aligned}\mathcal{I}(\ell(t), z_{\epsilon,\delta}(t)) + \int_0^t \mathcal{R}_{\epsilon,\delta}(z'_{\epsilon,\delta}(r)) + \mathcal{R}^*_{\epsilon,\delta}(-\mathrm{D}_z\mathcal{I}(\ell(r), z_{\epsilon,\delta}(r)))\,\mathrm{d}r \\ = \mathcal{I}(\ell(0), z_{\epsilon,\delta}(0)) - \int_0^t \langle \ell'(r), z_{\epsilon,\delta}(r)\rangle_{\mathcal{V}^*,\mathcal{V}}\,\mathrm{d}r \quad \textit{for all } t \in [0,T],\end{aligned} \tag{3.1.1}$$

and fulfills the a priori estimates

$$\begin{aligned}\int_0^T \mathcal{R}_{\epsilon,\delta}(z'_{\epsilon,\delta}(r)) + \mathcal{R}^*_{\epsilon,\delta}(-\mathrm{D}_z\mathcal{I}(\ell(r), z_{\epsilon,\delta}(r)))\,\mathrm{d}r \\ \le c_0 + C\Big(\mathcal{I}(\ell(0), z_0) + \|\ell\|^2_{H^1(0,T,\mathcal{V}^*)}\Big)\end{aligned} \tag{3.1.2}$$

$$\sup_{t\in[0,T]} \|z_{\epsilon,\delta}(t)\|_{\mathcal{Z}} \le C, \tag{3.1.3}$$

$$\epsilon\|z_{\epsilon,\delta}\|^2_{H^1(0,T;\mathcal{V})} \le C, \tag{3.1.4}$$

$$\delta\int_0^T |z'_{\epsilon,\delta}(r)|^2_{\mathcal{Z}}\,\mathrm{d}r \le C, \tag{3.1.5}$$

where $C > 0$ is independent of ϵ and δ and c_0 is the constant from (1.2.11).

Proof. First, by using standard arguments from convex analysis and that $\partial\mathcal{R}^*_{\epsilon,\delta}$ is single-valued, see Lemma A.6, we rewrite $(\mathrm{RIS}_{\epsilon,\delta})$ as

$$z'(t) = \partial\mathcal{R}^*_{\epsilon,\delta}\big(-\mathrm{D}_z\mathcal{I}(\ell(t), z(t))\big) \quad \text{f.a.a. } t \in (0,T), \quad z(0) = z_0. \tag{3.1.6}$$

Instead of investigating (3.1.6), we consider a slightly modified version thereof and show later that this modification is superfluous. For $M > 0$, let $\tilde{\Pi}_M$ be a regularized version of the orthogonal projection Π_M on the closed ball $\overline{B_{\mathcal{Z}}(0, M)}$ with radius M, which satisfies

$$\|\tilde{\Pi}_M(z)\|_{\mathcal{Z}} \leq 2M \ \forall z \in \mathcal{Z}, \qquad \tilde{\Pi}_M(z) = \Pi_M(z) \ \forall z \in \overline{B_{\mathcal{Z}}(0, M)}, \tag{3.1.7}$$

$$\sup_{z\in\mathcal{Z}} \|(\mathrm{D}_z\tilde{\Pi}_M(z))^*\|_{\mathcal{L}(\mathcal{Z}^*,\mathcal{Z}^*)} \leq C_1, \tag{3.1.8}$$

$$\sup_{z\in\mathcal{Z}} \|\mathrm{D}_z((\mathrm{D}_z\tilde{\Pi}_M(z))^*)\|_{\mathcal{L}(\mathcal{Z},\mathcal{L}(\mathcal{Z}^*,\mathcal{Z}^*))} \leq C_2. \tag{3.1.9}$$

A construction of $\tilde{\Pi}_M$ is given in Lemma C.2. With given $\tilde{\Pi}_M$, we consider the following modified ODE

$$\zeta'(t) = \partial\mathcal{R}^*_{\epsilon,\delta}\Big(- \mathrm{D}_z\tilde{\Pi}_M(\zeta(t))^*\mathrm{D}_z\mathcal{I}\big(\ell(t), \tilde{\Pi}_M(\zeta(t))\big)\Big), \quad \zeta(0) = z_0. \tag{3.1.10}$$

The inner mapping $\zeta \mapsto -\mathrm{D}_z\tilde{\Pi}_M(\zeta)^*\mathrm{D}_z\mathcal{I}(\ell(t), \tilde{\Pi}_M(\zeta))$ is globally Lipschitz continuous with respect to ζ, which can be seen as follows:

$$\begin{aligned}
&\|\mathrm{D}_z\tilde{\Pi}_M(\zeta)^*\mathrm{D}_z\mathcal{I}(\ell(t), \tilde{\Pi}_M(\zeta)) - \mathrm{D}_z\tilde{\Pi}_M(\eta)^*\mathrm{D}_z\mathcal{I}(\ell(t), \tilde{\Pi}_M(\eta))\|_{\mathcal{Z}^*}\\
\leq\ & \|(\mathrm{D}_z\tilde{\Pi}_M(\zeta)^* - \mathrm{D}_z\tilde{\Pi}_M(\eta)^*)\mathrm{D}_z\mathcal{I}(\ell(t), \tilde{\Pi}_M(\zeta))\|_{\mathcal{Z}^*}\\
&\quad + \|\mathrm{D}_z\tilde{\Pi}_M(\eta)^*(\mathrm{D}_z\mathcal{I}(\ell(t), \tilde{\Pi}_M(\eta)) - \mathrm{D}_z\mathcal{I}(\ell(t), \tilde{\Pi}_M(\eta))\|_{\mathcal{Z}^*}\\
\leq\ & \|\mathrm{D}_z\tilde{\Pi}_M(\zeta)^* - \mathrm{D}_z\tilde{\Pi}_M(\eta)^*\|_{\mathcal{L}(\mathcal{Z}^*,\mathcal{Z}^*)}\|\mathrm{D}_z\mathcal{I}(\ell(t), \tilde{\Pi}_M(\zeta))\|_{\mathcal{Z}^*}\\
&\quad + \|\mathrm{D}_z\tilde{\Pi}_M(\eta)^*\|_{\mathcal{L}(\mathcal{Z}^*,\mathcal{Z}^*)}\|\mathrm{D}_z\mathcal{I}(\ell(t), \tilde{\Pi}_M(\eta)) - \mathrm{D}_z\mathcal{I}(\ell(t), \tilde{\Pi}_M(\eta)\|_{\mathcal{Z}^*}\\
\leq\ & C_2\|\zeta - \eta\|_{\mathcal{Z}}\|\mathrm{D}_z\mathcal{I}(\ell(t), \tilde{\Pi}_M(\zeta))\|_{\mathcal{Z}^*} + C_1L_{\mathrm{D}_z\mathcal{I}}\|\zeta - \eta\|_{\mathcal{Z}},
\end{aligned}$$

where we used that $\mathrm{D}_z\mathcal{I}(\ell(t), \cdot)$ is Lipschitz continuous on $\overline{B_{\mathcal{Z}}(0, M)}$ with constant $L_{\mathrm{D}_z\mathcal{I}} > 0$, which follows from the fact that $A \in \mathcal{L}(\mathcal{Z}, \mathcal{Z}^*)$ and (1.2.5). Moreover, from the continuity of $A \in \mathcal{L}(\mathcal{Z}, \mathcal{Z}^*)$ and (1.2.5) we infer that $\mathrm{D}_z\mathcal{I}(\ell(t), \cdot)$ is bounded on $\overline{B_{\mathcal{Z}}(0, M)}$. Therefore,

$$\zeta \mapsto \partial\mathcal{R}^*_{\epsilon,\delta}\Big(- \mathrm{D}_z\tilde{\Pi}_M(\zeta)^*\mathrm{D}_z\mathcal{I}(\ell(t), \tilde{\Pi}_M(\zeta))\Big), \tag{3.1.11}$$

is also globally Lipschitz continuous w.r.t. ζ, where we exploited the Lipschitz continuity of $\partial\mathcal{R}^*_{\epsilon,\delta}$ from Lemma A.6. Hence, by the Picard-Lindelöf theorem, it follows the existence of a unique solution $\zeta \in W^{1,\infty}(0, T; \mathcal{Z})$ of (3.1.10). In order to prove the boundedness of the solution, we reformulate (3.1.10) equivalently to

$$\mathcal{R}_{\epsilon,\delta}(\zeta'(t)) + \mathcal{R}^*_{\epsilon,\delta}(-\mathrm{D}_z\tilde{\Pi}_M(\zeta(t))^*\mathrm{D}_z\mathcal{I}(\ell(t), \tilde{\Pi}_M(\zeta(t)))) \tag{3.1.12}$$

$$= \langle -\mathrm{D}_z\mathcal{I}(\ell(t), \tilde{\Pi}_M(\zeta(t))), \mathrm{D}_z\tilde{\Pi}_M(\zeta(t))\zeta'(t)\rangle_{\mathcal{Z}^*,\mathcal{Z}} \tag{3.1.13}$$

$$= -\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{I}(\ell(t), \tilde{\Pi}_M(\zeta(t))) - \langle \ell'(t), \tilde{\Pi}_M(\zeta(t))\rangle_{\mathcal{V}^*,\mathcal{V}} \tag{3.1.14}$$

for almost all $t \in (0,T)$ by making use of the Fenchel-Young equality and the chain rule for Sobolev functions. Now integration over $(0,t)$ leads to

$$\mathcal{I}(\ell(t), \tilde{\Pi}_M(\zeta(t))) + \int_0^t \mathcal{R}_{\epsilon,\delta}(\zeta'(r)) + \mathcal{R}^*_{\epsilon,\delta}(-\mathrm{D}_z\tilde{\Pi}_M(\zeta(r))^*\mathrm{D}_z\mathcal{I}(\ell(r), \tilde{\Pi}_M(\zeta(r)))\,\mathrm{d}r \\ = \mathcal{I}(\ell(0), \tilde{\Pi}_M(\zeta(0))) - \int_0^t \langle \ell'(r), \tilde{\Pi}_M(\zeta(r))\rangle_{\mathcal{V}^*,\mathcal{V}}\,\mathrm{d}r. \tag{3.1.15}$$

With Young's inequality and (1.2.9) at hand, the integral on the right hand side can be estimated by

$$\begin{aligned} &\int_0^t \left|\langle \ell'(r), \tilde{\Pi}_M(\zeta(r))\rangle_{\mathcal{V}^*,\mathcal{V}}\right| \mathrm{d}r \\ &\le \frac{1}{2}\bigg(\|\ell\|^2_{H^1(0,t;\mathcal{V}^*)} + \int_0^t \|\tilde{\Pi}_M(\zeta(r))\|^2_{\mathcal{Z}}\,\mathrm{d}r\bigg) \\ &\le \frac{1}{2}\bigg(\|\ell\|^2_{H^1(0,t;\mathcal{V}^*)} + \int_0^t \lambda\mathcal{I}(\ell(r), \tilde{\Pi}_M(\zeta(r))) + \mu\|\ell(r)\|^2_{\mathcal{V}^*}\,\mathrm{d}r\bigg) \\ &\le \frac{1}{2}\bigg((1+\mu)\|\ell\|^2_{H^1(0,t;\mathcal{V}^*)} + \int_0^t \lambda\mathcal{I}(\ell(r), \tilde{\Pi}_M(\zeta(r)))\,\mathrm{d}r\bigg). \end{aligned} \tag{3.1.16}$$

By exploiting the non-negativity of $\mathcal{R}_{\epsilon,\delta}$ and $\mathcal{R}^*_{\epsilon,\delta}$, equation (3.1.15) gives

$$\mathcal{I}(\ell(t), \tilde{\Pi}_M(\zeta(t))) \\ \le \mathcal{I}(\ell(0), z_0) + \frac{1}{2}\bigg((1+\mu)\|\ell\|^2_{H^1(0,t;\mathcal{V}^*)} + \int_0^t \lambda\mathcal{I}(\ell(r), \tilde{\Pi}_M(\zeta(r)))\,\mathrm{d}r\bigg) \tag{3.1.17}$$

such that applying Gronwall's inequality results in

$$\mathcal{I}(\ell(t), \tilde{\Pi}_M(\zeta(t))) \le e^{\frac{1}{2}\lambda T}\bigg(\mathcal{I}(\ell(0), z_0) + \frac{1+\mu}{2}\|\ell\|^2_{H^1(0,T;\mathcal{V}^*)}\bigg). \tag{3.1.18}$$

Eventually, we apply estimate (1.2.11) and end up with

$$\|\tilde{\Pi}_M(\zeta(t))\|_{\mathcal{Z}} \le e^{\frac{1}{2}\lambda T}\bigg(\mathcal{I}(\ell(0), z_0) + \frac{1+\mu}{2}\|\ell\|^2_{H^1(0,T;\mathcal{V}^*)}\bigg) + c_0 =: C. \tag{3.1.19}$$

Since the right hand side is independent of M, the solution of (3.1.10) is bounded by C for all $M > 0$. Therefore, if we choose $M > C$, the modification with the projection $\tilde{\Pi}_M$ becomes superfluous and the solution $\zeta \in W^{1,\infty}(0,T;\mathcal{Z})$ of (3.1.10) solves (3.1.6). Moreover, (3.1.19) directly gives the claimed estimate (3.1.3) and (3.1.1) follows from (3.1.15). After inserting (3.1.18) in (3.1.16) and exploiting (1.2.11), the energy equality (3.1.15) results in (3.1.2). Eventually, from (3.1.2), it follows by exploiting the non-negativity of $\mathcal{R}$, $|\cdot|_{\mathcal{Z}}$ and $\mathcal{R}^*_{\epsilon,\delta}$ that

$$\frac{\epsilon}{2}\int_0^T \|z'_{\epsilon,\delta}(r)\|^2_{\mathcal{V}}\,\mathrm{d}r \le c_0 + C\Big(\mathcal{I}(\ell(0), z_0) + \|\ell\|^2_{H^1(0,T,\mathcal{V}^*)}\Big),$$

which yields together with (3.1.3) the estimate (3.1.4). In the same way we deduce (3.1.5). Finally, we verify the improved regularity of the solution z of (3.1.6), which is $z \in H^2(0,T;\mathcal{Z})$. In accordance with the assumption regarding the energy $\mathcal{I}$, the fact that $z \in W^{1,\infty}(0,T;\mathcal{Z})$ and $\ell \in H^1(0,T;\mathcal{V}^*)$, the inner function of the right hand side in (3.1.6) is absolutely continuous. Therefore, the composition with the Lipschitz continuous function $\partial\mathcal{R}^*_{\epsilon,\delta} : \mathcal{Z}^* \to \mathcal{Z}$, cf. Lemma A.6, is absolutely continuous, too, and thus $z' \in W^{1,1}(0,T;\mathcal{Z})$. Hence, the second derivative exists almost everywhere and we obtain by exploiting the identity in (3.1.6) for almost all $t \in (0,T)$ that

$$
\begin{aligned}
\|z''(t)\|_{\mathcal{Z}} &= \lim_{h\searrow 0} \frac{\|z'(t+h)-z'(t)\|_{\mathcal{Z}}}{h} \\
&= \lim_{h\searrow 0} \frac{\big\|\partial\mathcal{R}^*_{\epsilon,\delta}\big(-\mathrm{D}_z\mathcal{I}(\ell(t+h),z(t+h))\big) - \partial\mathcal{R}^*_{\epsilon,\delta}\big(-\mathrm{D}_z\mathcal{I}(\ell(t),z(t))\big)\big\|_{\mathcal{Z}}}{h} \\
&\le \lim_{h\searrow 0} L\frac{\big\|-\mathrm{D}_z\mathcal{I}(\ell(t+h),z(t+h)) - \big(-\mathrm{D}_z\mathcal{I}(\ell(t),z(t))\big)\big\|_{\mathcal{Z}^*}}{h} \\
&= L\Big\|-\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{D}_z\mathcal{I}(\ell(t),z(t))\Big\|_{\mathcal{Z}^*}
\end{aligned}
$$

such that z'' is pointwisely bounded by a L^2-integrable function, which gives $z \in H^2(0,T;\mathcal{Z})$ as claimed. □

Remark 3.1.2. Note that the estimates (3.1.2)–(3.1.5) hold true for all external loads $\ell \in H^1(0,T;\mathcal{V}^*)$ satisfying (1.2.10) such that the constant $C = e^{\frac{1}{2}\lambda T}\big(\mathcal{I}(\ell(0),z_0) + \frac{1+\mu}{2}\|\ell\|^2_{H^1(0,T;\mathcal{V}^*)}\big) + c_0$ is not only independent of $\epsilon, \delta > 0$ but also independent of the loads we consider, i.e., only loads that comply with (1.2.10).

Assuming that the initial data complies with (1.2.25), which is $-\mathrm{D}_z\mathcal{I}(\ell(0),z_0) \in \partial\mathcal{R}(0)$, we deduce additional properties concerning the initial state of the solution of $(\mathrm{RIS}_{\epsilon,\delta})$.

Lemma 3.1.3. *Let the initial value satisfying* (1.2.25)*. Then the solution $z_{\epsilon,\delta}$ of* $(\mathrm{RIS}_{\epsilon,\delta})$ *fulfills*

$$
z'_{\epsilon,\delta}(0) = 0. \tag{3.1.20}
$$

Proof. Due to (1.2.25) and (1.2.18), we have

$$
\mathcal{R}(v) \ge \langle -\mathrm{D}_z\mathcal{I}(\ell(0),z_0), v\rangle_{\mathcal{Z}^*,\mathcal{Z}}
$$

such that the definition of the conjugate functional gives

$$
\begin{aligned}
\mathcal{R}^*_{\epsilon,\delta}(-\mathrm{D}_z\mathcal{I}(\ell(0),z_0)) &= \sup_{v\in\mathcal{Z}}\Big(\langle -\mathrm{D}_z\mathcal{I}(\ell(0),z_0), v\rangle_{\mathcal{Z}^*,\mathcal{Z}} - \mathcal{R}_{\epsilon,\delta}(v)\Big) \\
&\le \sup_{v\in\mathcal{Z}}\Big(\langle -\mathrm{D}_z\mathcal{I}(\ell(0),z_0), v\rangle_{\mathcal{Z}^*,\mathcal{Z}} - \mathcal{R}(v)\Big) \le 0.
\end{aligned}
$$

By exploiting the non-negativity of $\mathcal{R}^*_{\epsilon,\delta}$, we deduce $\mathcal{R}^*_{\epsilon,\delta}(-\mathrm{D}_z\mathcal{I}(\ell(0),z_0)) = 0$, which results in

$$\mathcal{R}^*_{\epsilon,\delta}(v) \geq \mathcal{R}^*_{\epsilon,\delta}(-\mathrm{D}_z\mathcal{I}(\ell(0),z_0)) + \langle 0, v - (-\mathrm{D}_z\mathcal{I}(\ell(0),z_0))\rangle_{\mathcal{Z}^*,\mathcal{Z}}$$

such that $0 \in \partial\mathcal{R}^*_{\epsilon,\delta}(-\mathrm{D}_z\mathcal{I}(\ell(0),z_0))$. Since $z'_{\epsilon,\delta}$ is continuous by the embedding $H^1(0,T;\mathcal{Z}) \hookrightarrow C([0,T];\mathcal{Z})$, (3.1.6) holds for all $t \in [0,T]$. In particular, $z'_{\epsilon,\delta}(0) = \partial\mathcal{R}^*_{\epsilon,\delta}(-\mathrm{D}_z\mathcal{I}(\ell(0),z_0)) = 0$ because $\partial\mathcal{R}^*_{\epsilon,\delta}$ is single-valued, cf. Lemma A.6. $\square$

The next lemma is concerned with an important estimate that will be useful later in order to verify the approximability of optimal control problems governed by rate-independent systems by means of simpler regularized problems.

Lemma 3.1.4. *For given $\epsilon,\delta > 0$ and $z_0 \in \mathcal{Z}$, the solution $z_{\epsilon,\delta}$ of* (RIS$_{\epsilon,\delta}$) *satisfies*

$$\begin{aligned}\epsilon\langle z''_{\epsilon,\delta}(t), z'_{\epsilon,\delta}(t)\rangle_{\mathcal{V}^*,\mathcal{V}} + \delta\langle Az''_{\epsilon,\delta}(t), z'_{\epsilon,\delta}(t)\rangle_{\mathcal{Z}^*,\mathcal{Z}}& \\ + \left\langle \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{D}_z\mathcal{I}(\ell(t), z_{\epsilon,\delta}(t)), z'_{\epsilon,\delta}(t)\right\rangle_{\mathcal{Z}^*,\mathcal{Z}} &= 0.\end{aligned} \tag{3.1.21}$$

for almost all $t \in (0,T)$. Furthermore, we have for all $t \in [0,T]$ that

$$\frac{\epsilon}{2}\|z'_{\epsilon,\delta}(t)\|^2_{\mathcal{V}} + \int_0^t \mathrm{D}^2_z\mathcal{E}(z_{\epsilon,\delta}(r))[z'_{\epsilon,\delta}(r), z'_{\epsilon,\delta}(r)]\,\mathrm{d}r \leq \int_0^t \langle \ell'(r), z'_{\epsilon,\delta}(r)\rangle_{\mathcal{V}^*,\mathcal{V}}\,\mathrm{d}r. \tag{3.1.22}$$

Proof. We start with proving (3.1.21). Therefore, according to Theorem 3.1.1, let $z_{\epsilon,\delta} \in H^2(0,T;\mathcal{Z})$ be the solution of (RIS$_{\epsilon,\delta}$). We define $h(t) := \epsilon z'_{\epsilon,\delta}(t) + \delta Az'_{\epsilon,\delta}(t) + \mathrm{D}_z\mathcal{I}(\ell(t), z_{\epsilon,\delta}(t)) \in H^1(0,T;\mathcal{Z}^*)$ such that (3.1.21) can be written as $\langle h'(t), z'_{\epsilon,\delta}(t)\rangle_{\mathcal{Z}^*,\mathcal{Z}} = 0$ for almost all $t \in (0,T)$. Then by exploiting the composition of $\mathcal{R}_{\epsilon,\delta}$, we obtain from (RIS$_{\epsilon,\delta}$)

$$-h(t) \in \partial\mathcal{R}(z'_{\epsilon,\delta}(t)) \text{ for all } t \in [0,T],$$

which yields

$$\begin{aligned}\langle -h(t), z'_{\epsilon,\delta}(t)\rangle_{\mathcal{Z}^*,\mathcal{Z}} &= \mathcal{R}(z'_{\epsilon,\delta}(t)) \\ \langle -h(s), z'_{\epsilon,\delta}(t)\rangle_{\mathcal{Z}^*,\mathcal{Z}} &\leq \mathcal{R}(z'_{\epsilon,\delta}(t))\end{aligned} \tag{3.1.23}$$

for all $t, s \in [0,T]$. Subtracting this inequality from the upper equation results in

$$\langle h(s) - h(t), z'_{\epsilon,\delta}(t)\rangle_{\mathcal{Z}^*,\mathcal{Z}} \geq 0. \tag{3.1.24}$$

Now let $t \in (0,S)$ be a differentiability point of h (almost all $t \in (0,T)$ are differentiability points since $h \in H^1(0,T;\mathcal{Z}^*)$) and $s := t \pm \tau$ with $\tau > 0$, then dividing (3.1.24) by τ gives

$$\frac{1}{\tau}\langle h(t \pm \tau) - h(t), z'_{\epsilon,\delta}(t)\rangle_{\mathcal{Z}^*,\mathcal{Z}} \geq 0.$$

By passing $\tau \searrow 0$, we obtain $\langle h'(t), z'_{\epsilon,\delta}(t)\rangle_{\mathcal{Z}^*,\mathcal{Z}} = 0$, which is (3.1.21). In the next step we verify (3.1.22). Integrating (3.1.21) over $(0,t)$ yields

$$
\begin{aligned}
0 &= \int_0^t \frac{\epsilon}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|z'_{\epsilon,\delta}(r)\|_{\mathcal{V}}^2 + \frac{\delta}{2}\frac{\mathrm{d}}{\mathrm{d}t}|z'_{\epsilon,\delta}(r)|_{\mathcal{Z}}^2 + \langle\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{D}_z\mathcal{I}(\ell(r), z_{\epsilon,\delta}(r)), z'_{\epsilon,\delta}(r)\rangle_{\mathcal{Z}^*,\mathcal{Z}}\,\mathrm{d}r \\
&= \frac{\epsilon}{2}\|z'_{\epsilon,\delta}(t)\|_{\mathcal{V}}^2 - \frac{\epsilon}{2}\|z'_{\epsilon,\delta}(0)\|_{\mathcal{V}}^2 + \frac{\delta}{2}|z'_{\epsilon,\delta}(t)|_{\mathcal{Z}}^2 - \frac{\delta}{2}|z'_{\epsilon,\delta}(0)|_{\mathcal{Z}}^2 \\
&\qquad + \int_0^t \mathrm{D}_z^2\mathcal{E}(z_{\epsilon,\delta}(r))[z'_{\epsilon,\delta}(r), z'_{\epsilon,\delta}(r)] - \langle\ell'(r), z'_{\epsilon,\delta}(r)\rangle_{\mathcal{V}^*,\mathcal{V}}\,\mathrm{d}r.
\end{aligned}
$$

Eventually, after exploiting (3.1.20) and the non-negativity of $|\cdot|_{\mathcal{Z}}$, we end up with (3.1.22). □

With this lemma at hand, we can derive the following improved a priori estimates concerned with the solutions $(z_{\epsilon,\delta})_{\delta>0}$ of $(\mathrm{RIS}_{\epsilon,\delta})$.

Theorem 3.1.5. *Let $\epsilon >$ be given. Then for all $\delta > 0$, the solutions $z_{\epsilon,\delta}$ of* $(\mathrm{RIS}_{\epsilon,\delta})$ *satisfy*

$$\epsilon\|z_{\epsilon,\delta}\|_{H^1(0,T;\mathcal{Z})}^2 \leq C \tag{3.1.25}$$

$$\epsilon\|z_{\epsilon,\delta}\|_{W^{1,\infty}(0,T;\mathcal{V})} \leq C \tag{3.1.26}$$

with $C > 0$ independent of ϵ, δ.

Proof. By exploiting (1.2.3), the coercivity of A, and the non-negativity of the norm, we infer from (3.1.22) that

$$
\begin{aligned}
\alpha\int_0^T \|z'_{\epsilon,\delta}(r)\|_{\mathcal{Z}}^2\,\mathrm{d}r &\leq \int_0^T \langle Az'_{\epsilon,\delta}(r), z'_{\epsilon,\delta}(r)\rangle_{\mathcal{Z}^*,\mathcal{Z}}\,\mathrm{d}r \\
&\leq -\int_0^T \mathrm{D}_z^2\mathcal{F}(z_{\epsilon,\delta}(r))[z'_{\epsilon,\delta}(r), z'_{\epsilon,\delta}(r)]\,\mathrm{d}r + \int_0^T \langle\ell'(r), z'_{\epsilon,\delta}(r)\rangle_{\mathcal{V}^*,\mathcal{V}}\,\mathrm{d}r
\end{aligned}
$$

Next, by means of (1.2.5) and (3.1.3), the first integral on the right hand side can be estimated by

$$
\begin{aligned}
&-\int_0^T \mathrm{D}_z^2\mathcal{F}(z_{\epsilon,\delta}(r))[z'_{\epsilon,\delta}(r), z'_{\epsilon,\delta}(r)]\,\mathrm{d}r \\
&\leq \int_0^T C(1 + \|z_{\epsilon,\delta}(r)\|_{\mathcal{Z}}^q)\|z'_{\epsilon,\delta}(r)\|_{\mathcal{Z}}\|z'_{\epsilon,\delta}(r)\|_{\mathcal{V}}\,\mathrm{d}r \\
&\leq \int_0^T \tilde{C}\Big(\rho\|z'_{\epsilon,\delta}(r)\|_{\mathcal{Z}}^2 + \frac{1}{4\rho}\|z'_{\epsilon,\delta}(r)\|_{\mathcal{V}}^2\Big)\,\mathrm{d}r,
\end{aligned}
$$

where we used Young's inequality in the last step with $\rho > 0$ arbitrary. Thus, if we choose $\rho = \frac{\alpha}{2\tilde{C}}$, we obtain

$$\frac{\alpha}{2}\int_0^T \|z'_{\epsilon,\delta}(r)\|_{\mathcal{Z}}^2\,\mathrm{d}r \leq \int_0^T \frac{\tilde{C}^2}{2\alpha}\|z'_{\epsilon,\delta}(r)\|_{\mathcal{V}}^2\,\mathrm{d}r + \int_0^T \langle\ell'(r), z'_{\epsilon,\delta}(r)\rangle_{\mathcal{V}^*,\mathcal{V}}\,\mathrm{d}r$$

$$\leq \frac{\tilde{C}^2}{2\alpha}\|z'_{\epsilon,\delta}\|^2_{L^2(0,T;\mathcal{V})} + \|\ell'\|_{L^2(0,T;\mathcal{V}^*)}\|z'_{\epsilon,\delta}\|_{L^2(0,T;\mathcal{V})},$$

where we applied Hölder's inequality in the last step. Finally, dividing by $\frac{\alpha}{2}$ and exploiting the $H^1(0,T;\mathcal{V})$-bound from (3.1.4) yields

$$\begin{aligned}\|z'_{\epsilon,\delta}\|^2_{L^2(0,T;\mathcal{Z})} &\leq C(\|z'_{\epsilon,\delta}\|^2_{L^2(0,T;\mathcal{V})} + \|z'_{\epsilon,\delta}\|_{L^2(0,T;\mathcal{V})}) \\ &\leq \tilde{C}\Big(\frac{1}{\epsilon} + \frac{1}{\sqrt{\epsilon}}\Big).\end{aligned}$$

Combining this with the uniform $L^\infty(0,T;\mathcal{Z})$-bound from (3.1.3) results in

$$\epsilon\|z_{\epsilon,\delta}\|^2_{H^1(0,T;\mathcal{Z})} \leq C(1+\sqrt{\epsilon}+\epsilon),$$

which is (3.1.25) since we are only interested in the case $\epsilon \searrow 0$, such that ϵ is bounded from above. In the next step we prove (3.1.26). For this purpose, let $t \in [0,T]$ be arbitrary. By integrating (3.1.21) over $(0,t)$, we deduce as in the proof of Lemma 3.1.4 that

$$\begin{aligned}0 &= \int_0^t \frac{\epsilon}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|z'_{\epsilon,\delta}(r)\|^2_{\mathcal{V}} + \frac{\delta}{2}\frac{\mathrm{d}}{\mathrm{d}t}|z'_{\epsilon,\delta}(r)|^2_{\mathcal{Z}} + \langle\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{D}_z\mathcal{I}(\ell(r),z_{\epsilon,\delta}(r)), z'_{\epsilon,\delta}(r)\rangle_{\mathcal{Z}^*,\mathcal{Z}}\,\mathrm{d}r \\ &= \frac{\epsilon}{2}\|z'_{\epsilon,\delta}(t)\|^2_{\mathcal{V}} - \frac{\epsilon}{2}\|z'_{\epsilon,\delta}(0)\|^2_{\mathcal{V}} + \frac{\delta}{2}|z'_{\epsilon,\delta}(t)|^2_{\mathcal{Z}} - \frac{\delta}{2}|z'_{\epsilon,\delta}(0)|^2_{\mathcal{Z}} \\ &\qquad + \int_0^t \mathrm{D}^2_z\mathcal{E}(z_{\epsilon,\delta}(r))[z'_{\epsilon,\delta}(r), z'_{\epsilon,\delta}(r)] - \langle\ell'(r), z'_{\epsilon,\delta}(r)\rangle_{\mathcal{V}^*,\mathcal{V}}\,\mathrm{d}r.\end{aligned}$$

Exploiting the coercivity of A, the non-negativity of the $|\cdot|_{\mathcal{Z}}$-norm and (3.1.20) gives

$$\begin{aligned}&\frac{\epsilon}{2}\|z'_{\epsilon,\delta}(t)\|^2_{\mathcal{V}} + \alpha\int_0^t \|z'_{\epsilon,\delta}(r)\|^2_{\mathcal{Z}}\,\mathrm{d}r \\ &\leq \int_0^t \Big|\mathrm{D}^2_z\mathcal{F}(z_{\epsilon,\delta}(r))[z'_{\epsilon,\delta}(r), z'_{\epsilon,\delta}(r)]\Big|\,\mathrm{d}r + \int_0^t \langle\ell'(r), z'_{\epsilon,\delta}(r)\rangle_{\mathcal{V}^*,\mathcal{V}}\,\mathrm{d}r \\ &\leq \int_0^t C(1+\|z_{\epsilon,\delta}(r)\|^q_{\mathcal{Z}})\|z'_{\epsilon,\delta}(r)\|_{\mathcal{Z}}\|z'_{\epsilon,\delta}(r)\|_{\mathcal{V}}\,\mathrm{d}r + \int_0^t \frac{1}{2}\Big(\|\ell'(r)\|^2_{\mathcal{V}^*} + \|z'_{\epsilon,\delta}(r)\|^2_{\mathcal{V}}\Big)\,\mathrm{d}r \\ &\leq \int_0^t \tilde{C}\Big(\rho\|z'_{\epsilon,\delta}(r)\|^2_{\mathcal{Z}} + \frac{1}{4\rho}\|z'_{\epsilon,\delta}(r)\|^2_{\mathcal{V}}\Big)\,\mathrm{d}r + \frac{1}{2}\|\ell'\|^2_{L^2(0,T;\mathcal{V}^*)} + \frac{1}{2}\|z'_{\epsilon,\delta}\|^2_{L^2(0,T;\mathcal{V})},\end{aligned}$$

where we used (1.2.5) and Young's inequality in the second last step and (3.1.3) together with Young's inequality (here with arbitrary $\rho > 0$) in the last step. By choosing $\rho = \frac{\alpha}{\tilde{C}}$, we infer

$$\epsilon\|z'_{\epsilon,\delta}(t)\|^2_{\mathcal{V}} \leq C\Big(1 + \|z'_{\epsilon,\delta}\|^2_{L^2(0,T;\mathcal{V})}\Big)$$

such that exploiting (3.1.4) yields

$$\epsilon\|z'_{\epsilon,\delta}(t)\|_{\mathcal{V}} \leq C\big(1+\sqrt{\epsilon}\big).$$

Since this holds true for all $t \in [0,T]$ and we are only interested in $\epsilon \searrow 0$, this in combination with (3.1.3) gives (3.1.26). □

Beside these a priori estimates that depend on ϵ, we can also derive estimates independent of ϵ, which will be a uniform $W^{1,1}(0,T;\mathcal{Z})$-bound. To this end, we first have the following lemma:

Lemma 3.1.6. *For all $\rho > 0$ there exists a constant $C_\rho > 0$ not depending on ϵ, δ such that*

$$\left|\langle \mathrm{D}_z^2\mathcal{F}(z_{\epsilon,\delta}(t))z'_{\epsilon,\delta}(t), z'_{\epsilon,\delta}(t)\rangle_{\mathcal{V}^*,\mathcal{V}}\right| \le \rho\|z'_{\epsilon,\delta}(t)\|_{\mathcal{Z}}^2 + C_\rho\mathcal{R}(z'_{\epsilon,\delta}(t))\|z'_{\epsilon,\delta}(t)\|_{\mathcal{V}} \tag{3.1.27}$$

for all $t \in [0,T]$.

Proof. Let $t \in [0,T]$ be arbitrarily given. We start with exploiting assumption (1.2.5) and the uniform boundedness from (3.1.3) in order to obtain

$$\begin{aligned}\left|\langle \mathrm{D}_z^2\mathcal{F}(z_{\epsilon,\delta}(t))z'_{\epsilon,\delta}(t), z'_{\epsilon,\delta}(t)\rangle_{\mathcal{V}^*,\mathcal{V}}\right| &\le C(1+\|z_{\epsilon,\delta}(t)\|_{\mathcal{Z}}^q)\|z'_{\epsilon,\delta}(t)\|_{\mathcal{Z}}\|z'_{\epsilon,\delta}(t)\|_{\mathcal{V}}\\ &\le \tilde{C}\|z'_{\epsilon,\delta}(t)\|_{\mathcal{Z}}\|z'_{\epsilon,\delta}(t)\|_{\mathcal{V}}\end{aligned}$$

Next, we first apply Ehrling's lemma and after that Young's inequality in order to obtain for all $\rho_1, \rho_2 > 0$ that

$$\begin{aligned}&\left|\langle \mathrm{D}_z^2\mathcal{F}(z_{\epsilon,\delta}(t))z'_{\epsilon,\delta}(t), z'_{\epsilon,\delta}(t)\rangle_{\mathcal{V}^*,\mathcal{V}}\right|\\ &\le \rho_1\|z'_{\epsilon,\delta}(t)\|_{\mathcal{Z}}^2 + C_{\rho_1}\|z'_{\epsilon,\delta}(t)\|_{\mathcal{Z}}\|z'_{\epsilon,\delta}(t)\|_{\mathcal{X}}\\ &\le (\rho_1+\rho_2)\|z'_{\epsilon,\delta}(t)\|_{\mathcal{Z}}^2 + C_{\rho_1,\rho_2}\|z'_{\epsilon,\delta}(t)\|_{\mathcal{X}}^2\\ &\le (\rho_1+\rho_2)\|z'_{\epsilon,\delta}(t)\|_{\mathcal{Z}}^2 + CC_{\rho_1,\rho_2}\mathcal{R}(z'_{\epsilon,\delta}(t))\|z'_{\epsilon,\delta}(t)\|_{\mathcal{V}},\end{aligned}$$

where we exploited the compactness of the embedding $\mathcal{Z} \hookrightarrow^c \mathcal{V}$ and the lower bound of (1.2.15) in the last step. Because this holds for all $\rho_1, \rho_2 > 0$, we end up with (3.1.27). □

From now on, we assume that $0 < \delta < \epsilon$, which is no restriction because we first pass δ to zero and after that we pass ϵ to zero. For the sake of simplicity, we introduce for given $0 < \delta < \epsilon$ the ϵ, δ-scalar product, which is defined by

$$\langle z_1, z_2\rangle_{\epsilon,\delta} := \langle z_1, z_2\rangle_{\mathcal{V}^*,\mathcal{V}} + \frac{\delta}{\epsilon}\langle Az_1, z_2\rangle_{\mathcal{Z}^*,\mathcal{Z}}, \quad z_1, z_2 \in \mathcal{Z} \tag{3.1.28}$$

and its the associated norm is

$$\|z\|_{\epsilon,\delta} := \Big(\langle z, z\rangle_{\epsilon,\delta}\Big)^{\frac{1}{2}} = \left(\|z\|_{\mathcal{V}}^2 + \frac{\delta}{\epsilon}|z|_{\mathcal{Z}}^2\right)^{\frac{1}{2}},\ z \in \mathcal{Z}. \tag{3.1.29}$$

One directly sees that

$$\|z\|_{\mathcal{V}} \le \|z\|_{\epsilon,\delta} \le \|z\|_{\mathcal{Z}} \tag{3.1.30}$$

for all $z \in \mathcal{Z}$.

Now that we have defined the ϵ, δ-norm, we show the following differentiability result.

Lemma 3.1.7. *Let $v \in H^1(0,T;\mathcal{Z})$ be given. Then the mapping $f : [0,T] \to \mathbb{R}$; $t \mapsto \|v(t)\|_{\epsilon,\delta}$ is differentiable for almost all $t \in (0,T)$ and almost everywhere in $\omega := \{t \in [0,T] : \|v(t)\|_{\epsilon,\delta} > 0\}$ the derivative is*

$$\frac{\mathrm{d}}{\mathrm{d}t} f(t) = \frac{\mathrm{d}}{\mathrm{d}t}\|v(t)\|_{\epsilon,\delta} = \left\langle v'(t), \frac{v(t)}{\|v(t)\|_{\epsilon,\delta}} \right\rangle_{\epsilon,\delta} \tag{3.1.31}$$

Moreover, it holds

$$\int_\omega \frac{\mathrm{d}}{\mathrm{d}t}\|v(t)\|_{\epsilon,\delta}\,\mathrm{d}t = \|v(T)\|_{\epsilon,\delta} - \|v(0)\|_{\epsilon,\delta}. \tag{3.1.32}$$

Proof. For all $s,t \in [0,T]$ we have by using (3.1.30)

$$|f(t) - f(s)| = \big|\|v(t)\|_{\epsilon,\delta} - \|v(s)\|_{\epsilon,\delta}\big| \leq \|v(t) - v(s)\|_{\epsilon,\delta} \leq \|v(t) - v(s)\|_{\mathcal{Z}}$$

such that the absolute continuity of v is transferred to f. Therefore, f is differentiable almost everywhere. Next, we verify that the derivative is given by (3.1.31). To this end, let $t \in \omega$ be a point, where z is differentiable, and let $0 \neq h \in \mathbb{R}$ be given. Then we have

$$\begin{aligned}
&\left| \frac{f(t) - f(t+h)}{h} - \left\langle z'(t), \frac{z(t)}{\|z(t)\|_{\epsilon,\delta}} \right\rangle_{\epsilon,\delta} \right| \\
&= \left| \frac{\|z(t)\|_{\epsilon,\delta} - \|z(t+h)\|_{\epsilon,\delta}}{h} - \left\langle z'(t), \frac{z(t)}{\|z(t)\|_{\epsilon,\delta}} \right\rangle_{\epsilon,\delta} \right| \\
&\leq \left| \frac{\|z(t)\|_{\epsilon,\delta} - \|z(t) + hz'(t)\|_{\epsilon,\delta}}{h} - \left\langle z'(t), \frac{z(t)}{\|z(t)\|_{\epsilon,\delta}} \right\rangle_{\epsilon,\delta} \right| \\
&\qquad\qquad + \left| \frac{\|z(t+h)\|_{\epsilon,\delta} - \|z(t) + hz'(t)\|_{\epsilon,\delta}}{h} \right| \\
&\to 0 \text{ as } h \to 0.
\end{aligned}$$

The first term tends to zero because the norm is differentiable outside of zero with derivative $\langle \frac{\mathrm{d}}{\mathrm{d}v}\|v\|_{\epsilon,\delta}, \eta\rangle_{\epsilon,\delta} = \langle \frac{v}{\|v\|_{\epsilon,\delta}}, \eta\rangle_{\epsilon,\delta}$ for $\eta \in \mathcal{Z}$ and $0 \neq v \in \mathcal{Z}$. The second term vanishes since z is differentiable in t. In order to prove (3.1.32), we assume that $[0,T] \setminus \omega$ has positive Lebesgue measure. Otherwise, the claim follows directly from the fundamental theorem of calculus for absolutely continuous functions. The idea is now to split ω in intervals such that the fundamental theorem of calculus is also applicable. Therefore, we define

$$t_1 := \inf\{t \in [0,T] : f(t) = 0\}, \quad t_2 := \sup\{t \in [0,T] : f(t) = 0\}$$

and the assumption ensures that $t_1, t_2 \in [0,T]$. Moreover, from the continuity of f we deduce $0 = f(t_1) = f(t_2)$ and that the set

$$S := \omega \cap \Big([0,T] \setminus \big([0,t_1] \cup [t_2,T]\big)\Big)$$

is open. By means of the representation theorem for open sets [Ber98, Lem. 5.7.1], S can be uniquely written as the union of countable many disjoint open intervals, i.e., $S = \bigcup_{k\in\mathbb{N}}(s_{2k-1}, s_{2k})$ with $t_1 \le s_1 < s_2 < ... \le t_2$. We directly obtain $f(s_k) = 0$ for all $k \in \mathbb{N}$ because otherwise $s_k \in S$, which contradicts the disjointness of the intervals. Eventually, from the fundamental theorem of calculus for absolutely continuous functions we infer

$$\begin{aligned}\int_\omega \frac{\mathrm{d}}{\mathrm{d}t} f(t)\mathrm{d}t &= \int_0^{t_1} \frac{\mathrm{d}}{\mathrm{d}t} f(t)\mathrm{d}t + \int_{t_2}^{T} \frac{\mathrm{d}}{\mathrm{d}t} f(t)\mathrm{d}t + \int_S \frac{\mathrm{d}}{\mathrm{d}t} f(t)\mathrm{d}t \\ &= \int_0^{t_1} \frac{\mathrm{d}}{\mathrm{d}t} f(t)\mathrm{d}t + \int_{t_2}^{T} \frac{\mathrm{d}}{\mathrm{d}t} f(t)\mathrm{d}t + \sum_{k\in\mathbb{N}} \int_{s_{2k-1}}^{s_{2k}} \frac{\mathrm{d}}{\mathrm{d}t} f(t)\mathrm{d}t \\ &= f(t_1) - f(0) + f(T) - f(t_2) + \sum_{k\in\mathbb{N}} f(s_{2k}) - f(s_{2k-1}) \\ &= f(T) - f(0).\end{aligned}$$

Note that this equation is also valid for $t_1 = 0$, $t_2 = T$ or $S = \emptyset$. All in all, this proves (3.1.32). $\square$

With this at hand, we are now able to derive a uniform $W^{1,1}(0,T;\mathcal{Z})$-bound for $z_{\epsilon,\delta}$ that does not depend on ϵ.

Theorem 3.1.8. *Let $0 < \delta \le \epsilon$ be given. Then there exists a constant $C > 0$ independent of ϵ, δ such that the solution $z_{\epsilon,\delta} \in H^2(0,T;\mathcal{Z})$ of* $(\mathrm{RIS}_{\epsilon,\delta})$ *fulfills*

$$\|z_{\epsilon,\delta}\|_{W^{1,1}(0,T;\mathcal{Z})} \le C. \tag{3.1.33}$$

Proof. We start with proving $\int_0^T \|z'_{\epsilon,\delta}(r)\|_{\mathcal{Z}}\,\mathrm{d}r \le C$. Therefore, it is sufficient to consider only the points $t \in [0,T]$, where the derivative does not vanish, i.e., $\omega = \{t \in [0,T] : \|z'_{\epsilon,\delta}(t)\|_{\epsilon,\delta} \neq 0\}$, since the others have no influence on the integral. In view of (3.1.21), we obtain by rearranging the terms and using the notation from (3.1.28) that

$$\begin{aligned}\langle A z'_{\epsilon,\delta}(t), z'_{\epsilon,\delta}(t)\rangle_{\mathcal{Z}^*,\mathcal{Z}} = -\epsilon\langle z''_{\epsilon,\delta}(t), z'_{\epsilon,\delta}(t)\rangle_{\epsilon,\delta} - \langle \mathrm{D}_z^2\mathcal{F}(z_{\epsilon,\delta}(t))z'_{\epsilon,\delta}(t), z'_{\epsilon,\delta}(t)\rangle_{\mathcal{V}^*,\mathcal{V}} \\ + \langle \ell'(t), z'_{\epsilon,\delta}(t)\rangle_{\mathcal{V}^*,\mathcal{V}}.\end{aligned}$$

Exploiting the assumed coercivity of A from (1.2.3) and estimate (3.1.27) with $\rho = \frac{\alpha}{2}$ yields

$$\frac{\alpha}{2}\|z'_{\epsilon,\delta}(t)\|^2_{\mathcal{Z}} \le -\epsilon\langle z''_{\epsilon,\delta}(t), z'_{\epsilon,\delta}(t)\rangle_{\epsilon,\delta} + C_\rho\mathcal{R}(z'_{\epsilon,\delta}(t))\|z'_{\epsilon,\delta}(t)\|_{\mathcal{V}} + \|\ell'(t)\|_{\mathcal{V}^*}\|z'_{\epsilon,\delta}(t)\|_{\mathcal{V}}$$

such that dividing by $\|z'_{\epsilon,\delta}(t)\|_{\epsilon,\delta} \neq 0$ results in

$$\frac{\alpha}{2}\|z'_{\epsilon,\delta}(t)\|_{\mathcal{Z}} \le -\epsilon\left\langle z''_{\epsilon,\delta}(t), \frac{z'_{\epsilon,\delta}(t)}{\|z'_{\epsilon,\delta}(t)\|_{\epsilon,\delta}}\right\rangle_{\epsilon,\delta} + C_\rho\mathcal{R}(z'_{\epsilon,\delta}(t)) + \|\ell'(t)\|_{\mathcal{V}^*},$$

where we used estimate (3.1.30). Now integrating over ω gives

$$\begin{aligned}
\frac{\alpha}{2}\int_0^T \|z'_{\epsilon,\delta}(r)\|_{\mathcal{Z}}\,\mathrm{d}r = \frac{\alpha}{2}\int_\omega \|z'_{\epsilon,\delta}(r)\|_{\mathcal{Z}}\,\mathrm{d}r \\
&\le -\epsilon \int_\omega \left\langle z''_{\epsilon,\delta}(r), \frac{z'_{\epsilon,\delta}(r)}{\|z'_{\epsilon,\delta}(r)\|_{\epsilon,\delta}}\right\rangle_{\epsilon,\delta} \mathrm{d}r + C_\rho \int_\omega \mathcal{R}(z'_{\epsilon,\delta}(r))\,\mathrm{d}r \\
&\qquad + \int_\omega \|\ell'(r)\|_{\mathcal{V}^*}\,\mathrm{d}r \\
&\le \quad -\epsilon\Big(\|z'_{\epsilon,\delta}(T)\|_{\epsilon,\delta} - \|z'_{\epsilon,\delta}(0)\|_{\epsilon,\delta}\Big) + C_\rho \int_0^T \mathcal{R}(z'_{\epsilon,\delta}(r))\,\mathrm{d}r \\
&\qquad + \|\ell'\|_{L^1(0,T;\mathcal{V}^*)} \\
&\le C
\end{aligned}$$

with $C > 0$ independent of ϵ and δ. Here we used (3.1.31) and (3.1.32) in the second last step and the uniform bound in the end follows from (3.1.20) and (3.1.2). This yields in combination with (3.1.3) the assertion. □

3.2 The once viscous regularized system $(\mathrm{RIS})_\epsilon$

This section is concerned with the once regularized rate-independent system, which reads as

$$0 \in \partial\mathcal{R}_\epsilon(z'(t)) + \mathrm{D}_z\mathcal{I}(\ell(t), z(t)) \quad \text{f.a.a. } t \in (0,T), \quad z(0) = z_0, \tag{RIS_ϵ}$$

where $\mathcal{R}_\epsilon(z) = \mathcal{R}(z) + \frac{\epsilon}{2}\|z\|_{\mathcal{V}}^2$. With the results from the last section at hand, we are able prove the existence of solutions of (RIS_ϵ). Additionally, we infer useful a priori estimates with the aim to extract a weakly converging subsequence after a suitable parametrization. The limit will then provide a so-called parametrized balanced viscosity solution. To be more precise, we will show that (RIS_ϵ) is uniquely solvable by a function $z_\epsilon \in H^1(0,T;\mathcal{Z})$. In particular, the solution z_ϵ is given by the limit of the solutions according the twice regularized system ($\mathrm{RIS}_{\epsilon,\delta}$), i.e., $z_{\epsilon,\delta} \rightharpoonup z_\epsilon$ in $H^1(0,T;\mathcal{Z})$ as $\delta \searrow 0$. But before we show that (RIS_ϵ) is solvable, we first verify the uniqueness of the solution, which is part of the following lemma.

Lemma 3.2.1. *Let $z_1, z_2 \in H^1(0,T;\mathcal{Z})$ be solutions of* (RIS_ϵ). *Then it holds $z_1 = z_2$.*

Proof. Since z_1, z_2 are both solutions of (RIS_ϵ), we have

$$-\mathrm{D}_z\mathcal{I}(\ell(t), z_1(t)) \in \partial\mathcal{R}_\epsilon(z_1'(t)), \quad -\mathrm{D}_z\mathcal{I}(\ell(t), z_2(t)) \in \partial\mathcal{R}_\epsilon(z_2'(t))$$

for almost all $t \in (0,T)$ such that the monotonicity of $\partial\mathcal{R}_\epsilon$, particularly (A.5) with $\xi_1 := -\mathrm{D}_z\mathcal{I}(\ell(t), z_1(t))$, $\xi_2 := -\mathrm{D}_z\mathcal{I}(\ell(t), z_2(t))$, $z_1 := z_1'(t)$, $z_2 := z_2'(t)$, gives

$$0 \ge \frac{\epsilon}{2}\|z_1'(t) - z_2'(t)\|_{\mathcal{V}}^2 + \langle \mathrm{D}_z\mathcal{I}(\ell(t), z_1(t)) - \mathrm{D}_z\mathcal{I}(\ell(t), z_2(t)), z_1'(t) - z_2'(t)\rangle_{\mathcal{Z}^*,\mathcal{Z}}.$$

By exploiting the concrete structure of the energy and the assumed symmetry of A from (1.2.3), we deduce

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\langle Az_1(t) - Az_2(t), z_1(t) - z_2(t)\rangle_{\mathcal{Z}^*,\mathcal{Z}} + \frac{\epsilon}{2}\|z_1'(t) - z_2'(t)\|_{\mathcal{V}}^2 \\
&\leq -\langle \mathrm{D}_z\mathcal{F}(z_1(t)) - \mathrm{D}_z\mathcal{F}(z_2(t)), z_1'(t) - z_2'(t)\rangle_{\mathcal{V}^*,\mathcal{V}} \\
&\leq C\|z_1'(t) - z_2'(t)\|_{\mathcal{V}}\|z_1(t) - z_2(t)\|_{\mathcal{Z}} \\
&\leq \frac{\epsilon}{2}\|z_1'(t) - z_2'(t)\|_{\mathcal{V}}^2 + \tilde{C}\|z_1(t) - z_2(t)\|_{\mathcal{Z}}^2
\end{aligned}$$

with $\tilde{C} > 0$. In the second last step we used the mean value theorem in combination with the boundedness of the second derivative of $\mathcal{F}$ by (1.2.5) and the fact that $z_1, z_2 \in L^\infty(0,T;\mathcal{Z})$. Further, the last step was obtained by applying Young's inequality. Eventually, integrating over $(0,s)$ and making use of the coercivity of A by assumption (1.2.3) leads to

$$\frac{\alpha}{2}\|z_1(s) - z_2(s)\|_{\mathcal{Z}}^2 \leq \int_0^s \tilde{C}\|z_1(t) - z_2(t)\|_{\mathcal{Z}}^2 \,\mathrm{d}t,$$

where we exploited the initial condition $z_1(0) = z_2(0) = z_0$. Finally, from Gronwall's inequality we infer that $z_1 = z_2$ since s was arbitrary. □

In order to prove that the limit of $(z_{\epsilon,\delta})_{\delta>0}$, the solutions of (RIS$_{\epsilon,\delta}$), solves (RIS$_\epsilon$), we have to show that the corresponding energies converge in an appropriate sense. Since the same argumentation will take place later again, we outsource this needed convergence result in the following lemma.

Lemma 3.2.2. *Let $(z_n)_{n\in\mathbb{N}} \subset H^1(0,T;\mathcal{Z})$ be a pointwise uniformly bounded sequence with fixed initial value $z_0 \in \mathcal{Z}$, i.e., $\sup_{n\in\mathbb{N}}\sup_{t\in[0,T]}\|z_n(t)\|_{\mathcal{Z}} < \infty$ and $z_n(0) = z_0$ for all $n \in \mathbb{N}$. Moreover, we assume that*

$$z_n \rightharpoonup z \text{ in } H^1(0,T;\mathcal{Z}) \text{ and } z_n(t) \rightharpoonup z(t) \text{ in } \mathcal{Z} \text{ for all } t \in [0,T]$$

by passing $n \to \infty$ with a limit $z \in H^1(0,T;\mathcal{Z})$. Then for all $t \in [0,T]$ and all $v \in L^2(0,T;\mathcal{Z})$ the following convergence results are valid:

$$\begin{aligned}
\liminf_{n\to\infty}\int_0^t \langle Az_n(r), z_n'(r) - v(r)\rangle_{\mathcal{Z}^*,\mathcal{Z}}\,\mathrm{d}r &\geq \int_0^t \langle Az(r), z'(r) - v(r)\rangle_{\mathcal{Z}^*,\mathcal{Z}}\,\mathrm{d}r \\
\lim_{n\to\infty}\int_0^t \langle \mathrm{D}_z\mathcal{F}(z_n(r)), z_n'(r) - v(r)\rangle_{\mathcal{Z}^*,\mathcal{Z}}\,\mathrm{d}r &= \int_0^t \langle \mathrm{D}_z\mathcal{F}(z(r)), z'(r) - v(r)\rangle_{\mathcal{Z}^*,\mathcal{Z}}\,\mathrm{d}r.
\end{aligned}$$

Proof. Let $t \in [0,T]$ and $v \in L^2(0,T;\mathcal{Z})$ be given. Then exploiting the symmetry of A, the weak lower semicontinuity of $|\cdot|_{\mathcal{Z}}$ and the assumed (pointwise) weak convergence results in

$$\liminf_{n\to\infty}\int_0^t \langle Az_n(r), z_n'(r) - v(r)\rangle_{\mathcal{Z}^*,\mathcal{Z}}\,\mathrm{d}r$$

$$
\begin{aligned}
&= \liminf_{n\to\infty} \int_0^t \langle Az_n(r), z_n'(r)\rangle_{\mathcal{Z}^*,\mathcal{Z}} - \langle Av(r), z_n(r)\rangle_{\mathcal{Z}^*,\mathcal{Z}} \,\mathrm{d}r \\
&= \liminf_{n\to\infty} \int_0^t \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|z_n(r)|_{\mathcal{Z}}^2 \,\mathrm{d}r - \int_0^t \langle Av(r), z(r)\rangle_{\mathcal{Z}^*,\mathcal{Z}} \,\mathrm{d}r \\
&= \liminf_{n\to\infty} \frac{1}{2}|z_n(t)|_{\mathcal{Z}}^2 - \frac{1}{2}|z_0|_{\mathcal{Z}}^2 - \int_0^t \langle Az(r), v(r)\rangle_{\mathcal{Z}^*,\mathcal{Z}} \,\mathrm{d}r \\
&\geq \frac{1}{2}|z(t)|_{\mathcal{Z}}^2 - \frac{1}{2}|z(0)|_{\mathcal{Z}}^2 - \int_0^t \langle Az(r), v(r)\rangle_{\mathcal{Z}^*,\mathcal{Z}} \,\mathrm{d}r \\
&= \int_0^t \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|z(r)|_{\mathcal{Z}}^2 \,\mathrm{d}r - \int_0^t \langle Az(r), v(r)\rangle_{\mathcal{Z}^*,\mathcal{Z}} \,\mathrm{d}r \\
&= \int_0^t \langle Az(r), z'(r) - v(r)\rangle_{\mathcal{Z}^*,\mathcal{Z}} \,\mathrm{d}r,
\end{aligned}
$$

which gives the first assertion. Thanks to the pointwise weak convergence, the weak continuity of $\mathcal{F}$ from (1.2.8) yields $\lim_{n\to\infty}\mathcal{F}(z_n(t)) = \mathcal{F}(z(t))$. Further, applying the mean value theorem and using (1.2.5) yields for all $n \in \mathbb{N}$ and $r \in [0,T]$

$$
\begin{aligned}
\|\mathrm{D}_z\mathcal{F}(z_n(r))\|_{\mathcal{V}^*} &\leq \|\mathrm{D}_z\mathcal{F}(0)\|_{\mathcal{V}^*} + \int_0^1 \|\mathrm{D}_z^2\mathcal{F}(\tau z_n(r))z_n(r)\|_{\mathcal{V}^*} \,\mathrm{d}\tau \\
&\leq \|\mathrm{D}_z\mathcal{F}(0)\|_{\mathcal{V}^*} + \int_0^1 C(1+\tau^q\|z_n(r)\|_{\mathcal{Z}}^q)\|z_n(r)\|_{\mathcal{Z}} \,\mathrm{d}\tau \leq \tilde{C},
\end{aligned}
\tag{3.2.1}
$$

where we exploited the assumed uniform boundedness of z_n, i.e., $\|z_n(r)\|_{\mathcal{Z}} \leq C$ independent of $n \in \mathbb{N}$ and $r \in [0,T]$. Hence, by $\mathcal{V}^* \hookrightarrow \mathcal{Z}^*$ continuously, we have that $\|\mathrm{D}_z\mathcal{F}(z_n(r))\|_{\mathcal{Z}^*} \leq C$. In combination with the weak-weak convergence of $\mathrm{D}_z\mathcal{F}$ by (1.2.6), which implies $\mathrm{D}_z\mathcal{F}(z_n(r)) \rightharpoonup \mathrm{D}_z\mathcal{F}(z(r))$ in $\mathcal{Z}^*$ for all $r \in [0,T]$, Lebesgue's dominated convergence theorem yields

$$
\lim_{n\to\infty} \int_0^t \langle \mathrm{D}_z\mathcal{F}(z_n(r)), v(r)\rangle_{\mathcal{Z}^*,\mathcal{Z}} \,\mathrm{d}r = \int_0^t \langle \mathrm{D}_z\mathcal{F}(z(r)), v(r)\rangle_{\mathcal{Z}^*,\mathcal{Z}} \,\mathrm{d}r
$$

All together, we have shown that

$$
\begin{aligned}
&\lim_{n\to\infty} \int_0^t \langle \mathrm{D}_z\mathcal{F}(z_n(r)), z_n'(r) - v(r)\rangle_{\mathcal{Z}^*,\mathcal{Z}} \,\mathrm{d}r \\
&= \lim_{n\to\infty} \int_0^t \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}(z_n(r)) - \langle \mathrm{D}_z\mathcal{F}(z_n(r)), v(r)\rangle_{\mathcal{Z}^*,\mathcal{Z}} \,\mathrm{d}r \\
&= \lim_{n\to\infty} \mathcal{F}(z_n(t)) - \mathcal{F}(z_0) - \int_0^t \langle \mathrm{D}_z\mathcal{F}(z_n(r)), v(r)\rangle_{\mathcal{Z}^*,\mathcal{Z}} \,\mathrm{d}r \\
&= \mathcal{F}(z(t)) - \mathcal{F}(z_0) - \int_0^t \langle \mathrm{D}_z\mathcal{F}(z(r)), v(r)\rangle_{\mathcal{Z}^*,\mathcal{Z}} \,\mathrm{d}r \\
&= \int_0^t \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}(z(r)) - \langle \mathrm{D}_z\mathcal{F}(z(r)), v(r)\rangle_{\mathcal{Z}^*,\mathcal{Z}} \,\mathrm{d}r
\end{aligned}
$$

$$= \int_0^t \langle \mathrm{D}_z \mathcal{F}(z(r)), z'(r) - v(r) \rangle_{\mathcal{Z}^*,\mathcal{Z}} \,\mathrm{d}r,$$

which completes the proof. □

Finally, we can now prove the existence of a solution of (RIS$_\epsilon$) as the limit of solutions of (RIS$_{\epsilon,\delta}$).

Theorem 3.2.3. *Let $\epsilon > 0$ be fixed and let $(z_{\epsilon,\delta})_{\delta>0}$ be a sequence solving the twice regularized systems* (RIS$_{\epsilon,\delta}$). *Then there exists $z_\epsilon \in H^1(0,T;\mathcal{Z}) \cap W^{1,\infty}(0,T;\mathcal{V})$ such that*

$$z_{\epsilon,\delta} \rightharpoonup z_\epsilon \text{ in } H^1(0,T;\mathcal{Z}), \quad z_{\epsilon,\delta} \rightharpoonup^* z_\epsilon \text{ in } W^{1,\infty}(0,T;\mathcal{V}), \tag{3.2.2}$$

and

$$z_{\epsilon,\delta}(t) \rightharpoonup z_\epsilon(t) \text{ in } \mathcal{Z}, \quad z_{\epsilon,\delta}(t) \to z_\epsilon(t) \text{ in } \mathcal{V} \text{ for all } t \in [0,T], \text{ as } \delta \searrow 0. \tag{3.2.3}$$

Moreover, the limit z_ϵ is a solution of (RIS$_\epsilon$).

Proof. Due to (3.1.25) and (3.1.26), the sequence $(z_{\epsilon,\delta})_{\delta>0}$ is uniformly bounded in $H^1(0,T;\mathcal{Z}) \cap W^{1,\infty}(0,T;\mathcal{V})$ by a constant not depending on δ. Therefore, by passing δ to zero, we can extract a weakly converging subsequence

$$z_{\epsilon,\delta_n} \rightharpoonup z_\epsilon \text{ in } H^1(0,T;\mathcal{Z}), \quad z_{\epsilon,\delta_n} \rightharpoonup^* z_\epsilon \text{ in } W^{1,\infty}(0,T;\mathcal{V}). \tag{3.2.4}$$

Furthermore, we have

$$z_{\epsilon,\delta_n}(t) \rightharpoonup z_\epsilon(t) \text{ in } \mathcal{Z}, \quad z_{\epsilon,\delta_n}(t) \to z_\epsilon(t) \text{ in } \mathcal{V} \text{ for all } t \in [0,T] \tag{3.2.5}$$

which can be seen as follows. From Aubin-Lions lemma, especially the compact embedding $H^1(0,T;\mathcal{Z}) \hookrightarrow^c C([0,T];\mathcal{V})$, we deduce the pointwise strong convergence in $\mathcal{V}$. Additionally, z_{ϵ,δ_n} is pointwisely bounded in $\mathcal{Z}$ due to (3.1.3). Thus, for all $t \in [0,T]$ it exists a weakly converging subsequence (denoted by the same symbol) and a limit $z(t) \in \mathcal{Z}$ such that $z_{\epsilon,\delta_n}(t) \rightharpoonup z(t)$ in $\mathcal{Z}$. Because weak limits are uniquely determined, the pointwise strong convergence in $\mathcal{V}$ yields that $z(t) = z_\epsilon(t)$. Since this holds true for all subsequences, we obtain the first convergence in (3.2.5) for the entire sequence. Now by exploiting this pointwise convergence, we obtain

$$z_0 = z_{\epsilon,\delta_n}(0) \to z_\epsilon(0) \text{ in } \mathcal{V}, \quad \delta_n \searrow 0, \tag{3.2.6}$$

such that the limit satisfies the initial condition $z_\epsilon(0) = z_0$. According to the definition of the subdifferential, the solution z_{ϵ,δ_n} fulfills

$$\begin{aligned} \mathcal{R}(v) \geq \mathcal{R}(z'_{\epsilon,\delta_n}(t)) - \epsilon \langle z'_{\epsilon,\delta_n}(t), v - z'_{\epsilon,\delta_n}(t) \rangle_{\mathcal{V}^*,\mathcal{V}} - \delta_n \langle A z'_{\epsilon,\delta_n}(t), v - z'_{\epsilon,\delta}(t) \rangle_{\mathcal{Z}^*,\mathcal{Z}} \\ - \langle \mathrm{D}_z \mathcal{I}(\ell(t), z_{\epsilon,\delta_n}(t)), v - z'_{\epsilon,\delta_n}(t) \rangle_{\mathcal{Z}^*,\mathcal{Z}} \end{aligned}$$

for all $v \in \mathcal{Z}$ and almost all $t \in (0,T)$, which implies by integrating

$$\begin{aligned}\int_0^t \mathcal{R}(v(r))\,\mathrm{d}r \geq \int_0^t \mathcal{R}(z'_{\epsilon,\delta_n}(r))\,\mathrm{d}r &+ \int_0^t \epsilon\langle z'_{\epsilon,\delta_n}(r), z'_{\epsilon,\delta_n}(r) - v(r)\rangle_{\mathcal{V}^*,\mathcal{V}}\,\mathrm{d}r \\ &+ \int_0^t \delta_n\langle Az'_{\epsilon,\delta_n}(r), z'_{\epsilon,\delta_n}(r) - v(r)\rangle_{\mathcal{Z}^*,\mathcal{Z}}\,\mathrm{d}r \\ &+ \int_0^t \langle \mathrm{D}_z\mathcal{I}(\ell(r), z_{\epsilon,\delta_n}(r)), z'_{\epsilon,\delta_n}(r) - v(r)\rangle_{\mathcal{Z}^*,\mathcal{Z}}\,\mathrm{d}r\end{aligned}$$

for all $t \in [0,T]$ and all $v \in L^2(0,T;\mathcal{Z})$. Now we discuss the terms on the right hand side separately. First, due to the weak convergence $z'_{\epsilon,\delta_n} \rightharpoonup z'_\epsilon$ in $L^2(0,T;\mathcal{Z})$ and the convexity and lower semicontinuity of $\mathcal{R}$ from (1.2.13), Lemma A.1 is applicable and yields

$$\liminf_{n\to\infty} \int_0^t \mathcal{R}(z'_{\epsilon,\delta_n}(r))\,\mathrm{d}r \geq \int_0^t \mathcal{R}(z'_\epsilon(r))\,\mathrm{d}r.$$

Regarding the second term, we exploit the weak lower semicontinuity of the norm and (3.2.4) in order to obtain

$$\begin{aligned}&\liminf_{n\to\infty} \int_0^t \epsilon\langle z'_{\epsilon,\delta_n}(r), z'_{\epsilon,\delta_n}(r) - v(r)\rangle_{\mathcal{V}^*,\mathcal{V}}\,\mathrm{d}r \\ =&\liminf_{n\to\infty} \int_0^t \epsilon\|z'_{\epsilon,\delta_n}(r)\|^2_{\mathcal{V}} - \epsilon\langle z'_{\epsilon,\delta_n}(r), v(r)\rangle_{\mathcal{V}^*,\mathcal{V}}\,\mathrm{d}r \\ \geq&\int_0^t \epsilon\|z'_\epsilon(r)\|^2_{\mathcal{V}} - \epsilon\langle z'_\epsilon(r), v(r)\rangle_{\mathcal{V}^*,\mathcal{V}}\,\mathrm{d}r \\ =&\int_0^t \epsilon\langle z'_\epsilon(r), z'_\epsilon(r) - v(r)\rangle_{\mathcal{V}^*,\mathcal{V}}\,\mathrm{d}r.\end{aligned}$$

Next, by using (3.1.5), the third integral can be estimated by

$$\begin{aligned}&\int_0^t \delta_n\langle Az'_{\epsilon,\delta_n}(r), z'_{\epsilon,\delta_n}(r) - v(r)\rangle_{\mathcal{Z}^*,\mathcal{Z}}\,\mathrm{d}r \\ =&\int_0^t \delta_n|z'_{\epsilon,\delta_n}(r)|^2_{\mathcal{Z}} - \delta_n\langle Az'_{\epsilon,\delta_n}(r), v(r)\rangle_{\mathcal{Z}^*,\mathcal{Z}}\,\mathrm{d}r \\ \geq&-\delta_n \int_0^t \langle Az'_{\epsilon,\delta_n}(r), v(r)\rangle_{\mathcal{Z}^*,\mathcal{Z}}\,\mathrm{d}r \to 0, \text{ as } n\to\infty.\end{aligned}$$

Eventually, exploiting the concrete structure of the energy and using Lemma 3.2.2, which is applicable due to (3.2.4), (3.2.5) and (3.1.3), we infer

$$\begin{aligned}&\liminf_{n\to\infty} \int_0^t \langle \mathrm{D}_z\mathcal{I}(\ell(r), z_{\epsilon,\delta_n}(r)), z'_{\epsilon,\delta_n}(r) - v(r)\rangle_{\mathcal{Z}^*,\mathcal{Z}}\,\mathrm{d}r \\ =&\liminf_{n\to\infty} \int_0^t \langle Az_{\epsilon,\delta_n}(r), z'_{\epsilon,\delta_n}(r) - v(r)\rangle_{\mathcal{Z}^*,\mathcal{Z}}\,\mathrm{d}r\end{aligned}$$

$$
\begin{aligned}
&\quad + \lim_{n\to\infty} \int_0^t \langle \mathrm{D}_z\mathcal{F}(z_{\epsilon,\delta_n}(r)), z'_{\epsilon,\delta_n}(r) - v(r)\rangle_{\mathcal{Z}^*,\mathcal{Z}}\,\mathrm{d}r \\
&\quad - \lim_{n\to\infty} \int_0^t \langle \ell(r), z'_{\epsilon,\delta_n}(r) - v(r)\rangle_{\mathcal{Z}^*,\mathcal{Z}}\,\mathrm{d}r \\
&\geq \int_0^t \langle Az_\epsilon(r), z'_\epsilon(r) - v(r)\rangle_{\mathcal{Z}^*,\mathcal{Z}}\,\mathrm{d}r + \int_0^t \langle \mathrm{D}_z\mathcal{F}(z_\epsilon(r)), z'_\epsilon(r) - v(r)\rangle_{\mathcal{Z}^*,\mathcal{Z}}\,\mathrm{d}r \\
&\qquad - \int_0^t \langle \ell(r), z'_\epsilon(r) - v(r)\rangle_{\mathcal{Z}^*,\mathcal{Z}}\,\mathrm{d}r \\
&= \int_0^t \langle \mathrm{D}_z\mathcal{I}(\ell(r), z_\epsilon(r)), z'_\epsilon(r) - v(r)\rangle_{\mathcal{Z}^*,\mathcal{Z}}\,\mathrm{d}r,
\end{aligned}
$$

where we used the weak convergence (3.2.4) in order to determine the limit of the integral concerned with the load ℓ. All in all, we end up with

$$
\begin{aligned}
\int_0^t \mathcal{R}(v(r))\,\mathrm{d}r \geq \liminf_{n\to\infty} \int_0^t \mathcal{R}(z'_{\epsilon,\delta_n}(r)) &+ \epsilon\langle z'_{\epsilon,\delta_n}(r), z'_{\epsilon,\delta_n}(r) - v(r)\rangle_{\mathcal{V}^*,\mathcal{V}} \\
&+ \delta_n\langle Az'_{\epsilon,\delta_n}(r), z'_{\epsilon,\delta_n}(r) - v(r)\rangle_{\mathcal{Z}^*,\mathcal{Z}} \\
&+ \langle \mathrm{D}_z\mathcal{I}(\ell(r), z_{\epsilon,\delta_n}(r)), z'_{\epsilon,\delta_n}(r) - v(r)\rangle_{\mathcal{Z}^*,\mathcal{Z}}\,\mathrm{d}r \\
\geq \int_0^t \mathcal{R}(z'_\epsilon(r)) + \epsilon\langle z'_\epsilon(r), z'_\epsilon(r) - v(r)\rangle_{\mathcal{V}^*,\mathcal{V}}& \\
&+ \langle \mathrm{D}_z\mathcal{I}(\ell(r), z_\epsilon(r)), z'_\epsilon(r) - v(r)\rangle_{\mathcal{Z}^*,\mathcal{Z}}\,\mathrm{d}r.
\end{aligned}
$$

Finally, we prove that this implies the subdifferential inclusion in (RIS_ϵ). Therefore, let $w \in \mathcal{Z}$ and $0 \neq \phi \in C_c^\infty(0,T)$ with $\phi(t) \geq 0$ for all $t \in [0,T]$ be arbitrary. We define $\phi_{\max} := \max_{t\in[0,T]} \phi(t)$ and $v(t) := \frac{\phi(t)}{\phi_{\max}} w + (1 - \frac{\phi(t)}{\phi_{\max}}) z'_\epsilon(t)$ such that inserting this v in the above inequality results in

$$
\begin{aligned}
\int_0^t \mathcal{R}\Big(\frac{\phi(r)}{\phi_{\max}} w + \Big(1 - \frac{\phi(r)}{\phi_{\max}}\Big) z'_\epsilon(r)\Big) - \mathcal{R}(z'_\epsilon(r)) + \epsilon\Big\langle z'_\epsilon(r), \frac{\phi(r)}{\phi_{\max}}(w - z'_\epsilon(r))\Big\rangle_{\mathcal{V}^*,\mathcal{V}} \\
+ \Big\langle \mathrm{D}_z\mathcal{I}(\ell(r), z_\epsilon(r)), \frac{\phi(t)}{\phi_{\max}}(w - z'_\epsilon(r))\Big\rangle_{\mathcal{Z}^*,\mathcal{Z}}\mathrm{d}r \geq 0
\end{aligned}
$$

Next, we exploit the convexity of $\mathcal{R}$ and multiply with $\phi_{\max} > 0$ to end up with

$$
\begin{aligned}
\int_0^t \phi(r)\Big(\mathcal{R}(w) - \mathcal{R}(z'_\epsilon(r)) + \epsilon\langle z'_\epsilon(r), w - z'_\epsilon(r)\rangle_{\mathcal{V}^*,\mathcal{V}} \\
+ \langle \mathrm{D}_z\mathcal{I}(\ell(r), z_\epsilon(r)), w - z'_\epsilon(r)\rangle_{\mathcal{Z}^*,\mathcal{Z}}\Big)\mathrm{d}r \geq 0.
\end{aligned}
$$

Since ϕ was arbitrary, the fundamental lemma of the calculus of variations yields

$$
\mathcal{R}(w) - \mathcal{R}(z'_\epsilon(t)) + \epsilon\langle z'_\epsilon(t), w - z'_\epsilon(t)\rangle_{\mathcal{V}^*,\mathcal{V}} + \langle \mathrm{D}_z\mathcal{I}(\ell(t), z_\epsilon(t)), w - z'_\epsilon(t)\rangle_{\mathcal{Z}^*,\mathcal{Z}} \geq 0
$$

for almost all $t \in (0,T)$ such that the definition of the subdifferential yields

$$0 \in \partial\mathcal{R}_\epsilon(z_\epsilon'(t)) + \mathrm{D}_z\mathcal{I}(\ell(t), z_\epsilon(t)) \text{ f.a.a. } t \in (0,T).$$

Thus, by taking (3.2.6) into account, z_ϵ is indeed a solution of (RIS_ϵ). In view of Lemma 3.2.1, we know that this solution is uniquely determined. Therefore, and since the argumentation from above holds for all subsequences, the entire sequence converges as claimed in (3.2.2) and (3.2.3). □

Now that we have proven the existence of solutions, we show that they also satisfy a priori estimates similar to those in the previous section.

Theorem 3.2.4. *The solution $z_\epsilon \in H^1(0,T;\mathcal{Z})$ of* (RIS_ϵ) *fulfills the energy identity*

$$\begin{aligned}\mathcal{I}(\ell(t), z_\epsilon(t)) + \int_0^t \mathcal{R}_\epsilon(z_\epsilon'(r)) + \mathcal{R}_\epsilon^*(-\mathrm{D}_z\mathcal{I}(\ell(r), z_\epsilon(r)))\,\mathrm{d}r \\ = \mathcal{I}(\ell(0), z_\epsilon(0)) - \int_0^t \langle \ell'(r), z_\epsilon(r)\rangle_{\mathcal{V}^*,\mathcal{V}}\,\mathrm{d}r\end{aligned} \tag{3.2.7}$$

for all $t \in [0,T]$. Additionally, we have the a priori estimates

$$\begin{aligned}\int_0^T \mathcal{R}_\epsilon(z_\epsilon'(r)) + \mathcal{R}_\epsilon^*(-\mathrm{D}_z\mathcal{I}(\ell(r), z_\epsilon(r)))\,\mathrm{d}r \\ \le c_0 + C\Big(\mathcal{I}(\ell(0), z_0) + \lambda\kappa T + \|\ell\|^2_{H^1(0,T,\mathcal{V}^*)}\Big)\end{aligned} \tag{3.2.8}$$

and

$$\sup_{t\in[0,T]} \|z_\epsilon(t)\|_{\mathcal{Z}} \le C, \tag{3.2.9}$$

$$\epsilon\|z_\epsilon\|_{W^{1,\infty}(0,T;\mathcal{V})} \le C, \tag{3.2.10}$$

$$\epsilon\|z_\epsilon\|^2_{H^1(0,T;\mathcal{Z})} \le C, \tag{3.2.11}$$

$$\|z_\epsilon\|_{W^{1,1}(0,T;\mathcal{Z})} \le C, \tag{3.2.12}$$

where $c_0 > 0$ is the constant from (1.2.11) *and $C > 0$ is independent of ϵ.*

Proof. The first part of the proof is similar to that of Theorem 3.1.1. In the same way, we derive the energy identity (3.2.7) and estimate (3.2.8). In order to prove (3.2.9), let $\epsilon > 0$ and $t \in [0,T]$ be arbitrarily given. Then the sequence $(z_{\epsilon,\delta})_{\delta>0}$ of solutions of ($\mathrm{RIS}_{\epsilon,\delta}$) converges to z_ϵ in the sense of (3.2.2) and (3.2.3). Thanks to the pointwise weak convergence (3.2.3) and the uniform boundedness (3.1.3), it follows that $\|z_\epsilon(t)\|_{\mathcal{Z}} \le C$, where $C > 0$ is from (3.1.3) and does not depend on ϵ. Since $t \in [0,T]$ was arbitrary, the boundedness holds for all $t \in [0,T]$ as claimed. The boundedness from (3.1.26) transfers to the weak* limit z_ϵ (cf. (3.2.2)) due to the compactness with respect to the weak* topology by the Banach-Alaoglu theorem, which gives (3.2.10). Further, (3.1.25) and (3.1.33) are maintained in the limit $\delta \searrow 0$, too. This follows by means of the weak lower semicontinuity of the norm and weak convergence from (3.2.2), such that (3.2.11) and (3.2.12) are verified. □

Remark 3.2.5. Analogously to Remark 3.1.2, the constant $C > 0$ in (3.2.9)–(3.2.12) can be chosen independently of the load $\ell \in H^1(0, T; \mathcal{V}^*)$ if ℓ satisfies (1.2.10).

The uniform estimate (3.2.10) together with (1.2.20) and the boundedness of $\partial\mathcal{R}(0) \subset \mathcal{V}^*$ by Lemma 1.2.5 directly gives the following estimate concerned with the derivative of the energy, which is required in the next chapter.

Corollary 3.2.6. *There exists a constant $C > 0$ not depending on ϵ such that*

$$\|\mathrm{D}_z\mathcal{I}(\ell(\cdot), z_\epsilon(\cdot))\|_{L^\infty(0,T;\mathcal{V}^*)} \leq C. \tag{3.2.13}$$

From the next lemma we infer that this estimate even holds for all $t \in [0, T]$.

Lemma 3.2.7. *Let $z \in C_{weak}([0, T]; \mathcal{Z})$ and $\ell \in H^1(0, T; \mathcal{V}^*)$ with*

$$\|\mathrm{D}_z\mathcal{I}(\ell(\cdot), z(\cdot))\|_{L^\infty(0,T;\mathcal{V}^*)} \leq C$$

be given. Then there holds

$$\sup_{t\in[0,T]} \|\mathrm{D}_z\mathcal{I}(\ell(t), z(t))\|_{\mathcal{V}^*} \leq C.$$

Proof. Let $t \in [0, T]$ be arbitrary. Then there exists a sequence $(t_n)_{n\in\mathbb{N}} \subset [0, T]$ with $t_n \to t$ as $n \to \infty$ and $\|\mathrm{D}_z\mathcal{I}(\ell(t_n), z(t_n))\|_{\mathcal{V}^*} \leq C$ for all $n \in \mathbb{N}$. This boundedness yields the existence of a (not relabeled) weakly converging subsequence, i.e., $\mathrm{D}_z\mathcal{I}(\ell(t_n), z(t_n)) \rightharpoonup \xi \in \mathcal{V}^*$. Further, the fact that $z(t_n) \rightharpoonup z(t)$ in $\mathcal{Z}$ and $\ell(t_n) \to \ell(t)$ in $\mathcal{V}^*$ by assumption and continuous embedding, respectively, yields in combination with (1.2.6) that

$$\begin{aligned}\mathrm{D}_z\mathcal{I}(\ell(t_n), z(t_n)) &= Az(t_n) + \mathrm{D}_z\mathcal{F}(z(t_n)) - \ell(t_n)\\ &\rightharpoonup Az(t) + \mathrm{D}_z\mathcal{F}(z(t)) - \ell(t) = \mathrm{D}_z\mathcal{I}(\ell(t), z(t)) \text{ in } \mathcal{Z}^*, \text{ as } n \to \infty.\end{aligned}$$

Due to the uniqueness of weak limits, $\mathrm{D}_z\mathcal{I}(\ell(t), z(t))$ has to coincide with $\xi \in \mathcal{V}^*$. Since this argumentation is applicable for all subsequence, we obtain

$$\mathrm{D}_z\mathcal{I}(\ell(t_n), z(t_n)) \rightharpoonup \mathrm{D}_z\mathcal{I}(\ell(t), z(t)) \text{ in } \mathcal{V}^*$$

along the entire sequence such that the weak lower semicontinuity of the norm gives the assertion. □

Because $z_\epsilon \in H^1(0, T; \mathcal{Z})$ and thus especially $z_\epsilon \in C_{weak}([0, T]; \mathcal{Z})$, we have the following slightly modified version of Corollary 3.2.6:

Corollary 3.2.8. *There exists a constant $C > 0$ not depending on ϵ such that*

$$\sup_{t\in[0,T]} \|\mathrm{D}_z\mathcal{I}(\ell(t), z_\epsilon(t))\|_{\mathcal{V}^*} \leq C. \tag{3.2.14}$$

3.3 The vanishing viscosity limit $\epsilon \searrow 0$

In this section we want to derive the definition of parametrized balanced viscosity solutions. For given solutions $z_\epsilon \in H^1(0,T;\mathcal{Z}) \cap W^{1,\infty}(0,T;\mathcal{V})$, $\epsilon > 0$, of the viscous regularized systems (RIS$_\epsilon$) we first parametrize their graphs by its arc length measured with respect to the dissipation and then we pass $\epsilon \searrow 0$. This parametrization is done by means of the vanishing viscosity contact potential, cf. [MRS12, Rem. 3.1], [MRS16, Sec. 3],

$$\mathfrak{p}(v,w) := \mathcal{R}(v) + \|v\|_{\mathcal{V}} \operatorname{dist}_{\mathcal{V}^*}(w, \partial\mathcal{R}(0)), \quad v \in \mathcal{Z}, w \in \mathcal{V}^*, \tag{3.3.1}$$

where the distance is given by

$$\operatorname{dist}_{\mathcal{V}^*}(w,\partial\mathcal{R}(0)) := \inf\big\{\|w-\eta\|_{\mathcal{V}^*} : \eta \in \partial\mathcal{R}(0)\big\}. \tag{3.3.2}$$

In Lemma A.4 it is shown that the infimum is attained for some $\eta \in \partial\mathcal{R}(0)$ such that it is in fact a minimum. By applying Young's inequality and the representation from (A.2), the vanishing viscosity contact potential satisfies

$$\begin{aligned}\mathfrak{p}(v,w) &= \mathcal{R}(v) + \|v\|_{\mathcal{V}} \operatorname{dist}_{\mathcal{V}^*}(w,\partial\mathcal{R}(0)) \\ &\le \mathcal{R}(v) + \frac{\epsilon}{2}\|v\|_{\mathcal{V}}^2 + \frac{1}{2\epsilon}\operatorname{dist}_{\mathcal{V}^*}(w,\partial\mathcal{R}(0))^2 \\ &= \mathcal{R}_\epsilon(v) + \mathcal{R}_\epsilon^*(w).\end{aligned} \tag{3.3.3}$$

To be more precise in terms of the parametrization, for a given solution z_ϵ of (RIS$_\epsilon$) we define

$$s_\epsilon(t) := t + \int_0^t \mathfrak{p}(z_\epsilon'(r), -\mathrm{D}_z\mathcal{I}(\ell(r), z_\epsilon(r)))\,\mathrm{d}r, \quad S_\epsilon := s_\epsilon(T). \tag{3.3.4}$$

Note that the above integral is well defined due to the fact that $-\mathrm{D}_z\mathcal{I}(\ell(\cdot), z_\epsilon(\cdot)) \in L^\infty(0,T;\mathcal{V}^*)$ by Corollary 3.2.6. Since $\mathfrak{p}(v,w) \ge 0$, this function is strictly monotonously increasing and its inverse function

$$\hat{t}_\epsilon := (s_\epsilon)^{-1} : [0,S_\epsilon] \to [0,T] \tag{3.3.5}$$

exists. Thanks to $s_\epsilon'(t) = 1 + \mathfrak{p}(z_\epsilon'(t), -\mathrm{D}_z\mathcal{I}(\ell(t), z_\epsilon(t))) \ge 1$ for almost all $t \in [0,T]$, the inverse function $\hat{t}_\epsilon$ is Lipschitz continuous with derivative

$$0 < \hat{t}_\epsilon'(s) = \frac{1}{s_\epsilon'(\hat{t}_\epsilon(s))} = \frac{1}{1 + \mathfrak{p}\Big(z_\epsilon'(\hat{t}_\epsilon(s)), -\mathrm{D}_z\mathcal{I}\big(\ell(\hat{t}_\epsilon(s)), z_\epsilon(\hat{t}_\epsilon(s))\big)\Big)} \le 1 \tag{3.3.6}$$

for almost all $s \in [0,S_\epsilon]$. To summarize, for all $\epsilon > 0$ the functions $\hat{t}_\epsilon$ satisfy

$$\begin{gathered}\hat{t}_\epsilon \in W^{1,\infty}(0,S_\epsilon), \quad \hat{t}_\epsilon(0) = 0, \quad \hat{t}_\epsilon(S_\epsilon) = T, \\ 0 < \hat{t}_\epsilon'(s) \le 1 \quad \text{f.a.a. } s \in (0,S_\epsilon).\end{gathered} \tag{3.3.7}$$

Eventually, we define the parametrized functions

$$\hat{z}_\epsilon := z_\epsilon \circ \hat{t}_\epsilon : [0, S_\epsilon] \to \mathcal{Z} \tag{3.3.8}$$

$$\hat{\ell}_\epsilon := \ell_\epsilon \circ \hat{t}_\epsilon : [0, S_\epsilon] \to \mathcal{V}^*. \tag{3.3.9}$$

The following lemma is concerned with the differentiability and limit behavior of these compositions. Here we assume that $S = S_\epsilon$ for all $\epsilon > 0$, which is no crucial restriction because we will later show S_ϵ is uniformly bounded. Thus, the functions of the sequence can be constantly extended to a larger finite interval such that all these functions have the same domain.

Lemma 3.3.1. *Let $(\ell_\epsilon)_{\epsilon>0} \subset H^1(0,T;\mathcal{V}^*)$ and $(\hat{t}_\epsilon)_{\epsilon>0} \subset W^{1,\infty}(0,S)$ be sequences with $\ell_\epsilon \rightharpoonup \ell$ in $H^1(0,T;\mathcal{V}^*)$, $\hat{t}_\epsilon \rightharpoonup^* \hat{t}$ in $W^{1,\infty}(0,S)$ and $\hat{t}_\epsilon(0) = 0$, $\hat{t}_\epsilon(S) = T$, $0 < \hat{t}'_\epsilon(s) \leq 1$ f.a.a. $s \in (0,S)$ and all $\epsilon > 0$. Then $\hat{\ell}_\epsilon := \ell_\epsilon \circ \hat{t}_\epsilon \in H^1(0,S;\mathcal{V}^*)$ for all $\epsilon > 0$ with derivative*

$$\hat{\ell}'_\epsilon(s) = \ell'_\epsilon(\hat{t}_\epsilon(s))\hat{t}'_\epsilon(s) \tag{3.3.10}$$

for almost all $s \in [0,S]$. Moreover, we have

$$\hat{\ell}_\epsilon = \ell_\epsilon \circ \hat{t}_\epsilon \rightharpoonup \ell \circ \hat{t} := \hat{\ell} \in H^1(0,S;\mathcal{V}^*) \tag{3.3.11}$$

and for all converging sequences $s_\epsilon \to s$ in $\mathbb{R}$ it holds

$$\hat{\ell}_\epsilon(s_\epsilon) \rightharpoonup \hat{\ell}(s) \text{ in } \mathcal{V}^*. \tag{3.3.12}$$

Proof. We follow the lines of [KT23, Lem. A.1] and start with proving (3.3.10). Here we suppress the subscript since the formula holds for all elements of the sequence. Because $\hat{t}$ and ℓ are in particular absolutely continuous, they are differentiable almost everywhere. Let $N_\ell \subset [0,T]$ and $N_{\hat{t}} \subset [0,S]$ denote the null sets, where ℓ and $\hat{t}$, respectively, are not differentiable. Then the pre-image $\hat{t}^{-1}(N_\ell) \subset [0,S]$ of N_ℓ has measure zero, too, which can directly be seen by

$$0 = \lambda(N_\ell) = \int_{N_\ell} 1 \mathrm{d}t = \int_{\hat{t}(\hat{t}^{-1}(N_\ell))} 1 \mathrm{d}t = \int_{\hat{t}^{-1}(N_\ell)} \hat{t}'(s) \, \mathrm{d}s$$

since $\hat{t}' > 0$ almost everywhere in $[0,S]$. Then for almost all $s \in [0,S]$, to be more precise all $s \in [0,S] \setminus \big(N_{\hat{t}} \cup \hat{t}^{-1}(N_\ell)\big)$, $\hat{t}$ is differentiable in s and ℓ is differentiable in $\hat{t}(s)$ such that the classical chain rule yields (3.3.10) for all these s. Next, we verify that the composition $\hat{\ell} = \ell \circ \hat{t}$ is indeed an element of $H^1(0,S;\mathcal{V}^*)$. From the continuous embedding $H^1(0,T;\mathcal{V}^*) \hookrightarrow C([0,T];\mathcal{V}^*)$ we infer that

$$\|\hat{\ell}\|_{L^\infty(0,S;\mathcal{V}^*)} = \|\ell\|_{L^\infty(0,S;\mathcal{V}^*)} \leq C\|\ell\|_{H^1(0,T;\mathcal{V}^*)}. \tag{3.3.13}$$

Moreover, the derivative (3.3.10) satisfies

$$\begin{aligned} \|\hat{\ell}'\|^2_{L^2(0,S;\mathcal{V}^*)} &= \int_0^S \|\ell'(\hat{t}(s))\hat{t}'(s)\|^2_{\mathcal{V}^*} \, \mathrm{d}s \leq \|\hat{t}'\|_{L^\infty(0,S)} \int_0^S \|\ell(\hat{t}(s))\|^2_{\mathcal{V}^*} |\hat{t}'(s)| \, \mathrm{d}s \\ &= \|\hat{t}'\|_{L^\infty(0,S)} \int_0^T \|\ell(t)\|^2_{\mathcal{V}^*} \, \mathrm{d}t = \|\hat{t}'\|_{L^\infty(0,S)} \|\ell\|^2_{L^2(0,T;\mathcal{V}^*)}, \end{aligned} \tag{3.3.14}$$

which gives the claim.

In order to prove (3.3.12), let $v \in \mathcal{V}$ be arbitrarily given. By means of compact embedding, we obtain the uniform convergence

$$\hat{t}_\epsilon \to \hat{t} \text{ in } C([0,S])$$

such that $\hat{t}_\epsilon(s_\epsilon) \to \hat{t}(s)$. The continuous embedding $H^1(0,T;\mathcal{V}^*) \hookrightarrow C^{0,\frac{1}{2}}([0,T];\mathcal{V}^*)$ gives pointwise weak convergence $\ell_\epsilon(t) \rightharpoonup \ell(t)$ in $\mathcal{V}^*$ and

$$\|\ell_\epsilon\|_{C^{0,\frac{1}{2}}([0,T];\mathcal{V}^*)} \le \tilde{C}\|\ell_\epsilon\|_{H^1(0,T;\mathcal{V}^*)} \le C$$

since $(\ell_\epsilon)_\epsilon$ converges weakly and is thus bounded. Therefore, we have

$$\begin{aligned} |\langle \hat{\ell}_\epsilon(s_\epsilon) - \hat{\ell}(s), v\rangle_{\mathcal{V}^*,\mathcal{V}}| &= |\langle \ell_\epsilon(\hat{t}_\epsilon(s_\epsilon)) - \ell(\hat{t}(s)), v\rangle_{\mathcal{V}^*,\mathcal{V}}| \\ &\le |\langle \ell_\epsilon(\hat{t}_\epsilon(s_\epsilon)) - \ell_\epsilon(\hat{t}(s)), v\rangle_{\mathcal{V}^*,\mathcal{V}}| + |\langle \ell_\epsilon(\hat{t}(s)) - \ell(\hat{t}(s)), v\rangle_{\mathcal{V}^*,\mathcal{V}}| \\ &\le \|\ell_\epsilon(\hat{t}_\epsilon(s_\epsilon)) - \ell_\epsilon(\hat{t}(s))\|_{\mathcal{V}^*}\|v\|_{\mathcal{V}} + |\langle \ell_\epsilon(\hat{t}(s)) - \ell(\hat{t}(s)), v\rangle_{\mathcal{V}^*,\mathcal{V}}| \\ &\le C|\hat{t}_\epsilon(s_\epsilon) - \hat{t}(s)|^{\frac{1}{2}}\|v\|_{\mathcal{V}} + |\langle \ell_\epsilon(\hat{t}(s)) - \ell(\hat{t}(s)), v\rangle_{\mathcal{V}^*,\mathcal{V}}| \\ &\to 0, \text{ as } \epsilon \searrow 0, \end{aligned}$$

which is (3.3.12). Due to (3.3.13) and (3.3.14), the sequence $(\hat{\ell}_\epsilon)_\epsilon$ is uniformly bounded in $H^1(0,S;\mathcal{V}^*)$. Thus, it exists a subsequence converging weakly to a limit $\tilde{\ell} \in H^1(0,S;\mathcal{V}^*)$, which in turn yields pointwise weak convergence in $\mathcal{V}^*$. The already proven pointwise convergence (3.3.12) implies that $\tilde{\ell}$ has to coincide with $\ell \circ \hat{t}$, which gives convergence for the entire sequence as claimed in (3.3.11). □

In the same way one can show that $\hat{z}_\epsilon \in H^1(0,S_\epsilon;\mathcal{Z})$ and that the derivative is given by

$$\hat{z}_\epsilon'(s) = z_\epsilon'(\hat{t}_\epsilon(s))\hat{t}_\epsilon'(s) \tag{3.3.15}$$

for almost all $s \in (0,S_\epsilon)$. Therefore, by exploiting the positive 1-homogeneity of $\mathcal{R}$ in combination with (3.3.7), we infer that for almost all $s \in (0,S_\epsilon)$

$$\begin{aligned} &\mathfrak{p}\big(\hat{z}_\epsilon'(s), -\mathrm{D}_z\mathcal{I}(\ell(\hat{t}_\epsilon(s)), \hat{z}_\epsilon(s))\big) \\ = \;& \mathcal{R}(\hat{z}_\epsilon'(s)) + \|\hat{z}_\epsilon'(s)\|_{\mathcal{V}} \operatorname{dist}_{\mathcal{V}^*}(-\mathrm{D}_z\mathcal{I}(\ell(\hat{t}_\epsilon(s)), \hat{z}_\epsilon(s)), \partial\mathcal{R}(0)) \\ = \;& \hat{t}_\epsilon'(s)\Big(\mathcal{R}(z_\epsilon'(\hat{t}_\epsilon(s))) + \|z_\epsilon'(\hat{t}_\epsilon(s))\|_{\mathcal{V}} \operatorname{dist}_{\mathcal{V}^*}(-\mathrm{D}_z\mathcal{I}(\ell(\hat{t}_\epsilon(s)), \hat{z}_\epsilon(s)), \partial\mathcal{R}(0))\Big) \\ = \;& \hat{t}_\epsilon'(s)\mathfrak{p}\big(z_\epsilon'(\hat{t}_\epsilon(s)), -\mathrm{D}_z\mathcal{I}(\ell(\hat{t}_\epsilon(s)), \hat{z}_\epsilon(s))\big) \end{aligned}$$

such that the equality from (3.3.6) yields

$$\hat{t}_\epsilon'(s) + \mathfrak{p}\big(\hat{z}_\epsilon'(s), -\mathrm{D}_z\mathcal{I}(\ell(\hat{t}_\epsilon(s)), \hat{z}_\epsilon(s))\big) = 1 \quad \text{f.a.a. } s \in (0,S_\epsilon) \tag{3.3.16}$$

As well as (3.3.6), this gives a uniform $W^{1,\infty}$-bound for time functions $\hat{t}_\epsilon$. Therefore, we can extract a weakly* converging subsequence and the limit will be part of the so-called parametrized solution. But concerning the state functions we do not have a similar a priori bound. The parametrized functions $\hat{z}_\epsilon$ only satisfy the uniform $W^{1,1}$-bound from (3.2.12) such that the limit of a subsequence will not be weakly differentiable anymore, which makes it difficult to formulate an energy dissipation balance like (3.2.7) for the parametrized limit. Nevertheless, the boundedness of $\mathcal{R}(\hat{z}'_\epsilon(\cdot))$ from (3.3.16) will be sufficient. This leads to the concept of $\mathcal{R}$-absolutely continuous functions and generalized metric derivatives, which we introduce in the following.

Definition 3.3.2. Let $1 \leq p \leq \infty$. Then we define the set of all p-absolutely continuous functions with respect to $\mathcal{R}$ by

$$\begin{aligned}\mathrm{AC}^p([0,T];\mathcal{R}) := \Big\{ z : [0,T] \to \mathcal{Z} : \exists\, m \in L^p(0,T), m \geq 0 \text{ a.e. with} \\ \mathcal{R}(z(t)-z(s)) \leq \int_s^t m(r)\,\mathrm{d}r \;\forall\, 0 \leq s < t \leq T \Big\}.\end{aligned} \tag{3.3.17}$$

Proposition 3.3.3. *Let $z \in \mathrm{AC}^p([0,T];\mathcal{R})$ be given. Then the limit*

$$\mathcal{R}[z'](t) := \lim_{h \searrow 0} \mathcal{R}\bigg(\frac{z(t+h)-z(t)}{h}\bigg) = \lim_{h \searrow 0} \mathcal{R}\bigg(\frac{z(t)-z(t-h)}{h}\bigg) \tag{3.3.18}$$

exists for almost all $t \in (0,T)$ and is called generalized metric derivative of z in t. Furthermore, the mapping $t \mapsto \mathcal{R}[z'](t)$ belongs to $L^p(0,T)$ and fulfills the condition in (3.3.17) *with minimality, which means, if there exists another $m \in L^p(0,T)$ satisfiying* (3.3.17)*, then $\mathcal{R}[z'](t) \leq m(t)$ for almost all $t \in (0,T)$.*

A proof of this can be found in [RMS08, Prop. 2.2]. One directly sees by exploiting the continuity of $\mathcal{R}$ on $\mathcal{V}$ that

$$\mathcal{R}[z'](t) = \lim_{h \searrow 0} \mathcal{R}\bigg(\frac{z(t+h)-z(t)}{h}\bigg) = \mathcal{R}(z'(t)) \tag{3.3.19}$$

if $z : [0,S] \to \mathcal{V}$ is classically differentiable in t.

In anticipation of the following limit analysis of the sequence $(\hat{t}_\epsilon, \hat{z}_\epsilon)_{\epsilon>0}$, we already define:

Definition 3.3.4 (Parametrized balanced viscosity (BV) solution)**.** For given data $z_0 \in \mathcal{Z}$ and $\ell \in H^1(0,T;\mathcal{V}^*)$ we call a triple $(S,\hat{t},\hat{z}) \in [T,\infty) \times W^{1,\infty}(0,T) \times \mathrm{AC}^\infty([0,S];\mathcal{R}) \cap L^\infty(0,S;\mathcal{Z})$ a *normalized, $\mathfrak{p}$-parametrized balanced viscosity (BV) solution* of the rate-independent system (RIS) if the set

$$\begin{aligned} G &:= \big\{ s \in [0,S] : \mathrm{dist}_{\mathcal{V}^*}(-\mathrm{D}_z\mathcal{I}(\hat{\ell}(s),\hat{z}(s)), \partial\mathcal{R}(0)) > 0 \big\} \\ &= \big\{ s \in [0,S] : -\mathrm{D}_z\mathcal{I}(\hat{\ell}(s),\hat{z}(s)) \notin \partial\mathcal{R}(0) \big\} \end{aligned} \tag{3.3.20}$$

is a relatively open subset of $[0,S]$ and

$$\begin{aligned} \hat{z} &\in W^{1,1}_{loc}(G;\mathcal{V}),\\ \mathrm{D}_z\mathcal{I}(\hat{\ell}(\cdot),\hat{z}(\cdot)) &\in L^\infty_{loc}(G;\mathcal{V}^*),\\ \|\hat{z}'(\cdot)\|_{\mathcal{V}}\,\mathrm{dist}_{\mathcal{V}^*}(-\mathrm{D}_z\mathcal{I}(\hat{\ell}(\cdot),\hat{z}(\cdot)),\partial\mathcal{R}(0)) &\in L^\infty(G;\mathbb{R}), \end{aligned}$$

where $\hat{\ell} := \ell\circ\hat{t} : [0,S]\to\mathcal{V}^*$. Moreover, the following conditions shall be satisfied:
Initial and end time condition:

$$\hat{z}(0)=z_0,\quad \hat{t}(0)=0,\quad \hat{t}(S)=T, \tag{3.3.21}$$

Complementarity condition:

$$\hat{t}'(s)\geq 0\quad \text{f.a.a. } s\in(0,S), \tag{3.3.22}$$

$$\hat{t}'(s)\,\mathrm{dist}_{\mathcal{V}^*}(-\mathrm{D}_z\mathcal{I}(\hat{\ell}(s),\hat{z}(s)),\partial\mathcal{R}(0))=0\quad \text{f.a.a. } s\in(0,S) \tag{3.3.23}$$

Normalization condition: For almost all $s\in(0,S)$ it holds

$$1=\begin{cases}\hat{t}'(s)+\mathcal{R}[\hat{z}'](s)+\|\hat{z}'(s)\|_{\mathcal{V}}\,\mathrm{dist}_{\mathcal{V}^*}(-\mathrm{D}_z\mathcal{I}(\hat{\ell}(s),\hat{z}(s)),\partial\mathcal{R}(0)), & \text{if } s\in G\\ \hat{t}'(s)+\mathcal{R}[\hat{z}'](s), & \text{if } s\in[0,S]\setminus G\end{cases} \tag{3.3.24}$$

Energy identity: For all $s\in[0,S]$ it holds

$$\begin{aligned}\mathcal{I}(\hat{\ell}(s),\hat{z}(s))+\int_0^s\mathcal{R}[\hat{z}'](r)\,\mathrm{d}r+\int_{(0,s)\cap G}\|\hat{z}'(r)\|_{\mathcal{V}}\,\mathrm{dist}_{\mathcal{V}^*}(-\mathrm{D}_z\mathcal{I}(\hat{\ell}(r),\hat{z}(r)),\partial\mathcal{R}(0))\,\mathrm{d}r\\ =\mathcal{I}(\hat{\ell}(0),z_0)-\int_0^s\langle\hat{\ell}'(r),\hat{z}(r)\rangle_{\mathcal{V}^*,\mathcal{V}}\,\mathrm{d}r.\end{aligned} \tag{3.3.25}$$

In view of the normalization condition (3.3.24), we can distinguish between three different regimes for normalized, $\mathfrak{p}$-parametrized balanced viscosity solution.

(i) *Sticking:* If $\hat{z}$ is constant, and thus $\mathcal{R}[\hat{z}']=0$ and $\hat{t}'=1$ almost everywhere, the potential force is to weak in order to change the state of the system.

(ii) *Rate-independent slip:* Here both the state and the time change, i.e., $\mathcal{R}[\hat{z}']>0$ and $\hat{t}'>1$, but in a way that the dissipation is strong enough to compensate the external force. According to the complementarity condition (3.3.23), the distance vanishes, which in turn yields that the state is locally stable and can be seen in physical time.

(iii) *Viscous jump:* In this case, the physical time stands still and the state is not locally stable, i.e., $\hat{t}'=0$ and $\mathrm{dist}_{\mathcal{V}^*}(-\mathrm{D}_z\mathcal{I}(\hat{\ell}(\cdot),\hat{z}(\cdot)),\partial\mathcal{R}(0))>0$ almost everywhere. The state changes from one region of local stability to another, which describes a jump in physical time.

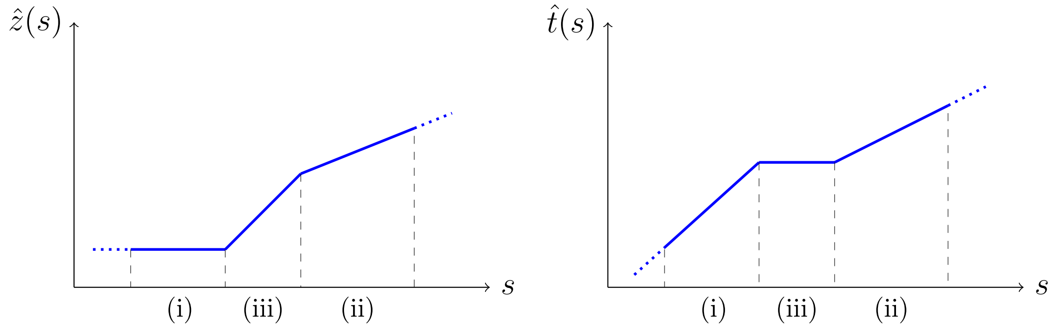

Figure 3.1: Different regimes of a normalized, $\mathfrak{p}$-parametrized balanced viscosity solution. (i) Sticking, (ii) rate-independent slip, (iii) viscous jump.

Lemma 3.3.5. *Assume that $\ell_n \rightharpoonup \ell$ in $\mathcal{V}^*$ and $z_n \rightharpoonup z$ in $\mathcal{Z}$ as $n \to \infty$ and $\sup_{n\in\mathbb{N}} \|\mathrm{D}_z\mathcal{I}(\ell_n, z_n)\|_{\mathcal{V}^*} < \infty$, then*

$$\mathrm{D}_z\mathcal{I}(\ell_n, z_n) \rightharpoonup \mathrm{D}_z\mathcal{I}(\ell, z) \text{ in } \mathcal{V}^*, \tag{3.3.26}$$

$$\mathrm{dist}_{\mathcal{V}^*}(-\mathrm{D}_z\mathcal{I}(\ell, z), \partial\mathcal{R}(0)) \leq \liminf_{n\in\mathbb{N}} \mathrm{dist}_{\mathcal{V}^*}(-\mathrm{D}_z\mathcal{I}(\ell_n, z_n), \partial\mathcal{R}(0)). \tag{3.3.27}$$

Proof. The assumptions on the energy, especially $A \in \mathcal{L}(\mathcal{Z}, \mathcal{Z}^*)$ and the weak continuity of $\mathrm{D}_z\mathcal{F}$ from (1.2.6) yield

$$Az_n + \mathrm{D}_z\mathcal{F}(z_n) \rightharpoonup Az + \mathrm{D}_z\mathcal{F}(z) \text{ in } \mathcal{Z}^* \tag{3.3.28}$$

such that exploiting the weak convergence $\ell_n \rightharpoonup \ell$ gives

$$\mathrm{D}_z\mathcal{I}(\ell_n, z_n) \rightharpoonup \mathrm{D}_z(\ell, z) \text{ in } \mathcal{Z}^*.$$

Eventually, using the boundedness of $\mathrm{D}_z\mathcal{I}(\ell_n, z_n)$ with respect to the $\mathcal{V}^*$-norm by assumption leads to a subsequence converging weakly in $\mathcal{V}^*$. Due to the uniqueness of weak limits, the limit has to be $\mathrm{D}_z\mathcal{I}(\ell, z)$ and we end up with (3.3.26) for the entire sequence. Finally, (3.3.27) follows from (3.3.26), since the distance is continuous and convex on $\mathcal{V}^*$ and thus weakly lower semicontinuous. □

Now we have everything together to prove the existence of a parametrized balanced viscosity solutions by the vanishing viscosity approach.

Theorem 3.3.6. *Let $(z_\epsilon)_{\epsilon>0}$ be a sequence of solutions of* (RIS$_\epsilon$) *associated with $\ell_\epsilon \in H^1(0,T;\mathcal{V}^*)$ and fixed initial value $z_0 \in \mathcal{Z}$. Moreover, let $S_\epsilon, \hat{t}_\epsilon, \hat{z}_\epsilon$ be defined by* (3.3.4), (3.3.5), (3.3.8) *and the loads are supposed to satisfy $\ell_\epsilon \rightharpoonup \ell$ in $H^1(0,T;\mathcal{V}^*)$ as $\epsilon \searrow 0$. Then there exists a subsequence $(\epsilon_n)_{n\in\mathbb{N}}$ converging to zero and a limit $(S, \hat{t}, \hat{z})$ with $S \in [T, \infty)$, $\hat{t} \in W^{1,\infty}(0,S)$ and $\hat{z} \in \mathrm{AC}^\infty([0,S];\mathcal{R}) \cap L^\infty(0,S;\mathcal{Z}) \cap C([0,S];\mathcal{V}) \cap C_{weak}([0,S];\mathcal{Z})$ such that*

$$S_{\epsilon_n} \to S, \tag{3.3.29}$$

$$\hat{t}_{\epsilon_n} \rightharpoonup^* \hat{t} \text{ in } W^{1,\infty}(0,S), \quad \hat{t}_{\epsilon_n} \to \hat{t} \text{ in } C([0,S]), \tag{3.3.30}$$
$$\hat{z}_{\epsilon_n} \rightharpoonup^* \hat{z} \text{ in } L^\infty(0,S;\mathcal{Z}), \quad \hat{z}_{\epsilon_n} \to \hat{z} \text{ in } C([0,S];\mathcal{V}), \tag{3.3.31}$$
$$\hat{z}_{\epsilon_n}(s_n) \rightharpoonup \hat{z}(s) \text{ in } \mathcal{Z} \text{ for all } s_n \to s, \tag{3.3.32}$$
$$\hat{z}_{\epsilon_n} \rightharpoonup^* \hat{z} \text{ in } W^{1,\infty}(a,b;\mathcal{V}) \tag{3.3.33}$$

for every connected component $[a,b] \subset G = \{s \in [0,S] : \mathrm{dist}_{\mathcal{V}^*}(-\mathrm{D}_z\mathcal{I}(\hat{\ell}(s),\hat{z}(s)), \partial\mathcal{R}(0)) > 0\}$*. Here the functions* $\hat{t}_{\epsilon_n}$*,* $\hat{z}_{\epsilon_n}$ *are constantly extended if* $S_{\epsilon_n} < S$*. Moreover, the limit* $(S,\hat{t},\hat{z})$ *is a normalized,* $\mathfrak{p}$*-parametrized BV solution associated with* ℓ *in the sense of Definition 3.3.4 and satisfies the pointwise uniform bounds*

$$\sup_{s\in[0,S]} \|\mathrm{D}_z\mathcal{I}(\hat{\ell}(s),\hat{z}(s))\|_{\mathcal{V}^*} < \infty, \tag{3.3.34}$$
$$\sup_{s\in[0,S]} \|\hat{z}(s)\|_{\mathcal{Z}} < \infty. \tag{3.3.35}$$

In the proof we follow the lines of [Tho22], where the existence of normalized, $\mathfrak{p}$-parametrized BV solutions is shown for fixed $\ell \in W^{1,\infty}(0,T;\mathcal{V}^*)$. Since we only consider loads $\ell \in H^1(0,T;\mathcal{V}^*)$ and additionally assumed a converging sequence of loads, too, we have to adjust the parts, where the reduced regularity and the convergence must be taken into account. Because the proof is pretty comprehensive, we split it into different steps.

Proof. **Step 1: Extracting a converging subsequence**
We start with proving the boundedness of the sequence $(S_\epsilon, \hat{t}_\epsilon, \hat{z}_\epsilon)$ in order to extract a (weakly*) converging subsequence satisfying (3.3.29)-(3.3.33). From the definition of S_ϵ, (3.2.8) and (3.3.3) we deduce

$$\begin{aligned} S_\epsilon &= T + \int_0^T \mathfrak{p}(z_\epsilon'(r), -\mathrm{D}_z\mathcal{I}(\ell_\epsilon(r), z_\epsilon(r)))\,\mathrm{d}r \\ &\leq T + \int_0^T \mathcal{R}_\epsilon(z_\epsilon'(r)) + \mathcal{R}_\epsilon^*(-\mathrm{D}_z\mathcal{I}(\ell_\epsilon(r), z_\epsilon(r)))\,\mathrm{d}r \\ &\leq T + c_0 + C\Big(\mathcal{I}(\ell_\epsilon(0), z_0) + \lambda\kappa T + \|\ell_\epsilon\|^2_{H^1(0,T,\mathcal{V}^*)}\Big). \end{aligned}$$

Since $\ell_\epsilon \rightharpoonup \ell$ in $H^1(0,T;\mathcal{V}^*)$ by assumption, we have that $\|\ell_\epsilon\|_{H^1(0,T,\mathcal{V}^*)} \leq C$. Moreover, $\ell_\epsilon(0)$ and thus $\mathcal{I}(\ell_\epsilon(0), z_0)$ is also bounded due to the continuous embedding $H^1(0,T;\mathcal{V}^*) \hookrightarrow C([0,T];\mathcal{V}^*)$. Therefore, S_ϵ is bounded independent of ϵ and we can extract a converging subsequence such that $S_{\epsilon_n} \to S$, as $\epsilon_n \searrow 0$. Let $\bar{S} = \sup_{\epsilon>0} S_\epsilon \in [T,\infty)$. Then we extend $\hat{t}_{\epsilon_n}$ and $\hat{z}_{\epsilon_n}$ constantly on $[0,\bar{S}]$ such that all these functions live on the same time horizon. By $\hat{t}_{\epsilon_n}(0) = 0$ and (3.3.7), we infer that $\|\hat{t}_{\epsilon_n}\|_{W^{1,\infty}(0,\bar{S})} \leq C$. So there exists another subsequence (also denoted by ϵ_n) and a limit $\hat{t} \in W^{1,\infty}(0,\bar{S})$ with $\hat{t}_{\epsilon_n} \rightharpoonup^* \hat{t}$ in $W^{1,\infty}(0,\bar{S})$. Due to compact embedding, we additionally obtain the uniform convergence in (3.3.30). By means of (3.2.9), we have $A := \sup_{n\in\mathbb{N}} \sup_{z\in[0,\bar{S}]} \|\hat{z}_{\epsilon_n}\|_{\mathcal{Z}} < \infty$ and from (3.3.16) and (3.3.19) we infer that $B := \sup_{n\in\mathbb{N}} \|\mathcal{R}[\hat{z}_{\epsilon_n}']\|_{L^\infty(0,\bar{S})} < \infty$. Hence, Lemma B.4 is applicable and we can extract a further subsequence (also denoted by ϵ_n) satisfying (3.3.31) and (3.3.32). In addition to that, we also deduce from Lemma B.4 that the limit $\hat{z}$ is an element of $C_{weak}([0,S];\mathcal{Z}) \cap \mathrm{AC}^\infty([0,S];\mathcal{R})$ and satisfies $\sup_{s\in[0,S]} \|\hat{z}(s)\|_{\mathcal{Z}} < \infty$. Lastly, due to continuous embedding, the weak convergence of $(\ell_{\epsilon_n})_n$ implies uniform pointwise boundedness independent of ϵ_n such that (3.2.14) gives

$$\sup_{n\in\mathbb{N}} \|\mathrm{D}_z\mathcal{I}(\ell_{\epsilon_n}(\hat{t}_{\epsilon_n}(s)), \hat{z}_{\epsilon_n}(s))\|_{\mathcal{V}^*} \leq C \tag{3.3.36}$$

for all $s \in [0,S]$. Hence, Lemma 3.3.5 is applicable and we obtain by means of (3.3.12)

$$\mathrm{D}_z\mathcal{I}(\hat{\ell}_{\epsilon_n}(s), \hat{z}_{\epsilon_n}(s)) \rightharpoonup \mathrm{D}_z\mathcal{I}(\hat{\ell}(s), \hat{z}(s)) \text{ in } \mathcal{V}^*, \tag{3.3.37}$$

$$\begin{aligned} &\mathrm{dist}_{\mathcal{V}^*}(-\mathrm{D}_z\mathcal{I}(\hat{\ell}(s), \hat{z}(s)), \partial\mathcal{R}(0)) \\ &\qquad \leq \liminf_{n\to\infty} \mathrm{dist}_{\mathcal{V}^*}(-\mathrm{D}_z\mathcal{I}(\hat{\ell}_{\epsilon_n}(s), \hat{z}_{\epsilon_n}(s)), \partial\mathcal{R}(0)) \end{aligned} \tag{3.3.38}$$

for all $s \in [0,S]$. Moreover, due to the weak lower semicontinuity of the norm, the pointwise bound with respect to the $\mathcal{V}^*$-norm (3.3.36) transfers to the limit, which is (3.3.34).

Step 2: Relative openness of G
Assuming that G is not open in $[0,S]$ yields the existence of $s \in G$ and a sequence $s_k \to s$, $k \to \infty$, with $s_k \notin G$, i.e., $\mathrm{dist}_{\mathcal{V}^*}(-\mathrm{D}_z\mathcal{I}(\hat{\ell}(s_k), \hat{z}(s_k)), \partial\mathcal{R}(0)) = 0$ for all $k \in \mathbb{N}$. From the continuity of $\hat{\ell}$ and the fact that $\hat{z} \in C_{weak}([0,S];\mathcal{Z})$ we infer

$$\hat{\ell}(s_k) \to \hat{\ell}(s) \text{ in } \mathcal{V}^*, \quad \hat{z}(s_k) \rightharpoonup \hat{z}(s) \text{ in } \mathcal{Z}$$

such that Lemma 3.3.5 together with (3.3.34) leads to

$$\mathrm{dist}_{\mathcal{V}^*}(-\mathrm{D}_z\mathcal{I}(\hat{\ell}(s), \hat{z}(s)), \partial\mathcal{R}(0)) \leq \liminf_{k\in\mathbb{N}} \mathrm{dist}_{\mathcal{V}^*}(-\mathrm{D}_z\mathcal{I}(\hat{\ell}(s_k), \hat{z}(s_k)), \partial\mathcal{R}(0)) = 0,$$

which contradicts $s \in G$.

Step 3: Initial and end time condition
The uniform convergence of $(\hat{z}_{\epsilon_n})_n$ (3.3.31) implies

$$z_0 = \hat{z}_{\epsilon_n}(0) \to \hat{z}(0) \text{ in } \mathcal{V},\ n \to \infty.$$

By exploiting the uniform convergence of $(\hat{t}_{\epsilon_n})_n$ from (3.3.30) together with (3.3.29), we infer

$$0 = \hat{t}_{\epsilon_n}(0) \to \hat{t}(0) \quad \text{and} \quad T = \hat{t}_{\epsilon_n}(S_\epsilon) \to \hat{t}(S).$$

Step 4: Improved regularity of $\hat{z}$ in G
Next, we want to prove (3.3.33). To this end, let a connected component $[a,b] \subset G$ be given and we define

$$c := \inf_{s\in[a,b]} \operatorname{dist}_{\mathcal{V}^*}(-\mathrm{D}_z\mathcal{I}(\hat{\ell}(s),\hat{z}(s)),\partial\mathcal{R}(0)).$$

Then $c > 0$, which can be seen as follows. Let $(s_k)_{k\in\mathbb{N}} \subset [a,b]$ be a sequence that realizes the infimum. Since $[a,b]$ is compact, there exists a (not relabeled) subsequence and a limit $s \in [a,b]$ with $s_k \to s$. Thanks to the continuity of $\hat{\ell}$ and $\hat{z} \in C_{weak}([0,S];\mathcal{Z})$, we deduce $\hat{\ell}(s_k) \to \hat{\ell}(s)$ and $\hat{z}(s_k) \rightharpoonup \hat{z}(s)$ in $\mathcal{Z}$ such that Lemma 3.3.5 yields

$$\begin{aligned} c &= \inf_{s\in[a,b]} \operatorname{dist}_{\mathcal{V}^*}(-\mathrm{D}_z\mathcal{I}(\hat{\ell}(s),\hat{z}(s)),\partial\mathcal{R}(0)) = \lim_{k\to\infty} \operatorname{dist}_{\mathcal{V}^*}(-\mathrm{D}_z\mathcal{I}(\hat{\ell}(s_k),\hat{z}(s_k)),\partial\mathcal{R}(0)) \\ &= \liminf_{k\to\infty} \operatorname{dist}_{\mathcal{V}^*}(-\mathrm{D}_z\mathcal{I}(\hat{\ell}(s_k),\hat{z}(s_k)),\partial\mathcal{R}(0)) \geq \operatorname{dist}_{\mathcal{V}^*}(-\mathrm{D}_z\mathcal{I}(\hat{\ell}(s),\hat{z}(s)),\partial\mathcal{R}(0)) > 0 \end{aligned}$$

because $s \in [a,b] \subset G$. Another use of Lemma 3.3.5, particularly the weak lower semicontinuity of the distance, together with (3.3.36) and (3.3.37) leads to

$$\liminf_{n\to\infty} \operatorname{dist}_{\mathcal{V}^*}(-\mathrm{D}_z\mathcal{I}(\hat{\ell}_{\epsilon_n}(s),\hat{z}_{\epsilon_n}(s)),\partial\mathcal{R}(0)) \geq \operatorname{dist}_{\mathcal{V}^*}(-\mathrm{D}_z\mathcal{I}(\hat{\ell}(s),\hat{z}(s)),\partial\mathcal{R}(0)) \geq c$$

for all $s \in [a,b]$. Therefore, we have

$$\forall s \in [a,b]\ \exists n_0 \in \mathbb{N}\ \forall n \geq n_0 : \ \operatorname{dist}_{\mathcal{V}^*}(-\mathrm{D}_z\mathcal{I}(\hat{\ell}_{\epsilon_n}(s),\hat{z}_{\epsilon_n}(s)),\partial\mathcal{R}(0)) \geq \frac{c}{2}$$

In fact, this $n_0 \in \mathbb{N}$ can be chosen independently of $s \in [a,b]$, which can be seen as follows. We assume that

$$\forall n_0 \in \mathbb{N}\ \exists n > n_0\ \exists s \in [a,b] : \ \operatorname{dist}_{\mathcal{V}^*}(-\mathrm{D}_z\mathcal{I}(\hat{\ell}_{\epsilon_n}(s),\hat{z}_{\epsilon_n}(s)),\partial\mathcal{R}(0)) < \frac{c}{2},$$

which yields after transition to a further subsequence the existence of a sequence $s_{n_k} \to s \in [a,b]$ with

$$\operatorname{dist}_{\mathcal{V}^*}(-\mathrm{D}_z\mathcal{I}(\hat{\ell}_{\epsilon_{n_k}}(s_{n_k}),\hat{z}_{\epsilon_{n_k}}(s_{n_k})),\partial\mathcal{R}(0)) < \frac{c}{2} \text{ for all } k \in \mathbb{N}.$$

By (3.3.12) and (3.3.32), respectively, we have that $\hat{\ell}_{\epsilon_{n_k}}(s_{n_k}) \rightharpoonup \hat{\ell}(s)$ in $\mathcal{V}^*$ and $\hat{z}_{\epsilon_{n_k}}(s_{n_k}) \rightharpoonup \hat{z}(s)$ in $\mathcal{Z}$ such that using Lemma 3.3.5 leads to

$$\begin{aligned} &\operatorname{dist}_{\mathcal{V}^*}(-\mathrm{D}_z\mathcal{I}(\hat{\ell}(s),\hat{z}(s)),\partial\mathcal{R}(0)) \\ &\qquad \leq \liminf_{k\to\infty} \operatorname{dist}_{\mathcal{V}^*}(-\mathrm{D}_z\mathcal{I}(\hat{\ell}_{\epsilon_{n_k}}(s_{n_k}),\hat{z}_{\epsilon_{n_k}}(s_{n_k})),\partial\mathcal{R}(0)) < \frac{c}{2}, \end{aligned}$$

which contradicts $s \in [a,b] \subset G$ and the definition of c. Thus, we have shown

$$\exists n_0 \in \mathbb{N}\ \forall n \geq n_0\ \forall s \in [a,b] : \operatorname{dist}_{\mathcal{V}^*}(-\mathrm{D}_z\mathcal{I}(\hat{\ell}_{\epsilon_n}(s),\hat{z}_{\epsilon_n}(s)),\partial\mathcal{R}(0)) \geq \frac{c}{2}$$

such that (3.3.16) yields

$$
\begin{aligned}
\frac{c}{2}\|\hat{z}'_{\epsilon_n}(s)\|_{\mathcal{V}} \leq \hat{t}'_{\epsilon_n}(s) + \mathcal{R}(\hat{z}'_{\epsilon_n}(s)) \\
+ \|\hat{z}'_{\epsilon_n}(s)\|_{\mathcal{V}} \operatorname{dist}_{\mathcal{V}^*}(-\mathrm{D}_z\mathcal{I}(\hat{\ell}_{\epsilon_n}(s), \hat{z}_{\epsilon_n}(s)), \partial\mathcal{R}(0)) = 1
\end{aligned}
\tag{3.3.39}
$$

for all $s \in [a, b]$ and all $n \geq n_0$, where we used the non-negativity of $\hat{t}'_{\epsilon_n}$ and $\mathcal{R}$. From this and (3.2.9) we conclude that for every subsequence it exists a further subsequence converging weakly* in $W^{1,\infty}(a, b; \mathcal{V})$ and the limit has to be $\hat{z}$ because of the already known convergence to $\hat{z}$ with respect to the weaker topology. Hence, the entire sequence converges to $\hat{z}$, which is (3.3.33). Since the bound in (3.3.39) depends on the chosen component of G, $\hat{z}_{\epsilon_n}$ is not uniformly bounded in $W^{1,\infty}(G; \mathcal{V})$. Nevertheless, according to the representation theorem for open sets, cf. [Ber98, Lem. 5.7.1], $G \setminus \{0, S\}$ can be written as a union of countable many pairwise disjoint open intervals, i.e.,

$$
G \setminus \{0, S\} = \bigcup_{k\in\mathbb{N}} (a_k, b_k),
\tag{3.3.40}
$$

and we define the compact sets $G_N \subset G \setminus \{0, S\}$ by

$$
G_N := \bigcup_{k=1}^{N} \left[a_k + \frac{1}{Nk^2}, b_k - \frac{1}{Nk^2} \right].
\tag{3.3.41}
$$

Obviously, they satisfy $G_N \subset G_{N+1}$ and $\lambda\big((G \setminus \{0, S\}) \setminus G_N\big) \leq \frac{2}{N} \sum_{k=1}^{N} \frac{1}{k^2} \to 0$, as $N \to \infty$. Thus, G_N is an exhausting sequence of $G \setminus \{0, S\}$, and due to the previous considerations $\hat{z} \in W^{1,\infty}(G_N; \mathcal{V})$ for all $N \in \mathbb{N}$ and $\hat{z}'$ exists almost everywhere in G, which gives $\hat{z} \in W^{1,\infty}_{loc}(G; \mathcal{V})$.

Step 5: Complementary condition

Firstly, the boundedness of $\hat{t}'_{\epsilon_n}$, i.e., $0 \leq \hat{t}'_{\epsilon_n}(s) \leq 1$ for all $s \in [0, \bar{S}]$, from (3.3.7) transfers to the weak* limit $\hat{t}'$ by means of the Banach-Alaoglu theorem, which is (3.3.22). Next, from (3.2.8) it follows

$$
\int_0^T \mathcal{R}^*_{\epsilon_n}(-\mathrm{D}_z\mathcal{I}(\ell_{\epsilon_n}(r), z'_{\epsilon_n}(r)))\,\mathrm{d}r \leq C
$$

with $C > 0$ independent of ϵ_n since $\|\ell_{\epsilon_n}\|_{H^1(0,T;\mathcal{V}^*)}$ is uniformly bounded according to the assumed weak convergence of $(\ell_{\epsilon_n})_n$. Eventually, by the representation of $\mathcal{R}^*_{\epsilon_n}$ via the distance by (A.2) and applying the formula of change of variables, the above estimate transfers to

$$
\int_0^{S_{\epsilon_n}} \hat{t}'_{\epsilon_n}(r) \operatorname{dist}_{\mathcal{V}^*}(-\mathrm{D}_z\mathcal{I}(\hat{\ell}_{\epsilon_n}(r), \hat{z}_{\epsilon_n}(r)), \partial\mathcal{R}(0))^2\,\mathrm{d}r \leq 2\epsilon_n C.
\tag{3.3.42}
$$

In order to verify that the left hand side converges to the expected limit, we first define for a given $\kappa > 0$

$$\begin{aligned} f_n^\kappa(s) &:= \min\{\mathrm{dist}_n(s), \mathrm{dist}(s), \kappa\}, \\ f^\kappa(s) &:= \min\{\mathrm{dist}(s), \kappa\} \end{aligned}$$

with the abbreviations

$$\begin{aligned} \mathrm{dist}_n(s) &:= \mathrm{dist}_{\mathcal{V}^*}(-\mathrm{D}_z\mathcal{I}(\hat{\ell}_{\epsilon_n}(s), \hat{z}_{\epsilon_n}(s)), \partial\mathcal{R}(0)), \\ \mathrm{dist}(s) &:= \mathrm{dist}_{\mathcal{V}^*}(-\mathrm{D}_z\mathcal{I}(\hat{\ell}(s), \hat{z}(s)), \partial\mathcal{R}(0)). \end{aligned}$$

Then the pointwise convergence

$$f_n^\kappa(s) \to f^\kappa(s) \text{ for all } s \in [0, \bar{S}] \tag{3.3.43}$$

holds because, if for given $s \in [0, \bar{S}]$ the minimum in f_n^κ is not attained in $\mathrm{dist}_n(s)$ for all $n \in \mathbb{N}$ except for finitely many, then the assertion follows directly. Hence, consider the indices n_k, where $f_{n_k}^\kappa(s) = \mathrm{dist}_{n_k}(s)$. Then Lemma 3.3.5 yields

$$\liminf_{k\to\infty} f_{n_k}^\kappa(s) = \liminf_{k\to\infty} \mathrm{dist}_{n_k}(s) \geq \mathrm{dist}(s) = f^\kappa(s).$$

On the other hand, we have

$$\limsup_{k\to\infty} f_{n_k}^\kappa(s) \leq \limsup_{k\to\infty} f^\kappa(s) = f^\kappa(s),$$

such that combining this leads to $\limsup_{k\to\infty} f_{n_k}^\kappa(s) \leq f(s) \leq \liminf_{k\to\infty} f_{n_k}^\kappa(s)$ and, thus, (3.3.43) holds true. Because the functions f_n^κ are also uniformly bounded by $g \equiv \kappa \in L^\infty(0, \bar{S})$ in addition to the pointwise convergence (3.3.43), Lebesgue's dominated convergence theorem yields

$$f_n^\kappa \to f^\kappa \text{ in } L^1(0, \bar{S}).$$

With this at hand and the weak* convergence $\hat{t}'_{\epsilon_n} \rightharpoonup^* \hat{t}'$ from (3.3.30), we deduce from (3.3.42)

$$\begin{aligned} 0 &\geq \liminf_{n\to\infty} \int_0^{S_{\epsilon_n}} \hat{t}'_{\epsilon_n}(r)\,\mathrm{dist}_{\mathcal{V}^*}(-\mathrm{D}_z\mathcal{I}(\hat{\ell}_{\epsilon_n}(r), \hat{z}_{\epsilon_n}(r)), \partial\mathcal{R}(0))^2\,\mathrm{d}r \\ &\geq \liminf_{n\to\infty} \int_0^{S_{\epsilon_n}} \hat{t}'_{\epsilon_n}(r) f_n^\kappa(r)\,\mathrm{d}r = \int_0^S \hat{t}'(r) f^\kappa(r)\,\mathrm{d}r. \end{aligned}$$

Finally, we drive $\kappa \to \infty$ such that $f^\kappa(s) \to \mathrm{dist}_{\mathcal{V}^*}(-\mathrm{D}_z\mathcal{I}(\hat{\ell}(s), \hat{z}(s)), \partial\mathcal{R}(0))$ for almost all $s \in [0, \bar{S}]$ since the distance is bounded according to (3.3.34). Thus, Fatou's lemma gives

$$0 \geq \lim_{\kappa\to\infty} \int_0^S \hat{t}'(r) f^\kappa(r)\,\mathrm{d}r \geq \int_0^S \hat{t}'(r)\,\mathrm{dist}_{\mathcal{V}^*}(-\mathrm{D}_z\mathcal{I}(\hat{\ell}(r), \hat{z}(r)), \partial\mathcal{R}(0))\,\mathrm{d}r$$

and the non-negativity of the integrand implies (3.3.23).

Step 6: Lower estimate in the energy equality
First of all, the representation of $\mathcal{R}^*_{\epsilon_n}$ via the distance from (A.2) and applying Young's inequality yields for almost all $s \in [0, S_{\epsilon_n}]$

$$\begin{aligned} &\|\hat{z}'_{\epsilon_n}(s)\|_{\mathcal{V}} \operatorname{dist}_{\mathcal{V}^*}(-\mathrm{D}_z\mathcal{I}(\hat{\ell}_{\epsilon_n}(s), \hat{z}_{\epsilon_n}(s)), \partial\mathcal{R}(0)) \\ \leq &\frac{\epsilon_n}{2\hat{t}'_{\epsilon_n}(s)}\|\hat{z}'_{\epsilon_n}(s)\|^2_{\mathcal{V}} + \frac{\hat{t}'_{\epsilon_n}(s)}{2\epsilon_n} \operatorname{dist}_{\mathcal{V}^*}(-\mathrm{D}_z\mathcal{I}(\hat{\ell}_{\epsilon_n}(s), \hat{z}_{\epsilon_n}(s)), \partial\mathcal{R}(0))^2 \\ = &\frac{\epsilon_n}{2\hat{t}'_{\epsilon_n}(s)}\|\hat{z}'_{\epsilon_n}(s)\|^2_{\mathcal{V}} + \hat{t}'_{\epsilon_n}(s)\mathcal{R}^*_{\epsilon_n}\big(-\mathrm{D}_z\mathcal{I}(\hat{\ell}_{\epsilon_n}(s), \hat{z}_{\epsilon_n}(s))\big). \end{aligned} \tag{3.3.44}$$

Note that the right hand side is well defined since $\hat{t}'_{\epsilon_n}(s) > 0$ for almost all $s \in [0, S_{\epsilon_n}]$ by (3.3.7). In order to prove (3.3.25), let $s \in [0, S]$ be arbitrarily given. We start from the energy identity (3.2.7) in physical time and choose $t := \hat{t}_{\epsilon_n}(s)$ in order to obtain by means of substitution, (3.3.10), (3.3.15) and $\hat{z}_{\epsilon_n}(0) = z_0$

$$\begin{aligned} &\mathcal{I}(\hat{\ell}_{\epsilon_n}(0), \hat{z}_0) - \mathcal{I}(\hat{\ell}_{\epsilon_n}(s), \hat{z}_{\epsilon_n}(s)) = \mathcal{I}(\ell_{\epsilon_n}(0), z_{\epsilon_n}(0)) - \mathcal{I}(\ell_{\epsilon_n}(\hat{t}_{\epsilon_n}(s)), z_{\epsilon_n}(\hat{t}_{\epsilon_n}(s))) \\ &= \int_0^{\hat{t}_{\epsilon_n}(s)} \mathcal{R}_{\epsilon_n}(z'_{\epsilon_n}(r)) + \mathcal{R}^*_{\epsilon_n}(-\mathrm{D}_z\mathcal{I}(\ell_{\epsilon_n}(r), z_{\epsilon_n}(r)))\mathrm{d}r + \int_0^{\hat{t}_{\epsilon_n}(s)} \langle \ell'_{\epsilon_n}(r), z_{\epsilon_n}(r)\rangle_{\mathcal{V}^*,\mathcal{V}}\,\mathrm{d}r \\ &= \int_0^s \hat{t}'_{\epsilon_n}(r)\Big(\mathcal{R}_{\epsilon_n}(z'_{\epsilon_n}(\hat{t}_{\epsilon_n}(r))) + \mathcal{R}^*_{\epsilon_n}\Big(-\mathrm{D}_z\mathcal{I}\big(\ell_{\epsilon_n}(\hat{t}_{\epsilon_n}(r)), z_{\epsilon_n}(\hat{t}_{\epsilon_n}(r))\big)\Big)\Big)\mathrm{d}r \\ &\qquad\qquad + \int_0^s \hat{t}'_{\epsilon_n}(r)\langle \ell'_{\epsilon_n}(\hat{t}_{\epsilon_n}(r)), z_{\epsilon_n}(\hat{t}_{\epsilon_n}(r))\rangle_{\mathcal{V}^*,\mathcal{V}}\,\mathrm{d}r \\ &= \int_0^s \mathcal{R}(\hat{z}'_{\epsilon_n}(r)) + \frac{\epsilon_n}{2\hat{t}'_{\epsilon_n}(r)}\|\hat{z}'_{\epsilon_n}(r)\|^2_{\mathcal{V}} + \hat{t}'_{\epsilon_n}(r)\mathcal{R}^*_{\epsilon_n}\big(-\mathrm{D}_z\mathcal{I}(\hat{\ell}_{\epsilon_n}(r), \hat{z}_{\epsilon_n}(r))\big)\mathrm{d}r \\ &\qquad\qquad + \int_0^s \langle \hat{\ell}'_{\epsilon_n}(r), \hat{z}_{\epsilon_n}(r)\rangle_{\mathcal{V}^*,\mathcal{V}}\,\mathrm{d}r, \end{aligned}$$

where we used the 1-homogeneity of $\mathcal{R}$ from (1.2.14) and the fact that $\hat{t}'_{\epsilon_n}(s) > 0$ almost everywhere, cf. (3.3.7), in the last step. Now, inserting estimate (3.3.44) and rearranging the terms results in

$$\begin{aligned} \mathcal{I}(\hat{\ell}_{\epsilon_n}(s), \hat{z}_{\epsilon_n}(s)) + \int_0^s \mathcal{R}(\hat{z}'_{\epsilon_n}(r)) + \|\hat{z}'_{\epsilon_n}(r)\|_{\mathcal{V}} \operatorname{dist}_{\mathcal{V}^*}(-\mathrm{D}_z\mathcal{I}(\hat{\ell}_{\epsilon_n}(r), \hat{z}_{\epsilon_n}(r)), \partial\mathcal{R}(0))\,\mathrm{d}r \\ \leq \mathcal{I}(\hat{\ell}_{\epsilon_n}(0), \hat{z}_{\epsilon_n}(0)) - \int_0^s \langle \hat{\ell}'_{\epsilon_n}(r), \hat{z}_{\epsilon_n}(r)\rangle_{\mathcal{V}^*,\mathcal{V}}\,\mathrm{d}r. \end{aligned}$$

We consider each term of this energy inequality separately and start with the integral on the left hand side. The lower semicontinuity of the $\mathcal{R}$-variation and its representation via the metric derivative from Lemma B.2 leads to

$$\begin{aligned} &\int_0^s \mathcal{R}[\hat{z}'](r)\mathrm{d}r = \operatorname{Var}_{\mathcal{R}}(\hat{z}; [0, s]) \\ &\qquad\qquad \leq \liminf_{n\to\infty} \operatorname{Var}_{\mathcal{R}}(\hat{z}_{\epsilon_n}; [0, s]) = \liminf_{n\to\infty} \int_0^s \mathcal{R}(\hat{z}'_{\epsilon_n}(r))\mathrm{d}r, \end{aligned} \tag{3.3.45}$$

where we used Lemma B.3 in the last step. Regarding the next term, analogously to (3.3.41), we define an exhausting sequence of $G \cap (0, s)$ by

$$G_N := \bigcup_{k=1}^{N} I_{k,N} := \bigcup_{k=1}^{N} \Big[a_k + \frac{1}{Nk^2}, b_k - \frac{1}{Nk^2} \Big].$$

With (3.3.33) and (3.3.38) at hand, we infer from Lemma C.5

$$\begin{aligned} &\int_{I_{k,N}} \|\hat{z}'(r)\|_{\mathcal{V}} \operatorname{dist}_{\mathcal{V}^*}(-\mathrm{D}_z\mathcal{I}(\hat{\ell}(r), \hat{z}(r)), \partial\mathcal{R}(0)) \,\mathrm{d}r \\ &\le \liminf_{n\to\infty} \int_{I_{k,N}} \|\hat{z}'_{\epsilon_n}(r)\|_{\mathcal{V}} \operatorname{dist}_{\mathcal{V}^*}(-\mathrm{D}_z\mathcal{I}(\hat{\ell}_{\epsilon_n}(r), \hat{z}_{\epsilon_n}(r)), \partial\mathcal{R}(0)) \,\mathrm{d}r \\ &\le \int_{I_{k,N}} 1 \,\mathrm{d}r, \end{aligned} \tag{3.3.46}$$

where the last estimate followed from (3.3.39). Because this holds for all $I_{k,N} \subset G$ and all intervals $I \subset I_{k,N}$, too, we have

$$\|\hat{z}'(r)\|_{\mathcal{V}} \operatorname{dist}_{\mathcal{V}^*}(-\mathrm{D}_z\mathcal{I}(\hat{\ell}(r), \hat{z}(r)), \partial\mathcal{R}(0)) \le 1 \text{ f.a.a. } r \in G. \tag{3.3.47}$$

Since the intervals $I_{k,N}$ are pairwise disjoint, we deduce from the first estimate in (3.3.46)

$$\begin{aligned} &\int_{G_N} \|\hat{z}'(r)\|_{\mathcal{V}} \operatorname{dist}_{\mathcal{V}^*}(-\mathrm{D}_z\mathcal{I}(\hat{\ell}(r), \hat{z}(r)), \partial\mathcal{R}(0)) \,\mathrm{d}r \\ &\le \liminf_{n\to\infty} \int_{G_N} \|\hat{z}'_{\epsilon_n}(r)\|_{\mathcal{V}} \operatorname{dist}_{\mathcal{V}^*}(-\mathrm{D}_z\mathcal{I}(\hat{\ell}_{\epsilon_n}(r), \hat{z}_{\epsilon_n}(r)), \partial\mathcal{R}(0)) \,\mathrm{d}r \\ &\le \liminf_{n\to\infty} \int_0^s \|\hat{z}'_{\epsilon_n}(r)\|_{\mathcal{V}} \operatorname{dist}_{\mathcal{V}^*}(-\mathrm{D}_z\mathcal{I}(\hat{\ell}_{\epsilon_n}(r), \hat{z}_{\epsilon_n}(r)), \partial\mathcal{R}(0)) \,\mathrm{d}r, \end{aligned}$$

where we exploited the non-negativity of the integrand in the last step. Eventually, the exhausting property of G_N yields the pointwise convergence

$$\begin{aligned} &\chi_{G_N}(r)\|\hat{z}'(r)\|_{\mathcal{V}} \operatorname{dist}_{\mathcal{V}^*}(-\mathrm{D}_z\mathcal{I}(\hat{\ell}(r), \hat{z}(r)), \partial\mathcal{R}(0)) \\ &\qquad\qquad \to \|\hat{z}'(r)\|_{\mathcal{V}} \operatorname{dist}_{\mathcal{V}^*}(-\mathrm{D}_z\mathcal{I}(\hat{\ell}(r), \hat{z}(r)), \partial\mathcal{R}(0)) \end{aligned}$$

almost everywhere in $G \cap (0, s)$ such that together with (3.3.47) it follows from Lebesgue's dominated convergence theorem

$$\begin{aligned} &\int_{(0,s)\cap G} \|\hat{z}'(r)\|_{\mathcal{V}} \operatorname{dist}_{\mathcal{V}^*}(-\mathrm{D}_z\mathcal{I}(\hat{\ell}(r), \hat{z}(r)), \partial\mathcal{R}(0)) \,\mathrm{d}r \\ &= \lim_{N\to\infty} \int_{G_N} \|\hat{z}'(r)\|_{\mathcal{V}} \operatorname{dist}_{\mathcal{V}^*}(-\mathrm{D}_z\mathcal{I}(\hat{\ell}(r), \hat{z}(r)), \partial\mathcal{R}(0)) \,\mathrm{d}r \\ &\le \liminf_{n\to\infty} \int_0^s \|\hat{z}'_{\epsilon_n}(r)\|_{\mathcal{V}} \operatorname{dist}_{\mathcal{V}^*}(-\mathrm{D}_z\mathcal{I}(\hat{\ell}_{\epsilon_n}(r), \hat{z}_{\epsilon_n}(r)), \partial\mathcal{R}(0)) \,\mathrm{d}r. \end{aligned} \tag{3.3.48}$$

Next, the pointwise strong convergence in $\mathcal{V}$ and weak convergence in $\mathcal{Z}$ of $\hat{z}_{\epsilon_n}$ from (3.3.31) and (3.3.32), respectively, together with the weak continuity of $\mathcal{F}$ from (1.2.8) and the pointwise weak convergence of $\hat{\ell}_{\epsilon_n}$ by (3.3.12) leads to

$$\mathcal{I}(\hat{\ell}(s), \hat{z}(s)) \le \liminf_{n\to\infty} \mathcal{I}(\hat{\ell}_{\epsilon_n}(s), \hat{z}_{\epsilon_n}(s)),$$
$$\mathcal{I}(\hat{\ell}(0), z_0) = \lim_{n\to\infty} \mathcal{I}(\hat{\ell}_{\epsilon_n}(0), \hat{z}_{\epsilon_n}(0)).$$

Lastly, from (3.3.11) and (3.3.31) we deduce

$$\int_0^s \langle \hat{\ell}'(r), \hat{z}(r)\rangle_{\mathcal{V}^*,\mathcal{V}} \,\mathrm{d}r = \lim_{n\to\infty} \int_0^s \langle \hat{\ell}'_{\epsilon_n}(r), \hat{z}_{\epsilon_n}(r)\rangle_{\mathcal{V}^*,\mathcal{V}} \,\mathrm{d}r.$$

All in all, we end up with

$$\begin{aligned}
&\mathcal{I}(\hat{\ell}(s), \hat{z}(s)) + \int_0^s \mathcal{R}[\hat{z}'](r)\,\mathrm{d}r + \int_{(0,s)\cap G} \|\hat{z}'(r)\|_{\mathcal{V}} \,\mathrm{dist}_{\mathcal{V}^*}(-\mathrm{D}_z\mathcal{I}(\hat{\ell}(r), \hat{z}(r)), \partial\mathcal{R}(0))\,\mathrm{d}r\\
&\le \liminf_{n\to\infty} \bigg(\mathcal{I}(\hat{\ell}_{\epsilon_n}(s), \hat{z}_{\epsilon_n}(s)) + \int_0^s \mathcal{R}(\hat{z}'_{\epsilon_n}(r))\\
&\qquad\qquad + \|\hat{z}'_{\epsilon_n}(r)\|_{\mathcal{V}} \,\mathrm{dist}_{\mathcal{V}^*}(-\mathrm{D}_z\mathcal{I}(\hat{\ell}_{\epsilon_n}(r), \hat{z}_{\epsilon_n}(r)), \partial\mathcal{R}(0))\,\mathrm{d}r\bigg)\\
&\le \limsup_{n\to\infty} \bigg(\mathcal{I}(\hat{\ell}_{\epsilon_n}(0), \hat{z}_{\epsilon_n}(0)) - \int_0^s \langle \hat{\ell}'_{\epsilon_n}(r), \hat{z}_{\epsilon_n}(r)\rangle_{\mathcal{V}^*,\mathcal{V}}\,\mathrm{d}r\bigg)\\
&= \mathcal{I}(\hat{\ell}(0), z_0) - \int_0^s \langle \hat{\ell}'(r), \hat{z}(r)\rangle_{\mathcal{V}^*,\mathcal{V}}\,\mathrm{d}r,
\end{aligned} \tag{3.3.49}$$

which gives "$\le$" in (3.3.25).

Step 7: Upper estimate in the energy equality
In order to prove the opposite inequality, we follow the lines of [KT23]. According to Lemma A.4, for all $r \in [0, S]$ let $\mu(r) \in \partial\mathcal{R}(0)$ such that

$$\| - \mathrm{D}_z\mathcal{I}(\hat{\ell}(r), \hat{z}(r)) - \mu(r)\|_{\mathcal{V}^*} = \mathrm{dist}(-\mathrm{D}_z\mathcal{I}(\hat{\ell}(r), \hat{z}(r)), \partial\mathcal{R}(0)). \tag{3.3.50}$$

Moreover, for $r \in [0, S)$ and $0 < h < h_r$ with $h_r > 0$ sufficiently small such that $r + h_r \in [0, S]$, we define $\Delta_h\hat{z}(r) := \hat{z}(r + h) - \hat{z}(r)$. Then estimate (1.2.24) with $z_1 = \hat{z}(r)$ and $z_2 = \hat{z}(r + h)$, which fulfill the required boundedness in Lemma 1.2.6 due to (3.3.35), implies

$$\begin{aligned}
\mathcal{I}(\hat{\ell}(r), \hat{z}(r + h)) - \mathcal{I}(\hat{\ell}(r), \hat{z}(r)) \ge{}& \langle \mathrm{D}_z\mathcal{I}(\hat{\ell}(r), \hat{z}(r)) + \mu(r), \Delta_h\hat{z}(r)\rangle_{\mathcal{Z}^*,\mathcal{Z}}\\
&- \langle \mu(r), \Delta_h\hat{z}(r)\rangle_{\mathcal{Z}^*,\mathcal{Z}} - C\mathcal{R}(\Delta_h\hat{z}(r))\|\Delta_h\hat{z}(r)\|_{\mathcal{V}}.
\end{aligned}$$

Next, rearranging the terms and exploiting (3.3.50) and (1.2.18) yields

$$\langle \hat{\ell}(r), \Delta_h\hat{z}(r)\rangle_{\mathcal{V}^*,\mathcal{V}} \le \mathcal{E}(\hat{z}(r + h)) - \mathcal{E}(\hat{z}(r)) + \langle -\mathrm{D}_z\mathcal{I}(\hat{\ell}(r), \hat{z}(r)) - \mu(r), \Delta_h\hat{z}(r)\rangle_{\mathcal{Z}^*,\mathcal{Z}}$$

$$
\begin{aligned}
&+ \langle \mu(r), \Delta_h \hat{z}(r)\rangle_{\mathcal{Z}^*,\mathcal{Z}} + C\mathcal{R}(\Delta_h \hat{z}(r))\|\Delta_h \hat{z}(r)\|_{\mathcal{V}} \\
\le\ & \mathcal{E}(\hat{z}(r+h)) - \mathcal{E}(\hat{z}(r)) + \|\Delta_h \hat{z}(r)\|_{\mathcal{V}}\| - \mathrm{D}_z\mathcal{I}(\hat{\ell}(r),\hat{z}(r)) - \mu(r)\|_{\mathcal{V}^*} \\
&+ \mathcal{R}(\Delta_h \hat{z}(r)) + C\mathcal{R}(\Delta_h \hat{z}(r))\|\Delta_h \hat{z}(r)\|_{\mathcal{V}} \\
\le\ & \mathcal{E}(\hat{z}(r+h)) - \mathcal{E}(\hat{z}(r)) + \|\Delta_h \hat{z}(r)\|_{\mathcal{V}} \operatorname{dist}(-\mathrm{D}_z\mathcal{I}(\hat{\ell}(r),\hat{z}(r)), \partial\mathcal{R}(0)) \\
&+ (1 + C\|\Delta_h \hat{z}(r)\|_{\mathcal{V}})\mathcal{R}(\Delta_h \hat{z}(r)).
\end{aligned}
$$

From now on let $(s_1, s_2) \subset [0, S]$ be an arbitrary interval and $h > 0$ is chosen such that $s_2 + h \le S$, which is not a restriction as we want to pass $h \searrow 0$ later. In the above inequality we divide both sides by $h > 0$ and integrate over (s_1, s_2) to obtain

$$
\begin{aligned}
\frac{1}{h}\int_{s_1}^{s_2} \langle \hat{\ell}(r), \Delta_h \hat{z}(r)\rangle_{\mathcal{V}^*,\mathcal{V}} \,\mathrm{d}r \le\ & \frac{1}{h}\int_{s_1}^{s_2} \mathcal{E}(\hat{z}(r+h)) - \mathcal{E}(\hat{z}(r)) \,\mathrm{d}r \\
&+ \int_{s_1}^{s_2} \Big\|\frac{1}{h}\Delta_h \hat{z}(r)\Big\|_{\mathcal{V}} \operatorname{dist}(-\mathrm{D}_z\mathcal{I}(\hat{\ell}(r),\hat{z}(r)), \partial\mathcal{R}(0)) \,\mathrm{d}r \\
&+ \int_{s_1}^{s_2} (1 + C\|\Delta_h \hat{z}(r)\|_{\mathcal{V}})\mathcal{R}\Big(\frac{1}{h}\Delta_h \hat{z}(r)\Big) \mathrm{d}r.
\end{aligned}
\tag{3.3.51}
$$

Now we pass $h \searrow 0$ and consider each term separately. We start with integral on the left hand side. Thanks to

$$
\begin{aligned}
\langle \hat{\ell}(r+h), \hat{z}(r+h)\rangle_{\mathcal{V}^*,\mathcal{V}} - \langle \hat{\ell}(r), \hat{z}(r)\rangle_{\mathcal{V}^*,\mathcal{V}} =\ & \langle \hat{\ell}(r+h) - \hat{\ell}(r), \hat{z}(r+h)\rangle_{\mathcal{V}^*,\mathcal{V}} \\
&+ \langle \hat{\ell}(r), \hat{z}(r+h) - \hat{z}(r)\rangle_{\mathcal{V}^*,\mathcal{V}},
\end{aligned}
$$

we can rewrite it as

$$
\begin{aligned}
\frac{1}{h}\int_{s_1}^{s_2} \langle \hat{\ell}(r), \Delta_h \hat{z}(r)\rangle_{\mathcal{V}^*,\mathcal{V}} \,\mathrm{d}r =\ & \frac{1}{h}\int_{s_1}^{s_2} \langle \hat{\ell}(r+h), \hat{z}(r+h)\rangle_{\mathcal{V}^*,\mathcal{V}} - \langle \hat{\ell}(r), \hat{z}(r)\rangle_{\mathcal{V}^*,\mathcal{V}} \,\mathrm{d}r \\
&- \int_{s_1}^{s_2} \frac{1}{h}\langle \Delta_h \hat{\ell}(r), \hat{z}(r+h)\rangle_{\mathcal{V}^*,\mathcal{V}} \,\mathrm{d}r \\
=\ & \frac{1}{h}\int_{s_2}^{s_2+h} \langle \hat{\ell}(r), \hat{z}(r)\rangle_{\mathcal{V}^*,\mathcal{V}} \,\mathrm{d}r - \frac{1}{h}\int_{s_1}^{s_1+h} \langle \hat{\ell}(r), \hat{z}(r)\rangle_{\mathcal{V}^*,\mathcal{V}} \,\mathrm{d}r \\
&- \int_{s_1}^{s_2} \frac{1}{h}\langle \Delta_h \hat{\ell}(r), \hat{z}(r+h)\rangle_{\mathcal{V}^*,\mathcal{V}} \,\mathrm{d}r.
\end{aligned}
$$

Since the mapping $s \mapsto \langle \hat{\ell}(r), \hat{z}(r)\rangle_{\mathcal{V}^*,\mathcal{V}} \in \mathbb{R}$ is continuous as $\hat{\ell}$ and $\hat{z}$ are continuous, the fundamental theorem of calculus yields

$$
\begin{aligned}
& \lim_{h\searrow 0} \frac{1}{h}\int_{s_2}^{s_2+h} \langle \hat{\ell}(r), \hat{z}(r)\rangle_{\mathcal{V}^*,\mathcal{V}} \,\mathrm{d}r - \frac{1}{h}\int_{s_1}^{s_1+h} \langle \hat{\ell}(r), \hat{z}(r)\rangle_{\mathcal{V}^*,\mathcal{V}} \,\mathrm{d}r \\
=\ & \langle \hat{\ell}(s_2), \hat{z}(s_2)\rangle_{\mathcal{V}^*,\mathcal{V}} - \langle \hat{\ell}(s_1), \hat{z}(s_1)\rangle_{\mathcal{V}^*,\mathcal{V}}.
\end{aligned}
$$

From Lemma C.4 we infer that $\frac{1}{h}\Delta_h \hat{\ell} \to \hat{\ell}'$ strongly in $L^2(0, S; \mathcal{V}^*)$. Together with $\hat{z}(\cdot + h) \to \hat{z}(\cdot)$ uniformly in $C([0, S]; \mathcal{V})$, as $h \searrow 0$, which follows directly from the fact that $\hat{z}$ is uniformly continuous as $\hat{z}$ is defined on a compact set, we obtain

$$\lim_{h\searrow 0}\int_{s_1}^{s_2}\frac{1}{h}\langle \Delta_h\hat{\ell}(r),\hat{z}(r+h)\rangle_{\mathcal{V}^*,\mathcal{V}}\,\mathrm{d}r=\int_{s_1}^{s_2}\langle\hat{\ell}'(r),\hat{z}(r)\rangle_{\mathcal{V}^*,\mathcal{V}}\,\mathrm{d}r.$$

Next, we show that the mapping $[0,S]\ni s\mapsto\mathcal{E}(\hat{z}(s))\in\mathbb{R}$ is continuous. Therefore, let $(s_n)_n\subset[0,S]$ be a converging sequence with a limit $s\in[0,S]$. Then the mean value theorem yields the existence of $\tau_n\in[0,1]$ such that

$$|\mathcal{E}(\hat{z}(s))-\mathcal{E}(\hat{z}(s_n))|=|\langle \mathrm{D}_z\mathcal{E}(\hat{z}(s)+\tau_n(\hat{z}(s_n)-\hat{z}(s)),\hat{z}(s)-\hat{z}(s_n)\rangle_{\mathcal{V}^*,\mathcal{V}}|.$$

We set $\xi_n=\hat{z}(s)+\tau_n(\hat{z}(s_n)-\hat{z}(s))$ and the fact that $\hat{z}\in C_{weak}([0,S];\mathcal{Z})$ gives $\xi_n\rightharpoonup\hat{z}(s)$ in $\mathcal{Z}$ such that exploiting (1.2.6) and the fact that $A\in\mathcal{L}(\mathcal{Z},\mathcal{Z}^*)$ results in

$$\mathrm{D}_z\mathcal{E}(\xi_n)\rightharpoonup\mathrm{D}_z\mathcal{E}(\hat{z}(s))\text{ in }\ \mathcal{Z}^*.$$

Additionally, $\mathrm{D}_z\mathcal{E}(\xi_n)$ is bounded in $\mathcal{V}^*$, which can be seen as follows. The concrete structure of $\mathcal{E}$ together with the linearity of A gives

$$\begin{aligned}\mathrm{D}_z\mathcal{E}(\xi_n)=\tau_n\mathrm{D}_z\mathcal{E}(\hat{z}(s))+(1-\tau_n)\mathrm{D}_z\mathcal{E}(\hat{z}(s_n))&\\ +\mathrm{D}_z\mathcal{F}(\xi_n)-\tau_n\mathrm{D}_z\mathcal{F}(\hat{z}(s))-(1-\tau_n)\mathrm{D}_z\mathcal{F}(\hat{z}(s_n))&\end{aligned}$$

From (3.3.34) we infer that the first two summands are bounded in $\mathcal{V}^*$, since $\hat{\ell}(s)$ is bounded, too, by continuous embedding. Similar to (3.2.1), by exploiting (1.2.5), we show that the other terms are also bounded in $\mathcal{V}^*$. Then by using this additional boundedness, we infer that even

$$\mathrm{D}_z\mathcal{E}(\xi_n)\rightharpoonup\mathrm{D}_z\mathcal{E}(\hat{z}(s))\text{ in }\mathcal{V}^* \tag{3.3.52}$$

because every subsequence of $\mathrm{D}_z\mathcal{E}(\xi_n)$ has a further subsequence that converges with respect to the weak topology on $\mathcal{V}^*$ and the limit has to be $\mathrm{D}_z\mathcal{E}(\hat{z}(s))$ due to the uniqueness of weak limits. Thus, the entire sequence has to converge as claimed. Then exploiting (3.3.52) and the continuity of $\hat{z}$ on $\mathcal{V}$ from (3.3.31) leads to

$$\begin{aligned}|\mathcal{E}(\hat{z}(s))-\mathcal{E}(\hat{z}(s_n))|&=|\langle\mathrm{D}_z\mathcal{E}(\xi_n),\hat{z}(s)-\hat{z}(s_n)\rangle_{\mathcal{V}^*,\mathcal{V}}|\\ &\to|\langle\mathrm{D}_z\mathcal{E}(\hat{z}(s)),0\rangle_{\mathcal{V}^*,\mathcal{V}}|=0,\ \text{ as } n\to\infty,\end{aligned}$$

which is the continuity of $\mathrm{D}_z\mathcal{E}(\hat{z}(\cdot))$. With this at hand, applying the fundamental theorem of calculus results in

$$\begin{aligned}\lim_{h\searrow 0}\frac{1}{h}\int_{s_1}^{s_2}\mathcal{E}(\hat{z}(r+h))-\mathcal{E}(\hat{z}(r))\,\mathrm{d}r&=\lim_{h\searrow 0}\frac{1}{h}\int_{s_2}^{s_2+h}\mathcal{E}(\hat{z}(r))\,\mathrm{d}r-\frac{1}{h}\int_{s_1}^{s_1+h}\mathcal{E}(\hat{z}(r))\,\mathrm{d}r\\ &=\mathcal{E}(\hat{z}(s_2))-\mathcal{E}(\hat{z}(s_1)).\end{aligned}$$

Next, since $\hat{z} \in \mathrm{AC}^\infty([0,S];\mathcal{R})$, the limit $\lim_{h\searrow 0}\mathcal{R}\big(\frac{1}{h}\Delta_h\hat{z}(r)\big) = \mathcal{R}[\hat{z}'](r)$ exists almost everywhere by Proposition 3.3.3. Further, $\|\Delta_h\hat{z}(r)\|_\mathcal{V} \to 0$, as $h \searrow 0$, because $\hat{z} \in C([0,S];\mathcal{V})$. Thus, we obtain the pointwise convergence

$$(1 + C\|\Delta_h\hat{z}(r)\|_\mathcal{V})\mathcal{R}\bigg(\frac{1}{h}\Delta_h\hat{z}(r)\bigg) \to \mathcal{R}[\hat{z}'](r).$$

Additionally, the integrand is also uniformly bounded due to the fact that $\hat{z} \in \mathrm{AC}^\infty([0,S];\mathcal{R})$, which can be seen by

$$\begin{aligned}(1 + C\|\Delta_h\hat{z}(r)\|_\mathcal{V})\mathcal{R}\bigg(\frac{1}{h}\Delta_h\hat{z}(r)\bigg) &\le \frac{1}{h}(1 + C\|\Delta_h\hat{z}(r)\|_\mathcal{V})\int_r^{r+h} m(\sigma)\,\mathrm{d}\sigma \\ &\le (1 + 2C\|\hat{z}\|_{L^\infty(0,S;\mathcal{V})})\|m\|_{L^\infty(0,S)} < \infty\end{aligned}$$

with $m \in L^\infty(0,S)$ from (3.3.17). Therefore, applying Lebesgue's dominated convergence theorem leads to

$$\lim_{h\searrow 0}\int_{s_1}^{s_2}(1 + C\|\Delta_h\hat{z}(r)\|_\mathcal{V})\mathcal{R}\bigg(\frac{1}{h}\Delta_h\hat{z}(r)\bigg)\mathrm{d}r = \int_{s_1}^{s_2}\mathcal{R}[\hat{z}'](r)\,\mathrm{d}r.$$

Since $\mathrm{dist}(-\mathrm{D}_z\mathcal{I}(\hat{\ell}(s),\hat{z}(s)),\partial\mathcal{R}(0)) = 0$ for almost all $s \in [0,S]\setminus G$, we have shown that passing $h \searrow 0$ in (3.3.51) yields

$$\begin{aligned}\langle\hat{\ell}(s_2),\hat{z}(s_2)\rangle_{\mathcal{V}^*,\mathcal{V}} - \langle\hat{\ell}(s_1),\hat{z}(s_1)\rangle_{\mathcal{V}^*,\mathcal{V}} &- \int_{s_1}^{s_2}\langle\hat{\ell}'(r),\hat{z}(r)\rangle_{\mathcal{V}^*,\mathcal{V}}\,\mathrm{d}r \\ &\le \mathcal{E}(\hat{z}(s_2)) - \mathcal{E}(\hat{z}(s_1)) + \int_{s_1}^{s_2}\mathcal{R}[\hat{z}'](r)\mathrm{d}r\end{aligned} \tag{3.3.53}$$

for all $[s_1,s_2] \subset [0,S]\setminus G$. Now let $(s_1,s_2) \subset G$ be a component of G, cf. (3.3.40), and $[s_1^n,s_2^n] \subset (s_1,s_2)$ a sequence of exhausting compact intervals. Then regarding the remaining integral in (3.3.51), we first observe that the integrand converges pointwisely almost everywhere in $[s_1^n,s_2^n]$ since $\hat{z} \in W^{1,\infty}_{loc}(G;\mathcal{V})$, which is

$$\begin{aligned}&\bigg\|\frac{1}{h}\Delta_h\hat{z}(s)\bigg\|_\mathcal{V}\mathrm{dist}(-\mathrm{D}_z\mathcal{I}(\hat{\ell}(s),\hat{z}(s)),\partial\mathcal{R}(0)) \\ &\qquad\to \|\hat{z}'(r)\|_\mathcal{V}\,\mathrm{dist}(-\mathrm{D}_z\mathcal{I}(\hat{\ell}(s),\hat{z}(s)),\partial\mathcal{R}(0)), \text{ as } h \searrow 0.\end{aligned}$$

Moreover, thanks to $\hat{z} \in W^{1,\infty}_{loc}(G;\mathcal{V})$, we have for all $h < \delta_n := \frac{s_2 - s_2^n}{2}$

$$\bigg\|\frac{1}{h}\Delta_h\hat{z}(s)\bigg\|_\mathcal{V} \le \frac{1}{h}\int_s^{s+h}\|\hat{z}'(r)\|_\mathcal{V}\,\mathrm{d}r \le \|\hat{z}'\|_{L^\infty(s_1^n,s_2^n+\delta_n;\mathcal{V})}$$

for all $s \in [s_1^n,s_2^n]$. Together with (3.3.34) along with Lemma 1.2.5 we obtain that the integrand is bounded by a $L^\infty(s_1^n,s_2^n;\mathbb{R})$-function. Hence, applying Lebesgue's dominated convergence theorem results in

$$\lim_{h\searrow 0}\int_{s_1^n}^{s_2^n}\bigg\|\frac{1}{h}\Delta_h\hat{z}(r)\bigg\|_\mathcal{V}\mathrm{dist}(-\mathrm{D}_z\mathcal{I}(\hat{\ell}(r),\hat{z}(r)),\partial\mathcal{R}(0))\,\mathrm{d}r$$

$$= \int_{s_1^n}^{s_2^n} \|\hat{z}'(r)\|_{\mathcal{V}} \operatorname{dist}(-\mathrm{D}_z\mathcal{I}(\hat{\ell}(r), \hat{z}(r)), \partial\mathcal{R}(0)) \,\mathrm{d}r.$$

Eventually, from (3.3.51) we infer

$$\begin{aligned} \langle \hat{\ell}(s_2^n), \hat{z}(s_2^n)\rangle_{\mathcal{V}^*,\mathcal{V}} - \langle \hat{\ell}(s_1^n), \hat{z}(s_1^n)\rangle_{\mathcal{V}^*,\mathcal{V}} - \int_{s_1^n}^{s_2^n} \langle \hat{\ell}'(r), \hat{z}(r)\rangle_{\mathcal{V}^*,\mathcal{V}} \,\mathrm{d}r \\ \le \mathcal{E}(\hat{z}(s_2^n)) - \mathcal{E}(\hat{z}(s_1^n)) + \int_{s_1^n}^{s_2^n} \mathcal{R}[\hat{z}'](r)\,\mathrm{d}r \\ + \int_{s_1^n}^{s_2^n} \|\hat{z}'(r)\|_{\mathcal{V}} \operatorname{dist}(-\mathrm{D}_z\mathcal{I}(\hat{\ell}(r), \hat{z}(r)), \partial\mathcal{R}(0)) \,\mathrm{d}r. \end{aligned} \tag{3.3.54}$$

Thanks to the continuity of the occurring terms, this holds for (s_1, s_2) instead of $[s_1^n, s_2^n]$, too. Finally, let $s \in [0, S)$ be given. Then except for a countable number of points the interval $[0, s]$ can be written as a countable union of disjoint open intervals, which are included in G, and remaining intervals in the complement of G. Then summing up the inequalities (3.3.53) and (3.3.54) for all these intervals results in

$$\begin{aligned} \mathcal{I}(\hat{\ell}(s), \hat{z}(s)) + \int_0^s \mathcal{R}[\hat{z}'](r)\mathrm{d}r + \int_{(0,s)\cap G} \|\hat{z}'(r)\|_{\mathcal{V}} \operatorname{dist}_{\mathcal{V}^*}(-\mathrm{D}_z\mathcal{I}(\hat{\ell}(r), \hat{z}(r)), \partial\mathcal{R}(0)) \,\mathrm{d}r \\ \ge \mathcal{I}(\hat{\ell}(0), z_0) - \int_0^s \langle \hat{\ell}'(r), \hat{z}(r)\rangle_{\mathcal{V}^*,\mathcal{V}} \,\mathrm{d}r. \end{aligned}$$

where we exploited the continuity of the energy mapping $s \mapsto \mathcal{I}(\hat{\ell}(s), \hat{z}(s)) = \mathcal{E}(\hat{z}(s)) - \langle \hat{\ell}(s), \hat{z}(s)\rangle_{\mathcal{V}^*,\mathcal{V}}$, which was already shown before, and the continuity of the occurring integrals. The continuity also yields that the inequality is valid for $s = S$ and we end up with "$\ge$" in (3.3.25).

Step 8: Normalization condition

Due to the equality in (3.3.25), the estimates in (3.3.49) are even valid with equality and the limit inferior is indeed a real limit. Therefore, in view of (3.3.45) and (3.3.48), we obtain from Lemma C.1 the improved convergences

$$\lim_{n\to\infty} \mathcal{I}(\hat{\ell}_{\epsilon_n}(s), \hat{z}_{\epsilon_n}(s)) = \mathcal{I}(\hat{\ell}(s), \hat{z}(s)) \tag{3.3.55}$$

and

$$\begin{aligned} \lim_{n\to\infty} \int_0^s \mathcal{R}(\hat{z}'_{\epsilon_n}(r)) + \|\hat{z}'_{\epsilon_n}(r)\|_{\mathcal{V}} \operatorname{dist}_{\mathcal{V}^*}(-\mathrm{D}_z\mathcal{I}(\hat{\ell}_{\epsilon_n}(r), \hat{z}_{\epsilon_n}(r)), \partial\mathcal{R}(0)) \,\mathrm{d}r \\ = \int_0^s \mathcal{R}[\hat{z}'](r)\,\mathrm{d}r + \int_{(0,s)\cap G} \|\hat{z}'(r)\|_{\mathcal{V}} \operatorname{dist}_{\mathcal{V}^*}(-\mathrm{D}_z\mathcal{I}(\hat{\ell}(r), \hat{z}(r)), \partial\mathcal{R}(0)) \,\mathrm{d}r \end{aligned} \tag{3.3.56}$$

Lastly, we want to prove (3.3.24). To this end, we insert (3.3.16) into (3.3.56) and exploit (3.3.30) in order to obtain

$$\begin{aligned}
&\int_0^s \mathcal{R}[\hat{z}'](r)\,\mathrm{d}r + \int_{(0,s)\cap G} \|\hat{z}'(r)\|_{\mathcal{V}} \operatorname{dist}_{\mathcal{V}^*}(-\mathrm{D}_z\mathcal{I}(\hat{\ell}(r),\hat{z}(r)),\partial\mathcal{R}(0))\,\mathrm{d}r \\
&= \lim_{n\to\infty}\int_0^s 1-\hat{t}'_{\epsilon_n}(r)\,\mathrm{d}r = \int_0^s 1-\hat{t}'(r)\,\mathrm{d}r.
\end{aligned}$$

Since this holds for all $s\in[0,S]$, the convention

$$\|\hat{z}'(r)\|_{\mathcal{V}} \operatorname{dist}_{\mathcal{V}^*}(-\mathrm{D}_z\mathcal{I}(\hat{\ell}(r),\hat{z}(r)),\partial\mathcal{R}(0)) := 0,\ \ r\in G$$

yields the pointwise equality

$$\mathcal{R}[\hat{z}'](r) + \|\hat{z}'(r)\|_{\mathcal{V}} \operatorname{dist}_{\mathcal{V}^*}(-\mathrm{D}_z\mathcal{I}(\hat{\ell}(r),\hat{z}(r)),\partial\mathcal{R}(0)) = 1-\hat{t}'(r)$$

for almost all $r\in[0,S]$, which is (3.3.24). □

Remark 3.3.7. Instead of using the above vanishing viscosity approach, the existence of parametrized balanced viscosity solutions can also be shown by applying a time discretization scheme. The procedure here is similar to that of (IMP) except that one has to search for local minimizer and take the parametrization into account. We refer here to [Kne19, Neg14, MS20, AM23] for more details.

Chapter 4

Optimal control of rate-independent systems

This chapter is concerned with optimal control problems governed by rate-independent systems. The optimization and optimal control of rate-independent systems has already been considered by various authors. Exemplary, we refer here to [Bro87, BK13, BK15, AO14, CHNM16, CP16, CEC18, Mü18, CCMN21], considering the sweeping process, and [Wac12, Wac15, Wac16, MMW20] dealing with an elastoplasticity framework. In contrast to this, a prototypical rate-independent system is for example considered in [SWW17, GW18]. What they all have in common is that they deal with a uniform convex energy. If the energy is not convex, the literature on this topic is rather sparse. The existence of optimal solutions in the case of a non-convex energy is addressed in [Rin09b, Ste12, EL14, KT23]. But we are also interested in the approximation of these optimal control problems by simplified problems. In [Rin09a, MR09] the authors use a time discretization approach to perform this approximation. However, there the underlying solution concept is that of global energetic solutions. More related to our work is the vanishing viscosity approach investigated in [MW21] for the notion of a quasi-static evolution for optimal control of perfect plasticity. Compared to the time discretization, this leads to optimal control problems, where the control-to-state map is single-valued since the viscous regularized system is uniquely solvable, cf. Lemma 3.2.1, which simplifies the solvability by numerical methods. In [KMS22] this idea has been extended to general prototypical rate-independent systems in finite dimensional spaces. Moreover, there the authors use the notion of $\mathfrak{p}$-parametrized balanced viscosity solutions as the underlying solution concept for the rate-independent system. We aim to transfer these results to our infinite dimensional setting.

To be more precise, in this chapter we investigate optimal control problems of the following form:

$$\left.\begin{aligned} \min \quad & J(S,\hat{z},\ell) := j(\hat{z}(S)) + \frac{\beta}{2}\,\|\ell\|^2_{H^1(0,T;\mathcal{V}^*)} \\ \text{s.t.} \quad & \ell \in H^1(0,T;\mathcal{V}^*), \quad (S,\hat{t},\hat{z}) \in \mathcal{L}(z_0,\ell), \\ & -\mathrm{D}_z\mathcal{I}(\ell(0),z_0) \in \partial\mathcal{R}(0), \quad -\mathrm{D}_z\mathcal{I}(\ell(T),\hat{z}(S)) \in \partial\mathcal{R}(0), \end{aligned}\right\} \tag{OCP}$$

where

$$\mathcal{L}(z_0,\ell) := \big\{(S,\hat{t},\hat{z}) \in [T,\infty) \times W^{1,\infty}(0,S) \times \mathrm{AC}^\infty([0,S];\mathcal{R}) \cap L^\infty(0,S;\mathcal{Z}) : \\ (S,\hat{t},\hat{z}) \text{ is a parametrized BV solution associated with } \ell\big\}.$$

Herein, the function $j : \mathcal{V} \to \mathbb{R}$ in the objective is supposed to be continuous and bounded from below, e.g., $j(z) = \|z - z_{des}\|^2_{\mathcal{V}}$ for a desired final state $z_{des} \in \mathcal{V}$, and $\beta > 0$ is a fixed Tikhonov parameter. That means we want to approximate the desired state as best as possible under minimal costs, which are described by means of the external load and weighted by $\beta > 0$. The final state and the load are linked due to the rate-independent system in the constraints, i.e., $\hat{z}$ is a parametrized solution associated with the load ℓ. Further, the initial condition in the constraints enforces the solution to be locally stable at the beginning. This will later be relevant in order to ensure the reverse approximation property, where we will construct a recovery sequence $(z_\epsilon,\ell_\epsilon)_{\epsilon>0}$ of feasible functions for the regularized optimal control problem whose objective value converges to that of a given solution of the original problem (OCP). Additionally, we require that in the end time the state is also locally stable, which means we do not end in a viscous jump such that the final state can be seen in physical time, which certainly makes sense from an application point of view. First, we clarify whether the problem (OCP) can be solved at all.

4.1 Existence of solutions of (OCP)

The existence of a globally optimal solution of (OCP) can be derived from the following compactness result concerning $\mathfrak{p}$-parametrized BV solutions from [KT23, Thm. 3.12]. For given $\rho > 0$ we define the set

$$M_\rho := \Big\{(S,\hat{t},\hat{z},\ell) : (S,\hat{t},\hat{z}) \in \mathcal{L}(z_0,\ell) \text{ and } (z_0,\ell) \text{ satisfies} \\ -\mathrm{D}_z\mathcal{I}(\ell(0),z_0) \in \mathcal{V}^* \text{ and } \|z_0\|_{\mathcal{Z}} + \|\ell\|_{H^1(0,T;\mathcal{V}^*)} \le \rho\Big\}. \tag{4.1.1}$$

Then M_ρ is compact in the following sense:

Theorem 4.1.1 ([KT23, Thm. 3.12])**.** *Let $z_0 \in \mathcal{Z}$ and $\rho > 0$ be given. Then for every sequence $(S_n,\hat{t}_n,\hat{z}_n,\ell_n)_{n\in\mathbb{N}} \subset M_\rho$ there exists a (not relabeled) subsequence and a limit $(S,\hat{t},\hat{z},\ell) \in M_\rho$ with*

$$S_n \to S \text{ in } \mathbb{R},\ \hat{t}_n \rightharpoonup^* \hat{t} \text{ in } W^{1,\infty}(0,S),\ \hat{t}(S) = T,\ \ell_n \rightharpoonup \ell \text{ in } H^1(0,T;\mathcal{V}^*), \tag{4.1.2}$$

$$\hat{z}_n \rightharpoonup^* \hat{z} \text{ in } L^\infty(0,S;\mathcal{Z}), \quad \hat{z}_n \to \hat{z} \text{ in } C([0,S];\mathcal{V}), \tag{4.1.3}$$
$$\hat{z}_n(S_n) \to \hat{z}(S) \text{ in } \mathcal{V}, \tag{4.1.4}$$
$$\mathrm{D}_z\mathcal{E}(\hat{z}_n(\cdot)) \rightharpoonup^* \mathrm{D}_z\mathcal{E}(\hat{z}(\cdot)) \text{ in } L^\infty(0,S;\mathcal{V}^*), \tag{4.1.5}$$

and for all $s \in [0,S]$ we have that

$$\hat{t}_n(s) \to \hat{t}(s) \text{ in } \mathbb{R}, \quad \hat{z}_n(s) \rightharpoonup \hat{z}(s) \text{ in } \mathcal{Z}, \quad \hat{z}_n(s) \to \hat{z}(s) \text{ in } \mathcal{V}, \tag{4.1.6}$$
$$\mathrm{D}_z\mathcal{E}(\hat{z}_n(s)) \rightharpoonup \mathrm{D}_z\mathcal{E}(\hat{z}(s)) \text{ in } \mathcal{V}^*. \tag{4.1.7}$$

Additionally, the mapping $s \mapsto \mathrm{D}_z\mathcal{E}(\hat{z}(s))$ is continuous with respect to the weak topology on $\mathcal{V}^*$.

The proof is similar to that dealing with the existence of $\mathfrak{p}$-parametrized BV solutions, which is Theorem 3.3.6, but requires substantial extensions of the analysis, in particular w.r.t. a priori estimates for $\mathfrak{p}$-parametrized BV solutions. Instead of considering solutions of the viscous regularized problem (RIS$_\epsilon$), we already start with a sequence of parametrized solutions. Hence, in order to extract a (weakly) converging subsequence, one needs a priori estimates for the parametrized solutions that correspond to those in section 3.2 and 3.3. With this at hand, one can derive the properties of a parametrized solution from Definition 3.3.4 for the limit functions similarly as in the proof of Theorem 3.3.6. Because this would be similarly extensive and thus exceeds the scope of this elaboration, we refer here to [KT23, Ch.3], where the authors have derived the required a priori estimates in detail and also presented a detailed proof of the compactness property.

Remark 4.1.2. In view of (4.1.4), we can even prove that

$$\mathrm{D}_z\mathcal{E}(\hat{z}_n(S_n)) \rightharpoonup \mathrm{D}_z\mathcal{E}(\hat{z}(S)) \text{ in } \mathcal{V}^*, \text{ as } n \to \infty. \tag{4.1.8}$$

The proof is similar to that concerning the convergence in (4.1.7). But instead of exploiting the weak convergence $\hat{z}_n(s) \rightharpoonup \hat{z}(s)$ in $\mathcal{Z}$, we use $\hat{z}_n(S_n) \rightharpoonup \hat{z}(S)$ in $\mathcal{Z}$, which follows by standard arguments from (4.1.4) in combination with the additional pointwise uniform boundedness of $\hat{z}_n$ in $\mathcal{Z}$ by (3.3.35). This enhanced convergence is important to ensure that the end time condition in (OCP) is also satisfied for the limit of a converging sequence of feasible parametrized solutions and loads, which will be used in the next theorem in order to prove existence of optimal solutions of (OCP).

Theorem 4.1.3. *Let $z_0 \in \mathcal{Z}$ and assume that* (1.2.26) *holds, i.e., there exists $\ell_0 \in \mathcal{V}^*$ such that $-\mathrm{D}_z\mathcal{I}(\ell_0, z_0) \in \partial\mathcal{R}(0)$. Then* (OCP) *has a globally optimal solution.*

Proof. First of all, the feasible set of (OCP) is non-empty because the constant function $z \equiv z_0$ together with $\hat{t} = id$ as the identity and $S = T$ is a parametrized solution associated with $\ell \equiv \ell_0$, i.e., $(S, \hat{t}, \hat{z}) \in \mathcal{L}(z_0, \ell)$. Thus, there is at least one feasible point. Now let $(S_n, \hat{t}_n, \hat{z}_n, \ell_n)_{n\in\mathbb{N}}$ be an infimal sequence, which is

$\lim_{n\to\infty} J(S_n, \hat{z}_n, \ell_n) = \inf J(S, \hat{z}, \ell) =: I$, where the infimum is considered over all feasible points of (OCP) and is finite since $j : \mathcal{V} \to \mathbb{R}$ is assumed to be bounded from below. Therefore, we have that $\sup_{n\in\mathbb{N}} \|\ell_n\|_{H^1(0,T;\mathcal{V}^*)} \le C$. Moreover, $-\mathrm{D}_z\mathcal{I}(\ell_n(0), z_0) \in \mathcal{V}^*$ for all $n \in \mathbb{N}$ is satisfied, too, which follows directly from $-\mathrm{D}_z\mathcal{I}(\ell_n(0), z_0) \in \partial\mathcal{R}(0)$ and Lemma 1.2.5. Hence, we are able to apply Theorem 4.1.1, which gives the existence of a converging (not relabeled) subsequence and a limit satisfying $(S, \hat{t}, \hat{z}) \in \mathcal{L}(z_0, \ell)$ and (4.1.2)-(4.1.7). From the continuous embedding $H^1(0,T;\mathcal{V}^*) \hookrightarrow C^{0,\frac{1}{2}}([0,T];\mathcal{V}^*)$ and the weak convergence $\ell_n \rightharpoonup \ell$ in $H^1(0,T;\mathcal{V}^*)$ we infer that ℓ_n converges pointwisely weakly, too. Especially, we have $\ell_n(0) \rightharpoonup \ell(0)$ in $\mathcal{V}^*$ such that $-\mathrm{D}_z\mathcal{I}(\ell(0), z_0) \in \partial\mathcal{R}(0)$ by exploiting the convexity and closedness of the subdifferential. In the same way we deduce from $\ell_n(T) \rightharpoonup \ell(T)$ in $\mathcal{V}^*$ and (4.1.8) that $-\mathrm{D}_z\mathcal{I}(\ell(T), \hat{z}(S)) \in \partial\mathcal{R}(0)$, too. Thus, the limit is feasible for (OCP). Thanks to the uniform convergence of $\hat{z}_n$ in $\mathcal{V}$ from (4.1.3), the supposed continuity of j and weak lower semicontinuity of the norm in combination with (4.1.2), we infer

$$J(S, \hat{z}, \ell) = j(\hat{z}(S)) + \frac{\beta}{2}\|\ell\|^2_{H^1(0,T;\mathcal{V}^*)} \le \liminf_{n\to\infty} j(\hat{z}_n(S_n)) + \frac{\beta}{2}\|\ell_n\|^2_{H^1(0,T;\mathcal{V}^*)} = I$$

such that $(S, \hat{t}, \hat{z}, \ell)$ is a minimizer of (OCP). □

4.2 Reverse approximation

The main goal of this chapter is to show that (OCP) can be approximated by simpler minimization problems, where the condition that $(S, \hat{t}, \hat{z})$ is a parametrized BV solution associated with ℓ is replaced by the uniquely solvable viscous regularized system (RIS$_\epsilon$), cf. Section 3.2. Therefore, we follow the lines of [KMS22], where the finite dimensional case with $\mathcal{X} = \mathcal{V} = \mathcal{Z} = \mathbb{R}^n$ is discussed, and adjust the procedure so that it also works for our infinite dimensional framework. One part to prove this approximability will be to construct a recovery sequence consisting of functions $(z_\epsilon, \ell_\epsilon)$, which comply with (RIS$_\epsilon$) and are thus feasible for the regularized optimal control problems such that the objective value converges to the minimum of the origin problem (OCP). Unfortunately, to this end we have to assume the following:

Assumption 4.2.1. *There exists at least one optimal solution* $(S, \hat{t}, \hat{z}, \ell) \in [T, \infty) \times W^{1,\infty}(0, S; \mathbb{R}) \times H^1(0, S; \mathcal{Z}) \times H^1(0, T; \mathcal{V}^*)$ *of* (OCP) *that satisfies*

$$\hat{t}'(s) \ge \delta \text{ for almost all } s \in (0, S) \tag{4.2.1}$$

with $0 < \delta \le 1$.

This is a pretty restrictive assumption. Besides the increased regularity of $\hat{z}$, the condition (4.2.1) implies that $\hat{t}$ is strictly monotone increasing. Hence, its inverse function $\sigma := \hat{t}^{-1} : [0, T] \to [0, S]$ exists and satisfies $1 \le \sigma'(t) < \frac{1}{\delta}$ for almost all

$t \in (0,T)$ since additionally $\hat{t}'(s) \leq 1$ almost everywhere by (3.3.24). Thus we have $\sigma \in W^{1,\infty}(0,T)$. Furthermore, $\tilde{z} := \hat{z} \circ \sigma$, which describes the state in physical time, is a differential solution of (RIS), i.e., $\tilde{z} \in H^1(0,T;\mathcal{Z})$ solves

$$0 \in \partial\mathcal{R}(\tilde{z}'(t)) + \mathrm{D}_z\mathcal{I}(\ell(t), \tilde{z}(t)), \quad \tilde{z}(0) = z_0. \tag{4.2.2}$$

This can be seen as follows: First, by construction of $\tilde{z}$, we obtain from (3.3.21) that $\tilde{z}(0) = z_0$. Analogously to Lemma 3.3.1, one shows that $\tilde{z} \in H^1(0,T;\mathcal{Z})$ with derivative $\tilde{z}'(t) = \hat{z}'(\sigma(t))\sigma'(t)$. Due to (4.2.1) and the complementarity condition (3.3.23), we have that

$$\mathrm{dist}_{\mathcal{V}^*}(-\mathrm{D}_z\mathcal{I}(\hat{\ell}(s), \hat{z}(s)), \partial\mathcal{R}(0)) = 0 \tag{4.2.3}$$

for almost all $s \in (0,S)$. Therefore, the energy identity (3.3.25) can be written as

$$\mathcal{I}(\hat{\ell}(s), \hat{z}(s)) + \int_0^s \mathcal{R}(\hat{z}'(r))\,\mathrm{d}r = \mathcal{I}(\hat{\ell}(0), z_0) - \int_0^s \langle \hat{\ell}'(r), \hat{z}(r)\rangle_{\mathcal{V}^*,\mathcal{V}}\,\mathrm{d}r \text{ for all } s \in [0,S],$$

where we used Lemma B.2 and Lemma B.3 in combination with the assumed regularity of $\hat{z}$. In view of $\hat{\ell} = \ell \circ \hat{t}$ and (1.2.14), the substitution $s = \sigma(t) = \hat{t}^{-1}(t)$ yields

$$\mathcal{I}(\ell(t), \tilde{z}(t)) + \int_0^t \mathcal{R}(\tilde{z}'(r))\,\mathrm{d}r = \mathcal{I}(\ell(0), z_0) - \int_0^t \langle \ell'(r), \tilde{z}(r)\rangle_{\mathcal{V}^*,\mathcal{V}}\,\mathrm{d}r \text{ for all } t \in [0,T]$$

and from (4.2.3) we infer

$$-\mathrm{D}_z\mathcal{I}(\ell(t), \tilde{z}(t)) \in \partial\mathcal{R}(0) \quad \text{f.a.a. } t \in (0,T). \tag{4.2.4}$$

Now rearranging the terms in the above energy equality and applying Lemma C.3 leads to

$$\begin{aligned}
\int_0^t \mathcal{R}(\tilde{z}'(r))\,\mathrm{d}r &= \mathcal{I}(\ell(0), z_0) - \mathcal{I}(\ell(t), \tilde{z}(t)) - \int_0^t \langle \ell'(r), \tilde{z}(r)\rangle_{\mathcal{V}^*,\mathcal{V}}\,\mathrm{d}r \\
&= \int_0^t -\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{I}(\ell(r), \tilde{z}(r)) - \langle \ell'(r), \tilde{z}(r)\rangle_{\mathcal{V}^*,\mathcal{V}}\,\mathrm{d}r \\
&= \int_0^t -\langle \mathrm{D}_z\mathcal{I}(\ell(r), \tilde{z}(r)), \tilde{z}'(r)\rangle_{\mathcal{Z}^*,\mathcal{Z}}\,\mathrm{d}r.
\end{aligned}$$

Since this holds for all $t \in [0,T]$, the equality is also pointwise fulfilled, i.e.,

$$\mathcal{R}(\tilde{z}'(t)) = -\langle \mathrm{D}_z\mathcal{I}(\ell(t), \tilde{z}(t)), \tilde{z}'(t)\rangle_{\mathcal{Z}^*,\mathcal{Z}} \quad \text{f.a.a. } t \in (0,T).$$

In view of (1.2.19), this in combination with (4.2.4) yields

$$-\mathrm{D}_z\mathcal{I}(\ell(t), \tilde{z}(t)) \in \partial\mathcal{R}(\tilde{z}'(t))$$

for almost all $t \in (0,T)$ such that $\tilde{z}$ is indeed a differential solution. As seen in Example 2.2.5, if the energy is not uniformly convex, one cannot guarantee the existence of a differential solution. Unfortunately, it is also unpromising to weaken this assumption, which will be discussed in detail in the next chapter.

The construction of a recovery sequence is based on viscous regularization and modifying the energy by adding an additional quadratic penalty term, which depends on the given differential solution $\tilde{z}$ and enforces the solution of the penalized viscous regularized system to converge to $\tilde{z}$. To this end, for a given $\eta > 0$ we define the penalized energy as

$$\mathcal{J}_\eta(t,\ell,z) = \frac{1}{2}\langle Az, z\rangle_{\mathcal{Z}^*,\mathcal{Z}} + \mathcal{F}(z) - \langle \ell, z\rangle_{\mathcal{V}^*,\mathcal{V}} + \frac{\eta}{2}\|z - \tilde{z}(t)\|_{\mathcal{V}}^2. \tag{4.2.5}$$

Then the rate-independent system with penalized energy reads

$$0 \in \partial\mathcal{R}(z'(t)) + \mathrm{D}_z\mathcal{J}_\eta(t,\ell(t),z(t)) \text{ f.a.a. } t \in (0,T), \quad z(0) = z_0. \tag{4.2.6}$$

and its viscous regularized version is

$$0 \in \partial\mathcal{R}_\epsilon(z_\epsilon'(t)) + \mathrm{D}_z\mathcal{J}_\eta(t,\ell(t),z_\epsilon(t)), \quad z_\epsilon(0) = z_0. \tag{4.2.7}$$

The constant $\eta > 0$ will be chosen sufficiently large such that the penalized energy is uniformly convex, cf. Lemma 4.2.2, which in turn implies uniqueness of differential solutions of the penalized system (4.2.6). For the sake of simplicity, we define the energy functionals

$$\mathcal{E}_\eta(z) = \mathcal{E}(z) + \frac{\eta}{2}\|z\|_{\mathcal{V}}^2, \quad \mathcal{I}_\eta(\ell,z) = \mathcal{I}(\ell,z) + \frac{\eta}{2}\|z\|_{\mathcal{V}}^2$$

such that

$$\begin{aligned}\mathrm{D}_z\mathcal{J}_\eta(t,\ell(t),z_\epsilon(t)) &= \mathrm{D}_z\mathcal{I}(\ell(t),z_\epsilon(t)) + \eta(z_\epsilon(t) - \tilde{z}(t))\\ &= \mathrm{D}_z\mathcal{E}(z(t)) - (\ell(t) + \eta\tilde{z}(t)) + \eta z_\epsilon(t)\\ &= \mathrm{D}_z\mathcal{I}(\ell(t) + \eta\tilde{z}(t), z_\epsilon(t)) + \eta z_\epsilon(t)\\ &= \mathrm{D}_z\mathcal{I}_\eta(\ell(t) + \eta\tilde{z}(t), z_\epsilon(t))\end{aligned} \tag{4.2.8}$$

Hence, (4.2.7) can be written as

$$0 \in \partial\mathcal{R}_\epsilon(z_\epsilon'(t)) + \mathrm{D}_z\mathcal{I}_\eta(\ell(t) + \eta\tilde{z}(t), z_\epsilon(t)), \quad z_\epsilon(0) = z_0. \tag{4.2.9}$$

Note that Assumption 4.2.1 yields $\tilde{z} \in H^1(0,T;\mathcal{Z})$ such that $\ell + \eta\tilde{z} \in H^1(0,T;\mathcal{V}^*)$. Thus, if we define $\mathcal{F}_\eta(z) := \mathcal{F}(z) + \frac{\eta}{2}\|z\|_{\mathcal{V}}^2$, then $\mathcal{F}_\eta$ complies with the requirements with regard to the non-linearity $\mathcal{F}$ from Section 1.2 such that the general results from Chapter 3 are applicable. In particular, we make use of the results of Section 3.1 regarding the twice regularized system, which reads with the penalized energy as

$$0 \in \partial\mathcal{R}_{\epsilon,\delta}(z_{\epsilon,\delta}'(t)) + \mathrm{D}_z\mathcal{J}_\eta(t,\ell(t),z_{\epsilon,\delta}(t)), \quad z_{\epsilon,\delta}(0) = z_0, \tag{4.2.10}$$

Thanks to (4.2.8), this is equivalent to

$$0 \in \partial\mathcal{R}_{\epsilon,\delta}(z'_{\epsilon,\delta}(t)) + \mathrm{D}_z\mathcal{I}_\eta(\ell(t) + \eta\tilde{z}(t), z_{\epsilon,\delta}(t)), \quad z_{\epsilon,\delta}(0) = z_0. \tag{4.2.11}$$

The existence of a solution of (4.2.10) is ensured by Theorem 3.1.1 and by passing $\delta \searrow 0$, we deduce from Theorem 3.2.3 that

$$z_{\epsilon,\delta} \rightharpoonup z_\epsilon \text{ in } H^1(0,T;\mathcal{Z}) \tag{4.2.12}$$

and for all $t \in [0,T]$ it holds

$$z_{\epsilon,\delta}(t) \rightharpoonup z_\epsilon(t) \text{ in } \mathcal{Z}, \quad z_{\epsilon,\delta}(t) \to z_\epsilon(t) \text{ in } \mathcal{V}, \quad \delta \searrow 0. \tag{4.2.13}$$

In particular, the limit z_ϵ solves the once regularized system (4.2.7). Additionally, $z_{\epsilon,\delta}$ and z_ϵ fulfill the a priori bounds from Theorem 3.1.1 and Theorem 3.2.4, respectively. But in this section we only need the pointwise bounds from (3.1.3) and (3.2.9), which are

$$\sup_{t\in[0,T]} \|z_{\epsilon,\delta}(t)\|_{\mathcal{Z}} \leq C \tag{4.2.14}$$

and

$$\sup_{t\in[0,T]} \|z_\epsilon(t)\|_{\mathcal{Z}} \leq C \tag{4.2.15}$$

with $C > 0$ independent of ϵ and δ. Moreover, in the current setting concerned with penalized energy the estimate (3.1.22) equals

$$\frac{\epsilon}{2}\|z'_{\epsilon,\delta}(t)\|_{\mathcal{V}}^2 + \int_0^t \mathrm{D}_z^2\mathcal{E}_\eta(z_{\epsilon,\delta}(r))[z'_{\epsilon,\delta}(r), z'_{\epsilon,\delta}(r)]\,\mathrm{d}r \leq \int_0^t \langle \ell'(r) + \eta\tilde{z}'(r), z'_{\epsilon,\delta}(r)\rangle_{\mathcal{V}^*,\mathcal{V}}\,\mathrm{d}r \tag{4.2.16}$$

For the further procedure we need the following lemma, which deals with the convexity of the penalized energy.

Lemma 4.2.2. *Let $r > 0$ be given. Then there exists $\eta > 0$ such that the penalized energy $\mathcal{E}_\eta$ is uniformly convex on $\overline{B_{\mathcal{Z}}(0,r)}$, i.e.,*

$$\mathrm{D}_z^2\mathcal{E}_\eta(z)[v,v] \geq \gamma\|v\|_{\mathcal{Z}}^2 \quad \textit{for all } v \in \mathcal{Z}, z \in \overline{B_{\mathcal{Z}}(0,r)} \tag{4.2.17}$$

with $\gamma > 0$.

Proof. Making use of property (1.2.5) gives

$$|\mathrm{D}_z^2\mathcal{F}(z)[v,v]| \leq C(1 + r^q)\|v\|_{\mathcal{Z}}\|v\|_{\mathcal{V}}$$

for all $v \in \mathcal{Z}, z \in \overline{B_{\mathcal{Z}}(0,r)}$ such that applying Young's inequality results in

$$|\mathrm{D}_z^2\mathcal{F}(z)[v,v]| \leq \frac{C(1+r^q)\kappa}{2}\|v\|_{\mathcal{Z}}^2 + \frac{C(1+r^q)}{2\kappa}\|v\|_{\mathcal{V}}^2$$

for all $\kappa > 0$. Hence, for $\kappa = \frac{\alpha}{C(1+r^q)}$ we obtain by exploiting (1.2.3)

$$\begin{aligned}
\mathrm{D}_z^2\mathcal{E}_\eta(z)[v,v] &= \langle Av, v\rangle_{\mathcal{Z}^*,\mathcal{Z}} + \mathrm{D}_z^2\mathcal{F}(z)[v,v] + \eta\|v\|_{\mathcal{V}}^2 \\
&\geq \alpha\|v\|_{\mathcal{Z}}^2 + \mathrm{D}_z^2\mathcal{F}(z)[v,v] + \eta\|v\|_{\mathcal{V}}^2 \\
&\geq \alpha\|v\|_{\mathcal{Z}}^2 - \frac{\alpha}{2}\|v\|_{\mathcal{Z}}^2 - \frac{(C(1+r^q))^2}{2\alpha}\|v\|_{\mathcal{V}}^2 + \eta\|v\|_{\mathcal{V}}^2 \\
&= \frac{\alpha}{2}\|v\|_{\mathcal{Z}}^2 + \left(\eta - \frac{(C(1+r^q))^2}{2\alpha}\right)\|v\|_{\mathcal{V}}^2,
\end{aligned}$$

Eventually, choosing $\eta > \frac{(C(1+r^q))^2}{2\alpha}$ gives the claim for $\gamma = \frac{\alpha}{2}$. □

In accordance with the uniform boundedness from (4.2.14), the previous lemma is applicable. Thus, from (4.2.16) we infer by exploiting the uniform convexity (4.2.17) and the non-negativity of the norm

$$\int_0^T \|z_{\epsilon,\delta}'(r)\|_{\mathcal{Z}}^2 \,\mathrm{d}r \leq \frac{1}{\gamma}\int_0^T \langle \ell'(r) + \eta\tilde{z}'(r), z_{\epsilon,\delta}'(r)\rangle_{\mathcal{V}^*,\mathcal{V}} \,\mathrm{d}r. \tag{4.2.18}$$

The application of Hölder's inequality for the right hand side, making use of the compact embedding $\mathcal{Z} \hookrightarrow^c \mathcal{V}$, in particular $\|z_{\epsilon,\delta}'(r)\|_{\mathcal{V}} \leq C\|z_{\epsilon,\delta}'(r)\|_{\mathcal{Z}}$, and dividing by $\|z_{\epsilon,\delta}'\|_{L^2(0,T;\mathcal{Z})}$ yields

$$\|z_{\epsilon,\delta}'\|_{L^2(0,T;\mathcal{Z})} \leq \tilde{C}\|\ell' + \eta\tilde{z}'\|_{L^2(0,T;\mathcal{V}^*)}. \tag{4.2.19}$$

By means of the weak lower semicontinuity of the norm, this transfers to the limit $z_\epsilon \in H^1(0,T;\mathcal{Z})$, cf. (4.2.12). Therefore, in combination with (4.2.14) and (4.2.15) we have that

$$\|z_{\epsilon,\delta}\|_{H^1(0,T;\mathcal{Z})} \leq C, \quad \|z_\epsilon\|_{H^1(0,T;\mathcal{Z})} \leq C \tag{4.2.20}$$

for all $\epsilon, \delta > 0$, where $C > 0$ does neither depend on ϵ nor on δ. In comparison to (3.1.25) and (3.2.11), this means that in the case of a uniformly convex energy, we obtain a $H^1(0,T;\mathcal{Z})$-bound independent of the parameters of the regularization. Moreover, from the convexity of the energy we can also derive improved a priori estimates of the form (3.1.26) and (3.2.10), respectively, where ϵ is replaced by $\sqrt{\epsilon}$. Surely, the upper bounds still depend on the regularization parameter ϵ, but despite this they will be useful later. Therefore, we consider (4.2.16) again and exploit (4.2.17) in order to obtain

$$\begin{aligned}
\frac{\epsilon}{2}\|z_{\epsilon,\delta}'(t)\|_{\mathcal{V}}^2 &\leq \int_0^t \langle \ell'(r) + \eta\tilde{z}'(r), z_{\epsilon,\delta}'(r)\rangle_{\mathcal{V}^*,\mathcal{V}} \,\mathrm{d}r \\
&\leq \|\ell' + \eta\tilde{z}'\|_{L^2(0,T;\mathcal{V}^*)}\|z_{\epsilon,\delta}'\|_{L^2(0,T;\mathcal{V})} \\
&\leq C\|\ell' + \eta\tilde{z}'\|_{L^2(0,T;\mathcal{V}^*)}^2,
\end{aligned}$$

where we used (4.2.19) in the last step. Thus, we have shown that

$$\|z'_{\epsilon,\delta}\|_{L^\infty(0,T;\mathcal{V})} \leq \frac{C}{\sqrt{\epsilon}} \tag{4.2.21}$$

and after passing $\delta \searrow 0$

$$\|z'_{\epsilon}\|_{L^\infty(0,T;\mathcal{V})} \leq \frac{C}{\sqrt{\epsilon}} \tag{4.2.22}$$

by the Banach-Alaoglu theorem with a constant $C > 0$ independent of ϵ and δ. Now that we have derived the required a priori estimates, the next steps deal with the expected convergence $z_\epsilon \to \tilde{z}$ by passing $\epsilon \searrow 0$. First, we show weak convergence in $H^1(0,T;\mathcal{Z})$, which is the following lemma:

Lemma 4.2.3. *The solution $z_\epsilon \in H^1(0,T;\mathcal{Z})$ of (4.2.7) satisfies*

$$z_\epsilon \rightharpoonup \tilde{z} \text{ in } H^1(0,T;\mathcal{Z}), \text{ as } \epsilon \searrow 0. \tag{4.2.23}$$

Proof. Due to the boundedness in (4.2.20), there exists a weakly converging subsequence (also denoted by z_ϵ) and a weak limit $z^* \in H^1(0,T;\mathcal{Z})$ with

$$z_\epsilon \rightharpoonup z^* \text{ in } H^1(0,T;\mathcal{Z}),\ \epsilon \searrow 0. \tag{4.2.24}$$

Moreover, we have that

$$z_\epsilon(t) \rightharpoonup z^*(t) \text{ in } \mathcal{Z} \text{ for all } t \in [0,T], \quad z_\epsilon \to z^* \text{ in } C([0,T];\mathcal{V}). \tag{4.2.25}$$

The second convergence follows directly from the compact embedding $H^1(0,T;\mathcal{Z}) \hookrightarrow^c C([0,T];\mathcal{V})$. According to (4.2.15), for every $t \in [0,T]$ it exists a subsequence (also denoted by $z_\epsilon(t)$) and a limit $z \in \mathcal{Z}$ with $z_\epsilon(t) \rightharpoonup z$ in $\mathcal{Z}$. Because weak limits are uniquely determined, z has to coincide with $z^*(t)$. Since this holds for every subsequence, we obtain the first convergence in (4.2.25) for the entire sequence. In the next step we will show that z^* is a differential solution of (4.2.6). In view of (4.2.8), this is equivalent to z^* solving

$$0 \in \partial\mathcal{R}(z^{*\prime}(t)) + \mathrm{D}_z\mathcal{I}_\eta(\ell(t) + \eta\tilde{z}(t), z^*(t)), \quad z^*(0) = z_0. \tag{4.2.26}$$

The argumentation is similar to the proof of Theorem 3.2.3. However, for convenience of the reader we present the arguments in detail. First of all, the pointwise convergence (4.2.25) ensures the initial condition $z^*(0) = 0$. According to the definition of the subdifferential, (4.2.9) can be equivalently written as

$$\begin{aligned}\mathcal{R}(z'_\epsilon(t)) - \mathcal{R}(v) + \epsilon\langle z'_\epsilon(t), z'_\epsilon(t) - v\rangle_{\mathcal{V}^*,\mathcal{V}}&\\ + \langle \mathrm{D}_z\mathcal{I}_\eta(\ell(t) + \eta\tilde{z}(t), z_\epsilon(t)), z'_\epsilon(t) - v\rangle_{\mathcal{Z}^*,\mathcal{Z}} &\leq 0\end{aligned}$$

for all $v \in \mathcal{Z}$ and almost all $t \in (0,T)$. Then integrating this inequality over $(0,T)$ yields

$$\int_0^T \mathcal{R}(z_\epsilon'(r)) - \mathcal{R}(v(r)) + \epsilon\langle z_\epsilon'(r), z_\epsilon'(r) - v(r)\rangle_{\mathcal{V}^*,\mathcal{V}} \\ + \langle \mathrm{D}_z\mathcal{I}_\eta(\ell(r) + \eta\tilde{z}(r), z_\epsilon(r)), z_\epsilon'(r) - v(r)\rangle_{\mathcal{Z}^*,\mathcal{Z}}\,\mathrm{d}r \leq 0$$

for all $v \in L^2(0,T;\mathcal{Z})$. From the weak convergence $z_\epsilon' \rightharpoonup z^{*\prime}$ in $L^2(0,T;\mathcal{Z})$ by (4.2.24) in combination with Lemma A.1 we deduce for the first term

$$\liminf_{\epsilon\searrow 0} \int_0^T \mathcal{R}(z_\epsilon'(r)) - \mathcal{R}(v(r))\,\mathrm{d}r \geq \int_0^T \mathcal{R}(z^{*\prime}(r)) - \mathcal{R}(v(r))\,\mathrm{d}r.$$

Concerning the second term, we exploit the boundedness from (4.2.20) in order to obtain

$$\int_0^T \epsilon\langle z_\epsilon'(r), z_\epsilon'(r) - v(r)\rangle_{\mathcal{V}^*,\mathcal{V}}\,\mathrm{d}r \to 0, \quad \epsilon \searrow 0.$$

By exploiting the explicit structure of the penalized energy, the last term can be written as

$$\begin{aligned}
&\int_0^T \langle \mathrm{D}_z\mathcal{I}_\eta(\ell(r) + \eta\tilde{z}(r), z_\epsilon(r)), z_\epsilon'(r) - v(r)\rangle_{\mathcal{Z}^*,\mathcal{Z}}\,\mathrm{d}r \\
&= \int_0^T \langle A z_\epsilon(r), z_\epsilon'(r) - v(r)\rangle_{\mathcal{Z}^*,\mathcal{Z}}\,\mathrm{d}r + \int_0^T \langle \mathrm{D}_z\mathcal{F}(z_\epsilon(r)), z_\epsilon'(r) - v(r)\rangle_{\mathcal{Z}^*,\mathcal{Z}}\,\mathrm{d}r \\
&\quad - \int_0^T \langle \ell(r) + \eta\tilde{z}(r), z_\epsilon'(r) - v(r)\rangle_{\mathcal{V}^*,\mathcal{V}} - \eta\langle z_\epsilon(r), z_\epsilon'(r) - v(r)\rangle_{\mathcal{V}^*,\mathcal{V}}\,\mathrm{d}r.
\end{aligned}$$

Because the sequence $(z_\epsilon)_{\epsilon>0}$ converges (pointwise) weakly to z^* by (4.2.25) and (4.2.24), respectively, and is additionally uniformly bounded due to (4.2.15), Lemma 3.2.2 implies for the first and second integral of the right hand side

$$\begin{aligned}
\liminf_{\epsilon\searrow 0} \int_0^T \langle A z_\epsilon(r), z_\epsilon'(r) - v(r)\rangle_{\mathcal{Z}^*,\mathcal{Z}}\,\mathrm{d}r &\geq \int_0^T \langle A z^*(r), z^{*\prime}(r) - v(r)\rangle_{\mathcal{Z}^*,\mathcal{Z}}\,\mathrm{d}r, \\
\lim_{\epsilon\searrow 0} \int_0^T \langle \mathrm{D}_z\mathcal{F}(z_\epsilon(r)), z_\epsilon'(r) - v(r)\rangle_{\mathcal{Z}^*,\mathcal{Z}}\,\mathrm{d}r &= \int_0^T \langle \mathrm{D}_z\mathcal{F}(z^*(r)), z^{*\prime}(r) - v(r)\rangle_{\mathcal{Z}^*,\mathcal{Z}}\,\mathrm{d}r.
\end{aligned}$$

Further, exploiting the strong convergence from (4.2.25) and the weak convergence (4.2.24) leads to

$$\begin{aligned}
&\lim_{\epsilon\searrow 0} \int_0^T \langle \ell(r) + \eta\tilde{z}(r), z_\epsilon'(r) - v(r)\rangle_{\mathcal{V}^*,\mathcal{V}} - \eta\langle z_\epsilon(r), z_\epsilon'(r) - v(r)\rangle_{\mathcal{V}^*,\mathcal{V}}\,\mathrm{d}r \\
&= \int_0^T \langle \ell(r) + \eta\tilde{z}(r), z^{*\prime}(r) - v(r)\rangle_{\mathcal{V}^*,\mathcal{V}} - \eta\langle z^*(r), z^{*\prime}(r) - v(r)\rangle_{\mathcal{V}^*,\mathcal{V}}\,\mathrm{d}r.
\end{aligned}$$

All together we have shown that

$$
\begin{aligned}
0 \geq \liminf_{\epsilon \searrow 0} \int_0^T & \mathcal{R}(z_\epsilon'(r)) - \mathcal{R}(v(r)) + \epsilon \langle z_\epsilon'(r), z_\epsilon'(r) - v(r) \rangle_{\mathcal{V}^*,\mathcal{V}} \\
& + \langle \mathrm{D}_z \mathcal{I}_\eta(\ell(r) + \eta \tilde{z}(r), z_\epsilon(r)), z_\epsilon'(r) - v(r) \rangle_{\mathcal{Z}^*,\mathcal{Z}} \,\mathrm{d}r \\
\geq \int_0^T & \mathcal{R}({z^*}'(r)) - \mathcal{R}(v(r)) + \langle \mathrm{D}_z \mathcal{I}_\eta(\ell(r) + \eta \tilde{z}(r), z^*(r)), {z^*}'(r) - v(r) \rangle_{\mathcal{Z}^*,\mathcal{Z}} \,\mathrm{d}r
\end{aligned}
$$

for all $v \in L^2(0,T;\mathcal{Z})$. If we choose $v(t) := \chi_{[t_1,t_2]}(t)v + \chi_{[0,T]\setminus[t_1,t_2]}{z^*}'(t)$ with $v \in \mathcal{Z}$, $0 \leq t_1 < t_2 \leq T$ arbitrary, where χ denotes the characteristic function of the associated set, this implies

$$
0 \geq \int_{t_1}^{t_2} \mathcal{R}({z^*}'(r)) - \mathcal{R}(v) + \langle \mathrm{D}_z \mathcal{I}_\eta(\ell(r) + \eta \tilde{z}(r), z^*(r)), {z^*}'(r) - v \rangle_{\mathcal{Z}^*,\mathcal{Z}} \,\mathrm{d}r.
$$

Since this is true for all t_1, t_2, the inequality holds pointwise almost everywhere, too, which is

$$
0 \geq \mathcal{R}({z^*}'(t)) - \mathcal{R}(v) + \langle \mathrm{D}_z \mathcal{I}_\eta(\ell(t) + \eta \tilde{z}(t), z^*(t)), {z^*}'(t) - v \rangle_{\mathcal{Z}^*,\mathcal{Z}}
$$

for all $v \in \mathcal{Z}$ and almost all $t \in (0,T)$, which in turn yields

$$
0 \in \partial \mathcal{R}({z^*}'(t)) + \mathrm{D}_z \mathcal{I}_\eta(\ell(t) + \eta \tilde{z}(t), z^*(t))
$$

by the definition of the subdifferential. Therefore, z^* is indeed a differential solution of (4.2.26).
Finally, we show that $z^* = \tilde{z}$. Thanks to construction of the penalized energy, we have

$$
\begin{aligned}
\mathrm{D}_z \mathcal{I}_\eta(\ell(t) + \eta \tilde{z}(t), \tilde{z}(t)) &= A\tilde{z}(t) + \mathrm{D}_z \mathcal{F}(\tilde{z}(t)) - (\ell(t) + \eta \tilde{z}(t)) + \eta \tilde{z}(t) \\
&= A\tilde{z}(t) + \mathrm{D}_z \mathcal{F}(\tilde{z}(t)) - \ell(t) \\
&= \mathrm{D}_z \mathcal{I}(\ell(t), \tilde{z}(t))
\end{aligned}
$$

such that $\tilde{z}$ is a differential solution of (4.2.26), too. Since the energy is uniformly convex by means of the additional penalization, differential solutions are uniquely determined, cf. Theorem 2.2.4, such that we end up with $z^* = \tilde{z}$. ☐

Before we are able to show that even $z_\epsilon \to \tilde{z}$ strongly in $H^1(0,T;\mathcal{Z})$, we need a further property of $\tilde{z}$, which is:

Lemma 4.2.4. *The differential solution $\tilde{z} \in H^1(0,T;\mathcal{Z})$ of (4.2.2) satisfies*

$$
\mathrm{D}_z^2 \mathcal{E}(\tilde{z}(t))[\tilde{z}'(t), \tilde{z}'(t)] - \langle \ell'(t), \tilde{z}'(t) \rangle_{\mathcal{V}^*,\mathcal{V}} = 0 \quad \textit{f.a.a. } t \in (0,T). \tag{4.2.27}
$$

Proof. Since $\partial\mathcal{R}(v) \subset \partial\mathcal{R}(0)$ for all $v \in \mathcal{Z}$ by (1.2.20) and $\tilde{z}$ is continuous, we have

$$-\mathrm{D}_z\mathcal{I}(\ell(t), \tilde{z}(t)) \in \partial\mathcal{R}(0)$$

for all $t \in [0, T]$. Then the definition of the subdifferential leads to

$$\mathcal{R}(\tilde{z}'(t)) = \langle -\mathrm{D}_z\mathcal{I}(\ell(t), \tilde{z}(t)), \tilde{z}'(t)\rangle_{\mathcal{Z}^*,\mathcal{Z}} \text{ f.a.a. } t \in (0, T), \tag{4.2.28}$$

$$\mathcal{R}(v) \geq \langle -\mathrm{D}_z\mathcal{I}(\ell(\tau), \tilde{z}(\tau)), v\rangle_{\mathcal{Z}^*,\mathcal{Z}} \text{ for all } v \in \mathcal{Z}, \tau \in [0, T]. \tag{4.2.29}$$

Let $t \in (0, T)$ be a Lebesgue point of $\tilde{z}'$ such that (4.2.28) is fulfilled, which is true for almost all $t \in (0, T)$. If we set $v = \tilde{z}'(t)$ and $\tau = t \pm h$ with $h > 0$, then (4.2.29) reads as

$$\mathcal{R}(\tilde{z}'(t)) \geq \langle -\mathrm{D}_z\mathcal{I}(\ell(t \pm h), \tilde{z}(t \pm h), \tilde{z}'(t)\rangle_{\mathcal{Z}^*,\mathcal{Z}}. \tag{4.2.30}$$

Subtracting (4.2.28) from (4.2.30) yields

$$\langle \mathrm{D}_z\mathcal{I}(\ell(t \pm h), \tilde{z}(t \pm h)) - \mathrm{D}_z\mathcal{I}(\ell(t), \tilde{z}(t)), \tilde{z}'(t)\rangle_{\mathcal{Z}^*,\mathcal{Z}} \geq 0$$

such that exploiting the explicit structure of the energy $\mathcal{I}$ and rearranging the terms leads to

$$\langle \mathrm{D}_z\mathcal{E}(\tilde{z}(t \pm h)) - \mathrm{D}_z\mathcal{E}(\tilde{z}(t)), \tilde{z}'(t)\rangle_{\mathcal{Z}^*,\mathcal{Z}} - \langle \ell(t \pm h) - \ell(t), \tilde{z}'(t)\rangle_{\mathcal{V}^*,\mathcal{V}} \geq 0.$$

Finally, dividing by $\pm h$, passing $h \searrow 0$ and using Lebesgue's differentiation theorem results in

$$\mathrm{D}_z^2\mathcal{E}(\tilde{z}(t))[\tilde{z}'(t), \tilde{z}'(t)] - \langle \ell'(t), \tilde{z}'(t)\rangle_{\mathcal{V}^*,\mathcal{V}} = 0$$

if $t \in (0, T)$ is additionally a Lebesgue point of ℓ'. Since this holds for almost all points $t \in (0, T)$, the proof is completed. □

In order to prove the strong convergence $z_\epsilon \to \tilde{z}$ in $H^1(0, T; \mathcal{Z})$, we need that the non-linearity $\mathcal{F} : \mathcal{Z} \to \mathbb{R}$ complies with the following assumption.

Assumption 4.2.5. *There exists an intermediate Banach space $\mathcal{W}$ such that $\mathcal{Z} \hookrightarrow^c \mathcal{W} \hookrightarrow \mathcal{V}$ and $D_z^2\mathcal{F}$ is supposed to be continuous on $\mathcal{W}$, i.e.,*

$$z_k \to z \textit{ in } \mathcal{W} \implies \mathrm{D}_z^2\mathcal{F}(z_k) \to \mathrm{D}_z^2\mathcal{F}(z) \textit{ in } \mathcal{L}(\mathcal{Z}, \mathcal{V}^*). \tag{4.2.31}$$

In Example 1.2.9 we considered a setting, which complies with this assumption.

Lemma 4.2.6. *The non-linearity $\mathcal{F} : \mathcal{Z} \to \mathbb{R}$ is supposed to satisfy Assumption 4.2.5. Then the solution $z_\epsilon \in H^1(0, T; \mathcal{Z})$ of* (4.2.7) *fulfills*

$$z_\epsilon \to \tilde{z} \textit{ in } H^1(0, T; \mathcal{Z}), \textit{ as } \epsilon \searrow 0. \tag{4.2.32}$$

Proof. First, we consider $z_{\epsilon,\delta}$, the solution of the twice regularized system (4.2.10), and infer required properties, which transfer to the limit z_ϵ by passing $\delta \searrow 0$. Therefore, we start with integrating (4.2.27) over $(0,T)$ and exploiting the concrete structure of $\mathcal{E}_\eta$ in order to infer

$$\int_0^T \mathrm{D}_z^2\mathcal{E}_\eta(\tilde{z}(r))[\tilde{z}'(r),\tilde{z}'(r)]\,\mathrm{d}r = \int_0^T \langle \ell'(r) + \eta\tilde{z}'(r), \tilde{z}'(r)\rangle_{\mathcal{V}^*,\mathcal{V}}\,\mathrm{d}r$$

Subtracting this from (4.2.16), exploiting the non-negativity of $\frac{\epsilon}{2}\|z'_{\epsilon,\delta}(t)\|^2_{\mathcal{V}}$ and rearranging the terms results in

$$\begin{aligned}
0 \geq &\int_0^T \mathrm{D}_z^2\mathcal{E}_\eta(z_{\epsilon,\delta}(r))[z'_{\epsilon,\delta}(r), z'_{\epsilon,\delta}(r)] - \mathrm{D}_z^2\mathcal{E}_\eta(\tilde{z}(r))[\tilde{z}'(r),\tilde{z}'(r)]\\
&- \langle \ell'(r) + \eta\tilde{z}'(r), z'_{\epsilon,\delta}(r) - \tilde{z}'(r)\rangle_{\mathcal{V}^*,\mathcal{V}}\,\mathrm{d}r\\
= &\int_0^T \mathrm{D}_z^2\mathcal{E}_\eta(z_{\epsilon,\delta}(r))[z'_{\epsilon,\delta}(r) - \tilde{z}'(r), z'_{\epsilon,\delta}(r) - \tilde{z}'(r)]\,\mathrm{d}r\\
&- \int_0^T \langle \ell'(r) + \eta\tilde{z}'(r), z'_{\epsilon,\delta}(r) - \tilde{z}'(r)\rangle_{\mathcal{V}^*,\mathcal{V}}\,\mathrm{d}r\\
&+ \int_0^T 2\mathrm{D}_z^2\mathcal{E}_\eta(z_{\epsilon,\delta}(r))[\tilde{z}'(r), z'_{\epsilon,\delta}(r)]\\
&\qquad - \mathrm{D}_z^2\mathcal{E}_\eta(\tilde{z}(r))[\tilde{z}'(r),\tilde{z}'(r)] - \mathrm{D}_z^2\mathcal{E}_\eta(z_{\epsilon,\delta}(r))[\tilde{z}'(r),\tilde{z}'(r)]\,\mathrm{d}r.
\end{aligned}$$

Next, using (4.2.17), the convexity of $\mathcal{E}_\eta$, leads to

$$\int_0^T \mathrm{D}_z^2\mathcal{E}_\eta(z_{\epsilon,\delta}(r))[z'_{\epsilon,\delta}(r) - \tilde{z}'(r), z'_{\epsilon,\delta}(r) - \tilde{z}'(r)]\,\mathrm{d}r \geq \int_0^T \gamma\|z'_{\epsilon,\delta}(r) - \tilde{z}'(r)\|^2_{\mathcal{Z}}\,\mathrm{d}r$$

such that the weak lower semicontinuity of the norm together with the weak convergence from (4.2.12) yields

$$\int_0^T \gamma\|z'_\epsilon(r) - \tilde{z}'(r)\|^2_{\mathcal{Z}}\,\mathrm{d}r \leq \liminf_{\delta\searrow 0}\int_0^T \gamma\|z'_{\epsilon,\delta}(r) - \tilde{z}'(r)\|^2_{\mathcal{Z}}\,\mathrm{d}r.$$

Another use of (4.2.12) gives

$$\int_0^T \langle \ell'(r) + \eta\tilde{z}'(r), z'_{\epsilon,\delta}(r) - \tilde{z}'(r)\rangle_{\mathcal{V}^*,\mathcal{V}}\,\mathrm{d}r \xrightarrow{\delta\searrow 0} \int_0^T \langle \ell'(r) + \eta\tilde{z}'(r), z'_\epsilon(r) - \tilde{z}'(r)\rangle_{\mathcal{V}^*,\mathcal{V}}\,\mathrm{d}r.$$

Next, we show that

$$\mathrm{D}_z^2\mathcal{E}_\eta(z_{\epsilon,\delta})\tilde{z}' \to \mathrm{D}_z^2\mathcal{E}_\eta(z_\epsilon)\tilde{z}' \text{ in } L^2(0,T;\mathcal{V}^*),\ \delta\searrow 0. \tag{4.2.33}$$

Making use of Assumption 4.2.5, i.e., the existence of a space $\mathcal{W}$ with $\mathcal{Z} \hookrightarrow^c \mathcal{W}$, and the pointwise weak convergence (4.2.13) yields $z_{\epsilon,\delta}(t) \to z_\epsilon(t)$ in $\mathcal{W}$ such that the continuity from (4.2.31) implies $\mathrm{D}_z^2\mathcal{E}_\eta(z_{\epsilon,\delta}(t))\tilde{z}'(t) \to \mathrm{D}_z^2\mathcal{E}_\eta(z_\epsilon(t))\tilde{z}'(t)$ in $\mathcal{V}^*$ for almost all $t \in [0,T]$. Note that, taking into account the structure of the energy $\mathcal{E}_\eta$, only $\mathrm{D}_z^2\mathcal{F}$ depends on $z_{\epsilon,\delta}$. Hence, exploiting estimate (1.2.5) in combination with the uniform bound (4.2.14) gives the existence of an upper bound $L^2(0,T;\mathbb{R})$-function, i.e., $\|\mathrm{D}_z^2\mathcal{F}(z_{\epsilon,\delta}(t))\tilde{z}'(t)\|_{\mathcal{V}^*} \le C\|\tilde{z}'(t)\|_{\mathcal{Z}}$ for almost all $t \in (0,T)$, since $\tilde{z} \in H^1(0,T;\mathcal{Z})$. Eventually, (4.2.33) follows from Lebesgue's dominated convergence theorem. Thus, we have by making use of (4.2.12)

$$\begin{aligned}
&\int_0^T 2\mathrm{D}_z^2\mathcal{E}_\eta(z_{\epsilon,\delta}(r))[\tilde{z}'(r), z'_{\epsilon,\delta}(r)] \\
&\qquad\qquad - \mathrm{D}_z^2\mathcal{E}_\eta(\tilde{z}(r))[\tilde{z}'(r),\tilde{z}'(r)] - \mathrm{D}_z^2\mathcal{E}_\eta(z_{\epsilon,\delta}(r))[\tilde{z}'(r),\tilde{z}'(r)]\,\mathrm{d}r \\
\to &\int_0^T 2\mathrm{D}_z^2\mathcal{E}_\eta(z_\epsilon(r))[\tilde{z}'(r), z'_\epsilon(r)] \\
&\qquad\qquad - \mathrm{D}_z^2\mathcal{E}_\eta(\tilde{z}(r))[\tilde{z}'(r),\tilde{z}'(r)] - \mathrm{D}_z^2\mathcal{E}_\eta(z_\epsilon(r))[\tilde{z}'(r),\tilde{z}'(r)]\,\mathrm{d}r,
\end{aligned}$$

as $\delta \searrow 0$. All in all, we obtain by passing $\delta \searrow 0$

$$\begin{aligned}
\int_0^T \gamma\|z'_\epsilon(r) - \tilde{z}'(r)\|_{\mathcal{Z}}^2\,\mathrm{d}r \le &\int_0^T \langle \ell'(r) + \eta\tilde{z}'(r), z'_\epsilon(r) - \tilde{z}'(r)\rangle_{\mathcal{V}^*,\mathcal{V}}\,\mathrm{d}r \\
&+ \int_0^T 2\mathrm{D}_z^2\mathcal{E}_\eta(z_\epsilon(r))[\tilde{z}'(r), z'_\epsilon(r)] - \mathrm{D}_z^2\mathcal{E}_\eta(\tilde{z}(r))[\tilde{z}'(r),\tilde{z}'(r)] \\
&\qquad\qquad - \mathrm{D}_z^2\mathcal{E}_\eta(z_\epsilon(r))[\tilde{z}'(r),\tilde{z}'(r)]\,\mathrm{d}r.
\end{aligned}$$

Finally, we can use the same argumentation for passing ϵ to zero because as well as $(z_{\epsilon,\delta})_\delta$ the sequence $(z_\epsilon)_\epsilon$ converges weakly in $H^1(0,T;\mathcal{Z})$ by (4.2.23). Since the limit with respect to $\epsilon \searrow 0$ is $\tilde{z}$, the right hand side in the above estimate tends to zero, which implies $z'_\epsilon \to \tilde{z}'$ in $L^2(0,T;\mathcal{Z})$. Eventually, from the Poincaré inequality and the fact that $z_\epsilon(0) = \tilde{z}(0) = z_0$ we infer $\|z_\epsilon - \tilde{z}\|_{L^2(0,T;\mathcal{Z})} \le C\|z'_\epsilon - \tilde{z}'\|_{L^2(0,T;\mathcal{Z})} \to 0$, as $\epsilon \searrow 0$, such that together $z_\epsilon \to \tilde{z}$ in $H^1(0,T;\mathcal{Z})$ as claimed. □

Before we formulate the main result of this section, the reverse approximation property, we prove the following lemma, whose statement deals with an end time property for the solutions of the penalized viscous regularized systems. This property ensures feasibility of the recovery sequence for the regularized optimal control problems, which will be part of the next section.

Lemma 4.2.7. *Let a parametrized BV solution that complies with Assumption 4.2.1 be given and denote the reparametrized function by $\tilde{z} = \hat{z} \circ \hat{t}^{-1}$. Then there exists a constant $C > 0$ not depending on ϵ such that the solution $z_\epsilon \in H^1(0,T;\mathcal{Z})$ of* (4.2.7) *satisfies*

$$\operatorname{dist}_{\mathcal{V}^*}(-\mathrm{D}_z\mathcal{E}(z_\epsilon(T)) + \ell(T) - \eta(z_\epsilon(T) - \tilde{z}(T)), \partial\mathcal{R}(0)) \le C\epsilon^{\frac{1}{4}}.$$

Proof. In view of (4.2.7), we have for almost all $t \in (0,T)$

$$-\mathrm{D}_z\mathcal{E}(z_\epsilon(t)) + \ell(t) - \eta(z_\epsilon(t) - \tilde{z}(t)) \in \partial\mathcal{R}_\epsilon(z'_\epsilon(t))$$

such that the Fenchel-Young equality gives

$$\mathcal{R}_\epsilon(z_\epsilon'(t)) + \mathcal{R}_\epsilon^*(-\mathrm{D}_z\mathcal{E}(z_\epsilon(t)) + \ell(t) - \eta(z_\epsilon(t) - \tilde{z}(t))) \\ = \langle -\mathrm{D}_z\mathcal{E}(z_\epsilon(t)) + \ell(t) - \eta(z_\epsilon(t) - \tilde{z}(t)), z_\epsilon'(t)\rangle_{\mathcal{Z}^*,\mathcal{Z}}$$

Now the non-negativity of $\mathcal{R}_\epsilon$, (A.2), (3.2.14) and (4.2.22) yield

$$\begin{aligned} \frac{1}{2\epsilon}\operatorname{dist}_{\mathcal{V}^*}(-\mathrm{D}_z\mathcal{E}(z_\epsilon(t)) + \ell(t) - \eta(z_\epsilon(t) - \tilde{z}(t)), \partial\mathcal{R}(0))^2 \\ &= \mathcal{R}_\epsilon^*(-\mathrm{D}_z\mathcal{E}(z_\epsilon(t)) + \ell(t) - \eta(z_\epsilon(t) - \tilde{z}(t))) \\ &\leq \langle -\mathrm{D}_z\mathcal{E}(z_\epsilon(t)) + \ell(t) - \eta(z_\epsilon(t) - \tilde{z}(t)), z_\epsilon'(t)\rangle_{\mathcal{V}^*,\mathcal{V}} \\ &\leq \| - \mathrm{D}_z\mathcal{E}(z_\epsilon(\cdot)) + \ell - \eta(z_\epsilon - \tilde{z})\|_{L^\infty(0,T;\mathcal{V}^*)}\|z_\epsilon'\|_{L^\infty(0,T;\mathcal{V})} \\ &\leq \frac{\tilde{C}}{\sqrt{\epsilon}}. \end{aligned}$$

for almost all $t \in (0,T)$. This estimate in fact holds true for all $t \in [0,T]$, which can be seen as follows. Let $t \in [0,T]$ and a sequence $t_n \to t$ in $\mathbb{R}$ be given such that the above estimate is valid for all t_n. Due the regularity of the occurring terms, (4.2.15) and (3.2.14), all requirements of Lemma 3.3.5 are satisfied such that the weak lower semicontinuity of the distance (3.3.27) implies that

$$\begin{aligned} &\frac{1}{2\epsilon}\operatorname{dist}_{\mathcal{V}^*}(-\mathrm{D}_z\mathcal{E}(z_\epsilon(t)) + \ell(t) - \eta(z_\epsilon(t) - \tilde{z}(t)), \partial\mathcal{R}(0))^2 \\ \leq\ &\liminf_{n\to\infty}\frac{1}{2\epsilon}\operatorname{dist}_{\mathcal{V}^*}(-\mathrm{D}_z\mathcal{E}(z_\epsilon(t_n)) + \ell(t_n) - \eta(z_\epsilon(t_n) - \tilde{z}(t_n)), \partial\mathcal{R}(0))^2 \\ \leq\ &\frac{\tilde{C}}{\sqrt{\epsilon}}. \end{aligned}$$

Finally, multiplying with 2ϵ and considering the square root gives the assertion. □

With this at hand, we are now able to prove the main result of this section:

Theorem 4.2.8 (Reverse approximation property)**.** *Let $(S, \hat{t}, \hat{z}) \subset [T,\infty)\times W^{1,\infty}(0,S;\mathbb{R})\times H^1(0,S;\mathcal{Z})$ be a $\mathfrak{p}$-parametrized BV solution that satisfies Assumption 4.2.1 such that the reparametrized function $\tilde{z} := \hat{z}\circ\hat{t}^{-1} \in H^1(0,T;\mathcal{Z})$ is a differential solution of* (RIS)*. Further, the non-linearity $\mathcal{F} : \mathcal{Z} \to \mathbb{R}$ has to comply with Assumption 4.2.5. Then there exists a sequence $(z_\epsilon, \ell_\epsilon)_{\epsilon>0} \subset H^1(0,T;\mathcal{Z})\times H^1(0,T;\mathcal{V}^*)$ such that z_ϵ is a solution of the viscous regularized problem* (RIS_ϵ) *with external load ℓ_ϵ, i.e.,*

$$0 \in \partial\mathcal{R}_\epsilon(z_\epsilon'(t)) + \mathrm{D}_z\mathcal{I}(\ell_\epsilon(t), z_\epsilon(t)) \text{ f.a.a. } t \in (0,T), \quad z_\epsilon(0) = z_0 \tag{4.2.34}$$

and

$$z_\epsilon \to \tilde{z} \text{ in } H^1(0,T;\mathcal{Z}), \quad \ell_\epsilon \to \ell \text{ in } H^1(0,T;\mathcal{V}^*), \text{ as } \epsilon \searrow 0. \tag{4.2.35}$$

Moreover, we have

$$\ell_\epsilon(0) = \ell(0), \quad \operatorname{dist}_{\mathcal{V}^*}(-\mathrm{D}_z\mathcal{I}(\ell_\epsilon(T), z_\epsilon(T)), \partial\mathcal{R}(0)) \leq C\epsilon^{\frac{1}{4}} \tag{4.2.36}$$

with $C > 0$ independent of ϵ.

Proof. We set $\ell_\epsilon := \ell - \eta(z_\epsilon - \tilde{z})$ with $\eta > 0$ sufficiently large such that the penalized energy $\mathcal{E}_\eta$ is uniformly convex, see Lemma 4.2.2. Then we have

$$\begin{aligned} \mathrm{D}_z\mathcal{I}(\ell_\epsilon(t), z_\epsilon(t)) &= Az_\epsilon(t) + \mathrm{D}_z\mathcal{F}(z_\epsilon(t)) - \ell_\epsilon(t) \\ &= Az_\epsilon(t) + \mathrm{D}_z\mathcal{F}(z_\epsilon(t)) - (\ell(t) + \eta\tilde{z}(t)) + \eta z_\epsilon(t) \\ &= \mathrm{D}_z\mathcal{I}_\eta(\ell(t) + \eta\tilde{z}(t), z_\epsilon(t)) \end{aligned}$$

such that (4.2.34) equals (4.2.9). Thus, Lemma 4.2.6 yields $z_\epsilon \to \tilde{z}$ in $H^1(0,T;\mathcal{Z})$ such that $\ell_\epsilon \to \ell$ in $H^1(0,T;\mathcal{V}^*)$. Eventually, exploiting $z_\epsilon(0) = \tilde{z}(0) = z_0$ and Lemma 4.2.7 gives (4.2.36), which completes the proof. □

4.3 Approximation by regularized optimal control problems $(\mathrm{vOCP})_\epsilon$

In the last section we have seen that differential solutions of (RIS) can be approximated by solutions of the viscous regularized systems (RIS_ϵ) if we additionally vary the external loads. This will be the main ingredient to show that the optimal control problem (OCP) can be approximated by simpler problems. Therefore, we consider the following viscous regularized optimal control problem:

$$\left.\begin{aligned} \min \quad & J_\epsilon(z_\epsilon, \ell) := j(z_\epsilon(T)) + \frac{\beta}{2}\,\|\ell\|^2_{H^1(0,T;\mathcal{V}^*)} \\ \text{s.t.} \quad & \ell \in H^1(0,T;\mathcal{V}^*), \quad z_\epsilon \in H^1(0,T;\mathcal{Z}), \\ & 0 \in \partial\mathcal{R}_\epsilon(z_\epsilon'(t)) + \mathrm{D}_z\mathcal{I}(\ell(t), z_\epsilon(t)), \quad z_\epsilon(0) = z_0, \\ & -\mathrm{D}_z\mathcal{I}(\ell(0), z_0) \in \partial\mathcal{R}(0), \\ & \mathrm{dist}_{\mathcal{V}^*}(-\mathrm{D}_z\mathcal{I}(\ell(T), z_\epsilon(T)), \partial\mathcal{R}(0)) \le \epsilon^{\frac{1}{8}}, \end{aligned}\right\} \qquad (\mathrm{vOCP}_\epsilon)$$

where $j : \mathcal{V} \to \mathbb{R}$ and $\beta > 0$ are the same as in (OCP). But the rate-independent system in the constraints is replaced by its viscous regularized version. Moreover, we require local stability at the beginning, too, and relax the end time condition such that the constructed sequence from the previous section will be feasible for (vOCP_ϵ), cf. (4.2.36). In Comparison to (4.2.36), we choose a decreased exponent such that the occurring constant is compensated for $\epsilon > 0$ sufficiently small, i.e., $\mathrm{dist}_{\mathcal{V}^*}(-\mathrm{D}_z\mathcal{I}(\ell_\epsilon(T), z_\epsilon(T)), \partial\mathcal{R}(0)) \le C\epsilon^{\frac{1}{4}} \le \epsilon^{\frac{1}{8}}$ for all $0 < \epsilon < \epsilon_0$. First, we show that ($\mathrm{vOCP}_\epsilon$) is solvable at all.

Theorem 4.3.1. *Let $\epsilon > 0$ be given. Then there exists a globally optimal solution $(z_\epsilon^*, \ell_\epsilon^*) \in H^1(0,T;\mathcal{Z}) \times H^1(0,T;\mathcal{V}^*)$ of* (vOCP_ϵ).

Proof. For the sake of simplicity and since $\epsilon > 0$ is fixed here, we suppress the subscript in this proof. First of of all, according to our standing assumption (1.2.26), the constant functions $z \equiv z_0$ and $\ell \equiv \ell_0$ are feasible for (vOCP_ϵ) such that the feasible set is non-empty. Now let $(z_n, \ell_n)_{n\in\mathbb{N}}$ be an infimal sequence of (vOCP_ϵ), i.e., $-\infty < \xi := \lim_{n\to\infty} J_\epsilon(z_n, \ell_n)$, where the infimum ξ is a real number because the objective is bounded from below by assumption. Therefore, we also have that $\|\ell_n\|_{H^1(0,T;\mathcal{V}^*)} \leq C$. Together with the a priori bound (3.2.11) (ϵ is fixed here) we can extract a weakly converging subsequence (denoted with the same index) with

$$(z_n, \ell_n) \rightharpoonup (z^*, \ell^*) \text{ in } H^1(0,T;\mathcal{Z}) \times H^1(0,T;\mathcal{V}^*). \tag{4.3.1}$$

In the next step we will verify the feasibility of (z^*, ℓ^*). This is pretty similar to the proof of Theorem 3.2.3. However, now the loads are also weakly converging and no longer fixed, thus, we present the proof in detail. According to (4.3.1), the weak limit (z^*, ℓ^*) has the desired regularity. Furthermore, the compact embedding $H^1(0,T;\mathcal{Z}) \hookrightarrow^c C([0,T];\mathcal{V})$ leads to pointwise strong convergence of z_n in $\mathcal{V}$ and exploiting the boundedness from (3.2.9) results in additional pointwise weak convergence in $\mathcal{Z}$ such that we have

$$z_n(t) \rightharpoonup z^*(t) \text{ in } \mathcal{Z}, \quad z_n(t) \to z^*(t) \text{ in } \mathcal{V}, \quad n \to \infty, \tag{4.3.2}$$

for all $t \in [0,T]$, which directly gives the initial condition $z^*(0) = z_0$. In order to prove the subdifferential inclusion, let $v \in \mathcal{Z}$ be arbitrary. Since z_n is a solution associated with ℓ_n, we have

$$\mathcal{R}(z_n'(t)) - \mathcal{R}(v) + \epsilon\langle z_n'(t), z_n'(t) - v\rangle_{\mathcal{V}^*,\mathcal{V}} + \langle \mathrm{D}_z\mathcal{I}(\ell_n(t), z_n(t)), z_n'(t) - v\rangle_{\mathcal{Z}^*,\mathcal{Z}} \leq 0$$

for almost all $t \in (0,T)$ and all $n \in \mathbb{N}$. Then integrating over $(0,T)$ yields

$$\begin{aligned}\int_0^T \mathcal{R}(z_n'(r)) - \mathcal{R}(v(r)) + \epsilon\langle z_n'(r), z_n'(r) - v(r)\rangle_{\mathcal{V}^*,\mathcal{V}}& \\ + \langle \mathrm{D}_z\mathcal{I}(\ell_n(r), z_n(r)), z_n'(r) - v(r)\rangle_{\mathcal{Z}^*,\mathcal{Z}} \,\mathrm{d}r &\leq 0\end{aligned} \tag{4.3.3}$$

for all $v \in L^2(0,T;\mathcal{Z})$. By driving $n \to \infty$, we obtain for the part of the left hand side concerned with the dissipation

$$\int_0^T \mathcal{R}(z^{*\prime}(r)) - \mathcal{R}(v(r)) \,\mathrm{d}r \leq \liminf_{n\to\infty} \int_0^T \mathcal{R}(z_n'(r)) - \mathcal{R}(v(r)) \,\mathrm{d}r,$$

where we exploited the weak convergence (4.3.1) and the weak lower semicontinuity result from Lemma A.1. Another use of (4.3.1) directly gives

$$\int_0^T \epsilon\langle z^{*\prime}(r), z^{*\prime}(r) - v(r)\rangle_{\mathcal{V}^*,\mathcal{V}} \,\mathrm{d}r \leq \liminf_{n\to\infty} \int_0^T \epsilon\langle z_n'(r), z_n'(r) - v(r)\rangle_{\mathcal{V}^*,\mathcal{V}} \,\mathrm{d}r$$

by means of the weak lower semicontinuity of the squared norm. In accordance with the concrete structure of the energy, the remaining term of integral in (4.3.3) can be written as

$$\begin{aligned}&\int_0^T \langle \mathrm{D}_z\mathcal{I}(\ell_n(r), z_n(r)), z_n'(r) - v(r)\rangle_{\mathcal{Z}^*,\mathcal{Z}} \,\mathrm{d}r \\ &= \int_0^T \langle A z_n(r), z_n'(r) - v(r)\rangle_{\mathcal{Z}^*,\mathcal{Z}} + \langle \mathrm{D}_z\mathcal{F}(z_n(r)), z_n'(r) - v(r)\rangle_{\mathcal{Z}^*,\mathcal{Z}} \\ &\qquad - \langle \ell_n(r), z_n'(r) - v(r)\rangle_{\mathcal{V}^*,\mathcal{V}} \,\mathrm{d}r.\end{aligned} \tag{4.3.4}$$

By using (4.3.1), (4.3.2) and (3.2.9), we infer from Lemma 3.2.2

$$\liminf_{n\to\infty} \int_0^T \langle Az_n(r), z_n'(r) - v(r)\rangle_{\mathcal{Z}^*,\mathcal{Z}} \,\mathrm{d}r \geq \int_0^T \langle Az^*(r), z^{*\prime}(r) - v(r)\rangle_{\mathcal{Z}^*,\mathcal{Z}} \,\mathrm{d}r,$$
$$\lim_{n\to\infty} \int_0^T \langle \mathrm{D}_z\mathcal{F}(z_n(r)), z_n'(r) - v(r)\rangle_{\mathcal{Z}^*,\mathcal{Z}} \,\mathrm{d}r = \int_0^T \langle \mathrm{D}_z\mathcal{F}(z^*(r)), z^{*\prime}(r) - v(r)\rangle_{\mathcal{Z}^*,\mathcal{Z}} \,\mathrm{d}r.$$

Next, we consider the last term of (4.3.4). By applying integration by parts, this term can be written as

$$\begin{aligned}
&\int_0^T \langle \ell_n(r), z_n'(r) - v(r)\rangle_{\mathcal{V}^*,\mathcal{V}} \,\mathrm{d}r \\
&= \int_0^T \langle \ell_n(r), z_n'(r)\rangle_{\mathcal{V}^*,\mathcal{V}} \,\mathrm{d}r - \int_0^T \langle \ell_n(r), v(r)\rangle_{\mathcal{V}^*,\mathcal{V}} \,\mathrm{d}r \\
&= \langle \ell_n(T), z_n(T)\rangle_{\mathcal{V}^*,\mathcal{V}} - \langle \ell_n(0), z_n(0)\rangle_{\mathcal{V}^*,\mathcal{V}} - \int_0^T \langle \ell_n'(r), z_n(r)\rangle_{\mathcal{V}^*,\mathcal{V}} \,\mathrm{d}r \\
&\qquad - \int_0^T \langle \ell_n(r), v(r)\rangle_{\mathcal{V}^*,\mathcal{V}} \,\mathrm{d}r.
\end{aligned}$$

The weak continuity of the point evaluation in combination with the weak convergence of $(\ell_n)_{n\in\mathbb{N}}$ in $H^1(0,T;\mathcal{V}^*)$ implies pointwise weak convergence, i.e.,

$$\ell_n(t) \rightharpoonup \ell^*(t) \text{ in } \mathcal{V}^* \text{ for all } t \in [0,T]. \tag{4.3.5}$$

Thus, thanks to the pointwise convergence of $(z_n)_{n\in\mathbb{N}}$ with respect to the strong topology in $\mathcal{V}$ by (4.3.2), we deduce

$$\langle \ell_n(T), z_n(T)\rangle_{\mathcal{V}^*,\mathcal{V}} - \langle \ell_n(0), z_n(0)\rangle_{\mathcal{V}^*,\mathcal{V}} \to \langle \ell^*(T), z^*(T)\rangle_{\mathcal{V}^*,\mathcal{V}} - \langle \ell^*(0), z^*(0)\rangle_{\mathcal{V}^*,\mathcal{V}}.$$

Furthermore, exploiting the strong convergence $z_n \to z^*$ in $L^2(0,T;\mathcal{V})$, which results from (4.3.1) and the compactness of the embedding $H^1(0,T;\mathcal{Z}) \hookrightarrow^c L^2(0,T;\mathcal{V})$, leads to

$$\begin{aligned}
&- \int_0^T \langle \ell_n'(r), z_n(r)\rangle_{\mathcal{V}^*,\mathcal{V}} \,\mathrm{d}r - \int_0^T \langle \ell_n(r), v(r)\rangle_{\mathcal{V}^*,\mathcal{V}} \,\mathrm{d}r \\
&\qquad \to - \int_0^T \langle \ell^{*\prime}(r), z^*(r)\rangle_{\mathcal{V}^*,\mathcal{V}} \,\mathrm{d}r - \int_0^T \langle \ell^*(r), v(r)\rangle_{\mathcal{V}^*,\mathcal{V}} \,\mathrm{d}r
\end{aligned}$$

such that we obtain after applying integrations by parts again

$$\int_0^T \langle \ell_n(r), z_n'(r) - v(r)\rangle_{\mathcal{V}^*,\mathcal{V}} \,\mathrm{d}r \to \int_0^T \langle \ell^*(r), z^{*\prime}(r) - v(r)\rangle_{\mathcal{V}^*,\mathcal{V}} \,\mathrm{d}r,\ n \to \infty.$$

Combining all these convergence results leads to

$$\begin{aligned}
\int_0^T &\mathcal{R}(z^{*\prime}(r)) - \mathcal{R}(v(r)) + \epsilon\langle z^{*\prime}(r), z^{*\prime}(r) - v(r)\rangle_{\mathcal{V}^*,\mathcal{V}} \\
&+ \langle \mathrm{D}_z\mathcal{I}(\ell^*(r), z^*(r)), z^{*\prime}(r) - v(r)\rangle_{\mathcal{Z}^*,\mathcal{Z}} \,\mathrm{d}r \leq 0
\end{aligned} \tag{4.3.6}$$

for all $v \in L^2(0,T;\mathcal{Z})$. Now, let $v \in \mathcal{Z}$ and $0 \le t_1 < t_2 \le T$ be given arbitrary. Then, analogous to the proof of Lemma 4.2.3, we define $v(t) := \chi_{[t_1,t_2]}(t)v + \chi_{[0,T]\setminus[t_1,t_2]}z^{*\prime}(t)$ such that inserting this test function in (4.3.6) yields

$$\begin{aligned}\int_{t_1}^{t_2} \mathcal{R}(z^{*\prime}(r)) - \mathcal{R}(v) + \epsilon\langle z^{*\prime}(r), z^{*\prime}(r) - v\rangle_{\mathcal{V}^*,\mathcal{V}} \\ + \langle \mathrm{D}_z\mathcal{I}(\ell^*(r), z^*(r)), z^{*\prime}(r) - v\rangle_{\mathcal{Z}^*,\mathcal{Z}}\,\mathrm{d}r \le 0.\end{aligned}$$

Since t_1, t_2 are arbitrary, this inequality holds also pointwise almost everywhere, i.e.,

$$\begin{aligned}\mathcal{R}(z^{*\prime}(t)) - \mathcal{R}(v(t)) + \epsilon\langle z^{*\prime}(t), z^{*\prime}(t) - v\rangle_{\mathcal{V}^*,\mathcal{V}} \\ + \langle \mathrm{D}_z\mathcal{I}(\ell^*(t), z^*(t)), z^{*\prime}(t) - v(r)\rangle_{\mathcal{Z}^*,\mathcal{Z}} \le 0 \text{ f.a.a. } t \in (0,T),\end{aligned}$$

which gives in accordance with the definition of the subdifferential

$$0 \in \partial\mathcal{R}_\epsilon(z^{*\prime}(t)) + \mathrm{D}_z\mathcal{I}(\ell^*(t), z^*(t)) \text{ f.a.a. } t \in (0,T)$$

such that z^* solves the regularized system. In addition to that, since ℓ_n converges pointwisely weakly, cf. (4.3.5), we especially have $\ell_n(0) \rightharpoonup \ell^*(0)$ and $\ell_n(T) \rightharpoonup \ell^*(T)$ in $\mathcal{V}^*$. Thus, using the weak closedness of $\partial\mathcal{R}(0)$ results in $-\mathrm{D}_z\mathcal{I}(\ell^*(0), z_0) \in \partial\mathcal{R}(0)$. Moreover, (4.3.2) and (3.2.14) in combination with the weak lower semicontinuity of the distance from Lemma 3.3.5 yield $\operatorname{dist}_{\mathcal{V}^*}(-\mathrm{D}_z\mathcal{I}(\ell^*(T), z^*(T)), \partial\mathcal{R}(0)) \le \epsilon^{\frac{1}{8}}$. Therefore, the limit $(\ell^*, z^*) \in H^1(0,T;\mathcal{V}^*) \times H^1(0,T;\mathcal{Z})$ is indeed feasible for (vOCP$_\epsilon$). Finally, by exploiting the continuity of $j : \mathcal{V} \to \mathbb{R}$ together with the pointwise strong convergence (4.3.2) and the weak lower semicontinuity of the squared norm, we obtain for the objective function

$$\begin{aligned} j(z^*(T)) + \frac{\beta}{2}\|\ell^*\|^2_{H^1(0,T;\mathcal{V})} &\le \liminf_{n\to\infty} j(z_n(T)) + \frac{\beta}{2}\|\ell_n\|^2_{H^1(0,T;\mathcal{V})} \\ &= \lim_{n\to\infty} j(z_n(T)) + \frac{\beta}{2}\|\ell_n\|^2_{H^1(0,T;\mathcal{V})} = \xi,\end{aligned}$$

where ξ was infimum of the objective values regarding all feasible points $(z,\ell) \in H^1(0,T;\mathcal{Z}) \times H^1(0,T;\mathcal{V}^*)$. Hence, the limit $(z^*, \ell^*) \in H^1(0,T;\mathcal{Z}) \times H^1(0,T;\mathcal{V}^*)$ is in fact a global minimizer of (vOCP$_\epsilon$), which completes the proof. □

Now that the solvability of (vOCP$_\epsilon$) has been clarified, we come to the main result, the approximation of global minimizers via vanishing viscosity.

Theorem 4.3.2. *(i) Let $(\ell^*_\epsilon, z^*_\epsilon)_{\epsilon>0} \subset H^1(0,T;\mathcal{V}^*) \times H^1(0,T;\mathcal{Z})$ be a sequence of global minimizers of* (vOCP$_\epsilon$). *Moreover, let $(S^*_\epsilon, \hat{t}^*_\epsilon, \hat{z}^*_\epsilon) \in [T,\infty) \times W^{1,\infty}(0,S^*_\epsilon;\mathbb{R}) \times H^1(0,S^*_\epsilon;\mathcal{Z})$ be defined according to* (3.3.4), (3.3.5) *and* (3.3.8), *respectively, and assume that $\ell^*_\epsilon \rightharpoonup \ell^*$ in $H^1(0,T;\mathcal{V}^*)$, as $\epsilon \searrow 0$. Then there exists a (not relabeled) subsequence satisfying*

$$S^*_\epsilon \to S^* \text{ in } \mathbb{R}, \quad \hat{t}^*_\epsilon \rightharpoonup^* \hat{t}^* \text{ in } W^{1,\infty}(0,S^*;\mathbb{R}), \quad \hat{z}^*_\epsilon \rightharpoonup^* \hat{z}^* \text{ in } L^\infty(0,S^*;\mathcal{Z}), \quad (4.3.7)$$

$$\hat{z}^*_\epsilon \to \hat{z}^* \text{ uniformly in } C([0, S^*]; \mathcal{V}), \quad (4.3.8)$$

$$\hat{z}^*_\epsilon(s_\epsilon) \rightharpoonup \hat{z}^*(s) \text{ in } \mathcal{Z} \text{ for all converging sequences } s_\epsilon \to s, \quad (4.3.9)$$

and the limit $(S^*, \hat{t}^*, \hat{z}^*)$ *is a* $\mathfrak{p}$*-parametrized balanced viscosity solution associated with* ℓ^**, i.e.,* $(S^*, \hat{t}^*, \hat{z}^*) \in \mathcal{L}(\ell^*)$*. Moreover, the limit is feasible for* (OCP) *and it holds*

$$\liminf_{\epsilon \searrow 0} J_\epsilon(z^*_\epsilon, \ell^*_\epsilon) \geq J(S^*, \hat{z}^*, \ell^*). \tag{4.3.10}$$

(ii) Let $(\bar{S}, \bar{t}, \bar{z}, \bar{\ell})$ *be a global minimizer of* (OCP)*, which satisfies Assumption 4.2.1, i.e., in particular,* $\bar{t}'(s) \geq \delta > 0$ *for almost all* $s \in (0, \bar{S})$*, and the non-linearity* $\mathcal{F} : \mathcal{Z} \to \mathbb{R}$ *complies with Assumption 4.2.5. Then there exists a sequence* $(\bar{\ell}_\epsilon, \bar{z}_\epsilon)_{\epsilon>0} \subset H^1(0, T; \mathcal{V}^*) \times H^1(0, T; \mathcal{Z})$ *of feasible points for* (vOCP$_\epsilon$) *satisfying*

$$\lim_{\epsilon \searrow 0} J_\epsilon(\bar{z}_\epsilon, \bar{\ell}_\epsilon) = J(\bar{S}, \bar{z}, \bar{\ell}). \tag{4.3.11}$$

Proof. (i) The existence of a converging subsequence with (4.3.8) and a parametrized balanced viscosity solution as the limit follows from Theorem 3.3.6. The assumed convergence $\ell^*_\epsilon \rightharpoonup \ell^*$ in $H^1(0, T; \mathcal{V}^*)$ implies pointwise weak convergence such that $\ell^*_\epsilon(0) \rightharpoonup \ell^*(0)$ in $\mathcal{V}^*$ and thus $-\mathrm{D}_z\mathcal{I}(\ell^*(0), z_0) \in \partial\mathcal{R}(0)$ by the weak closedness of $\partial\mathcal{R}(0)$. Further, concerning the end time condition, we use again the pointwise convergence of the loads, which is $\ell^*_\epsilon(T) \rightharpoonup \ell^*(T)$ in $\mathcal{V}^*$, and $\hat{z}^*_\epsilon(S^*_\epsilon) \rightharpoonup \hat{z}^*(S^*)$ in $\mathcal{Z}$ from (4.3.9) in order to infer from Lemma 3.3.5

$$\begin{aligned} \operatorname{dist}_{\mathcal{V}^*}(-\mathrm{D}_z\mathcal{I}(\ell^*(T), \hat{z}^*(S^*)), \partial\mathcal{R}(0)) &\leq \liminf_{\epsilon \searrow 0} \operatorname{dist}_{\mathcal{V}^*}(-\mathrm{D}_z\mathcal{I}(\ell^*_\epsilon(T), \hat{z}^*_\epsilon(S^*_\epsilon)), \partial\mathcal{R}(0)) \\ &\leq \liminf_{\epsilon \searrow 0} \epsilon^{\frac{1}{8}} = 0. \end{aligned}$$

Note that Lemma 3.3.5 was applicable due to the a priori bound (3.2.13). In the limit $\epsilon \searrow 0$ we thus have shown that $-\mathrm{D}_z\mathcal{I}(\ell^*(T), \hat{z}^*(S^*)) \in \partial\mathcal{R}(0)$. Therefore, the limit $(S^*, \hat{t}^*, \hat{z}^*, \ell^*)$ is feasible for (OCP). Furthermore, the uniform convergence of $\hat{z}^*_\epsilon$ from (4.3.8) yields $z^*_\epsilon(T) = \hat{z}^*_\epsilon(S^*_\epsilon) \to \hat{z}^*(S^*)$ strongly in $\mathcal{V}$ such that exploiting the continuity of $j : \mathcal{V} \to \mathbb{R}$ and the weak lower semicontinuity of the squared norm results in

$$\begin{aligned} \liminf_{\epsilon \searrow 0} J_\epsilon(z^*_\epsilon, \ell^*_\epsilon) &= \liminf_{\epsilon \searrow 0} j(z^*_\epsilon(T)) + \frac{\beta}{2}\|\ell^*_\epsilon\|^2_{H^1(0,T;\mathcal{V}^*)} \\ &\geq j(\hat{z}^*(S^*)) + \frac{\beta}{2}\|\ell^*\|^2_{H^1(0,T;\mathcal{V}^*)} = J(S^*, \hat{z}^*, \ell^*). \end{aligned}$$

(ii) Next we want to prove the second assertion. To this end, with $\epsilon > 0$ sufficiently small Theorem 4.2.8 yields the existence of sequence $(\bar{\ell}_\epsilon, \bar{z}_\epsilon)_{\epsilon>0} \subset H^1(0, T; \mathcal{V}^*) \times H^1(0, T; \mathcal{Z})$, which is feasible for regularized optimal control problems (vOCP$_\epsilon$). Moreover, from (4.2.35) we infer that $\bar{z}_\epsilon(T) \to \bar{z}(\bar{t}^{-1}(T)) = \bar{z}(\bar{S})$ strongly in $\mathcal{V}$ such that the continuity of j yields $j(\bar{z}_\epsilon(T)) \to j(\bar{z}(\bar{S}))$. Eventually, by exploiting the strong convergence $\bar{\ell}_\epsilon \to \bar{\ell}$ in $H^1(0,T;\mathcal{V}^*)$ from (4.2.35), we end up with

$$\begin{aligned}\lim_{\epsilon\searrow 0} J_\epsilon(\bar{z}_\epsilon, \bar{\ell}_\epsilon) &= \lim_{\epsilon\searrow 0} j(\bar{z}_\epsilon(T)) + \frac{\beta}{2}\|\bar{\ell}_\epsilon\|^2_{H^1(0,T;\mathcal{V}^*)} \\ &= j(\bar{z}(\bar{S})) + \frac{\beta}{2}\|\bar{\ell}\|^2_{H^1(0,T;\mathcal{V}^*)} = J(\bar{S}, \bar{z}, \bar{\ell}).\end{aligned}$$

□

Theorem 4.3.3. *Let* $(\ell^*_\epsilon, z^*_\epsilon)_{\epsilon>0} \subset H^1(0,T;\mathcal{V}^*) \times H^1(0,T;\mathcal{Z})$ *be a sequence of global minimizers of* (vOCP$_\epsilon$) *and* $(S^*_\epsilon, \hat{t}^*_\epsilon, \hat{z}^*_\epsilon)$ *are defined in accordance with* (3.3.4)*,* (3.3.5) *and* (3.3.8)*. Moreover,* $(\bar{S}, \bar{t}, \bar{z}, \bar{\ell})$ *is supposed to be a global minimizer of* (OCP)*, which complies with Assumption 4.2.1, and* $\mathcal{F} : \mathcal{Z} \to \mathbb{R}$ *satisfies Assumption 4.2.5. Then there exists a weak accumulation point in the sense that*

$$S^*_\epsilon \to S^* \text{ in } \mathbb{R}, \quad \hat{t}^*_\epsilon \rightharpoonup^* \hat{t}^* \text{ in } W^{1,\infty}(0, S^*; \mathbb{R}), \quad \ell^*_\epsilon \rightharpoonup \ell^* \text{ in } H^1(0,T;\mathcal{V}^*), \tag{4.3.12}$$

$$\hat{z}^*_\epsilon \rightharpoonup^* \hat{z}^* \text{ in } L^\infty(0, S^*; \mathcal{Z}), \quad \hat{z}^*_\epsilon \to \hat{z}^* \text{ uniformly in } C([0, S^*]; \mathcal{V}), \tag{4.3.13}$$

for a not relabeled subsequence and this accumulation point is a global minimizer of (OCP)*. Moreover, it even holds strong convergence for the external loads, which is* $\ell_\epsilon \to \ell^*$ *strongly in* $H^1(0,T;\mathcal{V}^*)$*.*

Proof. In accordance with the standing assumption (1.2.26), the constant functions $z \equiv z_0$ and $\ell \equiv \ell_0$ are feasible for (vOCP$_\epsilon$). Therefore, the objective is bounded independent of $\epsilon > 0$ by $J_\epsilon(z^*_\epsilon, \ell^*_\epsilon) \leq j(z_0) + \frac{\beta T}{2}\|\ell_0\|^2_{\mathcal{V}^*}$. Since j is supposed to be bounded from below, we have that $\|\ell^*_\epsilon\|_{H^1(0,T;\mathcal{V}^*)} \leq C$ with $C > 0$ not depending on $\epsilon > 0$, too. Thus, there exists a weakly converging (not relabeled) subsequence such that $\ell^*_\epsilon \rightharpoonup \ell^*$ in $H^1(0,T;\mathcal{V}^*)$. Now the first statement of Theorem 4.3.2 yields (4.3.12) and (4.3.13) for a further subsequence, which we also denote by the same symbol. Moreover, the limit is a parametrized solution associated with ℓ^*. Eventually, from (4.3.10), the assumed optimality of $(\ell^*_\epsilon, z^*_\epsilon)$ and the feasibility of $(\bar{\ell}_\epsilon, \bar{z}_\epsilon)$ for (vOCP$_\epsilon$) and (4.3.11), we infer

$$\begin{aligned}J(S^*, \hat{z}^*, \ell^*) &\leq \liminf_{\epsilon\searrow 0} J_\epsilon(z^*_\epsilon, \ell^*_\epsilon) \leq \limsup_{\epsilon\searrow 0} J_\epsilon(z^*_\epsilon, \ell^*_\epsilon) \\ &\leq \lim_{\epsilon\searrow 0} J_\epsilon(\bar{z}_\epsilon, \bar{\ell}_\epsilon) = J(\bar{S}, \bar{z}, \bar{\ell})\end{aligned}$$

such that $(S^*, \hat{z}^*, \ell^*)$ is indeed a global minimizer of (OCP). Additionally, all estimates hold in fact with equality such that the limit inferior and the limit superior coincide and the objective has to converge. Since $j(z^*_\epsilon(T)) \to j(\hat{z}^*(S^*))$ by (4.3.13) and the continuity of j, we obtain that

$$\begin{aligned}\|\ell^*_\epsilon\|^2_{H^1(0,T;\mathcal{V}^*)} &= \frac{2}{\beta}\big(J_\epsilon(z^*_\epsilon, \ell^*_\epsilon) - j(z^*_\epsilon(T))\big) \\ &\to \frac{2}{\beta}\big(J(S^*, \hat{z}^*, \ell^*) - j(\hat{z}^*(S^*))\big) = \|\ell^*\|^2_{H^1(0,T;\mathcal{V}^*)}.\end{aligned}$$

Finally, because $H^1(0,T;\mathcal{V}^*)$ is a Hilbert space, this in combination with the weak convergence of ℓ^*_ϵ yields strong convergence, which completes the proof. □

Chapter 5

Stability of parametrized BV solutions for discontinuous loads

In the last chapter we have seen that we can approximate the optimal control problem (OCP) by its viscous regularized versions (vOCP$_\epsilon$), where the state is not longer given as a parametrized BV solution associated with the external load ℓ but as a solution of the viscous regularized system (RIS$_\epsilon$). Unfortunately, this was only possible under the pretty restrictive assumption that there exists at least one optimal solution of (OCP) that complies with Assumption 4.2.1, which in turn yields that the into physical time reparametrized function $\tilde{z} = \hat{z} \circ \hat{t}^{-1}$ is a differential solution of (RIS), whose existence is not guaranteed in the case of a non-convex energy.

One part to prove this approximability was to construct a recovery sequence, which is feasible for (vOCP$_\epsilon$) and whose objective value converges to the minimum of (OCP). The construction was based on penalizing the energy in the way that the assumed differential solution appears in a quadratic penalty term, which enforces the solution of the penalized system to converge to $\tilde{z}$. Further, this penalty term was then included in the loads by $\ell_\epsilon := \ell - \eta(z_\epsilon - \tilde{z})$ in order to obtain the desired recovery sequence $(\ell_\epsilon, z_\epsilon)_{\epsilon>0}$. This is the part, where the restrictive continuity assumption comes into play because in accordance with our preceding analysis the load ℓ_ϵ has to be an element of $H^1(0,T;\mathcal{V}^*)$, which is not ensured for general parametrized BV solutions not satisfying Assumption 4.2.1. For example, if a viscous jump occurs, i.e., $\hat{t}' = 0$ almost everywhere in an interval (s_1, s_2), then one directly sees that the reparametrized function $\tilde{z} = \hat{z} \circ \hat{t}^{-1}$ jumps in $t = \hat{t}^{-1}(s_1) = \hat{t}^{-1}(s_2)$ such that $\ell_\epsilon := \ell - \eta(z_\epsilon - \tilde{z})$ jumps, too, and is no longer in $H^1(0,T;\mathcal{V}^*)$.

Now the natural idea to weaken this assumption is to allow discontinuous loads. Rate-independent systems with discontinuous data are already considered by various authors. For example, we refer to [KL09, Rec11, Rec16, RS18]. It also exists the notion of $\mathfrak{p}$-parametrized balanced viscosity solutions for discontinuous loads, which was considered in [KZ21]. Unfortunately, it will turn out that this solution concept is not suitable in the context of optimal control and we have to weaken the properties, which then leads to a relaxed solution concept. However, these solutions have other disadvantages, which we will illustrate by a selection of examples. The results in this chapter have already been published in [AM24]. We explicitly point out which theorem, lemma or other from [AM24] is involved.

Standing Assumptions

In this chapter we consider the simpler finite dimensional case that $\mathcal{Z} = \mathcal{V} = \mathcal{X} = \mathbb{R}^d$ since this slightly simplifies the analysis and is already sufficient to show the arising problems by dealing with discontinuous loads for rate-independent systems in the context of optimal control. We assume that the energy $\mathcal{I} : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ has the same structure as in Section 1.2, but now we also allow external loads $\ell \in BV([0,T];\mathbb{R}^d)$. For the convenience of the reader and since the spaces have changed, we briefly repeat our assumptions on energy and dissipation. For a given external load $\ell \in BV([0,T];\mathbb{R}^d)$, the energy functional reads as

$$\mathcal{I}(\ell(t), z) = \frac{1}{2}\langle Az, z\rangle + \mathcal{F}(z) - \langle \ell(t), z\rangle$$

and by denoting the time independent part as $\mathcal{E}(z) = \frac{1}{2}\langle Az, z\rangle + \mathcal{F}(z)$, the energy can be written as $\mathcal{I}(\ell(t), z) = \mathcal{E}(z) - \langle \ell(t), z\rangle$. Herein, $A \in \mathbb{R}^{d\times d}$ is supposed to be symmetric and positively definite and $\mathcal{F} : \mathbb{R}^d \to \mathbb{R}$ complies with (1.2.4) and (1.2.5), which is

$$\mathcal{F} \in C^2(\mathbb{R}^d;\mathbb{R}),\ \mathcal{F} \geq 0, \tag{5.0.1}$$

$$\|D^2\mathcal{F}(z)v\| \leq C(1 + \|z\|^q)\|v\| \ \forall v, z \in \mathbb{R}^d. \tag{5.0.2}$$

Moreover, we assume that the dissipation $\mathcal{R} : \mathbb{R}^d \to [0,\infty)$ satisfies (1.2.13)-(1.2.15), i.e., convexity, lower semicontinuity, positive 1-homogeneity and

$$c\|v\| \leq \mathcal{R}(v) \leq C\|v\| \quad \text{for all } v \in \mathbb{R}^d \tag{5.0.3}$$

with positive constants $c, C > 0$.

Here we understand $\ell \in BV([0,T];\mathbb{R}^d)$ as a pointwise defined function of bounded variation, i.e.,

$$\operatorname{Var}(\ell;[0,T]) := \sup\left\{ \sum_{i=1}^{n} \|\ell(t_i) - \ell(t_{i-1})\| \,:\, 0 = t_0 < \cdots < t_n = T,\ n \in \mathbb{N} \right\} < \infty.$$

We equip $BV([0,T];\mathbb{R}^d)$ with the norm

$$\|f\|_{BV([0,T];\mathbb{R}^d)} := \|f\|_{L^1(0,T;\mathbb{R}^d)} + \operatorname{Var}(f;[0,T])$$

such that $BV([0,T];\mathbb{R}^d)$ is a Banach space, cf. [ABD14, Prop. 1.10]. Otherwise, one could also consider the space of equivalence classes of functions of bounded variation on $(0,T)$, which we denote here by

$$\mathrm{BV}(0,T;\mathbb{R}^d) := \left\{ f \in L^1(0,T;\mathbb{R}^d) : \mathrm{D}f \in \mathfrak{M}(0,T;\mathbb{R}^d) \right\},$$

where $\mathrm{D}f$ denotes the distributional derivative and $\mathfrak{M}(0,T;\mathbb{R}^d)$ is the space of $\mathbb{R}^d$-valued regular Borel measures on $(0,T)$. In the counterexamples 5.1.2 and 5.2.11 we only consider left or right continuous functions, i.e., the "good" representatives of the associated equivalence class, and the statement does not depend on the representative such that it is also valid for considering equivalence classes. In general, we will see that the definition of $\mathfrak{p}$-parametrized balanced viscosity solutions (and also the later introduced relaxed solutions) with discontinuous loads does not depend on explicit pointwise representation of the load. Only when we consider the functions in physical time, instead of artificial time as in the case of parametrized solutions, we will make use of a concrete representative because here both, state and load function, can jump at the same time and the energy inequality provides different results depending on the respective representative, see Example 5.2.12 and 5.2.13.

5.1 Solution concepts for discontinuous loads in $BV(0,T)$

We start with following definition of normalized, $\mathfrak{p}$-parametrized balanced viscosity solutions for rate-independent systems with discontinuous loads.

Definition 5.1.1 ($\mathfrak{p}$-parametrized BV solutions)**.** Let $\ell \in BV([0,T];\mathbb{R}^d)$ be given. We call $(S,\hat{t},\hat{z},\hat{\ell}) \in (0,\infty)\times W^{1,\infty}(0,S)\times W^{1,\infty}(0,S;\mathbb{R}^d)\times BV([0,S];\mathbb{R}^d)$ a *normalized, $\mathfrak{p}$-parametrized balanced viscosity (BV) solution* of the rate-independent system (RIS) associated with ℓ, if

- the *initial and end time condition* hold
$$\hat{t}(0)=0,\ \hat{t}(S)=T,\ \hat{z}(0)=z_0, \tag{5.1.1}$$

- the *complementarity relations and normalization condition* are fulfilled for almost all $s\in(0,S)$
$$\hat{t}'(s)\geq 0,\ \hat{t}'(s)\,\mathrm{dist}(-\mathrm{D}_z\mathcal{I}(\hat{\ell}(s),\hat{z}(s)),\partial\mathcal{R}(0))=0, \tag{5.1.2}$$
$$\hat{t}'(s)+\mathcal{R}(\hat{z}'(s))+\|\hat{z}'(s)\|\,\mathrm{dist}(-\mathrm{D}_z\mathcal{I}(\hat{\ell}(s),\hat{z}(s)),\partial\mathcal{R}(0))=1, \tag{5.1.3}$$

- the *energy identity* is valid for all $s_1,s_2\in[0,S]$
$$\mathcal{E}(\hat{z}(s_2))+\int_{s_1}^{s_2}\mathcal{R}(\hat{z}'(r))+\|\hat{z}'(r)\|\,\mathrm{dist}(-\mathrm{D}_z\mathcal{I}(\hat{\ell}(r),\hat{z}(r)),\partial\mathcal{R}(0))\,\mathrm{d}r \\ =\mathcal{E}(\hat{z}(s_1))+\int_{s_1}^{s_2}\langle\hat{\ell}(r),\hat{z}'(r)\rangle\,\mathrm{d}r, \tag{5.1.4}$$

- the parametrized load $\hat{\ell}$ is compatible with ℓ in the following sense: for every $t_*\in[0,T]$, there exists $s_*\in\hat{t}^{-1}(t_*)$ such that for all $s\in\hat{t}^{-1}(t_*)$
$$\hat{\ell}(s)=\begin{cases}\ell(t_*-), & s<s_*,\\ \ell(t_*+), & s>s_*,\end{cases}\quad\text{and}\quad\hat{\ell}(s_*)\in\{\ell(t_*),\ell(t_*-),\ell(t_*+)\}, \tag{5.1.5}$$

where $\ell(t-) := \lim_{\tau \nearrow t} \ell(\tau)$ and $\ell(t+) := \lim_{\tau \searrow t} \ell(\tau)$ denote the one sided limits.

Here and in the following, we set $0- := 0$ and $T+ := T$.

The existence of such solutions is already shown in [KZ21], even in the more general infinite dimensional setting from Chapter 3. Compared to the definition of parametrized BV solutions for loads in $H^1(0,T;\mathcal{V}^*)$, (5.1.1)-(5.1.4) corresponds to (3.3.21)-(3.3.25). Note that the state function z now has an increased regularity, resulting from the finite dimensional setting considered here. This can be understood as follows. The normalization condition (3.3.16) along with the lower estimate from (1.2.15) yields

$$
\begin{aligned}
c\|\hat{z}_\epsilon'(s)\| &\le \mathcal{R}(\hat{z}_\epsilon'(s)) \\
&\le \hat{t}_\epsilon'(s) + \mathcal{R}(\hat{z}_\epsilon'(s)) + \|\hat{z}_\epsilon'(s)\| \operatorname{dist}(-\mathrm{D}_z\mathcal{I}(\ell(\hat{t}_\epsilon(s)), \hat{z}_\epsilon(s)), \partial\mathcal{R}(0)) = 1
\end{aligned}
$$

for almost all $s \in (0, S_\epsilon)$. Therefore, in combination with (3.2.9) we infer that $(\hat{z}_\epsilon)_{\epsilon>0}$ is uniformly bounded in $W^{1,\infty}(0,\bar{S};\mathbb{R}^d)$, where again $\bar{S} = \sup_{\epsilon>0} S_\epsilon$ such that all functions can be constantly extended on a common domain. Thus, the weak limit $\hat{z}$ is also weakly differentiable and we may consider the weak derivative instead of the metric derivative with respect to $\mathcal{R}$, which results in the handier complementary relation, normalization condition and energy identity. Moreover, the energy identity in (5.1.4) is expressed in terms of the potential energy $\mathcal{E}$. By means of integration by parts for Riemann-Stieltjes integrals, cf. [Apo74, Thm. 7.6], one can equivalently express the energy identity in terms of the total stored energy functional $\mathcal{I}$ as in (3.3.25). But the crucial difference is the composition of the external load in artificial time. Instead of $\hat{\ell} = \ell \circ \hat{t}$ we now have (5.1.5). This condition basically says that, if ℓ is discontinuous in a viscous jump, then it only attains its left and right limits during the viscous jump. Note that, if ℓ is continuous in t^*, then $\hat{\ell}(s) = \ell(\hat{t}(s))$ for all $s \in \hat{t}^{-1}(t^*)$.

A drawback of this solution concept is that it is not suitable for applications in optimal control. If we consider a minimization problem of the form (OCP), where the $H^1(0,T;\mathbb{R}^d)$-norm of the loads in the objective is replaced by the $BV([0,T];\mathbb{R}^d)$-norm and the feasible set is described by parametrized BV solutions in the sense of Definition 5.1.1, we are not able to show the existence of an optimal solution since by taking a weakly/pointwisely convergent subsequence of an infimal sequence, one cannot guarantee in general that the limit is also a parametrized BV solution associated with the limit of the loads according to Definition 5.1.1. This is illustrated by the following one-dimensional example, which even shows that this solution concept is not stable with respect to intermediate/strict convergence of a sequence of loads $(\ell_n)_{n\in\mathbb{N}}$ in $BV([0,T];\mathbb{R})$.

Example 5.1.2 ([AM24, Sec. 2.1])**.** We consider the following dissipation and energy:

$$
\mathcal{R}(z) = |z|, \qquad \mathcal{E}(z) = \frac{1}{2}z^2 - z, \qquad \mathcal{I}(\ell(t), z) = \mathcal{E}(z) - \ell(t)z, \tag{5.1.6}
$$

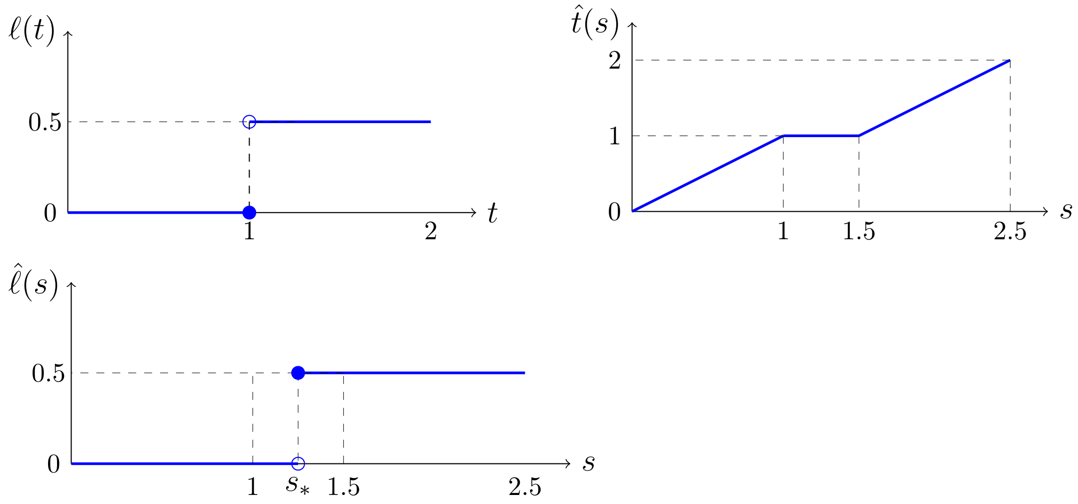

Figure 5.1: Example for an external load that is discontinuous in a viscous jump at time $t = 1$

where $\ell \in BV([0,2];\mathbb{R})$. The initial value and end time are set to $z_0 = 0$ and $T = 2$. The sequence of external loads is given by

$$\ell_n(t) = \begin{cases} 0, & t \in [0,1], \\ \frac{n}{2}t - \frac{n}{2}, & t \in (1, 1+\frac{1}{n}), \\ \frac{1}{2}, & t \in [1+\frac{1}{n}, 2]. \end{cases} \tag{5.1.7}$$

Then ℓ_n converges to

$$\ell = \begin{cases} 0, & t \in [0,1], \\ \frac{1}{2}, & t \in (1,2], \end{cases} \tag{5.1.8}$$

with respect to intermediate/strict convergence in $BV([0,2];\mathbb{R})$. By direct calculations one verifies that a normalized, $\mathfrak{p}$-parametrized BV solution associated with ℓ_n is given by

$$\hat{z}_n(s) = \begin{cases} 0, & s \in [0,1], \\ \frac{\frac{n}{2}}{1+\frac{n}{2}}s - \frac{\frac{n}{2}}{1+\frac{n}{2}}, & s \in (1, \frac{3}{2}+\frac{1}{n}), \\ \frac{1}{2}, & s \in [\frac{3}{2}+\frac{1}{n}, \frac{5}{2}], \end{cases} \tag{5.1.9}$$

$$\hat{t}_n(s) = \begin{cases} s, & s \in [0,1], \\ \frac{1}{1+\frac{n}{2}}s + \frac{\frac{n}{2}}{1+\frac{n}{2}}, & s \in (1, \frac{3}{2}+\frac{1}{n}), \\ s - \frac{1}{2}, & s \in [\frac{3}{2}+\frac{1}{n}, \frac{5}{2}], \end{cases} \tag{5.1.10}$$

along with $S_n = \frac{5}{2}$ and

$$\hat{\ell}_n(s) = \ell_n(\hat{t}_n(s)) = \begin{cases} 0, & s \in [0,1], \\ \frac{\frac{n}{2}}{1+\frac{n}{2}}s - \frac{\frac{n}{2}}{1+\frac{n}{2}}, & s \in (1, \frac{3}{2}+\frac{1}{n}), \\ \frac{1}{2}, & s \in [\frac{3}{2}+\frac{1}{n}, \frac{5}{2}]. \end{cases} \tag{5.1.11}$$

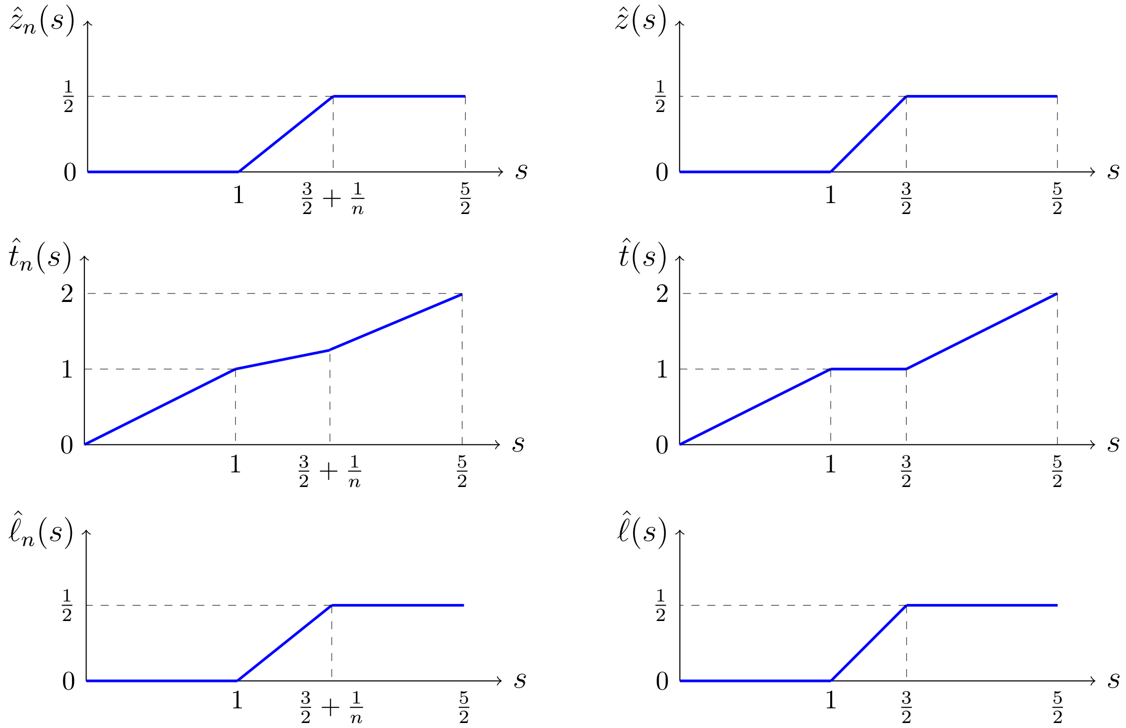

Figure 5.2: Graphs of the $\mathfrak{p}$-parametrized BV solution $(\hat{z}_n, \hat{t}_n, \hat{\ell}_n)$ from (5.1.9)-(5.1.11) associated with external load (5.1.7) and its corresponding limits $(\hat{z}, \hat{t}, \hat{\ell})$ from (5.1.12) and (5.1.13)

The pointwise-a.e. limits of these sequences for $n \to \infty$ read

$$\hat{z}(s) = \begin{cases} 0, & s \in [0,1], \\ s-1, & s \in (1, \frac{3}{2}), \\ \frac{1}{2}, & s \in [\frac{3}{2}, \frac{5}{2}], \end{cases} \quad \hat{t}(s) = \begin{cases} s, & s \in [0,1], \\ 1, & s \in (1, \frac{3}{2}), \\ s-\frac{1}{2}, & s \in [\frac{3}{2}, \frac{5}{2}], \end{cases} \tag{5.1.12}$$

$$\hat{\ell}(s) = \begin{cases} 0, & s \in [0,1], \\ s-1, & s \in (1, \frac{3}{2}), \\ \frac{1}{2}, & s \in [\frac{3}{2}, \frac{5}{2}]. \end{cases} \tag{5.1.13}$$

Note that $\hat{z}$ and $\hat{t}$ are not only the pointwise limits of $\hat{z}_n$ and $\hat{t}_n$, but also the strong limits in $H^1(0,S;\mathbb{R})$ and weak* limits in $W^{1,\infty}(0,S;\mathbb{R})$. Moreover, $\hat{\ell}$ is the limit of $\hat{\ell}_n$ with respect to intermediate/strict convergence in $BV([0,S];\mathbb{R})$. However, $(S, \hat{t}, \hat{z}, \hat{\ell})$ is no normalized, $\mathfrak{p}$-parametrized BV solution associated with the limit ℓ from (5.1.8) according to Definition 3.3.4 since condition (5.1.5) is violated in the viscous jump $(1, \frac{3}{2})$. Here $\hat{\ell}$ cannot be expressed by the left or right hand side limit of ℓ, i.e., $\hat{\ell}(s) \notin \{\ell(\hat{t}(s)-), \ell(\hat{t}(s)), \ell(\hat{t}(s)+)\}$ for all $s \in (1, \frac{3}{2})$.

This example shows that the limit of the composition $\ell_n \circ \hat{t}_n$ does in general not coincide with the composition of the separate limits $\ell \circ \hat{t}$ in jumps. Note that ℓ_n also

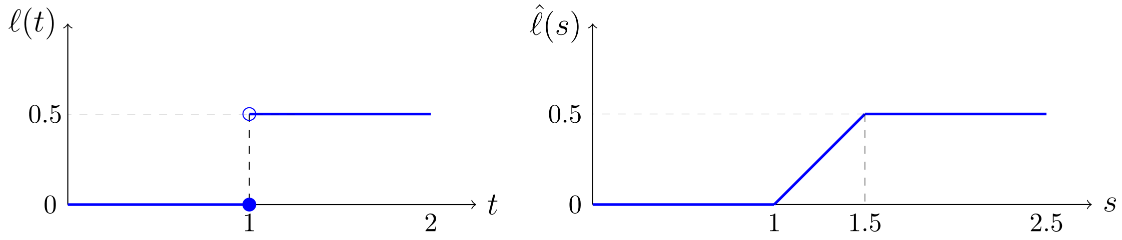

Figure 5.3: Violation of condition (5.1.5) in the viscous jump $(1, 1.5)$ for the limits (5.1.8) and (5.1.13)

converges with respect to weak* and intermediate/strict convergence in $\mathrm{BV}(0,T;\mathbb{R})$, the appropriate space of equivalence classes, so the statement is also valid in this setting. Hence, if we aim for a solution concept that is stable with respect to pointwise convergence of the external loads ℓ_n (or w.r.t. weak* convergence in $\mathrm{BV}(0,T;\mathbb{R}^d)$), we have to weaken the requirements on $\hat{\ell}$. To be more precise, the above example shows that one can hardly impose any condition on $\hat{\ell}$ in jumps. For this reason, we drop the compatibility condition in (5.1.5) and replace it by a less restrictive condition that does not provide any information on $\hat{\ell}$ in jumps. This leads to the following relaxed solution concept.

Definition 5.1.3 (Relaxed solution concept)**.** Let $\ell \in BV([0,T];\mathbb{R}^d)$ be given. We call $(S,\hat{t},\hat{z},\hat{\ell}) \in (0,\infty) \times W^{1,\infty}(0,S) \times W^{1,\infty}(0,S;\mathbb{R}^d) \times BV([0,S];\mathbb{R}^d)$ a *relaxed, normalized,* $\mathfrak{p}$*-parametrized BV solution* (or short simply *relaxed solution*) of the rate-independent system (RIS) associated with ℓ, if it satisfies (5.1.1)–(5.1.4) and, instead of the compatibility condition in (5.1.5), we just require

$$\hat{\ell}(s) = \ell(\hat{t}(s)) \text{ for almost all } s \in M, \tag{5.1.14}$$

where M is the set, where $\hat{t}$ is strictly increasing, i.e.,

$$M = \Big\{ s \in (0,S) : \hat{t}(s_1) < \hat{t}(s_2) \text{ for all } s_1, s_2 \in [0,S] \text{ with } s_1 < s < s_2 \Big\}. \tag{5.1.15}$$

Herein, we choose the continuous representative of $\hat{t}$ denoted by the same symbol for simplicity.

The next lemma states that this is indeed a generalization of the original Definition 5.1.1. Therefore, the existence of a relaxed solution is ensured by the existence of parametrized BV solution in the sense of Definition 5.1.1, which is even guaranteed in the more general infinite dimensional setting, cf. [KZ21, Thm. 4.5].

Lemma 5.1.4 ([AM24, Lem. 3.2])**.** *Let $\ell \in BV([0,T];\mathbb{R}^d)$ be given and suppose that $(S,\hat{t},\hat{z},\hat{\ell}) \in (0,\infty) \times W^{1,\infty}(0,S) \times W^{1,\infty}(0,S;\mathbb{R}^d) \times BV([0,S];\mathbb{R}^d)$ is a normalized, $\mathfrak{p}$-parametrized BV solution in the sense of Definition 5.1.1. Then it is also a relaxed solution in the sense of Defintion 5.1.3.*

Proof. We only have to verify the condition in (5.1.14). To this end, let $s_* \in M$ be arbitrary and set $t_* := \hat{t}(s_*)$. By definition of M, $\hat{t}^{-1}(t_*)$ is a singleton and therefore (5.1.5) implies $\hat{\ell}(s_*) \in \{\ell(t_*), \ell(t_*-), \ell(t_*+)\}$. Since ℓ is of bounded variation, it only has countably many jumps collected in the set $J(\ell) \subset [0,T]$ and is continuous elsewhere. Since $J(\ell)$ is countable and $\hat{t}$ is one-to-one on M, $\hat{t}^{-1}(J(\ell)) \cap M$ has zero measure and thus $\ell(\hat{t}(s)) = \ell(\hat{t}(s)-) = \ell(\hat{t}(s)+)$ f.a.a. $s \in M$, which implies (5.1.14). $\square$

Next, we verify that this concept of relaxed solutions has indeed the desired stability properties that were missing for solutions in the sense of Definition 3.3.4.

Theorem 5.1.5 ([AM24, Thm. 3.4])**.** *Let $(\ell_n)_{n\in\mathbb{N}}$ be a bounded sequence in $BV([0,T];\mathbb{R}^d)$ and consider a sequence $(S_n, \hat{t}_n, \hat{z}_n, \hat{\ell}_n)$ of relaxed solutions associated with ℓ_n. Assume moreover that the sequence $(\hat{\ell}_n)_{n\in\mathbb{N}}$ is bounded in $BV([0,S_n];\mathbb{R}^d)$. Then there exists a (not relabeled) subsequence such that*

$$S_n \to S, \quad \hat{z}_n \rightharpoonup^* \hat{z} \text{ in } W^{1,\infty}(0,S;\mathbb{R}^d), \quad \hat{t}_n \rightharpoonup^* \hat{t} \text{ in } W^{1,\infty}(0,S) \tag{5.1.16}$$

and

$$\hat{\ell}_n(s) \to \hat{\ell}(s) \quad \text{for all } s \in [0,S], \quad \ell_n(t) \to \ell(t) \quad \text{for all } t \in [0,T]. \tag{5.1.17}$$

Herein the functions are constantly extended if $S_n < S$. Furthermore, the limit $(S, \hat{t}, \hat{z}, \hat{\ell})$ is a relaxed solution associated with ℓ.

Proof. **Step 1: Convergence of a subsequence**
We start with the boundedness of the sequence of relaxed solutions. First of all, from (5.1.3) we infer that $\|\hat{z}'\|_{L^\infty(0,S_n;\mathbb{R}^d)} \leq \frac{1}{c}$ with $c > 0$ from (5.0.3). Next, we prove that the artificial end time is uniformly bounded. For this purpose, thanks to the boundedness of $\hat{z}_n'$ and $\hat{\ell}_n$ (by assumption), the energy identity (5.1.4) yields

$$\begin{aligned} \mathcal{E}(\hat{z}_n(S_n)) + \int_0^{S_n} \mathcal{R}(\hat{z}_n'(r)) + \|\hat{z}_n'(r)\| \operatorname{dist}(-\mathrm{D}_z\mathcal{I}(\hat{\ell}_n(r), \hat{z}_n(r)), \partial\mathcal{R}(0)) \,\mathrm{d}r \\ = \mathcal{E}(z_0) + \int_0^{S_n} \langle \hat{\ell}_n(r), \hat{z}_n'(r)\rangle \,\mathrm{d}r \leq \mathcal{E}(z_0) + \|\hat{z}_n'\|_{L^\infty(0,S_n;\mathbb{R}^d)} \|\hat{\ell}_n\|_{L^1(0,S_n;\mathbb{R}^d)} \leq C. \end{aligned}$$

In combination with (5.1.3), this implies

$$\begin{aligned} S_n &= \int_0^{S_n} \hat{t}_n'(r) + \mathcal{R}(\hat{z}_n'(r)) + \|\hat{z}_n'(r)\| \operatorname{dist}(-\mathrm{D}_z\mathcal{I}(\hat{\ell}_n(r), \hat{z}_n(r)), \partial\mathcal{R}(0)) \,\mathrm{d}r \\ &= T + \int_0^{S_n} \mathcal{R}(\hat{z}_n'(r)) + \|\hat{z}_n'(r)\| \operatorname{dist}(-\mathrm{D}_z\mathcal{I}_n(\hat{\ell}(r), \hat{z}_n(r)), \partial\mathcal{R}(0)) \,\mathrm{d}r \leq C, \end{aligned}$$

where we made use of the non-negativity of $\mathcal{E}(\hat{z}_n(S_n))$ by assumption (5.0.1) and the fact that A is positive definite. Therefore, there is a subsequence (denoted by the same index) such that $S_n \to S$. As in the statement of the theorem, we extend $\hat{t}_n$, $\hat{z}_n$, and $\hat{\ell}_n$ by constant continuation, if necessary, i.e., if $S_n < S$. Due to $\|\hat{z}_n'\|_{L^\infty(0,S;\mathbb{R}^d)} \leq 1/c$ and $\hat{z}_n(0) = z_0$ for all $n \in \mathbb{N}$, we have $\|\hat{z}_n\|_{W^{1,\infty}(0,S;\mathbb{R}^d)} \leq C$. Furthermore, due to (5.1.1)-(5.1.3), we conclude $\|\hat{t}_n\|_{W^{1,\infty}(0,S)} \leq C$. Thus, there exists a further subsequence (again denoted by the same symbol) such that (5.1.16) holds. The pointwise convergence of the sequences of loads along a further subsequence is an immediate consequence of the boundedness of the respective sequences by assumption, cf. [Nat75, Ch.8, Thm.(E.Helly)],

Step 2: Correlation between $\hat{\ell}$ and ℓ

Next, we show that (5.1.14) holds true in the limit. We argue by contradiction and assume that the set

$$G := \Big\{ s \in M : \hat{\ell}(s) \neq \ell(\hat{t}(s)) \Big\} \subset M \tag{5.1.18}$$

has positive measure. This yields the existence of a $\rho > 0$ such that

$$\widetilde{M}_\rho = M \cap \Big\{ s \in (0,S) : \|\hat{\ell}(s) - \ell(\hat{t}(s))\| \geq \rho \Big\}$$

has positive measure, too, since if not, then the sets $\{s \in M : \|\hat{\ell}(s) - \ell(\hat{t}(s))\| \geq \frac{1}{n}\}$ have measure zero for all $n \in \mathbb{N}$ and thus $G = \bigcup_{n\in\mathbb{N}} \{s \in M : \|\hat{\ell}(s) - \ell(\hat{t}(s))\| \geq \frac{1}{n}\}$ as a countable union of null sets, too, which contradicts the assumption. Since $\hat{t}_n, \hat{\ell}_n$ are part of the relaxed solution associated with ℓ_n, we have $\hat{\ell}_n(s) = \ell_n(\hat{t}_n(s))$ almost everywhere in

$$M_n := \Big\{ s \in (0,S) : \hat{t}_n(s_1) < \hat{t}_n(s_2) \text{ for all } s_1 < s < s_2 \Big\}.$$

We denote the corresponding null sets, where this is not satisfied, by N_n. Then

$$M_\rho := \widetilde{M}_\rho \setminus \bigcup_{n\in\mathbb{N}} N_n$$

still has positive measure. Now let $k \in \mathbb{N}$, $k > 1$, be arbitrary. Since M_ρ has positive measure, there exists points $\tilde{s}_1 < \tilde{s}_2 < \ldots < \tilde{s}_{2k} \in M_\rho$ and, by construction of M, there holds

$$\hat{t}(\tilde{s}_{2i}) < \hat{t}(\tilde{s}_{2(i+1)}) \quad \forall\, i = 1, ..., k-1.$$

Therefore, if we define $s_i := \tilde{s}_{2i}$, $i = 1, ..., k$, then $s_1 < s_2 < \ldots < s_k \in M_\rho$ holds true as well as $\hat{t}(s_1) < \hat{t}(s_2) < \ldots < \hat{t}(s_k)$. Thus, we obtain

$$\delta_k := \min_{i=1,\ldots,k-1} \frac{\hat{t}(s_{i+1}) - \hat{t}(s_i)}{4} > 0 \quad \text{and} \quad \mu_k := \min_{i=1,\ldots,k-1} \frac{s_{i+1} - s_i}{4} > 0. \tag{5.1.19}$$

Next, we verify that for all $i = 1, ..., k$

$$\forall\, \epsilon > 0\ \exists\, \bar{n}_i(\epsilon) \in \mathbb{N}\ \forall\, n \geq \bar{n}_i(\epsilon) :\ \lambda(M_n \cap B(s_i, \epsilon)) > 0, \tag{5.1.20}$$

which allows us to choose $s_i^n \in B(s_i, \epsilon)$ with $\hat{\ell}_n(s_i^n) = \ell_n(\hat{t}_n(s_i^n))$. Assuming that (5.1.20) is not valid implies the existence of an $\epsilon > 0$ and a subsequence $(n_j)_{j\in\mathbb{N}}$ such that $\lambda(M_{n_j} \cap B(s_i, \epsilon)) = 0$ for all $j \in \mathbb{N}$. Then, by the definition of M_{n_j}, we obtain for all $s \in B(s_i, \epsilon) \setminus M_{n_j}$ an interval $I_s \ni s$ with $\hat{t}_{n_j} = \text{const.}$ on I_s. As $\lambda(M_{n_j} \cap B(s_i, \epsilon)) = 0$, such an interval exists for almost all $s \in B(s_i, \epsilon)$ and thus, as $\hat{t}_{n_j}$ is absolutely continuous, $\hat{t}_{n_j}$ is constant on $B(s_i, \epsilon)$. Along with the uniform convergence $\hat{t}_{n_j} \to \hat{t}$, this implies

$$\hat{t}\Big(s_i - \frac{\epsilon}{2}\Big) = \lim_{j\to\infty} \hat{t}_{n_j}\Big(s_i - \frac{\epsilon}{2}\Big) = \lim_{j\to\infty} \hat{t}_{n_j}\Big(s_i + \frac{\epsilon}{2}\Big) = \hat{t}\Big(s_i + \frac{\epsilon}{2}\Big),$$

which contradicts $s_i \in M_\rho \subset M$ so that (5.1.20) is indeed true.

Therefore, if we now choose $\epsilon = \epsilon_k := \min\{\delta_k, \mu_k\}$ in (5.1.20), we obtain for all $n \geq \bar{n}_k = \max\{\bar{n}_i(\epsilon_k) : i = 1, \dots, k\}$, with $\bar{n}_i(\epsilon_k)$ from (5.1.20), and all $i = 1, \dots, k$ a point $s_i^n \in (B(s_i, \epsilon_k) \cap M_n) \setminus N_n$, which implies

$$\hat{\ell}_n(s_i^n) = \ell_n(\hat{t}_n(s_i^n)) \quad \forall\, n \geq \bar{n}_k \tag{5.1.21}$$

by the definition of M_n and N_n. Note that N_n has measure zero. Together with (5.1.21), the triangle inequality gives

$$\sum_{i=1}^k \|\hat{\ell}_n(s_i) - \ell_n(\hat{t}(s_i))\| \leq \sum_{i=1}^k \|\ell_n(\hat{t}_n(s_i^n)) - \ell_n(\hat{t}(s_i))\| + \sum_{i=1}^k \|\hat{\ell}_n(s_i^n) - \hat{\ell}_n(s_i)\| \tag{5.1.22}$$

for all $n \geq \bar{n}_k$.

Let us estimate the expressions on the right hand side of (5.1.22). We already know that the limit $\hat{t}$ satisfies $\hat{t} \in W^{1,\infty}(0, S)$ and hence it is Lipschitz continuous with constant $L > 0$ on $[0, S]$. Below we will see that $\hat{t}$ and $\hat{z}$ satisfy the normalization condition in (5.1.3) and thus, the Lipschitz constant of $\hat{t}$ is less or equal one. Moreover, due to the compact embedding $W^{1,\infty}(0, S) \hookrightarrow C([0, S])$, $\hat{t}_n$ converges uniformly to $\hat{t}$ in $[0, S]$ and hence there exists an index $\tilde{n}_k \in \mathbb{N}$ such that $\sup_{s\in[0,S]} |\hat{t}_n(s) - \hat{t}(s)| < \delta_k$ for all $n \geq \tilde{n}_k$. Together with the Lipschitz continuity of $\hat{t}$ with constant one and $s_i^n \in B(s_i, \epsilon_k) \subset B(s_i, \delta_k)$ for $n \geq \bar{n}_k$, this results in

$$|\hat{t}_n(s_i^n) - \hat{t}(s_i)| \leq |\hat{t}_n(s_i^n) - \hat{t}(s_i^n)| + |\hat{t}(s_i^n) - \hat{t}(s_i)| < 2\delta_k. \tag{5.1.23}$$

for all $n \geq n_k := \max\{\bar{n}_k, \tilde{n}_k\}$. In view of the definition of δ_k in (5.1.19), this estimate implies that the intervals $[\min\{\hat{t}_n(s_i^n), \hat{t}(s_i)\}, \max\{\hat{t}_n(s_i^n), \hat{t}(s_i)\}]$, $i = 1, \dots, k$, do not overlap and consequently

$$\sum_{i=1}^k \|\ell_n(\hat{t}_n(s_i^n)) - \ell_n(\hat{t}(s_i))\| \leq \operatorname{Var}(\ell_n; [0, T]) \quad \forall\, n \geq n_k. \tag{5.1.24}$$

Similarly, because of $s_i^n \in B(s_i, \epsilon_k) \subset B(s_i, \mu_k)$ and the definition of μ_k in (5.1.19), the intervals $[\min\{s_i^n, s_i\}, \max\{s_i^n, s_i\}]$, $i = 1, ..., k$, do not overlap and therefore

$$\sum_{i=1}^k \|\hat{\ell}_n(s_i^n) - \hat{\ell}_n(s_i)\| \leq \operatorname{Var}(\hat{\ell}_n; [0, S]) \quad \forall\, n \geq n_k. \tag{5.1.25}$$

In view of the boundedness of $(\mathrm{Var}(\ell_n;[0,T]))_n$ and $(\mathrm{Var}(\hat{\ell}_n;[0,S]))_n$ by assumption, inserting (5.1.24) and (5.1.25) in (5.1.22) yields

$$\sum_{i=1}^{k} \|\hat{\ell}_n(s_i) - \ell_n(\hat{t}(s_i))\| \leq \mathrm{Var}(\ell_n;[0,T]) + \mathrm{Var}(\hat{\ell}_n;[0,S]) \leq C \quad \forall\, n \geq n_k.$$

with $C > 0$ independent of n and k. Now using the pointwise convergences of $\hat{\ell}_n, \ell_n$ by (5.1.17) and passing to the limit $n \to \infty$ results in

$$\sum_{i=1}^{k} \|\hat{\ell}(s_i) - \ell(\hat{t}(s_i))\| \leq C.$$

On account of $s_i \in M_\rho$, however, the left hand side is larger than $k\,\rho$ and, as k was arbitrary, passing to the limit $k \to \infty$ finally leads to the desired contradiction. Hence, G as defined in (5.1.18) has indeed measure zero and consequently (5.1.14) is valid.

Proving the remaining conditions (5.1.1)–(5.1.4) for the limit $(S, \hat{t}, \hat{z}, \hat{\ell})$ is along the lines of the proof of stability for normalized, $\mathfrak{p}$-parametrized BV solutions for problems with smooth external loads, see e.g. [MRS12, Sec. 5]. However, for convenience of the reader, we present the arguments in detail.

Step 3: Initial and end time condition, monotony of $\hat{t}$
Since the set $\{f \in L^\infty(0,S) : f(s) \geq 0 \text{ f.a.a. } s \in (0,S)\}$ is weakly* closed, the limit fulfills $\hat{t}'(s) \geq 0$ alomst everywhere in $(0,S)$. Moreover, $W^{1,\infty}(0,S;\mathbb{R}^d)$ embeds compactly into $C(0,S;\mathbb{R}^d)$ and therefore, $(\hat{t}_n)_{n\in\mathbb{N}}$ and $(\hat{z}_n)_{n\in\mathbb{N}}$ converge uniformly. Therefore, $\hat{t}(0) = 0$ as well as $\hat{t}(S) = \lim_{n\to\infty} \hat{t}_n(S_n) = T$ and $\hat{z}(0) = \lim_{n\in\mathbb{N}} \hat{z}_n(0) = z_0$ hold true so that (5.1.1) is fulfilled.

Step 4: Energy equality
Let $s_1, s_2 \in [0,S]$ be given. Due to the pointwise convergence of $\hat{z}_n \to \hat{z}$ and the continuity of $\mathcal{E}$, we have $\mathcal{E}(\hat{z}_n(s_i)) \to \mathcal{E}(\hat{z}(s_i))$, $i = 1,2$. Along with the the lower

semicontinuity of $\mathfrak{p}(\cdot,\cdot)$ according to Lemma A.1 and Lemma C.5, it follows that

$$\begin{aligned}
&\mathcal{E}(\hat z(s_2)) + \int_{s_1}^{s_2} \mathcal{R}(\hat z'(r)) + \|\hat z'(s)\| \operatorname{dist}(-\mathrm{D}_z\mathcal{I}(\hat\ell(r),\hat z(r)),\partial\mathcal{R}(0))\,\mathrm{d}r \\
&\quad= \mathcal{E}(\hat z(s_2)) + \int_{s_1}^{s_2} \mathfrak{p}(\hat z'(r), -\mathrm{D}_z\mathcal{I}(\hat\ell(r),\hat z(r)))\,\mathrm{d}r \\
&\quad\le \liminf_{n\to\infty}\left(\mathcal{E}(\hat z_n(s_2)) + \int_{s_1}^{s_2} \mathfrak{p}(\hat z_n'(r), -\mathrm{D}_z\mathcal{I}(\hat\ell_n(r),\hat z_n(r)))\,\mathrm{d}r\right) \\
&\quad\le \limsup_{n\to\infty}\left(\mathcal{E}(\hat z_n(s_2)) + \int_{s_1}^{s_2} \mathfrak{p}(\hat z_n'(r), -\mathrm{D}_z\mathcal{I}(\hat\ell_n(r),\hat z_n(r)))\,\mathrm{d}r\right) \\
&\quad\le \limsup_{n\to\infty}\left(\mathcal{E}(\hat z_n(s_1)) + \int_{s_1}^{s_2} \langle\hat\ell_n(r),\hat z_n'(r)\rangle\,\mathrm{d}r\right) \\
&\quad= \mathcal{E}(\hat z(s_1)) + \int_{s_1}^{s_2} \langle\hat\ell(r),\hat z'(r)\rangle\,\mathrm{d}r,
\end{aligned} \tag{5.1.26}$$

where we exploited the strong convergence $\hat\ell_n \to \ell$ in $L^2(0,S;\mathbb{R}^d)$ by the compactness of $\mathrm{BV}(0,S;\mathbb{R}^d) \hookrightarrow^c L^2(0,S;\mathbb{R}^d)$ and the weak* convergence $\hat z_n' \rightharpoonup^* \hat z'$ in $L^\infty(0,S;\mathbb{R}^d)$ in the last step. The opposite inequality follows from Lemma C.6. Together, we end up with equality and since $s_1, s_2 \in [0,S]$ were arbitrary, (5.1.4) is shown.

Step 5: Complementarity condition
The convergences $\hat\ell_n \to \hat\ell$ in $L^2(0,S;\mathbb{R}^d)$ and $\hat z_n \rightharpoonup^* \hat z$ in $W^{1,\infty}(0,S;\mathbb{R}^d)$ lead to $\mathrm{D}_z\mathcal{I}(\hat\ell_n(\cdot),\hat z_n(\cdot)) \to \mathrm{D}_z\mathcal{I}(\hat\ell(\cdot),\hat z(\cdot))$ in $L^2(0,S;\mathbb{R}^d)$ and, by the Lipschitz continuity of the distance, we obtain

$$\operatorname{dist}\big(-\mathrm{D}_z\mathcal{I}(\hat\ell_n(\cdot),\hat z_n(\cdot)),\partial\mathcal{R}(0)\big) \to \operatorname{dist}\big(-\mathrm{D}_z\mathcal{I}(\hat\ell(\cdot),\hat z(\cdot)),\partial\mathcal{R}(0)\big) \text{ in } L^2(0,S).$$

Together with the weak* convergence $\hat t_n' \rightharpoonup^* \hat t'$ in $L^\infty(0,S)$, this results in

$$\begin{aligned}
&\int_0^S \hat t'(r) \operatorname{dist}(-\mathrm{D}_z\mathcal{I}(\hat\ell(r),\hat z(r)),\partial\mathcal{R}(0))\,\mathrm{d}r \\
&\quad= \lim_{n\to\infty}\int_0^S \hat t_n'(r) \operatorname{dist}(-\mathrm{D}_z\mathcal{I}(\hat\ell_n(r),\hat z_n(r)),\partial\mathcal{R}(0))\,\mathrm{d}r = 0
\end{aligned}$$

such that $\hat t'(s)\operatorname{dist}(-D_z\hat{\mathcal{I}}(s,\hat z(s)),\partial\mathcal{R}(0)) = 0$ for almost all $s \in (0,S)$ due to the non-negativity of the integrand.

Step 6: Normalization
The pointwise convergence $\mathcal{E}(\hat z_n(\cdot)) \to \mathcal{E}(\hat z(\cdot))$ in combination with the fact that all inequalities in (5.1.26) hold true with equality yields for all $s_1, s_2 \in [0,S]$ that

$$s_2 - s_1 = \int_{s_1}^{s_2} \hat t_n'(r) + \mathfrak{p}(\hat z_n'(r), -\mathrm{D}_z\mathcal{I}(\hat\ell_n(r),\hat z_n(r)))\,\mathrm{d}r$$

$$\to \int_{s_1}^{s_2} \hat{t}'(r) + \mathfrak{p}(\hat{z}'(r), -\mathrm{D}_z\mathcal{I}(\hat{\ell}(r), \hat{z}(r)))\,\mathrm{d}r, \ n \to \infty.$$

Thus, assuming the existence of a Lebesgue measurable set $E \subset (0,S)$ with $\hat{t}'(s) + \mathfrak{p}(\hat{z}'(s), -\mathrm{D}_z\mathcal{I}(\hat{\ell}(s), \hat{z}(s))) > 1$ for almost all $s \in E$ and $\lambda(E) > 0$ gives for every finite union $U \subset [0,S]$ of pairwise disjoint intervals with $E \subset U$

$$\begin{aligned}\lambda(U) &= \int_U \hat{t}'(r) + \mathfrak{p}(\hat{z}'(r), -\mathrm{D}_z\mathcal{I}(\hat{\ell}(r), \hat{z}(r)))\,\mathrm{d}r\\ &\geq \int_E \hat{t}'(r) + \mathfrak{p}(\hat{z}'(r), -\mathrm{D}_z\mathcal{I}(\hat{\ell}(r), \hat{z}(r)))\,\mathrm{d}r\\ &\geq \lambda(E) + \varepsilon\end{aligned}$$

for an $\varepsilon > 0$, only depending on E and not on U, which contradicts the regularity of the Lebesgue measure. Similarly, $\hat{t}'(s) + \mathfrak{p}(\hat{z}'(s), -\mathrm{D}_z\mathcal{I}(\hat{\ell}(s), \hat{z}(s))) < 1$ for almost all $s \in E$ with $\lambda(E) > 0$ would imply

$$\begin{aligned}S &= \int_0^S \hat{t}'(r) + \mathfrak{p}(\hat{z}'(r), -\mathrm{D}_z\mathcal{I}(\hat{\ell}(r), \hat{z}(r)))\,\mathrm{d}r\\ &= \int_E \hat{t}'(r) + \mathfrak{p}(\hat{z}'(r), -\mathrm{D}_z\mathcal{I}(\hat{\ell}(r), \hat{z}(r)))\,\mathrm{d}r\\ &\qquad\qquad + \int_{(0,S)\setminus E} \hat{t}'(r) + \mathfrak{p}(\hat{z}'(r), -\mathrm{D}_z\mathcal{I}(\hat{\ell}(r), \hat{z}(r)))\,\mathrm{d}r\\ &< \lambda(E) + \lambda((0,S)\setminus E) = S.\end{aligned}$$

Hence, we have $\hat{t}'(\cdot) + \mathfrak{p}(\hat{z}'(\cdot), -\mathrm{D}_z\mathcal{I}(\hat{\ell}(\cdot), \hat{z}(\cdot))) = 1$ almost everywhere in $(0,S)$, which is (5.1.3). □

Remark 5.1.6. Note that the statement of Theorem 5.1.5 is also valid for bounded loads in $\mathrm{BV}(0,T;\mathbb{R}^d)$ and $\mathrm{BV}(0,S;\mathbb{R}^d)$, respectively, the appropriate spaces of equivalence classes. This can be seen as follows. Let sequences with $\|\ell_n\|_{\mathrm{BV}(0,T;\mathbb{R}^d)} \leq C$ and $\|\hat{\ell}_n\|_{\mathrm{BV}(0,S;\mathbb{R}^d)} \leq C$ be given. Then there exists (not relabeled) weakly* converging subsequences, i.e.,

$$\hat{\ell}_n \rightharpoonup^* \hat{\ell} \quad \text{in } \mathrm{BV}(0,S;\mathbb{R}^d), \quad \ell_n \rightharpoonup^* \ell \quad \text{in } \mathrm{BV}(0,T;\mathbb{R}^d).$$

We choose the respective representatives (denoted by the same symbols) that minimize the pointwise variation such that $\|\ell_n\|_{BV([0,T];\mathbb{R}^d)} = \|\ell_n\|_{\mathrm{BV}(0,T;\mathbb{R}^d)}$ and $\|\hat{\ell}_n\|_{BV([0,S];\mathbb{R}^d)} = \|\ell_n\|_{\mathrm{BV}(0,S;\mathbb{R}^d)}$. Now we are in the setting of Theorem 5.1.5 and the pointwise limits of ℓ_n and $\hat{\ell}_n$ coincide almost everywhere with the respective weak* limit, or in other words, they are in the respective equivalence class. Thus, the concept of relaxed solutions is also stable with respect to weak* convergence of loads in $\mathrm{BV}(0,T;\mathbb{R}^d)$.

With this stability result for the relaxed solutions at hand, we are now able to ensure the existence of solutions to optimal control problems governed by rate-independent systems with discontinuous loads. Therefore, we consider

$$\left.\begin{aligned} \min \quad & J(S,\hat{z},\ell) := j(\hat{z}(S)) + \frac{\beta}{2}\,\|\ell\|^2_{BV([0,T];\mathbb{R}^d)} \\ \text{s.t.} \quad & \ell \in BV([0,T];\mathbb{R}^d), \quad (S,\hat{t},\hat{z},\hat{\ell}) \in \mathcal{L}(\ell), \\ & \|\hat{\ell}\|_{BV([0,S];\mathbb{R}^d)} \le K, \end{aligned}\right\} \qquad (\mathrm{OCP}_{BV})$$

where

$$\mathcal{L}(\ell) := \Big\{(S,\hat{t},\hat{z},\hat{\ell}) \in [T,\infty) \times W^{1,\infty}(0,S) \times W^{1,\infty}(0,S;\mathbb{R}^d) \times BV([0,S];\mathbb{R}^d) : \\ (S,\hat{t},\hat{z},\hat{\ell}) \text{ is a relaxed solution associated with } \ell\Big\}.$$

Furthermore, $\beta > 0$ and $K > T\|\ell_0\|$ is a given constants, where $\ell_0 \in \mathbb{R}^d$ satisfies the assumption to the initial data (1.2.26), i.e., $-\mathrm{D}_z\mathcal{I}(\ell_0, z_0) \in \partial\mathcal{R}(0)$. The function $j : \mathbb{R}^d \to \mathbb{R}$ in the objective is supposed to be continuous and bounded from below. In comparison to (OCP), we omit the local stability condition at the start and at the end. Since (5.1.2), (5.1.3), (5.1.4) and (5.1.14) are also satisfied for on null sets modified ℓ and $\hat{\ell}$, respectively, and ℓ and $\hat{\ell}$ are not necessarily continuous, we can just choose the value of the loads at the beginning and at final time such that the additional constraints are fulfilled, too. Surely, this increases the variation of ℓ and thus the objective value, which we want to minimize, too, but allows for possible solutions that actually do not comply with these constraints. Therefore, this would not be a real restriction on the feasible set. As an alternative one could demand, for example, that the right hand limit is locally stable at the beginning. But this will not carry over to the limit of the loads, which can easily be seen by the following example. Let $\ell_n \in BV([0,1];\mathbb{R})$ be given by

$$\ell_n(t) = \begin{cases} 0, & \text{if } 0 \le t \le \frac{1}{n}, \\ 1, & \text{if } \frac{1}{n} < t \le 1, \end{cases}$$

such that $\ell_n \to \ell$ pointwisely with limit

$$\ell(t) = \begin{cases} 0, & \text{if } t = 0, \\ 1, & \text{if } 0 < t \le 1. \end{cases}$$

In this example the right hand limit in zero jumps from zero to one such that a condition such as $-\mathrm{D}_z\mathcal{I}(\ell(0+), z_0) \in \partial\mathcal{R}(0)$ does not necessarily have to be fulfilled in the limit, although it holds along the entire sequence. Here, too, we see that the pointwise convergence and weak* convergence in $\mathrm{BV}(0,T;\mathbb{R}^d)$, respectively, is not sufficient in order to conclude feasibility for the limit. But without these additional restrictions, we can use Theorem 5.1.5 in order to prove the following existence result.

Corollary 5.1.7 ([AM24, Cor. 3.5])**.** *There exists at least one globally optimal solution*

$$\begin{aligned}&(S^*, \hat{t}^*, \hat{z}^*, \hat{\ell}^*, \ell^*)\\ &\qquad \in [T, \infty) \times W^{1,\infty}(0, S^*) \times W^{1,\infty}(0, S^*; \mathbb{R}^d) \times BV([0, S^*]; \mathbb{R}^d) \times BV([0, T]; \mathbb{R}^d)\end{aligned}$$

of (OCP_{BV})*.*

Proof. Due to the assumption $-Az_0 - \mathcal{F}(z_0) + \ell_0 \in \partial\mathcal{R}(0)$, the constant functions $\hat{z} \equiv z_0$ and $\hat{\ell} \equiv \ell_0$ together with $\hat{t} = \mathrm{id}$ and $S = T$ generate a normalized, $\mathfrak{p}$-parametrized BV solution associated with $\ell \equiv \ell_0$ in the sense of Definition 5.1.1. Thus, it is also a relaxed solution associated with $\ell \equiv \ell_0$. Moreover, $\hat{\ell}$ trivially satisfies the boundedness condition in the constraints. Therefore, the feasible set of (OCP_{BV}) is non-empty and we obtain an infimal sequence $(S_n, \hat{t}_n, \hat{z}_n, \hat{\ell}_n, \ell_n)$ such that $J(S_n, \hat{z}_n, \ell_n) \to J^*$, where

$$J^* := \inf\Big\{J(S, \hat{z}, \ell) : \ell \in BV([0, T]; \mathbb{R}^d),\ (S, \hat{t}, \hat{z}, \hat{\ell}) \in \mathcal{L}(\ell),\ \|\hat{\ell}\|_{BV([0,S];\mathbb{R}^d)} \leq K\Big\}.$$

Since j is bounded from below, the sequence $(\ell_n)_n$ is bounded in $BV([0, T]; \mathbb{R}^d)$. Moreover, according to the constraints, the sequence $(\|\hat{\ell}_n\|_{BV([0,S_n];\mathbb{R}^d)})_n$ is also bounded. Therefore, Theorem 5.1.5 gives the existence of a subsequence converging to a limit $(S^*, \hat{t}^*, \hat{z}^*, \hat{\ell}^*, \ell^*)$ such that $(S^*, \hat{t}^*, \hat{z}^*, \hat{\ell}^*) \in \mathcal{L}(\ell^*)$ in the sense of (5.1.16)–(5.1.17). Now, given an arbitrary partition $\{t_k\}$ of $[0, T]$, the pointwise convergence of ℓ_n from (5.1.17) implies

$$\sum_k \|\ell(t_k) - \ell(t_{k-1})\| = \lim_{n\to\infty} \sum_k \|\ell_n(t_k) - \ell_n(t_{k-1})\| \leq \liminf_{n\to\infty} \mathrm{Var}(\ell_n; [0, T]).$$

Since $\{t_k\}$ was arbitrary, the total variation is thus lower semicontinuous with respect to pointwise convergence and so is the norm in $BV([0, T]; \mathbb{R}^d)$ and $BV([0, S]; \mathbb{R}^d)$, respectively. Thus, the bound on the $BV([0, S]; \mathbb{R}^d)$-norm readily carries over to the limit $\hat{\ell}^*$ giving its feasibility. Finally, the compact embedding $W^{1,\infty}(0, S^*; \mathbb{R}^d) \hookrightarrow^c C([0, S^*]; \mathbb{R}^d)$ yields $\hat{z}_n \to \hat{z}^*$ uniformly so that exploiting the continuity of j and the lower semicontinuity of the $BV([0, T]; \mathbb{R}^d)$-norm results in

$$J(S^*, \hat{z}^*, \ell^*) \leq \liminf_{n\to\infty} J(S_n, \hat{z}_n, \ell_n) = J^*,$$

which means $(S^*, \hat{t}^*, \hat{z}^*, \hat{\ell}^*, \ell^*)$ is an optimal solution of (OCP_{BV}). □

5.2 Physical plausibility of the relaxed solution concept

Now that we have developed a suitable solution concept that has the required stability properties regarding the pointwise convergence and the weak* convergence in

$\mathrm{BV}(0,T;\mathbb{R}^d)$, respectively, of the external loads, let us rank this concept in the list of other solution concepts for rate-independent systems. We compare our relaxed solution concept with the local solutions, one of the weakest concept in this context. In contrast to the setting of Definition 2.2.9, the time dependent part of the energy is now discontinuous, too, such that z and ℓ can jump at the same time and the Lebesgue integral in the energy inequality (2.2.11) is not well defined. However, there exists other concepts of integrals, which allow to integrate a function of bounded variation with respect to another function of bounded variation. This leads to the notion of the Kurzweil integral.

Definition 5.2.1 (Kurzweil integral)**.** For a given positive function

$$\delta \in \Big\{\delta : [a,b] \to \mathbb{R} : \delta(t) > 0 \text{ for all } t \in [a,b]\Big\},$$

we call a partition

$$P = \Big\{(\tau_i, [t_{i-1}, t_i])_{i=1,\dots,n} : \tau_i \in [t_{i-1}, t_i] \text{ for all } i = 1,\dots,n,\ n \in \mathbb{N}\Big\}$$

δ-fine if $[t_{i-1}, t_i] \subset (\tau_i - \delta(\tau_i), \tau_i + \delta(\tau_i))$ for all $i = 1,\dots,n$. Let X be a Banach space and X^* its dual space. For given functions $f : [a,b] \to X^*$ and $g : [a,b] \to X$, we say that

$$\mathbb{R} \ni I := \int_a^b \langle f(t), \mathrm{d}g(t)\rangle_{X^*,X}$$

is the Kurzweil integral of f over $[a,b]$ with respect to g if for all $\epsilon > 0$ there exists a positive function $\delta : [a,b] \to \mathbb{R}_{>0}$ such that for all δ-fine partitions

$$\left| I - \sum_{i=1}^{n} \langle f(\tau_i), g(t_i) - g(t_{i-1})\rangle_{X^*,X} \right| < \epsilon. \tag{5.2.1}$$

Example 5.2.2. Let $f, g : [0,2] \to \mathbb{R}$ be given by

$$f(t) = \begin{cases} 0, & \text{if } 0 \le t < 1, \\ 1, & \text{if } 1 \le t \le 2, \end{cases} \quad \text{and} \quad g(t) = \begin{cases} 0, & \text{if } 0 \le t < 1, \\ 1, & \text{if } 1 \le t \le 2, \end{cases}$$

such that both functions jump at the same time. For arbitrary given $\epsilon > 0$, we define

$$\delta(t) = \begin{cases} \frac{1}{2}|t-1|, & \text{if } t \neq 1, \\ 1, & \text{if } t = 1. \end{cases}$$

Now let $(\tau_i, [t_{i-1}, t_i])_{i=1,\dots,n}$ be an arbitrary δ-fine partition of $[0,2]$. Then there exists an index $i \in \{1,\dots,n\}$ such that $1 \in [t_{i-1}, t_i]$. The associated $\tau_i \in [t_{i-1}, t_i]$ has to be 1 since otherwise $1 \notin (\tau_i - \delta(\tau_i), \tau_i + \delta(\tau_i))$ such that $[t_{i-1}, t_i] \not\subseteq (\tau_i - \delta(\tau_i), \tau_i + \delta(\tau_i))$ and the partition would not be δ-fine. According to the definition of g, the other points of the partition have no influence on the sum in (5.2.1). Thus, the integral has to be $I = \langle f(1), g(t_i) - g(t_{i-1})\rangle = f(1) = 1$.

The Kurzweil integral satisfies a variety of properties. We refer here to [KL09, Sec. 1] and [Tvr89, Sec. 2] for an overview about the concept of Kurzweil integrals. Important for our purposes is the following result from [KL09, Thm. 1.9, Prop. 1.12].

Theorem 5.2.3. *Let X be a Banach space and X^* its dual. Moreover, assume that $f \in BV([a,b];X^*)$ and $g \in BV([a,b];X)$ then the integrals $\int_a^b \langle f(x), \mathrm{d}g(t)\rangle_{X^*,X}$ and $\int_a^b \langle g(t), \mathrm{d}f(t)\rangle_{X,X^*}$ exist and satisfy the following formula for integration by parts*

$$\begin{aligned}\int_a^b \langle f(t), \mathrm{d}g(t)\rangle_{X^*,X} + \int_a^b \langle g(t), \mathrm{d}f(t)\rangle_{X,X^*} &= \langle f(b), g(b)\rangle_{X^*,X} - \langle f(a), g(a)\rangle_{X^*,X} \\ &+ \sum_{t\in[a,b]} \big(\langle f(t) - f(t-), g(t) - g(t-)\rangle_{X^*,X} - \langle f(t+) - f(t), g(t+) - g(t)\rangle_{X^*,X}\big).\end{aligned} \tag{5.2.2}$$

In comparison to the common integration by parts formula, we additionally have to take the jumps of f and g into account. Since both functions are of bounded variation, only a finite number of points t contributes to the value in the sum and the sum is finite. Analogously to Definition 2.2.9, we can now define local solutions for discontinuous loads.

Definition 5.2.4. Let $\ell \in BV([0,T];\mathbb{R}^d)$ be given. We call $z \in BV([0,T];\mathbb{R}^d)$ a *local solution* of (RIS) associated with ℓ, if

$$0 \in \partial\mathcal{R}(0) + \mathrm{D}_z\mathcal{I}(\ell(t), z(t)) \quad \text{f.a.a. } t \in [0,T], \tag{5.2.3}$$

$$\begin{aligned}&\mathcal{I}(\ell(t_2), z(t_2)) + \mathrm{Var}_{\mathcal{R}}(z;[t_1,t_2]) \\ &\qquad \le \mathcal{I}(\ell(t_1), z(t_1)) - \int_{t_1}^{t_2} \langle z(r), \mathrm{d}\ell(r)\rangle \quad \forall\, 0 \le t_1 \le t_2 \le T,\end{aligned} \tag{5.2.4}$$

where

$$\begin{aligned}&\mathrm{Var}_{\mathcal{R}}(z;[t_1,t_2]) := \\ &\quad \sup\Big\{ \sum_{i=1}^k \mathcal{R}(z(\xi_i) - z(\xi_{i-1})) \;\Big|\; t_1 = \xi_0 < \xi_1 < \cdots < \xi_n = t_2,\ n \in \mathbb{N}\Big\}\end{aligned} \tag{5.2.5}$$

and the integral on the right hand side is to be understood as a Kurzweil integral.

For external loads $\ell \in H^1(0,T;\mathbb{R}^d)$, the Kurzweil integral in (5.2.4) can be converted into the Lebesgue integral $\int_{t_1}^{t_2} \langle z(r), \ell'(r)\rangle \mathrm{d}r$, cf. [KL09, Prop. 1.10], so that the above definition coincides with Definition 2.2.9. However, this is not the only possible generalization of the notion of local solutions in the case of discontinuous loads. Starting with a given load $\ell \in H^1(0,T;\mathbb{R}^d)$, the energy inequality (2.2.11) can be equivalently written in terms of the potential energy $\mathcal{E}$, i.e.,

$$\mathcal{E}(z(t_2)) + \mathrm{Var}_{\mathcal{R}}(z;[t_1,t_2]) \le \mathcal{E}(z(t_1)) + \int_{t_1}^{t_2} \langle \ell(r), \mathrm{d}z(r)\rangle \quad \forall\, 0 \le t_1 \le t_2 \le T$$

by using the formula of integration by parts for Riemann-Stieltjes integrals from [Apo74, Thm. 7.6]. If we now want to generalize this inequality for discontinuous $\ell \in BV([0,T];\mathbb{R}^d)$, we replace the Riemann-Stieltjes integral on the right hand side by the Kurzweil integral. But this reformulation leads to another notion of local solutions, which is no longer equivalent to Definition 5.2.4 because applying the formula for integration by parts for Kurzweil integrals (5.2.2) leads to

$$\begin{aligned}\mathcal{I}(\ell(t_2), z(t_2)) + \mathrm{Var}_{\mathcal{R}}(z; [t_1, t_2]) - \sum_{t\in[t_1,t_2]} \Delta(\ell, z, t)& \\ \leq \mathcal{I}(\ell(t_1), z(t_1)) - \int_{t_1}^{t_2} \langle z(r), \mathrm{d}\ell(r)\rangle \quad \forall\, 0 \leq t_1 \leq t_2 \leq T&\end{aligned} \tag{5.2.6}$$

with the additional term $\sum_{t\in[t_1,t_2]} \Delta(\ell, z, t)$, which is given by

$$\Delta(\ell, z, t) = \langle \ell(t) - \ell(t-), z(t) - z(t-)\rangle - \langle \ell(t+) - \ell(t), z(t+) - z(t)\rangle.$$

This leads us to the following alternative definition of local solutions for discontinuous loads:

Definition 5.2.5. Let $\ell \in BV([0,T];\mathbb{R}^d)$ be given. We call $z \in BV([0,T];\mathbb{R}^d)$ *(alternative) local solution* of (RIS) associated with ℓ, if the local stability condition (5.2.3) and alternative energy inequality (5.2.6) are fulfilled.

Note that the additional term $\sum_{t\in[t_1,t_2]} \Delta(\ell, z, t)$ vanishes if ℓ is continuous such that both definitions coincide in case of a continuous external load. Furthermore, this effect does not arise in case of the parametrized solution concepts from Definition 5.1.1 and 5.1.3, respectively. Here, due to the continuity of $\hat{z}$, the energy inequality (5.1.4) can be written equivalently as

$$\begin{aligned}\mathcal{I}(\hat{\ell}(s_2), \hat{z}(s_2)) + \int_{s_1}^{s_2} \mathcal{R}(\hat{z}'(r)) + \|\hat{z}'(r)\|\,\mathrm{dist}(-\mathrm{D}_z\mathcal{I}(\hat{\ell}(r), \hat{z}(r)), \partial\mathcal{R}(0))\,\mathrm{d}r& \\ = \mathcal{I}(\hat{\ell}(s_1)\hat{z}(s_1)) - \int_{s_1}^{s_2} \langle \hat{z}(r), \mathrm{d}\hat{\ell}(r)\rangle \quad \text{for all } 0 \leq s_1 \leq s_2 \leq S&\end{aligned}$$

with a Riemann-Stieltjes integral on the right hand side and no additional term involving the jumps of $\hat{\ell}$ pops up.

In order to compare the relaxed solution concept with the local solutions, we first have to translate the relaxed solutions into physical time. For this purpose, assume that we are given a relaxed solution $(S, \hat{t}, \hat{z}, \hat{\ell})$. Then we define the set of projections of $(\hat{t}, \hat{z})$ as

$$\begin{aligned}&\mathfrak{P}(\hat{t}, \hat{z}) := \\ &\quad \Big\{z : [0,T] \to \mathbb{R}^d : \forall\, t \in [0,T]\ \exists\, s \in [0,S] \text{ with } (t, z(t)) = (\hat{t}(s), \hat{z}(s))\Big\}.\end{aligned} \tag{5.2.7}$$

Note that $\mathfrak{P}$ consists of all functions $z : [0,T] \to \mathbb{R}^d$, whose graph is subset of the solution trajectory $[0,S] \ni s \mapsto (\hat{t}(s), \hat{z}(s))$. The following theorem correlates local with relaxed solutions. Under suitable assumptions we can show that every local solution can be interpreted as a relaxed solution. However, the opposite is not true, as we will see in the example below. In the proof we will frequently make use of

$$\mathrm{Var}_{\mathcal{R}}(z;[t_1,t_2]) = \mathrm{Var}_{\mathcal{R}}(z;[0,t_2]) - \mathrm{Var}_{\mathcal{R}}(z;[0,t_1]) \tag{5.2.8}$$

for all $0 \le t_1 \le t_2 \le T$ provided that z is continuous in t_1.

Theorem 5.2.6 ([AM24, Thm. 4.3])**.** *Let $\ell \in BV([0,T];\mathbb{R}^d)$ and local solution $z \in BV([0,T];\mathbb{R}^d)$ in the sense of Definition 5.2.4 be given. Furthermore, we assume that $\mathcal{R}$ is symmetric, i.e.,*

$$\mathcal{R}(z_1) = \mathcal{R}(z_2) \iff \|z_1\| = \|z_2\| \quad \forall\, z_1, z_2 \in \mathbb{R}^d, \tag{5.2.9}$$

and that z only admits a finite number of jumps. Then there exists a parametrization and $\hat{\ell} \in BV([0,S];\mathbb{R}^d)$ such that $(S,\hat{t},\hat{z},\hat{\ell})$ is a relaxed solution and $z \in \mathfrak{P}(\hat{t},\hat{z})$.

Proof. **Step 1: Construction and regularity of $\hat{t}$ and $\hat{z}$**
We first need to construct a suitable parametrization $s \mapsto (\hat{t}(s),\hat{z}(s))$ of the solution trajectory as a candidate for a relaxed solution. In the literature on $\mathfrak{p}$-parametrized BV solutions, given a solution in physical time, the parametrized solution $\hat{z}$ is frequently designed in jumps in such a way that the dissipative distance is minimized. More precisely, if $t \in [0,T]$ is a jump point between the values z_1 and z_2, then $\hat{z}$ is constructed by means of the minimizer of

$$\left\{\begin{array}{ll} \min & \displaystyle\int_0^1 \mathfrak{p}(v'(r), -\mathrm{D}_z\mathcal{I}(\ell(t),v(r)))\mathrm{d}r \\ \text{s.t.} & v \in W^{1,1}(0,1;\mathbb{R}^d), \quad v(0) = z_1, \quad v(1) = z_2, \end{array}\right. \tag{5.2.10}$$

cf. [MR15, Sec. 3.8.2]. Here, we pursue a different strategy. Since, in the relaxed solution concept, $\hat{\ell}$ is an additional variable in jumps, we have certain freedom in the choice of $\hat{z}$. This allows us to use a simpler construction of $\hat{z}$ and the associated parametrization. The idea is to define $\hat{z}$ in a jump point t as an affin linear function connecting the value of z in t with its left or right hand side limit and then define $\hat{\ell}$ in dependence of $\hat{z}$ such that the normalization condition (5.1.3) and energy identity (5.1.4) are satisfied. For this purpose, let us define the following modified $\mathcal{R}$-variation

$$\mathrm{Var}_{\mathcal{R},0}(z;[0,t]) := \mathcal{R}(z(0+) - z_0) + \mathrm{Var}_{\mathcal{R}}(z;[0+,t]) \tag{5.2.11}$$

taking into account potential jumps at the initial time. Note that the local solution need not satisfy the initial condition. Given $\mathrm{Var}_{\mathcal{R},0}$, we set

$$s : [0,T] \to [0,S], \quad s(t) := \begin{cases} 0, & t = 0, \\ t + \mathrm{Var}_{\mathcal{R},0}(z;[0,t]), & t \in (0,T], \end{cases} \tag{5.2.12}$$

where $S := s(T)$. Note that, by (1.2.15),

$$\mathrm{Var}_{\mathcal{R}}(z;[0,T]) \le C\,\mathrm{Var}(z;[0,T]) < \infty,$$

so that S is finite. Let us investigate the regularity of s. We denote the jump points of z that are greater zero by $0 < t_1 < t_2 < \ldots < t_N \leq T$ and set $J(z) := \{t_1, \ldots, t_N\}$. According to its definition, $\mathrm{Var}_{\mathcal{R}}(z; [0+, \cdot])$ is continuous in intervals of continuity of z, i.e., (t_n, t_{n+1}) with $t_n, t_{n+1} \in J(z)$, $n = 1, \ldots, N$, as it inherits its continuity from z there. Thus, by construction, the jump points of s are $J(s) := \{0, t_1, \ldots, t_N\}$, if $z(0+) \neq z_0$, and $J(s) = J(z)$ otherwise. In the remaining intervals however, i.e., in $(0, t_1)$, (t_1, t_2), ..., (t_{N-1}, t_N), (t_N, T), the function s is continuous. Moreover, the non-negativity of $\mathcal{R}$ implies that s is strictly increasing and hence invertible. We denote its inverse function as

$$\hat{t} : [0, S] \setminus \bigcup_{n=0}^{N} I_n \to [0, T] \setminus J(s)$$

with

$$I_0 := [0, \mathcal{R}(z(0+) - z_0)] \quad \text{and} \quad I_n := [s(t_n-), s(t_n+)]. \tag{5.2.13}$$

Then $\hat{t}$ is monotonously increasing as an inverse function of an increasing function. Furthermore, $\hat{t}$ is Lipschitz continuous with Lipschitz constant $L \leq 1$ on all intervals of $[0, S] \setminus \bigcup_{n=0}^{N} I_n$, since (5.2.12) and (5.2.8) give for $s_1 < s_2$ from such an interval that

$$\begin{aligned} 0 \leq \hat{t}(s_2) - \hat{t}(s_1) &= s_2 - s_1 - \big(\mathrm{Var}_{\mathcal{R},0}(z; [0, \hat{t}(s_2)]) - \mathrm{Var}_{\mathcal{R},0}(z; [0, \hat{t}(s_1)])\big) \\ &= s_2 - s_1 - \mathrm{Var}_{\mathcal{R}}(z; [\hat{t}(s_1), \hat{t}(s_2)]) \leq s_2 - s_1. \end{aligned} \tag{5.2.14}$$

Therefore, after constant continuation on all I_n, $n = 0, ..., N$, i.e.,

$$\hat{t}(s) = 0 \quad \forall\, s \in I_0, \qquad \hat{t}(s) = t_n \quad \forall\, s \in I_n,\ n = 1, \ldots, N,$$

the function $\hat{t}$ is Lipschitz continuous with Lipschitz constant $L \leq 1$. So there holds $\hat{t} \in W^{1,\infty}(0, S)$, i.e., the required regularity of $\hat{t}$, and $\hat{t}'(s) \geq 0$ for almost all $s \in (0, S)$, i.e., the sign condition in (5.1.2). Moreover, by construction of $\hat{t}$, we obtain that M, the set where $\hat{t}$ is strictly increasing, see (5.1.15), is given by

$$M = (0, S) \setminus \bigcup_{n=0}^{N} I_n =: \bigcup_{m=1}^{M} G_m, \tag{5.2.15}$$

where $G_m \subset (0, S)$ are open intervals.

Given $\hat{t}$, we define $\hat{z}$ as composition of z and $\hat{t}$ in the parts, where z is continuous and in jumps we choose $\hat{z}$ as the linear function that connects the left hand and right hand limits of z such that $\hat{z}$ reads

$$\hat{z}(s) := \begin{cases} z(\hat{t}(s)), & s \in (0, S) \setminus \bigcup_{n=0}^{N} I_n, \\ z(t_n), & s = s(t_n),\ n = 1, \ldots, N, \\ z(T), & s = S, \\ z(t_n-) + \frac{z(t_n) - z(t_n-)}{s(t_n) - s(t_n-)}\,(s - s(t_n-)), & s \in I_n^-, \\ z(t_n) + \frac{z(t_n+) - z(t_n)}{s(t_n+) - s(t_n)}\,(s - s(t_n)), & s \in I_n^+, \\ z_0 + \frac{z(0+) - z_0}{\mathcal{R}(z(0+) - z_0)}\, s, & s \in I_0, \end{cases} \tag{5.2.16}$$

where $I_n^- := (s(t_n-), s(t_n))$, $I_n^+ := (s(t_n), s(t_n+))$, and I_0 as defined in (5.2.13). Note that, by construction, $\hat{z}(0) = z_0$, i.e., $\hat{z}$ satisfies the initial condition in (5.1.1).

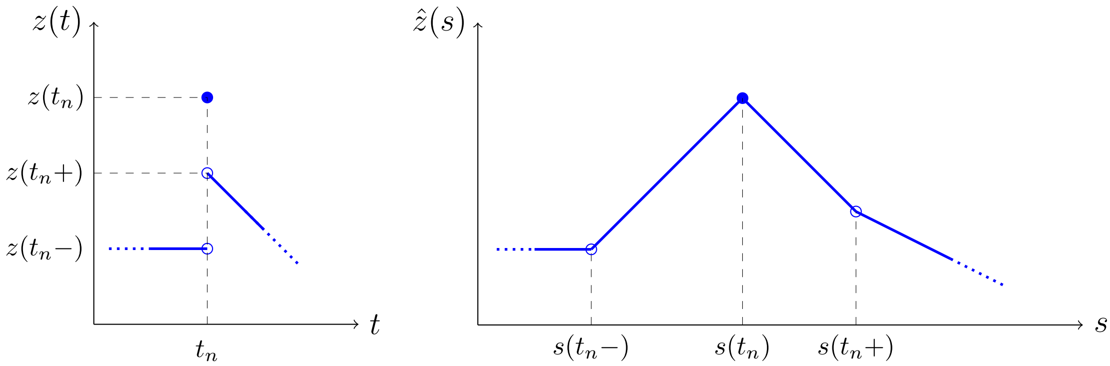

Figure 5.4: Illustration of the construction of $\hat{z}$ in a jump point t_n

Next, we show that $\hat{z}$ constructed in this way is Lipschitz continuous, too. Let us first consider an arbitrary interval G_m and let $s_1, s_2 \in G_m$ with $s_1 < s_2$ be arbitrary. Then, by definition of G_m, z is continuous on $[\hat{t}(s_1), \hat{t}(s_2)]$ and thus, similarly to (5.2.14), the construction of s in (5.2.12) yields

$$s(\hat{t}(s_2)) - \hat{t}(s_2) - s(\hat{t}(s_1)) + \hat{t}(s_1) = \mathrm{Var}_{\mathcal{R}}(z; [\hat{t}(s_1), \hat{t}(s_2)]). \tag{5.2.17}$$

With this at hand, assumption (5.0.3) and the construction of $\hat{z}$ in (5.2.16) yield

$$\begin{aligned} c\,\|\hat{z}(s_2) - \hat{z}(s_1)\| &\le \mathcal{R}(\hat{z}(s_2) - \hat{z}(s_1)) \\ &\le \mathrm{Var}_{\mathcal{R}}(\hat{z}; [s_1, s_2]) \\ &= \mathrm{Var}_{\mathcal{R}}(z; [\hat{t}(s_1), \hat{t}(s_2)]) \\ &= s_2 - \hat{t}(s_2) - s_1 + \hat{t}(s_1) \le s_2 - s_1, \end{aligned} \tag{5.2.18}$$

where we exploited the monotonicity of $\hat{t}$ in the last estimate. Next, let us consider an arbitrary interval I_n^- and arbitrary points $s_1, s_2 \in I_n^-$. Using the definition of s in (5.2.12) and $\mathrm{Var}_{\mathcal{R},0}$ in (5.2.11) and again (5.2.8), one obtains

$$\begin{aligned} s(t_n) - s(t_n-) &= \mathrm{Var}_{\mathcal{R}}(z; [0, t_n]) - \mathrm{Var}_{\mathcal{R}}(z; [0, t_n-]) \\ &= \mathcal{R}(z(t_n) - z(t_n-)). \end{aligned} \tag{5.2.19}$$

Together with the construction of $\hat{z}$ in jumps according to (5.2.16), this gives

$$\begin{aligned} \|\hat{z}(s_2) - \hat{z}(s_1)\| &= \frac{\|z(t_n) - z(t_n-)\|}{s(t_n) - s(t_n-)}\,|s_2 - s_1| \\ &\le \frac{\|z(t_n) - z(t_n-)\|}{\mathcal{R}(z(t_n) - z(t_n-))}\,|s_2 - s_1| \le \frac{1}{c}\,|s_2 - s_1|, \end{aligned} \tag{5.2.20}$$

where we again used (1.2.15) for the last estimate. Completely analogously, one derives the same estimate for arbitrary $s_1, s_2 \in I_n^+$. In I_0, we similarly obtain

$$\|\hat{z}(s_2) - \hat{z}(s_1)\| = \frac{\|z(0+) - z_0\|}{\mathcal{R}(z(0+) - z_0)} |s_2 - s_1| \leq \frac{1}{c} |s_2 - s_1| \quad \forall\, s_1, s_2 \in I_0.$$

Since $\hat{z}$ is additionally continuous by construction, see (5.2.16), this estimate together with (5.2.18) and (5.2.20) implies that $\hat{z}$ is Lipschitz continuous on $[0, S]$ with Lipschitz constant $1/c$. Consequently, we obtain the desired regularity of a relaxed solution, i.e., $\hat{z} \in W^{1,\infty}(0, S; \mathbb{R}^d)$. Furthermore, $\hat{t}, \hat{z}$ are constructed in such a way that $z \in \mathfrak{P}(\hat{t}, \hat{z})$.

Step 2: Complementary, normalization, and energy identity outside of jumps

Now that we have defined $\hat{t}$ and $\hat{z}$, we still need to define $\hat{\ell}$ outside the set M so that (5.1.2)–(5.1.4) are fulfilled. On M, however, $\hat{\ell}$ is fixed by (5.1.14) and set to $\hat{\ell} = \ell \circ \hat{t}$. To show (5.1.2), let us assume by contrary that there exists a set $E \subset M$ of positive measure such that

$$\operatorname{dist}(-\mathrm{D}_z\mathcal{I}(\hat{\ell}(s), \hat{z}(s)), \partial\mathcal{R}(0)) > 0 \quad \text{a.e. in } E. \tag{5.2.21}$$

For the image of E under $\hat{t}$, we obtain $|\hat{t}(E)| = \int_E |\hat{t}'(s)|\,\mathrm{d}s > 0$, since $\hat{t}$ is Lipschitz continuous and monotonically increasing on M. Then, in view of $\hat{z} = z \circ \hat{t}$ and $\hat{\ell} = \ell \circ \hat{t}$ on M, (5.2.21) implies $\operatorname{dist}(-\mathrm{D}_z\mathcal{I}(\ell(t), z(t)), \partial\mathcal{R}(0)) > 0$ for almost all $t \in \hat{t}(E)$, which contradicts (5.2.3). Thus, we obtain

$$\operatorname{dist}(-\mathrm{D}_z\mathcal{I}(\hat{\ell}(s), \hat{z}(s)), \partial\mathcal{R}(0)) = 0 \quad \text{f.a.a. } s \in M \tag{5.2.22}$$

Moreover, due to constant continuation outside of M, there holds $\hat{t}'(s) = 0$ for almost all $s \in (0, S) \setminus M$ and hence, (5.1.2) is valid independent of the choice of $\hat{\ell}$ on $(0, S) \setminus M$.

Next, we show that (5.1.3) is satisfied almost everywhere on M. By exploiting (5.2.22), this is

$$\hat{t}'(s) + \mathcal{R}(\hat{z}'(s)) = 1 \quad \text{f.a.a. } s \in M. \tag{5.2.23}$$

Because M is a finite union of intervals, see (5.2.15), this is in turn equivalent to

$$\int_{s_1}^{s_2} \hat{t}'(r) + \mathcal{R}(\hat{z}'(r))\,\mathrm{d}r = s_2 - s_1 \quad \forall\ s_1, s_2 \in G_m,\ m = 1, \ldots, M. \tag{5.2.24}$$

Using Lemma B.3, $\hat{z} = z \circ \hat{t}$ in M by (5.2.16), and (5.2.17) yields for arbitrary $s_1, s_2 \in G_m$

$$\int_{s_1}^{s_2} \hat{t}'(r) + \mathcal{R}(\hat{z}'(r))\,\mathrm{d}r = \hat{t}(s_2) - \hat{t}(s_1) + \operatorname{Var}_{\mathcal{R}}(\hat{z}; [s_1, s_2])$$

$$
\begin{aligned}
&= \hat{t}(s_2) - \hat{t}(s_1) + \mathrm{Var}_{\mathcal{R}}(z; [\hat{t}(s_1), \hat{t}(s_2)]) \\
&= s(\hat{t}(s_2)) - s(\hat{t}(s_1)) \\
&= s_2 - s_1,
\end{aligned}
$$

which is (5.2.24) such that (5.1.3) indeed holds almost everywhere in M.

Concerning the energy identity in (5.1.4), take an arbitrary interval G_m and let $s_1, s_2 \in G_m$, $s_1 < s_2$, be arbitrary, too. By construction of $\hat{\ell}$ and $\hat{z}$, there holds $\hat{\ell} = \ell \circ \hat{t}$ and $\hat{z} = z \circ \hat{t}$ in G_m and therefore, the energy inequality (5.2.4) with $t_1 = \hat{t}(s_1)$ and $t_2 = \hat{t}(s_2)$ yields

$$
\begin{aligned}
&\mathcal{I}(\hat{\ell}(s_2), \hat{z}(s_2)) + \mathrm{Var}_{\mathcal{R}}(\hat{z}; [s_1, s_2]) - \mathcal{I}(\hat{\ell}(s_1), \hat{z}(s_1)) \\
&\quad = \mathcal{I}(\ell(\hat{t}(s_1)), z(\hat{t}(s_2))) + \mathrm{Var}_{\mathcal{R}}(z; [\hat{t}(s_1), \hat{t}(s_2)]) - \mathcal{I}(\ell(\hat{t}(s_1)), z(\hat{t}(s_1))) \\
&\quad \leq - \int_{\hat{t}(s_1)}^{\hat{t}(s_2)} \langle z(t), \mathrm{d}\ell(t) \rangle \\
&\quad = - \int_{s_1}^{s_2} \langle (z \circ \hat{t})(r), \mathrm{d}(\ell \circ \hat{t})(r) \rangle = - \int_{s_1}^{s_2} \langle \hat{z}(r), \mathrm{d}\hat{\ell}(r) \rangle,
\end{aligned}
\tag{5.2.25}
$$

where we used the change of variables formula for Kurzweil integrals from [Ben09, Thm. 6.1], which is applicable due to the continuity of $\hat{t}$. For the $\mathcal{R}$-variation, Lemma B.3 along with (5.2.22) yields

$$
\mathrm{Var}_{\mathcal{R}}(\hat{z}; [s_1, s_2]) = \int_{s_1}^{s_2} \mathcal{R}(\hat{z}'(r)) + \|\hat{z}'(r)\| \operatorname{dist}(-\mathrm{D}_z \mathcal{I}(\hat{\ell}(r), \hat{z}(r)), \partial\mathcal{R}(0)) \, \mathrm{d}r. \tag{5.2.26}
$$

Since $\hat{z} \in W^{1,\infty}(0, S; \mathbb{R}^d)$ and thus $\hat{z} \in C([0, T]; \mathbb{R}^d)$ by continuous embedding, applying the formula of integration by parts (5.2.2) to the Kurzweil integral on the right hand side of (5.2.25) results in

$$
- \int_{s_1}^{s_2} \langle \hat{z}(r), \mathrm{d}\hat{\ell}(r) \rangle = \int_{s_1}^{s_2} \langle \hat{\ell}(r), \hat{z}'(r) \rangle \, \mathrm{d}r + \langle \hat{\ell}(s_2), \hat{z}(s_2) \rangle - \langle \hat{\ell}(s_1), \hat{z}(s_1) \rangle,
$$

where we have already rewritten the Kurzweil integral on the right hand side as a Lebesgue integral according to [KL09, Prop. 1.10] and the regularity of $\hat{z}$. Inserting this together with (5.2.26) in (5.2.25) implies in view of the definition of $\mathcal{E}$ in (1.2.2) that

$$
\begin{aligned}
\mathcal{E}(\hat{z}(s_2)) + \int_{s_1}^{s_2} \mathcal{R}(\hat{z}'(r)) + \|\hat{z}'(r)\| \operatorname{dist}(-\mathrm{D}_z \mathcal{I}(\hat{\ell}(r), \hat{z}(r)), \partial\mathcal{R}(0)) \, \mathrm{d}r& \\
\leq \mathcal{E}(\hat{z}(s_1)) + \int_{s_1}^{s_2} \langle \hat{\ell}(r), \hat{z}'(r) \rangle \, \mathrm{d}r.&
\end{aligned}
\tag{5.2.27}
$$

Lemma C.6 yields that (5.2.27) is even satisfied with equality and thus we obtain the desired energy identity for arbitrary $s_1, s_2 \in G_m$.

Step 3: Construction of $\hat{\ell}$ in jumps

To motivate our definition of $\hat{\ell}$ in jumps, let us take a closer look at the connection of the normalization condition (5.1.3) and the energy identity in jumps. As $\hat{t}$ is constant there, (5.1.3) reads

$$\mathcal{R}(\hat{z}'(s)) + \|\hat{z}'(s)\| \operatorname{dist}(-\mathrm{D}_z\mathcal{I}(\hat{\ell}(s), \hat{z}(s)), \partial\mathcal{R}(0)) = 1 \quad \text{f.a.a. } s \in \bigcup_{n=0}^{N} I_n. \tag{5.2.28}$$

The term on the left hand side is exactly the expression that also arises in the energy identity. So, if we consider an arbitrary jump interval I_n and arbitrary $s_1, s_2 \in I_n$ and if we suppose that $\hat{\ell}$ is chosen in I_n such that (5.2.28) holds, then the energy identity in (5.1.4) becomes

$$\begin{aligned}
s_2 - s_1 &= \int_{s_1}^{s_2} \mathcal{R}(\hat{z}'(r)) + \|\hat{z}'(r)\| \operatorname{dist}(-\mathrm{D}_z\mathcal{I}(\hat{\ell}(r), \hat{z}(r)), \partial\mathcal{R}(0))\,\mathrm{d}r \\
&= \mathcal{E}(\hat{z}(s_1)) - \mathcal{E}(\hat{z}(s_2)) + \int_{s_1}^{s_2} \langle \hat{\ell}(r), \hat{z}'(r)\rangle\,\mathrm{d}r && \text{(by (5.1.4))} \\
&= -\int_{s_1}^{s_2} \langle \mathrm{D}_z\mathcal{E}(\hat{z}(r)), \hat{z}'(r)\rangle\,\mathrm{d}r + \int_{s_1}^{s_2} \langle \hat{\ell}(r), \hat{z}'(r)\rangle\,\mathrm{d}r \\
&= -\int_{s_1}^{s_2} \langle \mathrm{D}_z\mathcal{I}(\hat{\ell}(r), \hat{z}(r)), \hat{z}'(r)\rangle\,\mathrm{d}r.
\end{aligned}$$

Since this equality must hold for every $s_1, s_2 \in I_n$, provided that $\hat{\ell}$ is such that the energy identity holds, we see that $\hat{\ell}$ satisfying the normalization condition (5.2.28) fulfills the energy identity, if and only if

$$\langle \mathrm{D}_z\mathcal{I}(\hat{\ell}(s), \hat{z}(s)), \hat{z}'(s)\rangle + 1 = 0 \quad \text{f.a.a. } s \in \bigcup_{n=0}^{N} I_n. \tag{5.2.29}$$

To summarize, in order to ensure (5.1.3) and (5.1.4) to hold in jumps, we must construct $\hat{\ell}$ such that (5.2.28) and (5.2.29) are fulfilled. In view of $\mathrm{D}_z\mathcal{I}(\ell, z) = Az + \mathrm{D}_z\mathcal{F}(z) - \ell$, (5.2.29) leads to the natural choice

$$\hat{\ell}(s) := A\hat{z}(s) + \mathrm{D}_z\mathcal{F}(\hat{z}(s)) + \frac{\hat{z}'(s)}{\|\hat{z}'(s)\|^2} \quad \forall\, s \in \mathring{I}\,, \tag{5.2.30}$$

where $\mathring{I} := \bigcup_{n=1}^{N} \left(I_n^- \cup I_n^+\right) \cup \operatorname{int}(I_0)$. Then (5.2.29) and thus the energy identity are trivially fulfilled then. Note that $\hat{z}'(s)$ is constant and different from zero in each component of $\mathring{I}$ such that $\hat{\ell}(s)$ is well defined and due to the regularity of $\hat{z}$ the respective restriction is an element of $W^{1,\infty}(\operatorname{int}(I_0); \mathbb{R}^d)$, $W^{1,\infty}(I_n^-; \mathbb{R}^d)$ and $W^{1,\infty}(I_n^+; \mathbb{R}^d)$, respectively, for all $n = 1, ..., N$. Note moreover that $I_n^- = \emptyset$ and $I_n^+ = \emptyset$, if $\hat{z}'$ vanishes there. It remains to prove (5.2.28) for $\hat{z}$ together with $\hat{\ell}$ as defined in (5.2.30). For that purpose, let us investigate the dissipation functional in the jumps. As seen in (5.2.19), in a jump point t_n, there holds $\mathcal{R}(z(t_n) - z(t_n-)) = s(t_n) - s(t_n-)$

and completely analogously, one shows $\mathcal{R}(z(t_n+) - z(t_n)) = s(t_n+) - s(t_n)$. Using the monotonicity of s along with the positive 1-homogeneity of $\mathcal{R}$ and the structure of $\hat{z}$ in the jump parts I_n^-, I_n^+, see (5.2.16), this implies

$$\mathcal{R}(\hat{z}'(s)) = \mathcal{R}\bigg(\frac{z(t_n) - z(t_n-)}{s(t_n) - s(t_n-)}\bigg) = 1 \quad \forall\, s \in I_n^- \tag{5.2.31}$$

as well as

$$\mathcal{R}(\hat{z}'(s)) = \mathcal{R}\bigg(\frac{z(t_n+) - z(t_n)}{s(t_n+) - s(t_n)}\bigg) = 1 \quad \forall\, s \in I_n^+, \tag{5.2.32}$$

while, on I_0, we have by construction of $\hat{z}$ in (3.3.8) that $\mathcal{R}(\hat{z}'(s)) = 1$ for all $s \in \operatorname{int}(I_0)$. Now let $s \in \mathring{I} = \bigcup_{n=1}^N \big(I_n^- \cup I_n^+\big) \cup \operatorname{int}(I_0)$ be arbitrary. Inserting (5.2.30) in the left hand side in (5.2.28) and employing Lemma A.8 yield

$$\begin{aligned}
&\mathcal{R}(\hat{z}'(s)) + \|\hat{z}'(s)\| \operatorname{dist}(-\mathrm{D}_z\mathcal{I}(\hat{\ell}(s), \hat{z}(s)), \partial\mathcal{R}(0)) \\
&\quad = \mathcal{R}(\hat{z}'(s)) + \|\hat{z}'(s)\| \operatorname{dist}\left(\frac{\hat{z}'(s)}{\|\hat{z}'(s)\|^2}, \partial\mathcal{R}(0)\right) \\
&\quad = \mathcal{R}(\hat{z}'(s)) + \sup_{v\in\mathbb{R}^d, \|v\|\le\|\hat{z}'(s)\|} \left(\Big\langle \frac{\hat{z}'(s)}{\|\hat{z}'(s)\|^2}, v\Big\rangle - \mathcal{R}(v)\right) \\
&\quad = \mathcal{R}(\hat{z}'(s)) + \sup_{0<\alpha<1} \alpha \sup_{v\in\mathbb{R}^d, \|v\|=\|\hat{z}'(s)\|} \left(\Big\langle \frac{\hat{z}'(s)}{\|\hat{z}'(s)\|^2}, v\Big\rangle - \mathcal{R}(v)\right) \\
&\quad = \mathcal{R}(\hat{z}'(s)) + \sup_{0<\alpha<1} \alpha \sup_{v\in\mathbb{R}^d, \|v\|=\|\hat{z}'(s)\|} \left(\Big\langle \frac{\hat{z}'(s)}{\|\hat{z}'(s)\|^2}, v\Big\rangle - \mathcal{R}(\hat{z}'(s))\right) \\
&\quad = 1
\end{aligned}$$

where we exploited (5.2.9), i.e., the symmetry of $\mathcal{R}$ by assumption, the 1-homogeneity of $\mathcal{R}$, and (5.2.31) and (5.2.32). Thus, we have verified (5.2.28), which, together with (5.2.29), which holds by construction of $\hat{\ell}$ according to (5.2.30), implies (5.1.2) and (5.1.4) in the jumps, as explained above. Therefore, the energy identity (5.1.4) holds for all $s_1, s_2 \in I_n$, $n = 0, \ldots, N$, and all $s_1, s_2 \in G_m$, $m = 1, \ldots, M$, and consequently, the continuity of $\hat{z}$ implies that it holds for all $s_1, s_2 \in [0, S]$.

Step 4: Regularity of $\hat{\ell}$
To summarize, we have seen that with $\hat{\ell}$ defined by

$$\hat{\ell}(s) = \begin{cases} \ell(\hat{t}(s)), & s \in [0, S] \setminus \bigcup_{n=0}^N I_n, \\ A\hat{z}(s) + \mathrm{D}_z\mathcal{F}(\hat{z}(s)) + \frac{\hat{z}'(s)}{\|\hat{z}'(s)\|^2}, & s \in \mathring{I}, \end{cases} \tag{5.2.33}$$

and one sided continuously extended such that $\hat{\ell}$ is everywhere defined in $[0, S]$, all conditions of a relaxed solution from Definition 5.1.3 are fulfilled. It remains to verify the required smoothness of $\hat{\ell}$. Due to $\ell \in BV([0,T]; \mathbb{R}^d)$ and the monotonicity of $\hat{t}$, the composition $\ell \circ \hat{t}$ is a function in $BV([0, S]; \mathbb{R}^d)$. Moreover, since $\hat{z}'$ is constant and non-zero in each component of $\mathring{I}$, $A\hat{z}(\cdot) + \mathrm{D}_z\mathcal{F}(\hat{z}(\cdot)) + \frac{\hat{z}'(\cdot)}{\|\hat{z}'(\cdot)\|^2}$ inherits the regularity of $\hat{z}$ and is therefore continuous. Thus, as a piecewise composition of finitely many functions of bounded variation, $\hat{\ell}$ is a function of bounded variation, too, i.e., $\hat{\ell} \in BV([0,S];\mathbb{R}^d)$ as required. $\square$

The following remarks show that we cannot omit the additional requirements of the theorem, namely the symmetry of $\mathcal{R}$ and that the local solution z only jumps finitely many times.

Remark 5.2.7 ([AM24, Rem. 4.4])**.** The symmetry of $\mathcal{R}$ is necessary to fulfill the normalization condition (5.1.3) with $\hat{\ell}$ as chosen in (5.2.30). This can be seen as follows. Let $\mathcal{R}:\mathbb{R}^2 \to \mathbb{R}$ be defined by $\mathcal{R}(z) = \frac{1}{2}|z_1| + |z_2|$ and suppose that there is a jump interval I with constant $\hat{z}'$ given by $\hat{z}'(s) = (1, \frac{1}{2})^\top$ for all $s \in I$. Then we have $\mathcal{R}(\hat{z}'(s)) = 1$ and $\frac{\hat{z}'(s)}{\|\hat{z}'(s)\|^2} = (\frac{4}{5}, \frac{2}{5})^\top$ for all $s \in I$. For the distance to $\partial\mathcal{R}(0)$, we thus obtain in light of (5.2.30) that

$$\operatorname{dist}(-\mathrm{D}_z\mathcal{I}(\hat{\ell}(s), \hat{z}(s)), \partial\mathcal{R}(0)) = \operatorname{dist}\Big(\frac{\hat{z}(s)}{\|\hat{z}'(s)\|^2}, [-\tfrac{1}{2}, \tfrac{1}{2}] \times [-1,1]\Big) = \frac{3}{10} \quad \forall\, s \in I.$$

Due to $\hat{t}'(s) = 0$ for all $s \in I$, we end up with

$$\hat{t}'(s) + \mathcal{R}(\hat{z}'(s)) + \|\hat{z}'(s)\| \operatorname{dist}(-\mathrm{D}_z\mathcal{I}(\hat{\ell}(s), \hat{z}(s)), \partial\mathcal{R}(0)) = 1 + \sqrt{\frac{5}{4}}\frac{3}{10} > 1 \quad \forall\, s \in I$$

such that (5.1.3) is indeed violated.

Remark 5.2.8 ([AM24, Rem. 4.5])**.** The symmetry of $\mathcal{R}$ together with (1.2.13) and (1.2.14) automatically implies that $\mathcal{R}$ is a scaled version of the norm, i.e., there exists an $\alpha \geq 0$ such that $\mathcal{R}(\cdot) = \alpha\|\cdot\|$. This can be seen as follows. Let $e \in \mathbb{R}^d$ be a unit vector and define $\alpha := \mathcal{R}(e)$. Then, for $y \neq 0$, the symmetry condition in (5.2.9) implies $\mathcal{R}\big(\frac{y}{\|y\|}\big) = \mathcal{R}(e) = \alpha$ such that $\mathcal{R}(y) = \alpha\|y\|$ for all $y \in \mathbb{R}^d$ due to the positive homogeneity of $\mathcal{R}$.

Remark 5.2.9 ([AM24, Rem. 4.6])**.** The reason why we assume in Theorem 5.2.6 that z has only finitely many jumps is to guarantee the required regularity of $\hat{\ell}$, i.e., $\hat{\ell} \in BV([0,S];\mathbb{R}^d)$. Note that, according to (5.2.33), $\hat{\ell}$ is continuous except for the points in the set $\bigcup_{n=0}^N \{s(t_n-), s(t_n), s(t_n+)\}$ and thus, if there are infinitely many of those points, the total variation of $\hat{\ell}$ may become infinite. At least in one dimension, i.e., if $d = 1$, there is however no alternative to the choice of $\hat{\ell}$ in (5.2.33), as it is easily seen that this construction of $\hat{\ell}$ in the jumps is the only way to guarantee (5.2.29) and equivalently the energy identity, while $\hat{\ell}$ is determined by $\ell \circ \hat{t}$ outside the jumps. Therefore, in order to allow for countable infinite many jumps of z without destroying the regularity of $\hat{\ell}$, we need to modify the construction of $\hat{z}$ in the jumps. Let us again consider the one dimensional case. If one aims to avoid a jump of $\hat{\ell}$ at $s = s(t_n-)$, one has to choose $\hat{z}$ in the jump from $s(t_n-)$ to $s(t_n)$ such that $\ell(\hat{t}(s)) = A\hat{z}(s+) + \mathrm{D}_z\mathcal{F}(\hat{z}(s+)) + \frac{\hat{z}'(s+)}{|\hat{z}'(s+)|^2}$. Thanks to the continuity of

$\hat{z}$, this can be reformulated as $\mathrm{D}_z\mathcal{I}(\ell(\hat{t}(s)),\hat{z}(s)) + \frac{\hat{z}'(s+)}{|\hat{z}'(s+)|^2} = 0$, which results in the following condition for $\hat{z}$ at the beginning of the jump

$$\hat{z}'(s+) = \frac{-\mathrm{D}_z\mathcal{I}(\ell(\hat{t}(s)),\hat{z}(s))}{|\mathrm{D}_z\mathcal{I}(\ell(\hat{t}(s)),\hat{z}(s))|^2}, \tag{5.2.34}$$

provided that $\mathrm{D}_z\mathcal{I}(\ell(\hat{t}(s)),\hat{z}(s)) \neq 0$. The condition in (5.2.34) will in general not be fulfilled by the linear interpolant from (5.2.16), but, of course, one can find other Lipschitz continuous functions satisfying (5.2.34) as well as $\hat{z}(s(t_n-)) = z(t_n-)$ and $\hat{z}(s(t_n)) = z(t_n)$. However, in view of (5.2.30) and the required boundedness of $\hat{\ell}$ as a function of bounded variation, $\hat{z}'$ must be bounded away from zero in $(s(t_n-), s(t_n))$. This is however not possible, if $\hat{z}'$ is smooth and $-\mathrm{D}_z\mathcal{I}(\hat{t}(s),\hat{z}(s))$ and $z(\hat{t}(s)) - z(\hat{t}(s)-)$ have a different sign. If $\hat{z}'$ is not smooth, an additional discontinuity in $\hat{\ell}$ arises, cf. (5.2.30), contradicting again the required regularity of $\hat{\ell}$, if this happens more than finitely many times with a jump height that does not tend to zero sufficiently fast. All in all, we see that the construction of $\hat{\ell}$ in the presence of countably many jumps of z is very involved and seems hardly to be possible in general.

A close inspection of the proof of Theorem 5.2.6 shows that its statement also holds for local solutions in the sense of Definition 5.2.5. To see this, observe that the energy equality is established by a distinction of cases in the proof of Theorem 5.2.6. First, in step 2 of the proof, an arbitrary interval $[s_1, s_2] \subset G_m$ is considered. In $[\hat{t}(s_2), \hat{t}(s_2)]$, however, z is continuous such that the additional term involving $\Delta(\ell, z, t)$ vanishes and the energy inequality in (5.2.6) coincides with the one in (5.2.4) such that the arguments from step 2 are unaffected and apply to the alternative definition of a local solution, too. Furthermore, in step 3 of the proof, where the jumps are considered, $\hat{z}$ as well as $\hat{\ell}$ are explicitly constructed such that the energy identity is fulfilled and the energy inequality from (5.2.4) and (5.2.6), respectively, is not used for this at all. Thus, the arguments from step 3 also apply for the alternative definition of a local solution and we arrive at the following.

Corollary 5.2.10 ([AM24, Cor. 4.8])**.** *Suppose that $\ell \in BV([0,T];\mathbb{R}^d)$ is given and that $z \in BV([0,T];\mathbb{R}^d)$ is a local solution in the sense of Definition 5.2.5 that only admits a finite number of jumps. Assume moreover that the dissipation functional fulfills the symmetry condition from* (5.2.9)*. Then there exists a parametrization and $\hat{\ell} \in BV([0,S];\mathbb{R}^d)$ such that $(S,\hat{t},\hat{z},\hat{\ell})$ is a relaxed solution which fulfills $z \in \mathfrak{P}(\hat{t},\hat{z})$.*

The next example disproves the reverse statement of Theorem 5.2.6. This means that there exists a relaxed solution $(S,\hat{t},\hat{z},\hat{\ell})$ in the sense of Definition 5.1.3 that does not provide a local solution z, which satisfies $z \in \mathfrak{P}(\hat{t},\hat{z})$. Moreover, we will see that the concepts of differential, global energetic and local solution are all not stable with respect to the pointwise and weak* convergence in $\mathrm{BV}(0,T;\mathbb{R}^d)$ of the loads.

Example 5.2.11 ([AM24, Sec. 4.2])**.** As in Example 5.1.2, the initial value and end time are set to $z_0 = 0$ and $T = 2$ and energy and dissipation are given by

$$\mathcal{R}(z) = |z|, \qquad \mathcal{E}(z) = \frac{1}{2}z^2 - z, \qquad \mathcal{I}(\ell(t), z) = \mathcal{E}(z) - \ell(t)z. \tag{5.2.35}$$

But now we consider a slightly modified version of the sequence of loads, which reads as

$$\ell_n(t) = \begin{cases} 0, & t \in [0,1], \\ \frac{n}{2}t - \frac{n}{2}, & t \in (1, 1+\frac{1}{n}), \\ 0, & t \in [1+\frac{1}{n}, 2]. \end{cases} \tag{5.2.36}$$

This sequence converges pointwise everywhere to $\ell \equiv 0$. Moreover, it converges weakly* in $\mathrm{BV}(0, T, \mathbb{R})$ to zero, too, but not in the intermediate sense because the variation jumps from two to zero in the limit. By direct calculations one verifies that a relaxed solution associated with ℓ_n is given by

$$\hat{z}_n(s) = \begin{cases} 0, & s \in [0,1], \\ \frac{\frac{n}{2}}{1+\frac{n}{2}}s - \frac{\frac{n}{2}}{1+\frac{n}{2}}, & s \in (1, \frac{3}{2}+\frac{1}{n}), \\ \frac{1}{2}, & s \in [\frac{3}{2}+\frac{1}{n}, \frac{5}{2}], \end{cases} \tag{5.2.37}$$

$$\hat{t}_n(s) = \begin{cases} s, & s \in [0,1], \\ \frac{1}{1+\frac{n}{2}}s + \frac{\frac{n}{2}}{1+\frac{n}{2}}, & s \in (1, \frac{3}{2}+\frac{1}{n}), \\ s - \frac{1}{2}, & s \in [\frac{3}{2}+\frac{1}{n}, \frac{5}{2}], \end{cases} \tag{5.2.38}$$

$$\hat{\ell}_n(s) = \ell_n(\hat{t}_n(s)) = \begin{cases} 0, & s \in [0,1], \\ \frac{\frac{n}{2}}{1+\frac{n}{2}}s - \frac{\frac{n}{2}}{1+\frac{n}{2}}, & s \in (1, \frac{3}{2}+\frac{1}{n}), \\ 0, & s \in [\frac{3}{2}+\frac{1}{n}, \frac{5}{2}], \end{cases} \tag{5.2.39}$$

where the final artificial time is $S_n = \frac{5}{2}$. Note that this also defines a normalized $\mathfrak{p}$-parametrized BV solution in the sense of Definition 5.1.1 since no jump occurs, i.e., $\hat{t}(s) > 0$ everywhere, such that both concepts coincide in this case. The limits of $\hat{z}$, $\hat{t}$ and $\hat{\ell}$ with respect to pointwise convergence as well as weak* convergence in $W^{1,\infty}(0, S)$ and intermediate/strict convergence in $BV([0, S]; \mathbb{R})$, respectively, are given by

$$\hat{z}(s) = \begin{cases} 0, & s \in [0,1], \\ s-1, & s \in (1, \frac{3}{2}), \\ \frac{1}{2}, & s \in [\frac{3}{2}, \frac{5}{2}], \end{cases} \quad \hat{t}(s) = \begin{cases} s, & s \in [0,1], \\ 1, & s \in (1, \frac{3}{2}), \\ s - \frac{1}{2}, & s \in [\frac{3}{2}, \frac{5}{2}], \end{cases} \tag{5.2.40}$$

$$\hat{\ell}(s) = \begin{cases} 0, & s \in [0,1], \\ s-1, & s \in (1, \frac{3}{2}), \\ 0, & s \in [\frac{3}{2}, \frac{5}{2}]. \end{cases} \tag{5.2.41}$$

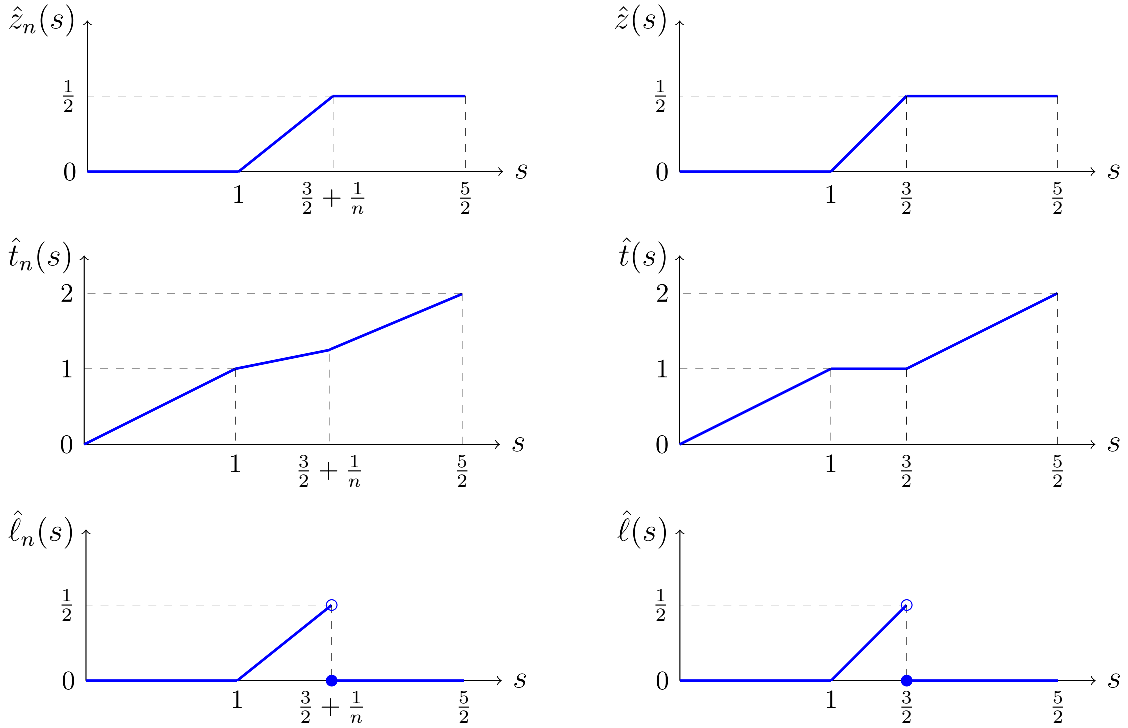

Figure 5.5: Graphs of the $\mathfrak{p}$-parametrized BV solution $(\hat{z}_n, \hat{t}_n, \hat{\ell}_n)$ from (5.2.37)-(5.2.39) associated with external load (5.2.36) and its corresponding limits $(\hat{z}, \hat{t}, \hat{\ell})$ from (5.2.40) and (5.2.41)

As in Example 5.1.2, the limit is no normalized, $\mathfrak{p}$-parametrized BV solution associated with ℓ, since $\hat{\ell}$ from (5.2.41) is not compatible with $\ell \equiv 0$ in the viscous jump $(1, \frac{3}{2})$ according to the condition in (5.1.5). However, as all other requirements in Definition 5.1.1 are satisfied, i.e., (5.1.1)–(5.1.4), and, since $\hat{\ell} = \ell \circ \hat{t}$ in $M = (0, \frac{5}{2}) \setminus (1, \frac{3}{2})$, the limit is indeed a relaxed solution in accordance with our findings in Theorem 5.1.5. Note in this context that $\hat{\ell}$ and $\hat{z}$ satisfy (5.2.30) in the viscous jump $(1, \frac{3}{2})$.

Let us now consider this example in physical time. By direct calculations one sees that $z_n \in W^{1,\infty}(0,2)$ given by

$$z_n(t) = \begin{cases} 0, & t \in [0,1], \\ \frac{n}{2}t - \frac{n}{2}, & t \in (1, 1+\frac{1}{n}), \\ \frac{1}{2}, & t \in [1+\frac{1}{n}, 2], \end{cases} \tag{5.2.42}$$

satisfies (RIS) almost everywhere in $(0,2)$, where energy and dissipation are given by (5.2.35) and ℓ_n is as defined in (5.2.36). Therefore, z_n is a differentiable solution and consequently a global energetic and local solution, too, since the energy is convex cf. Lemma 2.1.1. If we now pass $n \to \infty$, then z_n converges to

$$\tilde{z}(t) = \begin{cases} 0, & t \in [0,1], \\ \frac{1}{2} & t \in (1,2]. \end{cases} \tag{5.2.43}$$

Surely, this is not a differential solution associated with the limit of loads $\ell \equiv 0$ since z is not even continuous. Due to the uniform convexity of the energy in (5.2.35), the differential solution is unique, see Theorem 2.2.4. Therefore, the only differential solution associated with the limit $\ell \equiv 0$ is $z \equiv 0$. As the initial state $z_0 = 0$ is locally stable, this is the only physically meaningful solution in the absence of external loading. Furthermore, the limit z is also not a local solution and thus not a global energetic solution, too, because the energy inequality (5.2.4) is violated. To see this, let $t_1 > 1$ such that (5.2.4) reads

$$\frac{1}{8} = \mathcal{I}(\ell(t_2), \tilde{z}(t_2)) + \mathrm{Var}_{\mathcal{R}}(\tilde{z}; [0, t_2]) > \mathcal{I}(\ell(0), \tilde{z}(0)) - \int_0^{t_2} \langle \tilde{z}(r), \mathrm{d}\ell(r) \rangle = 0. \tag{5.2.44}$$

This effect is closely related to the lack of continuity of the Kurzweil integral with respect to the weak* and intermediate/strict convergence in $\mathrm{BV}(0,T) \times \mathrm{BV}(0,T)$. Note that, although z_n converges to $\tilde{z}$ in the intermediate topology and $\ell_n \rightharpoonup^* \ell \equiv 0$, there holds

$$\begin{aligned} -\mathcal{I}(\ell_n(2), z_n(2)) - \mathrm{Var}_{\mathcal{R}}(z_n; [0,2]) + \mathcal{I}(\ell_n(0), z_n(0)) &= \int_0^2 \langle z_n(r), \mathrm{d}\ell_n(r) \rangle \\ &= -\frac{1}{8} \neq 0 = \int_0^2 \langle \tilde{z}(r), \mathrm{d}\ell(r) \rangle \end{aligned}$$

for all $n \in \mathbb{N}$. A graphic interpretation of this observation is as follows: Due to the external force ℓ_n, the state z_n is raised to a higher energy level. As n is increased, this shifting is accelerated such that, in the limit, the state remains on the higher energy level although the load vanishes. As a consequence, this increase of energy is not compensated by ℓ and the energy inequality is violated. This example shows that not only the notion of normalized, $\mathfrak{p}$-parametrized BV solutions, but even the concepts of global energetic and local solutions are not stable with respect to a sequence of loads converging pointwise and weakly* in $\mathrm{BV}(0,T)$, respectively.

Let us finally investigate if there is another local solution that coincides with the limit $(\hat{z}, \hat{t})$ in the sense that $\bar{z} \in \mathfrak{P}(\hat{t}, \hat{z})$, cf. (5.2.7). For this purpose, we transform the limit $\hat{z}$ back into physical time, which yields

$$\bar{z}(t) \in \hat{z}(\hat{t}^{-1}(t)) = \begin{cases} 0, & t \in [0,1), \\ [0, \frac{1}{2}], & t = 1, \\ \frac{1}{2}, & t \in (1,2], \end{cases}$$

where $\hat{t}^{-1}$ is the set-valued inverse of $\hat{t}$. Thus, for every $\bar{z} \in \mathfrak{P}(\hat{t}, \hat{z})$, the inequality in (5.2.44) holds true, which shows that no element of $\mathfrak{P}(\hat{t}, \hat{z})$ is a local solution.

Even if we consider a modified load corresponding to the transformation of $\hat{\ell}$ from (5.2.41) into physical time, i.e.,

$$\bar{\ell}(t) \in \hat{\ell}(\hat{t}^{-1}(t)) = \begin{cases} 0, & t \neq 1, \\ [0, \frac{1}{2}], & t = 1, \end{cases}$$

the inequality in (5.2.44) will still be valid, since the Kurzweil integral on the right side of (5.2.44) is always zero, no matter how one chooses $\bar{\ell}$ in $t = 1$, see [Tvr89, Corollary 2.14]. This shows that, in general, one cannot interpret a relaxed solution $(S, \hat{t}, \hat{z}, \hat{\ell})$ as a local solution.

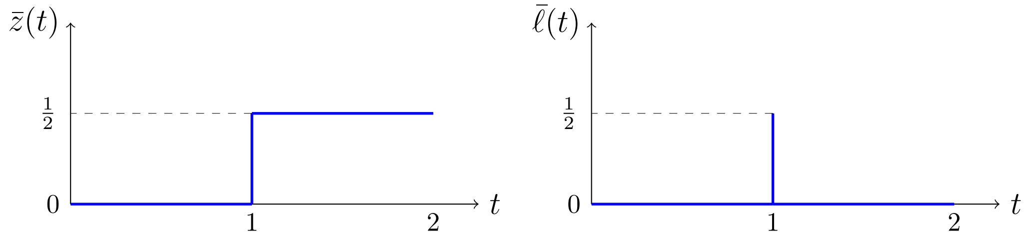

Figure 5.6: Graphs of the reparametrized set-valued functions according to (5.2.7)

The above example illustrates the following: the price one has to pay for a solution concept that is stable with respect to pointwise or weak* convergence of the loads is that "solutions" are allowed that are not meaningful from a physical perspective. As already mentioned, the only reasonable solution of (RIS) without external loads (i.e., $\ell \equiv 0$) and with a uniformly convex energy and a locally stable initial state z_0 is $z \equiv z_0$. The relaxed solution however jumps at $t = 1$ from $z_0 = 0$ to $1/2$ without any impact of an external load, which makes of course no sense at all.

We already have seen that the concept of local solutions is not stable with respect to pointwise and weak* convergence of the loads. The next examples show that this is also true for loads converging with respect to intermediate/strict convergence in $BV([0, T]; \mathbb{R})$. This concerns both definitions of local solutions in Definition 5.2.4 and 5.2.5, respectively, regardless whether the energy inequality is given as in (5.2.4) or in (5.2.6).

Example 5.2.12 ([AM24, Sec. 4.3])**.** We again consider the setting from Example 5.1.2, where energy, dissipation, initial state, and end time were given by

$$\mathcal{R}(z) = |z|, \quad \mathcal{E}(z) = \frac{1}{2}z^2 - z, \quad \mathcal{I}(t, z) = \mathcal{E}(z) - \ell(t)z, \quad z_0 = 0, \quad T = 2,$$

while the sequence of loads was

$$\ell_n(t) = \begin{cases} 0, & t \in [0, 1], \\ \frac{n}{2}t - \frac{n}{2}, & t \in (1, 1 + \frac{1}{n}), \\ \frac{1}{2}, & t \in [1 + \frac{1}{n}, 2]. \end{cases} \tag{5.2.45}$$

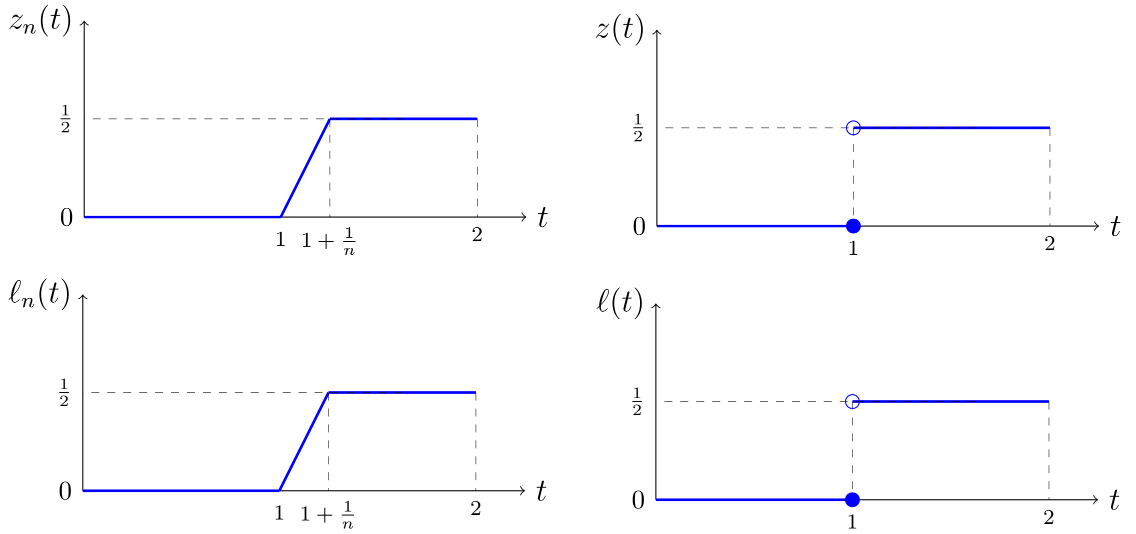

Figure 5.7: Graphs of the local solution z_n from (5.2.46) associated with load ℓ_n (5.2.45) and its limits z, ℓ (5.2.47)

However, instead of investigating the behavior of the normalized, $\mathfrak{p}$-parametrized BV solutions, we consider now solutions in physical time. One easily verifies that a differential and thus local solution is given by

$$z_n(t) = \begin{cases} 0, & t \in [0,1], \\ \frac{n}{2}t - \frac{n}{2}, & t \in (1, 1+\frac{1}{n}), \\ \frac{1}{2}, & t \in [1+\frac{1}{n}, 2]. \end{cases} \tag{5.2.46}$$

With respect to intermediate/strict convergence in $BV([0,T];\mathbb{R})$ the functions ℓ_n and z_n converge to

$$z(t) = \begin{cases} 0, & t \in [0,1], \\ \frac{1}{2} & t \in (1,2], \end{cases} \qquad \ell(t) = \begin{cases} 0, & t \in [0,1], \\ \frac{1}{2} & t \in (1,2]. \end{cases} \tag{5.2.47}$$

However, for any $0 \le t_1 < 1 < t_2 \le 2$, the energy inequality (5.2.6) is not fulfilled, since

$$\mathcal{I}(t_2, z(t_2)) + \mathrm{Var}_{\mathcal{R}}(z; [t_1,t_2]) - \sum_{t \in [t_1,t_2]} \Delta(\ell, z, t) = \frac{1}{8}$$
$$> 0 = \mathcal{I}(t_1, z(t_1)) - \int_{t_1}^{t_2} \langle z(r), \mathrm{d}\ell(r) \rangle.$$

Note that we again observe the same lack of continuity of the Kurzweil integral as in the example before, as $\int_0^2 \langle z_n(r), \mathrm{d}\ell_n(r) \rangle = -\frac{1}{8} \neq 0 = \int_0^2 \langle z(r), \mathrm{d}\ell(r) \rangle$ for all $n \in \mathbb{N}$.

In order construct a counterexample for the notion of solutions according to Definition 5.2.4, we need to slightly modify the setting.

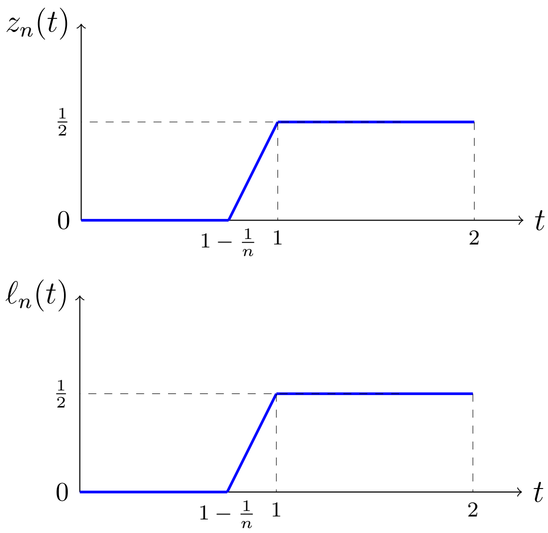

Figure 5.8: Graphs of the local solution z_n associated with loads ℓ_n and its limits z, ℓ from Example 5.2.13

Example 5.2.13 ([AM24, Sec. 4.3])**.** Let energy, dissipation, initial state and end time unchanged, but the sequence of loads is now set to

$$\ell_n(t) = \begin{cases} 0, & t \in [0, 1-\frac{1}{n}], \\ \frac{n}{2}t - \frac{n}{2} + \frac{1}{2}, & t \in (1-\frac{1}{n}, 1), \\ \frac{1}{2}, & t \in [1,2]. \end{cases}$$

The associated differential (and thus also local) solutions read

$$z_n(t) = \begin{cases} 0, & t \in [0, 1-\frac{1}{n}], \\ \frac{n}{2}t - \frac{n}{2} + \frac{1}{2}, & t \in (1-\frac{1}{n}, 1), \\ \frac{1}{2}, & t \in [1,2]. \end{cases}$$

As limits with respect to intermediate/strict convergence, we obtain the right continuous functions

$$z(t) = \begin{cases} 0, & t \in [0,1), \\ \frac{1}{2}, & t \in [1,2], \end{cases} \qquad \ell(t) = \begin{cases} 0, & t \in [0,1), \\ \frac{1}{2}, & t \in [1,2]. \end{cases} \tag{5.2.48}$$

In this case the energy inequality (5.2.4) is not valid for any $0 \le t_1 < 1 < t_2 \le 2$, since

$$\mathcal{I}(t_2, z(t_2)) + \mathrm{Var}_{\mathcal{R}}(z; [t_1, t_2]) = -\frac{1}{8} > -\frac{1}{4} = \mathcal{I}(t_1, z(t_1)) - \int_{t_1}^{t_2} \langle z(r), \mathrm{d}\ell(r)\rangle.$$

As before, one again verifies that the Kurzweil integrals involving the loads do not converge, as $\int_0^2 \langle z_n(r), \mathrm{d}\ell_n(r)\rangle = \frac{1}{8} \neq \frac{1}{4} = \int_0^2 \langle z(r), \mathrm{d}\ell(r)\rangle$ for all $n \in \mathbb{N}$.

The above examples show that local solutions are indeed even not stable with respect to intermediate/strict convergence of the loads $\ell \in BV([0,T];\mathbb{R})$. Moreover, we observe that the integrals $\int_0^2 \langle z(r), \mathrm{d}\ell(r) \rangle$, where on the one hand z, ℓ are given by (5.2.47) and on the other hand by (5.2.48), differs, although the functions z, ℓ are one sided continuous and only differ in $t = 1$. Here we see, in view of the Kurzweil integral, which depends on the pointwise definition of the considered functions, it makes a difference whether we consider equivalence classes or pointwise defined functions of bounded variation. The concept of equivalence classes seems here not to be meaningful since the validity of the energy inequality depends on the chosen representative and is even varying for the left and right hand continuous representative.

Chapter 6

Conclusion and Outlook

To summarize, in Chapter 3 we have seen how the existence of $\mathfrak{p}$-parametrized balanced viscosity solutions for rate-independent systems can be shown by using the vanishing viscosity approach. After a suitable parametrization, the uniquely determined solutions of the viscous regularized systems provide a weakly converging subsequence and the limit is a $\mathfrak{p}$-parametrized BV solution. This property is also helpful in the context of optimal control of rate-independent systems because, under suitable additional assumptions, even problems of the form (OCP) can be approximated by regularized problems (vOCP$_\epsilon$), where we replaced the feasible set of parametrized BV solutions associated with given external load as the control variable by the solution of the viscous regularized system (RIS$_\epsilon$). The advantage of this is that now the control-to-state operator is single-valued, which simplifies the application of numerical methods for solving these problems. To show this approximation result, we were able to adopt some ideas from [KMS22], where a finite dimensional setting was considered, and had to modify the underlying analysis substantially to make the approximability work in our infinite dimensional setting as well. However, we also had to make additional assumptions on at least one solution of the original problem (OCP) in order to construct a recovery sequence that is feasible for the regularized problems (vOCP$_\epsilon$) and has the necessary convergence properties to obtain the desired convergence of the objective value. We have seen that this assumption is pretty restrictive since it requires the existence of a differential solution of (RIS), which is only guaranteed in the case of a uniformly convex energy and not in the energy-dissipation framework that we consider. However, it was necessary because otherwise the load ℓ_ϵ constructed for the recovery sequence that includes this differential solution would not satisfy the required regularity, i.e., $\ell_\epsilon \notin H^1(0,T;\mathcal{V}^*)$.

The obvious approach to weaken this assumption is then to also allow systems with discontinuous loads, which was investigated in Chapter 5. We have seen that the solution concept of $\mathfrak{p}$-parametrized balanced viscosity solutions for loads in $BV(0,T)$ from [KZ21] is not suitable in the context of optimal control of rate-independent systems since it does not have the required stability properties needed to ensure the existence of an optimal solution. For a pointwise or weakly* in $\mathrm{BV}(0,T)$ convergent sequence of loads, the associated parametrized solutions can converge to a limit that is no longer a parametrized solution. Thus we have no compactness result regarding the feasible set of the optimal control problem in terms of the pointwise/weak* convergence. This can be remedied by the weaker solution concept of the relaxed solutions. Unfortunately, these have disadvantages with regard to the physical plausibility because they allow exemplary solutions, where the state can be raised to a higher energy level without any impact of the external force as seen in Example 5.2.11. This also contradicts the solution concepts of global energetic and local solutions as the energy (in)equality is violated, which in turn means that the relaxed solutions are even weaker than local solutions. In addition, we have also seen that it is generally difficult to define local solutions in this context since there exists different procedures to extend this notion to the broader class of discontinuous loads that do not have to coincide. Example 5.2.11 shows that, if one is interested in physically meaningful solution concepts that also allow for discontinuous loads, one has to assume stronger properties than in the case of relaxed solutions. But the price one has to pay for this is the the non-existent compactness property regarding pointwise/weak* convergence of the loads. Now, of course, the question arises whether there is a physically plausible solution concept between the relaxed solutions and the $\mathfrak{p}$-parametrized balanced viscosity solutions for loads in $BV(0,T)$ that is stable with respect to a stronger notion of convergence in $BV(0,T)$.

A further aspect for future research is to transfer our results to a more general energy-dissipation framework that does not require the semilinear structure of the energy functional or also allows for additional state dependent dissipation potentials, i.e., $\mathcal{R}(z,z')$, so that the results are applicable for a broader class of physical problems. In addition to that, our approximation results for (OCP) lead to the natural question, if knowledge gained here can be used to calculate an optimal solution by solving the regularized systems numerically. To this end, one option is to apply an additional smoothing of the dissipation potential that results in a differentiable control-to-state operator. Then one ends up in a smooth optimal control problem and can use standard adjoint-based methods for solving it. The smoothing parameter must then be driven to zero, possibly simultaneously with the regularization parameter $\epsilon > 0$, in order to end up with a solution of the regularized problem (vOCP_ϵ) or directly of the original problem (OCP), respectively. For the case of a quadratic energy, by means of this additional smoothing of the dissipation potential, the authors in [SWW17] have shown differentiability of the control-to-state operator and have derived optimality conditions for the regularized optimal control problem. However, they worked with the twice regularized system ($\mathrm{RIS}_{\epsilon,\delta}$) instead of the once regularized (RIS_ϵ) as we did. So one could also ask whether our approximation results are also valid if we replace (RIS_ϵ) by ($\mathrm{RIS}_{\epsilon,\delta}$) in the constraints of (vOCP_ϵ) and pass both regularization parameter δ, ϵ simultaneously to zero. Of course, this list of open questions is by far not complete, but hopefully it gives a good outlook on possible future research in this field.

Appendix A

Elements of convex analysis

Lemma A.1. *Let X be a Banach space and $f : X \to \mathbb{R} \cup \{\infty\}$ non-negative, proper, convex and lower semicontinuous. Then the functional*

$$F : L^1(a,b;X) \to \mathbb{R} \cup \{\infty\}, \qquad v \mapsto \begin{cases} \int_a^b f(v(x)) \, \mathrm{d}x, & \textit{if } f \circ v \in L^1(a,b), \\ \infty, & \textit{else}, \end{cases} \tag{A.1}$$

is proper, convex and lower semicontinuous, too.

Proof. We follow the lines of [Sie20, Lem. A.3.5]. Since f is proper, there exists $v_0 \in X$ with $f(v_0) < \infty$ such that the constant function $v \equiv v_0 \in L^1(a,b;X)$ fulfills $F(v_0) < \infty$ and F is proper, too. In order to prove the convexity of F, we assume that $v, w \in \mathrm{dom}(F)$. Then exploiting the convexity of f gives for almost all $x \in (a,b)$ and $\lambda \in [0,1]$

$$f(\lambda v(x) + (1-\lambda) w(x)) \leq \lambda f(v(x)) + (1-\lambda) f(w(x)).$$

Because f is lower semicontinuous and therefore measurable, the composition $f \circ (\lambda v + (1-\lambda) w)$ is measurable, too, and we can integrate the above inequality over (a,b) and obtain

$$\begin{aligned} F(\lambda v + (1-\lambda) w) &= \int_a^b f(\lambda v(x) + (1-\lambda) w(x)) \, \mathrm{d}x \\ &\leq \int_a^b \lambda f(v(x)) + (1-\lambda) f(w(x)) \, \mathrm{d}x = \lambda F(v) + (1-\lambda) F(w) \end{aligned}$$

such that F is convex. If one function $v, w \notin \mathrm{dom}(F)$, then the right hand side of the above inequality equals infinity and thus the inequality is certainly fulfilled. It still remains to show the lower semicontinuity of F. Therefore, we will verify the equivalent condition that the sublevel sets $\mathcal{N}(F,\alpha) = \{v \in L^1(a,b;X) : F(v) \leq \alpha\}$ are closed for every $\alpha \in \mathbb{R}$, cf. [ET99, Prop. 2.3]. Hence, let a sequence $\{v_n\}_{n \in \mathbb{N}} \subset \mathcal{N}(F,\alpha)$ with $v_n \to v$ in $L^1(a,b;X)$ be given. Then there exists a subsequence that converges pointwise almost everywhere, i.e.,

$$v_{n_k}(x) \to v(x) \quad \text{f.a.a. } x \in (a,b).$$

Now using Fatou's lemma, where the non-negativity of f is required, and the lower semicontinuity of f leads to

$$F(v) = \int_a^b f(v(x))\,\mathrm{d}x \leq \int_a^b \liminf_{k\to\infty} f(v_{n_k}(x))\,\mathrm{d}x \leq \liminf_{k\to\infty} \int_a^b f(v_{n_k}(x))\,\mathrm{d}x \leq \alpha$$

such that the limit satisfies $v \in \mathcal{N}(F, \alpha)$ and the sublevel set is indeed closed. $\square$

Lemma A.2. *Let $\mathcal{R} : \mathcal{V} \to [0, \infty)$ comply with* (1.2.13)-(1.2.15). *Further, $\partial^{\mathcal{Z}}\mathcal{R}(v) \subset \mathcal{Z}^*$ denotes the subdifferential with respect to the $\mathcal{Z}$-$\mathcal{Z}^*$-duality and $\partial^{\mathcal{V}}\mathcal{R}(v) \subset \mathcal{V}^*$ is the subdifferential concerning the $\mathcal{V}$-$\mathcal{V}^*$-duality. Then it holds for all $v \in \mathcal{Z}$ that $\partial^{\mathcal{Z}}\mathcal{R}(v) = \partial^{\mathcal{V}}\mathcal{R}(v)$.*

Proof. We start with exploiting the homogeneity and convexity of $\mathcal{R}$ in order to obtain for all $v, w \in \mathcal{Z}$

$$\frac{1}{2}\mathcal{R}(w) = \mathcal{R}\left(\frac{1}{2}w\right) = \mathcal{R}\left(\frac{1}{2}(w - v) + \frac{1}{2}v\right) \leq \frac{1}{2}\mathcal{R}(w - v) + \frac{1}{2}\mathcal{R}(v),$$

which directly implies $\mathcal{R}(w) - \mathcal{R}(v) \leq \mathcal{R}(w - v)$. Then for $\eta \in \partial^{\mathcal{Z}}\mathcal{R}(v) \subset \mathcal{Z}^*$ we have

$$\langle \eta, w - v\rangle_{\mathcal{Z}^*,\mathcal{Z}} \leq \mathcal{R}(w) - \mathcal{R}(v) \leq \mathcal{R}(w - v)$$

for all $w \in \mathcal{Z}$. Now choosing $w = v + \xi$ with arbitrary $\xi \in \mathcal{Z}$ yields along (1.2.15)

$$\langle \eta, \xi\rangle_{\mathcal{Z}^*,\mathcal{Z}} \leq \mathcal{R}(\xi) \leq C\|\xi\|_{\mathcal{V}}.$$

Therefore, by means of the density of $\mathcal{Z} \subset \mathcal{V}$, η can be extended an element of $\mathcal{V}^*$. Moreover, the density implies that for all $w \in \mathcal{V}$ it exists a sequence $(w_n)_n \subset \mathcal{Z}$ with $w_n \to w$ in $\mathcal{V}$, as $n \to \infty$. Thus, the continuity of $\mathcal{R}$ on $\mathcal{V}$ yields

$$\mathcal{R}(w) - \mathcal{R}(v) = \lim_{n\to\infty} \mathcal{R}(w_n) - \mathcal{R}(v) \geq \lim_{n\to\infty} \langle \eta, w_n - v\rangle_{\mathcal{V}^*,\mathcal{V}} = \langle \eta, w - v\rangle_{\mathcal{V}^*,\mathcal{V}}$$

such that $\eta \in \partial^{\mathcal{V}}\mathcal{R}(v) \subset \mathcal{V}^*$. Since $\mathcal{Z} \subset \mathcal{V}$, the opposite inclusion follows directly. $\square$

Lemma A.3. *Let $\mathcal{R}_\epsilon : \mathcal{Z} \to [0, \infty)$ be given by $\mathcal{R}_\epsilon(z) = \mathcal{R}(z) + \frac{\epsilon}{2}\|z\|_{\mathcal{V}}^2$. Then for all $z \in \mathcal{Z}$ it holds*

$$\partial^{\mathcal{Z}}\mathcal{R}_\epsilon(z) = \partial^{\mathcal{V}}\mathcal{R}_\epsilon(z) = \partial\mathcal{R}(z) + \partial\Big(\frac{\epsilon}{2}\|\cdot\|_{\mathcal{V}}^2\Big)(z) \subset \mathcal{V}^*.$$

Proof. Let $\eta \in \partial^{\mathcal{Z}}\Big(\frac{\epsilon}{2}\|\cdot\|_{\mathcal{V}}^2\Big)(z) \subset \mathcal{Z}^*$ be given. Then the definition of the subdifferential yields for all $v \in \mathcal{Z}$

$$\langle \eta, v - z\rangle_{\mathcal{Z}^*,\mathcal{Z}} \leq \frac{\epsilon}{2}\big(\|v\|_{\mathcal{V}}^2 - \|z\|_{\mathcal{V}}^2\big).$$

Inserting $v = w + z$ with arbitrary $w \in \mathcal{Z}$ and applying the Cauchy-Schwarz inequality results in

$$\langle \eta, w \rangle_{\mathcal{Z}^*,\mathcal{Z}} \leq \frac{\epsilon}{2}\big(\|w+z\|_{\mathcal{V}}^2 - \|z\|_{\mathcal{V}}^2\big) \leq \frac{\epsilon}{2}\big(\|w\|_{\mathcal{V}}^2 + 2\|w\|_{\mathcal{V}}\|z\|_{\mathcal{V}}\big).$$

Thus, $\eta \in \mathcal{Z}^*$ can be extended to an element of $\mathcal{V}^*$ because $\mathcal{Z}$ is densely embedded in $\mathcal{V}$. Additionally, the continuity of the norm implies that $\eta \in \partial^{\mathcal{V}}\Big(\frac{\epsilon}{2}\|\cdot\|_{\mathcal{V}}^2\Big)(z)$. Clearly, the inclusion $\partial^{\mathcal{V}}\Big(\frac{\epsilon}{2}\|\cdot\|_{\mathcal{V}}^2\Big)(z) \subset \partial^{\mathcal{Z}}\Big(\frac{\epsilon}{2}\|\cdot\|_{\mathcal{V}}^2\Big)(z)$ holds since $\mathcal{Z} \subset \mathcal{V}$. Finally, the sum rule for convex subdifferential, cf. [ET99, Prop. 5.6], along with Lemma A.2 gives the assertion. □

Lemma A.4. *Let* $w \in \mathcal{V}^*$ *and* $\operatorname{dist}_{\mathcal{V}^*}(w, \partial\mathcal{R}(0)) = \inf_{\eta\in\partial\mathcal{R}(0)} \|w-\eta\|_{\mathcal{V}^*}$. *Then there exists* $\eta \in \partial\mathcal{R}(0)$ *such that* $\operatorname{dist}_{\mathcal{V}^*}(w, \partial\mathcal{R}(0)) = \|w-\eta\|_{\mathcal{V}^*}$. *Hence, the infimum is in fact a minimum.*

Proof. Let $\{\eta_n\}_{n\in\mathbb{N}} \subset \partial\mathcal{R}(0)$ be an infimal sequence, i.e., $\lim_{n\to\infty} \|w-\eta_n\|_{\mathcal{V}^*} = \operatorname{dist}_{\mathcal{V}^*}(w, \partial\mathcal{R}(0))$. Therefore, $\|w-\eta_n\|_{\mathcal{V}^*}$ is bounded and we obtain a weakly convergent subsequence $w - \eta_{n_k} \rightharpoonup \xi$ in $\mathcal{V}^*$, which implies $\eta_{n_k} \rightharpoonup w - \xi$ in $\mathcal{V}^*$. Since $\partial\mathcal{R}(0) \subset \mathcal{V}^*$ (cf. Lemma 1.2.5) is closed and convex, and therefore, in particular, weakly closed, the weak limit $w-\xi$ is part of $\partial\mathcal{R}(0)$, too. Eventually, exploiting the weak lower semicontinuity of the norm results in

$$\|w-(w-\xi)\|_{\mathcal{V}^*} = \|\xi\|_{\mathcal{V}^*} \leq \liminf_{k\to\infty} \|w-\eta_{n_k}\|_{\mathcal{V}^*},$$

such that the infimum is indeed a minimum. □

Lemma A.5. *Let* $w \in \mathcal{V}^*$ *be given. Then the convex conjugate of the regularized dissipation satisfies*

$$\mathcal{R}_\epsilon^*(w) = \frac{1}{2\epsilon}\operatorname{dist}_{\mathcal{V}^*}(w, \partial\mathcal{R}(0))^2. \tag{A.2}$$

Proof. First, we consider the function $f : \mathcal{V} \to \mathbb{R}$, $v \mapsto \frac{\epsilon}{2}\|v\|_{\mathcal{V}}^2$ and want to determine its convex conjugate, which is generally given by

$$f^*(\eta) = \sup_{v\in\mathcal{V}} \langle \eta, v\rangle_{\mathcal{V}^*,\mathcal{V}} - \frac{\epsilon}{2}\|v\|_{\mathcal{V}}^2.$$

For all $v \in \mathcal{V}$ we have

$$\langle \eta, v\rangle_{\mathcal{V}^*,\mathcal{V}} - \frac{\epsilon}{2}\|v\|_{\mathcal{V}}^2 \leq \|\eta\|_{\mathcal{V}^*}\|v\|_{\mathcal{V}} - \frac{\epsilon}{2}\|v\|_{\mathcal{V}}^2 \leq \frac{1}{2\epsilon}\|\eta\|_{\mathcal{V}^*}^2$$

because the function $\mathbb{R}_{\geq 0} \ni x \mapsto \|\eta\|_{\mathcal{V}^*}x - \frac{\epsilon}{2}x^2$ attains it maximum in $x = \frac{1}{\epsilon}\|\eta\|_{\mathcal{V}^*}$. Otherwise, since $\mathcal{V}$ is reflexive, it exists $\tilde{v} \in \mathcal{V}$ with $\|\tilde{v}\|_{\mathcal{V}} = 1$ such that $\|\eta\|_{\mathcal{V}^*} = \langle \eta, \tilde{v}\rangle_{\mathcal{V}^*,\mathcal{V}}$, cf. [Kab18, Thm. 9.10]. Now we define $v := \frac{\|\eta\|_{\mathcal{V}^*}}{\epsilon}\tilde{v} \in \mathcal{V}$ and obtain

$$\langle \eta, v\rangle_{\mathcal{V}^*,\mathcal{V}} - \frac{\epsilon}{2}\|v\|_{\mathcal{V}}^2 = \frac{1}{\epsilon}\|\eta\|_{\mathcal{V}^*}^2 - \frac{1}{2\epsilon}\|\eta\|_{\mathcal{V}^*}^2 = \frac{1}{2\epsilon}\|\eta\|_{\mathcal{V}^*}^2.$$

Thus, we have that $f^*(\eta) = \frac{1}{2\epsilon}\|\eta\|_{\mathcal{V}^*}^2$. Now applying the inf-convolution formula from [Att84, Prop. 3.4] and using (1.2.21) yields for the convex conjugate of $\mathcal{R}_\epsilon = \mathcal{R} + f$

$$\mathcal{R}_\epsilon^*(w) = \inf_{\eta \in \mathcal{V}^*} (\mathcal{R}^*(\eta) + f^*(w - \eta)) = \inf_{\eta \in \partial\mathcal{R}(0)} \frac{1}{2\epsilon}\|w - \eta\|_{\mathcal{V}^*}^2 = \frac{1}{2\epsilon}\operatorname{dist}_{\mathcal{V}^*}(w, \partial\mathcal{R}(0))^2,$$

which gives the claim. □

Lemma A.6. *Let $\mathcal{R}_{\epsilon,\delta} : \mathcal{Z} \to [0, \infty)$ be given by $\mathcal{R}_{\epsilon,\delta}(z) = \mathcal{R}(z) + \frac{\epsilon}{2}\|z\|_{\mathcal{V}}^2 + \frac{\delta}{2}|z|_{\mathcal{Z}}^2$, where $\mathcal{R}$ satisfies* (1.2.13)-(1.2.15). *Then $\partial\mathcal{R}_{\epsilon,\delta}^* : \mathcal{Z}^* \rightrightarrows \mathcal{Z}$ is single-valued and globally Lipschitz continuous.*

Proof. Due to the convexity of $\mathcal{R}$, the regularized function $\mathcal{R}_{\epsilon,\delta}$ is uniformly convex, i.e.,

$$\begin{aligned}\mathcal{R}_{\epsilon,\delta}(\lambda z_1 + (1-\lambda)z_2) \leq \lambda\mathcal{R}_{\epsilon,\delta}(z_1) + (1-\lambda)\mathcal{R}_{\epsilon,\delta}(z_2)& \\ - \lambda(1-\lambda)\frac{\epsilon}{2}\|z_1 - z_2\|_{\mathcal{V}}^2 - \lambda(1-\lambda)\frac{\delta}{2}|z_1 - z_2|_{\mathcal{Z}}^2&\end{aligned}$$

for all $z_1, z_2 \in \mathcal{Z}$ and $\lambda \in [0, 1]$. Let $\xi_1 \in \partial\mathcal{R}_{\epsilon,\delta}(z_1), \xi_2 \in \partial\mathcal{R}_{\epsilon,\delta}(z_2)$ such that

$$\begin{aligned}\mathcal{R}_{\epsilon,\delta}\Big(\frac{1}{2}(z_1 + z_2)\Big) &\geq \mathcal{R}_{\epsilon,\delta}(z_1) + \frac{1}{2}\langle\xi_1, z_2 - z_1\rangle_{\mathcal{Z}^*,\mathcal{Z}}, \\ \mathcal{R}_{\epsilon,\delta}\Big(\frac{1}{2}(z_1 + z_2)\Big) &\geq \mathcal{R}_{\epsilon,\delta}(z_2) + \frac{1}{2}\langle\xi_1, z_1 - z_2\rangle_{\mathcal{Z}^*,\mathcal{Z}}.\end{aligned}$$

By exploiting the uniform convexity, we obtain

$$\langle\xi_1, z_2 - z_1\rangle_{\mathcal{Z}^*,\mathcal{Z}} \leq \mathcal{R}_{\epsilon,\delta}(z_2) - \mathcal{R}_{\epsilon,\delta}(z_1) - \frac{\epsilon}{4}\|z_1 - z_2\|_{\mathcal{V}}^2 - \frac{\delta}{4}|z_1 - z_2|_{\mathcal{Z}}^2$$

and

$$\langle\xi_2, z_1 - z_2\rangle_{\mathcal{Z}^*,\mathcal{Z}} \leq \mathcal{R}_{\epsilon,\delta}(z_1) - \mathcal{R}_{\epsilon,\delta}(z_2) - \frac{\epsilon}{4}\|z_1 - z_2\|_{\mathcal{V}}^2 - \frac{\delta}{4}|z_1 - z_2|_{\mathcal{Z}}^2.$$

Adding these estimates yields

$$\langle\xi_1 - \xi_2, z_1 - z_2\rangle_{\mathcal{Z}^*,\mathcal{Z}} \geq \frac{\epsilon}{2}\|z_1 - z_2\|_{\mathcal{V}}^2 + \frac{\delta}{2}|z_1 - z_2|_{\mathcal{Z}}^2 \geq \frac{\min\{\epsilon, \delta\}}{2}\|z_1 - z_2\|_{\mathcal{Z}}^2 \tag{A.3}$$

such that $\partial\mathcal{R}_{\epsilon,\delta}^*$ is a strictly monotone operator. If $\xi_1 = \xi_2$, (A.3) implies $z_1 = z_2$ such that $\partial\mathcal{R}_{\epsilon,\delta}^*$ is single-valued. Moreover, (A.3) along with the Cauchy-Schwarz inequality gives the Lipschitz continuity

$$\|\partial\mathcal{R}_{\epsilon,\delta}^*(\xi_1) - \partial\mathcal{R}_{\epsilon,\delta}^*(\xi_2)\|_{\mathcal{Z}} = \|z_1 - z_2\|_{\mathcal{Z}} \leq \frac{2}{\min\{\epsilon, \delta\}}\|\xi_1 - \xi_2\|_{\mathcal{Z}^*}. \tag{A.4}$$

□

The same statement holds true for the once regularized dissipation $\mathcal{R}_\epsilon$, which is the following lemma.

Lemma A.7. *Let* $\mathcal{R}_\epsilon : \mathcal{Z} \to [0,\infty)$ *be given by* $\mathcal{R}_\epsilon(z) = \mathcal{R}(z) + \frac{\epsilon}{2}\|z\|_{\mathcal{V}}^2$. *Then* $\partial\mathcal{R}_\epsilon^* : \mathcal{V}^* \rightrightarrows \mathcal{V}$ *is single-valued and globally Lipschitz continuous.*

Proof. First of all, Lemma A.3 yields that $\partial\mathcal{R}_\epsilon(z) \subset \mathcal{V}^*$ for all $z \in \mathcal{Z}$. Analogously to the proof of Lemma A.6, we derive for all $z_1, z_2 \in \mathcal{Z}$ and $\xi_1 \in \partial\mathcal{R}_\epsilon(z_1) \subset \mathcal{V}^*, \xi_2 \in \partial\mathcal{R}_\epsilon(z_2) \subset \mathcal{V}^*$

$$\langle \xi_1 - \xi_2, z_1 - z_2\rangle_{\mathcal{V}^*,\mathcal{V}} = \langle \xi_1 - \xi_2, z_1 - z_2\rangle_{\mathcal{Z}^*,\mathcal{Z}} \geq \frac{\epsilon}{2}\|z_1 - z_2\|_{\mathcal{V}}^2. \tag{A.5}$$

Thus, $\partial\mathcal{R}_\epsilon^*$ is a strictly monotone operator and single-valued. Finally, applying the Cauchy-Schwarz inequality yields the Lipschitz continuity

$$\|\partial\mathcal{R}_\epsilon^*(\xi_1) - \partial\mathcal{R}_\epsilon^*(\xi_2)\|_{\mathcal{V}} = \|z_1 - z_2\|_{\mathcal{V}} \leq \frac{2}{\epsilon}\|\xi_1 - \xi_2\|_{\mathcal{V}^*}.$$

□

Lemma A.8. *Let* $\mathcal{R} : \mathbb{R}^d \to [0,\infty)$ *comply with* (1.2.13)–(1.2.15) *and* $\tau > 0$ *be given. Then the following identity holds*

$$\tau \operatorname{dist}(\eta, \partial\mathcal{R}(0)) = \sup_{v\in\mathbb{R}^d,\, \|v\|\leq\tau} \big(\langle \eta, v\rangle - \mathcal{R}(v)\big)$$

for all $\eta \in \mathbb{R}^d$.

Proof. Let I_τ denote the indicator functional of $\overline{B(0,\tau)}$, the closed ball of radius τ centered at zero, i.e.,

$$I_\tau(z) := \begin{cases} 0, & \text{if } \|z\| \leq \tau, \\ \infty, & \text{else.} \end{cases}$$

Then the inf-convolution formula from [Att84, Prop. 3.4] yields

$$\big(\mathcal{R} + I_\tau\big)^*(\eta) = \inf_{v\in\mathbb{R}^d} \Big(\mathcal{R}^*(v) + I_\tau^*(\eta - v)\Big). \tag{A.6}$$

From (1.2.21) we already have that $\mathcal{R}^* = I_{\partial\mathcal{R}(0)}$ and

$$I_\tau^*(\xi) = \sup_{v\in\mathbb{R}^d} \langle \xi, v\rangle - I_\tau(v) = \sup_{v\in\mathbb{R}^d, \|v\|\leq\tau} \langle \xi, v\rangle = \tau \sup_{v\in\mathbb{R}^d, \|v\|\leq 1} \langle \xi, v\rangle = \tau\|\xi\|, \quad \xi \in \mathbb{R}^d$$

shows that the conjugate of I_τ is given by $I_\tau^*(\cdot) = \tau\|\cdot\|$. Inserting this in (A.6) yields

$$\big(\mathcal{R} + I_\tau\big)^*(\eta) = \inf_{v\in\mathbb{R}^d} \Big(I_{\partial\mathcal{R}(0)}(v) + \tau\|\eta - v\|\Big) = \tau \inf_{v\in\partial\mathcal{R}(0)} \|\eta - v\| = \tau \operatorname{dist}(\eta, \partial\mathcal{R}(0)),$$

which gives the claim. □

Appendix B

The $\mathcal{R}$-variation

Definition B.1. For a function $z : [0,T] \to \mathcal{Z}$ we define the total $\mathcal{R}$-variation on the interval $[a,b] \subset [0,T]$ by

$$\mathrm{Var}_{\mathcal{R}}(z;[a,b]) := \sup\Big\{ \sum_{i=1}^{N} \mathcal{R}(z(t_i) - z(t_{i-1})) : a = t_0 < t_1 < \cdots < t_N = b, N \in \mathbb{N} \Big\}.$$

Lemma B.2. *Let* $\mathcal{R} : \mathcal{Z} \to [0,\infty)$ *comply with* (1.2.13)–(1.2.15) *and* $z \in \mathrm{AC}^p([0,T];\mathcal{R})$*,* $p \in (1,\infty]$*, with* $\sup_{t\in[0,T]} \|z(t)\|_{\mathcal{Z}} < \infty$*, then*

$$\mathrm{Var}_{\mathcal{R}}(z;[a,b]) = \int_a^b \mathcal{R}[z'](s)\,\mathrm{d}s \quad \text{for all } 0 \le a < b \le T. \tag{B.1}$$

Moreover, the $\mathcal{R}$*-variation is lower semicontinuous, i.e.,*

$$\mathrm{Var}_{\mathcal{R}}(z;[a,b]) \le \liminf_{n\to\infty} \mathrm{Var}_{\mathcal{R}}(z_n;[a,b]) \quad \text{for all} \;\; 0 \le a < b \le T.$$

with a sequence $(z_n)_n \subset \mathrm{AC}^p([0,T];\mathcal{R})$ *satisfying* $z_n(t) \rightharpoonup z(t)$ *in* $\mathcal{Z}$ *for all* $t \in [0,T]$*.*

Proof. Let $\{t_i\}_{i=0,\dots,n}$ be an arbitrary partition of $[a,b]$. Since $z \in \mathrm{AC}^p([0,T];\mathcal{R})$, the metric derivative

$$\mathcal{R}[z'](s) = \lim_{h\searrow 0} \mathcal{R}\left(\frac{z(t+h) - z(t)}{h}\right) = \lim_{h\searrow 0} \mathcal{R}\left(\frac{z(t) - z(t-h)}{h}\right) \tag{B.2}$$

exists for almost all $s \in [a,b]$ and fulfills (3.3.17) with minimality, cf. [RMS08, Prop. 2.2]. Thus, we have

$$\mathcal{R}(z(t_i) - z(t_{i-1})) \le \int_{t_{i-1}}^{t_i} \mathcal{R}[z'](s)\,\mathrm{d}s$$

for all $i = 1,\dots,n$, such that summing up yields

$$\sum_{i=1}^{n} \mathcal{R}(z(t_i) - z(t_{i-1})) \le \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \mathcal{R}[z'](s)\,\mathrm{d}s = \int_a^b \mathcal{R}[z'](s)\,\mathrm{d}s. \tag{B.3}$$

Because this holds true for all partitions, we deduce

$$\mathrm{Var}_{\mathcal{R}}(z;[a,b]) \leq \int_a^b \mathcal{R}[z'](s)\,\mathrm{d}s.$$

In order to prove the reverse estimate, let a sequence of partitions $(\{t_i^h\}_{i=0}^{n_h})_h$ with $h := \max_{i=1,\dots,n_h} t_i^h - t_{i-1}^h \to 0$ and $\lim_{h\searrow 0}\sum_{i=1}^{n_h}\mathcal{R}(z(t_i^h)-z(t_{i-1}^h)) = \mathrm{Var}_{\mathcal{R}}(z;[a,b])$ be given. We define the piecewise linear interpolant of z by

$$\xi_h(t) := z(t_{i-1}) + \frac{t-t_{i-1}^h}{t_i^h - t_{i-1}^h}(z(t_i^h)-z(t_{i-1}^h)),\ t\in[t_{i-1}^h,t_i^h).$$

Since $z\in \mathrm{AC}^p([0,T];\mathcal{R})$, z is in particular continuous with respect to the $\mathcal{X}$-norm by (1.2.15), and we deduce

$$\|z(t)-\xi_h(t)\|_{\mathcal{X}} \leq \frac{t-t_{i-1}^h}{t_i^h-t_{i-1}^h}\|z(t)-z(t_i^h)\|_{\mathcal{X}} + \left(1-\frac{t-t_{i-1}^h}{t_i^h-t_{i-1}^h}\right)\|z(t)-z(t_{i-1}^h)\|_{\mathcal{X}} \to 0,$$

as $h\searrow 0$, such that $\xi_h\to z$ pointwisely in $\mathcal{X}$. In addition to that, ξ_h is also pointwisely bounded in $\mathcal{Z}$ as z is, too. Thus, by the standard argumentation by means of subsequences, we deduce that $\xi_h \rightharpoonup z$ pointwisely in $\mathcal{Z}$. Moreover, from the 1-homogeneity of $\mathcal{R}$ and Hölder's inequality we infer for $p\in(1,\infty)$ with q defined by $\frac{1}{p}+\frac{1}{q}=1$

$$\begin{aligned}
\int_{t_{i-1}^h}^{t_i^h}\mathcal{R}(\xi_h'(s))^p\,\mathrm{d}s =& \int_{t_{i-1}^h}^{t_i^h}\frac{1}{(t_i^h-t_{i-1}^h)^p}\mathcal{R}(z(t_i^h)-z(t_{i-1}^h))^p\,\mathrm{d}s\\
\leq& (t_i^h-t_{i-1}^h)^{1-p}\left(\int_{t_{i-1}^h}^{t_i^h}\mathcal{R}[z'](s)\,\mathrm{d}s\right)^p\\
=& \left(\int_{t_{i-1}^h}^{t_i^h}(t_i^h-t_{i-1}^h)^{\frac{1-p}{p}}\mathcal{R}[z'](s)\,\mathrm{d}s\right)^p\\
\leq& \left(\int_{t_{i-1}^h}^{t_i^h}(t_i^h-t_{i-1}^h)^{\frac{(1-p)q}{p}}\,\mathrm{d}s\right)^{\frac{p}{q}}\int_{t_{i-1}^h}^{t_i^h}\mathcal{R}[z'](s)^p\,\mathrm{d}s\\
=& (t_i^h-t_{i-1}^h)^{\left(\frac{(1-p)q}{p}+1\right)\frac{p}{q}}\int_{t_{i-1}^h}^{t_i^h}\mathcal{R}[z'](s)^p\,\mathrm{d}s = \int_{t_{i-1}^h}^{t_i^h}\mathcal{R}[z'](s)^p\,\mathrm{d}s.
\end{aligned}$$

Eventually, summing up yields $\|\mathcal{R}(\xi_h')\|_{L^p(a,b)} \leq \|\mathcal{R}[z']\|_{L^p(a,b)}$. If $p=\infty$, we also have $\|\mathcal{R}(\xi_h')\|_{L^\infty(a,b)} \leq \|\mathcal{R}[z']\|_{L^\infty(a,b)}$, which directly follows from

$$\mathcal{R}(\xi_h'(s)) = \frac{1}{t_i^h-t_{i-1}^h}\mathcal{R}(z(t_i^h)-z(t_{i-1}^h)) \leq \frac{1}{t_i^h-t_{i-1}^h}\int_{t_{i-1}^h}^{t_i^h}\mathcal{R}[z'](s)\,\mathrm{d}s \leq \|\mathcal{R}[z']\|_{L^\infty(a,b)}.$$

Thus, there exists a (not relabeled) weakly* converging subsequence and $v\in L^p(a,b)$ with $\mathcal{R}(\xi_h') \rightharpoonup^* v$ in $L^p(a,b)$. The pointwise weak convergence of ξ_h in $\mathcal{Z}$ together with the lower semicontinuity of $\mathcal{R}$ from (1.2.13) results in

$$\mathcal{R}(z(t)-z(s)) \leq \liminf_{h\searrow 0} \mathcal{R}(\xi_h(t)-\xi_h(s)) \leq \limsup_{h\searrow 0} \int_s^t \mathcal{R}(\xi_h'(s))\,\mathrm{d}s = \int_s^t v(s)\,\mathrm{d}s$$

for all $a \leq s < t \leq b$, where we used Jensen's inequality for the second estimate. Therefore, choosing $t = s+h$, dividing by $h>0$ and passing $h \searrow 0$ leads to

$$\mathcal{R}[z'](s) = \lim_{h\searrow 0} \mathcal{R}\Big(\frac{z(s+h)-z(s)}{h}\Big) \leq \lim_{h\searrow 0} \frac{1}{h}\int_s^{s+h} v(r)\,\mathrm{d}r = v(s)$$

for almost all $s \in (a,b)$, where the last equality follows from Lebesgue's differentiation theorem. Finally, by integration over (a,b), we obtain

$$\begin{aligned}\int_a^b \mathcal{R}[z'](s)\,\mathrm{d}s \leq \int_a^b v(s)\,\mathrm{d}s = \lim_{h\searrow 0}\int_a^b \mathcal{R}(\xi_h'(s))\,\mathrm{d}s&\\ = \lim_{h\searrow 0}\sum_{i=1}^{n_h}\int_{t_{i-1}^h}^{t_i^h}\mathcal{R}\Big(\frac{z(t_i^h)-z(t_{i-1}^h)}{t_i^h-t_{i-1}^h}\Big)\,\mathrm{d}s = \lim_{h\searrow 0}\sum_{i=1}^{n_h}\mathcal{R}(z(t_i^h)-z(t_{i-1}^h))&\\ =\mathrm{Var}_{\mathcal{R}}(z;[a,b]),&\end{aligned}$$

which gives the claim. The lower semicontinuity of the variation follows from the lower semicontinuity of $\mathcal{R}$ from (1.2.13). This can be seen as follows: Let $\{t_i\}_{i=1}^N$ be an arbitrary partition of $[a,b]$. Then the lower semicontinuity of $\mathcal{R}$ yields

$$\begin{aligned}\liminf_{n\to\infty}\mathrm{Var}_{\mathcal{R}}(z_n;[a,b]) &\geq \liminf_{n\to\infty}\sum_{i=1}^N \mathcal{R}(z_n(t_i)-z_n(t_{i-1}))\\ &\geq \sum_{i=1}^N \liminf_{n\to\infty}\mathcal{R}(z_n(t_i)-z_n(t_{i-1})) \geq \sum_{i=1}^N \mathcal{R}(z(t_i)-z(t_{i-1})).\end{aligned}$$

Since this estimate is valid for all partitions of $[a,b]$, we end up with

$$\liminf_{n\to\infty}\mathrm{Var}_{\mathcal{R}}(z_n;[a,b]) \geq \mathrm{Var}_{\mathcal{R}}(z;[a,b]),$$

which completes the proof. □

Lemma B.3. *Let $z \in W^{1,p}(0,T;\mathcal{Z})$ with $p \in (1,\infty]$. Then it holds*

$$\mathrm{Var}_{\mathcal{R}}(z;[a,b]) = \int_a^b \mathcal{R}(z'(s))\,\mathrm{d}s$$

for all $0 \leq a < b \leq T$.

Proof. Let an arbitrary partition $a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b$ be given. Then applying Jensen's inequality gives

$$\mathcal{R}(z(t_i)-z(t_{i-1})) = \mathcal{R}\Big(\int_{t_{i-1}}^{t_i} z'(s)\,\mathrm{d}s\Big) \leq \int_{t_{i-1}}^{t_i}\mathcal{R}(z'(s))\,\mathrm{d}s$$

for all $i = 1, \ldots, n$. Summing up results in $\sum_{i=1}^n \mathcal{R}(z(t_i) - z(t_{i-1})) \leq \int_a^b \mathcal{R}(z'(s))\,\mathrm{d}s$ and, as the partition was arbitrary, this gives $\mathrm{Var}_{\mathcal{R}}(z; [a,b]) \leq \int_a^b \mathcal{R}(z'(s))\,\mathrm{d}s$. On the other hand, the lower semicontinuity of $\mathcal{R}$ implies that for almost all $s \in (0,T)$ it holds

$$\mathcal{R}[z'](s) = \lim_{h \searrow 0} \mathcal{R}\bigg(\frac{z(s+h) - z(s)}{h}\bigg) = \liminf_{h \searrow 0} \mathcal{R}\bigg(\frac{z(s+h) - z(s)}{h}\bigg) \geq \mathcal{R}(z'(s)).$$

Therefore, from Lemma B.2, which is applicable due to $z \in W^{1,p}(0,T;\mathcal{Z}) \subset \mathrm{AC}^p([0,T];\mathcal{R}) \cap L^\infty(0,T;\mathcal{Z})$ by (1.2.15), we infer

$$\int_a^b \mathcal{R}(z'(s))\,\mathrm{d}s \leq \int_a^b \mathcal{R}[z'](s)\,\mathrm{d}s = \mathrm{Var}_{\mathcal{R}}(z; [a,b]).$$

□

The following lemma is slightly modified version of [KT23, Prop. D.1].

Lemma B.4. *The dissipation* $\mathcal{R} : \mathcal{Z} \to [0, \infty)$ *is supposed to comply with* (1.2.13)-(1.2.15)*. Then the following assertions hold:*

(a) Let $z \in \mathrm{AC}^1([a,b];\mathcal{R})$ *with* $\sup_{t \in [a,b]} \|z(t)\|_{\mathcal{Z}} < \infty$*, then* $z \in C([a,b];\mathcal{V})$ *and there exists* $C > 0$ *such that*

$$\|z\|_{C([a,b];\mathcal{V})} \leq C \|z\|_{L^\infty(a,b;\mathcal{Z})}. \tag{B.4}$$

for all $z \in \mathrm{AC}^1([a,b];\mathcal{R})$ *with* $\sup_{t \in [a,b]} \|z(t)\|_{\mathcal{Z}} < \infty$.

(b) Let $(z_n)_n \subset \mathrm{AC}^\infty([a,b];\mathcal{R}) \cap L^\infty(a,b;\mathcal{Z})$ *be a sequence satisfying*

$$A := \sup_{n \in \mathbb{N}} \sup_{t \in [a,b]} \|z_n(t)\|_{\mathcal{Z}} < \infty \text{ and } B := \sup_{n \in \mathbb{N}} \|\mathcal{R}[z_n']\|_{L^\infty(a,b)} < \infty. \tag{B.5}$$

Then there exists $z \in \mathrm{AC}^\infty([a,b];\mathcal{R})$ *with* $\sup_{t \in [a,b]} \|z(t)\|_{\mathcal{Z}} < \infty$ *and a (not relabeled) subsequence* $(z_n)_n$ *such that*

$$z_n \to z \text{ in } C([a,b];\mathcal{V}) \tag{B.6}$$

and for all converging sequences $[a,b] \ni t_n \to t$ *it holds*

$$z_n(t_n) \rightharpoonup z(t) \text{ in } \mathcal{Z}. \tag{B.7}$$

Moreover, the limit satisfies $z \in C_{weak}([a,b];\mathcal{Z})$.

Proof. Let $s, t \in [a,b]$ and $z \in \mathrm{AC}^1([a,b];\mathcal{R})$ with $\sup_{t \in [a,b]} \|z(t)\|_{\mathcal{Z}} < \infty$ be given. Then using Ehrling's lemma yields that for every $\mu > 0$ there exists a constant $C_\mu > 0$ such that

$$\|z(s) - z(t)\|_{\mathcal{V}} \leq \mu \|z(s) - z(t)\|_{\mathcal{Z}} + C_\mu \|z(s) - z(t)\|_{\mathcal{X}}$$

$$\leq 2\mu\|z\|_{L^\infty(a,b;\mathcal{Z})} + \tilde{C}_\mu \int_s^t \mathcal{R}[z'](r)\,\mathrm{d}r,$$

where we used (1.2.15) in the last step. Since $\mu > 0$ was arbitrary, the above estimate gives $z \in C([a,b];\mathcal{V})$. Additionally, (B.4) follows directly from the continuity of the embedding $\mathcal{Z} \hookrightarrow \mathcal{V}$. Now let a sequence $(z_n)_n \subset \mathrm{AC}^\infty([a,b];\mathcal{R}) \cap L^\infty(a,b;\mathcal{Z})$ satisfying (B.5) be given. Using Ehrling's lemma again yields that for every $\mu > 0$ there exists a constant $C_\mu > 0$ such that for all $s,t \in [a,b]$ and all $n \in \mathbb{N}$

$$\begin{aligned}\|z_n(s) - z_n(t)\|_{\mathcal{V}} &\leq \mu\|z_n(s) - z_n(t)\|_{\mathcal{Z}} + C_\mu\|z_n(s) - z_n(t)\|_{\mathcal{X}} \\ &\leq 2\mu\|z_n\|_{L^\infty(a,b;\mathcal{Z})} + \tilde{C}_\mu \int_s^t \mathcal{R}[z_n'](r)\,\mathrm{d}r \\ &\leq 2\mu A + \tilde{C}_\mu B|s-t|.\end{aligned}$$

Then, for given $\epsilon > 0$, choosing $\mu < \frac{\epsilon}{4A}$ and $\delta < \frac{\epsilon}{2B\tilde{C}_\mu}$ implies $\|z(s) - z(t)\|_{\mathcal{V}} < \epsilon$ for all $s,t \in [a,b]$ with $|s-t| < \delta$. Hence, the sequence $(z_n)_n \subset C([a,b];\mathcal{V})$ is equicontinuous. Furthermore, by assumption we have that $(z_n(t))_n$ is bounded in $\mathcal{Z}$ for all $t \in [a,b]$. Thus, by means of the compact embedding $\mathcal{Z} \hookrightarrow^c \mathcal{V}$, the sets $\{z_n(t) : n \in \mathbb{N}\} \subset \mathcal{V}$, $t \in [a,b]$, are relatively compact. Now applying the Arzelà-Ascoli theorem, cf. [Kab18, Thm. 2.5] and the remark below, yields the existence of a subsequence $(z_n)_n \subset C([a,b];\mathcal{V})$ satisfying (B.6). Since, additionally $\|z_n(t_n)\|_{\mathcal{Z}} \leq A$ by assumption, there exists a weakly converging subsequence $z_{n_k}(t_{n_k}) \rightharpoonup \tilde{z} \in \mathcal{Z}$ and in view of (B.6) and the uniqueness of limits, we obtain $\tilde{z} = z(t)$. Because this holds true for all subsequences, we end up with (B.7) for the entire sequence. Surely, the boundedness with respect to the $\mathcal{Z}$-norm transfers to the limit since the norm is weakly lower semicontinuous and, moreover, the lower semicontinuity of $\mathcal{R}$ yields

$$\mathcal{R}(z(s) - z(t)) \leq \liminf_{n\to\infty} \mathcal{R}(z_n(s) - z_n(t)) \leq \int_s^t B\,\mathrm{d}r$$

such that $z \in \mathrm{AC}^\infty([a,b];\mathcal{R}) \cap L^\infty((a,b);\mathcal{Z})$. With the same arguments as above, which are exploiting the strong convergence in $\mathcal{V}$ and the pointwise boundedness in $\mathcal{Z}$ for all $t \in [a,b]$, we infer that $z \in C_{weak}([a,b];\mathcal{Z})$. $\square$

Appendix C

Auxiliary results

Lemma C.1. *Let* $(a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ *be sequences satisfying* $\liminf_{n\to\infty} a_n \geq a$, $\liminf_{n\to\infty} b_n \geq b$ *and* $\lim_{n\to\infty} a_n + b_n = a + b$, *then* $\lim_{n\to\infty} a_n = a$ *and* $\lim_{n\to\infty} b_n = b$.

Proof. From the assumptions we infer

$$a + b = \lim_{n\to\infty} a_n + b_n = \liminf_{n\to\infty} a_n + b_n \geq \liminf_{n\to\infty} a_n + \liminf_{n\to\infty} b_n$$

such that subtracting b on both sides yields

$$a \geq \liminf_{n\to\infty} a_n + \liminf_{n\to\infty} b_n - b \geq \liminf_{n\to\infty} a_n,$$

which implies $\liminf_{n\to\infty} a_n = a$. In the same way, we conclude $\liminf_{n\to\infty} b_n = b$ such that

$$\begin{aligned}
\limsup_{n\to\infty} a_n &= \limsup_{n\to\infty} a_n + b_n - b_n \leq \limsup_{n\to\infty} a_n + b_n + \limsup_{n\to\infty} -b_n \\
&= \limsup_{n\to\infty} a_n + b_n - \liminf_{n\to\infty} b_n = a + b - b = a.
\end{aligned}$$

and analogously $\limsup_{n\to\infty} b_n = b$. Thus, limit inferior and limit superior coincide, which gives the claim. □

Lemma C.2. *Let* $M > 0$ *be given and* $\Pi_M : \mathcal{Z} \to \mathcal{Z}$ *denotes to orthogonal projection onto the closed ball* $\overline{B_{\mathcal{Z}}(0, M)}$ *with radius* M *centered in zero. Then there exists a mapping* $\tilde{\Pi}_M : \mathcal{Z} \to \mathcal{Z}$ *with*

$$\|\tilde{\Pi}_M(z)\|_{\mathcal{Z}} \leq 2M \ \forall z \in \mathcal{Z}, \qquad \tilde{\Pi}_M(z) = \Pi_M(z) \ \forall z \in \overline{B_{\mathcal{Z}}(0, M)}, \tag{C.1}$$

$$\sup_{z\in\mathcal{Z}} \|(\mathrm{D}_z\tilde{\Pi}_M(z))^*\|_{\mathcal{L}(\mathcal{Z}^*,\mathcal{Z}^*)} \leq C_1 \tag{C.2}$$

$$\sup_{z\in\mathcal{Z}} \|\mathrm{D}_z((\mathrm{D}_z\tilde{\Pi}_M(z))^*)\|_{\mathcal{L}(\mathcal{Z},\mathcal{L}(\mathcal{Z}^*,\mathcal{Z}^*))} \leq C_2. \tag{C.3}$$

Proof. Let $f : \mathbb{R} \to \mathbb{R}$ be a monotonically increasing polynomial function that satisfies the properties

$$f(M) = M, \quad f'(M) = 1, \quad f''(M) = 0,$$
$$f(2M) = \frac{3}{2}M, \quad f'(2M) = 0, \quad f''(2M) = 0.$$

Then we define $\tilde{\Pi}_M : \mathcal{Z} \to \mathcal{Z}$ by

$$\tilde{\Pi}_M(z) = \begin{cases} z, & \text{if } \|z\| \leq M, \\ f(\|z\|_{\mathcal{Z}})\frac{z}{\|z\|_{\mathcal{Z}}}, & \text{if } M < \|z\|_{\mathcal{Z}} < 2M, \\ \frac{3}{2}M\frac{z}{\|z\|_{\mathcal{Z}}}, & \text{else.} \end{cases} \tag{C.4}$$

One directly sees that $\tilde{\Pi}_M$ complies with (C.1). The derivative of $\tilde{\Pi}_M$ is given by

$$\mathrm{D}_z\tilde{\Pi}_M(z)(h) = \begin{cases} h, & \text{if } \|z\| \leq M, \\ f'(\|z\|_{\mathcal{Z}})\langle z, h\rangle_{\mathcal{Z}}\frac{z}{\|z\|^2_{\mathcal{Z}}} & \\ \qquad + \frac{f(\|z\|_{\mathcal{Z}})}{\|z\|^3_{\mathcal{Z}}}(\|z\|^2_{\mathcal{Z}}h - z\langle z, h\rangle_{\mathcal{Z}}), & \text{if } M < \|z\|_{\mathcal{Z}} < 2M, \\ \frac{3}{2}M\frac{1}{\|z\|^3_{\mathcal{Z}}}(\|z\|^2_{\mathcal{Z}}h - z\langle z, h\rangle_{\mathcal{Z}}), & \text{else.} \end{cases}$$

By passing $\|z\|_{\mathcal{Z}} \to M$ and $\|z\|_{\mathcal{Z}} \to 2M$, respectively, we see that $\mathrm{D}_z\tilde{\Pi}_M$ is continuous with respect to z. Therefore, $\tilde{\Pi}_M$ is also differentiable in $z \in \mathcal{Z}$ satisfying $\|z\|_{\mathcal{Z}} = M$ or $\|z\|_{\mathcal{Z}} = 2M$ since $\tilde{\Pi}_M$ is continuous, too. We denote the Riesz isomorphism, which maps $\mathcal{Z}$ into its dual space $\mathcal{Z}^*$, by $\mathcal{J}_{\mathcal{Z}} : \mathcal{Z} \to \mathcal{Z}^*$. Then the adjoint operator $(\mathrm{D}_z\tilde{\Pi}_M(z))^* : \mathcal{Z}^* \to \mathcal{Z}^*$ is given by

$$(\mathrm{D}_z\tilde{\Pi}_M(z))^*(h) = \begin{cases} h, & \text{if } \|z\| \leq M, \\ \left\langle h, \frac{f'(\|z\|_{\mathcal{Z}})}{\|z\|^2_{\mathcal{Z}}}z\right\rangle_{\mathcal{Z}^*,\mathcal{Z}}\mathcal{J}_{\mathcal{Z}}(z) & \\ \qquad + \frac{f(\|z\|_{\mathcal{Z}})}{\|z\|^3_{\mathcal{Z}}}\Big(\|z\|^2_{\mathcal{Z}}h - \langle h, z\rangle_{\mathcal{Z}^*,\mathcal{Z}}\mathcal{J}_{\mathcal{Z}}(z)\Big), & \text{if } M < \|z\|_{\mathcal{Z}} < 2M, \\ \frac{3}{2}M\frac{1}{\|z\|^3_{\mathcal{Z}}}\Big(\|z\|^2_{\mathcal{Z}}h - \langle h, z\rangle_{\mathcal{Z}^*,\mathcal{Z}}\mathcal{J}_{\mathcal{Z}}(z)\Big), & \text{else.} \end{cases}$$

Now we derive again in order to obtain for $h_1 \in \mathcal{Z}^*$, $h_2 \in \mathcal{Z}$

1. $\|z\|_{\mathcal{Z}} \leq M$:

$$\mathrm{D}_z((\mathrm{D}_z\tilde{\Pi}_M(z))^*)[h_1, h_2] = 0$$

2. $M < \|z\|_{\mathcal{Z}} < 2M$:

$$\mathrm{D}_z((\mathrm{D}_z\tilde{\Pi}_M(z))^*)[h_1, h_2] =$$
$$\left\langle h_1, \frac{f''(\|z\|_{\mathcal{Z}})}{\|z\|^3_{\mathcal{Z}}}\langle z, h_2\rangle_{\mathcal{Z}} + f'(\|z\|_{\mathcal{Z}})\left(\frac{h_2}{\|z\|^2_{\mathcal{Z}}} - 2\frac{\langle z, h\rangle_{\mathcal{Z}}z}{\|z\|^4_{\mathcal{Z}}}\right)\right\rangle_{\mathcal{Z}^*,\mathcal{Z}}\mathcal{J}_{\mathcal{Z}}(z)$$

$$+ \left\langle h_1, \frac{f'(\|z\|_{\mathcal{Z}})}{\|z\|_{\mathcal{Z}}^2} z \right\rangle_{\mathcal{Z}^*,\mathcal{Z}} \mathcal{J}_{\mathcal{Z}}(z)$$
$$+ \left(\frac{f'(\|z\|_{\mathcal{Z}})}{\|z\|_{\mathcal{Z}}^4} \langle z, h_2 \rangle_{\mathcal{Z}} - 3 \frac{f(\|z\|_{\mathcal{Z}})}{\|z\|_{\mathcal{Z}}^5} \langle z, h_2 \rangle_{\mathcal{Z}} \right) \Big(\|z\|_{\mathcal{Z}}^2 h_1 - \langle h_1, z \rangle_{\mathcal{Z}^*,\mathcal{Z}} \mathcal{J}_{\mathcal{Z}}(z) \Big)$$
$$+ \frac{f(\|z\|_{\mathcal{Z}})}{\|z\|_{\mathcal{Z}}^3} \Big(2\langle z, h_2 \rangle_{\mathcal{Z}} h_1 - \langle h_1, h_2 \rangle_{\mathcal{Z}^*,\mathcal{Z}} \mathcal{J}_{\mathcal{Z}}(z) - \langle h_1, z \rangle_{\mathcal{Z}^*,\mathcal{Z}} \mathcal{J}_{\mathcal{Z}}(h_2) \Big)$$

3. $2M \leq \|z\|_{\mathcal{Z}}$*:*

$$\begin{aligned} &\mathrm{D}_z((\mathrm{D}_z \tilde{\Pi}_M(z))^*)[h_1, h_2] = \\ &\qquad \frac{3}{2} M \Bigg[- \frac{3}{\|z\|_{\mathcal{Z}}^3} \langle z, h_2 \rangle_{\mathcal{Z}} h_1 + \frac{3}{\|z\|_{\mathcal{Z}}^5} \langle z, h_2 \rangle_{\mathcal{Z}} \langle h_1, z \rangle_{\mathcal{Z}^*,\mathcal{Z}} \mathcal{J}_{\mathcal{Z}}(z) \\ &\qquad + \frac{1}{\|z\|_{\mathcal{Z}}^3} \Big(2\langle z, h_2 \rangle_{\mathcal{Z}} h_1 - \langle h_1, h_2 \rangle_{\mathcal{Z}^*,\mathcal{Z}} \mathcal{J}_{\mathcal{Z}}(z) - \langle h_1, z \rangle_{\mathcal{Z}^*,\mathcal{Z}} \mathcal{J}_{\mathcal{Z}}(h_2) \Big) \Bigg], \end{aligned}$$

With the same arguments as above, we obtain the differentiability of $(\mathrm{D}_z \tilde{\Pi}_M(z))^*$ with $\|z\|_{\mathcal{Z}} \in \{M, 2M\}$. Finally, the boundedness of f and its first and second derivative give (C.2) and (C.3). □

Lemma C.3. *Let* $z \in W^{1,1}(0,T;\mathcal{Z})$ *and* $\ell \in H^1(0,T;\mathcal{V}^*)$ *be given and the energy* $\mathcal{I}$ *is supposed to comply with the assumptions in Section 1.2. Then the mapping* $t \mapsto \mathcal{I}(\ell(t), z(t)) = \frac{1}{2}\langle Az(t), z(t) \rangle_{\mathcal{Z}^*,\mathcal{Z}} + \mathcal{F}(z(t)) - \langle \ell(t), z(t) \rangle_{\mathcal{V}^*,\mathcal{V}}$ *is differentiable almost everywhere with derivative*

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \mathcal{I}(\ell(t), z(t)) = \langle Az(t), z'(t) \rangle_{\mathcal{Z}^*,\mathcal{Z}} + \langle \mathrm{D}_z \mathcal{F}(z(t)), z'(t) \rangle_{\mathcal{Z}^*,\mathcal{Z}} \\ - \langle \ell(t), z'(t) \rangle_{\mathcal{V}^*,\mathcal{V}} - \langle \ell'(t), z(t) \rangle_{\mathcal{V}^*,\mathcal{V}}. \end{aligned} \tag{C.5}$$

In particular, we have for all $0 \leq t_1 < t_2 \leq T$

$$\begin{aligned} &\mathcal{I}(\ell(t_2), z(t_2)) - \mathcal{I}(\ell(t_1), z(t_1)) \\ &\qquad = \int_{t_1}^{t_2} \langle Az(r), z'(r) \rangle_{\mathcal{Z}^*,\mathcal{Z}} + \langle \mathrm{D}_z \mathcal{F}(z(r)), z'(r) \rangle_{\mathcal{Z}^*,\mathcal{Z}} \\ &\qquad\qquad - \langle \ell(r), z'(r) \rangle_{\mathcal{V}^*,\mathcal{V}} - \langle \ell'(r), z(r) \rangle_{\mathcal{V}^*,\mathcal{V}} \,\mathrm{d}r. \end{aligned} \tag{C.6}$$

Proof. The density of $C^\infty([0,T];\mathcal{Z}) \subset W^{1,1}(0,T;\mathcal{Z})$ and $C^\infty([0,T];\mathcal{V}^*) \subset H^1(0,T;\mathcal{V}^*)$ allows us to select a sequences $(z_n)_{n\in\mathbb{N}} \subset C^\infty([0,T];\mathcal{Z})$, $(\ell_n)_{n\in\mathbb{N}} \subset C^\infty([0,T];\mathcal{V}^*)$ such that $z_n \to z$ in $W^{1,1}(0,T;\mathcal{Z})$ and $\ell_n \to \ell$ in $H^1(0,T;\mathcal{V}^*)$ as $n \to \infty$. By means of continuous embedding, we obtain the pointwise convergences $z_n(t) \to z(t)$ in $\mathcal{Z}$ and $\ell_n(t) \to \ell(t)$ in $\mathcal{V}^*$ for all $t \in [0,T]$ such that

$$Az_n(t) + \mathrm{D}_z \mathcal{F}(z_n(t)) \to Az(t) + \mathrm{D}_z \mathcal{F}(z(t)) \text{ in } \mathcal{Z}^*$$

due to the continuity of A and $\mathrm{D}_z \mathcal{F}$. Furthermore, the term is uniformly bounded in $\mathcal{Z}^*$, i.e., $\|Az_n(t) + \mathrm{D}_z \mathcal{F}(z_n(t))\|_{\mathcal{Z}^*} \leq C$ for all $t \in [0,T]$ and all $n \in \mathbb{N}$, which can be seen as follows. Firstly, z_n is uniformly bounded due to the assumed convergence $z_n \to z$ in $W^{1,1}(0,T;\mathcal{Z})$ and the continuity of the embedding $W^{1,1}(0,T;\mathcal{Z}) \hookrightarrow C([0,T];\mathcal{Z})$. The mean value theorem in combination with (1.2.5) yields for all $n \in \mathbb{N}$ and $t \in [0,T]$

$$\begin{aligned}\|\mathrm{D}_z\mathcal{F}(z_n(t))\|_{\mathcal{V}^*} &\le \|\mathrm{D}_z\mathcal{F}(0)\|_{\mathcal{V}^*} + \int_0^1 \|\mathrm{D}_z^2\mathcal{F}(\tau z_n(t))z_n(t)\|_{\mathcal{V}^*}\,\mathrm{d}\tau \\ &\le \|\mathrm{D}_z\mathcal{F}(0)\|_{\mathcal{V}^*} + \int_0^1 C\big(1+\tau^q\|z_n(t)\|_{\mathcal{Z}}^q\big)\|z_n(t)\|_{\mathcal{Z}}\,\mathrm{d}\tau \le \tilde{C},\end{aligned}$$

where we exploited the uniform boundedness of z_n in the last estimate. Hence, by $\mathcal{V}^* \hookrightarrow \mathcal{Z}^*$ continuously and $\|Az_n(t)\|_{\mathcal{Z}^*} \le \|A\|_{\mathcal{L}(\mathcal{Z},\mathcal{Z}^*)}\|z_n(t)\|_{\mathcal{Z}} \le C$, we have that $\|Az_n(t)+\mathrm{D}_z\mathcal{F}(z_n(t))\|_{\mathcal{Z}^*} \le C$. Thus, in combination with the pointwise convergence Lebesgue's dominated convergence theorem yields

$$Az_n + \mathrm{D}_z\mathcal{F}(z_n) \to Az + \mathrm{D}_z\mathcal{F}(z) \text{ in } L^\infty(0,T;\mathcal{Z}^*).$$

Eventually, by exploiting the assumed convergences $z_n' \to z'$ in $L^1(0,T;\mathcal{Z})$ and $\ell_n \to \ell$ in $H^1(0,T;\mathcal{V}^*)$ we obtain for all $0 \le t_1 < t_2 \le T$

$$\begin{aligned}\mathcal{I}(\ell_n(t_2),z_n(t_2)) - \mathcal{I}(\ell_n(t_1),z_n(t_1)) &= \int_{t_1}^{t_2} \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{I}(\ell_n(r),z_n(r))\,\mathrm{d}r \\ = \int_{t_1}^{t_2} \langle Az_n(r),z_n'(r)\rangle_{\mathcal{Z}^*,\mathcal{Z}} &+ \langle \mathrm{D}_z\mathcal{F}(z_n(r)),z_n'(r)\rangle_{\mathcal{Z}^*,\mathcal{Z}} \\ &- \langle \ell_n(r),z_n'(r)\rangle_{\mathcal{V}^*,\mathcal{V}} - \langle \ell_n'(r),z_n(r)\rangle_{\mathcal{V}^*,\mathcal{V}}\,\mathrm{d}r \\ \to \int_{t_1}^{t_2} \langle Az(r),z'(r)\rangle_{\mathcal{Z}^*,\mathcal{Z}} &+ \langle \mathrm{D}_z\mathcal{F}(z(r)),z'(r)\rangle_{\mathcal{Z}^*,\mathcal{Z}} \\ &- \langle \ell(r),z'(r)\rangle_{\mathcal{V}^*,\mathcal{V}} - \langle \ell'(r),z(r)\rangle_{\mathcal{V}^*,\mathcal{V}}\,\mathrm{d}r,\end{aligned}$$

where we used the chain rule for classically differentiable functions. Finally, the pointwise convergences of z_n and ℓ_n along with the continuity of A and $\mathcal{F}$ yields

$$\lim_{n\to\infty} \mathcal{I}(\ell_n(t_2),z_n(t_2)) - \mathcal{I}(\ell_n(t_1),z_n(t_1)) = \mathcal{I}(\ell(t_2),z(t_2)) - \mathcal{I}(\ell(t_1),z(t_1)),$$

which gives the claim. □

Lemma C.4 ([KT23, Lem. A.2]). *Let $v \in H^1(0,S;\mathcal{V}^*)$ and $h > 0$ be given. We define the by zero extended derivative of v (denoted with the same symbol) by*

$$v'(s) := \begin{cases} v'(s), & \text{if } s \in (0,S), \\ 0, & \text{if } s \in (S,S+h). \end{cases}$$

Then the mapping $[0,S+h] \ni s \mapsto v(s) := v(0) + \int_0^s v'(r)\,\mathrm{d}r$ belongs to $H^1(0,S+h;\mathcal{V}^*)$ and coincide with the given function $v \in H^1(0,S;\mathcal{V}^*)$ on $[0,S]$. Thus, we *are able to define* $L_h : H^1(0,S;\mathcal{V}^*) \to L^2(0,S;\mathcal{V}^*)$, $s \mapsto L_h v(s) := \frac{1}{h}(v(s+h) - v(s))$, *where* v *is interpreted as the extended function if necessary. Then* $L_h : H^1(0,S;\mathcal{V}^*) \to L^2(0,S;\mathcal{V}^*)$ *is a linear and continuous operator and satisfies*

$$\|L_h v\|_{L^2(0,S;\mathcal{V}^*)} \leq \|v'\|_{L^2(0,S;\mathcal{V}^*)}. \tag{C.7}$$

Furthermore, it holds $L_h v \to v'$ *strongly in* $L^2(0,S;\mathcal{V}^*)$ *as* $h \searrow 0$.

Proof. The linearity of L_h follows immediately due to its construction and (C.7) yields the continuity. For given $v \in H^1(0,S;\mathcal{V}^*)$ the fundamental theorem of calculus, the Cauchy-Schwarz inequality and Fubini's theorem lead to

$$\begin{aligned}
\|L_h v\|^2_{L^2(0,S;\mathcal{V}^*)} &= \int_0^S \|L_h v(s)\|^2_{\mathcal{V}^*} \,\mathrm{d}s = \int_0^S \left\| \int_0^1 v'(s+rh)\mathrm{d}r \right\|^2_{\mathcal{V}^*} \mathrm{d}s \\
&\leq \int_0^S \int_0^1 \|v'(s+rh)\|^2_{\mathcal{V}^*} \,\mathrm{d}r\,\mathrm{d}s = \int_0^1 \int_0^S \|v'(s+rh)\|^2_{\mathcal{V}^*} \,\mathrm{d}s\,\mathrm{d}r \\
&\leq \int_0^1 \|v'\|^2_{L^2(0,S;\mathcal{V}^*)} \,\mathrm{d}r = \|v'\|^2_{L^2(0,S;\mathcal{V}^*)},
\end{aligned}$$

which implies (C.7). In order to prove the strong convergence in $L^2(0,S;\mathcal{V}^*)$, assume first that $v \in C^\infty([0,S];\mathcal{V}^*)$. Then we have

$$\lim_{h\searrow 0} \|L_h v - v'\|^2_{L^2(0,S;\mathcal{V}^*)} = \lim_{h\searrow 0} \int_0^S \left\| \frac{1}{h}(v(s+h) - v(s)) - v'(s) \right\|^2_{\mathcal{V}^*} \mathrm{d}s = 0$$

by means of Lebesgue's dominated convergence theorem because the integrand converges pointwise to zero and is uniformly bounded by $2\|v'\|^2_{L^\infty(0,S;\mathcal{V}^*)}$. Eventually, for given $\epsilon > 0$ and $v \in H^1(0,S;\mathcal{V}^*)$ choose by means of density a function $\tilde{v} \in C^\infty([0,S];\mathcal{V}^*)$ with $\|v - \tilde{v}\|_{H^1(0,S;\mathcal{V}^*)} < \frac{\epsilon}{3}$. Further, let $h_0 > 0$ such that $\|L_h\tilde{v} - \tilde{v}'\|^2_{L^2(0,S;\mathcal{V}^*)} < \frac{\epsilon}{3}$ for all $0 < h < h_0$. Then we infer from the linearity of L_h and (C.7)

$$\begin{aligned}
\|L_h v - v'\|_{L^2(0,S;\mathcal{V}^*)} &\leq \|L_h v - L_h\tilde{v}\|_{L^2(0,S;\mathcal{V}^*)} + \|L_h\tilde{v} - \tilde{v}'\|_{L^2(0,S;\mathcal{V}^*)} + \|\tilde{v}' - v'\|_{L^2(0,S;\mathcal{V}^*)} \\
&\leq 2\|\tilde{v}' - v'\|_{L^2(0,S;\mathcal{V}^*)} + \|L_h\tilde{v} - \tilde{v}'\|_{L^2(0,S;\mathcal{V}^*)} < \epsilon
\end{aligned}$$

for all $0 < h < h_0$, which yields the assertion. □

The following Lemma is a slightly modified version of [MRS09, Lem. 3.1], but can also be found in [KZ21, Lem. C.1].

Lemma C.5. *Assume that* $v_n, v \in L^\infty(0,S;\mathcal{V})$ *with* $v_n \rightharpoonup^* v$ *in* $L^\infty(0,S;\mathcal{V})$ *and* $\delta_n, \delta \in L^1(0,S;[0,\infty))$ *with* $\delta(s) \leq \liminf_{n\to\infty} \delta_n(s)$ *for almost all* $s \in (0,S)$, *then we have*

$$\int_0^S \|v(r)\|_{\mathcal{V}}\, \delta(r)\,\mathrm{d}r \leq \liminf_{n\to\infty} \int_0^S \|v_n(r)\|_{\mathcal{V}}\, \delta_n(r)\,\mathrm{d}r.$$

Proof. We define $\tilde{\delta}_n(r) := \min\{\delta_n(r), \delta(r)\}$. Then for almost all $r \in (0,S)$ it holds

$$\delta(r) \le \liminf_{n\to\infty} \tilde{\delta}_n(r) \le \limsup_{n\to\infty} \tilde{\delta}_n(r) \le \limsup_{n\to\infty} \delta(r) = \delta(r)$$

such that $\tilde{\delta}_n(r) \to \delta(r)$ almost everywhere in $(0,S)$. By exploiting the pointwise almost everywhere boundedness $0 \le \tilde{\delta}_n \le \delta \in L^1(0,S)$ by construction of $\tilde{\delta}_n$, Lebesgue's dominated convergence theorem yields $\tilde{\delta}_n \to \delta$ in $L^1(0,S)$. Besides, we have

$$\begin{aligned} \int_0^S \|v_n(r)\|_{\mathcal{V}}\, \delta_n(r)\, \mathrm{d}r &\ge \int_0^S \|v_n(r)\|_{\mathcal{V}}\, \tilde{\delta}_n(r)\, \mathrm{d}r \\ &\ge \int_0^S \|v_n(r)\|_{\mathcal{V}}\, \delta(r)\, \mathrm{d}r - \int_0^S \|v_n(r)\|_{\mathcal{V}} |\tilde{\delta}_n(r) - \delta(r)|\, \mathrm{d}r. \end{aligned} \tag{C.8}$$

With regard to the second integral on the right hand side, applying Hölder's inequality and exploiting the boundedness of $\|v_n\|_{L^\infty(0,S;\mathcal{V})}$ by the assumed weak* convergence results in

$$\int_0^S \|v_n(r)\|_{\mathcal{V}} |\tilde{\delta}_n(r) - \delta(r)|\, \mathrm{d}r \le \|v_n\|_{L^\infty(0,S;\mathcal{V})} \|\tilde{\delta}_n - \delta\|_{L^1(0,S)} \to 0, \quad n \to \infty.$$

In order to determine the limit behavior of the other integral, let $\xi \in L^\infty(0,S;\mathcal{V}^*)$ defined via $\|\xi(r)\|_{\mathcal{V}^*} = 1$ and $\langle \xi(r), v(r)\rangle_{\mathcal{V}^*,\mathcal{V}} = \|v(r)\|_{\mathcal{V}}$ for almost all $r \in (0,S)$, cf. [Kab18, Sec. 9.2, Eqn. (9.7)]. It is easy to see that $\xi(r)$ is given by the into $\mathcal{V}^*$ embedded version of $\frac{v(r)}{\|v(r)\|_{\mathcal{V}}}$. Thus, ξ inherits its measurability from v and is indeed an element of $L^\infty(0,S;\mathcal{V}^*)$ Then another use of [Kab18, Sec. 9.2, Eqn. (9.7)] and exploiting the weak* convergence $v_n \rightharpoonup^* v$ in $L^\infty(0,S;\mathcal{V})$ in combination with the fact that $\delta\xi \in L^1(0,S;\mathcal{V}^*)$ gives

$$\begin{aligned} \int_0^S \|v_n(r)\|_{\mathcal{V}}\, \delta(r)\, \mathrm{d}r &= \int_0^S \sup_{\xi\in\mathcal{V}^*, \|\xi\|_{\mathcal{V}^*}=1} \langle \xi, v_n(r)\rangle_{\mathcal{V}^*,\mathcal{V}}\, \delta(r)\, \mathrm{d}r \\ &\ge \int_0^S \langle \xi(r), v_n(r)\rangle_{\mathcal{V}^*,\mathcal{V}}\, \delta(r)\, \mathrm{d}r \\ &\to \int_0^S \langle \xi(r), v(r)\rangle_{\mathcal{V}^*,\mathcal{V}}\, \delta(r)\, \mathrm{d}r = \int_0^S \|v(r)\|_{\mathcal{V}}\, \delta(r)\, \mathrm{d}r. \end{aligned}$$

Finally, considering the limit inferior in (C.8) and exploiting the two convergence results from above leads to

$$\begin{aligned} &\liminf_{n\to\infty} \int_0^S \|v_n(r)\|_{\mathcal{V}}\, \delta_n(r)\, \mathrm{d}r \\ &\ge \liminf_{n\to\infty} \int_0^S \|v_n(r)\|_{\mathcal{V}}\, \delta(r)\, \mathrm{d}r - \lim_{n\to\infty} \int_0^S \|v_n(r)\|_{\mathcal{V}} |\tilde{\delta}_n(r) - \delta(r)|\, \mathrm{d}r \\ &= \int_0^S \|v(r)\|_{\mathcal{V}}\, \delta(r)\, \mathrm{d}r, \end{aligned}$$

which yields the assertion. $\square$

Lemma C.6. *For every* $(S, \hat{t}, \hat{z}, \hat{\ell}) \in (0,\infty)\times W^{1,\infty}(0,S)\times W^{1,\infty}(0,S;\mathbb{R}^d)\times BV([0,S];\mathbb{R}^d)$ *and all* $0 \le s_1 \le s_2 \le S$ *there holds*

$$\begin{aligned}\mathcal{E}(\hat{z}(s_2)) + \int_{s_1}^{s_2} \mathcal{R}(\hat{z}'(r)) + \|\hat{z}'(r)\| \operatorname{dist}(-\mathrm{D}_z\mathcal{I}(\hat{\ell}(r), \hat{z}(r)), \partial\mathcal{R}(0)) \,\mathrm{d}r \\ \ge \mathcal{E}(\hat{z}(s_1)) + \int_{s_1}^{s_2} \langle \hat{\ell}(r), \hat{z}'(r)\rangle \,\mathrm{d}r.\end{aligned} \tag{C.9}$$

Proof. Lemma A.4 implies that for all $r \in (s_1, s_2)$ there exists $w_r \in \partial\mathcal{R}(0)$ such that $\operatorname{dist}(-\mathrm{D}_z\mathcal{I}(\hat{\ell}(r), \hat{z}(r)), \partial\mathcal{R}(0)) = \| - \mathrm{D}_z\mathcal{I}(\hat{\ell}(r), \hat{z}(r)) - w_r\|$. From this we conclude

$$\begin{aligned}\langle -\mathrm{D}_z\mathcal{I}(\hat{\ell}(r), \hat{z}(r)), \hat{z}'(r)\rangle &= \langle -\mathrm{D}_z\mathcal{I}(\hat{\ell}(r), \hat{z}(r)) - w_r, \hat{z}'(r)\rangle + \langle w_r, \hat{z}'(r)\rangle \\ &\le \| - \mathrm{D}_z\mathcal{I}(\hat{\ell}(r), \hat{z}(r)) - w_r\| \, \|\hat{z}'(r)\| + \langle w_r, \hat{z}'(r)\rangle \\ &\le \|\hat{z}'(r)\| \operatorname{dist}(-\mathrm{D}_z\mathcal{I}(\hat{\ell}(r), \hat{z}(r)), \partial\mathcal{R}(0)) + \mathcal{R}(\hat{z}'(r)),\end{aligned}$$

where we used $w_r \in \partial\mathcal{R}(0)$ for the last inequality. Now integrating over (s_1, s_2) yields

$$\begin{aligned}&\int_{s_1}^{s_2} \|\hat{z}'(r)\| \operatorname{dist}(-\mathrm{D}_z\mathcal{I}(\hat{\ell}(r), \hat{z}(r)), \partial\mathcal{R}(0)) + \mathcal{R}(\hat{z}'(r)) \,\mathrm{d}r \\ &\quad \ge \int_{s_1}^{s_2} \langle -\mathrm{D}_z\mathcal{I}(\hat{\ell}(r), \hat{z}(r)), \hat{z}'(r)\rangle \,\mathrm{d}r \\ &\quad = \int_{s_1}^{s_2} \langle -\mathrm{D}_z\mathcal{E}(\hat{z}(r)), \hat{z}'(r)\rangle + \langle \hat{\ell}(r), \hat{z}'(r)\rangle \,\mathrm{d}r \\ &\quad = \mathcal{E}(\hat{z}(s_1)) - \mathcal{E}(\hat{z}(s_2)) + \int_0^s \langle \hat{\ell}(r), \hat{z}'(r)\rangle \,\mathrm{d}r,\end{aligned}$$

which gives the claim. $\square$

Index

R

S

T

U

V

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