Gold Price Dynamics: Geometric Random Walk and ARIMAX Models[1]

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Abstract
This paper presents a unified analysis of gold price dynamics by integrating two complementary studies: one applying the Geometric Random Walk (GRW) model with ARIMA techniques, and the other investigating the relationship between gold prices and global GDP using ARIMAX models. Together, these approaches offer a comprehensive understanding of gold price behavior, uncertainty, and predictive modeling from 1970 to 2024, with forecasts to 2030.

The first part explores gold price movements through the lens of a GRW model, focusing on five distinct historical phases since the collapse of the Bretton Woods system in 1971. Gold returns and growth rates reveal varying structural behaviors across these phases, including periods of rapid increase and prolonged stability. ARIMA models are applied to each phase, with parameters and diagnostics presented in tabular form. Forecasts based on a Geometric Linear Regression (GLR) model project annual gold price growth between 7.9% and 10.6% through 2030. The GRW model also serves as a benchmark for evaluating expert forecasts, offering a statistical foundation to assess their reliability and degree of overconfidence.

The second part uses ARIMAX (AutoRegressive Integrated Moving Average with eXogenous variables models) to examine the link between gold prices and global GDP. Two specifications are compared: a log-level ARMAX(1,0,1) model and a first-difference logarithmic model. Both are estimated over the 1970–2023 period. The log-level model demonstrates a strong, statistically significant relationship, suggesting that a 1% increase in global GDP corresponds to approximately a 1% rise in gold prices. Although it involves non-stationary time series, this model provides more stable and interpretable forecasts. The differenced model addresses stationarity concerns but results in greater uncertainty, as reflected in wider prediction intervals and reduced forecast clarity.

By combining these two modeling approaches, the poster highlights how gold prices are shaped by both internal market dynamics and external macroeconomic drivers. The results underscore the role of time series models in managing forecast uncertainty and evaluating market expectations. This work contributes to the understanding of gold as a financial asset and a global economic indicator.

Keywords: Gold prices, ARIMA, ARIMAX, global GDP, Gold Price Forecasting, Gold Price Returns, Geometric Random Walk with Drift, Expert Predictions, Statistical Modeling, Uncertainty.

[1] This paper is the written version of the poster I presented at the JSM 2025 Contributed Poster Presentations, Business and Economic Statistics Section, held in Nashville, Tennessee, on August 4. It is currently in draft form, with further refinements and developments planned. ChatGPT aided me in refining the text and generating R code snippets.

Part 1

Assessing Gold Price Forecasts: A Random Walk Model Approach to Evaluating Expert Predictions and Measuring Uncertainty

1. Introduction

Gold price forecasting is a well-studied area, with numerous papers employing ARIMA models to predict future prices[2]. However, forecasts that do not use ARIMA models often result in point forecasts, which provide limited insight into the associated uncertainties. This paper shifts the focus to using a simple geometric random walk model with drift to evaluate expert forecasts. By presenting prediction intervals, we aim to illustrate the potential variability in future prices and help investors better assess the risks and rewards of their investments. Prediction intervals are crucial for understanding the range of possible gold price outcomes and the risks associated with any forecast. Investors should be cautious of point forecasts that do not provide this information.

The gold price has undergone dramatic shifts since the breakdown of the gold standard, reflecting changes in global economic and geopolitical conditions. Under the Bretton Woods system, the U.S. dollar was fixed at $35 per ounce of gold, ensuring relative price stability. For example, in January 1971, the official gold price was $37.44 per ounce. The difference between $35 and $37.44 per ounce reflects adjustments made to the official gold price under the Bretton Woods system to account for changes in market conditions and inflation before the U.S. suspended the dollar's convertibility into gold. However, this stability ended in August 1971 when U.S. President Richard Nixon announced the suspension of the U.S. dollar's convertibility into gold, effectively dismantling the Bretton Woods system. This marked a turning point, transitioning gold from a fixed price to a freely traded commodity, leading to significant volatility and long-term growth in prices.

Figure 1 illustrates the daily gold prices from 1971 to 2024, highlighting notable peaks. The first major surge occurred on February 24, 1975, with gold reaching $185.25 per ounce, followed by a sharp increase to $850 on January 21, 1980. Subsequent peaks include $1,895 on September 5, 2011; $2,067.20 on August 6, 2020; $2,078.40 on December 28, 2023; and a record high of $2,777.80 on October 30, 2024. These milestones reflect the dynamic interplay between economic events, financial crises, and gold's enduring role as a safe-haven asset.

Gold price trends can be divided into six distinct phases:

1. **Initial Stability and Gradual Increase (1968–1971):**
   During the late 1960s and early 1970s, gold prices were stable, fluctuating within a

[2] See, e.g., Jöckel and Pflaumer (1981b) proposing ARIMA(0,1,1) with drift and ARIMA(0,2,2) based on monthly prices between 1968 and 1979, or recent literature such as Bunnag (2024) with an extensive literature review, and Madhika et al. (2023) focusing also on ARIMA(0,1,1).

narrow range. This phase was marked by the influence of the Bretton Woods system, which tied gold prices to fixed exchange rates.

2. **Sharp Increase (1971–1980):**
   The collapse of the Bretton Woods system in 1971 triggered a steep upward trend in gold prices. High inflation, the oil crisis, and geopolitical tensions fueled demand for gold, pushing prices to unprecedented levels, culminating in the 1980 peak of $850.
3. **Volatility and Decline (1981–2000):**
   Following the 1980 peak, gold prices entered a period of volatility and gradual decline. Stabilized global markets and reduced inflationary pressures contributed to a downward trend, with prices reaching their lowest levels in decades by the late 1990s.
4. **Steady Rise and New Peaks (2001–2012):**
   The early 2000s marked a period of sustained growth in gold prices, driven by global economic uncertainty and the 2008 financial crisis. By 2011, gold reached a new high of $1,895 per ounce, reflecting heightened demand as a hedge against risk.
5. **Moderation and Stability (2013–2018):**
   Gold prices stabilized between 2013 and 2018, fluctuating within a relatively narrow range. This phase represented a period of calm in global financial markets following the sharp increases of the prior decade.
6. **Renewed Increase and Record Highs (2019–2024):**
   Since 2019, gold prices have resumed their upward trajectory, driven by economic uncertainties, the COVID-19 pandemic, and geopolitical tensions. In October 2024, gold reached an all-time high of $2,777.80 per ounce.

Figure 2 provides an overview of the annual minimum, maximum, and difference in gold prices from 1971 to 2024. The data reveal several notable trends:

1. **Volatility Growth Over Time:**
   While gold price differences (maximum minus minimum) were relatively small during the early years of the series (e.g., $6.59 in 1971), they began to increase significantly after the collapse of the Bretton Woods system, peaking at $792.7 in 2024.
2. **Periods of Stability and Spikes:**
   The 1980s and 1990s exhibit lower differences, with annual ranges often under $100, reflecting a period of relative price stability. In contrast, the 2000s and beyond show larger differences, particularly during times of economic or geopolitical uncertainty.
3. **Major Peaks in Differences:**
   Large differences occurred in years of global economic stress or financial crises, such as 1980 ($368.5), 2011 ($576), and 2020 ($592.9). The highest difference, $792.7, was recorded in 2024, indicating unprecedented volatility.
4. **Cyclical Patterns:**
   The differences occasionally show a cyclical pattern, with periods of rapid growth followed by stabilization, suggesting alternating phases of heightened and reduced investor uncertainty.

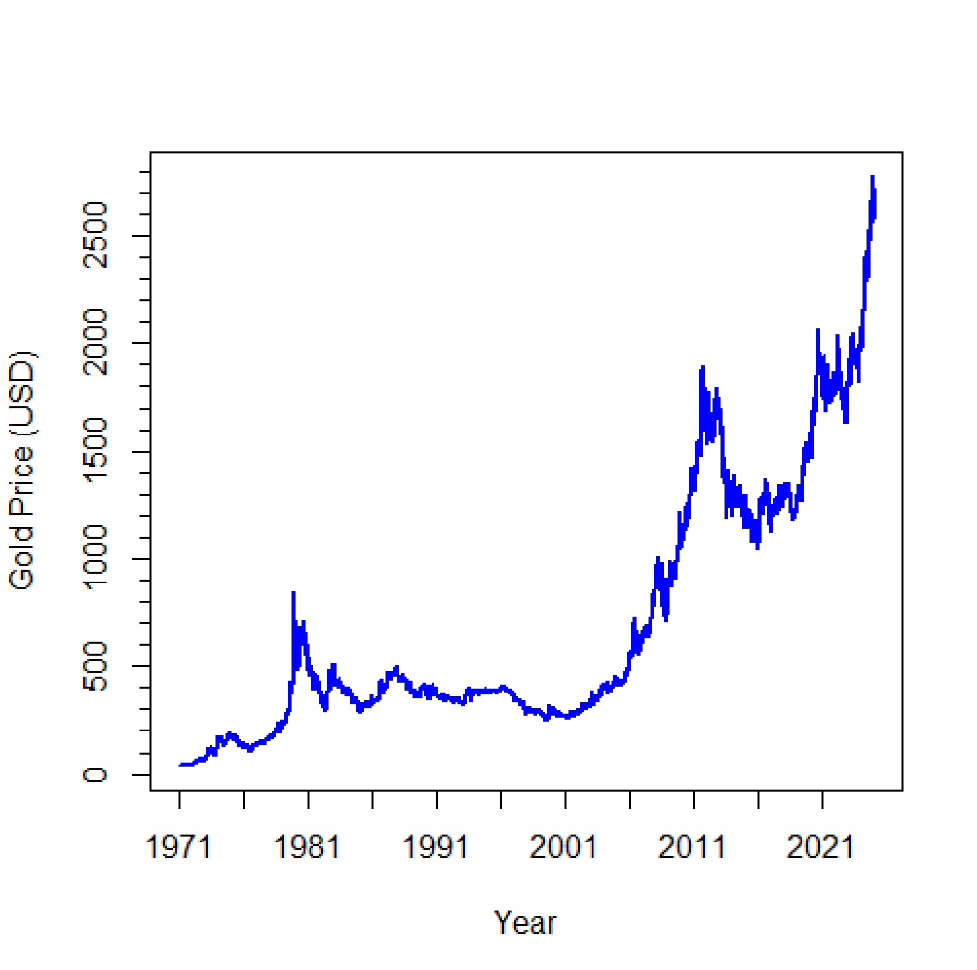

Figure 1: Daily Gold Price from January 4, 1971 to December 30, 2024 (14023 obs.)
Peaks: 1975-02-24: 185.25; 1980-01-21: 850; 2011-09-05 1895; 2020-08-06: 2067.20; 2023-12-28 2078.40; 2024-10-30: 2777.80

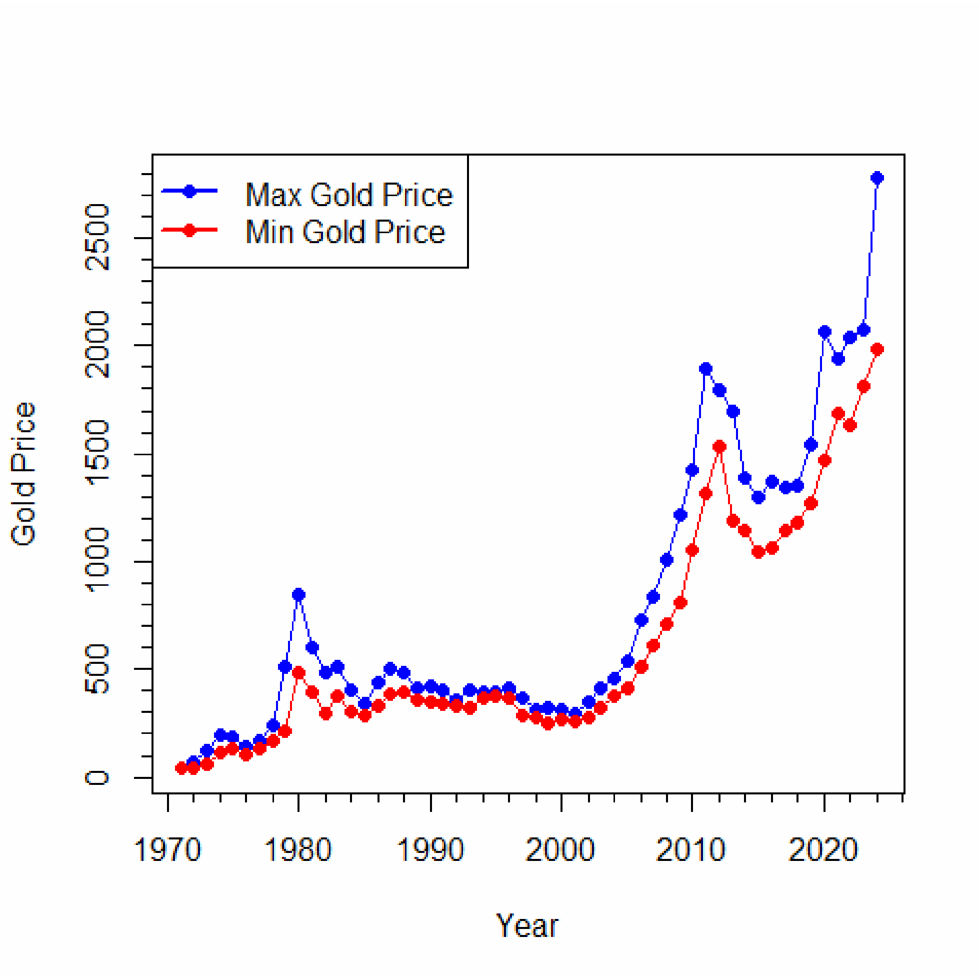

Figure 2: Annual Maximum and Minimum Gold Prices 1971 to 2024 (see also the table in the Appendix)

2. Data and Methods

2.1 Data and Software

Data Set: Daily gold prices from January 1, 1971, to December 31, 2024, based on average closing prices in USD, collected from various online sources.

Number of Daily Trading Days: Mostly between 260 and 261 per year

Analysis performed using R.

2.2 Geometric Random Walk: Model Representation and Evolution

Initially, the process is represented as a Geometric Random Walk (GRW), where the first differences in logarithmic scale follow an ARIMA(0,1,0) process

$$\ln S_t = \ln S_{t-1} + \mu + u_t .$$

Here:
• S_t: Original series (e.g., stock price, gold price, oil price)
• ln S_t: Natural logarithm of S_t,
• μ: Drift term (long-term average growth rate),
• u_t: White noise error term ($u_t \sim N(0, \sigma^2)$).

The introduction of a moving average component modifies the process into an ARIMA(0,1,1) model:

$$\ln S_t = \ln S_{t-1} + \mu + u_t + \beta u_{t-1} .$$

This adjustment accounts for short-term autocorrelation in the changes of the logarithmic series, with β representing the dependence on the previous error term.

Taking the first difference ($\Delta \ln S_t = \ln S_t - \ln S_{t-1}$):

$$\Delta \ln S_t = \mu + u_t \qquad \text{(ARIMA(0,1,0)).}$$

After the inclusion of the moving average term:

$$\Delta \ln S_t = \mu + u_t + \beta u_{t-1} \qquad \text{(ARIMA(0,1,1)).}$$

Exponentiating ln S_t expresses the series in its original scale:

$$S_t = S_{t-1} \cdot \exp\left(\mu + u_t\right) \qquad \text{(ARIMA(0,1,0)),}$$

and, after including the moving average component:

$$S_t = S_{t-1} \cdot \exp\left(\mu + u_t + \beta u_{t-1}\right) \text{ (ARIMA(0,1,1)).}$$

Key Properties

Mean

For both models, the expected value of the log-differences is:

E[Δ ln St] = μ.

This indicates a constant average growth rate in logarithmic terms.

Variance

For ARIMA(0,1,0): $Var\left(\Delta \ln S_t\right) = \sigma_u^2$.

For ARIMA(0,1,1): $Var\left(\Delta \ln S_t\right) = \sigma_u^2 \left(1 + \beta^2\right)$.

Autocorrelation

For ARIMA(0,1,0), the log-differences are uncorrelated: $Corr\left(\Delta \ln S_t, \Delta \ln S_{t-k}\right) = 0 \quad k \neq 0$.

For ARIMA(0,1,1), the first-order autocorrelation is: $Corr\left(\Delta \ln S_t, \Delta \ln S_{t-1}\right) = \dfrac{-\beta}{1+\beta^2}$.

Applications

These models are widely applied in:
• Finance: Modeling stock or gold prices or returns,
• Economics: Analyzing growth rates with short-term dependencies,
• Energy Forecasting: Predicting proportional changes in energy metrics.

2.3 Logarithmic Growth Rates and Returns

The first difference of the natural logarithm, $\Delta \ln S_t$, represents the logarithmic growth rate of the series S_t:

$$\Delta \ln S_t = \ln S_t - \ln S_{t-1}$$

This measure approximates the proportional change in S_t and is commonly referred to as the logarithmic return or growth rate.

Relationship Between Logarithmic and Simple Returns

The relationship between the simple or discrete return R_t and the logarithmic return $r_t = \Delta \ln S_t$ is given by:

$$R_t = \frac{S_t}{S_{t-1}} - 1 = \exp\left(r_t\right) - 1 .$$

For small values of r_t, this relationship simplifies due to the approximation $\exp\left(r_t\right) \approx 1 + r_t$, leading to: $R_t \approx r_t$.

This approximation is often used in practice for series with small proportional changes.

3. Growth Rates and Returns of Gold Price

3.1 Growth Rates

From the daily gold prices, monthly averages between 1971 and 2024 have been calculated, and the log-transformed gold prices are plotted in Figure 3.

A log-scale presentation of gold prices reveals distinct differences in growth rates across historical phases. For example, the sharp increase phase (1972–1980) is characterized by a steeper slope in the log plot compared to the steady rise phase (2001–2012), indicating faster exponential growth during the earlier period.

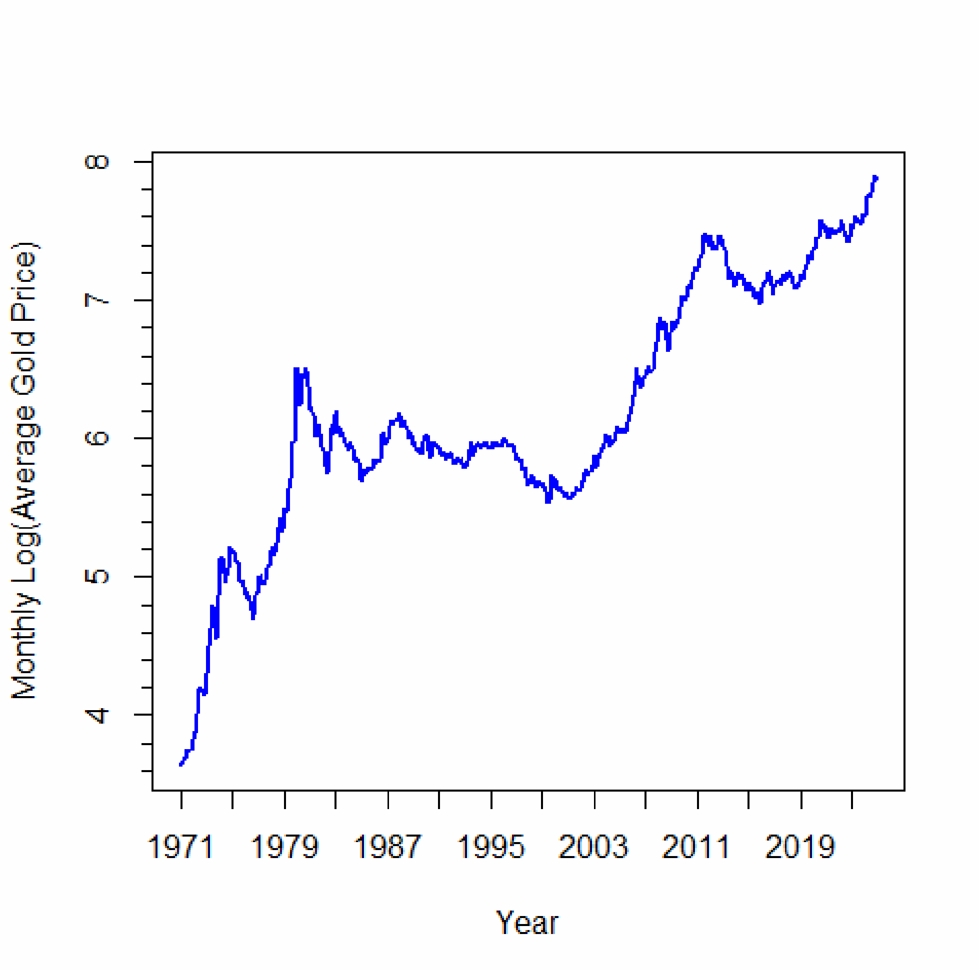

Figure 3: Monthly Logarithmic Gold Price 1971 to 2024

In Figure 4, the log differences, which represent the log growth rates or log returns, are shown for the entire period. The log differences reveal the characteristic fluctuations and variability of the dataset.

Figure 5 highlights the log differences across different time phases, providing a closer examination of how these fluctuations vary over specific intervals. This phased analysis helps identify potential structural changes or shifts in the underlying dynamics of the dataset.

Given that log differences have been calculated, the natural models to explore are Generalized Random Walks (GRWs), specifically ARIMA(0,1,0) or ARIMA(0,1,1). These models provide a robust framework to capture the stochastic behavior observed in the data, while accounting for potential dependencies and random shocks. The upcoming chapter explores the theoretical and practical aspects of GRWs, offering insights into their application in modeling such dynamics.

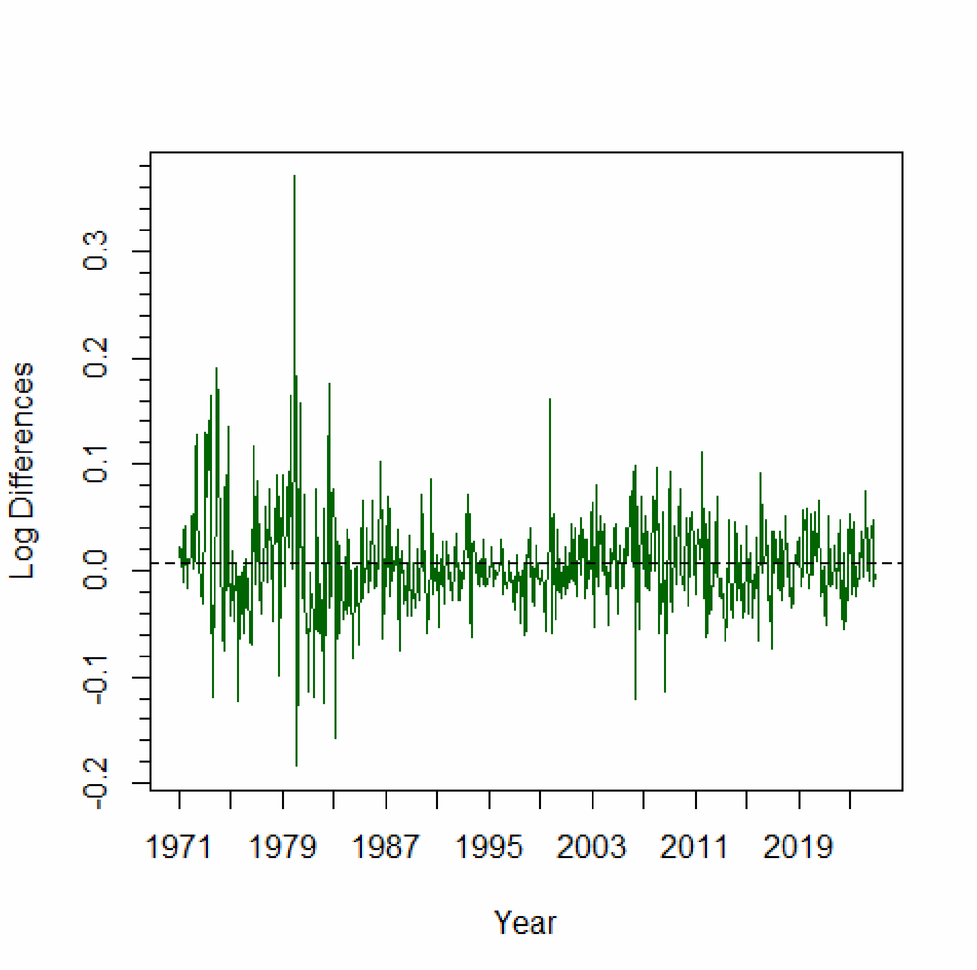

Figure 4: Log Differences of Monthly Average Gold Prices 1971 to 2024

Based on the development of gold prices described in the introduction, we divide the time series from 1971 into five distinct phases:

Phase	**From**	**To**
Phase 1	1971-01	1980-09
Phase 2	1980-10	2001-09
Phase 3	2001-10	2012-10
Phase 4	2012-11	2018-04
Phase 5	2018-05	2024-12

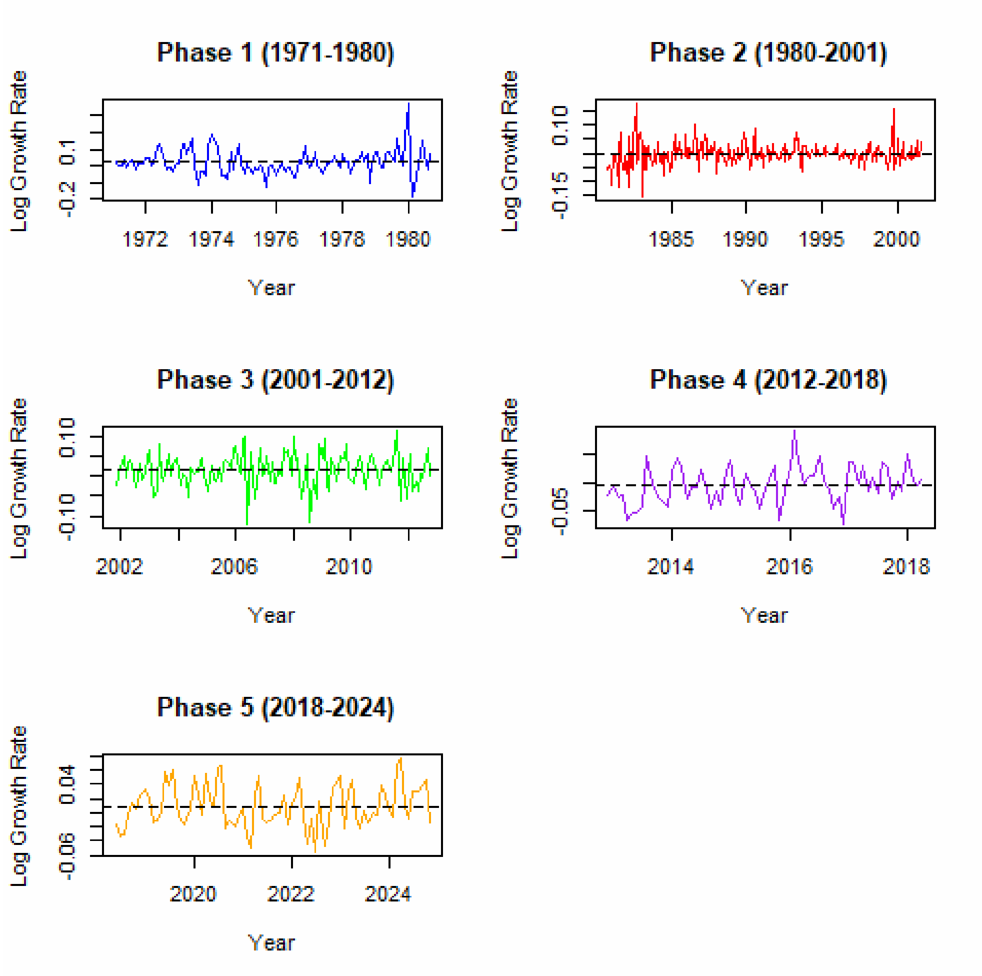

Figure 5: Log Differences of Monthly Average Gold Prices in Different Time Phases
(see also Table 1 for mean and standard deviation)

For each phase, we calculate the mean and standard deviation (volatility) of the log growth rates. The monthly metrics are then transformed into yearly metrics, and the log returns are converted to discrete returns. The results are presented in Table 1 and Table 2:

Table 1: Means and Standard Deviations (Volatilities) for Log Growth Rates

			Monthly		Yearly	
	From	to	Mean	Sdv	Mean	Sdv
Phase 1	1971-01	1980-09	0.0248	0.0724	0.2978	0.2509
Phase 2	1980-10	2001-09	-0.0034	0.0385	-0.0405	0.1335
Phase 3	2001-10	2012-10	0.0138	0.0401	0.1654	0.1388
Phase 4	2012-11	2018-04	-0.0039	0.033	-0.047	0.1142
Phase 5	2018-05	2024-12	0.0091	0.0303	0.1092	0.1048
All Phases	1971-01	2024-12	0.0066	0.0466	0.0787	0.1615

Table 2: Means and Standard Deviations (Volatilities) for Discrete Growth Rates

			Monthly		Yearly	
	From	to	Mean	Sdv	Mean	Sdv
Phase 1	1971-01	1980-09	0.0251	0.0751	0.3469	0.2852
Phase 2	1980-10	2001-09	-0.0034	0.0393	-0.0397	0.1428
Phase 3	2001-10	2012-10	0.0139	0.0409	0.1799	0.1489
Phase 4	2012-11	2018-04	-0.0039	0.0335	-0.0459	0.121
Phase 5	2018-05	2024-12	0.0091	0.0307	0.1154	0.1105
All Phases	1971-01	2024-12	0.0066	0.0477	0.0819	0.1753

Log Growth Rates

- **Phase 1 (1971-1980)**: This phase shows the highest mean monthly and yearly growth rates, at 0.0248 and 0.2978, respectively, indicating a period of rapid growth in gold prices. The high volatility (0.0724 monthly and 0.2509 yearly) reflects the significant market fluctuations during this time, likely driven by global economic events such as the oil crisis and inflation.
- **Phase 2 (1980-2001)**: The negative growth rates in Phase 2 (-0.0034 monthly and -0.0405 yearly) suggest stagnation in gold prices. The volatility decreases considerably compared to Phase 1, with a monthly standard deviation of 0.0385 and a yearly standard deviation of 0.1335, reflecting a more stable but unremarkable period for gold.
- **Phase 3 (2001-2012)**: This phase exhibits a resurgence in gold prices, with positive growth rates (0.0138 monthly and 0.1654 yearly). The volatility increases slightly compared to Phase 2, indicating more fluctuations, though still not as high as during Phase 1.
- **Phase 4 (2012-2018)**: Phase 4 shows negative growth rates again (-0.0039 monthly and -0.0470 yearly), mirroring a period of relative stagnation or even decline in gold prices. Volatility remains lower compared to earlier phases, reflecting more subdued market conditions.
- **Phase 5 (2018-2024)**: This most recent phase shows a moderate positive growth rate (0.0091 monthly and 0.1092 yearly), indicating gradual appreciation in gold prices. The volatility has decreased further, suggesting that the market has entered a more stable phase.
- **All Phases**: Over the entire period from 1971 to 2024, the average monthly and yearly growth rates are positive (0.0066 monthly and 0.0787 yearly), reflecting a long-term upward trend. However, the volatility (0.0466 monthly and 0.1615 yearly) shows that there have been significant fluctuations, especially during the sharp rise in gold prices during the 1970s and the subsequent periods of stagnation or slower growth.

Discrete Growth Rates

- **Phase 1**: The discrete growth rates reflect the same pattern as the log growth rates but show even higher returns, with a monthly mean of 0.0251 and a yearly mean of 0.3469. This phase is characterized by significant returns and very high volatility, emphasizing the market’s high risk and potential during this period.

- **Phase 2**: As with the log growth rates, the discrete growth rates are negative in this phase (-0.0034 monthly and -0.0397 yearly), reflecting a period of stagnation. Volatility is similar to the log growth rates but slightly lower (monthly S.D. = 0.0393, yearly S.D. = 0.1428).
- **Phase 3**: Positive discrete growth rates (0.0139 monthly and 0.1799 yearly) are observed here, suggesting a recovery in gold prices, although the volatility is somewhat lower than in Phase 1.
- **Phase 4**: Similar to the log returns, discrete returns in this phase are negative, indicating a period of decline in gold prices (-0.0039 monthly and -0.0459 yearly). Volatility, though reduced, still reflects some market uncertainty.
- **Phase 5**: The discrete growth rates again reflect moderate positive returns (0.0091 monthly and 0.1154 yearly), with lower volatility (0.0307 monthly and 0.1105 yearly), suggesting a more stable but positive performance in recent years.
- **All Phases**: The overall discrete growth rates mirror the log growth rates in terms of trends, with an average monthly return of 0.0066 and a yearly return of 0.0819. The volatility (0.0477 monthly and 0.1753 yearly) is slightly higher than the log growth rates, primarily due to the earlier phases of large fluctuations in gold prices.

Summary

- The periods of rapid growth in the 1970s (Phase 1) and steady rise in the 2000s (Phase 3) are clearly visible in both the log and discrete growth rates, with corresponding volatility peaks.
- The 1980–2001 period (Phase 2) is marked by negative growth, reflecting a period of stagnation for gold prices.
- Recent phases (Phases 4 and 5) indicate more moderate growth and lower volatility, signaling a transition to a more stable market environment.

3.2 The Impact of Timing on Gold Investment Returns: Analyzing the 1971–2024 Period

The overall discrete return on gold from 1971 to 2024 is approximately 8.2%. This period can be divided into three phases (1, 3, and 5) with positive returns, though these returns have been decreasing over time: 35% in Phase 1, 18% in Phase 3, and 11.5% in Phase 5. These positive return trends are interrupted by two phases (2 and 4) with negative returns, reflecting periods of stagnation or decline in gold prices.

Interestingly, the higher the gold price, the lower the positive returns, suggesting that as gold prices rise, the potential for high returns diminishes. In Phase 1, the official gold price was likely undervalued, contributing to the strong surge in gold prices as it adjusted toward its true market value.

In general, the profitability of an investment in gold is highly dependent on the timing of the investment. For instance, someone who purchased gold at the peak of 2011 ($1,895 per ounce) would have seen a modest return of about 2.4% annually over the following 13 years (from 2011 to 2024), calculated as ln(2,607/1,895)/13.25. This illustrates how market conditions at the time of purchase can significantly influence long-term returns.

In general, the return reflects that gold, while often seen as a safe-haven asset, may not always offer high growth compared to other assets over long periods. Its value tends to be more stable during times of economic uncertainty, but in periods of growth, other investments might outperform. For instance, equity markets or real estate investments often offer higher returns over extended periods.

This lower return emphasizes that timing is crucial when investing in gold, and holding periods significantly affect the profitability of the investment.

4. Application of Geometric Random Walk Using ARIMA Models

In this chapter, we apply the Geometric Random Walk (GRW) model to monthly gold price data using ARIMA modeling techniques. A detailed analysis is conducted for the entire period, examining the autocorrelation functions (ACF), partial autocorrelation functions (PACF), model estimation, and residual analysis. For the five identified phases of gold price movement, we provide a tabular summary of the results, focusing on key metrics without conducting an in-depth analysis. The reader is assumed to be familiar with the fundamentals of ARIMA modeling, as this chapter focuses on the application and results of the model rather than its theoretical underpinnings. This approach allows us to capture the overall dynamics of gold price trends and volatility, as well as the influence of the MA term and drift in the GRW model, across different market phases.

Autocorrelation and Partial Autocorrelation Functions (ACF and PACF)

The Figures 6a and 6b display the autocorrelation functions (ACF) and partial autocorrelation functions (PACF) for the log-transformed gold price data and its first differences over the entire period. ACF measures the correlation between the series and its lagged values, providing insight into the overall structure and persistence of price movements. The PACF, on the other hand, isolates the correlation at specific lags by accounting for the influence of intervening lags, allowing for a clearer understanding of direct dependencies. These plots serve as key diagnostic tools for determining the appropriate ARIMA model order by revealing patterns of autocorrelation and potential dependencies in the data.

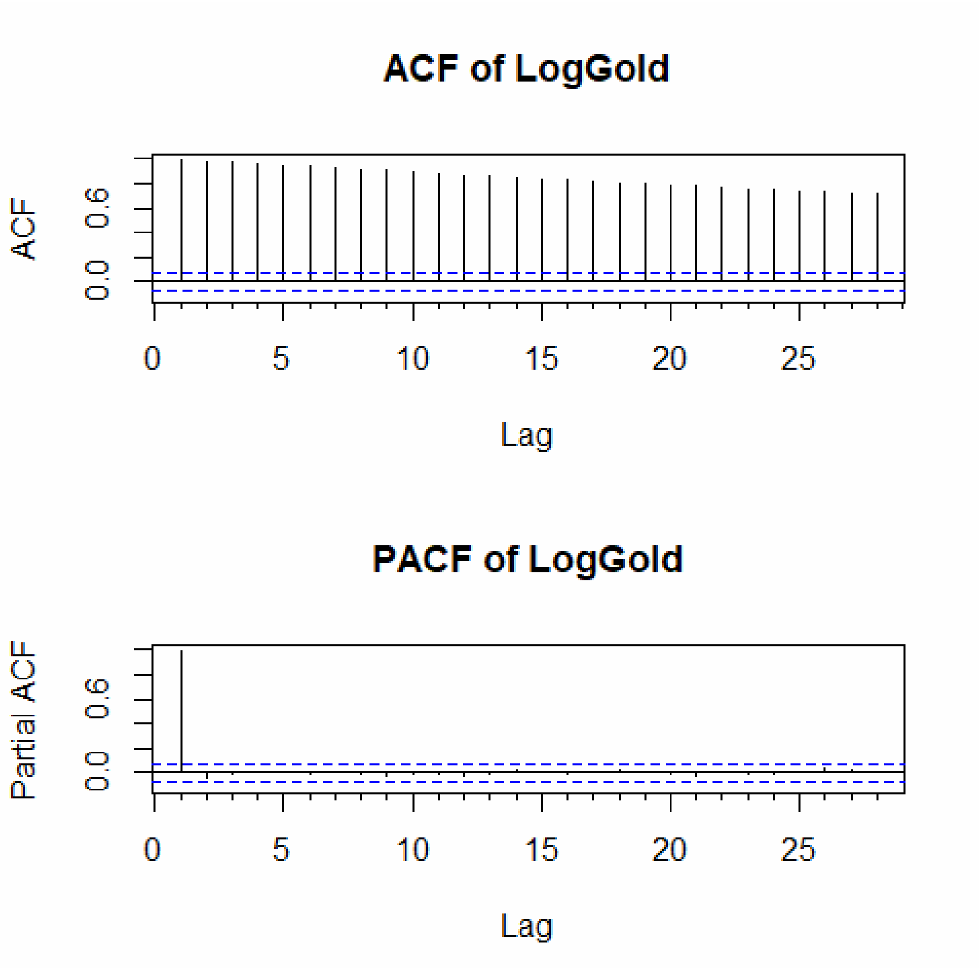

Figure 6a: Autocorrelation Function (ACF) and Partial Autocorrelation Function (PACF) of Monthly LogGold (1971–2024)

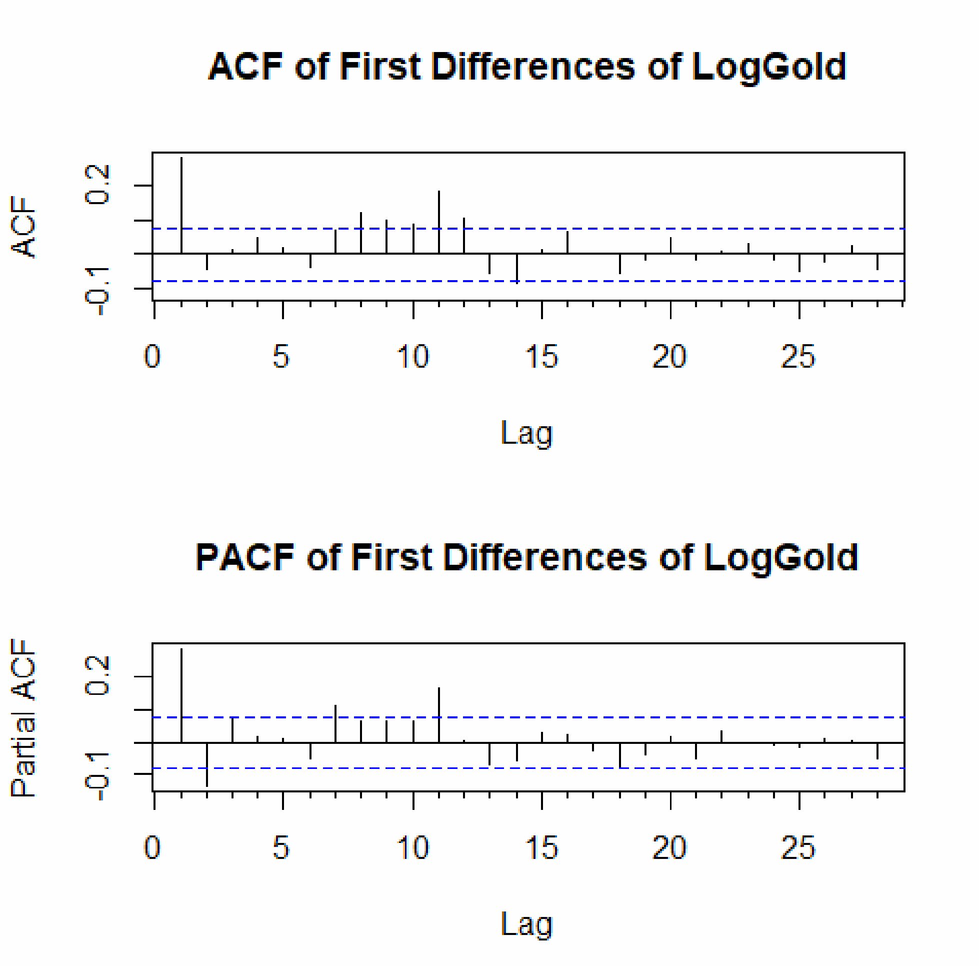

Figure 6b: Autocorrelation Function (ACF) and Partial Autocorrelation Function (PACF) of First Differences of Monthly LogGold (1971–2024)

The **ACF** of the log-transformed gold prices decays slowly, indicating a lack of short-term independence and suggesting non-stationarity in the series. The **PACF** shows a significant spike at lag 1, which is characteristic of an ARIMA(0,1,1) model or a generalized random walk (GRW) process. These features align with the expected behavior of a series that follows a random walk with some dependency structure.

After taking the **first differences**, the ACF exhibits behavior close to white noise, with most autocorrelations falling within the confidence bounds. However, there are a few significant spikes, particularly at short lags, indicating potential short-term dependencies. The PACF also shows residual structure but is largely consistent with the properties of a GRW.

These patterns in the ACF and PACF reinforce the suitability of the ARIMA(0,1,1) model, as identified by `auto.arima` in the next estimation step. This model effectively captures the underlying stochastic dynamics of the log-transformed gold prices, accounting for random shocks and the observed drift in the data. The GRW framework will be further explored in the upcoming chapter, highlighting its theoretical and practical applications.

Estimation with auto.arima

The estimation of ARIMA models for the gold price data was performed using the `auto.arima` function of R, which automatically selects the optimal model order based on criteria such as the Akaike Information Criterion (AIC). This process simplifies model identification by efficiently determining the best-fitting ARIMA model for the data, taking into account both the autoregressive (AR) and moving average (MA) components, as well as any necessary differencing (d) to achieve stationarity.

Estimation Results:

auto.arima(monthly_avg_gold$LogGold, seasonal = TRUE)

```
Series: monthly_avg_gold$LogGold
ARIMA(0,1,1) with drift

Coefficients:
         ma1   drift
      0.3438  0.0066
s.e.  0.0389  0.0023

sigma^2 = 0.001967:  log likelihood = 1098.63
AIC=-2191.27   AICc=-2191.23   BIC=-2177.85

   ME                RMSE        MAE         MPE       MAPE
0.0000007524921 0.04425232 0.03155952 0.006164356 0.5294356

   MASE        ACF1
0.9783736 -0.01616304
```

Comment on Estimation

The `auto.arima` function identifies an **ARIMA(0,1,1) model with drift** as the most suitable for the log-transformed gold prices. The estimated **moving average coefficient (ma1)** is 0.34380 with a standard error of 0.03890, indicating its significance. The drift term is estimated at 0.00660 with a standard error of 0.00230, capturing the upward trend in the series.

The model's fit statistics further support its suitability:

- **σ^2:** 0.0019670
- **Log-likelihood:** 1098
- **AIC:** −2191.27
- **AICc:** −2191.23
- **BIC:** −2177.85

Figure 7 shows the residual analysis of the ARIMA(0,1,1) model, including the Ljung-Box test with Q*=16.263Q, degrees of freedom (df) = 9, and a p-value of 0.06159.

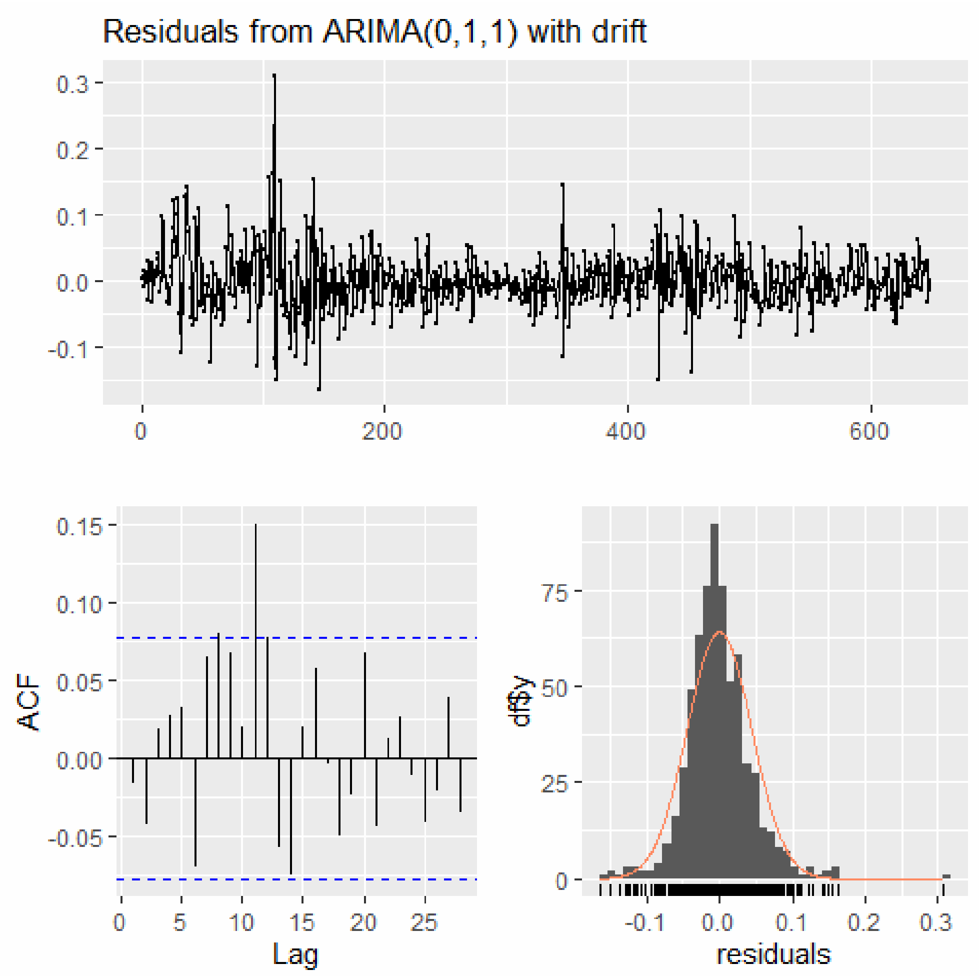

Figure 7: Residual Analysis of the ARIMA(0,1,1) Model
Ljung-Box test
Q* = 16.263, df = 9, p-value = 0.06159

Residual Analysis

The residual analysis (Figure 7) indicates that the model performs well:

- **ME (Mean Error):** Close to zero, suggesting unbiased estimates.
- **RMSE (Root Mean Square Error) and MAE (Mean Absolute Error):** Small, indicating accurate predictions.
- **ACF1 (Autocorrelation of Residuals at Lag 1):** −0.016, close to zero, confirming little autocorrelation in residuals.

The **Ljung-Box test** results (Q*=16.263, df=9, p-value = 0.061590) suggest that the residuals are not significantly autocorrelated, though the ACF reveals a single significant spike at lag 12. This spike may warrant further investigation, particularly concerning seasonal effects.

The ARIMA(0,1,1) model is well-suited to capture the stochastic behavior of log-transformed gold prices, including both their random fluctuations and the underlying drift (or trend) over time. The residual diagnostics confirm that the model assumptions are largely met, aligning with the characteristics of a **Geometric Random Walk (GRW)**. The single significant spike at lag 12 in the residual ACF could suggest potential periodicity, which might be explored in a seasonal ARIMA framework.

Summary of ARIMA Model Estimation Results for Gold Price Across Phases

Table 3 summarizes the estimation results for the gold price time series across different phases, highlighting key ARIMA model parameters, statistical metrics, and their alignment with the observed trends. The column terms in Table 3 represent key ARIMA model parameters and evaluation metrics, including the moving average coefficient (ma1), drift, residual variance (σ^2), Akaike Information Criterion (AIC), Root Mean Square Error (RMSE), log-likelihood, Ljung-Box test p-value (L-B p), mean log gold price, and whether the model was selected using the auto.arima function.

Table 3: Estimation Results for the Different Periods (Phases)

Phases		ma1	Drift	sig^2	AIC	RMSE	logLikeli.	L-B (p)	mean	auto.arima
all	ARIMA(0,1,1)	0.3438	0.0066	0.00197	-2191.27	0.0443	1098.63	0.062	0.0066	yes
		(0.0389)	(0.0023)							
1	ARIMA(0,1,1)	0.5438	0.0251	0.00414	-303.11	0.0635	154.56	0.597	0.0248	yes
		(0.0763)	(0.0091)							
2	ARIMA(2,1,2)			0.00145	-922.42	0.0377	466.21	0.059		yes
2-a	ARIMA(0,1,1)	0.1671	-0.0034	0.00146	-922.95	0.0380	464.47	0.021	-0.0034	no
		(0.0692)	(0.0028)							
2-b	ARIMA(0,1,1)	0.1729		0.00146	-923.5	0.0381	463.75	0.020		no
		(0.0687)								
3	ARIMA(0,1,0)		0.0138	0.00161	-471.81	0.0398	237.9	0.552	0.0138	yes
			(0.0035)							
4	ARIMA(0,1,1)	0.2632		0.00102	-259.98	0.0315	131.99	0.188		yes
		(0.1109)								
4-a	ARIMA(0,1,1)	0.2574	-0.004	0.00103	-258.62	0.0314	132.31	0.190	-0.0039	no
		(0.1119)	(0.0049)							
5	ARIMA(0,1,1)	0.3638	0.0089	0.00081	-333.71	0.0280	169.86	0.937	0.0091	yes
		(0.1057)	(0.0043)							

Remarks: ma1 and drift estimators with s.e. in parantheses; L-B (p): Ljung-Box test, p-value; mean (see Table 1)

Phase 2 : ARIMA(2,1,2): Coefficients:

```
         ar1     ar2      ma1      ma2
      0.2378  0.5764  -0.0665  -0.6973
s.e.  0.2361  0.1635   0.2287   0.1770
```

Key Observations from the Gold Price Estimation Results

1. **Upward Phases (Phases 1, 3, 5):**
    - Models like **ARIMA(0,1,1)** with drift (GRW with drift) are effective in capturing the trends during periods of rising gold prices.
    - The drift values are significant and align closely with the average means.
    - In these upward phases, the **auto.arima** function is sufficient to identify appropriate models without requiring manual intervention.
2. **Downward or Stagnating Phases (Phases 2 and 4):**
    - During periods of declining or stagnating gold prices, the drift becomes **insignificant**, making models without drift (GRW without drift) more appropriate.
    - **Phase 2:**
        - The **ARIMA(2,1,2)** model initially chosen by **auto.arima** captures the log-price dynamics but introduces complexity.
        - Manual adjustments (e.g., **Phase 2-a** and **Phase 2-b**) simplify the model, adhering to the principle of parsimony (Ockham's Razor), without significantly affecting metrics like AIC or RMSE.
    - **Phase 4:**
        - The ARIMA(0,1,1) model captures this period adequately. However, in **Phase 4-a**, the manual specification introduces a drift term, which proves to be insignificant.
3. **General Trends Across Phases:**
    - Drift values decrease across the phases, paralleling the trend observed in the mean values.
    - Models incorporating an **MA(1)** component often improve fit, particularly in upward phases.
    - Metrics such as AIC, RMSE, and Ljung-Box p-values (for residual autocorrelation) confirm that the models fit well overall.

Summary of Modeling Approach

- **GRW with drift** (potentially including an MA(1) component) is an adequate model for upward phases of gold prices.
- **GRW without drift** is preferable for stagnating or declining phases, as the drift becomes statistically insignificant.
- The **auto.arima** function is generally reliable for upward phases but requires manual adjustment in downward phases to maintain parsimony without deteriorating key model metrics.

Remarks on Positive MA Terms:

In all phases, the MA term is positive, which means that the price movements exhibit a short-term momentum effect. This suggests that any initial shock—such as an economic event or a shift in demand—has a persistent influence on the price. The positive MA term indicates that the shock's effect is carried over into subsequent periods, reinforcing the direction of the price movement (e.g., upward or downward). As a result, there is a higher likelihood of continued price adjustments in the short term following the initial shock, leading to increased volatility compared to a model without a positive MA term.

5. Forecasting Gold Price Up to 2030 Using the Geometric Linear Regression Model

In this study, the forecast of monthly gold prices in USD up to 2030 is made using the Geometric Linear Regression (GLR) model. The GLR model is built by fitting a linear regression to the historical logarithmic growth rates of gold prices, with the drift term from the ARIMA(0,1,1) model serving as the key driver for projecting future growth. This drift term, which is closely aligned with the observed growth rates, is used to estimate future price trends. The GLR model is applied to two time frames: the entire data period (1971–2024) and the most recent phase (2018–2024). By comparing these forecasts, we assess how well the model adapts to both the historical trends and the more volatile recent market conditions. The forecast includes point estimates along with 80% and 95% prediction intervals to reflect the uncertainty in the projections (see Figure 8 and Table 4).

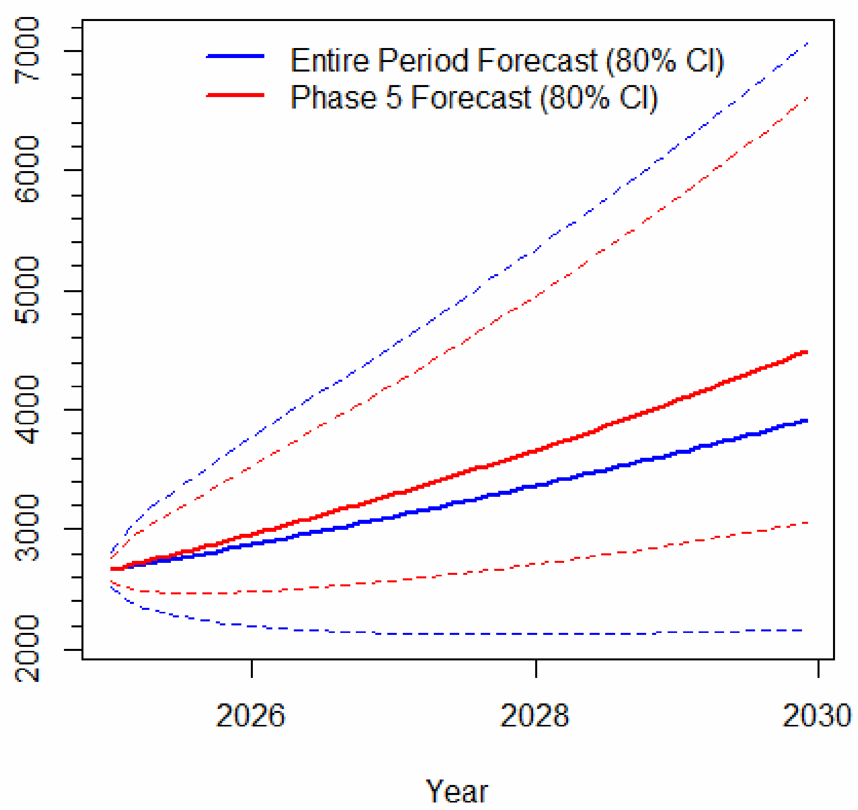

Figure 8: Gold Price Forecasts to 2030 with 80% Prediction Intervals, comparing the entire data period with Phase 5 to assess the model's adaptation to changing market conditions.

Table 4: Gold Price Forecasts to 2030: Comparison Between Last Period (Phase 5) and Entire Historical Period (1971-2024)

	Last Period	**2018-2024**	**(Phase 5)**		
months ahead	**Point Forec.**	**Lo 80**	**Hi 80**	**Lo 95**	**Hi 95**
12	2935	2478	3477	2265	3803
24	3265	2563	4158	2255	4726
36	3631	2697	4888	2304	5721
48	4038	2863	5696	2387	6833
60	4492	3057	6600	2494	8091
	Entire Period	**1971-2024**			
months ahead	**Point Forec.**	**Lo 80**	**Hi 80**	**Lo 95**	**Hi 95**
12	2856	2203	3703	1920	4249
24	3090	2133	4477	1753	5448
36	3344	2120	5273	1666	6711
48	3618	2136	6127	1617	8097
60	3914	2171	7058	1589	9643

The forecast results for the gold price, extending up to 60 months ahead, show varying degrees of uncertainty based on the period considered, with the 80% confidence intervals (CIs) displayed for simplicity. The **last period (2018–2024, Phase 5)** and the **entire historical period (1971–2024)** serve as the basis for these forecasts.

For the **last period (2018–2024, Phase 5)**, forecasts exhibit a steady increase in the point forecast values, with the 80% confidence interval widening over time. For example, at 12 months ahead, the point forecast is 2935 USD, with an 80% confidence interval ranging from 2478 to 3477 USD. This trend continues with forecasts for 24 months (3265 USD) and 60 months (4492 USD), suggesting a continued upward trajectory in gold prices, albeit with increased uncertainty as the forecast horizon extends.

For the **entire period (1971–2024)**, the forecasts are slightly lower than those for Phase 5. The point forecast at 12 months ahead is 2856 USD, with an 80% confidence interval ranging from 2203 to 3703 USD. As the forecast horizon increases, the uncertainty grows, with the 80% confidence intervals expanding further, reaching up to 7058 USD at 60 months. This indicates that, while the gold price has shown long-term growth, the recent period (2018–2024) demonstrates a more pronounced upward trend, likely due to current economic factors or market conditions.

These forecasts, with their associated 80% confidence intervals, highlight both the overall growth in gold price movements and the influence of market dynamics across different historical periods. The widening confidence intervals emphasize the uncertainty inherent in forecasting gold prices, with both periods showing significant variability in the projected values.

The forecast growth rates for gold prices until 2030, based on the most recent monthly gold price of 2639 USD, are notably high, with rates ranging from 7.9% to 10.6% per year. These growth rates may appear elevated at first glance, but they are consistent with the model's reflection of the historical growth trends, particularly in the more recent period (2018–2024, Phase 5), which has experienced a pronounced upward movement in gold prices.

The model captures the momentum of the past growth rates, which have been influenced by factors such as economic uncertainty, inflation, and shifts in investor demand for gold as a safe-haven asset. These influences are reflected in the forecasts, suggesting that the positive trend could persist into the coming years. While the forecast growth rates seem high, they align with the historical growth observed in the recent period, and this pattern of sustained growth is projected to continue in the short to medium term.

It's important to note that while the forecasts suggest relatively strong growth, they come with significant uncertainty, as reflected in the wide 80% prediction intervals. This range of uncertainty acknowledges the potential volatility in the gold market, influenced by both global economic conditions and unforeseen events. Thus, while the forecast growth rates are a reflection of past trends, they should be interpreted with caution, as future gold price movements may diverge depending on evolving market dynamics.

The forecast growth rates for gold prices in this study align closely with those of other institutions for the next few years. For example, various market forecasts suggest a continued upward trend in gold prices, with projections ranging between $2,795 and $4,151 per ounce by 2025, reflecting expectations of economic uncertainty and increased demand for gold as a safe-haven asset. However, it's important to highlight that the 80% prediction intervals in our model convey significant uncertainty, which is sometimes not fully captured in institutional forecasts that often focus solely on point forecasts. Our approach explicitly incorporates this uncertainty, providing a more comprehensive outlook that accounts for the variability in future gold price movements.

6. The Importance of the Geometric Random Walk Model in Evaluating Expert Predictions

Chapter 6 explores the critical role of the Geometric Random Walk (GRW) model in assessing expert predictions and measuring uncertainty in forecasting gold prices. The GRW model serves as a benchmark for evaluating the accuracy and reliability of expert forecasts, given its assumption that gold prices follow a random walk with a potential drift. By comparing expert predictions to the GRW baseline, we can evaluate whether experts provide valuable insights beyond what would be expected from random fluctuations. This chapter highlights the importance of using the GRW model to understand the predictive power of experts, test for overconfidence, and quantify the uncertainty inherent in forecasting, especially in the context of volatile markets.

The Geometric Random Walk (GRW) model plays a crucial role in assessing the accuracy and reliability of expert predictions, particularly in the context of forecasting gold prices. Financial markets, including gold, are inherently stochastic, meaning that future price movements are influenced by random factors that cannot be perfectly predicted. As such, the GRW model, with its assumption of price changes being random with a potential drift, serves as a useful benchmark for evaluating the quality of expert forecasts.

The GRW model captures the stochastic nature of financial time series, allowing for the modeling of price movements in a way that accounts for both randomness and underlying trends. By comparing expert forecasts to the GRW baseline, it becomes possible to assess whether experts' predictions provide valuable insights beyond what would be expected from a purely random process. This comparison offers a clear measure of the accuracy of expert predictions and highlights whether their insights align with observable market dynamics.

Furthermore, the GRW model is valuable for measuring uncertainty and volatility. Its simplicity enables the estimation of the inherent volatility of price movements, which can be contrasted with expert forecasts that may either overestimate or underestimate market fluctuations. This comparison is particularly useful in understanding the degree of uncertainty embedded in expert predictions and assessing how well these predictions account for market variability.

The GRW model also facilitates the evaluation of long-term trends, represented by the drift term, in the gold price. This aspect is particularly relevant for expert predictions that may include a long-term bias toward certain market outcomes, such as a sustained increase or decrease in gold prices. The GRW model allows for the identification of such trends and provides a benchmark against which the validity of expert projections can be tested. By assessing whether experts account for these trends in their forecasts, we can determine whether their predictions align with historical price movements or exhibit biases not supported by the data.

In addition, the GRW model offers a transparent and simple framework for testing expert overconfidence. Experts may sometimes provide deterministic forecasts with narrow confidence intervals, implicitly assuming a higher degree of predictability than is justified by the data. The GRW model, by contrast, emphasizes the randomness of market movements, highlighting any overconfidence in expert predictions that fail to incorporate the full extent of market uncertainty.

Finally, the GRW model serves as a tool for evaluating the predictive power of experts. By comparing expert forecasts with the GRW baseline, it is possible to assess whether expert predictions exhibit any real forecasting ability or if they are merely capturing patterns that could be explained by random walk behavior. This evaluation helps to identify whether expert judgments provide value beyond what would be expected from chance, offering a more rigorous assessment of their forecasting capabilities.

In summary, the GRW model is an essential tool for evaluating expert predictions in the context of gold price forecasting. It provides a benchmark for testing the accuracy, reliability, and predictive power of forecasts, while also offering insights into the uncertainty and volatility embedded in the market. By comparing expert predictions to the GRW model, we can better understand the effectiveness of expert insights and the degree to which they contribute to forecasting accuracy.

Appendix for Part 1

Minimum and Maximum Gold Prices (USD) by Year with Corresponding Dates (1971–2024)

Year	MinGold	MaxGold	MinGold	MinDate	MaxGold	MaxDate
1971	37.39	43.98	37.4	07.01.1971	44	06.12.1971
1972	44	70	44	03.01.1972	70	02.08.1972
1973	63.9	127	63.9	18.01.1973	127	06.07.1973
1974	116.5	195.25	116	02.01.1974	195	30.12.1974
1975	128.75	185.25	129	23.09.1975	185	24.02.1975
1976	103.5	140.35	104	25.08.1976	140	02.01.1976
1977	129.75	167.95	130	11.01.1977	168	11.11.1977
1978	165.7	242.75	166	05.01.1978	243	30.10.1978
1979	216.9	512	217	15.01.1979	512	28.12.1979
1980	481.5	850	482	18.03.1980	850	21.01.1980
1981	391.3	599.3	391	04.08.1981	599	06.01.1981
1982	296.8	481	297	21.06.1982	481	07.09.1982
1983	374.3	509.3	374	21.11.1983	509	15.02.1983
1984	307.5	405.9	308	20.12.1984	406	05.03.1984
1985	284.3	340.9	284	25.02.1985	341	19.08.1985
1986	326.3	438.1	326	02.01.1986	438	08.10.1986
1987	388.8	499.8	389	01.01.1987	500	14.12.1987
1988	395.3	484.1	395	26.09.1988	484	01.01.1988
1989	355.8	415.8	356	15.09.1989	416	24.11.1989
1990	345.9	423.8	346	14.06.1990	424	07.02.1990
1991	344.3	403	344	13.09.1991	403	16.01.1991
1992	330.4	359.6	330	10.11.1992	360	28.07.1992
1993	326.1	405.6	326	10.03.1993	406	02.08.1993
1994	369.7	396.3	370	22.04.1994	396	28.09.1994
1995	372.4	395.6	372	09.01.1995	396	19.04.1995
1996	367.4	414.8	367	03.12.1996	415	05.02.1996
1997	283	369.3	283	12.12.1997	369	01.01.1997
1998	273.4	313.2	273	28.08.1998	313	24.04.1998
1999	252.8	325.5	253	20.07.1999	326	05.10.1999
2000	263.8	312.7	264	27.10.2000	313	07.02.2000
2001	256	293.3	256	02.04.2001	293	17.09.2001
2002	276.5	349.3	276	01.01.2002	349	27.12.2002
2003	319.9	416.3	320	07.04.2003	416	30.12.2003
2004	375	454.2	375	10.05.2004	454	02.12.2004
2005	411.1	536.5	411	08.02.2005	536	12.12.2005
2006	513	725	513	02.01.2006	725	12.05.2006
2007	608.4	841.1	608	10.01.2007	841	08.11.2007
2008	712.5	1011.3	712	24.10.2008	1011	17.03.2008
2009	810	1212.5	810	15.01.2009	1212	02.12.2009
2010	1058	1421	1058	05.02.2010	1421	09.11.2010
2011	1319	1895	1319	28.01.2011	1895	05.09.2011
2012	1531	1791.8	1531	02.01.2012	1792	04.10.2012
2013	1192	1693.8	1192	28.06.2013	1694	02.01.2013

2014	1142	1385	1142	05.11.2014	1385	14.03.2014
2015	1049.4	1295.8	1049	17.12.2015	1296	22.01.2015
2016	1060	1366.3	1060	01.01.2016	1366	06.07.2016
2017	1145.9	1346.3	1146	02.01.2017	1346	08.09.2017
2018	1178.4	1355	1178	17.08.2018	1355	25.01.2018
2019	1269.5	1546.1	1270	23.04.2019	1546	04.09.2019
2020	1474.3	2067.2	1474	19.03.2020	2067	06.08.2020
2021	1684	1943.2	1684	30.03.2021	1943	04.01.2021
2022	1628.8	2039.1	1629	03.11.2022	2039	08.03.2022
2023	1811	2078.4	1811	24.02.2023	2078	28.12.2023
2024	1985.1	2777.8	1985	14.02.2024	2778	30.10.2024

Part 2:

The Role of Global GDP in Explaining and Forecasting Gold Prices: An ARIMAX Approach

1. Introduction

Building on the univariate time series models applied in Part 1 to explore gold price behavior, Part 2 takes the analysis a step further by incorporating global GDP as an exogenous variable. In this section, we employ the ARIMAX (AutoRegressive Integrated Moving Average with eXogenous variables) model to examine the dynamic relationship between gold prices and global GDP, enhancing our understanding of both past price movements and the influence of macroeconomic factors on future price trends.

The relationship between gold prices and macroeconomic indicators has long been of interest to researchers and practitioners. However, only recently has the gold price been explicitly linked to global GDP (gross domestic product). A groundbreaking study by Baur, O'Connor, Jerret, and Palmberg (2024) established this relationship using a simple regression framework, demonstrating that the gold price is positively influenced by global GDP. Their findings highlighted the importance of macroeconomic conditions in driving the value of gold, which functions as both a store of value and a barometer of global economic activity.

Incorporating this relationship, the ARIMAX model employed in this study expands the analysis by including both global GDP as an exogenous variable and the current and lagged values of gold prices as regressors. This model allows for the capture of the dynamic structure of the time series, addressing both short-term and long-term interactions between these variables. By accounting for the influence of past gold price movements alongside external shocks from global GDP, the ARIMAX framework offers a more comprehensive understanding of fluctuations in gold prices.

By combining the theoretical framework of gold price behaviour with an advanced time series modeling technique, this paper provides a nuanced perspective on the drivers of gold prices and emphasizes the importance of accounting for forecast uncertainty. This study aims to contribute to both the theoretical understanding and practical forecasting of gold prices in the context of global economic dynamics.

2. Yearly Gold Prices from 1970 to 2024

Figure 1 illustrates the development of yearly gold prices from 1970 to 2024 (refer to Table A1 in the Appendix of Part 2). For an analysis of daily gold prices, see Part 1. The historical evolution of gold prices can be divided into several distinct phases:

Initial Stability and Gradual Increase (1968–1971):

During the late 1960s and early 1970s, gold prices remained relatively stable with minor fluctuations. This period reflected a calm phase before significant market changes, as the gold price was still largely tied to the Bretton Woods system.

Sharp Increase (1971–1980):

After the collapse of the Bretton Woods system in 1971, gold prices began a steep upward trend. This phase was characterized by high inflation, the oil crisis, and geopolitical tensions, which drove demand for gold as a safe-haven asset. By the end of this period, gold prices had risen dramatically, reaching record levels by 1980.

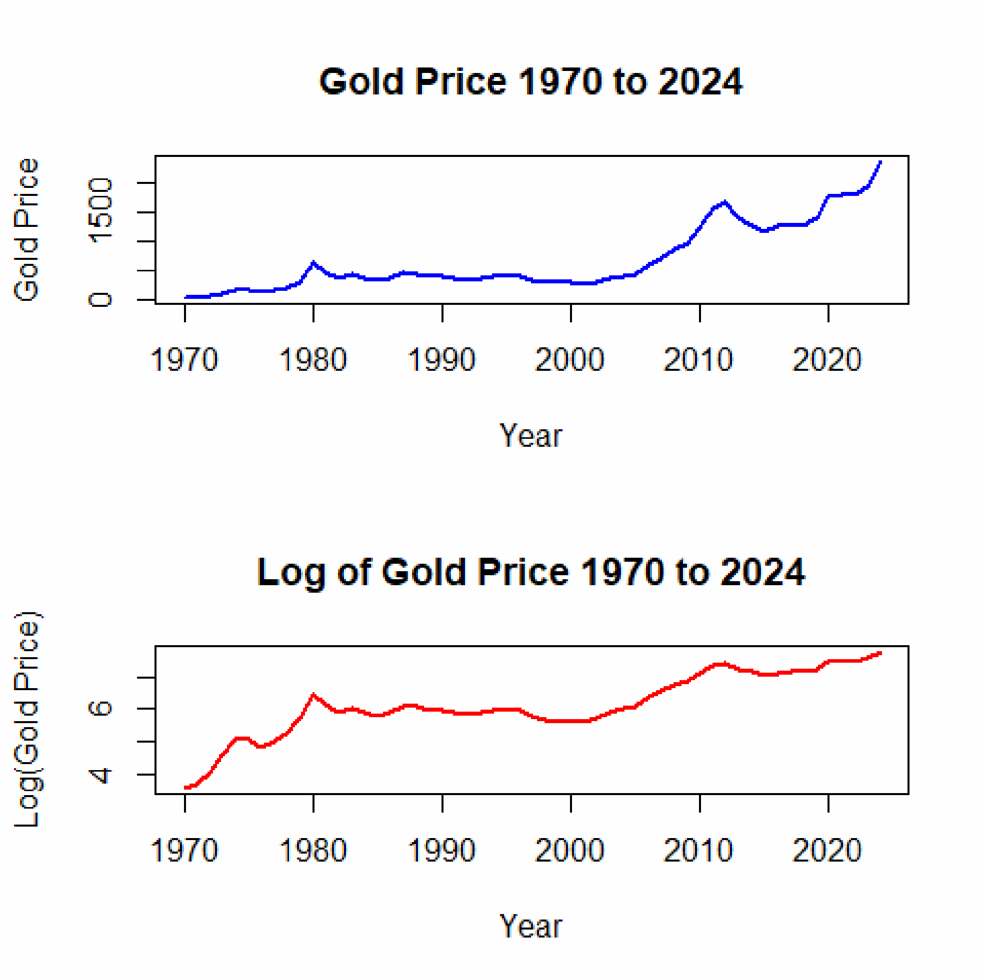

Figure 1: Yearly Gold Prices (USD per Ounce)

Volatility and Decline (1981–2000):

Following the peak in 1980, gold prices entered a period of volatility and gradual decline. Despite fluctuations, prices generally trended downward throughout the 1980s and 1990s, reflecting reduced inflationary pressures and stabilized global markets. By the late 1990s, gold prices were near their lowest levels in decades.

Steady Rise and New Peaks (2001–2012):

Starting in the early 2000s, gold prices entered a sustained period of growth. This phase was driven by global economic uncertainty, including the 2008 financial crisis, which increased demand for gold as a hedge against risk. Prices reached new all-time highs during this period.

Moderation and Stability (2013–2018):

Between 2013 and 2018, gold prices stabilized after the sharp increases of the previous decade. While there were minor fluctuations, prices generally remained within a relatively narrow range, reflecting a period of relative calm in global financial markets.

Renewed Increase and Record Highs (2019–2024):

From 2019 onwards, gold prices resumed their upward trajectory, driven by a combination of factors, including economic uncertainties, the COVID-19 pandemic, and geopolitical tensions. By 2024, gold prices had reached record levels, reflecting heightened demand for gold as a hedge against economic uncertainty.

Additional Note on Growth Rates

A log-scale presentation of gold prices reveals distinct differences in growth rates across historical phases. The **sharp increase phase (1972–1980)** is characterized by a steeper slope in the log plot compared to the **steady rise phase (2001–2012)**, reflecting faster exponential growth during the earlier period.

- **Sharp Increase Phase (1972–1980):**
  The gold price exhibited rapid exponential growth during this period, driven by high inflation, geopolitical crises, and economic instability. The steep slope in the log plot underscores the unprecedented pace of growth, with gold prices rising more than tenfold.
- **Steady Rise Phase (2001–2012):**
  This period shows a more moderate but sustained growth trajectory, with gold prices increasing approximately sixfold. The log plot reflects a smoother, less steep slope, indicating slower but consistent growth driven by heightened demand during the financial crisis and increased investor interest.
- **Recent Increase Phase (2019–2024):**
  The log plot for this phase shows a renewed but comparatively slower growth rate. While the trend remains positive, it is less pronounced than during earlier phases.

In summary, the log presentation highlights the differences in exponential growth rates, with the sharp increase phase demonstrating the fastest growth, followed by the steady rise and recent increase phases.

3. Methods, Modelling Framework and Results

3.1 Fundamentals

Baur et al. (2024) identify GDP as the primary driver of gold prices in the long run. We adapt their use of GDP as a proxy for the economic component of gold price determination while replacing their financial component—"capitalization of global equity and bond markets"—with the **gold price from the previous year** (lagged gold price). This substitution aligns with the **principle of parsimony**, which emphasizes simplicity in modelling, and the practical advantage that only one exogenous variable (GDP) needs to be forecast for prediction.

The rationale for using GDP as the key exogenous variable lies in its comprehensive composition. GDP reflects:

- **Price levels (inflation):** Rising inflation often triggers increased demand for gold as a hedge.
- **Real per capita income and population size:** Both components are expected to grow over time, contributing positively to gold demand.
- **Economic fundamentals:** Increasing national debt and expanding money supply which influence prices, further drive gold demand as a safe-haven asset.

As long as no new large gold deposits are discovered and brought onto the market, production and mining will continue to have only a marginal influence on gold prices, as noted by Baur et al. (2024, p. 5).

Data
Our analysis employs yearly data from 1970 to 2023, using the logarithms of gold prices (logY) and global GDP (logX). The dataset, along with its sources, is provided in Table A1 in the Appendix of Part 2.

Methods
This study employs **models** to investigate and forecast the relationship between gold prices and GDP. (AutoRegressive Integrated Moving Average with eXogenous variables) extends traditional ARIMA models by incorporating external regressors, allowing the model to:

- Capture the dynamics of the dependent variable (gold prices) through lagged values and residuals (ARIMA component).
- Account for the influence of external factors (e.g., GDP) on the dependent variable (exogenous variable component).

The framework is particularly useful for time series that exhibit both internal structure (e.g., trends and seasonality) and external influences. These models are defined as:

$$Y_t = c + \phi_1 \cdot Y_{t-1} + \ldots \phi_p Y_{t-p} + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2} + \ldots . \theta_{t-q} \varepsilon_{t-q} + \beta X_t + \varepsilon_t \, .$$

Where:

- Y_t: Dependent variable (e.g., gold prices).
- X_t: Exogenous variable (e.g., GDP).
- ϕ and θ: Coefficients of the autoregressive and moving average components.
- β: Coefficient of the exogenous variable.
- ϵ_t: Residuals (white noise).

The acronym ARIMA(p,d,q) stands for **AutoRegressive Integrated Moving Average**, where:

- **p** is the order of the autoregressive (AR) term, which models the relationship between an observation and a specified number of lagged observations.
- **d** is the degree of differencing required to make the time series stationary, reflecting the number of times the data needs to be differenced to achieve stationarity.
- **q** is the order of the moving average (MA) term, which models the relationship between an observation and a residual error from a moving average model applied to lagged observations

We implemented these models using the function from the R package *forecast* (Hyndman et al., 2020).

To ensure optimal model selection, the **auto.arima** function from the same package was employed. This function automates the selection of ARIMA model parameters (e.g., p, d, q) based on criteria such as the Akaike Information Criterion (AIC), balancing model complexity with forecasting accuracy. The resulting models provided point forecasts and prediction intervals, offering insights into the range of potential future gold prices.

3.2 Estimation Results

Model Overview: ARIMA(1,0,1) with Log Gold Price (Average Close Price)

The model with logarithmic transformation of the gold price and GDP data was estimated using an ARIMA(1,0,1) structure. The results are as follows:

Coefficients:

- **AR(1)**: 0.8404
- **MA(1)**: 0.5434
- **Intercept**: 2.7561
- **xreg (log GDP)**: 1.0140

Standard Errors:

- AR(1): 0.0811
- MA(1): 0.1417
- Intercept: 0.5282
- xreg (log GDP): 0.1538

Model Fit:

- σ^2=0.02405
- Log likelihood: 24.94
- AIC: -39.88
- BIC: -29.94
- RMSE: 0.14922

Residual Diagnostics:

The **Ljung-Box test** was applied to check for autocorrelation in the residuals (see Figure 2):

- ***Q statistic***: 6.5415
- **Degrees of freedom (df)**: 8
- **p-value**: 0.5868

Since the p-value is greater than 0.05, we conclude that there is no significant autocorrelation in the residuals. This indicates that the model adequately captures the structure of the data.

The **AIC** (Akaike Information Criterion), **BIC** (Bayesian Information Criterion), and **log likelihood** values are commonly used to assess model fit and compare competing models. Lower values of AIC and BIC indicate better-fitting models, while the log likelihood provides a measure of the likelihood of the model given the data. For further details on these metrics, see Hyndman and Athanasopoulos (2018).

Model Interpretation of $Y_t = 2.7561 + 0.8404 \cdot Y_{t-1} + 0.5435 \cdot \varepsilon_{t-1} + 1.0140 \cdot X_t + \varepsilon_t$

1. ARIMA Model Terms:

- **AR(1) (Autoregressive Term)**: The coefficient of 0.8404 suggests that the gold price is positively correlated with its previous value, meaning that changes in the gold price are influenced by its own past values.
- **MA(1) (Moving Average Term)**: The coefficient of 0.5434 indicates that the model accounts for random shocks or innovations affecting the gold price, with the impact of past residuals included in the model.
- **Intercept**: The intercept value of 2.7561 represents the baseline level of the log-transformed gold price in the absence of other factors.
- **xreg (Exogenous Variable – log GDP)**: The coefficient of 1.0140 indicates that changes in global GDP are a significant driver of gold price movements. Specifically, a 1% increase in global GDP (log-transformed) leads to an approximate 1.01% increase in the gold price, assuming other factors remain constant.

2. Economic Implications:

The positive relationship between **log(Y)** (gold price) and **log(X)** (GDP) suggests a strong economic linkage. A 1% increase in global GDP is associated with a 1.01% increase in the gold price, holding other factors constant. This implies that as the global economy expands, the demand for gold increases, likely due to its role as a hedge against inflation and economic instability.

3. Residual Diagnostics:

The Ljung-Box test shows that there is no significant autocorrelation in the residuals (p-value = 0.5868). This indicates that the ARIMA(1,0,1) model has successfully accounted for serial dependence in the data. Therefore, the model provides a good fit, and the residuals appear to be randomly distributed, without any patterns left unexplained by the model.

Key Points:

- The ARIMAX model effectively captures both the autoregressive and moving average processes in the gold price data.
- The inclusion of **log GDP** as an exogenous variable significantly explains gold price movements, with a notable elasticity of 1.01% for a 1% increase in GDP.
- The residual diagnostics confirm that the model adequately addresses serial dependence in the data.

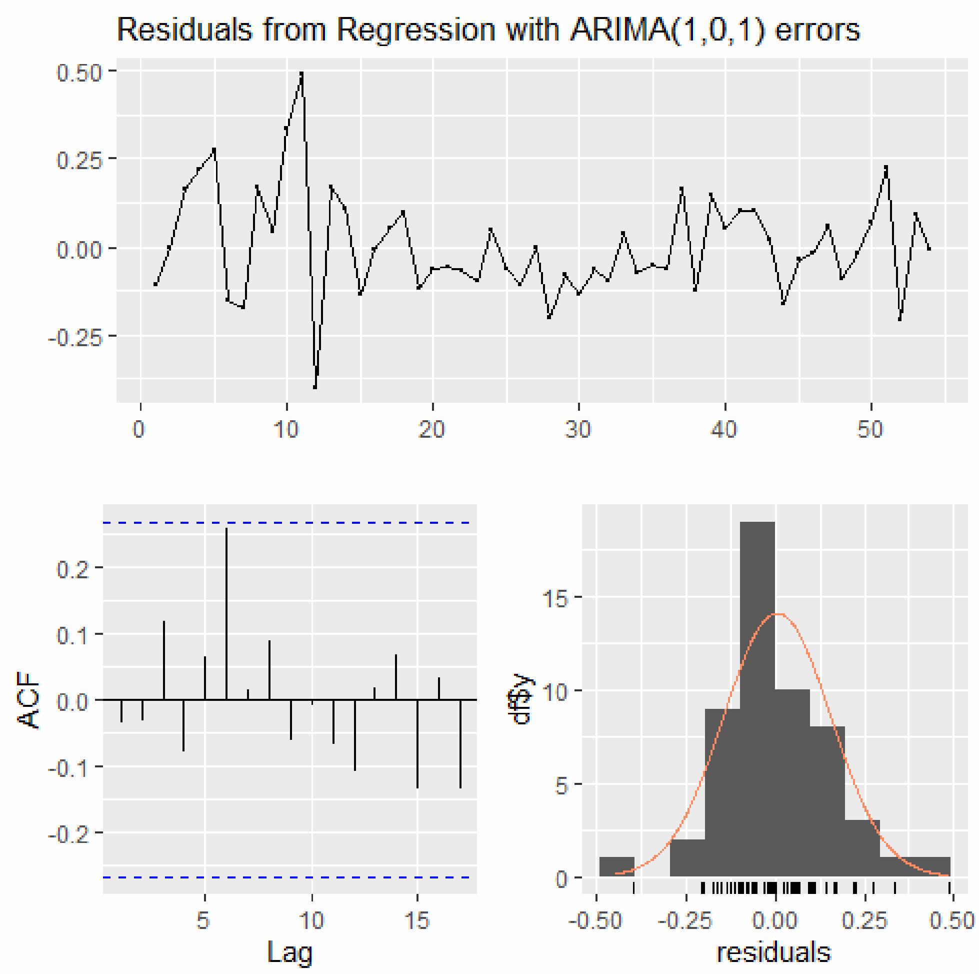

Figure 2: Diagnostic Plot of Residuals: Assessing Model Assumptions

Figure 3 shows the plot of actual and fitted log-transformed gold prices, alongside the plot of actual and fitted gold prices after transforming the log values back to the original level in Figure 4**.** Both graphs demonstrate a good fit, with the fitted values closely tracking the actual values. This indicates that the chosen model effectively captures the underlying relationship between gold prices and its predictors. The model provides a reliable representation of the gold price dynamics, both in the log-transformed form and after converting back to the original scale.

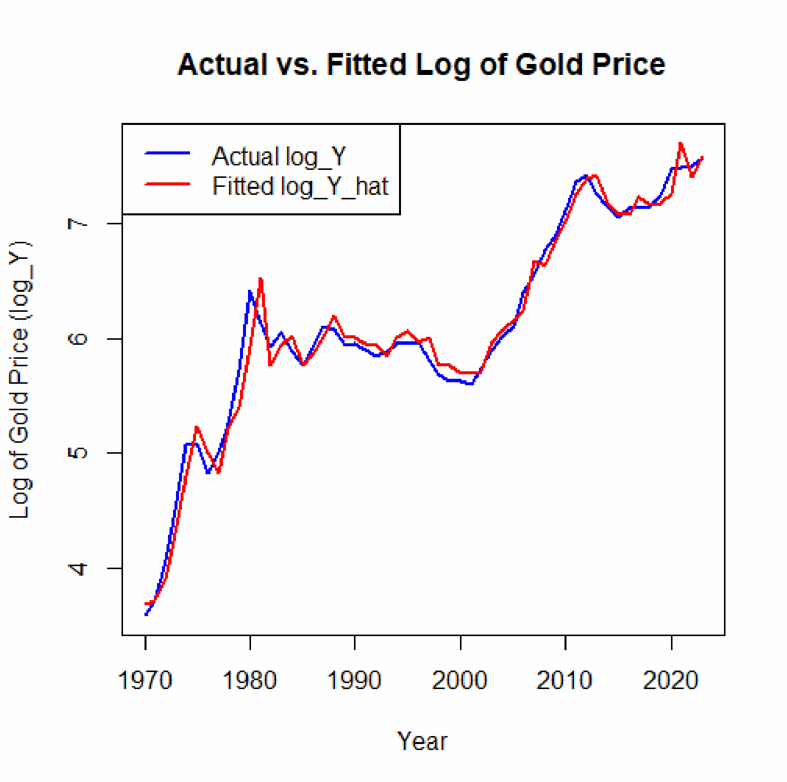

Figure 3: Actual vs. Fitted Log-Transformed Gold Prices

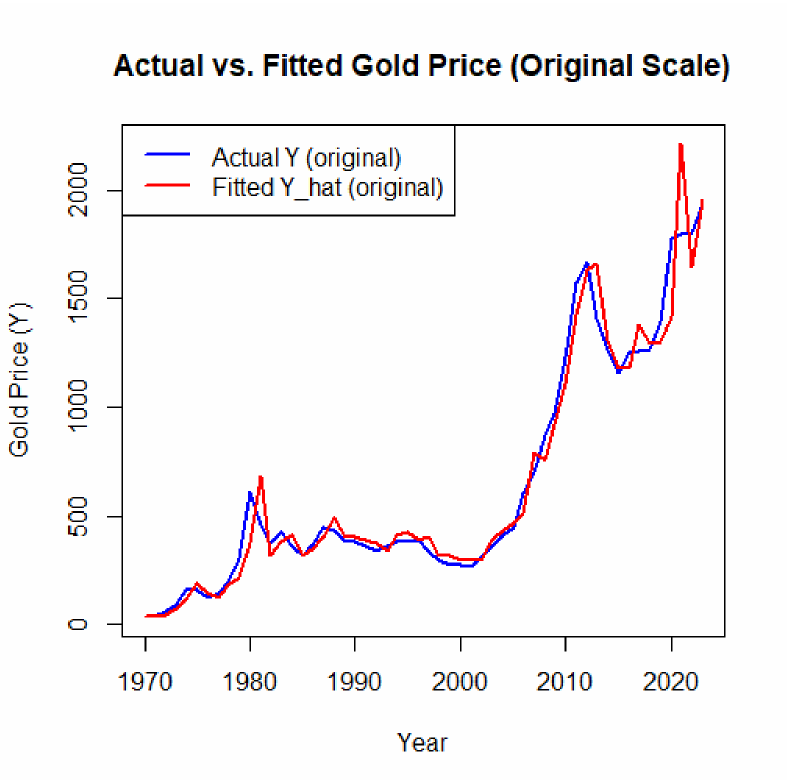

Figure 4: Actual vs. Fitted Gold Prices (Original Scale)

Table 1: Gold Price Forecasts with 95% Confidence Intervals and Yearly Growth Rates

year	Mean forecast	95%-CI-Low	95%-CI-High	Yearly growth
2023	1,943			
2024	2,005	1,480	2,717	0.032
2025	2,082	1,239	3,498	0.038
2026	2,166	1,156	4,057	0.040
2027	2,256	1,127	4,519	0.042
2028	2,354	1,125	4,924	0.043
2029	2,459	1,141	5,296	0.044
2030	2,571	1,170	5,649	0.045
2031	2,690	1,207	5,993	0.046
2032	2,816	1,252	6,335	0.047
2033	2,950	1,303	6,681	0.048

Table 1 presents the mean forecast, lower and upper 95% confidence intervals (CI), and the yearly growth rate under the assumption that the yearly growth rate of global GDP will remain at 5%—which represents the average growth rate over the past 20 years.

For the year 2033, the forecast mean gold price is approximately $2,950, with the 95% confidence interval ranging from $1,303 to $6,681. This indicates that while the model predicts an average annual growth rate of nearly 5% for gold prices, there is substantial uncertainty about the future development of gold prices. Despite the positive relationship between gold prices and GDP, the wide prediction intervals highlight the significant variability and the challenges in forecasting long-term trends for gold prices.

The results from Table 1 are presented graphically in Figure 5, which more dramatically illustrates the uncertainty in gold price forecasting by showing the extremely wide confidence intervals.

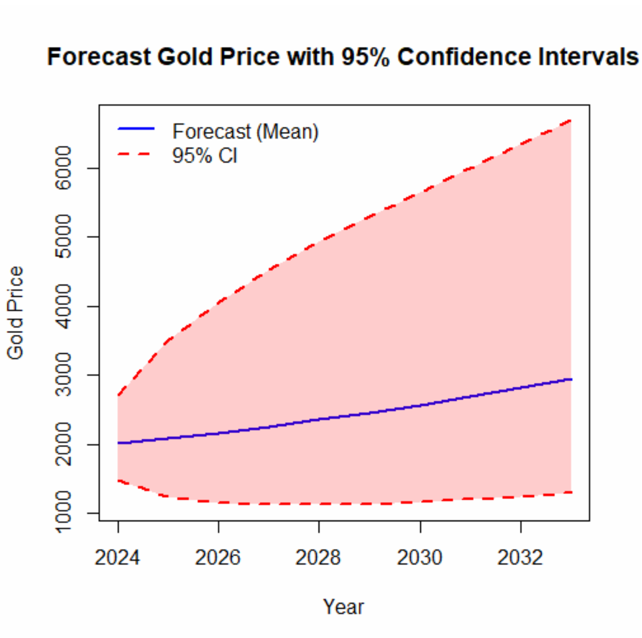

Figure 5: Uncertainty of Gold Price Forecasts

3.3 Justification for Using Nonstationary Time Series

1. **Theoretical Foundation:**
    - The relationship between gold prices and nominal GDP is expected to evolve over the long run, with both variables exhibiting persistent growth trends. These trends are influenced by factors such as inflation, population growth, and changes in per capita income.
    - Using the series in their levels allows for capturing these long-term trends and the equilibrium relationship between the two variables. If the data were differenced, these long-term trends would be lost, potentially obscuring the true economic relationship.
2. **Long-Run Focus:**
    - The primary objective of this study is to model the long-run relationship between GDP and gold prices. In economic theory, these variables are expected to move together over time, and their relationship is more accurately captured in levels rather than differences.
    - Differencing would remove the trends and hinder the study's goal of understanding the long-term dynamics between gold prices and GDP.
3. **Model Fit and Diagnostic Testing:**
    - Residual diagnostics from the ARIMAX model reveal that the residuals are stationary, suggesting that the model effectively captures the relationship between the variables despite their nonstationarity.
    - The close alignment between the observed and estimated gold prices further supports the use of nonstationary series, as it demonstrates that the model is appropriately capturing the underlying economic connection.
4. **Spurious Regression Concerns:**
    - Although nonstationary series can lead to spurious regression, this risk is mitigated by the fact that the residuals of the model are stationary, confirming that the observed relationship is genuine and not spurious.
    - To further ensure robustness, alternative models using first differences and first log differences were analyzed for comparison, offering a check against potential issues with nonstationarity.
5. **Intuitive Interpretation:**
    - Level models, which use nonstationary series, provide a more intuitive and easily interpretable representation of the long-term relationship between gold prices and GDP. This aligns with the study's objective of understanding and visualizing the long-run economic relationship in a straightforward manner.

3.4 Growth Rate Model

By differencing logY (log-transformed gold price) and logX (log-transformed GDP), the resulting series become stationary. This stationarity was confirmed through analysis of the **ACF** and **PACF**, though the plots are not shown here for brevity.

Estimation Results

The differencing process eliminates the influence of lagged values of Y, as evidenced by the following ARIMAX model results:

Model Specification:

- **Series:** ΔlogY (growth rate of the gold price)
- **Model:** ARIMA(0,0,1) with exogenous regressors (ΔlogX)
- **Coefficients:**
  - MA(1): 0.4373 (s.e. 0.1746)
  - xreg (ΔlogX): 1.3029 (s.e. 0.3229)
- **Model Fit:**
  - σ^2=0.02495
  - Log likelihood: 23.52
  - AIC: -41.03
  - BIC: -35.12
  - RMSE: 0.15495

Residual Diagnostics

- **Ljung-Box Test:**
  - Q=6.0277Q, degrees of freedom = 9, p=0.7371.
  - The high p-value indicates no significant autocorrelation in the residuals, confirming that the model adequately captures the structure of the data.

The results show a statistically significant positive relationship between log gold prices and log GDP, supporting the hypothesis that gold prices increase as global GDP grows. The prediction intervals calculated for this model are wider than those in the level model, reflecting the increased uncertainty inherent in forecasting gold prices.

Interpretation of Results

- Differencing the series removes the influence of lagged Y (log gold price), as expected, and simplifies the model to focus on the relationship between the growth rates of gold prices and GDP.
- The estimated coefficient for ΔlogX (log growth of GDP) is 1.3029, indicating that a 1% growth in GDP is associated with approximately a 1.30% increase in the gold price growth rate.

Visualizations

- Figure 6 shows the actual and fitted growth rates of the gold price. The close alignment of the two series suggests that the relationship between gold prices and GDP is not merely due to shared trends but reflects a **causal relationship**.

- Figure 7 presents the forecast for the growth rate of gold prices, assuming a **5% annual growth in GDP**. The forecast growth rate is approximately **6.5% per year**, with **95% confidence intervals** ranging from −27.3% to 40.3%.

Dependence on the Last Observation

As this is an ARIMAX model with a moving average (MA(1)) component, the forecast growth rate depends on the last observed value of the gold price, specifically the **average closing price for 2023**. This dependence highlights the sensitivity of the model to recent data and reinforces the importance of accurate and up-to-date input values for robust forecasting.

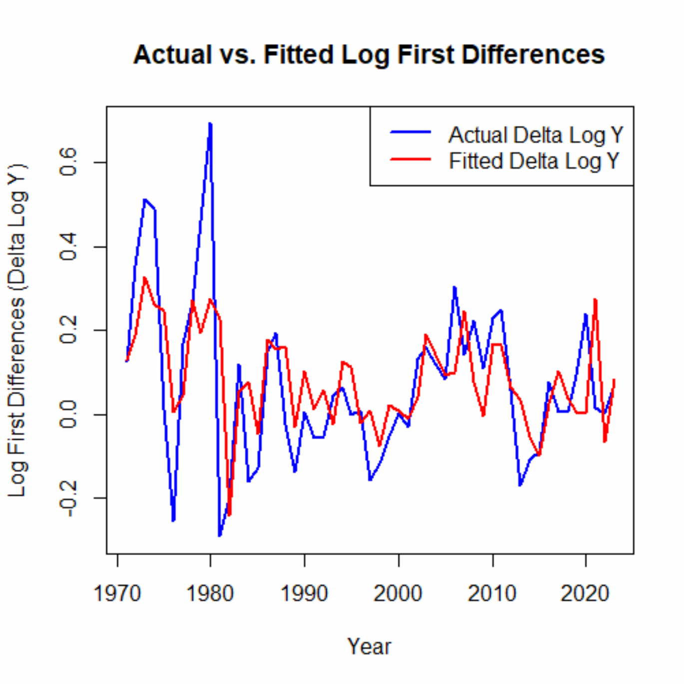

Figure 6: Actual and Fitted Growth Rates of Gold Prices

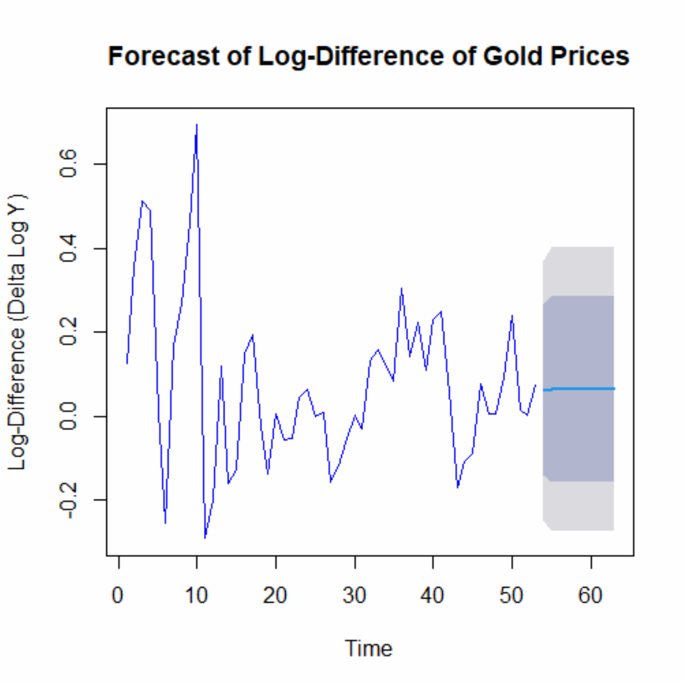

Figure 7: Forecast of the Growth Rates of the Gold Prices with 80%- and 95%- CIs

Comparison of Level and Growth Rate Models

The level model and the growth rate model offer complementary perspectives on the relationship between gold prices and GDP:

- **Level Model**: Captures the long-term relationship between gold prices and GDP in their original levels. This model is intuitive and easily interpretable, making it suitable for understanding overall trends. However, as the level model is based on nonstationary time series, it is more prone to overfitting and may yield narrower prediction intervals, which could underestimate uncertainty.

- **Growth Rate Model**: By differencing the data, this model removes long-term trends and focuses on short-term changes in gold prices and GDP. While the growth rate model is less intuitive and produces wider prediction intervals, it addresses stationarity concerns and reduces the risk of spurious regression. It is particularly useful for identifying causal relationships and making robust short-term forecasts.

In summary, the level model excels in providing a straightforward explanation of long-term dynamics, while the growth rate model offers a more statistically rigorous approach to analyzing short-term changes. The choice between these models depends on the specific focus of the analysis—whether it prioritizes understanding long-term trends or addressing stationarity concerns.

Key Findings

1. **Log-Level Model:**
The log-level model reveals a clear and statistically significant relationship between gold prices and global GDP, showing that gold prices increase as global GDP grows. This model is intuitive and captures the persistence in gold prices effectively through the inclusion of the lagged term.

2. **Prediction Intervals**:
While the log-level model generates meaningful forecasts, the wide confidence intervals underscore the uncertainty in future gold price movements. Notably, the gold price for 2024 approaches the upper bound of the prediction interval, demonstrating the model's limitations when extreme events or structural changes occur.

3. **Comparison First Difference Model:**
The first-difference logarithmic model offers a statistically valid alternative by addressing stationarity concerns. However, it is less intuitive and yields wider prediction intervals, making it less suitable for long-term forecasting or communicating findings to non-specialist audiences.

Preference for Log-Level Model

Despite its limitations, the log-level model is preferred for its simplicity, interpretability, and ability to capture long-term relationships between gold prices and global GDP. However, users must exercise caution, particularly for long-term forecasts, as the wide prediction intervals highlight the inherent uncertainty in projecting future gold price movements.

Conclusion

This study presents a comprehensive analysis of gold price behavior from both theoretical and practical perspectives, utilizing univariate time series models and advanced ARIMAX modeling. In Part 1, we examined the dynamics of gold price movements using a univariate GRW model, identifying key trends and volatility patterns. These findings provided the foundation for the next stage of analysis.

In Part 2, we extended the analysis by incorporating global GDP as an exogenous variable through the ARIMAX model. This approach allowed us to capture both short-term and long-term interactions between gold prices and macroeconomic factors, enhancing the robustness of our forecasts. Notably, both the GRW model in Part 1 and the ARIMAX model in Part 2 produced comparable results in terms of point forecasts and confidence intervals, which reflect the forecast uncertainty. However, the ARIMAX model is particularly sensitive to the forecast of global GDP, as it relies on this exogenous variable to produce accurate predictions.

Together, these two approaches offer complementary insights into the dynamics driving gold price fluctuations and provide a comprehensive framework for forecasting in the context of global economic conditions. The study also accounts for forecast uncertainty, offering valuable insights for investors, policymakers, and researchers.

Future research could further refine these models by incorporating additional macroeconomic indicators or exploring advanced forecasting techniques, which would enhance the precision and reliability of gold price predictions.

Final Remarks as of August 21, 2025

The analysis of gold price behavior through August 2025 reveals an **extraordinary year**, marked by significant growth and increased volatility (see Figure 7). The data from January 1 to August 21, 2025, shows that the gold market is experiencing a period of substantial upward movement, potentially signaling the beginning of a new market phase (**Phase 6**).

1. Forecast Comparison and Confidence Intervals:

The forecasts made at the end of 2024 anticipated continued growth in gold prices for 2025, but the actual values observed so far have approached the **upper limits** of the **95% confidence intervals** established in those forecasts. This indicates that the market has exceeded initial expectations and is following a trajectory of **stronger-than-anticipated growth**.

2. Extraordinary Growth:

The **mean log growth rate** of 0.00137 per day corresponds to an **annualized growth rate** of **0.3551** (35.51%), which leads to a **discrete annual growth rate** of **42.64%** ($\exp(0.3551)-1=0.4264$).

- The **discrete growth rate** between January 1, 2025, and August 21, 2025, has been **25.45%**.
- **Growth from the First Value to the All-Time High**: The **discrete growth rate** from the first price value in 2025 ($2,662.57 on January 1) to the **all-time high of $3,432.56** on April 21, 2025, is **28.61%**.

The **year-to-date growth** far surpasses any growth seen in the previous five phases, highlighting the **exceptional nature** of the 2025 gold price trajectory.

3. Increased Volatility:

The **volatility** of gold prices has also increased in 2025. The **daily standard deviation** is **0.01126**, leading to an **annualized volatility** of **0.1816**. While this is significant, it remains **second highest** compared to **Phase 1 (1971-1980)**, which had much higher volatility.
The rise in volatility in 2025 indicates a **turbulent market**, potentially driven by **speculation** or **external shocks** influencing investor sentiment and driving demand for gold.

4. Phase 6: Beginning of a New Phase?

Given the substantial change in the **mean growth rate** and **volatility**, 2025 could mark the start of **Phase 6**. This new phase, characterized by strong growth and high volatility, has witnessed the highest **mean growth rate** across all phases (35.51% annualized), and the second-highest **annualized volatility**.

5. Historical Context:

When compared to previous phases (see Table 1 in Part 1):

- **Phase 1 (1971-1980)** exhibited the highest growth and volatility, with a **yearly mean of 29.78%** and a **volatility of 25.09%**.
- **Phase 5 (2018-2024)** showed a more moderate **annual growth of 10.92%**, with a **volatility of 10.48%**.

The sharp rise in 2025 suggests that **gold prices may be undergoing a fundamental shift**, possibly driven by a combination of **geopolitical uncertainty**, **inflation concerns**, and **market speculation**.

6. Implications for Future Gold Prices:

While the **2025 growth** is impressive, such rapid increases may not be sustainable. The higher-than-expected growth, combined with increased volatility, suggests that the market may be **overvalued** at current levels. A **correction** may be expected if the drift reverts to a more typical level or if the speculative forces driving the price surge subside.

7. Conclusion:

The developments in **2025** highlight a significant shift in the gold market. The high growth and volatility observed could mark the beginning of **Phase 6**, a new phase characterized by **elevated risk and return expectations**. However, it might be premature to definitively classify this as a new phase after only eight months of data. **Further observation** over several months, or even years, may be necessary to confirm whether this is indeed a **long-term trend**.

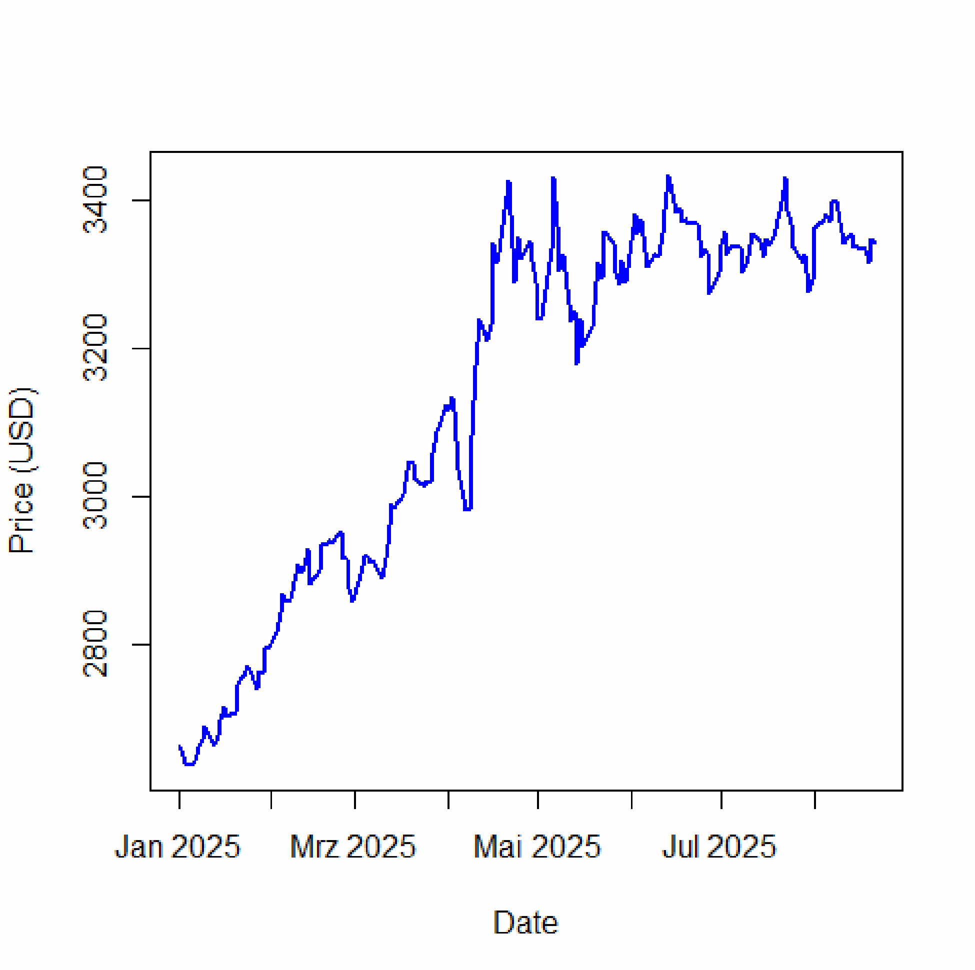

Figure 7: Daily Gold Prices from 2025-01-01 to 2025-08-21

Appendix for Part 2

Table A1: Data

year	Av_Clos_Price USD	Global GDPx10^12 USD
1970	36	3.00156
1971	40.8	3.316061
1972	58.2	3.832675
1973	97.1	4.662491
1974	158.8	5.358934
1975	160.9	5.990674
1976	124.8	6.508276
1977	147.8	7.356785
1978	193.6	8.700846
1979	307	10.113431
1980	614.8	11.419164
1981	459.2	11.796483
1982	376.1	11.648086
1983	423.7	11.972211
1984	360.7	12.445989
1985	317.4	13.023993
1986	368.2	15.35169
1987	446.8	17.485079
1988	436.8	19.547533
1989	381.3	20.324144
1990	383.7	22.82261
1991	362.3	23.823552
1992	343.9	25.468013
1993	360.1	25.959419
1994	384.2	27.938851
1995	384.1	31.14061
1996	387.7	31.857694
1997	331	31.752297
1998	294.1	31.697044
1999	278.9	32.730998
2000	279.3	33.839387
2001	271.2	33.62696
2002	310.1	34.917697
2003	363.8	39.152139
2004	409.5	44.116245
2005	445	47.760321
2006	604.3	51.749809
2007	696.4	58.314868
2008	872.4	64.072296
2009	973.7	60.718263

2010	1226.7	66.514175
2011	1573.2	73.957793
2012	1668.9	75.603977
2013	1409.5	77.751368
2014	1266.1	79.894386
2015	1158.9	75.359657
2016	1251.9	76.58803
2017	1260.4	81.550956
2018	1268.9	86.686871
2019	1393.3	87.945574
2020	1773.7	85.577718
2021	1798.9	97.527033
2022	1801.9	101.22506
2023	1943	105.43504

Sources:

Gold Price (Average Closing Price): macrotrend.net

GDP: Gross Domestic Product for World, World Bank, Gross Domestic Product for World [NYGDPMKTPCDWLD], retrieved from FRED, Federal Reserve Bank of St. Louis; https://fred.stlouisfed.org/series/NYGDPMKTPCDWLD, November 15, 2024.

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