A REMARK ON PREDICTION PROBLEMS IN REGRESSION ANALYSIS

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1 Introduction

In his paper, Kibria (1996) investigated the prediction problem in presence of uncertain prior information

$$H\beta = h \tag{1.1}$$

about the parameter vector β of the linear regression model

$$Y = X\beta + e. \tag{1.2}$$

Here Y is an $n \times 1$ vector of observations, β is a $p \times 1$ vector of unknown regression coefficients, X is an $n \times p$ known design matrix of rank p and e is an $n \times 1$ random error vector which follows a normal distribution with $N(0, \sigma^2 I)$, $\sigma^2 > 0$ unknown. The $q \times p$ matrix H is assumed to be of full row rank and h is a $q \times 1$ vector of constants, both H and h being known.

Suppose X_0 is a fixed $n_0 \times p$ matrix of additional observations on the regressor matris that is used to predict the future development of

$$Y_0 = X_0\beta. \tag{1.3}$$

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For this purpose, Kibria (1996) investigated the class of predictors

$$\hat{Y}_0 = X_0\hat{\beta}^*, \tag{1.4}$$

where $\hat{\beta}^*$ is one of the estimators described below. When comparing these predictors with bad other in terms of their mean square error (MSE) matrices under the condition $H_i : H\beta \neq h$ this author inverted matrices which are potentially singular. Subsequently we shall resume and correct his analysis.

2 Predictors and their mean square error matrices

We shall consider the following four predictors for $Y_0 = X_0\beta$:

1. Unrestricted predittor (URP)

$$\tilde{Y}_0 = X_0\tilde{\beta}_n, \tag{2.1}$$

where $\tilde{\beta}_n = C^{-1\prime}X'Y$ is the unrestricted least squares estimator (URLSE) of β, $C = X'X$.

2. Restricted predictor

$$\hat{Y}_0 = X_0\hat{\beta}_n, \tag{2.2}$$

where $\hat{\beta}_n = \tilde{\beta}_n I_{(L_n > F_{1-\alpha})} + \hat{\beta}_n I_{(L_n \leq F_{1-\alpha})}$ is the preliminary test estimator (PTLSD) for β, $I_{(S)}$ is the indicator function of the set S, $F_{1-\alpha}$ is the upper 100α percentile of the central F–distribution with $(p, (n-q)$ degrees of freedom and L_n is the well–known test statistic for testing the null hypothesis

$$H_0 : H\beta = h \quad \text{vs.} \quad H_1 : H\beta \neq h.$$

3 shrinkage predictor

The shrinkage predictor of $Y_0 = X_0\beta$ is defined by

$$\hat{Y}^{SP} = X_0\hat{\beta}_n^{SE}, \tag{3.1}$$

where $\hat{\beta}_n^{SE} = \hat{\beta}_n + (1 - uL_n^{-1})(\tilde{\beta}_n - \hat{\beta}_n)$ with

$$n = \frac{(q-2)(n-p)}{q(n-p+2)} \quad (q \geq 3)$$

being the shrinkage constant.

Let $M_i, i = 1, \ldots, 4$ denote the mean square error matrices of the four estimators introduced above. Under H_0, it was shown in Kibria (1996) that for example

$$M_1 - M_2 = \sigma^2 X_0 A X_0', \tag{3.2}$$

where $A = C^{-1}H'(HC^{-1}H')^{-1}HC^{-1}$. Unfortunately, it was also confirmed there that A and $M_1 - M_2$ are positive dfeinite (p.d.) matrices. This is, however, not the case. Both matrices can only shown to be nonnegative definite. Also the other mean square error matrix differences in section 4 of Kibria (1996) can namely be identified as n.n.d. matrices. The dominance ranking of the predictors under this weaker criterion fortunately remains valid.

The comparisons of section 5 of Kibira (1996) are based on the nonsingularity of mean square error matricess which will be done in the next section, where we use some results achived by Baksalary and Kala (1983).

4 Comparison of the predictors under H_1

Assume that $H_1 : H\beta \neq h$ is valid. Then the four predictors, except $\tilde{Y}_0 = X_0\tilde{\beta}_n$, will be biased. When comparing them, Kibria (1996) inverted $X_0 A X_0'$. This is not permitted, since A is not p.d., and if it were, X_0AX_0' need not be nonsingular. Hence we have to use an alternative method.

Lemma 1: (Baksalary and Kala, 1983)
Suppose A is a symmetric $n \times n$ matrix, a is an $a \times 1$ vector and α is a positive scalar. Then the following statements are equivalent

(i) $\alpha A - aa'$ is n.n.d.

(ii) A is n.n.d., $a \in \mathcal{R}(A)$ and $a'A^-a \leq \alpha$ where A^- is any generalized inverse of A, i.e. $AA^-A = A$ and $\mathcal{R}(\cdot)$ denotes the column space of a matrix. This result enables us to perform the following comparisons.

<u>Comparison between URP and RP</u>
the differenc eof the MSE matrices of $\tilde{Y}_0$ and $\hat{Y}_0$ is

$$M_1 - M_2 = \sigma^2 X_0AX_0' - X_0yy'X_0', \tag{4.1}$$

where

$$\nu = C^{-1}H'(HC^{-1}H')^{-1}(H\beta - h). \tag{4.2}$$

By Lemma 1, $M_1 - M_2$ is n.n.d. if and only if

a) X_0AX_0' is n.n.d.,

b) $X_0\nu \in \mathcal{R}(X_0AX_0')$,

c) $\nu'X_0'(X_0AX_0')^-X_0\nu \leq \sigma^2$, wher $(X_0AX_0')^-$ is any generalized inverse of X_0AX_0'.

Obviously condition a) is fulfilled. To show b) observe that $\mathcal{R}(X_0AX_0') = \mathcal{R}(X_0A)$. Hence if suffices that $\nu \in \mathcal{R}(A)$. Since H is of full row rank we have $HH^+ = I$ and consequently $\nu = ACH^+(H\beta - h) \in \mathcal{R}(A)$, where H^+ denotes the Moore–Penrose inverse of H. Thus $M_1 - M_2$ is n.n.d. if and only if condition c) is satisfied. This corresponds to condition (5.4) in Kibria (1996), where the invers of X_0AX_0' has to be replaced by a g–inverse.

We shall not perform the other MSE–matrix comparisons in detail. The MSE–matrix differences under H_1 considered further in Kibria's paper are easily seen to be of the form

$$M_i - M_j = \gamma X_0 A X_0' - \delta X_0 \nu \nu' X_0', \tag{4.3}$$

where γ and δ are positive scalars. Proceeding as in the comparison between URP and RP and applying Lemma 1 we can readily derive the dominance criteria corresponding to those in Kibria's paper where, however, $(X_0 A X_0')^-$. With one exception: The comparison between URP and SP in formula (5.17) seems to be completely incorrect since it is based on the maximum characteristic root of the "positive definite" matrix $[X_0'(X_0 A X_0')^{-1} X_0] C^{-1}$.

References

Baksalary / Kala (1983):

Kibira (1996):