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Investigating the Logical Structure of FUZZY IF-THEN Rule Bases using Concepts of Mathematical Logic and of Functional Analysis

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Investigating the Logical Structure of FUZZY IF-THEN Rule Bases using Concepts of Mathematical Logic and of Functional Analysis*

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Abstract
In developing a semantics for a Fuzzy If-Then Rule Base we in principle distinguish the following two approaches. Firstly, a Fuzzy If-Then Rule Base is considered as a Fuzzy Knowledge Base describing (time independent) situations by means of fuzzy logic. Secondly, a Fuzzy If-Then Rule Base describes the "inner part" of a fuzzy controller, therefore the conclusions of If-Then rules must be interpreted as assignments. In the paper presented we discuss only the first approach. To this end following TARSKI we define a suitable concept of model and semantic entailment for Fuzzy If-Then Rule Bases. Furthermore, we adopt the concept of fact from logic programming.

1 Introduction

By $\langle 0, 1\rangle$ we denote the set of all real numbers r with $0 \leqq r \leqq 1$. If U is an arbitrary set by a fuzzy set F on U we understand a mapping $F\colon U \to \langle 0, 1\rangle$, i. e. we do not distinguish between a fuzzy set F and its membership function μ_F. The set of all fuzzy sets on U is denoted by $F\mathbb{P}(U)$, furthermore, $\mathbb{P}(U)$ denotes the classical power set of U. Generally, Λ terms the empty set.

Let m and n be integers with $m, n \geqq 1$. Furthermore we are given non-empty sets $U_1, \dots,$ U_n, V called universe.

Now we assume that $X_1, \dots, X_n$, and Y are variables for fuzzy sets on $U_1, \dots, U_n$ and V, respectively, also called linguistic variables on the concerning universe.

Using the terminology of the (usual classical) Logic Programming we call constructs of the form

$$X_\nu = F_\nu \quad \text{and} \quad Y = G$$

FACTS on U_ν and FACTS on V, respectively, where $\nu = 1, \dots, n$, F_ν is a fuzzy set on U_ν, and G is a fuzzy set on V. By $\mathrm{FAC}(U_\nu)$ and $\mathrm{FAC}(Y)$ we denote the set of all facts on U_ν and V, respectively.

Constructs of the form

$$\text{IF } X_1 = F_1, \dots, \text{and } X_n = F_n \text{ THEN } Y = G$$

are termed as RULES on $[U_1, \dots, U_n; V]$. The set of all rules on $[U_1, \dots, U_n; V]$ is denoted by $\mathrm{RUL}(U_1, \dots, U_n; V)$.

Now we assume that

$F_{1\nu}, \dots, F_{m\nu}$ are fuzzy sets on U_ν where $\nu = 1, \dots, n$

and

$G_1, \dots, G_m$ are fuzzy sets on Y.

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A scheme $\mathfrak{R}$ of the form

$$\mathfrak{R}: \quad \begin{array}{l} \text{IF } X_1 = F_{11}, \dots, \text{and } X_n = F_{1n} \text{ THEN } Y = G_1 \\ \vdots \\ \text{IF } X_1 = F_{m1}, \dots, \text{and } X_n = F_{mn} \text{ THEN } Y = G_m \end{array}$$

is called FUZZY IF-THEN RULE BASE on $[U_1, \dots, U_n; V]$.

With respect to the semantic interpretation, $\mathfrak{R}$ will be considered as a "LOGICAL" FUZZY KNOWLEDGE BASE in contrast to the interpretation of $\mathfrak{R}$ as an IMPERATIVE ASSIGNMENT PROGRAMM in the area of fuzzy control.

This distinction between the "logical approach" and the "imperative approach" is decisive and fundamental for all considerations in the paper presented and in forthcoming papers dealing with fuzzy IF-THEN rule bases.

In the following of this paper we will consider only the logical approach.

2 The Concept of Model and of Semantic Entailment for logical Fuzzy IF-THEN Rule Bases

In developing the concepts of model and of semantic entailment we go back to the classical definition of the concept of semantic entailment in logical calculi.

Assume that E is an expression and $\mathfrak{E}$ is a set of expressions. Furthermore assume that we are given a set $\mathfrak{J}$ of interpretations and there is defined for an interpretation $I \in \mathfrak{J}$ the relation "I is a model of $\mathfrak{E}$" holds.

Now, we remember that in mathematical logic one defines "$\mathfrak{E}$ semantically entails E with respect to $\mathfrak{J}$" to be if and only if for every $I \in \mathfrak{J}$, if I is a model of $\mathfrak{E}$ then I is a model of $\{E\}$.

Remark
In some publications the word "semantically" is replaced by "model based" and for the complete formulation "$\mathfrak{E}$ semantically (model based) entails E with respect to $\mathfrak{J}$" is replaced by "E is a semantic (model based) consequence of $\mathfrak{E}$ with respect to $\mathfrak{J}$".

Now, we are going to formulate these fundamental definitions in the case of a given logical Fuzzy IF-THEN Rule Base $\mathfrak{R}$ as specified in the previous section.

Definition 1
Φ is said to be an interpretation on $[U_1, \dots, U_n; V]$
$=_{\text{def}}$ $\Phi: F\mathbb{P}(U_1) \times \cdots \times F\mathbb{P}(U_n) \to F\mathbb{P}(V)$.

Obviously, this definition means that Φ is a functional operator which generates a image function $G = \Phi(F_1, \dots, F_n)$ from the given n-tuple $[F_1, \dots, F_n]$ argument functions $F_1, \dots, F_n$ where for every $\nu \in \{1, \dots, n\}$, $F_\nu: U_\nu \to \langle 0, 1\rangle$ and $G: V \to \langle 0, 1\rangle$.

The set of all interpretations on $[U_1, \dots, U_n; V]$ is denoted by $\mathrm{INT}(U_1, \dots, U_n; V)$.

Definition 2
Φ is said to be a model of $\mathfrak{R}$
$=_{\text{def}}$
1. *Φ is an interpretation on $[U_1, \dots, U_n; V]$ and*
2. $\Phi(F_{11}, \dots, F_{1n}) = G_1$
   $\vdots$
   $\Phi(F_{m1}, \dots, F_{mn}) = G_m$

This definition says that for every rule

$$R_\mu\colon\ \mathrm{IF}\, X_1 = F_{\mu 1},\ \ldots\ ,\text{and}\quad X_n = F_{\mu n}\,\mathrm{THEN}\, Y = G_\mu$$

($\mu = 1, \ldots, m$) the functional operator Φ the n-tuple $[F_{\mu 1}, \ldots, F_{\mu n}]$ of premises converts into the conclusion G_μ, i. e. that the equation

$$\Phi(F_{\mu 1}, \ldots, F_{\mu n}) = G_\mu$$

holds.

In order to define an entailment relation we fix a set $\mathfrak{J}$ of interpretations of $\mathfrak{R}$. Expressions E are fuzzy if-then rules of the form $E = R$ where

$$R\colon\ \mathrm{IF}\, X_1 = F_1,\ \ldots,\ \text{and}\ X_n = F_n\,\mathrm{THEN}\, Y = G$$

Thus, Fuzzy IF-THEN Rule Base are sets $\mathfrak{E}$ of “expressions” E.

Remark
With respect to the conception of model state that the “order” of the rules in $\mathfrak{R}$ does not play any role, thus, we can consider $\mathfrak{R}$ as a usual (unordered) set.

Definition 3
$\mathfrak{R}$ semantically entails R with respect to $\mathfrak{J}$ (shortly denoted by $\mathfrak{R} \Vdash_{\mathfrak{J}} R$)
$=_{\mathrm{def}}$ *For every $\Phi \in \mathfrak{J}$, if Φ is a model of $\mathfrak{R}$ then Φ is a model of $\{R\}$.*

Obviously, in definitions 2 and 3 the fact that $\mathfrak{R}$ is a finite set does not play any role, therefore we admit that $\mathfrak{R}$ is an arbitrary set, i. e. $\mathfrak{R}$ can be finite, or infinite, or even empty.

Finally, we define the following entailment operator ENT

Definition 4
$\mathrm{ENT}(\mathfrak{R}, \mathfrak{J}) =_{\mathrm{def}} \{R \,|\, \mathfrak{R} \Vdash_{\mathfrak{J}} R\}$

3 Investigating and Applying the Entailment Operator

Adopting the “philosophy” of mathematical logic we are faced with the problem to investigate the behavior of the operator ENT depending on $\mathfrak{R}$ and $\mathfrak{J}$, in particular, to characterize $\mathrm{ENT}(\mathfrak{R}, \mathfrak{J})$ by deduction rules if possible or to show that this is not possible, respectively.

We start the investigation of $\mathrm{ENT}(\mathfrak{R}, \mathfrak{J})$ with the following two simple theorems.

Theorem 1
For every $\mathfrak{R}, \mathfrak{R}' \subseteqq \mathrm{RUL}(U_1, \ldots, U_n; V)$ and for every $\mathfrak{J} \subseteqq \mathrm{INT}(U_1, \ldots, U_n; V)$,

1. $\mathfrak{R} \subseteqq \mathrm{ENT}(\mathfrak{R}, \mathfrak{J})$
2. $\mathrm{ENT}(\mathrm{ENT}(\mathfrak{R}, \mathfrak{J}), \mathfrak{J}) \subseteqq \mathrm{ENT}(\mathfrak{R}, \mathfrak{J})$
3. *if* $\mathfrak{R} \subseteqq \mathfrak{R}'$ *then* $\mathrm{ENT}(\mathfrak{R}, \mathfrak{J}) \subseteqq \mathrm{ENT}(\mathfrak{R}', \mathfrak{J})$.
4. $\mathrm{ENT}(\Lambda, \mathfrak{J}) = \Lambda$

Obviously, this theorem means that for every fixed $\mathfrak{I}$ the operator $\mathrm{ENT}'_{\mathfrak{I}}$ defined by

$$\mathrm{ENT}'_{\mathfrak{I}}(\mathfrak{R}) =_{\mathrm{def}} \mathrm{ENT}(\mathfrak{R}, \mathfrak{I})$$

with $R \subseteqq \mathrm{RUL}(U_1, \ldots, U_m; V)$ is a closure operator in the sense of classical algebra with the sealing condition $\mathrm{ENT}'(\Lambda) = \Lambda$.

Theorem 2

For every $\mathfrak{R} \subseteqq \mathrm{RUL}(U_1, \ldots, U_n; V)$ and every $\mathfrak{I}, \mathfrak{I}' \subseteqq \mathrm{INT}(U_1, \ldots, U_n; V)$,

1. *if $\mathfrak{I} \subseteqq \mathfrak{I}'$ then $\mathrm{ENT}(\mathfrak{R}, \mathfrak{I}') \subseteqq \mathrm{ENT}(\mathfrak{R}, \mathfrak{I})$*
2. $\mathrm{ENT}(\mathfrak{R}, \Lambda) = \mathrm{RUL}(U_1, \ldots, U_n; V)$

If we define for every fixed $\mathfrak{R} \subseteqq \mathrm{RUL}(U_1, \ldots, U_n; V)$ the operator $\mathrm{ENT}''_{\mathfrak{R}}$ by

$$\mathrm{ENT}''_{\mathfrak{R}}(\mathfrak{I}) = \mathrm{ENT}(\mathfrak{R}, \mathfrak{I}),$$

where $\mathfrak{I} \subseteqq \mathrm{INT}(\mathfrak{R}, \mathfrak{I})$, then $\mathrm{ENT}''_{\mathfrak{R}}(\mathfrak{I})$ is comonotone with the scaling condition $\mathrm{ENT}(\Lambda) = \mathrm{RUL}(U_1, \ldots, U_n; V)$.

From mathematical logic we adopt the following fundamental definition.

Definition 5

1. ENT *is said to be compact in the first argument with respect to the fixed set $\mathfrak{I}$ of interpretations*
   $=_{\mathrm{def}}$ *For every $\mathfrak{R} \subseteqq \mathrm{RUL}(U_1, \ldots, U_n; V)$*
   and for every $R \in \mathrm{RUL}(U_1, \ldots, U_n; V)$,
   if $R \in \mathrm{ENT}(\mathfrak{R}, \mathfrak{I})$ then there exists an $\mathfrak{R}_{fin} \subseteqq \mathfrak{R}$ such that $\mathfrak{R}_{fin}$ is a finite set and $R \in \mathrm{ENT}(\mathfrak{R}_{fin}, \mathfrak{I})$.
2. ENT *is said to be cocompact in the second argument with respect to the fixed set $\mathfrak{R}$ of rules*
   $=_{\mathrm{def}}$ *For every $\mathfrak{I} \subseteqq \mathrm{INT}(U_1, \ldots, U_n; V)$*
   and for every $R \in \mathrm{RUL}(U_1, \ldots, U_n; V)$,
   if $R \in \mathrm{ENT}(\mathfrak{R}, \mathfrak{I})$ then there exists an $\mathfrak{I}_{cofin} \supseteqq \mathfrak{I}$ such that $\mathrm{INT}(U_1, \ldots, U_n; V) \setminus \mathfrak{I}_{cofin}$ is finite and $R \in \mathrm{ENT}(\mathfrak{R}, \mathfrak{I}_{cofin})$.

In mathematical logic the compactness of operators like $\mathrm{ENT}'_{\mathfrak{I}}$ plays a decisive role in investigating and solving the problem whether the set $\mathrm{ENT}'_{\mathfrak{I}}(\mathfrak{R})$ can be characterized by (recursive) deduction rules.

Now, in the present case to this problem one could counter that in practise infinite fuzzy IF–THEN rule bases $\mathfrak{R}$ do not occur. But we are of the opinion that this point of view is to narrow because one could use "parameterized" rule sets with an infinite parameter set and this fact could mean that the rule base considered is infinite.

The co-compactness defined for the operator $\mathrm{ENT}''_{\mathfrak{R}}$ seems to be strange, but as we have shown in [4] this modification of the "classical" compactness must be used for comonotonic operators.

Open problem

Up to now it is still an open problem in which cases for $\mathfrak{R}$ and $\mathfrak{I}$ the operators $\mathrm{ENT}'_{\mathfrak{I}}$ and $\mathrm{ENT}'_{\mathfrak{R}}$ are compact and cocompact, respectively. This problem will be investigated in a forthcoming paper.

Remark

As we have shown in [5, 6] the entailment operators defined can be applied in order to define concepts like consistency (inconsistency), completeness (incompleteness), dependency (independency) and equivalence of rule bases $\mathfrak{R}$.

4 Methods for generating interpretations and restricting models

The concepts of interpretation, model, and semantic entailment are very general and also very flexible such that in special situations or for special applications by "fine tuning" the definitions can be adapted to the problematic nature considered.

We offer two methods of adapting, firstly, the restriction of the concept of interpretation, and secondly, the restriction of the concept of model.

We begin with considering the first approach. Referring to definition 3 one could think that by fixing a subset $\mathfrak{J}$ of interpretations from $\mathrm{INT}(U_1, \ldots, U_n; V)$ and using this set in order to define the entailment relation $\mathfrak{R} \Vdash_{\mathfrak{J}} R$ the claim for restricting the concept of interpretation is sufficiently satisfied. But that this is not the case will be shown further down by applications. Strictly speaking, the applications require that the set $\mathfrak{J}$ of interpretation used depends on the given rule base $\mathfrak{R}$.

Thus, we formulate the following definition.

Definition 6
J is said to be an interpretation-selecting operator on $\mathrm{RUL}(U_1, \ldots, U_n; V)$ *and* $\mathrm{INT}(U_1, \ldots, U_n; V)$
$=_{\mathrm{def}} J : \mathbb{P}(\mathrm{RUL}(U_1, \ldots, U_n; V)) \to \mathbb{P}(\mathrm{INT}(U_1, \ldots, U_n; V))$.

If $\mathfrak{R} \subseteqq \mathrm{RUL}(U_1, \ldots, U_n; V)$ is a rule base then $\mathfrak{J} = J(\mathfrak{R})$ is to interpret as the set of interpretations which will be used in definition of $\mathfrak{R} \Vdash_{\mathfrak{J}} R$ and $\mathrm{ENT}(\mathfrak{R}, \mathfrak{J})$. According to this approach we define

Definition 7

1. $\mathfrak{R} \Vdash^*_J R =_{\mathrm{def}} \mathfrak{R} \Vdash_{J(\mathfrak{R})} R$
2. $\mathrm{ENT}^*(\mathfrak{R}, J) = \mathrm{ENT}(\mathfrak{R}, J(\mathfrak{R}))$

With respect to this more general approach, theorem 1 must be modified as follows.

Theorem 3
For every $\mathfrak{R}, \mathfrak{R}' \subseteqq \mathrm{RUL}(U_1, \ldots, U_n; V)$ *and every interpretation-selecting operator J on* $\mathrm{RUL}(U_1, \ldots, U_n; V)$ *and* $\mathrm{INT}(U_1, \ldots, U_n; V)$,

1. $\mathfrak{R} \subseteqq \mathrm{ENT}^*(\mathfrak{R}, J)$
2. *if* $J(\mathfrak{R}) \subseteqq J(\mathrm{ENT}^*(\mathfrak{R}, J))$ *then* $\mathrm{ENT}^*(\mathrm{ENT}^*(\mathfrak{R}, J), J) \subseteqq \mathrm{ENT}^*(\mathfrak{R}, J)$
3. *if* $\mathfrak{R} \subseteqq \mathfrak{R}'$ *and* $J(\mathfrak{R}' \subseteqq J(\mathfrak{R})$ *then* $\mathrm{ENT}^*(\mathfrak{R}, J) \subseteqq \mathrm{ENT}^*(\mathfrak{R}', J)$
4. $\mathrm{ENT}^*(\Lambda, J) = \Lambda$

After theorem 1 we defined the entailment operator $\mathrm{ENT}'_{\mathfrak{J}}$ and stated that $\mathrm{ENT}'_{\mathfrak{J}}$ is a closure operator satisfying the scaling condition $\mathrm{ENT}'_{\mathfrak{J}}(\Lambda) = \Lambda$.

Analogously for a fixed interpretation-selecting operator J we define

$$\mathrm{ENT}^1_J(\mathfrak{R}) =_{\mathrm{def}} \mathrm{ENT}^\star(\mathfrak{R}, J).$$

where $\mathfrak{R} \in \mathrm{RUL}(U_1, \ldots, U_n; V)$.

Trivially, we get the scaling condition $\mathrm{ENT}^1_J(\Lambda) = \Lambda$ but with respect to theorem 3 we can not conclude that ENT^1_J is a closure operator for an arbitrary (fixed) J.

Furthermore, theorem 2 must be modified as follows.

Theorem 4
For every $\mathfrak{R} \subseteqq \mathrm{RUL}(U_1, \dots, U_n; V)$ and every interpretation-selecting operator J and J' on $\mathrm{RUL}(U_1, \dots, U_n; V)$ and $\mathrm{INT}(U_1, \dots, U_n; V)$,

1. *if $J(\mathfrak{R}) \subseteqq J'(\mathfrak{R})$ then $\mathrm{ENT}^*(\mathfrak{R}, J) \subseteqq \mathrm{ENT}^*(\mathfrak{R}, J')$*
2. *if $J(\mathfrak{R}) = \Lambda$ then $\mathrm{ENT}^*(\mathfrak{R}, J) = \mathrm{RUL}(U_1, \dots, U_n; V)$.*

After theorem 2 we defined the operator $\mathrm{ENT}''_{\mathfrak{R}}$ and stated that $\mathrm{ENT}''_{\mathfrak{R}}$ and is comonotone and satisfies the scaling condition $\mathrm{ENT}''_{\mathfrak{R}}(\Lambda) = \mathrm{RUL}(U_1, \dots, U_n; V)$.

Now, for arbitrary interpretation-selecting operators J and J' on $\mathrm{RUL}(U_1, \dots, U_n; V)$ and $\mathrm{INT}(U_1, \dots, U_n; V)$ we define

$$J \subseteqq J' =_{\mathrm{def}} \text{ For every } \mathfrak{R} \subseteqq \mathrm{RUL}(U_1, \dots, U_n; V),\ J(\mathfrak{R}) \subseteqq J'(\mathfrak{R}).$$

Furthermore, for a fixed rule base $\mathfrak{R} \subseteqq \mathrm{RUL}(U_1, \dots, U_n; V)$ and arbitrary interpretation-selecting operator J we define

$$\mathrm{ENT}^2_{\mathfrak{R}}(J) =_{\mathrm{def}} \mathrm{ENT}^\star(\mathfrak{R}, J).$$

Then we can state that $\mathrm{ENT}^2_{\mathfrak{R}}$ is comonotone and satisfies the scaling condition $\mathrm{ENT}^2_{\mathfrak{R}}(\Lambda) = \mathrm{RUL}(U_1, \dots, U_n; V)$.

Remark
The definition of compactness of ENT^* with respect to the first argument can be adopted from definition 5 for ENT. It remains as a problem how the cocompactness of ENT^* with respect to the second argument can be defined.

In the following of this section we define and investigate several methods for generating special sets of interpretations for a given rule set $\mathfrak{R}$, i. e. methods for constructing special interpretation-selecting operators J. The methods will be based on concept of continuity of the functional operators considered, on the principles FATI and FITA, and on fuzzy relations interpreted as solution of systems of fuzzy relational equations.

4.1 The concept of continuity

We assume that $F\mathbb{P}(U_1), \dots, F\mathbb{P}(U_n)$, and $F\mathbb{P}(V)$ are topological spaces. Assume $\mathfrak{R} \subseteqq \mathrm{RUL}(U_1, \dots, U_n; V)$.

Definition 8
$J_{gc}(\mathfrak{R}) =_{\mathrm{def}} \{\Phi | \Phi \in \mathrm{INT}(U_1, \dots, U_n; V) \text{ and } \Phi \text{ is continuous}\}$

Obviously, J_{gc} does not depend on $\mathfrak{R}$ because we have supposed the continuity of Φ in its whole domain. According to this, we have used the index gc for J (as "globally continuous").

Because J_{gc} is constant we can state that this operator satisfies the conditions (see theorem 3)

$$J_{gc}(\mathfrak{R}) \subseteqq J_{gc}(\mathrm{ENT}^\star(\mathfrak{R}, J_{gc}))$$

and if $\mathfrak{R} \subseteqq \mathfrak{R}'$ then $J_{gc}(\mathfrak{R}') \subseteqq J_{gc}(\mathfrak{R})$, hence we have

Proposition 5
$\mathrm{ENT}^1_{J_{gc}}$ is a closure operator.

Furthermore, we have to remark that J_{gc}, of course, depends on the topologies given in $F\mathbb{P}(U_1), \dots, F\mathbb{P}(U_n)$, and $F\mathbb{P}(V)$ and even on the concept of continuity defined for Φ.

For instance, the topologies considered can be generated by suitable metrics, maybe by the well-known CHEBYSHEV distance d_{CH}, on $F\mathbb{P}(U)$ for arbitrary $F, G : U \to \langle 0, 1\rangle$ defined by

$$d_{CH}(F,G) =_{\text{def}} \operatorname{Sup}\{|F(x) - G(x)| \,|\, x \in U\} \ .$$

One can be of the opinion that to suppose the continuity of Φ in its whole domain is too strong, hence the class of admitted interpretations would be too small, hence the class of entailed rules would be too large.

Therefore we can claim that for a given rule base $\mathfrak{R}$ an operator Φ is continuous only at "points" of the form $[F_1, \dots, F_n]$, where $F_1 \in F\mathbb{P}(U_1), \dots, F_n \in F\mathbb{P}(U_n)$ and there is a rule $R \in \mathfrak{R}$ of the form $R =$ IF $X_1 = F_1, \dots, X_n = F_n$ THEN G.

Accordingly, we formulate where the index lc means "locally continuous".

Definition 9

$$J_{lc}(\mathfrak{R}) =_{\text{def}} \left\{ \Phi \;\middle|\; \begin{array}{l} \Phi \in \mathrm{INT}(U_1, \dots, U_n; V) \textit{ and for every rule } R \in \mathfrak{R}, \textit{ if } R \textit{ has the form} \\ R = \text{IF } X_1 = F_1, \dots, \textit{ and } X_n = F_n \text{ THEN } Y = G \\ \textit{then } \Phi \textit{ is continuous at the point } [F_1, \dots, F_n] \end{array} \right\}$$

Obviously, J_{lc} satisfies the following proposition (see theorem 3).

Proposition 6
For every $\mathfrak{R}, \mathfrak{R}' \subseteqq \mathrm{RUL}(U_1, \dots, U_n; V)$, if $\mathfrak{R} \subseteqq \mathfrak{R}'$ then $J_{lc}(\mathfrak{R}') \subseteqq J_{lc}(\mathfrak{R})$.

Using theorem 3 from proposition 6 we get that $\mathrm{ENT}^1_{J_{lc}}$ is monotonic. We underline that $J_{lc}(\mathfrak{R}) \subseteqq J_{lc}(\mathrm{ENT}^\star(\mathfrak{R}, J_{lc}))$ does not hold, in general, hence $\mathrm{ENT}^1_{J_{lc}}$ does not satisfy $\mathrm{ENT}^1_{J_{lc}}(\mathrm{ENT}^1_{J_{lc}}(\mathfrak{R})) \subseteqq \mathrm{ENT}^1_{J_{lc}}(\mathfrak{R})$, in general.

4.2 The principle FATI

We are given a rule base of the form

$$\mathfrak{R}: \quad \begin{array}{l} \text{IF } X_1 = F_{11}, \dots, \text{and } X_n = F_{1n} \text{ THEN } Y = G_1 \\ \vdots \\ \text{IF } X_1 = F_{m1}, \dots, \text{and } X_n = F_{mn} \text{ THEN } Y = G_m \end{array}$$

We define FATI($\mathfrak{R}$) $=_{\text{def}}$ The set of all FAT where FAT has the form
$FAT = [S_1, \dots, S_m, \alpha, \kappa, Q]$ and satisfies

1. for every $\mu \in \{1, \dots, m\}$, $S_\mu : U_1 \times \cdots \times U_n \times V \to \langle 0, 1\rangle$
2. $\alpha : \langle 0, 1\rangle^m \to \langle 0, 1\rangle$
3. $\kappa : \langle 0, 1\rangle^{n+1} \to \langle 0, 1\rangle$
4. $Q : \mathfrak{P}\langle 0, 1\rangle \to \langle 0, 1\rangle$

Starting with a $FAT = [S_1, \dots, S_m, \alpha, \kappa, Q]$ we define a functional operator Φ_{FAT} as follows:

- for every $\mu \in \{1, \ldots, m\}$, the $(n+1)$-ary relation S_μ is considered as an interpretation of the fuzzy if-then rule

$$R_\mu : \text{IF} X_1 = F_{\mu 1}, \ldots, \text{and} \quad X_n = F_{\mu n} \, \text{THEN} \, Y = G_\mu,$$

- by using the function α we define the superrelation S where for $x_1 \in U_1, \ldots, x_n \in U_n,$ and $y \in V$

$$S(x_1, \ldots, x_n, y) =_{\text{def}} \alpha(S_1(x_1, \ldots, x_n, y), \ldots, S_n(x_1, \ldots, x_n, y))$$

- by using κ and Q we define for arbitrary arguments $F_1 \in F\mathbb{P}(U_n), \ldots, F_n \in F\mathbb{P}(U_n)$ the value $\Phi(F_1, \ldots, F_n)$ of the operator Φ_{FAT} as follows where $y \in V$:

$$\Phi_{FAT}(F_1, \ldots, F_n)(y) =_{\text{def}} Q \left\{ \kappa(F_1(x_1), \ldots, F_n(x_n), S(x_1, \ldots, x_n, y)) \;\middle|\; \begin{array}{c} x_1 \in U_1 \\ \vdots \\ x_n \in U_n \end{array} \right\}$$

Finally, we define the set $J_{\text{FATI}}(\mathfrak{R})$ of all interpretations $\Phi \in \text{INT}(U_1, \ldots, U_n; V)$ which are selected by the principle FATI as follows

$$J_{\text{FATI}}(\mathfrak{R}) =_{\text{def}} \{\Phi_{FAT} | FAT \in \text{FATI}(\mathfrak{R})\}$$

4.3 The principle FITA

We start with the same rule base $\mathfrak{R}$ as specified in section 4.2. We define $\text{FITA}(\mathfrak{R}) =_{\text{def}}$ the set of all FIT where FIT has the form

$$FIT = [S_1, \ldots, S_m, \kappa_1, \ldots, \kappa_m, Q_1, \ldots, Q_m, \beta]$$

and

1. for every $\mu \in \{1, \ldots, m\}$, $S_\mu : U_1 \times \ldots \times U_n \times V \to \langle 0, 1 \rangle$
2. for every $\mu \in \{1, \ldots, m\}$, $\kappa_\mu : \langle 0, 1 \rangle^{n+1} \to \langle 0, 1 \rangle$
3. for every $\mu \in \{1, \ldots, m\}$, $Q_\mu : \mathfrak{P}\langle 0, 1 \rangle \to \langle 0, 1 \rangle$
4. $\beta : \langle 0, 1 \rangle^m \to \langle 0, 1 \rangle$

Starting with

$$FIT = [S_1, \ldots, S_m, \kappa_1, \ldots, \kappa_m, Q_1, \ldots, Q_m, \beta]$$

we define a functional operator Φ_{FIT} with

$$\Phi_{FIT} : F\mathbb{P}(U_1) \times \cdots \times F\mathbb{P}(U_n) \to F\mathbb{P}(V)$$

as follows:

- as in section 4.2 for every $\mu \in \{1, \ldots, m\}$, the $(n+1)$-ary relation S_μ is considered as an interpretation of the fuzzy if-then rule

$$R_\mu : \text{IF} X_1 = F_{\mu 1}, \ldots, \text{and} \quad X_n = F_{\mu n} \, \text{THEN} \, Y = G_\mu,$$

- using κ_μ and Q_μ we define the logical operator

$$\Phi_\mu(F_1, \ldots, F_n)(y) =_{\text{def}} Q_\mu \left\{ \kappa_\mu(F_1(x_1), \ldots, F_n(x_n), S_\mu(x_1, \ldots, x_n, y)) \;\middle|\; \begin{array}{c} x_1 \in U_1 \\ \vdots \\ x_n \in U_n \end{array} \right\}$$

- using the aggregation function β we define the final result

$$\Phi_{FIT}(F_1, \dots, F_n)(y) =_{\text{def}} \beta(\Phi_1(F_1, \dots, F_n)(y), \dots, \Phi_m(F_1, \dots, F_n)(y))$$

Finaly, we define the set $J_{\text{FITA}}(\mathfrak{R})$ of all interpretations selected by the principle FITA as follows

$$J_{\text{FITA}}(\mathfrak{R}) =_{\text{def}} \{\Phi_{FIT} | FIT \in \text{FITA}(\mathfrak{R})\}$$

We have to mention that the conditions formulated in theorem 3, point 2 and 3, for J are not satisfied by J_{FATI} and J_{FITA}, in general. If we define

$$J^{\star}_{\text{FATI}}(\mathfrak{R}) =_{\text{def}} \bigcup_{\gamma \subseteq \mathfrak{R}} J_{\text{FATI}}(\gamma)$$

and

$$J^{\star}_{\text{FITA}}(\mathfrak{R}) =_{\text{def}} \bigcup_{\gamma \subseteq \mathfrak{R}} J_{\text{FITA}}(\gamma)$$

then $J^{\star}_{\text{FATI}}$ and $J^{\star}_{\text{FITA}}$ fulfill the comonotonicity (see point 3 of theorem 3 and proposition 6).

4.4 Extensional (truth functional) generation of fuzzy relations

We obtain a further restriction of the generating procedures described in section 4.2 for FATI and in section 4.3 for FITA if for each fixed rule base $\mathfrak{R}$ (like in section 4.2 specified) there exists a vector $[\pi_1, \dots, \pi_m]$ of real functions

$$\pi_\mu : \langle 0, 1\rangle^{n+1} \to \langle 0, 1\rangle$$

($\mu \in \{1, \dots, m\}$) and if we define the relations $S_1, \dots, S_n$ used in the definitions of FATI and FITA as follows, where $x_1 \in U_1, \dots, x_n \in U_n, y \in V$

Definition 10
$S_\mu(x_1, \dots, x_n) =_{\text{def}} \pi_\mu(F_{\mu 1}(x_1), \dots, F_{\mu n}(x_n), G_\mu(y))$

This definition means that the membership value $S_\mu(x_1, \dots, x_n, y)$ of the vector $[x_1, \dots, x_n, y]$ only depends on the membership values $F_{\mu 1}(x_1), \dots, F_{\mu n}(x_n), G_\mu(y)$ of the elements $x_1, \dots, x_n, y$, but it is independent of these elements themselves. This fact is often denoted by the word “truth functional” or “extensionel”.

4.5 Methods for restricting models

The methods for restricting interpretations can be partly used for restricting models, of course. Therefore we do not discuss this approach.

In the following we introduce two methods for restricting models which are strictly “model based”, i. e. which can only be applied to restrict models but not to restrict interpretations.

These methods are adopted from nonmonotonic logic, especially from the theory of circumscription, and therefore they are based on the concept of minimal (and maximal, respectively) model.

For formulating the concept of minimal and maximal model we define where F, F' are fuzzy sets on U and Φ, Φ' are interpretations on $[U_1, \dots, U_n; V]$.

Definition 11

1. $F \subseteqq F' =_{\text{def}} \forall x(x \in U \to F(x) \leqq F'(x)$
2. $\Phi \subseteqq \Phi' =_{\text{def}} \forall F_1, \ldots, F_n(F_1 : U_1 \to \langle 0,1\rangle \wedge \cdots \wedge F_n : U_n \to \langle 0,1\rangle$
$\to \Phi(F_1, \ldots, F_n) \subseteqq \Phi'(F_1, \ldots, F_n))$

Definition 12

Φ is said to be a minimal (maximal) model of $\mathfrak{R}$

$=_{\text{def}}$ 1. *Φ is a model of $\mathfrak{R}$ and*
2. *for every interpretation Φ' on $[U_1, \ldots, U_n; V]$, if Φ' is a model of $\mathfrak{R}$ and $\Phi' \subseteqq \Phi$ ($\Phi \subseteqq \Phi'$) then $\Phi = \Phi'$.*

Now, referring to definition 6 let J be an interpretation-selecting operator on $\text{RUL}(U_1, \ldots, U_n; V)$ and $\text{INT}(U_1, \ldots, U_n; V)$.

Definition 13

1. *$\mathfrak{R}$ semantically entails R with respect to J based on minimal (maximal) models (shortly denoted by $\mathfrak{R} \Vdash_J^{min} R$ and $\mathfrak{R} \Vdash_J^{max} R$, respectively)*
$=_{\text{def}}$ *For every $\Phi \in J(\mathfrak{R})$, if Φ is a minimal (maximal) model of $\mathfrak{R}$ then Φ is a model of $\{R\}$.*
2. $\text{ENTMIN}(\mathfrak{R}, J) =_{\text{def}} \left\{R \middle| \mathfrak{R} \Vdash_J^{min} R\right\}$
3. $\text{ENTMAX}(\mathfrak{R}, J) =_{\text{def}} \left\{R \middle| \mathfrak{R} \Vdash_J^{max} R\right\}$

Proposition 7

1. If Φ is an minimal (maximal) model of $\mathfrak{R}$ and $\mathfrak{R}' \subseteqq \mathfrak{R}$ then Φ is a model of $\mathfrak{R}'$, but Φ is not a minimal (maximal) model of $\mathfrak{R}'$, in general.
2. The operators ENTMIN and ENTMAX are not monotone, in general.

Let ET be an arbitrary mapping

$$\text{ET} : \mathbb{P}(\text{RUL}(U_1, \ldots, U_n; V)) \to \mathbb{P}(\text{RUL}(U_1, \ldots, U_n; V))$$

Definition 14

ET *is said to be the cumulatively monotone*

$=_{\text{def}} \forall \mathfrak{R} \forall \mathfrak{R}'(\mathfrak{R} \subseteqq \text{RUL}(U_1, \ldots, U_n; V) \wedge \mathfrak{R}' \subseteqq \text{ET}(\mathfrak{R}) \to \text{ET}(\mathfrak{R}) = \text{ET}(\mathfrak{R} \cup \mathfrak{R}'))$

Problem

Which conditions are sufficient that ET is cumulatively monotone?

5 Incorporating Facts

In this section we solve the question how facts can be incorporated into the concepts developed above.

We choose sets of facts $FACTS_1 \subseteqq \text{FAC}(U_1), \ldots,$ and $FACTS_n \subseteqq \text{FAC}(U_n)$ and define

$$FACTS =_{\text{def}} FACTS_1 \cup \cdots \cup FACTS_n.$$

Then we enlarge the given fuzzy If-Then Rule Base $\mathfrak{R}$ to $\mathfrak{S}$ as follows

$$\mathfrak{S} =_{\text{def}} \mathfrak{R} \cup FACTS$$

The following definition is inspired by the concepts of logic programming and it is fundamental for the further developement of the theory.

Let $Y = G$ be an arbitrary fact from $\mathrm{FAC}(Y)$.

Definition 15

$\mathfrak{S}$ semantically entails $Y = G$ with respect to J

$=_{\text{def}}$ *There exist facts $X_1 = F_1 \in FACTS_1, \ldots, X_n = F_n \in FACTS_n$ such that $\mathfrak{R}$ semantically entails the rule* $\mathrm{IF}\, X_1 = F_1, \ldots, X_n = F_n\, \mathrm{THEN}\, Y = G$ *with respect to J.*

Remark

All further definitions of entailment formulated in the sections 2, 3 and 4 can be enlarged corresponding to definition 15.

6 Concluding remark

The considerations made above have predominate conceptional character. In forthcoming papers a theory will be worked out which consists of more extensive investigations of the logical structure of If-Then Rule Bases using the concepts in the paper presented.

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