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Extended formulations for binary optimal control problems

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Abstract
Extended formulations are an important tool in polyhedral combinatorics. Many combinatorial optimization problems require an exponential number of inequalities when modeled as a linear program in the natural space of variables. However, by adding artificial variables, one can often find a small linear formulation, i.e., one containing a polynomial number of variables and constraints, such that the projection to the original space of variables yields a perfect linear formulation. Motivated by binary optimal control problems with switching constraints, we show that a similar approach can be useful also for optimization problems in function space, in order to model the closed convex hull of feasible controls in a compact way. More specifically, we present small extended formulations for switches with bounded variation and for dwell-time constraints. For general linear switching point constraints, we devise an extended model linearizing the problem, but show that a small extended formulation that is compatible with discretization cannot exist unless P = NP.

Keywords Binary optimal control · Switching time optimization · Convexification · Extended formulations

Mathematics Subject Classification 49M25 · 90C27 · 90C57 · 90C60

1 Introduction

Optimal control problems with discrete-valued control variables are a rather recent topic in infinite-dimensional optimization. The joint consideration of ODE- or PDE-constraints with combinatorial restrictions leads to new challenges and insights both on the optimal control and on the discrete optimization side. In this paper, we focus on the setting where a binary control function varies over a continuous time horizon $[0, T]$ and

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assume that a set of admissible controls U is given, containing all feasible switching patterns $u: [0, T] \to \{0, 1\}$. Moreover, we assume throughout that all $u \in U$ have bounded variation, i.e., that u switches between 0 and 1 only a finite number of times. However, unlike in other approaches, we do not discretize the problem a priori, so that every point in $[0, T]$ remains a potential switching point. A rigorous formulation of these assumptions requires a formal introduction of the function spaces in which we model the set U, for this we refer to Sect. 2.

While many approaches presented in the literature for solving optimal control problems over such a set U are heuristic and depend on a predefined discretization [13–15], we recently proposed a branch-and-bound algorithm for solving such problems to global optimality in function space [5–7]. The core of the latter approach is the understanding of the closed convex hull $\overline{\text{conv}}(U)$ of the feasible set U and its outer description by linear inequalities in function space. Depending on the structure of U, the separation problem for $\overline{\text{conv}}(U)$ may be intractable even for a given discretization. For two classes of constraints, namely for the case of bounded variation and for the so-called minimum dwell-time constraints, we however exploit that the separation problem after discretization can be solved in polynomial time. Both types of constraints have been considered in the literature before. The former constraint just bounds the number of switchings by a constant [13, 14], while the latter requires that two switchings do not follow each other too closely in time [15].

From a combinatorial optimization perspective, this situation suggests a closer look at the complexity of a given class of switching constraints. It turns out, however, that a simple extension of the NP-hardness concept of finite-dimensional optimization to function spaces is not possible, since the complexity of the discretized problem may actually depend on the discretization, or, more precisely, on the number of grid cells used; see Example 6.5. Nevertheless, the tractability of relevant classes of switching constraints in finite dimension paves the way for the transfer of well-studied tools from discrete optimization to the infinite-dimensional setting.

One such tool are extended formulations. Many combinatorial optimization problems require an exponential number of inequalities when modeled as a linear program in the natural space of variables. However, by adding artificial variables, one can often find a small linear formulation, i.e., one containing a polynomial number of variables and constraints, such that the projection to the original space of variables yields the convex hull of the original feasible set; see [9] for a survey containing both examples and abstract results. In the present paper, we argue that the same approach can be useful also for optimization problems in function space. More specifically, our aim is to devise small extended formulations in function space for some relevant types of constraints U, such that the projection to the original space of variables agrees with $\overline{\text{conv}}(U)$; see Sect. 3 for a precise definition.

For the above-mentioned cases of bounded variation and dwell-time constraints, we show that such small extended formulations indeed exist; see Sects. 4 and 5. Both proofs are based on corresponding results on the existence of extended formulations in the finite-dimensional case. Roughly speaking, the extended formulations in function space can be viewed as limits of the finite-dimensional formulations when the number of grid cells goes to infinity. In particular, the formulations are compatible with discretization in a certain sense. The use of these extended formulations within the branch-and-bound algorithm of [7] could be an alternative for the outer approximation algorithm presented in [6], which requires the repeated dynamic separation of violated cutting planes and subsequent re-optimizations. The extended formulations lead to exactly the same dual bounds without the need of any separation loop.

It follows that, for the two mentioned types of constraints, discretization leads to extended formulations in finite dimension that are already known in the literature. However, the advantage of having at hand an extended formulation in function space, as opposed to the first-discretize-then-optimize paradigm, is twofold. Firstly, it shows that "in the limit" the model is consistent, so that artifacts arising only from the discretization are avoided. E.g., it may happen that all discretizations of an optimal control problem admit optimal solutions while the original problem in function space does not. This shows the necessity of having a well-defined model in function space in the first place before considering its discretizations. Secondly, discretization leads to a smaller feasible region and thus only allows to derive a primal bound for the problem in function space. Instead, an extended formulation in function space could, e.g., be dualized in function space and then discretized, so that a safe *dual* bound could be computed, which would be much more valuable within a branch-and-bound algorithm or similar approaches.

The situation is different for general switching point constraints as considered in Sect. 6. Here, we parametrize the binary control by the finitely many switching points and require that these switching points satisfy certain linear constraints; this generalizes the minimum dwell-time constraints. For this class, we show that a small extended formulation that is compatible with discretization cannot exist unless P = NP. This is not only the case for a fixed discretization, but also when the discretization may be refined. This result is essentially obtained by showing that the corresponding sets U after discretization become intractable, i.e., it is NP-hard to optimize a given linear objective function over the latter sets.

2 Function Spaces and Discretization

In order to show our main results on the existence of extended formulations, or even to define the concept of extended formulations in infinite dimension, we first need to introduce some notation concerning appropriate function spaces and discretization. However, we will limit ourselves to essential definitions and observations here. For more details, we refer the reader to the monographs [1, 4] or [2].

2.1 The space L^2 and distributional derivatives

Given an open subset $\Omega \subseteq \mathbb{R}$, we address optimization problems in the reflexive Banach space $L^2(\Omega)$ consisting of all Lebesgue measurable functions $u\colon \Omega \to \mathbb{R}$ such that $|u|^2$ is Lebesgue integrable, equipped with the norm

$$||u||_{L^2(\Omega)} := \Big(\int_\Omega |u(t)|^2 \, \mathrm{d}t \Big)^{1/2} .$$

In $L^2(\Omega)$, two functions are identified if they only differ on a Lebesgue null set, i.e., elements of $L^2(\Omega)$ are formally defined as equivalence classes of functions. In particular, pointwise evaluations are not well-defined for $u \in L^2(\Omega)$, and all constraints on u discussed in the following can only be required almost everywhere (a.e.) in Ω. Nevertheless, as is common, we will frequently specify an element $u \in L^2(\Omega)$ by a pointwise definition of a representative of the equivalence class u.

We write $u_n \to u$ if u_n converges strongly to u in $L^2(\Omega)$, i.e., if $||u - u_n||_{L^2(\Omega)} \to 0$ for $n \to \infty$. For $u \in L^2(\Omega)$, the distributional derivative Du is the linear functional on $\mathcal{C}_c^\infty(\Omega)$ defined by

$$Du(\varphi) := -\int_\Omega u(t)\varphi'(t)\,\mathrm{d}t \quad \forall \varphi \in \mathcal{C}_c^\infty(\Omega)\ ,$$

where $\mathcal{C}_c^\infty(\Omega)$ denotes the set of all smooth functions $\varphi\colon \Omega \to \mathbb{R}$ with compact support. If there exists a function $w \in L^2(\Omega)$ such that

$$Du(\varphi) = \int_\Omega w(t)\varphi(t)\,\mathrm{d}t \quad \forall \varphi \in \mathcal{C}_c^\infty(\Omega)\ ,$$

it is called the weak derivative of u and denoted by u'. If u is a differentiable function in the classical sense, the weak derivative u' agrees with the usual deriative, which follows from integration by parts.

In the following, we will write $Du \geq 0$ if

$$Du(\varphi) \geq 0 \quad \forall \varphi \in \mathcal{C}_c^\infty(\Omega), \varphi \geq 0\ .$$

In this case, the function u is monotonically increasing (outside a null set). The following observation shows that non-negativity of the distributional derivative is a closed condition.

Lemma 2.1 *Let $u_n \in L^2(\Omega)$ with $Du_n \geq 0$ for all $n \in \mathbb{N}$. If u_n converges strongly to u in $L^2(\Omega)$, then $Du \geq 0$.*

Proof Given any fixed $\varphi \in \mathcal{C}_c^\infty(\Omega)$, first note that

$$\left| \int_\Omega u_n(t)(-\varphi'(t))\,\mathrm{d}t - \int_\Omega u(t)(-\varphi'(t))\,\mathrm{d}t \right| \leq ||u_n - u||_{L^2(\Omega)} \cdot ||\varphi'||_{L^2(\Omega)} \to 0$$

for $n \to \infty$ since $||u_n - u||_{L^2(\Omega)} \to 0$ by assumption. If $\varphi \geq 0$, we then have

$$Du(\varphi) = \int_\Omega u(t)(-\varphi'(t))\,\mathrm{d}t = \lim_{n\to\infty} \int_\Omega u_n(t)(-\varphi'(t))\,\mathrm{d}t = \lim_{n\to\infty} Du_n(\varphi) \geq 0\ ,$$

which shows $Du \geq 0$. □

To conclude this subsection, we mention a technical statement that will be helpful in some of the following proofs. Here, we use the Sobolev space $H_0^1(\Omega)$, which can be defined as the closure of $\mathcal{C}_c^\infty(\Omega)$ with respect to the norm

$$||\varphi||_{H^1(\Omega)} := \left(||\varphi||^2_{L^2(\Omega)} + ||\varphi'||^2_{L^2(\Omega)}\right)^{1/2},$$

where φ' denotes the weak derivative of φ defined above. The latter definition, together with the definition of $Du \geq 0$ given above, immediately implies

Lemma 2.2 *Let $u \in L^2(\Omega)$ with $Du \geq 0$ and let $\varphi \in H_0^1(\Omega)$ such that $\varphi \geq 0$. Then we have $-\int_\Omega u(t)\varphi'(t)\,\mathrm{d}t \geq 0$.*

For a detailed introduction to Sobolev spaces, we refer the reader to the literature, eg., [1, 2, 4].

2.2 Functions of bounded variation

Throughout this paper, we will consider functions of bounded variation defined on Ω. For a precise definition, consider the seminorm on $L^2(\Omega)$ given by

$$|u|_{BV(\Omega)} := \sup_{\substack{\varphi \in \mathcal{C}_c^\infty(\Omega) \\ ||\varphi||_\infty \leq 1}} \int_\Omega u(t)\varphi'(t)\,\mathrm{d}t\ .$$

We then define

$$BV(\Omega) := \{u \in L^2(\Omega) \colon |u|_{BV(\Omega)} < \infty\}\ .$$

The distributional derivative Du of a function $u \in BV(\Omega)$ can be represented by a finite signed regular Borel measure on Ω. More formally, for each $u \in BV(\Omega)$ there exists such a measure μ with

$$Du(\varphi) = \int_\Omega \varphi\,\mathrm{d}\mu \quad \forall \varphi \in \mathcal{C}_c^\infty(\Omega)\ ,$$

which we will denote by ∂u in the following. The Jordan decomposition theorem then allows to write $\partial u = (\partial u)^+ - (\partial u)^-$, where $(\partial u)^+$ and $(\partial u)^-$ are non-negative measures on Ω, and this decomposition is unique. Setting $|\partial u| := (\partial u)^+ + (\partial u)^-$ and using [1, Theorem 6.26] we obtain $|u|_{BV(\Omega)} = |\partial u|(\Omega)$, i.e., the variation of u is the total variation of the measure ∂u.

In our proofs, we will make extensive use of $u^+, u^- \in L^2(\Omega)$ defined by

$$u^+(t) := (\partial u)^+\big(\Omega \cap (-\infty, t]\big),\ u^-(t) := (\partial u)^-\big(\Omega \cap (-\infty, t]\big)\ .$$

Since u^+ is Lebesgue measurable and $||u^+||_{L^\infty(\Omega)} \le (\partial u)^+(\Omega)$, we can use Fubini's theorem and obtain

$$\begin{aligned}(Du^+)(\varphi) &= -\int_\Omega u^+(t)\varphi'(t)\,\mathrm{d}t = -\int_\Omega \Big(\int_\Omega \chi_{(-\infty,t]}(\tau)\,\mathrm{d}(\partial u)^+(\tau)\Big)\varphi'(t)\,\mathrm{d}t \\ &= -\int_\Omega \Big(\int_\Omega \underbrace{\chi_{(-\infty,t]}(\tau)}_{=\chi_{[\tau,\infty)}(t)}\varphi'(t)\,\mathrm{d}t\Big)\,\mathrm{d}(\partial u)^+(\tau) = \int_\Omega \varphi(\tau)\,\mathrm{d}(\partial u)^+(\tau)\end{aligned}$$

for all $\varphi \in \mathcal{C}_c^\infty(\Omega)$, so that Du^+ is represented by the measure $(\partial u)^+$ and analogously Du^- is represented by $(\partial u)^-$. Moreover, it follows that $|u^+|_{BV(\Omega)} + |u^-|_{BV(\Omega)} = |u|_{BV(\Omega)}$ and hence $u^+, u^- \in BV(\Omega)$ with $\partial(u^+) = (\partial u)^+ \ge \partial u$ and $\partial(u^-) = (\partial u)^- \ge -\partial u$.
In this paper, we are mostly interested in functions $u \in BV(\Omega)$ with $u(t) \in \{0, 1\}$ for almost all $t \in \Omega$. Up to null sets, these functions can thus be viewed as binary switches that change only a finite number of times in Ω. In our formulations, we will assume moreover that the considered time horizon is $[0, T]$, for $T \in \mathbb{Q}_+$, and that the switch is zero at the beginning of this time horizon. For modeling this, given any $S \subseteq \mathbb{R}$, we introduce the notation

$$BV_\star(0, T; S) := \{u \in BV(-\infty, T) \colon u = 0 \text{ a.e. in } (-\infty, 0),\ u \in S \text{ a.e. in } (0, T)\}\ .$$

Here and in the following, we shortly write that u has some property a.e. in Ω if $u(t)$ has this property for almost all $t \in \Omega$. Now $u \in BV_\star(0, T; S)$ ensures that $\partial u\big((-\infty, 0)\big) = 0$ and

$$|u|_{BV(-\infty,T)} = |\partial u|(\{0\}) + |u|_{BV(0,T)}$$

for all $u \in L^2(-\infty, T)$, i.e., the initial value of u in the time horizon $[0, T]$ is taken into account. Analogously, we will use the notation

$$L^2_\star(0, T; S) := \{u \in L^2(-\infty, T) \colon u = 0 \text{ a.e. in } (-\infty, 0),\ u \in S \text{ a.e. in } (0, T)\}$$

and shortly write $L^2_\star(0, T)$ and $BV_\star(0, T)$ for $L^2_\star(0, T; \mathbb{R})$ and $BV_\star(0, T; \mathbb{R})$, respectively. By these definitions, each function $u \in BV_\star(0, T)$ now has a representative $t \mapsto \partial u\big([0, t]\big)$.

2.3 Discretization and approximation by piecewise constant functions

Since we are mostly interested in binary functions, a natural approach for discretization uses piecewise constant functions. For simplicity, we will concentrate on equidistant grids throughout the paper. Given a number $N \in \mathbb{N}$ of grid cells and $v \in \mathbb{R}^N$, let $\overline{v} \in L^2_\star(0, T)$ be the piecewise constant function defined by

$$\overline{v} = v_i \text{ on } \big((i-1)\tfrac{T}{N}, i\tfrac{T}{N}\big),\ i = 1, \dots, N\ .$$

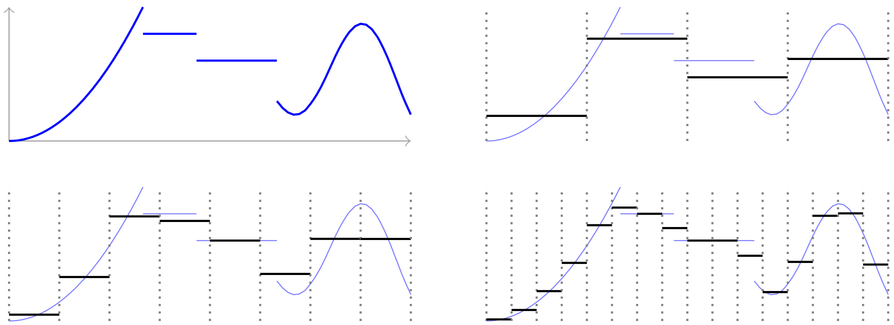

Fig. 1 Approximation of a function of bounded variation by its piecewise averages according to Lemma 2.5: a given function $u \in BV_\star(0,T)$ (upper left) and the piecewise averages $u_{[4]}$, $u_{[8]}$, and $u_{[16]}$

Conversely, given $u \in L^2_\star(0,T)$, let $u_N \in \mathbb{R}^N$ be defined by the piecewise averages

$$(u_N)_i := \frac{N}{T}\int_{(i-1)\frac{T}{N}}^{i\frac{T}{N}} u(t)\,\mathrm{d}t$$

for $i = 1,\dots,N$. Clearly, we then have $(\overline{v})_N = v$ for all $v \in \mathbb{R}^N$. For the following, we further introduce the notation $u_{[N]} := \overline{(u_N)} \in L^2_\star(0,T)$ for $u \in L^2_\star(0,T)$, i.e., $u_{[N]}$ arises from u by replacing its function values by their piecewise averages on the intervals $\big((i-1)\frac{T}{N}, i\frac{T}{N}\big)$.

Definition 2.3 Let $U \subseteq L^2_\star(0,T)$ and $N \in \mathbb{N}$. Then the *discretization* U_N of U is defined as the subset $U \cap \{\bar{u} \colon u \in \mathbb{R}^N\}$ of U.

By this definition, the discretization U_N consists of piecewise constant functions. However, we will sometimes identify U_N with a subset of $\mathbb{R}^N$, namely via the correspondence between a vector $v \in \mathbb{R}^N$ and the piecewise constant function $\bar{v}$.

Lemma 2.4 *For every* $U \subseteq L^2_\star(0,T)$*, we have* $\operatorname{conv}(U_N) \subseteq \operatorname{conv}(U)_N$.

Proof Let $u \in \operatorname{conv}(U_N) \subseteq L^2_\star(0,T)_N$ and choose $\lambda_i \geq 0$ and $u^i \in U_N \subseteq U$ for $i = 1,\dots,k$ such that $\sum_{i=1}^k \lambda_i = 1$ and $u = \sum_{i=1}^k \lambda_i u^i$. Then $u \in \operatorname{conv}(U)$ and hence $u \in \operatorname{conv}(U)_N$. □

Note that $\operatorname{conv}(U_N) \neq \operatorname{conv}(U)_N$ in general. For a simple example, let $U = \{\chi_{[0,1]}, \chi_{[1,2]}\}$ with $T = 2$, where χ_A denotes the characteristic function of $A \subseteq \mathbb{R}$. Then $U_N = \emptyset$ for all odd $N \in \mathbb{N}$, since each $u \in U_N$ must be constant on $[\frac{N-1}{N}, \frac{N+1}{N}]$, but then $u \notin U$. On the other hand, we have $\frac{1}{2}\chi_{[0,2]} \in \operatorname{conv}(U)_N$ for all N.
The following observation is a central ingredient in our proofs presented in the subsequent sections. It shows that a function $u \in L^2(0,T)$ can be approximated by its piecewise averages provided that u has bounded variation; see Fig. 1 for an illustration.

Lemma 2.5 *Let* $u \in BV_\star(0,T)$*. Then the piecewise averages* $u_{[N]}$ *strongly converge to* u *in* $L^2_\star(0,T)$ *for* $N \to \infty$.

Proof Given $N \in \mathbb{N}$, we first claim that $||u - (u_N)_i||_{L^\infty(I_i)} \leq |u|_{BV(I_i)}$ for all $i = 1, \dots, N$, where we set $I_i := \big((i-1)\frac{T}{N}, i\frac{T}{N}\big)$. Indeed, for almost all $t \in I_i$ we have

$$\begin{aligned}
|u(t) - (u_N)_i| &= \tfrac{N}{T}\Big|\int_{(i-1)\frac{T}{N}}^{t} \underbrace{u(t) - u(\tau)}_{=\partial u((\tau,t])}\,\mathrm{d}\tau + \int_t^{i\frac{T}{N}} \underbrace{u(t) - u(\tau)}_{=-\partial u((t,\tau])}\,\mathrm{d}\tau\Big| \\
&\leq \tfrac{N}{T}\Big(\int_{(i-1)\frac{T}{N}}^{t} \underbrace{|\partial u|\big((\tau,t]\big)}_{\leq|\partial u|(I_i)}\,\mathrm{d}\tau + \int_t^{i\frac{T}{N}} \underbrace{|\partial u|\big((t,\tau]\big)}_{\leq|\partial u|(I_i)}\,\mathrm{d}\tau\Big) \\
&\leq |\partial u|(I_i) = |u|_{BV(I_i)},
\end{aligned}$$

where $\partial u\big((t_1, t_2]\big) = \partial u\big([0, t_2]\big) - \partial u\big(([0, t_1]\big) = u(t_2) - u(t_1)$ for almost all $t_1, t_2 \in I_i$, $t_1 < t_2$, using the representative $t \mapsto \partial u\big([0, t]\big)$ of u mentioned above. We now derive

$$\begin{aligned}
||u - u_{[N]}||^2_{L^2(0,T)} &= \int_0^T |u(t) - u_{[N]}(t)|^2\,\mathrm{d}t \\
&= \sum_{i=1}^N \int_{I_i} |u(t) - (u_N)_i|^2\,\mathrm{d}t \\
&\leq \sum_{i=1}^N \tfrac{T}{N}||u - (u_N)_i||^2_{L^\infty(I_i)} \\
&\leq \sum_{i=1}^N \tfrac{T}{N}|u|^2_{BV(I_i)} \;\leq\; \tfrac{T}{N}|u|^2_{BV(-\infty,T)},
\end{aligned}$$

and the latter expression converges to zero for $N \to \infty$ since $|u|_{BV(-\infty,T)}$ is finite. The first inequality in this chain follows from Hölder's inequality, the second from the claim shown above, and the last one from the fact that the intervals I_i form a partition of $(0, T)$. □

We finally note that, if $Du \geq 0$, the monotonicity of u is also reflected in the discretization:

Lemma 2.6 *Let $u \in L^2_\star(0, T)$ with $Du \geq 0$. Then $(u_N)_N \geq (u_N)_{N-1} \geq \cdots \geq (u_N)_1 \geq 0$.*

Proof For $i = 0, \dots, N$, define $\varphi_i : \mathbb{R} \to \mathbb{R}$ by

$$\varphi_i(t) = \begin{cases} 0 & \text{for } t \in (-\infty, (i-1)\frac{T}{N}]\,, \\ t - (i-1)\frac{T}{N} & \text{for } t \in [(i-1)\frac{T}{N}, i\frac{T}{N}]\,, \\ \frac{T}{N} & \text{for } t \in [i\frac{T}{N}, \infty)\,. \end{cases}$$

Then, for $i = 1, \dots, N$, the function $\varphi_{i-1} - \varphi_i$ is non-negative and belongs to $H^1_0(-\infty, T)$, since it vanishes at $-\frac{T}{N}$ and T. Using $Du \geq 0$, Lemma 2.2 now yields

$$\int_{-\infty}^{T} (\varphi_i'(t) - \varphi_{i-1}'(t))u(t)\,\mathrm{d}t \geq 0\,.$$

This implies

$$(u_N)_i = \tfrac{N}{T} \int_{-\infty}^{T} \varphi_i'(t)u(t)\,\mathrm{d}t \geq \tfrac{N}{T} \int_{-\infty}^{N} \varphi_{i-1}'(t)u(t)\,\mathrm{d}t$$

and the latter integral is $(u_N)_{i-1}$ for $i = 2, \ldots, N$ and zero for $i = 1$. □

3 Extended formulations

Our aim is to devise extended formulations for binary switching constraints on a continuous time horizon $[0, T]$. More precisely, we investigate specific subsets $U \subseteq BV_\star(0, T; \{0, 1\})$ that are bounded in $BV_\star(0, T; \{0, 1\})$, meaning that there exists a uniform bound $\sigma \in \mathbb{Q}_+$ such that $|u|_{BV(-\infty,T)} \leq \sigma$ for all $u \in U$. Using the lower semicontinuity of $|\cdot|_{BV(-\infty,T)}$, one can show that this assumption guarantees that $\overline{\mathrm{conv}}(U)$ is still contained in $BV_\star(0, T)$. Here and in the following, all closures are taken in $L^2_\star(0, T)$.

Definition 3.1 An *extended formulation* of $U \subseteq BV_\star(0, T)$ is a set $U^{\mathrm{ext}} \subseteq L^2_\star(0, T)^{d+1}$ satisfying the following conditions:

(e1) The projection of U^{ext} to the first coordinate agrees with $\overline{\mathrm{conv}}(U)$.
(e2) The formulation is *linear*, i.e., the elements $(u, z_1, \ldots, z_d) \in U^{\mathrm{ext}}$ can be characterized by finitely many constraints, each of which is either of the form

$$L(u, z_1, \ldots, z_d) \geq 0 \text{ a.e. in } (0, T)$$

or of the form

$$DL(u, z_1, \ldots, z_d) \geq 0$$

for a continuous affine-linear operator $L\colon L^2_\star(0, T)^{d+1} \to L^2_\star(0, T)$.

The extended formulation is called *small* if the number of controls $d+1$ as well as the number of constraints describing U^{ext} according to (e2) are polynomial in the input size.

Herein, the projection of U^{ext} to the first coordinate is defined as the set

$$\{u \in L^2_\star(0, T)\colon \exists z_1, \ldots, z_d \in L^2_\star(0, T) \text{ s.t. } (u, z_1, \ldots, z_d) \in U^{\mathrm{ext}}\}\,.$$

Polynomiality in the input size is a very common requirement in combinatorial optimization. Nonetheless, in the infinite-dimensional setting, it calls for some clarification. Obviously, the concept is only well-defined when the sets U under consideration belong to a class that can be finitely parametrized, i.e., an individual instance U

from the class is fully determined by a finite input. To emphasize this, all input parameters except for T will be part of the notation for all problem classes investigated in the subsequent sections.

In finite dimension, extended formulations of polynomial size are usually called "compact". However, in our context, this term would be ambiguous, since it could also refer to the compactness of the set U^{ext} in the Banach space $L^2_\star(0,T)^{d+1}$. For this reason, we use the notion of smallness instead of compactness throughout this paper.

The next result shows that every extended formulation is closed in $L^2_\star(0,T)^{d+1}$. Moreover, the projection in Definition 3.1 is always closed if the extended formulation is bounded.

Lemma 3.2 *Every extended formulation U^{ext} is convex and closed in $L^2_\star(0,T)^{d+1}$. Moreover, if U^{ext} is bounded in $L^2_\star(0,T)^{d+1}$, then the projection of U^{ext} to the first coordinate is convex and closed in $L^2_\star(0,T)$.*

Proof Clearly, all constraints described in Condition (e2) of Definition 3.1 are convex. To show closedness, let $L\colon L^2_\star(0,T)^{d+1} \to L^2_\star(0,T)$ be any continuous and affine-linear operator. Then constraints of type $L(u, z_1, \ldots, z_d) \geq 0$ are obviously closed, while constraints of type $DL(u, z_1, \ldots, z_d) \geq 0$ are closed by Lemma 2.1.

Now assume that U^{ext} is bounded in $L^2_\star(0,T)^{d+1}$ and let U' be the projection of U^{ext} to the first coordinate. Then clearly U' is convex. To show closedness, let u_n be a sequence in U' that strongly converges to some $u \in L^2_\star(0,T)$. For $n \in \mathbb{N}$, choose $(z_1)_n, \ldots, (z_d)_n$ in $L^2_\star(0,T)$ with $(u_n, (z_1)_n, \ldots, (z_d)_n) \in U^{\text{ext}}$. Since U^{ext} is bounded in the reflexive Banach space $L^2_\star(0,T)^{d+1}$, there exists a subsequence of $(u_n, (z_1)_n, \ldots, (z_d)_n)$ converging weakly to some $(u', z_1, \ldots, z_d) \in U^{\text{ext}}$. But u_n converges strongly to u by assumption, hence we must have $u' = u$, so that u belongs to U'. Thus U' is closed in $L^2_\star(0,T)$. □

In finite dimension, the main relevance of small extended formulations stems from the fact that they provide efficient and elegant approaches to solving combinatorial optimization problems, by reformulating them as polynomial-size linear programs. In order to derive a similar result in function space, it is necessary that the extended formulations carry over to the discretization of the problem. This is formalized in the following definition.

Definition 3.3 An extended formulation U^{ext} of U is *compatible with discretization* if for all $N \in \mathbb{N}$ the following conditions hold:

(c1) the projection of $(U^{\text{ext}})_N$ to the first coordinate agrees with $\text{conv}(U_N)$,
(c2) the affine-linear operators L defining U^{ext} map $L^2_\star(0,T)^{d+1}_N$ to $L^2_\star(0,T)_N$, and
(c3) the coefficients of the restrictions $L|_{L^2_\star(0,T)^{d+1}_N}$ can be computed in polynomial time.

Note that the restrictions mentioned in (c3) map $L^2_\star(0,T)^{d+1}_N$ to $L^2_\star(0,T)_N$ according to (c2), so that they can be identified with affine-linear functions $\mathbb{R}^{(d+1)N} \to \mathbb{R}^N$, which are defined by polynomially many coefficients.

Using Definition 3.3, we now have

Theorem 3.4 *Assume that U admits a small extended formulation U^{ext} that is compatible with discretization. Then, for fixed $N \in \mathbb{N}$, the discretization U_N of U is tractable, i.e., for any $c \in L^2_\star(0,T)$ given by its piecewise averages c_N, the linear objective function*

$$\int_0^T c(t)u(t)\,\mathrm{d}t \tag{3.1}$$

can be minimized over $u \in U_N$ in polynomial time.

Proof Given $(u, z_1, \dots, z_d) \in L^2_\star(0,T)_N$, we have $(u, z_1, \dots, z_d) \in (U^{\text{ext}})_N$ if and only if the constraints in Condition (e2) of Definition 3.1 are satisfied by $(u, z_1, \dots, z_n)$. By Condition (c1), minimizing (3.1) over $u \in U_N$ or, equivalently, over $u \in \text{conv}(U_N)$, thus reduces to minimizing the same linear function over $(u, z_1, \dots, z_d) \in (U^{\text{ext}})_N$. Identifying elements of $L^2_\star(0,T)_N$ with vectors in $\mathbb{R}^N$, the latter in turn reduces to the minimization of $c_N^\top u$ subject to constraints of type $L(\overline{u}, \overline{z_1}, \dots, \overline{z_d}) \geq 0$ and $DL(\overline{u}, \overline{z_1}, \dots, \overline{z_d}) \geq 0$, where the variables are $u, z_1, \dots, z_d \in \mathbb{R}^N$. By the smallness of the extended formulation and by (c2), the latter constraints translate to polynomially many affine-linear constraints in $(u, z_1, \dots, z_d)$, the coefficients of which can be computed efficiently by Condition (c3). In summary, the minimization problem reduces to the solution of a polynomial-size linear program in dimension $(d+1)N$. □

Note that the running time for minimization over U_N obtained in the above proof is only pseudopolynomial in N, as N enters the dimension of the resulting linear program.
Formally, the set $(U^{\text{ext}})_N$ is *not* an extended formulation of U_N when interpreting elements of U_N and $(U^{\text{ext}})_N$ as piecewise constant functions rather than finite-dimensional vectors, as in Definition 2.3. Indeed, the linear description derived from U^{ext} does not guarantee that the functions are piecewise constant, i.e., it does not completely describe $(U^{\text{ext}})_N$.
Theorem 3.4 states that small and compatible extended formulations immediately lead to tractable discretizations, independent of the chosen number N of grid cells. However, depending on the context, it may be enough from a practical point of view to obtain a tractable discretization after a suitable refinement of the grid. As we will see in the following, it may well happen that a discretization turns tractable only after increasing the number of grid cells. For this reason, we also consider the following weaker condition:

Definition 3.5 An extended formulation U^{ext} of U is *weakly compatible with discretization* if for all $M \in \mathbb{N}$ one can efficiently compute some $\ell \in \mathbb{N}$ which is polynomial in M such that Conditions (c1) to (c3) of Definition 3.3 hold for $N = \ell M$.

To clarify the polynomiality condition in this definition, it is required that ℓ can be computed in polynomial time from the input of the problem, i.e., the input defining the instance U within the given class of feasible sets, and from M. Moreover, the value of ℓ must be polynomial in the value of M. This is crucial since the dimension of the linear programs resulting from discretization is linear in the number of grid cells. For a fixed problem input, the size of the linear program thus increases only polynomially by the refinement.

If U admits a small and weakly compatible extended formulation, it follows directly from Theorem 3.4 that for any $M \in \mathbb{N}$ one can efficiently compute some $\ell \in \mathbb{N}$ polynomial in M such that $U_{\ell M}$ is tractable. It depends on the given setting which of the two concepts of compatibility is appropriate: if the discretization can be arbitrarily refined by the user, a weakly compatible extended formulation is enough to obtain tractable discretizations.

4 Fixed bound on the variation

Our first goal is to devise an extended formulation for the case where a fixed bound on the variation is the only restriction on the set of feasible controls; this case has been investigated in [5, 13, 14]. We thus consider the set

$$U(\sigma) := \{u \in BV_\star(0, T; \{0, 1\}) \colon |u|_{BV(-\infty,T)} \leq \sigma\}$$

for given $\sigma \in \mathbb{N}$. A natural convex relaxation of $U(\sigma)$ arises from omitting the binarity constraint on the control, we then obtain

$$U(\sigma)^{\text{rel}} := \left\{u \in BV_\star(0, T; [0, 1]) \colon \ |u|_{BV(-\infty,T)} \leq \sigma\right\} .$$

Note that the set $U(\sigma)^{\text{rel}}$ is convex and, by the lower semicontinuity of $|\cdot|_{BV(0,T)}$ and the boundedness of $U(\sigma)^{\text{rel}}$ in $BV(0, T)$, also closed in $L^2_\star(0, T)$. We now show that $U(\sigma)^{\text{rel}}$ can be obtained as the projection of a very small extended formulation, where a single additional function z keeps track of the accumulated variation of u.

Theorem 4.1 *The set*

$$U(\sigma)^{\text{rel,ext}} := \left\{u \in L^2_\star(0, T; [0, 1]),\ z \in L^2_\star(0, T; [0, \sigma]) \colon Dz \geq Du,\ Dz \geq -Du\right\}$$

is a small extended formulation of $U(\sigma)^{\text{rel}}$ which is compatible with discretization.

Proof In the format of Definition 3.1, the constraints in the above formulation explicitly read $u \geq 0$, $-u + \mathbf{1} \geq 0$, $z \geq 0$, $-z + \boldsymbol{\sigma} \geq 0$, $D(z - u) \geq 0$, and $D(z + u) \geq 0$, where $\mathbf{1}$ and $\boldsymbol{\sigma}$ denote the functions being constantly 1 and σ, respectively. Condition (e2) of Definition 3.1 as well as smallness are thus obvious, so it suffices to show Condition (e1) and compatibility with discretization.

We first show Condition (e1). Given $(u, z) \in U(\sigma)^{\text{rel,ext}}$, we observe that z is essentially bounded by σ and monotonically increasing by $Dz \geq 0$. Thus $z \in BV_\star(0, T)$, such that Dz can be represented by a regular Borel measure ∂z satisfying $\partial z\big((-\infty, T)\big) \leq \sigma$. Now for every $\varphi \in \mathcal{C}^\infty_c(-\infty, T)$, we have $\varphi_{\max}, \varphi_{\min} \in H^1_0(-\infty, T)$ for $\varphi_{\max} := \max\{\varphi, 0\}$ and $\varphi_{\min} := -\min\{\varphi, 0\}$ and the weak derivatives satisfy $\varphi' = \varphi'_{\max} - \varphi'_{\min}$. Using $Dz \geq Du$ and $Dz \geq -Du$ together with

Lemma 2.2, we derive

$$\begin{aligned}\int_{-\infty}^{T} \varphi'(t)u(t)\,\mathrm{d}t &= -\int_{-\infty}^{T} \varphi'_{\max}(t)(-u(t))\,\mathrm{d}t - \int_{-\infty}^{T} \varphi'_{\min}(t)u(t)\,\mathrm{d}t \\ &\leq -\int_{-\infty}^{T} \varphi'_{\max}(t)z(t)\,\mathrm{d}t - \int_{-\infty}^{T} \varphi'_{\min}(t)z(t))\,\mathrm{d}t \\ &= \int_{-\infty}^{T} \varphi_{\max}\,\mathrm{d}\partial z + \int_{-\infty}^{T} \varphi_{\min}\,\mathrm{d}\partial z \\ &= \int_{-\infty}^{T} |\varphi|\,\mathrm{d}\partial z \;\leq\; \sigma\,||\varphi||_{\infty}.\end{aligned}$$

Since φ was arbitrary, this implies $u \in BV_\star(0, T)$ with $|u|_{BV(-\infty,T)} \leq \sigma$, thus $u \in U(\sigma)^{\mathrm{rel}}$. For the other direction, let $u \in U(\sigma)^{\mathrm{rel}}$ and define $z \in L^2_\star(0, T)$ by $z(t) := |\partial u|\big((-\infty, t)\big)$ for $t \in [0, T]$. Then clearly $(u, z) \in U(\sigma)^{\mathrm{rel,ext}}$ and hence u belongs to the projection of $U(\sigma)^{\mathrm{rel,ext}}$ to the first coordinate.
To show compatibility with discretization, first consider Condition (c1) of Definition 3.3 and let $u \in (U(\sigma)^{\mathrm{rel}})_N \subseteq U(\sigma)^{\mathrm{rel}}$. Defining $z(t) = |\partial u|\big((-\infty, t)\big)$ again, we obtain that z is piecewise constant on the same grid as u. Thus $z \in L^2_\star(0, T)_N$ and $(u, z) \in (U(\sigma)^{\mathrm{rel,ext}})_N$, hence u belongs to the projection of $(U(\sigma)^{\mathrm{rel,ext}})_N$ to the first component. For the reverse inclusion, let u belong to the latter projection. Choose $z \in L^2_\star(0, T)_N$ with $(u, z) \in (U(\sigma)^{\mathrm{rel,ext}})_N$. Then $u \in U(\sigma)^{\mathrm{rel}}$ follows from Condition (e1) already shown above and $u \in L^2_\star(0, T)_N$ follows from the choice of u, thus $u \in U(\sigma)^{\mathrm{rel}}_N$. This shows (c1).
Finally, the remaining conditions (c2) and (c3) of Definition 3.3 are obviously satisfied: given $N \in \mathbb{N}$, the discretized constraints read $u_i \geq 0, -u_i + 1 \geq 0, z_i \geq 0, -z_i + \sigma \geq 0$, $(z_i - u_i) - (z_{i-1} - u_{i-1}) \geq 0$, and $(z_i + u_i) - (z_{i-1} + u_{i-1}) \geq 0$ for $i = 1, \dots, N$. □

By Theorem 4.1, the projection of $U(\sigma)^{\mathrm{rel,ext}}$ to the first coordinate yields a relaxation for the set $U(\sigma)$. However, it can be shown that $\overline{\mathrm{conv}}(U(\sigma))$ is *strictly* contained in $U(\sigma)^{\mathrm{rel}}$ in general; see [5, Counterexample 3.1]. In other words, the formulation in Theorem 4.1 is *not* an extended formulation for $U(\sigma)$ in general. Our next goal is thus to devise an extended formulation that instead projects to $\overline{\mathrm{conv}}(U(\sigma))$.
We construct such a formulation by first considering the discretized problem, for which an extended formulation has been presented in [8, Section 4.5]: assuming that σ is even, the projection of

$$\begin{aligned}(U(\sigma)_N)^{\mathrm{ext}} := \big\{&u \in [0, 1]^N,\ z \in \mathbb{R}^N : \\ &u_i - u_{i-1} \leq z_i - z_{i-1} \quad \text{for } i = 1, \dots, N, \\ &0 \leq z_1 \leq z_2 \leq \dots \leq z_N \leq \tfrac{\sigma}{2}\big\}\end{aligned}$$

to the u-space coincides with $\mathrm{conv}(U(\sigma)_N)$, where again we identify piecewise constant functions with finite-dimensional vectors. This follows from [8, Lemma 4.29] using the substitution $z_i \leftarrow \sum_{j=1}^{i} z_j$. In this model, we define $z_0 = u_0 = 0$, but do

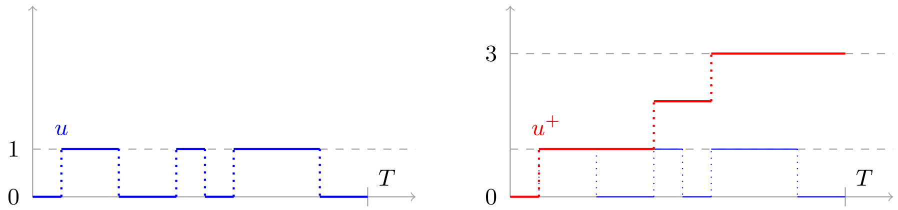

Fig. 2 Illustration of the extended formulation $U(\sigma)^{\text{ext}}$ for $\sigma = 6$. A feasible control u is shown on the left hand side, the control $z := u^+$ is added on the right hand side

not eliminate these variables for sake of a simpler formulation. The first constraint thus reduces to $u_1 \leq z_1$.

We now turn our attention back to the infinite-dimensional setting and show that $\overline{\text{conv}}(U(\sigma))$ can be described by an extended formulation that is inspired by the formulation $(U(\sigma)_N)^{\text{ext}}$. For this, define

$$U(\sigma)^{\text{ext}} := \left\{u \in L^2_\star(0,T;[0,1]),\ z \in L^2_\star(0,T;[0,\tfrac{\sigma}{2}]) \colon Dz \geq Du,\ Dz \geq 0\right\}.$$

For an illustration of this construction, see Fig. 2. Using this, we now obtain an extended formulation of $U(\sigma)$.

Theorem 4.2 *Let σ be even. Then $U(\sigma)^{\text{ext}}$ is a small extended formulation of $U(\sigma)$ which is compatible with discretization.*

Proof Again, Condition (e2) of Definition 3.1 and smallness are easily verified, so it suffices to show Condition (e1) as well as compatibility. First assume that $(u,z) \in U(\sigma)^{\text{ext}}$. By Lemma 2.5, we then have $(u_{[N]}, z_{[N]}) \to (u,z)$ for $N \to \infty$. By construction, it follows that $u_N \in [0,1]^N$ and $z_N \in [0,\frac{\sigma}{2}]^N$ for all $N \in \mathbb{N}$. Moreover, applying Lemma 2.6 to both z and $z-u$ yields $(z_N)_i - (z_N)_{i-1} \geq 0$ and $(z_N)_i - (z_N)_{i-1} \geq (u_N)_i - (u_N)_{i-1}$. In summary, we thus have $(u_N, z_N) \in (U(\sigma)_N)^{\text{ext}}$. Hence $u_N \in \text{conv}(U(\sigma)_N)$ and thus $u_{[N]} \in \text{conv}(U(\sigma))$, using the finite-dimensional extended formulation. This shows $u \in \overline{\text{conv}}(U(\sigma))$.

Now let $u \in \overline{\text{conv}}(U(\sigma))$. For showing that u belongs to the projection of $U(\sigma)^{\text{ext}}$ to the first coordinate, we may assume $u \in U(\sigma)$ by Lemma 3.2. Define $z := u^+$. Then $z \in L^2_\star(0,T)$ with $Dz \geq Du$ and $Dz \geq 0$, thus also $z \geq 0$ almost everywhere. Moreover,

$$\partial u^+\big([0,T)\big) + \partial u^-\big([0,T)\big) \leq |u|_{BV(-\infty,T)} \leq \sigma$$

and, since $u \leq 1$ almost everywhere, we have

$$\partial u^+\big([0,T)\big) - \partial u^-\big([0,T)\big) = \partial u\big([0,T)\big) \leq 1\,.$$

Summing up, we derive $\partial z\big([0,T)\big) = \partial u^+\big([0,T)\big) \leq \frac{\sigma+1}{2}$. Since $\partial z\big([0,T)\big)$ is integer and σ is even, this implies $\partial z\big([0,T)\big) \leq \frac{\sigma}{2}$ and hence $z \leq \frac{\sigma}{2}$ almost everywhere.

Finally, Condition (c1) of Definition 3.3 follows directly from the fact that $(U(\sigma)^{\text{ext}})_N$ corresponds to the finite-dimensional extended formulation $(U(\sigma)_N)^{\text{ext}}$ of $U(\sigma)_N$, while (c2) and (c3) are again obvious. □

The preceding proof is a blueprint for similar results on extended formulations in function space, provided that an extended formulation for the discretized problem is known. The compatibility with discretization follows easily in this situation. The first inclusion of (e1) is shown by approximating an element of the supposed extended formulation by piecewise constant functions according to Lemma 2.5 and then using the finite-dimensional result, while for the other inclusion an explicit construction of $z_1, \dots, z_d$ from u is proposed.
Note that the exact extended formulation $U(\sigma)^{\text{ext}}$ of $U(\sigma)$ is defined by the same number of constraints as the extended formulation $U(\sigma)^{\text{ext,rel}}$ of the relaxation $U(\sigma)^{\text{rel}}$. The difference between the two formulations is that the exact formulation only counts the positive variation and bounds it by $\frac{\sigma}{2}$, while the relaxed formulation bounds the total variation by σ.
In the case of odd σ, one can use [8, Lemma 4.30] to show that an extended formulation of $U(\sigma)$ is given by

$$U(\sigma)^{\text{ext}} := \left\{u \in L^2_\star(0,T;[0,1]),\ z \in L^2_\star(0,T;[0,\tfrac{\sigma-1}{2}]) \colon Dz \geq -Du,\ Dz \geq 0\right\}.$$

The proof is very similar to the proof of Theorem 4.2. Instead of bounding the positive variation by $\frac{\sigma}{2}$, as in the case of even σ, we now bound the negative variation by $\frac{\sigma-1}{2}$.

5 Minimum dwell-time constraints

We next consider the important class of minimum dwell-time constraints, which are also called min-up/min-down constraints in the literature; see [3, 11, 12] for the finite-dimensional and [15] for the infinite-dimensional case. Given $L, l \in \mathbb{Q}_+$ with $L+l > 0$ as input, the feasible set $U(L,l)$ now consists of all $u \in BV_\star(0,T;\{0,1\})$ such that, when switching up u at time τ, no switching down occurs in $[\tau, \tau+L)$, and analogously, when switching down u at τ, no switching up occurs in $[\tau, \tau+l)$. Rajan and Takriti [12] have devised the following extended formulation for $U(L,l)_N$ for the case $L, l \in \mathbb{N}$, where we have applied the same substitution of z-variables as in the previous section:

$$\begin{aligned}
(U(L,l)_N)^{\text{ext}} := \big\{u \in [0,1]^N,\ z \in \mathbb{R}^N_+ \colon &\\
z_i - z_{i-L} &\leq u_i && \text{for } i = L+1, \dots, N,\\
z_i - z_{i-l} &\leq 1 - u_{i-l} && \text{for } i = l+1, \dots, N,\\
z_i - z_{i-1} &\geq u_i - u_{i-1} && \text{for } i = 1, \dots, N,\\
z_i - z_{i-1} &\geq 0 && \text{for } i = 1, \dots, N\big\}.
\end{aligned}$$

Here z_0 and u_0 are again defined as zero. In order to exploit this model for general $L, l \in \mathbb{Q}$, the number of grid cells N must be chosen such that $NL/T, Nl/T \in \mathbb{N}$, as we will see in the proof of Theorem 5.1 below (which in practice represents a fairly severe restriction).

To obtain an extended formulation for the infinite-dimensional set $\overline{\text{conv}}(U(L,l))$, we first define a continuous linear map

$$V_r\colon L^2_\star(0,T) \to L^2_\star(0,T)$$

for given $r \in \mathbb{R}_+$ by $V_r(z)(t) = z(t-r)$, i.e., V_r shifts z to the right by r. Note that, by the definition of $L^2_\star(0,T)$, this implies $V_r(z) = 0$ a.e. in $(-\infty, r)$. Now consider the following formulation, inspired by the extended formulation of Rajan and Takriti:

$$\begin{aligned} U(L,l)^{\text{ext}} := \big\{ & u \in L^2_\star(0,T;[0,1]),\ z \in L^2_\star(0,T;[0,\sigma])\colon \\ & z - V_L(z) \le u, && \text{(5.1a)} \\ & z - V_l(z) \le \mathbf{1} - V_l(u), && \text{(5.1b)} \\ & Dz \ge Du,\ Dz \ge 0 \big\}\,. && \text{(5.1c)} \end{aligned}$$

Here we define $\sigma := \lceil 2T/L+l \rceil$. This value of σ is large enough to make the upper bound on z redundant, it is only used to ensure that $U(L,l)^{\text{ext}}$ is bounded in $L^2_\star(0,T)^2$.

Theorem 5.1 *The set $U(L,l)^{\text{ext}}$ is a small extended formulation of $U(L,l)$ which is weakly compatible with discretization.*

Proof Again, Condition (e2) of Definition 3.1 as well as smallness are easily verified, so it suffices to show Condition (e1) and weak compatibility. So let $(u,z) \in U(L,l)^{\text{ext}}$. We again have

$$(u_{[N]}, z_{[N]}) \to (u,z) \text{ for } N \to \infty$$

by Lemma 2.5. Now choose $\ell \in \mathbb{N}$ such that $\ell L/T$ and $\ell l/T$ are both integer and consider the subsequence given by indices $N_k := k\ell$, $k \in \mathbb{N}$. We claim that

$$(u_{N_k}, z_{N_k}) \in (U(N_k L/T, N_k l/T)_{N_k})^{\text{ext}} \text{ for all } k \in \mathbb{N}\,. \tag{5.2}$$

First note that $u_{N_k} \in [0,1]^{N_k}$ and $z_{N_k} \in \mathbb{R}^{N_k}_+$ by construction. Moreover, Lemma 2.6 again yields $(z_{N_k})_i - (z_{N_k})_{i-1} \ge (u_{N_k})_i - (u_{N_k})_{i-1}$ and $(z_{N_k})_i - (z_{N_k})_{i-1} \ge 0$ using (5.1c). Finally,

$$\begin{aligned} (z_{N_k})_i - (z_{N_k})_{i-N_k L/T} &= \frac{N_k}{T}\int_{(i-1)\frac{T}{N_k}}^{i\frac{T}{N_k}} z(t)\,\mathrm{d}t - \frac{N_k}{T}\int_{(i-N_k L/T-1)\frac{T}{N_k}}^{(i-N_k L/T)\frac{T}{N_k}} z(t)\,\mathrm{d}t \\ &= \frac{N_k}{T}\int_{(i-1)\frac{T}{N_k}}^{i\frac{T}{N_k}} z(t)\,\mathrm{d}t - \frac{N_k}{T}\int_{(i-1)\frac{T}{N_k}}^{i\frac{T}{N_k}} V_L(z)(t)\,\mathrm{d}t \\ &= \frac{N_k}{T}\int_{(i-1)\frac{T}{N_k}}^{i\frac{T}{N_k}} \big(z(t) - V_L(z)(t)\big)\,\mathrm{d}t \\ &\underset{(5.1a)}{\le} \frac{N_k}{T}\int_{(i-1)\frac{T}{N_k}}^{i\frac{T}{N_k}} u(t)\,\mathrm{d}t = (u_{N_k})_i\,, \end{aligned}$$

and the remaining constraint can be shown analogously using (5.1b). This concludes the proof of (5.2). The result of Rajan and Takriti now shows $u_{N_k} \in \operatorname{conv}(U(N_k L/T, N_k l/T)_{N_k})$ for all $k \in \mathbb{N}$, yielding $u_{[N_k]} \in \operatorname{conv}(U(L,l))$, which implies $u \in \overline{\operatorname{conv}}(U(L,l))$.
For showing the other inclusion, we may assume $u \in U(L,l)$ by Lemma 3.2. Define $z := u^+$. Then we have $Dz \geq Du$ and $Dz \geq 0$. Moreover, the linear inequality (5.1a) follows from the definition of $U(L,l)$ then. Indeed, we obtain $z - V_L(z) = u^+ - V_L(u^+) \in \{0,1\}$ a.e. from the definition of $U(L,l)$, since u can switch up at most once in any time interval of length L. If $u^+ - V_L(u^+) = 1$ a.e. in some interval (τ_1, τ_2), we derive that $\partial u^+(\{\tau\}) = 1$ for some $\tau \in (\tau_2 - L, \tau_1)$. From the definition of $U(L,l)$, we obtain $u = 1$ a.e. in (τ_1, τ_2), showing (5.1a). The constraint (5.1b) follows by a similar reasoning. Thus (u,z) satisfies all conditions of $U(L,l)^{\text{ext}}$, which shows that u belongs to the projection of $U(L,l)^{\text{ext}}$ to the first coordinate.
From the proof so far, it follows that the extended formulation $U(L,l)^{\text{ext}}$ is compatible with discretization provided that N is a multiple of ℓ. A feasible value for ℓ can be computed efficiently from the rational numbers L, l, and T, e.g., by multiplying the numerator of T with the denominators of L and l. We thus obtain that the extended formulation is weakly compatible with discretization. □

As seen in the proof of Theorem 5.1, the refinement factor ℓ required to obtain a compatible discretization does not depend on N, but on the input consisting of L, l, and T. Even when fixing N, the number of necessary grid cells thus grows by a factor that is polynomial in these numbers. In terms of L, l, and T, we thus only obtain a pseudopolynomial algorithm for solving the refined discretizations. However, at least for the extended formulation $U(L,l)^{\text{ext}}$ considered here, this is unavoidable, since even Condition (c2) of Definition 3.3 is not satisfied unless T/N is an integer multiple of L and l.

6 Linear switching point constraints

The minimum dwell-time constraints considered in the previous section form a special case of linear switching point constraints. Since we assume U to be bounded in $BV_\star(0,T;\{0,1\})$, each function $u \in U$ is defined by its finitely many switching points $t_1, \ldots, t_\sigma$, i.e., the points where the value of u changes; see [5, Section 3.2] for a formal definition. We assume again that u starts being switched off and consider the set $U(A,b) \subseteq BV_\star(0,T;\{0,1\})$ consisting of all functions u such that its switching points satisfy given linear inequalities $At \leq b$ for $A \in \mathbb{Q}^{m\times\sigma}$ and $b \in \mathbb{Q}^m$. More precisely, taking into account that functions in $L^2_\star(0,T)$ are only defined up to null sets, we let u belong to $U(A,b)$ if any only if there exists any representative of u with switching points $t_1, \ldots, t_\sigma$ satisfying $At \leq b$.
For simplicity, we assume that the constraints $At \leq b$ imply $0 \leq t_1 \leq \cdots \leq t_\sigma \leq T$, so that u switches up at t_i for odd i and down for even i. However, we do not require that any of the inequalities $0 \leq t_1 \leq \cdots \leq t_\sigma \leq T$ be strict. It is thus allowed to switch up immediately at zero, to switch multiple times at the same time point (so that the switchings neutralize each other), or to leave some of the switchings to time point T

(so that they become irrelevant). In particular, while the switching points uniquely determine $u \in BV_\star(0,T;\{0,1\})$, the vector of switching points belonging to some given $u \in BV_\star(0,T;\{0,1\})$ may not be unique.

6.1 Linearization

Different from the cases considered in the previous sections, there is no obvious linear formulation of $U(A,b)$ in the original space, due to the non-linear connection between the values of the function u and the switching points of u. The following model describing $U(A,b)$ by the use of additional controls can thus be seen as a linearization of $U(A,b)$:

$$
\begin{aligned}
U(A,b)^{\text{lin}} := \Big\{ & u \in L^2_\star(0,T;\{0,1\}),\ z^{(1)},\dots,z^{(\sigma)} \in L^2_\star(0,T;[0,1]): \\
& Dz^{(i)} \geq 0 \qquad \text{for all } i = 1,\dots,\sigma, && \text{(6.1a)}\\
& z^{(i)} \leq z^{(i-1)} \quad \text{for all } i = 2,\dots,\sigma, && \text{(6.1b)}\\
& u = \sum_{i=1}^{\sigma} (-1)^{i+1} z^{(i)}, && \text{(6.1c)}\\
& \sum_{i=1}^{\sigma} a_{ji} \int_0^T (1 - z^{(i)}(t))\,\mathrm{d}t \leq b_j \text{ for all } j = 1,\dots,m \Big\}. && \text{(6.1d)}
\end{aligned}
$$

Note that the constraints (6.1d) fit into the framework of Condition (e2) of Definition 3.1 by considering the left and right hand side as constant functions. However, this formulation contains binarity constraints on u. Still, we can show

Theorem 6.1 *The projection of $U(A,b)^{\text{lin}}$ to the first coordinate agrees with $U(A,b)$.*

Proof Given $u \in U(A,b)$, let $t_1,\dots,t_\sigma$ be a vector of switching points of u satisfying $At \leq b$. Define $z^{(i)} := \chi_{(t_i,T)}$ for all $i = 1,\dots,\sigma$. Then $z^{(1)},\dots,z^{(\sigma)} \in BV_\star(0,T;[0,1])$ and (6.1a), (6.1b), as well as (6.1c) are obviously satisfied. Moreover, we obtain

$$\textstyle\int_0^T (1 - z^{(i)}(t))\,\mathrm{d}t = t_i$$

for all $i = 1,\dots,\sigma$ by construction, so that (6.1d) reduces to $At \leq b$. In summary, we thus have $(u, z^{(1)},\dots,z^{(\sigma)}) \in U(A,b)^{\text{lin}}$.
For showing the other direction, assume that $(u, z^{(1)},\dots,z^{(\sigma)}) \in U(A,b)^{\text{lin}}$. First note that constraint (6.1a), together with $z^{(i)} \in [0,1]$ a.e. in $(-\infty,T)$, implies $|z^{(i)}|_{BV(-\infty,T)} \leq 1$, so that $|u|_{BV(-\infty,T)} \leq \sigma$ by (6.1c). From the binarity of u, it follows that u has finitely many switching points $0 \leq t_1 < \cdots < t_r < T$ with $r \leq \sigma$. Define $t_i := T$ for $i = r+1,\dots,\sigma$. It remains to show that $At \leq b$.
For this, we first show by induction that $z^{(i)} = 0$ a.e. in $(-\infty, t_i)$ for $i = \sigma,\dots,1$. If i is even, we have $z^{(j)} = 0$ a.e. in (t_{i-1}, t_i) for $j = i+1,\dots,\sigma$ by the induction

hypothesis. So from (6.1c) and (6.1b) we obtain

$$1 = u = \underbrace{z^{(1)}}_{\leq 1} \underbrace{-z^{(2)} + z^{(3)}}_{\leq 0} - \cdots - \underbrace{z^{(i-2)} + z^{(i-1)}}_{\leq 0} - z^{(i)} \leq 1 - z^{(i)} \text{ a.e. in } (t_{i-1}, t_i)$$

and hence $z^{(i)} = 0$ a.e. in (t_{i-1}, t_i). Using (6.1a), this implies $z^{(i)} = 0$ a.e. in $(-\infty, t_i)$. Now let i be odd. Then

$$0 = u = \underbrace{z^{(1)} - z^{(2)}}_{\geq 0} + \cdots + \underbrace{z^{(i-2)} - z^{(i-1)}}_{\geq 0} + z^{(i)} \geq z^{(i)} \text{ a.e. in } (t_{i-1}, t_i)$$

and hence $z^{(i)} = 0$ a.e. in (t_{i-1}, t_i), which again implies $z^{(i)} = 0$ a.e. in $(-\infty, t_i)$ by (6.1a).

We next show $z^{(i)} = \chi_{(t_i,T)}$ inductively for $i = 1, \ldots, \sigma$. First, let i be odd. By (6.1c), we have $1 = u = z^{(i)}$ a.e. in (t_i, t_{i+1}), since by the induction hypothesis $z^{(j)} = 1$ for $j < i$ and we have already shown $z^{(j)} = 0$ for $j > i$. Thus $z^{(i)} = \chi_{(t_i,T)}$. Similarly, for even i we obtain $0 = u = 1 - z^{(i)}$ on (t_i, t_{i+1}) and hence again $z^{(i)} = 1$ a.e. in (t_{i-1}, t_i), showing again that $z^{(i)} = \chi_{(t_i,T)}$.

In summary, we have $\int_0^T (1 - z^{(i)}(t))\,\mathrm{d}t = t_i$ for all $i = 1, \ldots, \sigma$, so that (6.1d) implies $At \leq b$ and hence $u \in U(A, b)$. □

A closer look at this proof reveals that the main difficulty was to derive the binarity of the auxiliary controls $z^{(1)}, \ldots, z^{(\sigma)}$, yielding $z^{(i)} = \chi_{(t_i,T)}$. If binarity of $z^{(1)}, \ldots, z^{(\sigma)}$ is required directly in the model, then the above result follows much more easily.
In terms of Definition 3.1, the set $U(A, b)^{\text{lin,rel}}$, resulting from replacing $\{0, 1\}$ by $[0, 1]$ in $U(A, b)^{\text{lin}}$, satisfies Condition (e2). Moreover, the resulting model is small, assuming that A and b are part of the problem input, and compatible with discretization, since each $u \in U(A, b)_N$ can have switching points only in $\{0, \frac{T}{N}, \ldots, (N-1)\frac{T}{N}\}$, so that the construction in the proof of Theorem 6.1 yields $z^{(1)}, \ldots, z^{(\sigma)} \in L^2_\star(0, T)_N$. Hence, an obvious question is whether $U(A, b)^{\text{lin,rel}}$ is an extended formulation for $\overline{\text{conv}}(U(A, b))$, i.e., also satisfies Condition (e1). The answer to this question is negative. In fact, even for the special case of minimum dwell-time constraints considered in the previous section, the model (6.1) does not yield a complete description.

Example 6.2 To see this, consider the minimum dwell-time instance defined by $T = 4$, $L = 2$, and $l = 0$, so that (6.1d) essentially reduces to $\int_0^T (z^{(1)}(t) - z^{(2)}(t))\,\mathrm{d}t \geq 2$. Let $u = z^{(1)} - z^{(2)}$ with $z^{(1)} = \chi_{(1,T)}$ and $z^{(2)} = \frac{1}{2}\chi_{(2,T)}$. Then $(u, z^{(1)}, z^{(2)})$ satisfies all constraints in (6.1) except for the binarity of u, but $u \notin \overline{\text{conv}}(U(L, l))$, since all elements of the latter set satisfy $u(2\frac{1}{4}) \geq u(1\frac{1}{4}) - u(\frac{3}{4})$. □

However, it is possible to model the constraints $z - V_L(z) \leq u$ and $z - V_l(z) \leq \mathbf{1} - V_l(u)$ of the extended formulation $U(L, l)^{\text{lin}}$ in the variable space of (6.1): using $z = \sum_{i \text{ odd}} z^{(i)}$ and hence $z - u = \sum_{i \text{ even}} z^{(i)}$, we obtain

$$\sum_{i \text{ even}} z^{(i)} \leq \sum_{i \text{ odd}} V_L(z^{(i)}), \quad \sum_{i \text{ odd}} z^{(i)} \leq \mathbf{1} + \sum_{i \text{ even}} V_l(z^{(i)})$$

as another extended formulation for the case of dwell-time constraints.
In the following section, we try to convince the reader that a small extended formulation which is compatible with discretization most likely does not exist for general linear switching point constraints, even in the weak sense of Definition 3.5.

6.2 Negative results

We now show that a small and compatible extended formulation cannot exists for general linear switching point constraints unless P = NP. Using Theorem 3.4, it suffices to show that it is NP-complete to decide whether $U(A, b)_N \neq \emptyset$ for given A and b. For all hardness proofs, we use reductions from the following elementary decision problem:

(BPF) Given $B \in \mathbb{Q}^{m\times n}$ and $d \in \mathbb{Q}^m$, does there exist some $x \in \{0, 1\}^n$ with $Bx \leq d$?

It is well-known that (BPF) is NP-complete [10]. We first show

Theorem 6.3 *Assume that $T \in \mathbb{Q}_+$, $\sigma, m \in \mathbb{N}$, $A \in \mathbb{Q}^{m\times\sigma}$, $b \in \mathbb{Q}^m$, and $N \in \mathbb{N}$ are part of the input. Then it is NP-complete to decide whether $U(A, b)_N \neq \emptyset$.*

Proof We show the statement by reduction from (BPF). For this, we set $T = n$ and $\sigma = 2n$. For all $i = 1, \dots, n$, we add the switching point constraints $t_{2i-1} = i-1$ and $i - 1 \leq t_{2i} \leq i$. In words, the control u switches up at 0, then down again between 0 and 1, up again at 1, and so on. So far, all these switchings are independent. We will model the variable x_i by $t_{2i} - (i - 1) \in [0, 1]$. Substituting all x_i in $Bx \leq d$ by these expressions, we obtain another set of linear constraints in t, which we add to the switching point constraints. This concludes the construction of A and b, which can obviously be done in polynomial time.
Now let $N = n$. Then all switching points of u belong to $\{0, \dots, n\}$, hence $t_{2i} - (i - 1) \in \{0, 1\}$ and $t_{2i} - (i - 1) = u_i$. Thus, by construction, the given instance of (BPF) has a feasible solution if and only if $U(A, b)_N \neq \emptyset$. Clearly, the problem of deciding whether $U(A, b)_N$ is non-empty belongs to NP, the certificate being an element of $U(A, b)_N$. □

Now using Theorem 6.3 and Theorem 3.4, considering, e.g, the objective function $c = \mathbf{0}$, we immediately obtain

Corollary 6.4 *A small extended formulation of $U(A, b)$ that is compatible with discretization does not exist unless P = NP.*

The proof of Theorem 6.3 relies on the possibility to choose the grid size N in the reduction, i.e., on the fact that N is part of the input. When keeping the same instance but considering finer grids, it is no longer true that u must be constant between $i - 1$ and i. Indeed, assuming that N is a multiple of T, the function u can switch at any point $(i - 1) + j\frac{T}{N}$ for $j = 0, \dots, \frac{N}{T}$. Thus $t_{2i} - (i - 1)$ is no longer binary, but can take any value in $\{j\frac{T}{N} : j = 0, \dots, \frac{N}{T}\}$. Hence, for large enough N, any vertex of the polytope $\{x \in [0, 1]^n : Bx \leq d\}$ can be represented by $x_i = t_{2i} - (i - 1)$ since A and b

are rational, so that the decision whether $U(A,b)_N \neq \emptyset$ reduces to deciding feasibility of a linear program, which can be done in polynomial time. In other words, Theorem 6.3 does not rule out the existence of a small extended formulation for $U(A,b)$ that is only *weakly* compatible with discretization.

Example 6.5 It can happen that different discretizations of the same problem are tractable or NP-hard depending on the choice of N, and the two situations may even alternate. As an example, consider the following fractional version of the vertex cover problem: given a simple graph $G=(V,E)$, $\gamma \in \mathbb{N}$ and $K \in \mathbb{Q}$, decide whether there exists a solution $x \in \mathbb{R}^V$ of

$$\begin{aligned} \textstyle\sum_{v\in V} x_v &\le K \\ x_v + x_w &\ge 1 \qquad \forall (v,w) \in E \\ x_v &\in [0,1] \;\forall v \in V \end{aligned}$$

such that all entries of x are integer multiples of $1/\gamma$. Theorem 6.3 shows that this problem can be reduced to deciding whether $U(A,b)_N = \emptyset$ with $N = \gamma n$, for appropriate A and b. Since the vertex cover polytope is half-integral, meaning that all vertices have entries being multiples of $1/2$, the above problem reduces to a linear program for even γ. For the problem constructed in the proof of Theorem 6.3, it can thus be decided in polynomial time whether $U(A,b)_N \neq \emptyset$ whenever N is an even multiple of n. However, the same decision problem turns NP-complete when N is an odd multiple of n. For this, it suffices to show that the above fractional vertex cover problem is NP-complete for all odd values of γ, which is done in Appendix A. □

However, by adding an objective function to the problem, we can show that even a weakly compatible extended formulation for $U(A,b)$ cannot exist in general.

Theorem 6.6 *A small extended formulation for $U(A,b)$ that is weakly compatible with discretization cannot exists unless P = NP.*

Proof Consider the problem

$$\left.\begin{aligned} \min \quad & \int_0^T c(t)u(t)\,\mathrm{d}t \\ \text{s.t.} \quad & u \in U(A,b) \end{aligned}\right\} \tag{6.2}$$

for the function $c \in L^2(0,T)$ defined by $c(t) = \frac{1}{2} - (t - \lfloor t \rfloor)$. Then c_N can be computed efficiently for each N and the discretized problem reads

$$\left.\begin{aligned} \min \quad & \tfrac{1}{N}\sum_{i=1}^{N} (c_N)_i u_i \\ \text{s.t.} \quad & u \in \mathbb{R}^N,\ \overline{u} \in U_N(A,b)\ . \end{aligned}\right\} \tag{6.3}$$

It suffices to show that (BPF) can be polynomially reduced to Problem (6.3) for $N = \ell(n)n$ whenever $\ell(n)$ is polynomial in n. Indeed, under the assumption that a small and weakly compatible extended formulation exists, this yields an efficient algorithm for deciding (BPF) as follows: First, choose $M = n$ and efficiently compute some ℓ as in Definition 3.3. Since ℓ is required to be polynomial in M, we would then have that (BPF) can be polynomially reduced to Problem (6.3) for $N = \ell M$, and by the compatibility assumption, Problem (6.3) can be solved in polynomial time for this N. This implies that (BPF) can be solved in polynomial time and hence P = NP.

In order to construct the desired polynomial reduction, let an instance of (BPF) be given by $B \in \mathbb{Q}^{m\times n}$ and $d \in \mathbb{Q}^m$. We define T, σ, and the switching point constraints exactly as in the first part of the proof of Theorem 6.3. We now claim that the given instance of (BPF) is feasible if and only if the constructed instance of Problem (6.3) for $N = \ell(n)n$ has an optimal value of zero. For this, the objective value of $u \in \mathbb{R}^N$ with $\overline{u} \in U_N(A, b)$ can be computed as follows: for $i = 1, \dots, n$, we have

$$\begin{aligned}\sum_{j=1}^{\ell(n)} (c_N)_{(i-1)\ell(n)+j} u_{(i-1)\ell(n)+j} &= \sum_{j=1}^{(t_{2i}-(i-1))\ell(n)} (c_N)_{(i-1)\ell(n)+j} \\ &= \int_{i-1}^{t_{2i}} c(t)\,\mathrm{d}t \\ &= \int_{0}^{t_{2i}-(i-1)} (\tfrac{1}{2} - t)\,\mathrm{d}t \\ &= \tfrac{1}{2}\big(t_{2i} - (i-1)\big)\big(1 - (t_{2i} - (i-1))\big),\end{aligned}$$

so that

$$\begin{aligned}c_N^\top u &= \sum_{i=1}^{n} \sum_{j=1}^{\ell(n)} (c_N)_{(i-1)\ell(n)+j} u_{(i-1)\ell(n)+j} \\ &= \frac{1}{2} \sum_{i=1}^{n} \big(t_{2i} - (i-1)\big)\big(1 - (t_{2i} - (i-1))\big).\end{aligned}$$

The latter expression is always non-negative, since $t_{2i} - (i-1) \in [0, 1]$ for all $i = 1, \dots, n$. It follows that all $u \in U(A, b)_N$ have a non-negative objective value in the constructed instance, and the objective value is zero if and only if $t_{2i} - (i-1) \in \{0, 1\}$ for all $i = 1, \dots, n$. This concludes the proof. □

A closer look at the proof of Theorem 6.6 reveals that the difficulty of linear optimization over $U(A, b)_N$ does *not* stem from the binarity of the switch u, but from the non-convex relation between the switching points of u and the value of u at a given point $t \in [0, T]$. This however does not rule out the existence of small and compatible extended formulations in special cases, as shown by Theorem 5.1.

A Fractional vertex cover

We claim that for odd $\gamma \in \mathbb{N}$, the following decision problem is NP-complete: given a simple graph $G = (V, E)$ and $K \in \mathbb{Q}$, decide whether there exists a solution $x \in \mathbb{Q}^V$ of

$$\left.\begin{aligned} \textstyle\sum_{v\in V} x_v &\leq K \\ x_v + x_w &\geq 1 \qquad \forall (v, w) \in E \\ x_v &\in [0,1]\ \forall v \in V \\ x_v &\in \tfrac{1}{\gamma}\mathbb{Z} \quad\ \forall v \in V\ . \end{aligned}\right\} \tag{γ-VC}$$

To show this claim, we reduce the NP-complete decision variant of the (ordinary) vertex cover problem to the above problem. So given an instance of the vertex cover problem, i.e., a graph $G' = (V', E')$ and $K' \in \mathbb{N}$, we construct $G = (V, E)$ from G' as follows: for each $v \in V'$, we add three new vertices $v^{(1)}, v^{(2)}, v^{(3)}$ and four new edges

$$(v, v^{(1)}),\ (v^{(1)}, v^{(2)}),\ (v^{(2)}, v^{(3)}),\ (v^{(3)}, v^{(1)})\ .$$

In words, we add a triangle T_v for each vertex $v \in V'$ and connect it to v by a bridge. We now claim that G' has a vertex cover of size at most K' if and only if (γ-VC) has a solution for G and $K := \frac{1}{\gamma}K' + 2|V'|$.

First assume that $S \subseteq V'$ is a vertex cover of G' with $|S| \leq K'$. Then the following vector is a solution of (γ-VC): set $x_v = \frac{\gamma+1}{2\gamma}$ for $v \in S \cup \{v^{(1)}, v^{(2)} : v \in V'\}$ and $x_v = \frac{\gamma-1}{2\gamma}$ otherwise. Indeed, it is easy to verify that x satisfies the covering constraints. For the cardinality constraint, we have

$$\sum_{v\in V} x_v = \tfrac{\gamma+1}{2\gamma}(|S| + 2|V'|) + \tfrac{\gamma-1}{2\gamma}(|V'| - |S| + |V'|) = \tfrac{1}{\gamma}|S| + 2|V'| \leq K\ .$$

For the other direction, let $x \in \mathbb{Q}^V$ solve (γ-VC). We may assume $x_{v^{(1)}} = \frac{\gamma+1}{2\gamma}$ for all $v \in V'$. Indeed, using a smaller value does not allow to decrease the costs of covering T_v while contributing less to cover $(v, v^{(1)})$. On the other hand, using a larger value increases the costs of covering T_v by at least as much as it would cost to increase x_v instead. So we can assume $x_v \geq \frac{\gamma-1}{2\gamma}$ for all $v \in V$ now. In particular, it suffices to choose $x_v \leq \frac{\gamma+1}{2\gamma}$ to cover all edges in E, hence

$$x_v \in \{\tfrac{\gamma-1}{2\gamma}, \tfrac{\gamma+1}{2\gamma}\} \quad \forall v \in V'$$

without loss of generality. Then it is easy to verify that the set $S := \{v \in V : x_v = \frac{\gamma+1}{2\gamma}\}$ is a vertex cover of G' of size at most K'. □

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