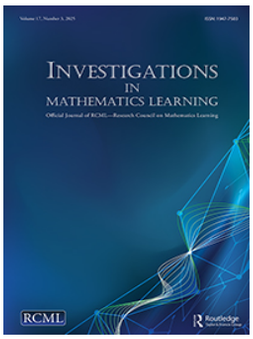

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The role of strategy keys in enhancing heuristics and self-regulation in mathematical problem-solving: A qualitative, explorative, and type-building study with primary school students

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ABSTRACT

Mathematical problem-solving is a 21st-century skill and is demanded in curricula globally. It is a highly complex competence that needs to be supported in the learning process. A common material to differentiate and support within (mathematics) lessons is aid cards. Yet, it remains unclear how students work with such cards qualitatively. In this qualitative, explorative, and type-building study, so-called strategy keys (eight general strategic cards with heuristics on them) have been developed to investigate how primary school students work with strategy keys when solving mathematical problems without prior training and to what extent this supports or hinders their problem-solving process. 16 primary school students were interviewed and videotaped while working on non-routine mathematical problems. The data were coded as product-oriented regarding success and as process-oriented regarding the types of episodes, observable heuristics, and interactions with the strategy keys. Qualitative analyses illustrate four types of processes in which students work with strategy keys. Type 1 and 2 do not benefit from key interactions, while type 3 and 4 benefit from them by showing new heuristics and/or a change of episodes. Implications for future studies and educators are made.

KEYWORDS

Aid cards; heuristics; primary school; problem solving; self-regulation

Introduction

Critical thinking, collaboration, communication, creativity, and problem-solving are core skills in the 21st century (Szabo et al., 2020; Trilling & Fadel, 2009). Mathematical problem-solving is understood as a non-routine problem in which an individual encounters a barrier when working on it (Dörner, 1979; Pólya, 1945; Rott, 2013; Schoenfeld, 1992). It is a central competence in German and many international curricula for secondary (KMK, 2022b; NCTM, 2000) and primary school students (KMK, 2022a; MoE Ministry of Education Singapore, 2012). Research has shown that students may find it challenging to work on and solve mathematical problems (Groß, 2013; Klang et al., 2021; Sturm, 2019). This is due to multiple challenges, inter alia difficulties in understanding a problem (Sinaga et al., 2023), in self-regulation (Dignath et al., 2008), and the choice, appliance, and appropriate use of heuristics (Kaitera & Harmoinen, 2022; Sturm et al., 2015). Thus, supporting the students' use of general problem-solving heuristics is one possible way (Groß et al., 2015; Klang et al., 2021) – also for primary school students (Dignath et al., 2008) that can be realized by using aid cards (also called prompts).

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Aid cards, often paper cards with written or visual hints on them, are common materials for differentiating in mathematics (Eichholz & Selter, 2018) and other subjects (Koenen & Emden, 2016; Schmidt & Weidig, 2010), at various ages and levels of achievement (Pozas et al., 2020; Russo et al., 2020). They are among others used to support learning processes (Kovács et al., 2022) or to work out new content (Elsner, 2010). With the use of aid cards, teachers aim to support their heterogeneous students and help them learn to work independently and autonomously (Russo et al., 2020). Although aid cards are widely distributed, we do not know much about them from a research perspective. We know that aid cards can support students in learning skills in general (Braun, 2020; Koenen & Emden, 2016; Kovács et al., 2022) and that students are willing to work with them (Russo et al., 2020). However, it is understudied how students work with such cards from a qualitative perspective and how it supports or hinders their learning processes.

In the context of this study, eight general, strategic problem-solving aid cards were developed. Those are called *strategy keys* – cards in the shape of keys naming one problem-solving heuristic for each key (Figure 1): make a table, draw a picture, find an example, start with a small number, use different colors, work backward, read the task again, and look for a pattern or rule (Herold-Blasius, 2017).

This qualitative, explorative, and type-building study aims to explore self-regulation and the use of heuristics in mathematical problem-solving by investigating the following research questions: How do primary school students use strategy keys when solving mathematical problems without prior instruction, and if they use those keys, which way do they support (or hinder) the students' mathematical problem-solving process (e.g., regarding heuristics or self-regulatory activities)?

Theoretical Background

Mathematical Problem-Solving

To clarify the term problem-solving, a mathematical problem is a non-routine task comprising three components: an undesired starting condition, a desired end condition, and a barrier that hinders the transformation from the starting to the end condition (Dörner, 1979; Pólya, 1945; Rott, 2013, 2014; Schoenfeld, 1985, 1992). When someone works on a mathematical problem and encounters a barrier, it is necessary to combine existing knowledge with heuristics to overcome this barrier (Bruder & Collet, 2011; Rott, 2013, 2014).

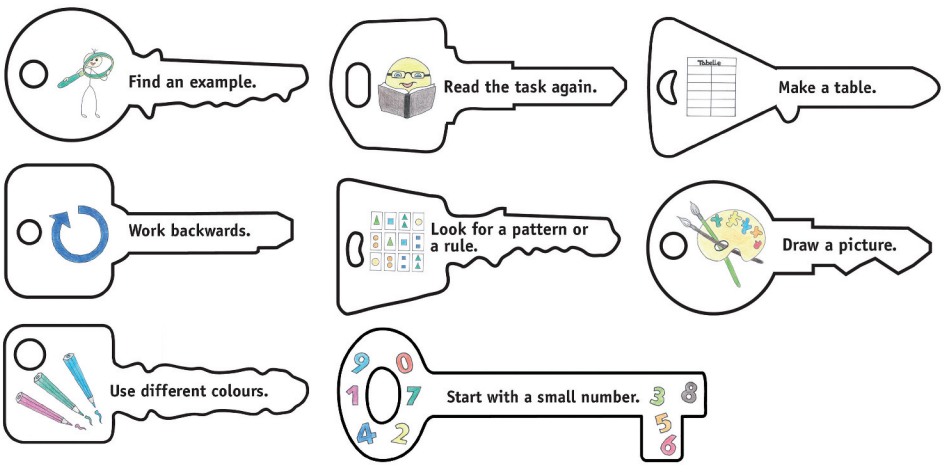


Figure 1. Overview of the eight strategy keys used in this study.

To work on problems successfully and investigate mathematical problem-solving, Schoenfeld (1985), pp. 12–14; 1992) found a set of four necessary components:

(1) Resources: "a foundation of basic mathematical knowledge"
(2) Heuristics: "general problem-solving techniques"
(3) Control: "general issues of how one selects and deploys the resources at one's disposal (often referred to as self-regulation)
(4) Belief systems: "perception of [one's own] knowledge, derived from [one's own] experiences with mathematics. That is, their beliefs about mathematics – consciously held or not – establish the psychological context within which they do mathematics."

Primary school students may encounter barriers, such as difficulties in understanding a problem (Salle et al., 2020; Sinaga et al., 2023) or in using appropriate heuristics (Kaitera & Harmoinen, 2022; Sturm, 2019), and thus, experience difficulties when working on mathematical problems. Groß (2013) asked and observed 3rd and 4th graders to solve five mathematical problems. He found that their success varied percentual from 0% to 15% for 3rd graders and from 0% to 30% for 4th graders only.

To improve students' problem-solving success, the use of heuristics and self-regulation is important (Dignath et al., 2008; Fuchs et al., 2003; Kaitera & Harmoinen, 2022; Sturm, 2019) as both can support to overcoming barriers (Bruder & Müller, 1990; Rott, 2013, 2014). For instance, heuristics in the form of visualizations such as drawings or tables may foster the understanding of a problem and support finding a solution (Hembree, 1992; Sturm, 2019). Applying self-regulation was found to be difficult for primary school students (Dignath et al., 2008; Gürtler et al., 2002). In a meta-analysis of self-regulation training programs, Dignath et al. (2008) found that self-regulation trainings proved to be effective. The training in strategy use was revealed to be even more beneficial for primary school students than for older students. Instruction of problem-solving strategies showed the highest effects (Dignath et al., 2008). As both heuristics and self-regulation seem especially promising for successful problem-solving, both will be focused on in this study and be explained in more detail.[1]

Problem-Solving Heuristics

Heuristics are "mental operations typically useful when solving problems" (Pólya, 1945, p. 155). This is a broad understanding of heuristics and includes problem-solving strategies such as "What is (un) known?" or "Draw a figure." Teaching experience with higher secondary school students (grades 11 and 12) showed that the use of heuristics can compensate for a deficit in problem-solving (Bruder & Müller, 1990). From the 1970s until today, significant amounts of heuristic training have been developed, ranging from demonstrating a heuristic to explicit training, and from single observations to weekly training over months (Bruder & Collet, 2011; Koichu et al., 2007; Rott & Gawlick, 2014). Remarkably, all the cited researchers and many others agree that heuristics can be taught. This is also true for primary school students (Dignath et al., 2008; Groß et al., 2015; Sturm, 2019). However, most of these studies showed moderate results. Thus, we are still looking for more efficient ways to teach problem-solving.

Self-Regulation

Self-regulation can be observed with certain activities: "[. . .] assessing one's knowledge, formulating a plan of attack, selecting strategies, and monitoring and evaluating progress" (Goos & Galbraith, 1996, p. 230). In general, self-regulation is described as "an active, constructive process whereby learners set goals for their learning and then attempt to monitor, regulate, and control their cognition, motivation, and behavior, guided and constrained by their goals and contextual features in the

[1]Belief systems are excluded from this study as the students are motivated and interested in mathematics (see "Methodology"). Thus, I assume a positive attitude toward mathematics and problem-solving. Resources will not be addressed explicitly as the problems selected are suitable for this age group, and the required knowledge can be assumed.

environment" (Pintrich, 2000, p. 453). Subsequently, the learner is responsible for setting and pursuing goals. Learning is regulated to achieve this goal. Self-regulation refers to short intervals, such as when solving mathematical problems (Schoenfeld, 1992; Stoppel, 2019). In contrast, self-regulated learning refers to longer periods during which several self-regulations can occur (Stoppel, 2019).

In the context of mathematical problem-solving, self-regulation can be observed and investigated when looking at phases, or more precisely, episodes (Rott, 2013; Schoenfeld, 1992). From his own experience as a mathematician, Pólya (1945) described four phases of problem-solving: 1) Understanding the problem, 2) Devising a plan, 3) Carrying out the plan, and 4) Looking back. This normative model can be employed to better understand the circular and repetitive processes of problem-solving, yet it is not that much to describe it (Rott, 2014). Building on Pòlya's ideas, Schoenfeld (1985) observed university students, worked out verbal protocols, videotaped their problem-solving processes, and identified six episodes within them: 1) reading, 2) analyzing, 3) exploring, 4) planning, 5) implementing, and 6) verifying. Schoenfeld (1985) interprets the change between these episodes as an act of self-regulation. This implies that a lack of self-regulation can be identified using these episodes. The *wild-goose-chase* process is described as: „ [. . .] they [the students] read the problem, picked a particular direction to work on, and pursued that direction until they ran out of time" (Schoenfeld, 1992, p. 191). Thus, a wild-goose-chase process is characterized by the occurrence of two or three different types of episodes: reading, exploring, and possibly analyzing (Rott, 2013; Schoenfeld, 1985, 1992).

Within his thesis, Rott (2013) expanded on Schoenfeld's episodes, identifying episodes that did not develop the solution process content-wise but could occupy longer periods during a problem-solving process, e.g., *writing* down what was said or thought before or an organization that includes drawing sketches or making tables. These types of episodes do not show new thoughts and thus, do not carry any new content.

In summary, mathematical problem-solving is a core competence that (primary school) students should and can learn (Dignath et al., 2008; Sinaga et al., 2023; Sturm, 2019). Yet, this is challenging as it is highly individualized and characterized by obstacles within the problem-solving process. In the next sections, support formats are described and illustrated.

Formats of Support When Solving Problems

Several support formats have been developed worldwide to support mathematical problem-solving. Five formats are described in the following paragraphs.

First, researchers use Pólya's (1945) problem-solving phases as a tool to observe processes and to impart knowledge about these processes. In Singapore, the curriculum is based on the four phases and some leading questions (MoE Ministry of Education Singapore, 2012). Thus, students should be able to monitor their processes and help themselves (Hoong et al., 2014; Lam et al., 2011). Nieuwoudt (2015) developed a similar approach for primary schools which is structured in phases and enriched with cooperative working in groups and the use of different heuristics.

Second, scaffolding is an approach to language learning and describes an intervention of a teaching person to enable a child to solve a problem that it would not have solved without support. During a period, a certain "scaffold" is built up to support a learning process. When a learner becomes more confident and knows what to do, this scaffold can be slowly withdrawn or fade out (Puntambekar & Hübscher, 2005; van de Pol et al., 2010; Wood et al., 1976). Ideally, the learner has now internalized the scaffolds learned before (van de Pol et al., 2010). Within the last decade, this approach has been applied to mathematics, particularly in multilingual contexts (Ge & Land, 2004; Prediger & Pöhler, 2015). Prediger and Pöhler (2015) offered a six-level structural scaffold by introducing a percentage bar when dealing with percentages. Passing through these levels, from informal thinking to the flexible use of concepts and strategies, leads secondary school students to lexical and conceptual knowledge of percentages. In the context of a classroom, the strategy keys could be introduced as a scaffold.

When students have internalized the heuristics written on the keys, they can fade out (Herold-Blasius & Rott, 2021; Herold-Blasius et al., 2023).

Third, various explicit instruction programs have been developed over the past few decades; some focus on heuristics (Bruder & Collet, 2011; Fülöp, 2015), and others address self-regulatory or metacognitive strategies (Kramarski et al., 2002; Mevarech & Kramarski, 1997; Popham et al., 2020). Wilburne and Dause (2017) developed a program to introduce self-regulated learning strategies. They present some evidence that low-achieving fourth-grade students could improve their problem-solving skills by self-regulating their learning, through self-monitoring and goal setting. Generally, self-regulation training programs proved effective, even for primary school students (Dignath et al., 2008).

Fourth, prompts are external stimuli, often within a labor setting, with a computer that provides hints depending on fixed components (e.g., time or a certain action) (Hasselhorn & Gold, 2017). In various (primarily psychological) studies, prompts have been found to function as strategy activators for university students in the context of metacognition and self-regulation (Bannert, 2009; Hübner et al., 2006). This is one reason why prompts have been transferred to school settings (Downton & Sullivan, 2017; Russo et al., 2020). One way of transferring prompts to a school setting can be aid cards – paper cards with written or visual hints on them. These cards can be used for various purposes:

- To stabilize behavior within regular lessons (Wills et al., 2010)
- To work out new content (Braun, 2020; Koenen & Emden, 2016; Kovács et al., 2022)
- To practice known content (Elsner, 2010; Kostka, 2012)
- To support learning processes visually and memorize more easily (Elsner, 2010; Kovács et al., 2022)
- To use as an encyclopedia (Schmidt & Weidig, 2010)
- To find methodological advice (Franke-Braun et al., 2008).

Most are textbooks recommending aid cards or practical articles with school reports. The few studies found show a consistent picture: Aid cards can improve (primary school) students' skills (Braun, 2020; Kovács et al., 2022). Kovács et al. (2022) compared the effectiveness of card use (teaching cards with pictures) to traditional teacher-led approaches or combined methods. They taught Basic Life Support (BLS) to 263 primary school students (aged 6 to 10), finding long-term skill improvement (Kovács et al., 2022). Braun (2020) used task-specific aid cards to improve about 50 primary students' way of dealing with contextual mathematical tasks. More than 50% of the students did not read the aid cards, and only 0,6% to 11% of them found the aid cards helpful (range depends on the task). Braun (2020) explains this by the fact that students expected content-specific aids instead of general-strategical ones. To the best of our knowledge, there is no qualitative study investigating the use of aid cards in the context of mathematical problem-solving with primary school students.

Fifth, strategy keys combine the components of these previous formats. Herold-Blasius et al. (2017) developed eight strategy keys – cards in the shape of keys naming one problem-solving heuristic for each key (Figure 1): make a table, draw a picture, find an example, start with a small number, use different colors, work backwards, read the task again, and look for a pattern or rule (Herold-Blasius, 2017, 2021; Kaitera & Harmoinen, 2022). In a previous data analysis, it was found that strategy keys could trigger heuristics. Moreover, students used the keys in nine different ways that were interpreted as pre-, while-, and post-self-regulatory activities (SR activities) (Herold-Blasius, 2021) (Table 1).

These strategy keys combine the advantages of the aforementioned support formats. First, they are broadly applicable within school, in mathematics and other subjects, and with a wide range of students from primary to secondary school. Second, they trigger heuristics, at least for short-term, and possibly for long-term usage. Third, if the strategy keys are implemented as scaffolds, teachers may introduce one key after the other or a set of two to three keys to build a bunch of keys. As soon as the students internalize the keys, the fading-out process begins. Kaitera and Harmoinen (2022) showed that fifth-grade students improved their skills in explaining their thinking, increased their variety of heuristics, and chose more sophisticated heuristics when solving different tasks. However, we do not know yet

Table 1. List of ways to use strategy keys.

Way to use strategy keys (*kind of* SR activity)	Description
To clarify the problem (*while*-SR activity)	Students use one or more strategy key(s) to better understand the problem
To generate a new strategy (*pre*-SR activity)	After a key interaction, a new heuristic is used. In contrast to the previous strategy, no heuristic was used before the key interaction.
To change the previous strategy (*pre*-SR activity)	The student has applied a heuristic. After a key interaction they choose a new one with no discernible connection to the previous heuristic.
To maintain the previous strategy (*while*-SR activity)	The student has already applied use a heuristic and does not change anything in the previous approach after the key interaction.
To refine the previous strategy (*while*-SR activity)	If a student already uses one heuristic and combines it with another, the former is, in a sense, further developed and refined.
To name the strategy (*pre*- or *post*-SR activity)	There are two scenarios here. First, a heuristic was used and named with the help of strategy keys (post-actional strategy naming). Second, a strategy key was selected to refer to a heuristic even before the action. This heuristic was then executed (pre-reactional strategy naming).
To motivate oneself (*while*-SR)	Students use strategy keys. For example, they did not give up and continued to work on tasks. Here, the strategy keys take over the teacher's motivating function in the short term.
To use as a checklist (*while*-SR activity)	Students check what they have already done and what else could potentially be helpful in the problem-solving process.
To take a break of thinking (*while*-SR activity)	If a student does not know what to do and cannot find a suitable heuristic, the strategy keys offer a break in thinking, for example, by having the student draw a picture. This pause allows a certain distance from the problem without doing nothing, and thus creates new possibilities for working on the problem.

how students work with these cards or exactly how they support their working process. Thus, the following two research questions guided this study:

RQ 1: How do primary school students use strategy keys when solving mathematical problems without prior instruction?

RQ 2: If primary school students use strategy keys, in which way do the keys support (or hinder) their mathematical problem-solving process (e.g., regarding heuristics or self-regulatory activities)?

Methodology

In this study, qualitative video data was collected with primary school students working on mathematical problems. The data was coded based on qualitative content analysis and analyzed by employing a type-building analysis.

Participants and Context of the Study

To answer the research questions, 16 primary school students (aged 7–10 years, grades 3–4, 7 girls and 9 boys) from an after-school club at University Duisburg-Essen (Germany) called "Maths for clever foxes" (Böttinger, 2016) were interviewed and videotaped while solving mathematical problems. None of the students were tested to attend this after-school club or for this study. Hence, no information on their mathematical achievement levels is available.

Data Collection

Each student was videotaped with one camera and asked to think aloud while working independently (Maher & Sigley, 2014; Sandmann, 2014). This method was chosen "as a tool for understanding students' thinking processes" (Pugalee, 2004, p. 27), knowing that it might promote metacognition.

However, all students, independent of their key usage, were asked to think aloud, which may neglect the potential interaction effect. The strategy keys lay visible on the table at all times and were introduced by the interviewer as follows: "I brought very difficult tasks for you today. If you get stuck, these keys may help you." Each key was then read out loud by the interviewer to diminish language misunderstandings. The interviewer was with the child during the interview and, if necessary, reminded the child to think aloud. Whenever the interviewer had difficulty understanding the students' thoughts, she asked clarifying questions, such as: "What are you thinking right now? How do you know? How did you do it?" (Goldin, 2000; Maher & Sigley, 2014). After each problem-solving process, students were asked to describe what they had done and why and were interviewed about their use of keys (e.g. "Did you use a strategy key? If so, how did it help you?").

Once the process and interview were completed, each student could decide whether they wanted to work on another problem. Four students worked on up to four problems (see Table 6 for details on the distribution) within 45–60 minutes. Finally, 41 problem-solving processes of 16 students were videotaped.

Selection of Mathematical Problems

The primary focus of this study was to investigate the use of strategy keys. Thus, students had to encounter barriers while working and possibly use strategies to overcome them. Thus, non-routine problems were chosen by the author and had to fulfill the following four criteria (Herold-Blasius, 2021; Herold-Blasius et al., 2017):

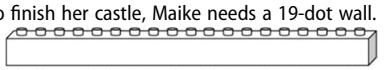
(1) Be curricular appropriate for primary school students (aged 7–11 years)
(2) Be open concerning the number of possible ways to obtain the solution, not necessarily the number of different solutions (Büchter & Leuders, 2011; Schoenfeld, 1985)
(3) Provide low-threshold access for all ability levels and should therefore require as little prior knowledge as possible (Leuders, 2011)
(4) Problems must be solvable by applying the heuristics offered with the strategy keys. At least two of the keys should be potentially helpful.

By analyzing the mathematical problems, promising heuristics, barriers, and, on this basis, potential strategy keys were identified and verified by a group of experts. Six mathematical non-routine problems were selected by applying these criteria (Table 2 and S2 in the supplement). Most problems address arithmetic or combinatorics, and only one geometry. This range was selected to observe not only problem-specific but also general heuristics.

Data Coding and Analysis

Methodologically, this qualitative, explorative, and type-building study (Döring & Bortz, 2016) was carried out in two steps. First, data was coded by applying qualitative content analysis with inductive and deductive categories (Mayring, 2000). Each problem-solving process was coded in four ways to describe it in as much detail as possible: 1) episodes within the process (deductive categories), 2) heuristics used in the process (deductive categories), 3) self-regulatory activities within the process (inductive categories), and 4) success of the process (deductive categories). Coding 1, 2, and 3 are process-oriented (i.e., using video data), whereas coding 4 is product-oriented (i.e., using students' written worksheets) (Herold-Blasius, 2021). Each coding, including the name of the code and its coding rule, is explained in the subsequent sections. Second, each coding is used as a dimension for comparing the 41 problem-solving processes. By applying these dimensions, grouping similar problem-solving processes leads to characterizing distinct types (Kelle & Kluge, 2010). This procedure is legitimated as "qualitative content analysis can be combined with other qualitative procedures" (Mayring, 2000, p. 8), and contrasting cases when building types may be realized by coding the data first (Kelle & Kluge, 2010).

Table 2. Two of the selected mathematical problems, appropriate heuristics, and potential strategy keys.

Mathematical problem	Appropriate heuristics for primary school students	Potential strategy keys
Farm: On a farm is an open-air enclosure for chickens. In this enclosure also live rabbits. Jens stands by the fence and counts 20 animals with 70 legs in total. How many chickens are there? (Collet, 2009; Pólya, 1962, p. 23)	Generate examples, systematic trying, draw a table or a sketch, identify a pattern	"Draw a picture," "Make a table," "Find an example," "Look for a pattern"
Bag of smarties: Jenny gets a bag full of smarties from her grandma. On the first day, she eats half of the smarties and one more. On the second day, she eats half of the remaining smarties and again one more. After that, she has six smarties left. How many smarties were in the bag at the beginning? (Aßmus, 2010, p. 137)	Work backwards or forwards, draw a table or a sketch	"Work backwards," "Find an example," "Draw a picture," "Make a table"
Lego bricks: Maike builds a castle using Lego bricks. She has three different kinds of bricks: three dots, four dots, and six dots.  To finish her castle, Maike needs a 19-dot wall.  Find all possibilities to build a wall with 19-dots using three-, four-, and six-dot bricks. How many three-, four-, and six-dot bricks do you need for each possibility? (Besser et al., 2015)	Generate examples, systematic trying, draw a table, use elements for aid (e.g. by drawing lines in the bricks)	"Find an example," "Use different colors," "Make a table," "Draw a picture"

Process Coding: Episodes

Schoenfeld (1985, 1992) identified six episodes. Rott (2013) expanded these types of episodes, identifying episodes that do not develop the solution process content-wise but may occupy longer periods of time during a problem-solving process, such as writing or organization. All eight episodes were coded according to the description in Table 3.

Process Coding: Heuristics

The appearances of heuristic techniques, such as drawing a figure, generating examples, or trying systematically, were coded using a manual based on the ideas of Koichu et al. (2007) and Rott (2013). The codes used in this study are listed in Table 4.

Table 3. Codes to identify types of episodes.

Code (Abbreviation)	Coding rule
Reading or rereading	The problem is read or re-read. It also contains the silent period after reading.
Analyzing the problem (in a coherent and structured way)	The problem solver tries to understand the problem, e.g. by thinking about what is sought and what is given.
Exploring aspects of the problem (in a much less structured way than in Analysis)	The problem solver investigates the problem which might be more or less chaotic. In this episode, typically many heuristics are implemented.
Planning all or part of a solution	The problem solver may think about possible ways to address the problem or explain how to implement a specific heuristic. This episode can be more or less explicit.
Implementing a plan	The problem solver proceeds in a targeted manner to reach the overall goal – a (correct) solution.
Verifying a solution	The problem solver looks back at the problem-solving process and reviews it. This episode often occurs at the end of a process; it may also be within the process if checking the results.
Writing	The problem solver does not work on the problem itself but documents what was said or thought before. New thought would automatically indicate another type of episode, e.g. exploration or implementation.
Organization	The problem solver realizes an organizational activity, such as drawing sketches or making tables, without having any new thoughts.

Table 4. Codes to identify heuristics.

Code (Abbreviation)	Description
Sought and given (S & G)	Checking for information given in the problem and what is searched for
Read again (R)	Reading again the problem or parts of it
Aid for systematization (AfS)	Introducing structuring elements that help to carry out activities
Example (Ex)	Generating, stating or recording an example. In case the same example is stated, the code is allocated only once.
Systematic trying (SyT)	Testing several values with the aim to approximate to the solution. Systematic trying often occurs with techniques that help to consider all values
Working backwards (WoBa)	Focusing the aim of the problem/the sought and working from there to the starting point; sometimes mathematical operations must be reversed

Process Coding: Key Interactions

Using the given data, an inductive, multistep, and circular procedure was applied according to qualitative content analysis (Mayring, 2000). First, all sequences identified in the all situations in which students or the interviewer work, talk about, read, or look at strategy keys. Second, these sequences were clustered and categorized into three categories (Table 5) (Herold-Blasius, 2021).

Table 5. Codes to identify key interactions.

Code (Abbreviation)	Description
Reading a strategy key	A student reads the strategy keys. It is not considered whether they are read out loud or the eye movement indicates that the strategy keys are being read.
Strategy key as a checklist	A student looks through the keys one by one and checks what has already been done and what could still help.
Naming a strategy key	A student makes clear which strategy key they choose by naming a certain key and/or by tapping on it.

Product Coding: Solution

Students' solution products were graded into four categories of success, which were customized for each problem with examples for each category (Herold-Blasius, 2021; Rott, 2013): (1) *No access*: missing understanding of the problem or meaningful work; (2) *Basic access*: main understanding of the problem and a basic approach; (3) *Advanced access*: proper understanding of the problem and solution to most parts; (4) *Full access*: correct solution.

The process- and product-oriented coding was employed by watching each problem-solving process at least three times to set time stamps first for episodes, second for heuristics, and third for key interactions. If video data was not unambiguous to code, for instance, heuristics, the students' solution was considered to clarify the coding. After the coding was completed, the three process-oriented codings were layered on top of each other and supplemented by the product-oriented coding – consequently, they built a condensed process diagram (Figure 2). This way, all processes are prepared similarly and, thus, comparable to each other (Herold-Blasius, 2021).

Mayring (2000) recommends checking codings for interrater reliability to ensure research criteria of reliability and validity. These four codings were validated by determining the interrater reliability. 12 of 41 problem-solving processes (29.27%) were coded by two independent raters (the author and a student assistant). The interrater reliability for *episodes* was calculated by using R^2 and, as recommended, for a time-unit kappa (Bakeman et al., 2009). For this coding, the threshold for temporal overlap was set at 80%. As tolerance for temporal deviations, 10 seconds was applied. The coding of episodes ($k = 0.82$) showed almost perfect agreement (Landis & Koch, 1977). The interrater reliability

[2]R Core Team (2023): *R: A Language and Environment for Statistical Computing*, R Foundation for Statistical Computing, Vienna, Austria, URL https://www.R-project.org/.

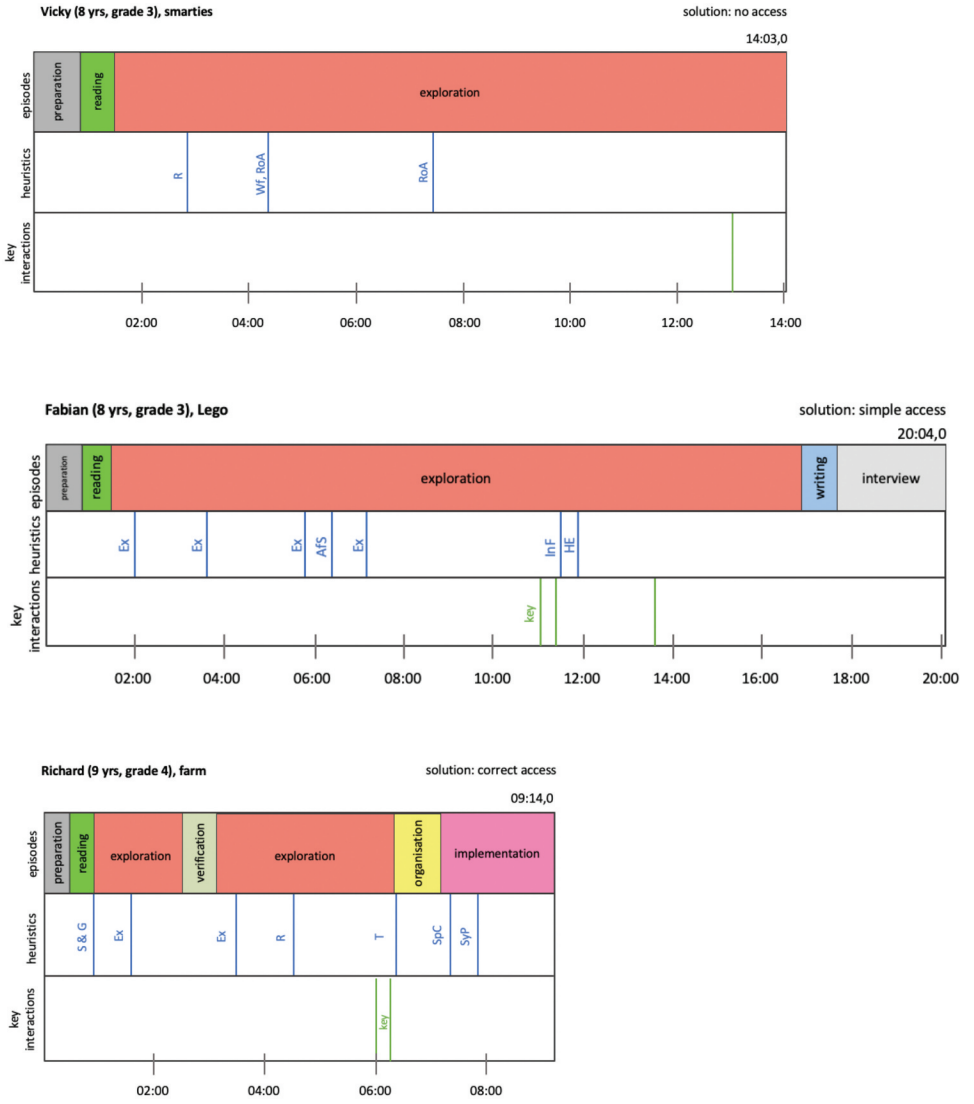


Figure 2. Problem-solving process of Vicky's smarties (top), Fabian's lego (middle), and Richard's farm (bottom) problem; AfS: aid for systematization, ex: example, HE: helping element, InF: informative figure, R: read again, RoA: routine task, S & G: look for the sought and given, SpC: special case, SyT: trying systematically, T: make a table, Wf: work forward.

for *heuristics* and *key interactions* was calculated, as recommended, for an event-unit kappa (Bakeman et al., 2009). These calculations were carried out using R[1] and additionally, the R-package cat to carry out the Iterative Proportional Fitting.[3] The coding of heuristics ($k = 0.681$) and key interactions ($k = 0.690$) showed both substantial agreement (Landis & Koch, 1977) (Table S6 in the supplement). A consensual validation was then carried out (Döring & Bortz, 2016) due to the complexity of the coding as the time stamp and name of the event had to be identified. The interrater reliability for *solution* ($k = 0.89$) showed almost perfect agreement (Landis & Koch, 1977).

[3]Harding T., Tusell F. (2023): *cat: Analysis and Imputation of Categorical-Variable Datasets with*, R package version 0.0–9, Ported to R by Ted Harding and Fernando Tusell. Original by Joseph L. Schafer., URL https://CRAN.R-project.org/package.

Data Analysis: Type Building

For the qualitative analysis, type building was used (Kelle & Kluge, 2010). A typology results from a grouping process in which objects, in this case a problem-solving process, are clustered according to one or more features, here the four codings. A type is a subgroup with similar features that can be described and characterized using those features (Kelle & Kluge, 2010). The data basis consists of 41 videos of the problem-solving processes, transcripts, student solutions, and process- and product-oriented coding. The analysis was realized in four steps (Bikner-Ahsbahs, 2015, p. 138; Kelle & Kluge, 2010):

(1) *"Re-constructing the cases:"* Each process is described based on the process diagrams. Relevant sequences concerning the research questions are identified and analyzed.
(2) *"Grouping of the cases according to maximum homogeneity within each group and heterogeneity among the groups*: The process diagrams of each problem-solving process are compared, contrasted, and grouped."
(3) *"Building ideal types as pure cases:"* The main features of the groups are idealized, and less important aspects are neglected.
(4) *"Building theoretical knowledge:"* Ideal types are compared, and empirical cases are analyzed – in this study, each problem-solving process. (Bikner-Ahsbahs, 2015, p. 138)

In the subsequent section, ideal types of problem-solving processes are described based on the process diagrams and qualitative insight into relevant sequences.

Results

16 students worked on up to four mathematical problems, resulting in 41 problem-solving processes. Table 6 provides an overview of the students, the problems they worked on, and their success. According to the product coding, four students were successful (Category 2 and 3), six were rather unsuccessful (Category 0 and 1) and the remaining six students showed mixed results depending on the specific problem. Thus, none of the problems were too easy or difficult and could be understood as non-routine problem-solving tasks.

The students worked on solving the problems for between 3:03 minutes (Simon, farm) and 40:24 minutes (Julius, seven gates). On average, each problem-solving process took 13:34 minutes. Almost 10 hours of video data formed the basis of this study.

Table 6. Overview of all 16 students (names are pseudonymized), the problems they worked on, and their level of success; gray cells imply wild-goose-chase processes, blank cells without a number imply that the student did not choose this problem.

Problem / Student	Farm	31 cent	Lego	Seven gates	Smarties	Chess	Total of problems
Anja	3				3		**2**
Anke	1		1				**2**
Anne	3						**1**
Christin	3			2		1	**3**
Collin	3	2		1		1	**4**
Fabian			1				**1**
Felix			3		2	1	**3**
Hannes	3						**1**
Julius		2		1			**2**
Laura	3	2			2	1	**4**
Markus			1		1		**2**
Prisha	1	1		0		0	**4**
Richard	3	2		2			**3**
Simon	3	1		2		0	**4**
Til	1	1					**2**
Vicky	0		1		0		**3**
Total of students	**12**	**7**	**5**	**6**	**5**	**6**	**41**

In Table 6, the gray cells signify processes in which the types of episodes were coded and identified as *wild-goose-chase* processes. This kind of process is characterized by a specific sequence of episodes and indicates a lack of self-regulation (Rott, 2013; Schoenfeld, 1985, 1992). Only three kinds of episodes can be observed in these processes: reading, exploration, and sometimes analysis. To identify wild-goose-chase processes, a process profile using 5-tuples was created for all the processes, consisting of A for analysis, E for exploration, P for planning, I for implementation, and V for verification (Rott, 2013). If an episode occurred during the problem-solving process, its existence was marked by one of these letters if there was no zero. Accordingly, the profile of a wild-goose-chase process is either A, E, 0, 0, 0 or 0, E, 0, 0, 0. In this study, 30 of the 41 processes (approx. 75%) were wild-goose-chase processes. This is more than Rott's (2013) finding when analyzing 110 fifth and sixth graders. He found only 34 wild-goose-chase processes (approx. 31%). One explanation for this might be the age of the students. Students in primary schools have self-regulatory strategies (Harnishfeger & Bjorklund, 2015), but lack the ability to apply them purposefully (Gürtler et al., 2002).

Within the given data, 83 key interactions were identified. All students interacted at least once with the strategy keys during their problem-solving processes. This is likely because the students were introduced to the keys for the first time in this setting and were curious about the material. The 83 key interactions are distributed 49 times naming a strategy key, 32 times reading the keys, and twice using the strategy keys as a checklist (see section "Data coding and analysis" for details concerning the coding).

Qualitative Analysis: Ideal Types and Exemplary Problem-Solving Processes

After analyzing all processes, four types of problem-solving processes were identified which will be illustrated in the following, each with a student's process. Simon's (farm) and Vicky's (farm) processes represent problem-solving processes without key interactions. Vicky's process (Smarties) is a typical process with key interactions, but no benefit from the keys. Fabian (Lego) exemplifies processes with key interactions and new heuristics after that. Richard (farm) represents the ideal type of key interaction – a problem-solving process with key interaction and subsequent behavior concerning heuristics, and self-regulatory activities.

Type 1 (Simon and Vicky, Farm): Processes without Key Interaction and No Benefit from the Strategy Keys

Overall, no key interactions occurred in 14 of 41 problem-solving processes. Two main scenarios were observed in this context (Herold-Blasius, 2021):

(A) The students encountered no barriers because they knew how to work on the problem. Students with such processes had an idea, perhaps even a plan, carried out the plan, and solved the problem. As no obstacles occurred, there was no need for key interactions (e.g., Simon, farm).
(B) The students encountered no barriers because they reduced the complexity of the problem. Students with such processes did not see the necessity for key interactions, even though it could have been potentially helpful (e.g., Vicky, farm).

This study aims to investigate how students use strategy keys and determine how the keys support or hinder their problem-solving process. Hence, this study did not analyze processes in which no interactions with keys could be observed.

Type 2 (Vicky, Smarties): Processes with Key Interaction and No Benefit from the Strategy Keys

Vicky worked on three mathematical problems (Table 7). The problem "smarties" was her first one. She worked on solving it for approx. 14 minutes without finding a solution (0: no access) (Figure 2). The first coding track shows that she read and explored the problem. As only these episodes were coded, she encountered a wild-goose-chase process. Three heuristics were coded on the second coding

track. During her work process, she read the problem again, created two routine tasks by combining given numbers randomly (e.g., $22 - 6 = 16$), and worked forward. The third coding track showed a key interaction after 13 minutes, initiated by the interviewer (Line 1 in Table 7). Vicky read the keys silently (Line 4) and chose the key "Find an example" (Line 5). She then said that the key would help her (Line 9). One minute later, the problem-solving process ended (Figure 2).

This process shows that there was no change concerning episodes or heuristics after a key interaction. Though Vicky chose a key, she did not find a way to apply or integrate it into the solution process. Examining her other problem-solving processes (Table 7), she found solutions coded with no or simple access. As in the given example, she mostly reduced the complexity of the problem by using the numbers given in the problem itself (routine task). Also, if she interacted with strategy keys, she would not find a purposeful way to implement them. Thus, there appears to be a discrepancy between choosing and applying a key.

Table 7. Vicky (smarties), transcript of key interaction (I: interviewer, S: student).

00:12:36–00:13:37 (Exploration)	
1	I: Shall we look at the keys?
2	*(S nods in agreement.)*
3	I: Maybe one of them reveals us something. Let's have a look.
4	*(S bends over the keys and reads them silently. Duration: 10 sec.)*
5	*(S taps on the key "Find an example.")*
6	I: Hmm . . .
7	S: Find an example.
8	I: Hmm . . . *(I put the key next to the other keys on the table.)*
9	S: Doesn't help me any further.
10	I: Why not? *(7 sec.)* Why doesn't it help you any further? *(5 sec.)* You don't know?
11	*(S shakes her head in denial.)*

Type 3 (Fabian, Lego): Processes with Key Interaction and Subsequent Behavior Concerning Heuristics

Fabian worked on the problem "Lego" only for approx. 17 minutes and resulted in a solution coded as simple access (Figure 2). The first coding track revealed him starting the problem-solving process by reading the problem and then exploring it for approx. 15 minutes, which is almost the entire process. This implied a wild-goose-chase process (see "Self-regulation"). He finished by writing down: "There are four solutions." The second coding track illustrates seven heuristics, including four different ones. Fabian generated four examples (1 three-dot and 4 four-dot bricks; 2 six-dot bricks, 1 three-dot, and 1 four-dot brick; 5 three-dot bricks and 1 four-dot brick; 1 three-dot brick, 2 six-dot bricks, and 1 four-dot brick), used aid for systematization and helping elements (as he drew lines in the 19-dot brick), and drew an informative figure. The third coding track shows the key interactions between minutes 11 and 14 (Figure 2). After the interviewer's suggestion (Line 1 in Table 8), Fabian found a new idea by interacting with the strategy keys. He drew an informative figure (InF) and marked the bricks by adding lines (HE) (Figure 2). This behavior can be interpreted as the generation of a new strategy. However, he rejected this idea shortly after. After this, he monitored his work process using keys as a checklist. Thus, new heuristics can be observed after key interactions. Yet, it remains difficult to choose and apply a key purposefully.

Type 4 (Richard, Farm): Processes with Key Interaction and Subsequent Behavior Concerning Heuristics and Self-Regulatory Activities

The first coding track shows that Richard first read and then explored the task (Figure 2). After three minutes, he checked his previous procedure (verification phase) and switched back to the exploration phase. After approx. seven minutes, the organization phase started, which was followed by an implementation. After approx. nine minutes, the process concluded with the correct solution. On

Table 8. Fabian (lego), transcript of key interaction (I: interviewer, S: student).

00:10:57–00:14:18 (Exploration)	
1	I: Shall we check if one of the keys may give you an idea?
2	*(S looks at the keys. Duration: 18 sec.)*
3	S: I think 'Draw a picture' could be good, couldn't it?
4	*(I put the key next to the other keys on the table.)*
5	*(S draws a Lego brick on the back of his sheet and marks each single Lego brick by adding lines.)*
6	S: No. I've had this one already *(S crosses out the drawing and thinks. Duration: 22 sec.)*
7	I: But you do not convince me yet that you are really sure. How can you be sure?
8	*(S pulls up his shoulders.)*
9	S: Don't know.
10	I: Is there another key that may help you?
11	S: Well, this *(S shows to the key "Start with a small number.")* cannot help me at the moment.
12	I: Hmm . . .
13	S: This *(S shows to the key "Work backwards.")* cannot help me. This *(S shows to the key "Use different colors.")* would not help me. This *(S shows to the key "Make a table.")* not either. This *(S shows to the key "Read the task again")* also not. This one not. This *(S shows to the key "Look for a pattern or a rule.")* also not. *(5 sec.)* This *(S shows to the key "Find an example.")* also not.
14	I: Okay. None.

the second coding track, the various heuristics used by Richard in his problem-solving process are marked. Seven heuristics, including six different ones, were identified. He first considered what was given and sought in the problem, and then generated two examples. After five minutes, he read the problem again and then created a table. Shortly after, Richard generated a special case that he then attempted systematically.

On the third coding track, after nearly 6 minutes Richard reached for the strategy keys after the interviewer suggested (Line 3 in Table 9). He looked at the keys and found the idea to draw a table by interacting with the key "Make a table." Shortly after, he switched to a new episode (organization) and applied the new heuristic "table" (T in Figure 2).

Thus, strategy keys may provide the impulse for this change at the level of heuristics and episode types. Richard's problem-solving process concerns the ideal case that strategy keys can prompt. A student works on a mathematical problem, becomes stuck, reads through the keys, and chooses one or two. They then show a heuristic they did not use before and even change the type of episode. Ideally, the student would achieve a correct result.

Table 9. Richard (farm), transcript of key interaction (I: interviewer, S: student).

00:05:47–00:07:37 (Exploration)	
1	I: If you look at these two, the once you have here *(I shows to the two examples generated before.)*. Now we have another look.
2	S: Yes.
3	I: Maybe a key could help again?
4	*(S looks at the strategy keys.)*
5	S: Hm. Maybe (. . .) one should draw a table.
6	I: Mhm. Let's take this one out. *(I moves the key "Make a table" next to the other keys.)*
7	*(S draws a table. Duration: 51 sec.)*
8	S: Now I have a lot of space to just try out.
9	I: Then get started.
10	I: Well, 20 rabbits. No. 20 rabbits are 80 already.
11	I: Write down already.
12	*(S writes down.)*

Discussion and Conclusion

This study investigated how students work with strategy keys and how this hinders or supports their problem-solving processes. In summary, data analysis revealed four types of problem-solving processes when working with strategy keys.

First, processes in which students do not interact with the keys and, thus, do not benefit from them. Two scenarios occurred: A) Students do not encounter barriers and know what to do and how to solve problems (e.g., Simon, farm). B) Students do not encounter barriers and reduce the complexity of the problem to the extent that routine procedures are sufficient (e.g., Vicky, farm). Yet, they did not understand the problem as intended and instead used the given numbers randomly (Stern, 1992). Thus, their main difficulty was understanding the problem, which is a common barrier when working on non-routine problems (Salle et al., 2020; Sinaga et al., 2023). Possibly, training students on specific strategy keys (e.g., "Read the task again" or "Draw a picture") could support their understanding of the problem.

Second, processes in which students interact with the keys but do not benefit from them (e.g., Vicky, Smarties). In those processes, students encountered a barrier and reached for aid by interacting with the strategy keys but then failed to integrate them into their problem-solving processes purposefully. They seemed to be successful in finding a potentially helpful key but did not know how to use the key or adapt it to that specific problem. It, therefore, seems that a purposeful use of the keys is not self-explanatory and that it may be easier to choose a strategy than to apply it successfully (Hasselhorn & Gold, 2017). It may benefit those students to explain each key and its purposeful use, e.g., integrated into an instruction concept.

Third, processes in which students interact with the keys and show subsequent behavior concerning heuristics (e.g., Fabian, Lego). In these, the keys worked as prompts and triggered heuristics (Hasselhorn & Gold, 2017), suggesting that strategy activation is possible for students at university (Bannert, 2009) and primary school (Russo et al., 2020).

Fourth, processes in which students interact with the keys and show subsequent behavior concerning heuristics and self-regulatory activities (e.g., Richard, farm). This type illustrates how successfully the interaction with strategy keys may support problem-solving processes by enhancing heuristics and self-regulatory activities. Both are essential factors for successful problem-solving (Schoenfeld, 1985) and have been reported in other areas (Koenen & Emden, 2016) but not with a qualitative perspective on mathematical problem-solving.

Limitations

Three main limitations of this study should be noted. First, only 16 primary school students participated, providing qualitative insight instead of generalizing claims. Second, task-based interviews with single-working students increased interaction with the interviewer, potentially influencing their problem-solving processes (Pugalee, 2004), especially since the interviewer suggested key interactions sometimes (e.g., Fabian, Lego). Hence, either reducing the interviewer's involvement or working in pairs or groups should be considered in future studies. Third, the strategy keys were introduced only at the beginning of the interview, which may have led to more key interactions due to curiosity and the feeling of social expectations toward the interviewer. Depending on the research interest, an instruction concept may reduce this curiosity.

Implications for practice

In the data provided, most students were unsuccessful in solving more than one problem, regardless of key interaction (type 1 B or 2). Thus, merely having the keys did not help the students solve the problems, and students with type 2 processes could not effectively integrate the keys into their processes. Yet, aid cards are commonly offered without a proper introduction (Koenen & Emden, 2016; Russo et al., 2020), independent of the student's decision to use them. In sum, the interaction with aid cards exemplified by strategy keys is not beneficial for all but for some students. For those, it is worth continuing to use aid cards in school and mathematics (Kaitera & Harmoinen, 2022).

Yet, some students need support in effectively integrating these cards into their processes and overcoming barriers as successfully as e.g. Richard (type 4) did. This implies that students need

a proper introduction to the strategy keys, preferably integrated into an instruction concept that could be designed differently:

- Introduce all strategy keys at once by applying them to a non-routine problem. In a series of problems, students encounter barriers, use the keys, and continuously reflect on their heuristic use (Herold-Blasius & Rott, 2021; Kaitera & Harmoinen, 2022).
- Introduce single keys with appropriate problems (Table 2). This way, students can learn strategies and then transfer them to new problems.
- Introduce each strategy key with an explanatory video and a worked example to illustrate a possible appliance of each heuristic. Students would then be encouraged to apply those heuristics to various non-routine problems (Herold-Blasius et al., 2023).

None of these training concepts has been applied in a primary school context so far, but they should be transferred to this age group.

Implications for research

The coding procedure of this study is unique as it allows a holistic view of problem-solving processes with low complexity reduction. This combination of codings provided detailed insights into students' interaction with strategy keys and their impact on the problem-solving process. Replicating this study with other age groups, in broader contexts, and with larger cohorts would be interesting. An intervention study using a pre-posttest design and video data over a longer period could reconstruct students' scaffolding process using these keys. Besides, the different training concepts mentioned should be developed and evaluated by applying design research approaches. Additionally, it remains unclear how to support students who struggle when working on mathematical problems. Research should among others focus on this struggling group of students by addressing questions such as how may these students be supported, or develop their problem-solving ability over a longer period.

This study contributes to understanding students' use of aid cards when solving non-routine problems. It provides qualitative insight into how such cards may support students in mathematical non-routine problems. The findings are not surprising but illustrate that heuristic aid cards, exemplified by strategy keys, are a promising tool to support non-routine problem-solving processes by overcoming barriers concerning heuristics and self-regulation. Yet, it remains unclear how to support students who fail to integrate such cards into their problem-solving processes. Instruction concepts are suggested, but research on unsuccessful and possibly low-achieving students in mathematical problem-solving is still needed.

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Data Availability Statement

The participants of this study did not give written consent for their data to be shared publicly, so due to the sensitive nature of the research, supporting data is not available.

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