# Matrix measures and random walks

Holger Dette
Ruhr-Universität Bochum
Fakultät für Mathematik
44780 Bochum, Germany
e-mail: holger.dette@rub.de
FAX: +49 234 3214 559

Bettina Reuther
Ruhr-Universität Bochum
Fakultät für Mathematik
44780 Bochum, Germany
e-mail: bettina.reuther@rub.de

W. J. Studden
Department of Statistics
Purdue University
West-Lafayette IN 47907-1399
USA
e-mail: studden@stat.purdue.edu

M. Zygmunt
Institute of Mathematics
Wroclaw University
pl. Grunwaldzki 2/4
50-384 Wroclaw, Poland
e-mail: marcin.zygmunt@vp.pl

July 22, 2005

**Abstract**

In this paper we study the connection between matrix measures and random walks with a tridiagonal block transition matrix. We derive sufficient conditions such that the blocks of the $n$-step transition matrix of the Markov chain can be represented as integrals with respect to a matrix valued spectral measure. Several stochastic properties of the processes are characterized by means of this matrix measure. In many cases this measure is supported in the interval $[-1, 1]$. The results are illustrated by several examples including random walks on a grid and the embedded chain of a queuing system.

AMS Subject Classification: 60J10, 42C05

Keywords and Phrases: Markov chain, block tridiagonal transition matrix, spectral measure, matrix measure, quasi birth and death processes, canonical moments, Chebyshev matrix polynomials

## 1 Introduction

Consider a homogeneous Markov chain with state space

$$\mathcal{C}_d = \{(i, j) \in \mathbb{N}_0 \times \mathbb{N} \mid 1 \le j \le d\} \tag{1.1}$$

and block tridiagonal transition matrix

$$P = \begin{pmatrix} B_0 & A_0 & & & 0 \\ C_1^T & B_1 & A_1 & & \\ & C_2^T & B_2 & A_2 & \\ 0 & & \ddots & \ddots & \ddots \end{pmatrix}, \tag{1.2}$$

where $d \in \mathbb{N}, A_0, A_1, \ldots, B_0, B_1, \ldots, C_1, C_2, \ldots$ are $d \times d$ matrices containing the probabilities of one-step transitions (here and throughout this paper $C^T$ denotes the transpose of the matrix $C$). If the one-step transition matrix is represented by

$$P = (P_{ii'})_{i,i'=0,1,\ldots} \tag{1.3}$$

with $d \times d$ block matrices $P_{ii'}$, the probability of going in one step from state $(i,j)$ to $(i',j')$ is given by the element in the position $(j,j')$ of the matrix $P_{ii'}$. Some illustrative examples will be given below. Block tridiagonal matrices of the form (1.2) typically appear in the analysis of the embedded Markov chains of continuous-time Markov processes with state space (1.1) and block tridiagonal infinitesimal generator [see e.g. the monographs of Neuts (1981) and Neuts (1989) or the recent work of Marek (2003) and Dayar and Quessette (2002) among many others] and these models have significant applications in the performance evalutation of communication systems [see e.g. Ost (2001)].

Matrices of the form (1.2) are also closely related to a sequence of matrix polynomials recursively defined by

$$xQ_n(x) = A_n Q_{n+1}(x) + B_n Q_n(x) + C_n^T Q_{n-1}(x), \; n \in \mathbb{N}_0, \tag{1.4}$$

where $Q_{-1}(x) = 0$ and $Q_0(x) = I_d$ denotes the $d \times d$ identity matrix. If $A_n = C_{n+1}$ and $B_n$ is symmetric it follows that there exists a matrix measure $\Sigma = \{\sigma_{ij}\}_{i,j=1,\ldots,d}$ on the real line (here $\sigma_{ij}$ are signed measures such that for any Borel set $A \subset \mathbb{R}$ the matrix $\Sigma(A)$ is nonnegative definite), such that the polynomials $Q_j(x)$ are orthonormal with respect to a left inner product, i.e.

$$\langle Q_i, Q_j \rangle = \int_{\mathbb{R}} Q_i(x) d\Sigma(x) Q_j^T(x) = \delta_{ij} I_d \tag{1.5}$$

[see e.g. Sinap and Van Assche (1996), or Duran (1995)]. In recent years several authors have studied properties of matrix orthonormal polynomials [see e.g. Rodman (1990), Duran and Van Assche (1995), Duran (1996, 1999), Dette and Studden (2001) among many others].

In the present paper we are interested in the relation between Markov chains with state space $\mathcal{C}_d$ defined in (1.1) and block tridiagonal transition matrix (1.2) and the polynomials $Q_j(x)$ defined by the recursive relation (1.4). In the case $d = 1$ this problem has been studied extensively in the literature [see Karlin and McGregor (1959), Whitehurst (1982), Woess (1985), Van Doorn and Schrijner (1993, 1995), Dette (1996) among many others], but the case $d > 1$ is more difficult, because in this case a system of matrix polynomials $\{Q_j(x)\}_{j \geq 0}$ satisfying a recurrence relation of the form (1.4) is not necessarily orthogonal with respect to an inner product induced by a matrix measure. In Section 2 we characterize the transition matrices of the form (1.2) such that there exists an integral representation for the corresponding $n$-step transition probabilities in terms of the matrix measure and corresponding orthogonal matrix polynomials, i.e.

$$P_{ij}^n = \Big( \int x^n Q_i(x) d\Sigma(x) Q_j^T(x) \Big) \Big( \int Q_j(x) d\Sigma(x) Q_j^T(x) \Big)^{-1},$$

where $P_{ij}^n$ denotes the $d \times d$ block of the $n$-step transition matrix $P^n$ in the position $(i,j)$. In other words: the element in the position $(k,l)$ in the expression on the right hand side is an integral representation for the probability of going in $n$ steps from state $(i,k)$ to $(j,l)$. We also derive a sufficient condition such that the spectral (matrix) measure $\Sigma$ (if it exists) is supported on the interval $[-1,1]$. In Section 3 we discuss several illustrative examples where this condition is satisfied including some examples from queuing theory. Section 4 continues our more theoretical discussion and some consequences of the integral representation are derived. In particular we present a characterization of recurrence by properties of the blocks of the transition matrix, which generalizes the classical characterization of recurrence of a birth and death chain [see Karlin and Taylor (1975)].

## 2 Random walk matrix polynomials

A matrix measure $\Sigma$ is a $d \times d$ matrix $\Sigma = \{\sigma_{ij}\}_{i,j=1,\dots,d}$ of finite signed measures $\sigma_{ij}$ on the Borel field of the real line $\mathbb{R}$ or of an appropriate subset. It will be assumed here that for each Borel set $A \subset \mathbb{R}$ the matrix $\Sigma(A) = \{\sigma_{ij}(A)\}_{i,j=1,\dots,d}$ is symmetric and nonnegative definite, i.e. $\Sigma(A) \geq 0$. The moments of the matrix measure $\Sigma$ are given by the $d \times d$ matrices

$$S_k = \int t^k d\Sigma(t) \qquad k = 0, 1, \cdots, \tag{2.1}$$

and only measures for which all relevant moments exist will be considered throughout this paper. Let $A_i$ $(i = 0, \dots, n)$ denote $d \times d$ matrices, then a matrix polynomial is defined by $P(t) = \sum_{i=0}^n A_i t^i$. The inner product of two matrix polynomials, say $P$ and $Q$, is defined by

$$\langle P, Q \rangle \ = \ \int P(t) \Sigma(dt) Q^T(t) \ , \tag{2.2}$$

where $Q^T(t)$ denotes the transpose of the matrix $Q(t)$. Sinap and Van Assche (1996) call this the 'left' inner product. Orthogonal polynomials are defined by orthogonalizing the sequence $I_p, tI_p, t^2 I_p, \cdots$ with respect to the above inner product. If $S_0, S_1, \dots$ is a given sequence of matrices such that the block Hankel matrices

$$\underline{H}_{2m} = \begin{pmatrix} S_0 & \cdots & S_m \\ \vdots & & \vdots \\ S_m & \dots & S_{2m} \end{pmatrix} \tag{2.3}$$

are positive definite, it is well known [see e.g. Marcellán and Sansigre (1993)] that a matrix measure $\Sigma$ with moments $S_j$ $(j \in \mathbb{N}_0)$ and a corresponding infinite sequence of orthogonal matrix polynomials with respect to $d\Sigma(x)$ exist. Moreover, these matrix polynomials satisfy a three term recurrence relation.

Let $\{Q_j(x)\}_{j\geq 0}$ denote a sequence of matrix polynomials defined by the recurrence relationship (1.4), where the matrices $C_j$ $(j \in \mathbb{N})$, $A_j$ $(j \in \mathbb{N}_0)$ in (1.2) are assumed to be non-singular. The following results characterizes the existence of a matrix measure $\Sigma$ such that the polynomials $Q_j(x)$ are orthogonal with respect to $d\Sigma(x)$ in the sense (2.2).

**Theorem 2.1.** *Assume that the matrices $A_n\,(n \in \mathbb{N}_0)$ and $C_n\,(n \in \mathbb{N})$ in the one-step transition matrix (1.2) are non-singular. There exists a matrix measure $\Sigma$ on the real line with positive definite Hankel matrices $\underline{H}_{2m}$ $(m \in \mathbb{N}_0)$ such that the polynomials $\{Q_n(x)\}_{n\in\mathbb{N}_0}$ defined by (1.4) are orthogonal with respect to the measure $d\Sigma(x)$ if and only if there exists a sequence of non-singular matrices $\{R_n\}_{n\in\mathbb{N}_0}$ such that the following relations are satisfied*

$$(2.4)\qquad \begin{aligned} &R_n B_n R_n^{-1} \text{ is symmetric } \ \forall\, n \in \mathbb{N}_0, \\ &R_n^T R_n = C_n^{-1} \cdots C_1^{-1} (R_0^T R_0) A_0 \cdots A_{n-1} \ \ \forall\, n \in \mathbb{N}. \end{aligned}$$

**Proof.** Assume that the polynomials $\{Q_n(x)\}_{n\in\mathbb{N}_0}$ are orthogonal with respect to the measure $d\Sigma(x)$, that is

$$(2.5)\qquad \int_{\mathbb{R}} Q_i(x) d\Sigma(x) Q_j^T(x) = 0,$$

whenever $i \neq j$ and

$$(2.6)\qquad \int_{\mathbb{R}} Q_i(x) d\Sigma(x) Q_i^T(x) = F_i > 0 \quad (i \in \mathbb{N}_0),$$

where we use the notation $F_i > 0$ for a positive definite matrix $F_i \in \mathbb{R}^{d\times d}$ (the fact that the matrix $F_i$ is positive definite follows from a straightforward calculation using the assumption that $\underline{H}_{2m}$ is positive definite for all $m \in \mathbb{N}_0$). Define $R_n = F_n^{-1/2}$ and $\tilde{Q}_n(x) = R_n Q_n(x)$, then it is easy to see that the polynomials $\{\tilde{Q}_n(x)\}_{n\in\mathbb{N}_0}$ are orthonormal with respect to the measure $d\Sigma(x)$. Therefore it follows from Sinap and Van Assche (1996) that there exist $d \times d$ non-singular matrices $\{D_n\}_{n\in\mathbb{N}}$ and symmetric matrices $\{E_n\}_{n\in\mathbb{N}_0}$ such that the recurrence relation

$$(2.7)\qquad x\tilde{Q}_n(x) = D_{n+1}\tilde{Q}_{n+1}(x) + E_n\tilde{Q}_n(x) + D_n^T\tilde{Q}_{n-1}(x)$$

is satisfied for all $n \in \mathbb{N}_0$, $(\tilde{Q}_{-1}(x) = 0, \tilde{Q}_0(x) = R_0)$. On the other hand we obtain from (1.4) and the representation $\tilde{Q}_n(x) = R_n Q_n(x)$ the recurrence relation

$$(2.8)\qquad x\tilde{Q}_n(x) = R_n A_n R_{n+1}^{-1}\tilde{Q}_{n+1}(x) + R_n B_n R_n^{-1}\tilde{Q}_n(x) + R_n C_n^T R_{n-1}^{-1}\tilde{Q}_{n-1}(x),$$

and a comparison of (2.7) and (2.8) yields

$$(2.9)\qquad D_{n+1} = R_n A_n R_{n+1}^{-1}, \quad E_n = R_n B_n R_n^{-1}\ , \quad D_n^T = R_n C_n^T R_{n-1}^{-1},$$

where the matrix $E_n$ is symmetric. Now a straightforward calculation gives

$$R_n A_n R_{n+1}^{-1} = (R_{n+1} C_{n+1}^T R_n^{-1})^T = (R_n^T)^{-1} C_{n+1} R_{n+1}^T,$$

or equivalently

$$R_{n+1}^T R_{n+1} = C_{n+1}^{-1} (R_n^T R_n) A_n.$$

This yields by an induction argument

$$R_n^T R_n = C_n^{-1} \cdots C_1^{-1} R_0^T R_0 A_0 \cdots A_{n-1}, \ n \in \mathbb{N}$$

and proves the first part of Theorem 3.1.

For the converse assume that the relations in (2.4) are satisfied and consider the polynomials $\tilde{Q}_n(x) = R_n Q_n(x)$. These polynomials satisfy the recurrence relation (2.8) and from (2.4) it follows that the matrices

$$E_n = R_n B_n R_n^{-1}$$

are symmetric ($n \in \mathbb{N}_0$), while

$$D_{n+1} = R_n A_n R_{n+1}^{-1} = (R_{n+1} C_{n+1}^T R_n^{-1})^T$$

by the second assumption in (2.4). Therefore the recurrence relation for the polynomials $\tilde{Q}_n(x)$ is of the form (2.7) and by the discussion following Theorem 3.1 in Sinap and van Assche (1996) these polynomials are orthonormal with respect to a matrix measure $d\Sigma(x)$. This also implies the orthogonality of the polynomials $Q_n(x) = R_n^{-1} \tilde{Q}_n(x)$ with respect to the measure $d\Sigma(x)$.

Because the polynomials $\underline{Q}_n(t) = R_0^{-1} D_1 \ldots D_n \tilde{Q}_n(t)$ have leading coefficient $I_d$ we obtain that the matrix

$$\langle \underline{Q}_n, \underline{Q}_n \rangle = \int \underline{Q}_n(t) d\Sigma(t) \underline{Q}_n^T(t) = R_0^{-1} D_1 \ldots D_n D_n^T \ldots D_1^T (R_0^T)^{-1} \tag{2.10}$$

is non-singular. On the other hand it follows from Dette and Studden (2001) that the left hand side of (2.10) is equal to the Schur complement, say $S_{2n} - S_{2n}^-$, of $S_{2n}$ in $\underline{H}_{2n}$. Because the matrix $\underline{H}_{2n}$ is positive definite if and only if $\underline{H}_{2n-2}$ and the Schur complement of $S_{2n}$ in $\underline{H}_{2n}$ are positive definite it follows by an induction argument that all Hankel matrices obtained from the moments of the matrix measure $\Sigma$ are positive definite. □

**Remark 2.2.** Throughout this paper a matrix measure $\Sigma$ with corresponding orthogonal matrix polynomials $Q_i(x)$ is called a random walk matrix measure or spectral measure and the polynomials $Q_i(x)$ will be called random walk matrix polynomials if the assumptions of Theorem 2.1 are satisfied. Because the polynomials $\tilde{Q}_i(x) = R_i Q_i(x)$ defined in the proof of Theorem 2.1 are orthonormal with respect to the measure $d\Sigma(x)$ it follows that

$$I_d = \langle \tilde{Q}_0, \tilde{Q}_0 \rangle = \int \tilde{Q}_0(x) d\Sigma(x) \tilde{Q}_0^T \ = \ R_0 S_0 R_0^T, \tag{2.11}$$

or equivalently

$$R_0^{-1}((R_0^T)^{-1}) \ = \ (R_0^T R_0)^{-1} = S_0, \tag{2.12}$$

where $S_0$ is the 0th moment of the matrix measure $\Sigma$ [see (2.1)]. We finally note that the matrices $R_n$ in Theorem 2.1 are not unique. If $\{R_n\}_{n\in\mathbb{N}_0}$ is a sequence of matrices satisfying (2.4), these relations are also fulfilled for the sequence $\{\tilde{R}_n\}_{n\in\mathbb{N}_0} = \{U_nR_n\}_{n\in\mathbb{N}_0}$ where $U_n$ $(n \in \mathbb{N}_0)$ are arbitrary orthogonal matrices.

Before we present some examples, where the conditions of Theorem 2.1 are satisfied we derive some consequences of the existence of a random walk measure. For this let $Q(x) = (Q_0^T(x), Q_1^T(x), \ldots)^T$ denote the vector of matrix polynomials defined by the recursion relation (1.4), then it is easy to see that the recurrence relation (1.4) is equivalent to

$$xQ(x) = PQ(x), \tag{2.13}$$

which gives (by iteration)

$$x^nQ(x) = P^nQ(x). \tag{2.14}$$

Therefore

$$\int x^nQ(x)d\Sigma(x)Q_j^T(x) = P^n\int Q(x)d\Sigma(x)Q_j^T(x), \tag{2.15}$$

and from the orthogonality of the random walk polynomials we obtain the representation

$$P_{ij}^n = \Big(\int x^nQ_i(x)d\Sigma(x)Q_j^T(x)\Big)\Big(\int Q_j(x)d\Sigma(x)Q_j^T(x)\Big)^{-1} \tag{2.16}$$

for the block in the position $(i, j)$ of the $n$-step transition matrix $P^n$.

**Theorem 2.3.** *If the assumptions of Theorem 2.1 are satisfied, the block $P_{ij}^n$ in the position $(i, j)$ of the $n$-step transition matrix $P^n$ of the random walk can be represented in the form (2.16), where $\Sigma$ denotes a random walk measure corresponding to the one-step transition matrix $P$.*

**Remark 2.4.** Note that the random walk measure is not necessarily uniquely determined by the random walk on the grid $\mathcal{C}_d$. However, using the case $i = j = 0$ in (2.16) it follows for the moments of the random walk measure

$$P_{00}^n = S_nS_0^{-1} \quad (n \in \mathbb{N}_0), \tag{2.17}$$

where $P_{00}^n$ is the first block in the $n$-step transition matrix of the random walk. Therefore the moments of a random walk measure are essentially uniquely determined. In the following we will derive a sufficient condition such that the random walk measure (if it exists) is supported on the interval $[-1, 1]$. In this case the measure is determined by its moments.

**Theorem 2.5.** *Assume that the conditions of Theorem 2.1 are satisfied and define the block diagonal matrix $R =$ diag$(R_0, R_1, R_2, \ldots)$. If the matrix $R$ is symmetric and the matrix*

$$\tilde{P} = R^TPR^{-1} \tag{2.18}$$

*has non-negative entries then the random walk matrix measure* $\Sigma = \{\sigma_{ij}\}_{i,j=1,\ldots,d}$ *corresponding to the polynomials in (1.4) is supported on the interval* $[-1,1]$, *that is*

$$supp(\sigma_{ij}) \subset [-1,1] \quad \forall\, i,j = 1,\ldots,d.$$

**Proof.** Note that the matrix in (2.18) is symmetric (because the assumptions of Theorem 2.1 are satisfied) and that the entries of $\tilde{P}$ are non-negative, by the assumptions of the theorem. According to Schur's test [see Halmos and Sunder (1978), Theorem 5.2 ] it follows that

$$\|\tilde{P}\|_2 \leq 1, \tag{2.19}$$

if we can find two vectors, say $v, w$, with positive components such that

$$\tilde{P}v \leq w \quad \text{and} \quad \tilde{P}w \leq v$$

(where the symbol $\leq$ means here inequality in each component). If

$$R = \operatorname{diag}(R_0, R_1, \ldots)$$

denotes the infinite dimensional diagonal block matrix with $d \times d$ blocks from Theorem 2.1 we obtain from the representation (2.18) with $v = w = R1$ (here 1 denotes the infinite dimensional vector with all elements equal to one) that

$$\tilde{P}v = \tilde{P}R1 = R^T P 1 \leq R^T 1,$$

which shows that (2.19) is indeed satisfied. Now let

$$\Pi_j = C_j^{-1} \ldots C_1^{-1} R_0^T R_0 A_0 \ldots A_{j-1} = R_j^T R_j,$$

and consider the inner product

$$\langle x, y \rangle_\Pi = \sum_{j=0}^{\infty} x_j^T \Pi_j y_j$$

[with $x = (x_0^T, x_1^T, \ldots); y = (y_0^T, y_1^T, \ldots); x_j \in \mathbb{R}^d, y_j \in \mathbb{R}^d$] and its corresponding norm, say $\|\cdot\|_\Pi$. Define

$$\ell^\infty = \{x = (x_0^T, x_1^T, \ldots) \mid \|x\|_\Pi^2 = \sum_{i=0}^{\infty} x_i^T \Pi_i x_i < \infty\}.$$

From the definition of $P$ and $\Pi_j$ it is easy to see that $\Pi_i P_{ij} = P_{ji}^T \Pi_j$ (for all $i, j \in \mathbb{N}_0$), which implies that $P$ is a selfadjoint operator with respect to the inner product $\langle \cdot, \cdot \rangle_\Pi$. Moreover, we have for any $x$

$$\begin{aligned} \|Px\|_\Pi &= x^T P^T \Pi P x = x^T R^T \tilde{P}^T \tilde{P} R x = \|\tilde{P} R x\|_2 \\ &\leq \|\tilde{P}\|_2 \|Rx\|_2 \leq x^T R^T R x = x^T \Pi x = \|x\|_\Pi, \end{aligned}$$

where we used the representation $\Pi = R^T R$ and (2.19). Consequently, $\|P\|_\Pi \leq 1$, which proves the theorem. $\square$

We note that there are many examples, where the assumptions of Theorem 2.5 are satisfied and we conjecture in fact that a random walk measure is always supported in the interval $[-1, 1]$. In the case $d = 1$ this property holds because in this case the assumptions of the Theorem 2.1 and 2.5 are obviously satisfied. This was shown before by Karlin and McGregor (1959), and an alternative proof can be found in Dette and Studden (1997), Chap. 8.

Our next result gives a relation between the Stieltjes transforms of two random walk measures, say $\Sigma$ and $\tilde{\Sigma}$, where only the matrices $B_0$ and $\tilde{B}_0$ differ in the corresponding one-step transition matrices $P$ and $\tilde{P}$.

**Theorem 2.6.** *Consider the one-step transition matrix $P$ in (1.2) and the matrix*

$$\tilde{P} = \begin{pmatrix} \tilde{B}_0 & A_0 & & & 0 \\ C_1^T & B_1 & A_1 & & \\ & C_2^T & B_2 & A_2 & \\ 0 & & \ddots & \ddots & \ddots \end{pmatrix}, \tag{2.20}$$

*and assume that there exists a random walk measure $\Sigma$ corresponding to the one-step transition matrix $P$ such that the matrix $R_0\tilde{B}_0R_0^{-1}$ is symmetric, where $R_0$ is a matrix such that (2.4) is satisfied. Then there exists also a random walk measure $\tilde{\Sigma}$ corresponding to the matrix $\tilde{P}$. If $\Sigma$ and $\tilde{\Sigma}$ are determinate, then the Stieltjes transforms of both matrix measures are related by*

$$\int \frac{d\Sigma(t)}{z-t} = \Big\{ \Big( \int \frac{d\tilde{\Sigma}(t)}{z-t} \Big)^{-1} - S_0^{-1}(B_0 - \tilde{B}_0) \Big\}^{-1}. \tag{2.21}$$

**Proof.** Because the matrix $R_0\tilde{B}_0R_0^{-1}$ is symmetric and the matrices $P$ and $\tilde{P}$ differ only by the element in the first block, the sequence of matrices $R_0, R_1, \ldots$ can be used to symmetrize the matrices $P$ and $\tilde{P}$ simultaneously [see the proof of Theorem 2.1]. Consequently, there exists a random walk measure corresponding to the random walk with one-step transition matrix $\tilde{P}$. Let $\{Q_n(x)\}_{n\in\mathbb{N}_0}$ denote the system of matrix orthogonal polynomials defined by the recursive relation (1.4) and define $\{\tilde{Q}_n(x)\}_{n\in\mathbb{N}_0}$ by the same recursion, where the matrix $B_0$ has been replaced by $\tilde{B}_0$. A straightforward calculation shows that the difference polynomials

$$R_j(x) = \tilde{Q}_j(x) - Q_j(x)$$

also satisfy the recursion (1.4) with initial conditions $R_0(x) = 0, R_1(x) = A_0^{-1}(B_0 - \tilde{B}_0)$. In particular these polynomials are "proportional" to the first associated orthogonal matrix polynomials

$$Q_n^{(1)}(x) = \int \frac{Q_n(x) - Q_n(t)}{x-t} d\Sigma(t) \quad (n \in \mathbb{N}_0), \tag{2.22}$$

that is

$$R_n(x) = Q_n^{(1)}(x) R_0^T R_0 (B_0 - \tilde{B}_0). \tag{2.23}$$

Recall from the proof of Theorem 2.1 that the systems of polynomials $\{R_n Q_n(x) R_0^{-1}\}_{n\in\mathbb{N}_0}$ and $\{R_n \tilde{Q}_n(x) R_0^{-1}\}_{n\in\mathbb{N}_0}$ are orthonormal with respect to the random walk measures $d\mu(x) = R_0 d\Sigma(x) R_0^T$ and $d\tilde{\mu} = R_0 d\tilde{\Sigma}(x) R_0^T$, respectively, and that $\mu$ and $\tilde{\mu}$ are determinate. Therefore we obtain from Markov's theorem for matrix orthogonal polynomials [see Duran (1996)] that

$$
\begin{aligned}
(2.24)\qquad \int \frac{d\tilde{\Sigma}(t)}{z-t} &= R_0^{-1} \int \frac{d\tilde{\mu}(t)}{z-t} (R_0^T)^{-1} \\
&= \lim_{n\to\infty} R_0^{-1} (R_n \tilde{Q}_n(z) R_0^{-1})^{-1} (R_n \tilde{Q}_n^{(1)}(z) R_0^T)(R_0^T)^{-1} \\
&= \lim_{n\to\infty} (\tilde{Q}_n(z))^{-1} \tilde{Q}_n^{(1)}(z) \\
&= \lim_{n\to\infty} \{Q_n(z) + Q_n^{(1)}(z) R_0^T R_0 (B_0 - \tilde{B}_0)\}^{-1} Q_n^{(1)}(z) \\
&= \lim_{n\to\infty} \{\{(Q_n(z))^{-1} Q_n^{(1)}(z)\}^{-1} + R_0^T R_0 (B_0 - \tilde{B}_0)\}^{-1} \\
&= \lim_{n\to\infty} \{R_0^T \{(R_n Q_n(z) R_0^{-1})^{-1} R_n Q_n^{(1)}(z) R_0^T\}^{-1} R_0 + R_0^T R_0 (B - \tilde{B}_0)\}^{-1} \\
&= \Big\{ R_0^T \Big( \int \frac{d\mu(t)}{z-t} \Big)^{-1} R_0 + R_0^T R_0 (B_0 - \tilde{B}_0) \Big\}^{-1} \\
&= \Big\{ \Big( \int \frac{d\Sigma(t)}{z-t} \Big)^{-1} + R_0^T R_0 (B_0 - \tilde{B}_0) \Big\}^{-1} \\
&= \Big\{ \Big( \int \frac{d\Sigma(t)}{z\ -t} \Big)^{-1} + S_0^{-1} (B_0 - \tilde{B}_0) \Big\}^{-1},
\end{aligned}
$$

where $\tilde{Q}_n^{(1)}(x)$ denotes the first associated orthogonal matrix polynomial obtained by the analogue of (2.22) from $\tilde{Q}_n(x)$ and we have used the fact that $\tilde{Q}_n^{(1)}(x) = Q_n^{(1)}(x)$ for the third equality (note that this identity is obvious from the definition of $P$ and $\tilde{P}$ in (1.2) and (2.20), respectively).

□

# 3 Examples

In this section we present several examples where the conditions of Theorem 2.1 are satisfied.

## 3.1 Random walks on the integers

Consider the classical random walk on $\mathbb{Z}$ [see e.g. Feller (1950)] with one-step up-, down- and holding transition probabilities $p_i, q_i$ and $r_i$ (respectively), where $p_i + q_i + r_i \le 1;\ \ i \in \mathbb{Z}$. By the one-to-one mapping

$$\psi : \begin{cases} \mathbb{Z} & \rightarrow & \mathcal{C}_2 \\ i & \rightarrow & \begin{cases} (i,1) & \text{if } i \in \mathbb{N}_0 \\ (-i-1,2) & \text{else} \end{cases} \end{cases}$$

this process can be interpreted as a process on the grid $\mathcal{C}_2$, where transitions from the first to the second row are only possible if the process is in state $(0,1)$. The transition matrix of this process is given by (1.2) with $2 \times 2$ blocks

$$B_0 = \begin{pmatrix} r_0 & q_0 \\ p_{-1} & r_{-1} \end{pmatrix}; \quad B_n = \begin{pmatrix} r_n & 0 \\ 0 & r_{-n-1} \end{pmatrix} \tag{3.1}$$

$$A_n = \begin{pmatrix} p_n & 0 \\ 0 & q_{-n-1} \end{pmatrix}; \quad C_n^T = \begin{pmatrix} q_n & 0 \\ 0 & p_{-n-1} \end{pmatrix}. \tag{3.2}$$

It is easy to see that the conditions of Theorem 2.1 are satisfied with the matrices

$$R_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{\frac{q_0}{p_{-1}}} \end{pmatrix}; \quad R_n = \begin{pmatrix} \sqrt{\frac{p_0 \cdots p_{n-1}}{q_1 \cdots q_n}} & 0 \\ 0 & \sqrt{\frac{q_0 q_{-1} \cdots q_{-n}}{p_{-1} p_{-2} \cdots p_{-n-1}}} \end{pmatrix}, \tag{3.3}$$

and consequently there exist a random walk matrix measure corresponding to this process, say $\Sigma$, which is supported in the interval $[-1,1]$ (see Theorem 2.5). For the calculation of the Stieltjes transform of this measure we use Theorem 2.6 and obtain

$$\Phi(z) = \int \frac{d\Sigma(t)}{z-t} = \left\{ \tilde{\Phi}^{-1}(z) - R_0^T R_0 (B_0 - \tilde{B}_0) \right\}^{-1}. \tag{3.4}$$

Here $\tilde{\Phi}$ is the Stieltjes transform of a random walk measure $\tilde{\Sigma}$ with transition matrix (1.2), where the matrix $B_0$ in (3.1) has been replaced by

$$\tilde{B}_0 = \begin{pmatrix} r_0 & 0 \\ 0 & r_{-1} \end{pmatrix},$$

and the matrix $B_0 - \tilde{B}_0$ is given by

$$B_0 - \tilde{B}_0 = \begin{pmatrix} 0 & q_0 \\ p_{-1} & 0 \end{pmatrix}.$$

Note that the matrix $\tilde{\Phi}$ is diagonal and if $\tilde{\Phi}^+$ and $\tilde{\Phi}^-$ denote the corresponding diagonal elements, we obtain from (3.4) the representation

$$\begin{aligned} \Phi(z) &= \int \frac{d\Sigma(t)}{z-t} = \begin{pmatrix} 1/\tilde{\Phi}^+(z) & -q_0 \\ -q_0 & 1/\tilde{\Phi}^-(z) \end{pmatrix}^{-1} \\ &= \frac{1}{1 - q_0^2 \tilde{\Phi}^+(z) \tilde{\Phi}^-(z)} \begin{pmatrix} \tilde{\Phi}^+(z) & q_0 \tilde{\Phi}^-(z) \tilde{\Phi}^+(z) \\ q_0 \Phi^-(z) \tilde{\Phi}^+(z) & \tilde{\Phi}^-(z) \end{pmatrix}. \end{aligned}$$

In particular for the classical random walk ($p_i = p, q_i = q, r_i = 0 \ \forall \ i \in \mathbb{Z}$) we have

$$\tilde{\Phi}^+(z) = -\frac{z - \sqrt{z^2 - 4pq}}{2pq}; \quad \tilde{\Phi}^-(z) = \frac{p}{q}\Phi^+(z),$$

and a straightforward calculation gives the result

$$\Phi(z) = \begin{pmatrix} \frac{-1}{\sqrt{z^2-4pq}} & \frac{1}{2q}\left(1 - \frac{z}{\sqrt{z^2-4pq}}\right) \\ \frac{1}{2q}\left(1 - \frac{z}{\sqrt{z^2-4pq}}\right) & \frac{p}{q}\frac{-1}{\sqrt{z^2-4pq}} \end{pmatrix},$$

which was also obtained by Karlin and McGregor (1959) by a probabilistic argument.

## 3.2 An example from queuing theory

In a recent paper Dayar and Quessette (2002) considered a system of two independent queues, where queue 1 is an $M/M/1$ and queue 2 is an $M/M/1/d-1$. Both queues have a Poisson arrival process with rate $\lambda_i$ $(i = 1, 2)$ and exponential service distributions with rates $\mu_i\,(i = 1, 2)$. It is easy to see that the embedded random walk corresponding to the quasi birth and death process representing the length of queue 1 (which is unbounded) and the length of queue 2 (which varies between $0, 1, \ldots, d-1$) has a one step transition matrix of the form (1.2), where the blocks $B_i, A_i$ and $C_i$ are given by

$$(3.5)\, B_0 = \begin{pmatrix} 0 & \frac{\lambda_2}{\lambda_1+\lambda_2} & & & \\ \frac{\mu_2}{\gamma-\mu_1} & 0 & \frac{\lambda_2}{\gamma-\mu_1} & & \\ & \ddots & \ddots & \ddots & \\ & & \frac{\mu_2}{\gamma-\mu_1} & 0 & \frac{\lambda_2}{\gamma-\mu_1} \\ & & & \frac{\mu_2}{\lambda_1+\mu_2} & 0 \end{pmatrix}, \quad B_i = \begin{pmatrix} 0 & \frac{\lambda_2}{\gamma-\mu_2} & & & \\ \frac{\mu_2}{\gamma} & 0 & \frac{\lambda_2}{\gamma} & & \\ & \ddots & \ddots & \ddots & \\ & & \frac{\mu_2}{\gamma} & 0 & \frac{\lambda_2}{\gamma} \\ & & & \frac{\mu_2}{\gamma-\lambda_2} & 0 \end{pmatrix},$$

$(i \geq 1)$,

$$(3.6)\;\; A_0 = \begin{pmatrix} \frac{\lambda_1}{\lambda_1+\lambda_2} & & & & \\ & \frac{\lambda_1}{\gamma-\mu_1} & & & \\ & & \ddots & & \\ & & & \frac{\lambda_1}{\gamma-\mu_1} & \\ & & & & \frac{\lambda_1}{\lambda_1+\mu_2} \end{pmatrix}, \;\; A_i = \begin{pmatrix} \frac{\lambda_1}{\gamma-\mu_2} & & & & \\ & \frac{\lambda_1}{\gamma} & & & \\ & & \ddots & & \\ & & & \frac{\lambda_1}{\gamma} & \\ & & & & \frac{\lambda_1}{\gamma-\lambda_2} \end{pmatrix},$$

$(i \geq 1)$ and

$$(3.7) \qquad C_i = \begin{pmatrix} \frac{\mu_1}{\gamma-\mu_2} & & & & \\ & \frac{\mu_1}{\gamma} & & & \\ & & \ddots & & \\ & & & \frac{\mu_1}{\gamma} & \\ & & & & \frac{\mu_1}{\gamma-\lambda_2} \end{pmatrix}, (i \geq 1)$$

respectively, and $\gamma = \lambda_1 + \lambda_2 + \mu_1 + \mu_2, \lambda_1 < \mu_1$. A straightforward calculation shows that the assumptions of Theorem 2.1 are satisfied, where the matrices $R_\ell$ are diagonal and given by

$$
\begin{aligned}
R_0 &= \operatorname{diag}\Big(\frac{\sqrt{(\lambda_1+\lambda_2)\mu_2}}{\sqrt{(\gamma-\mu_1)\lambda_2}}, 1, \frac{\sqrt{\lambda_2}}{\sqrt{\mu_2}}, \frac{\lambda_2}{\mu_2}, \dots, (\frac{\sqrt{\lambda_2}}{\sqrt{\mu_2}})^{d-3}, \frac{\sqrt{(\lambda_1+\mu_2)\lambda_2^{d-2}}}{\sqrt{(\gamma-\mu_1)\mu_2^{d-2}}}\Big), \\
R_1 &= \operatorname{diag}\Big(\frac{\sqrt{\lambda_1(\gamma-\mu_2)\mu_2}}{\sqrt{\lambda_2(\gamma-\mu_1)\mu_1}}, \frac{\sqrt{\gamma\lambda_1}}{\sqrt{(\gamma-\mu_1)\mu_1}}, \dots, \frac{\sqrt{\gamma\lambda_1\lambda_2^{d-3}}}{\sqrt{(\gamma-\mu_1)\mu_1\mu_2^{d-3}}}, \frac{\sqrt{\lambda_1(\gamma-\lambda_2)\lambda_2^{d-2}}}{\sqrt{(\gamma-\mu_1)\mu_1\mu_2^{d-2}}}\Big) \\
R_i &= (\sqrt{\frac{\lambda_1}{\mu_1}})^{i-1} R_1, i \geq 2.
\end{aligned}
$$

It also follows from Theorem 2.5 that the corresponding random walk matrix measure is supported in the interval $[-1, 1]$.

### 3.3 The simple random walk on the grid

Consider the random walk on the grid $\mathcal{C}_d$, where the probabilities of going from state $(i, j)$ to $(i, j+1)$, $(i, j-1)$, $(i-1, j)$, $(i+1, j)$ are given by $u, v, \ell, r$, respectively, where $u + v + \ell + r = 1$. In this case it follows that $A_i = rI_d\,(i \geq 0)$, $C_i = \ell I_d\,(i \geq 1)$,

$$
B_i = \begin{pmatrix} 0 & u & & & & \\ v & 0 & u & & & \\ & v & 0 & u & & \\ & & \ddots & \ddots & \ddots & \\ & & & v & 0 & u \\ & & & & v & 0 \end{pmatrix}, i \geq 0,
$$

and it is easy to see that the conditions of Theorem 2.1 are satisfied with

$$
R_0 = diag(1, \sqrt{\frac{u}{v}}, \sqrt{\frac{u^2}{v^2}}, ..., \sqrt{\frac{u^{d-1}}{v^{d-1}}}),\ R_i = (\sqrt{\frac{r}{\ell}})^i R_0,\ i \geq 1.
$$

It now follows from Theorem 2.5 that the corresponding random walk matrix measure is supported in the interval $[-1, 1]$. For the identification of the Stieltjes transform of the spectral measure we note that the orthonormal polynomials defined by (2.7) have constant coefficients given by $D = D_n = \sqrt{r\ell} I_d$,

$$
E = E_n = \begin{pmatrix} 0 & \sqrt{vu} & & & & \\ \sqrt{vu} & 0 & \sqrt{vu} & & & \\ & \sqrt{vu} & 0 & \sqrt{vu} & & \\ & & \ddots & \ddots & \ddots & \\ & & & \sqrt{vu} & 0 & \sqrt{vu} \\ & & & & \sqrt{vu} & 0 \end{pmatrix}. \tag{3.8}
$$

Therefore it follows from the work of Duran (1999) that the Stieltjes transform of the random walk measure is given by

$$\int \frac{d\Sigma(t)}{z-t} = \frac{1}{2r\ell}\Big\{zI_d - E - \Big\{(zI_d-E)^2 - 4r\ell I_d\Big\}^{1/2}\Big\}.$$

From the same reference we obtain that the support of the random walk measure is given by the set

$$\text{supp}(\Sigma) = \{x \in \mathbb{R} \mid xI_d - E \text{ has an eigenvalue in } [-2\sqrt{r\ell}, 2\sqrt{r\ell}]\}. \tag{3.9}$$

It is well known [see Basilevsky (1983)] that the eigenvalues of the matrix $E$ in (3.8) are given by

$$2\sqrt{uv}\cos(\frac{j\pi}{d+1}) \quad , j = 1, \dots, d,$$

with corresponding normalized eigenvectors

$$x_j = \sqrt{\frac{2}{d+1}}\Big(\sin(\ell\frac{\pi j}{d+1})\Big)_{\ell=1}^{d}.$$

Therefore it follows from (3.9) that

$$\text{supp}(\Sigma) = \Big[-2\sqrt{r\ell} + 2\sqrt{uv}\cos(\frac{\pi d}{d+1}), 2\sqrt{r\ell} + 2\sqrt{uv}\cos(\frac{\pi}{d+1})\Big]$$

(note that $\text{supp}(\Sigma) \subset [-1,1]$). For the calculation of the random walk measure we determine the spectral decomposition of the matrix

$$\begin{aligned} -H(x) &= 4I_d - D^{-1/2}(xI_d - E)D^{-1}(xI_d - E)D^{-1/2} \\ &= \frac{1}{r\ell}\Big\{4r\ell I_d - (xI_d - E)^2\Big\}. \end{aligned}$$

The eigenvalues of this matrix are given by

$$\lambda_j(x) = \frac{1}{r\ell}\Big\{4r\ell - (x - 2\sqrt{vu}\cos(\frac{\pi j}{d+1}))^2\Big\},$$

and by the results in Duran (1999) the weight of the matrix measure is given by

$$d\Sigma(x) = \frac{1}{2\pi\sqrt{r\ell}} U\Lambda(x)U^T dx,$$

where the matrix $\Lambda(x)$ is defined by

$$\Lambda(x) = \Big\{\text{diag}(\max(\lambda_1(x), 0), \dots, \max(\lambda_d(x), 0))\Big\}^{1/2},$$

and the elements of the matrix $U = \{u_{j\ell}\}_{j,\ell=1,\dots,d}$ are given by

$$u_{j\ell} = \sqrt{\frac{2}{d+1}}\sin(\ell\frac{j\pi}{d+1}).$$

### 3.4 A random walk on a graph

Consider a graph with $d$ rays which are connected at one point, the origin. On each ray the probability of moving away from the origin is $p$ and moving in one step towards to the origin is $q$, where $p+q=1$. From the origin the probability of going to the $i$th ray is $d_i > 0\,(i=1,\ldots,d)$ [see Figure 1, where the case $d=4$ is illustrated]. It is easy to see that this process corresponds to a

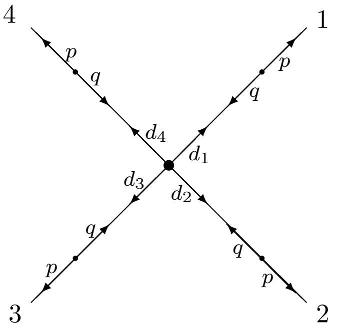


Figure 1: *A random walk on a graph*

random walk on the grid $\mathcal{C}_d$ with block tridiagonal transition matrix $P$ in (1.2), where $B_i = 0$ if $i \geq 1$, $C_i = qI_d \;\forall\; i \geq 1$, $A_0 = \operatorname{diag}(d_1, p, \ldots, p)$, $A_i = pI_d \;\forall\; i \geq 1$, and

$$B_0 = \begin{pmatrix} 0 & d_2 & \cdots & \cdots & d_d \\ q & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & & & \vdots \\ q & 0 & \cdots & \cdots & 0 \end{pmatrix},$$

where $\sum_{i=1}^d d_i = 1$. Moreover, this matrix clearly satisfies the assumptions of Theorem 2.1 with

$$R_0 = \operatorname{diag}\left(1, \sqrt{\frac{d_2}{q}}, ..., \sqrt{\frac{d_d}{q}}\right),\; R_1 = \operatorname{diag}\left(\sqrt{\frac{d_1}{q}}, \sqrt{\frac{d_2 p}{q^2}}, \ldots, \sqrt{\frac{d_d p}{q^2}}\right),\; R_i = \left(\sqrt{\frac{p}{q}}\right)^{i-1} R_1,\; i \geq 2.$$

By an application of Theorem 2.6 and the inversion formula for Stieltjes transforms we obtain for the corresponding random walk measure

$$d\Sigma(x) = \begin{bmatrix} a(x) & b_2(x) & b_3(x) & \ldots & b_d(x) \\ b_2(x) & f_2(x) & e_{2,3}(x) & \ldots & e_{2,d}(x) \\ b_3(x) & e_{2,3}(x) & f_3(x) & \ldots & e_{3,d}(x) \\ \vdots & \vdots & \vdots & & \vdots \\ b_{d-1}(x) & e_{2,d-1}(x) & e_{3,d-1}(x) & \ldots & e_{d-1,d}(x) \\ b_d(x) & e_{2,d}(x) & e_{3,d}(x) & \ldots & f_d(x) \end{bmatrix} dx,$$

where functions $a, b_i, e_{k,\ell}$ and $f_k$ are given by

$$\begin{aligned}
a(x) &= \frac{(\sum_{i=2}^d d_i^2 d_1 + d_1^2 q - (d_1-p)x^2)\sqrt{4pq-x^2}}{2p\pi((\sum_{i=2}^d d_i^2 + d_1 q)^2 - (\sum_{i=2}^d d_i^2 + (d_1-p)q)x^2)}, \\
b_k(x) &= -\frac{d_k x\sqrt{4pq-x^2}}{2\pi((\sum_{i=2}^d d_i^2 + d_1 q)^2 - (\sum_{i=2}^d d_i^2 + (d_1-p)q)x^2)}, \quad k = 2, \dots, d, \\
e_{k,\ell}(x) &= \frac{d_k d_\ell \sqrt{4pq-x^2}(px^2 - \sum_{i=2}^d d_j^2 d_1 - d_1^2 q)}{2\pi(d_1^2 q - (d_1-p)x^2)((\sum_{i=2}^d d_i^2 + d_1 q)^2 - (\sum_{i=2}^d d_i^2 + (d_1-p)q)x^2)}, \\
&\quad k = 2, \dots, d-1, \ \ell = 3, \dots, d, \\
f_k(x) &= \frac{(d_1(\sum_{i=2}^d d_i^2 + d_1 q)(\sum_{i=2, i\neq k}^d d_i^2 + d_1 q)\sqrt{4pq-x^2}}{2\pi(d_1^2 q - (d_1-p)x^2)((\sum_{i=2}^d d_i^2 + d_1 q)^2 - (\sum_{i=2}^d d_i^2 + (d_1-p)q)x^2)} \\
&\quad + \frac{-((\sum_{i=2}^d d_i^2(d_1-p) + \sum_{i=2, i\neq k}^d d_i^2 p + d_1(d_1-p)q)x^2)\sqrt{4pq-x^2}}{2\pi(d_1^2 q - (d_1-p)x^2)((\sum_{i=2}^d d_i^2 + d_1 q)^2 - (\sum_{i=2}^d d_i^2 + (d_1-p)q)x^2)}, \\
&\quad k = 2, \dots d.
\end{aligned}$$

Note that the random walk measure is supported in the interval $[-2\sqrt{pq}, 2\sqrt{pq}]$.

# 4 Further discussion

In the present section we derive further consequences of the existence of a random walk measure corresponding to the transition matrix (1.2). Throughout this section we assume that the conditions of Theorem 2.1 are satisfied and that the corresponding random walk measure is supported in the interval $[-1, 1]$.

## 4.1 Recurrence

We denote by

$$H_{ij}(z) = \sum_{n=0}^{\infty} (P_{ij}^n) z^n = \Big( \int \frac{Q_i(x) d\Sigma(x) Q_j^T(x)}{1-xz} \Big) \Big( \int Q_j(x) d\Sigma(x) Q_j^T(x) \Big)^{-1} \tag{4.1}$$

the (matrix) generating function of the block $(i, j)$, where the last identity follows from Theorem 2.3 and Lebesgue's Theorem. Therefore we obtain that a state $(i, \ell) \in \mathcal{C}_d$ is recurrent if and only if

$$\sum_{n=0}^{\infty} e_\ell^T P_{ii}^n e_\ell = \lim_{z \to 1} e_\ell^T H_{ii}(z) e_\ell$$

(4.2)

$$= \; e_\ell^T \Big(\int \frac{Q_i(x)d\Sigma(x)Q_i^T(x)}{1-x}\Big)\Big(\int Q_i(x)d\Sigma(x)Q_i^T(x)\Big)^{-1} e_\ell = \infty,$$

where $e_\ell^T = (0,\dots,0,1,0,\dots 0)^T$ denotes the $\ell$th unit vector in $\mathbb{R}^d$. We summarize this observation in the following Lemma.

**Corollary 4.1.** *Assume that the conditions of Theorem 2.1 are satisfied for the transition matrix $P$ in (1.2) corresponding to a random walk on $\mathcal{C}_d$ and that the corresponding spectral measure is supported in the interval $[-1,1]$. A state $(i,\ell) \in \mathcal{C}_d$ is recurrent if and only if condition (4.2) is satisfied. Moreover, if the random walk is irreducible it is recurrent if and only if the condition*

$$e_j^T \int_{-1}^{1} \frac{d\Sigma(x)}{1-x} S_0^{-1} e_j = \infty \tag{4.3}$$

*is satisfied for some $j \in \{1,\dots,d\}$ (in this case it is satisfied for any $j \in \{1,\dots,d\}$).*

**Corollary 4.2.** *Assume that the conditions of Theorem 2.1 are satisfied for the matrix $P$ in (1.2) corresponding to an irreducible random walk on $\mathcal{C}_d$ and that the corresponding spectral measure is supported in the interval $[-1,1]$. The random walk is positive recurrent if and only if one of the measures $d\tau_\ell(x) = e_\ell^T d\Sigma(x) S_0^{-1} e_\ell$ ($\ell = 1,\dots,d$) has a jump at the point 1. In this case all measures $d\tau_\ell(x)$ ($\ell = 1,\dots,d$) have a jump at the point 1.*

**Proof.** Let $d\tau_\ell(x) = e_\ell^T d\Sigma(x) S_0^{-1} e_\ell$, then the probability of returning from state $(0,\ell)$ to $(0,\ell)$ in $k$ steps is given by

$$\alpha_k = e_\ell^T (P_{00}^k) e_\ell = e_\ell^T \int_{-1}^{1} x^k d\Sigma(x) S_0^{-1} e_\ell = \int_{-1}^{1} x^k d\tau_\ell(x).$$

The random walk is positive recurrent if and only if $\alpha = \lim_{k\to\infty} \alpha_k$ exists and is positive. Considering the sequence $\alpha_{2n}$ it follows by the dominated convergence theorem that this is the case if and only if $\tau_\ell$ has a jump at $x = -1$ or $x = 1$. If $\tau_\ell$ has no jump at $x = 1$ we obtain

$$\begin{aligned} \tau_\ell(-1) &= \lim_{n\to\infty} \Big\{ -\int_{-1}^{1} x^{2n+1} d\tau_\ell(x) + \int_{-1^-}^{1} x^{2n+1} d\tau_\ell(x) \Big\} \\ &= -\lim_{n\to\infty} P_{00}^{2n+1} \le 0 \end{aligned}$$

and consequently $\tau_\ell$ has no jump at $x = -1$. Therefore the random walk is positive recurrent if and only if $\tau_\ell$ has a jump at $x = 1$.

□

**Remark 4.3.** For an irreducible random walk with a random walk measure $\Sigma$ satisfying $S_0 = I_d$ the properties of recurrence and positive recurrence are characterized by the diagonal elements of the corresponding random walk measure $\Sigma$ .

## 4.2 Canonical moments and random walk measures

In this section we will represent the Stieltjes transform of a random walk matrix measure $\Sigma$ which is supported in the interval $[-1, 1]$ in terms of its canonical moments, which were recently introduced by Dette and Studden (2001) in the context of matrix measures. We will use this representation to derive a characterization of recurrence of the process in terms of blocks of the matrix $P$.

**Theorem 4.4.** *The Stieltjes transform of a random walk measure $\Sigma$ which is supported in the interval $[-1, 1]$ has the following continued fraction expansions*

$$\begin{aligned}\int \frac{d\Sigma(x)}{z-x} &= \lim_{n\to\infty} S_0^{1/2}\Big\{zI_d + I_d - 2\zeta_1^T - \Big\{zI_d + I_d - 2\zeta_2^T - 2\zeta_3^T - \Big\{zI_d + I_d - 2\zeta_4^T - 2\zeta_5^T \dots \\ &\qquad \dots - \Big\{zI_d + I_d - 2\zeta_{2n}^T - 2\zeta_{2n+1}^T\Big\}^{-1} 4\zeta_{2n}^T\zeta_{2n-1}^T\Big\}^{-1} \dots 4\zeta_4^T\zeta_3^T\Big\}^{-1} 4\zeta_2^T\zeta_1^T\Big\}^{-1} S_0^{1/2} \\ &= \lim_{n\to\infty} S_0^{1/2}\Big\{(z+1)I_d - \Big\{I_d - \Big\{(z+1)I_d - \\ &\qquad \dots - \Big\{(z+1)I_d - 2\zeta_{2n+1}^T\Big\}^{-1} 2\zeta_{2n}^T\Big\}^{-1} \dots\Big\}^{-1} 2\zeta_2^T\Big\}^{-1} 2\zeta_1^T\Big\}^{-1} S_0^{1/2},\end{aligned}$$

*where the quantities $\zeta_j \in \mathbb{R}^{d\times d}$ are defined by $\zeta_0 = 0$, $\zeta_1 = U_1$, $\zeta_j = V_{j-1}U_j$ if $j \geq 2$ and the sequences $\{U_j\}$ and $\{V_j\}$ are the canonical moments of the random walk measure $\Sigma$. The convergence is uniform on compact subsets of $\mathbb{C}$ with positive distance from the interval $[-1, 1]$. In particular the following representation holds*

$$\int \frac{d\Sigma(x)}{1-x} = \frac{1}{2} S_0^{1/2}\Big[I_d + \sum_{l=1}^{\infty} (V_1^T)^{-1} \dots (V_l^T)^{-1} U_l^T \dots U_1^T\Big] S_0^{1/2}. \tag{4.4}$$

**Proof.** Let $\underline{P}_n(t)$ denote the $n$th monic orthogonal polynomial with respect to the matrix measure $d\Sigma(t)$, then it follows from Dette and Studden (2001) that $\underline{P}_n(t)$ can be calculated recursively as

$$\underline{P}_{n+1}(t) = \Big\{(t+1)I_d - 2\zeta_{2n+1}^T - 2\zeta_{2n}^T\Big\}\underline{P}_n(t) - 4\zeta_{2n}^T\zeta_{2n-1}^T\underline{P}_{n-1}(t), \tag{4.5}$$

where $\underline{P}_0(t) = I_d, \underline{P}_{-1}(t) = 0$, the quantities $\zeta_j \in \mathbb{R}^{d\times d}$ are defined by $\zeta_0 = 0$, $\zeta_1 = U_1$, $\zeta_j = V_{j-1}U_j$ if $j \geq 2$ and the sequences $\{U_j\}$ and $\{V_j\}$ are the canonical moments of the random walk measure $\Sigma$. Note that Dette and Studden (2001) define the canonical moments for matrix measures on the interval $[0, 1]$, but the canonical moments are invariant with respect to transformations of the measure. More precisely, it can be shown that measures related by an affine transformation $t \to a + (b-a)t$ $(a, b \in \mathbb{R},\ a < b)$ have the same canonical moments. The results for the corresponding orthogonal polynomials can also easily be extended to matrix measures on the interval $[-1, 1]$. The quantities

$$\Delta_{2n} := < \underline{P}_n, \underline{P}_n > = 2^{2n}(S_0\zeta_1 \dots \zeta_{2n})^T \tag{4.6}$$

are positive definite [see Dette and Studden (2001)] and consequently the polynomials

$$P_n(z) = \Delta_{2n}^{-1/2} \underline{P}_n(z)$$

are orthonormal with respect to the measure $d\Sigma(x)$. Now a straightforward calculation shows that these polynomials satisfy the recurrence relation

$$tP_k(t) = A_{k+1}P_{k+1}(t) + B_kP_k(t) + A_k^T P_{k-1}(t),\ k = 0, 1, ... \tag{4.7}$$

with initial conditions

$$P_{-1}(t) = 0,\ P_0(t) = S_0^{-1/2} \tag{4.8}$$

and coefficients

$$\begin{aligned} A_{n+1} &= \Delta_{2n}^{-1/2}\Delta_{2n+2}^{1/2}, && (4.9)\\ B_n &= -\Delta_{2n}^{-1/2}(I_d - 2\zeta_{2n}^T - 2\zeta_{2n+1}^T)\Delta_{2n}^{1/2}, && (4.10)\\ A_n^T &= 4\Delta_{2n}^{-1/2}\zeta_{2n}^T\zeta_{2n-1}^T\Delta_{2n-2}^{1/2} && (4.11) \end{aligned}$$

(note that the matrix $\Delta_{2n} = 4\Delta_{2n-2}\zeta_{2n-1}\zeta_{2n}$ is symmetric and therefore the two representations in (4.9) and (4.11) for the matrix $A_n$ are in fact identical). If $P_n^{(1)}(z)$ denotes the first associated orthogonal polynomial corresponding to $P_n(z)$ we obtain from Zygmunt (2002) the representation

$$\begin{aligned} F_n(z) = (P_{n+1}(z))^{-1}P_{n+1}^{(1)}(z) &= S_0\{zI_d - B_0 - A_1\{zI_d - B_1 - A_2\{zI_d - B_2 - \dots \\ &\qquad \dots - A_n\{zI_d - B_n\}^{-1}A_n^T\}^{-1}\dots A_1^T\}^{-1}. \end{aligned} \tag{4.12}$$

Now a straightforward application of (4.9) - (4.11) yields

$$\begin{aligned} F_n(z) &= S_0^{1/2}[zI_d + I_d - 2\zeta_1^T - [zI_d + I_d - 2\zeta_2^T - 2\zeta_3^T - [zI_d + I_d - 2\zeta_4^T - 2\zeta_5^T \dots \\ &\quad \dots - [zI_d + I_d - 2\zeta_{2n}^T - 2\zeta_{2n+1}^T]^{-1}4\zeta_{2n}^T\zeta_{2n-1}^T]^{-1}\dots 4\zeta_4^T\zeta_3^T]^{-1}4\zeta_2^T\zeta_1^T]^{-1}S_0^{1/2}, \end{aligned} \tag{4.13}$$

and an iterative application of the matrix identity

$$I_d + A^{-1}B = (I_d - (B + A)^{-1}B)^{-1}$$

and Markov's theorem [see Duran (1996)] give

$$\begin{aligned} \int \frac{d\Sigma(x)}{z - x} &= \lim_{n\to\infty} S_0^{1/2}\Big\{(z+1)I_d - \Big\{I_d - \Big\{(z+1)I_d - \\ &\qquad \dots - \Big\{(z+1)I_d - 2\zeta_{2n+1}^T\Big\}^{-1}2\zeta_{2n}^T\Big\}^{-1}\dots\Big\}^{-1}2\zeta_2^T\Big\}^{-1}2\zeta_1^T\Big\}^{-1}S_0^{1/2} \end{aligned}$$

(note that this transformation is essentially a contraction). This proves the first part of the theorem. For the second part we put $z = 1$ and use formula (1.3) in Fair (1971) to obtain

$$\begin{aligned} \int \frac{d\Sigma(x)}{1 - x} &= \lim_{n\to\infty} \frac{1}{2}S_0^{1/2}\Big\{I_d - \Big\{I_d - \Big\{I_d - \dots - \Big\{I_d - \zeta_{2n+1}^T\Big\}^{-1}\zeta_{2n}^T\Big\}^{-1}\dots\Big\}^{-1}\zeta_1^T\Big\}^{-1}S_0^{1/2} \\ &= \lim_{n\to\infty} \frac{1}{2}S_0^{1/2}\sum_{j=0}^{n+1} X_{j+1}^{-1}\zeta_j^T X_{j-1}X_j^{-1}\zeta_{j-1}^T X_{j-2}X_{j-1}^{-1}\dots X_1X_2^{-1}\zeta_1^T S_0^{1/2}, \end{aligned} \tag{4.14}$$

where $X_0 = I_d, X_1 = I_d$,

$$X_{n+1} = X_n - \zeta_n^T X_{n-1} \quad (n \geq 1).$$

Now a straightforward induction argument shows that $X_{n+1} = V_n^T ... V_1^T$ and (4.14) reduces to (4.4), which proves the remaining assertion of the theorem.

□

Our next result generalizes the famous characterization of recurrence in an irreducible birth and death chain to the matrix case.

**Theorem 4.5.** *Assume that the conditions of Theorem 2.1 are satisfied for the transition matrix of a random walk and that the corresponding spectral measure is supported in the interval* $[-1, 1]$. *The state* $(0, \ell)$ *is recurrent if and only if*

$$e_\ell^T S_0^{1/2} \sum_{i=0}^{\infty} T_{i+1}^{-1} A_i^{-1} C_i^T T_{i-1} T_i^{-1} A_{i-1}^{-1} C_{i-1}^T T_{i-2} T_{i-1}^{-1} \ldots T_1 T_2^{-1} A_1^{-1} C_1^T T_0 T_1^{-1} A_0^{-1} T_0 S_0^{-1/2} e_\ell = \infty,$$

*where* $T_i = Q_i(1)$ $(i \in \mathbb{N}_0)$ *and* $Q_i(x)$ *denotes the ith random walk polynomial defined by (1.4). In particular, an irreducible random walk on the grid* $\mathcal{C}_d$ *is recurrent if and only if one of the diagonal elements of the matrix*

$$S_0^{1/2} \sum_{i=0}^{\infty} T_{i+1}^{-1} A_i^{-1} C_i^T T_{i-1} T_i^{-1} A_{i-1}^{-1} C_{i-1}^T T_{i-2} T_{i-1}^{-1} \ldots T_1 T_2^{-1} A_1^{-1} C_1^T T_0 T_1^{-1} A_0^{-1} T_0 S_0^{-1/2}$$

*is infinite (in this case all diagonal elements of this matrix have this property).*

**Proof.** A combination of Corollary 4.1 and Theorem 4.4 shows that the state $(0, \ell)$ is recurrent if and only if

$$t = \frac{1}{2} e_\ell^T S_0^{1/2} [I_d + \sum_{j=1}^{\infty} (V_j^T)^{-1} ... (V_l^T)^{-1} U_j^T \ldots U_1^T] S_0^{-1/2} e_\ell = \infty, \tag{4.15}$$

where $U_1, U_2, \ldots$ are the canonical moments of the random walk measure $\Sigma$ and $V_j = I_d - U_j$ $(j \geq 1)$. In the following we express the right hand side in terms of the blocks of the one-step transition matrix $P$ corresponding to the random walk. For this consider the recurrence relation (1.4) and define $T_n = Q_n(1)$. Note that the polynomials $\underline{Q}_n(t) = A_0 \ldots A_{n-1} Q_n(t)$ are monic and satisfy the recurrence relation

$$\begin{aligned} \underline{Q}_{n+1}(t) = t\underline{Q}_n(t) \quad &- \quad A_0 \ldots A_{n-1} B_n A_{n-1}^{-1} \ldots A_0^{-1} \underline{Q}_n(t) \\ &- A_0 \ldots A_{n-1} C_n^T A_{n-2}^{-1} \ldots A_0^{-1} \underline{Q}_{n-1}(t). \end{aligned}$$

Therefore a comparison with (4.5) yields

$$\begin{aligned} A_0 \ldots A_{n-1} B_n A_{n-1}^{-1} \ldots A_0^{-1} &= -I_d + 2\zeta_{2n}^T + 2\zeta_{2n+1}^T \\ A_0 \ldots A_{n-1} C_n^T A_{n-2} \ldots A_0 &= 4\zeta_{2n}^T \zeta_{2n-1}^T. \end{aligned} \tag{4.16}$$

Using these representations and the fact $U_kV_k = V_kU_k$ [see Dette and Studden (2001), Theorem 2.7] it is easy to see that

$$T_n = Q_n(1) = 2^n A_{n-1}^{-1} \dots A_0^{-1} V_{2n-1}^T V_{2n-2}^T \dots V_1^T \ ,$$

and it follows from the same reference that these matrices are non-singular for all $n \in \mathbb{N}_0$. Therefore we can define

$$\hat{Q}_n(x) = T_n^{-1} Q_n(x), \tag{4.17}$$

and it is easy to see that these polynomials satisfy the recurrence relation

$$x\hat{Q}_n(x) = \hat{A}_n \hat{Q}_{n+1}(x) + \hat{B}_n \hat{Q}_n(x) + \hat{C}_n^T \hat{Q}_{n-1}(x), \tag{4.18}$$

where

$$\hat{A}_n = T_n^{-1} A_n T_{n+1}, \quad \hat{B}_n = T_n^{-1} B_n T_n, \quad \hat{C}_n^T = T_n^{-1} C_n^T T_{n-1} \tag{4.19}$$

(note that $\hat{A}_n + \hat{B}_n + \hat{C}_n^T = I_d$). Combining (4.16) with (4.19) we obtain

$$\begin{aligned}
\hat{A}_0 \dots \hat{A}_{n-1} \hat{B}_n \hat{A}_{n-1}^{-1} \dots \hat{A}_0^{-1} &= -I_d + 2\zeta_{2n}^T + 2\zeta_{2n+1}^T, \\
\hat{A}_0 \dots \hat{A}_{n-1} \hat{C}_n^T \hat{A}_{n-2}^{-1} \dots \hat{A}_0^{-1} &= 4\zeta_{2n}^T \zeta_{2n-1}^T,
\end{aligned}$$

and by an induction argument (noting that $\hat{A}_n + \hat{B}_n + \hat{C}_n^T = I_d$) it follows that

$$\begin{aligned}
2U_{2n}^T U_{2n-1}^T &= \hat{A}_0 \dots \hat{A}_{n-1} \hat{C}_n^T \hat{A}_{n-1}^{-1} \dots \hat{A}_0^{-1}, \\
2V_{2n+1}^T V_{2n}^T &= \hat{A}_0 \dots \hat{A}_{n-1} \hat{A}_n \hat{A}_{n-1}^{-1} \dots \hat{A}_0^{-1}.
\end{aligned}$$

Finally, we obtain for the left hand side of (4.15)

$$\begin{aligned}
t &= \frac{1}{2} e_\ell^T S_0^{1/2} \sum_{j=0}^{\infty} \Big\{ (V_1^T)^{-1} \dots (V_{2j}^T)^{-1} U_{2j}^T \dots U_1^T + (V_1^T)^{-1} \dots (V_{2j+1}^T)^{-1} U_{2j+1}^T \dots U_1^T \Big\} S_0^{-1/2} e_\ell \\
&= e_\ell^T S_0^{1/2} \sum_{j=0}^{\infty} \hat{A}_j^{-1} \hat{C}_j^T \hat{A}_{j-1}^{-1} \dots \hat{A}_1^{-1} \hat{C}_1^T \hat{A}_0^{-1} S_0^{-1/2} e_\ell \\
&= e_\ell^T S_0^{1/2} \sum_{j=0}^{\infty} T_{j+1}^{-1} A_j^{-1} C_j^T T_{j-1} T_j^{-1} A_{j-1}^{-1} C_{j-1}^T T_{j-2} T_{j-1}^{-1} \dots T_1 T_2^{-1} A_1^{-1} C_1^T T_0 T_1^{-1} A_0^{-1} T_0 S_0^{-1/2} e_\ell
\end{aligned}$$

with $T_\ell = Q_\ell(1)$ $(\ell \in \mathbb{N}_0)$, which proves the assertion of the theorem.

□

**Remark 4.6.** It is interesting to note the condition in Theorem 4.5 simplifies substantially if the matrices $T_i, A_i, C_i$ are communicating. In this case a irreducible random walk is recurrent if and only if

$$e_\ell^T S_0^{1/2} \sum_{i=0}^{\infty} T_{i+1}^{-1} T_i^{-1} (C_1 \dots C_i)^T (A_0 \dots A_i)^{-1} S_0^{-1/2} e_\ell = \infty$$

for some $\ell \in \{1, \ldots, d\}$.

**Example 4.7.** Consider the random walk on the graph introduced in Section 3.4. By Corollary 4.1 the state $(0,1)$ (which corresponds to the origin) is recurrent if and only if

$$\infty = e_1^T \Big( \int \frac{d\Sigma(x)}{1-x} \Big) \Big( \int d\Sigma(x) \Big)^{-1} e_1 = \int_{2\sqrt{pq}}^{2\sqrt{pq}} \frac{a(x)}{1-x} dx$$

where the functions $a$ is defined in Section 3.4 and we have used the fact that $\int d\Sigma(x) = S_0 = (R_0^T R_0)^{-1}$ [see Remark 2.2]. Because the support of the spectral measure is given by the interval $[-\sqrt{4pq}, -\sqrt{4pq}]$ it follows that the condition $p = q = \frac{1}{2}$ is necessary for the recurrence of the random walk. Now a straightforward calculation shows that the state $(0,1)$ (i.e. the center of the graph) is recurrent if and only if the condition $2\sum_{i=2}^d d_i^2 = \sum_{i=2}^d d_i$ is satisfied (in all other cases the integral is finite).

**Acknowledgements.** This work was done while H. Dette was visiting the Department of Statistics, Purdue University, and this author would like to thank the department for its hospitality. The work of H. Dette was supported by the Deutsche Forschungsgemeinschaft (De 502/22-1, SFB 475, Komplexitätsreduktion in multivariaten Datenstrukturen). The authors would like to thank Isolde Gottschlich, who typed this paper with considerable technical expertise.

# References

A. Basilevsky (1983). Applied Matrix Algebra in the Statistical Sciences. North-Holland.

T. Dayar, F. Quessette (2002). Quasi-birth and death processes with level geometric distribution. SIAM J. Matrix Anal. Appl. 24, 281-291.

H. Dette (1996). On the generating functions of a random walk on the nonnegative integers. J. Appl. Prob. 33, 1033-1052.

H. Dette, W.J. Studden (1997). The Theory of Canonical Moments with Applications in Statistics, Probability, and Analysis, Wiley.

H. Dette, W.J. Studden (2001). Matrix measures, moment spaces and Favard's theorem for the interval $[0,1]$ and $[0,\infty)$. Linear Algebra Appl. 345, 163 - 193.

A.J. Duran (1995). On orthogonal polynomials with respect to a positive definite matrix of measures, Can. J. Math. 47, 88-112.

A.J. Duran (1996). Markov's Theorem for orthogonal matrix polynomials, Can. J. Math. 48, 1180-1195.

A.J. Duran (1999). Ratio asymptotics for orthogonal matrix polynomials. Journal of Approximation Theory 100, 304-344.

A. J. Duran, W. Van Assche (1995). Orthogonal matrix polynomials and higher-order recurrence relations. Linear Algebra and its Applications 219, 261-280.

W. Fair (1971). Noncommutative continued fractions. SIAM J. Math. Anal. 2, 226-232.

W. Feller (1950). An introduction to probability theory and its applications. Vol. I, N.Y.

P.R. Halmos, V.S. Sunder (1978). Bounded Integral Operators on $L^2$-Spaces. Springer, N.Y.

S. Karlin, J. McGregor (1959). Random walks. Illionis J. Math. 3, 66-81.

S. Karlin, H.M. Taylor (1975). A First Course In Stochastic Processes. Academic Press, N.Y.

F. Marcellán, G. Sansigre (1993). On a class of matrix orthogonal polynomials on the real line. Linear Algebra Appl. 181, 97-109 .

I. Marek (2003). Quasi-birth and death processes, level geometric distributions. An aggregation / disaggregation approach. J. Comp. Appl. Math. 152.

M.F. Neuts (1981). Matrix Geometric Solutions in Stochastic Models. An Algorithmic Approach. The John Hopkins University Press, Baltimore, MD.

M.F. Neuts (1989). Structured Stochastic Matrices of M/G/1 Type and their Applications. Marcel Dekker. New York.

L. Rodman (1990). Orthogonal matrix polynomials. In: P. Nevai (ed.), Orthogonal polynomials: theory and practice. NATO ASI Series C, Vol. 295; Kluwer, Dordrecht.

A. Sinap, W. Van Assche (1996). Orthogonal Matrix polynomials and applications. Jour. Computational and Applied Math. 66: 27-52.

E.A. van Doorn, P. Schrijner (1993). Random walk polynomials and random walk measures. J. Comput. Appl. Math. 49, 289-296.

E.A. van Doorn, P. Schrijner (1995). Geometric ergodicity and quasi-stationarity in discrete-time birth-death processes. J. Aust. Math. Soc., Ser. B 37, 121-144.

T.A. Whitehurst (1982). An application of orthogonal polynomials to random walks. Pacific J. Math. 99, 205-213.

W. Woess (1985). Random walks and periodic continued fractions. Adv. Appl. Probab. 17, 67-84.

M.J. Zygmunt (2002). Matrix Chebyshev polynomials and continued fractions. Linear Algebra and its Applications 340, 155-168.