Confidence Intervals for the Between Group Variance in the Unbalanced One–Way Random Effects Model of Analysis of Variance

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Abstract

A confidence interval for the between group variance is proposed which is deduced from Wald's exact confidence interval for the ratio of the two variance components in the one–way random effects model and the exact confidence interval for the error variance resp. an unbiased estimator of the error variance. In a simulation study the confidence coefficients for these two intervals are compared with the confidence coefficients of two other commonly used confidence intervals. There, the confidence interval derived here yields confidence coefficients which are always greater than the prescribed level.

Key Words: variance components, unbalanced one–way nested model, Wald's confidence interval

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1 Introduction

In the present paper we consider confidence intervals for the between group variance in the unbalanced one–way random effects model of analysis of variance (ANOVA). In the case of a balanced design one method for constructing a confidence interval for the between group variance was independently proposed by Tukey (1951) and Williams (1962). The Tukey–Williams–method is based on two quadratic forms in normal variables which are exactly distributed as multiples of χ^2–distributed random variables and the expectations of these quadratic forms are parametric functions of the between group variance. Thus, for these two parametric functions of the between group variance exact $(1-\kappa)$–confidence intervals can be calculated and by solving the intersection of these two confidence intervals, a confidence interval of the between group variance is given which has a confidence coefficient at least as great as $1-2\kappa$ due to Bonferroni's inequality. The results of simulation studies conducted by Boardman (1974) indicated that the confidence coefficient of the Tukey–Wiliams–interval is near $1-\kappa$ (cf. also Graybill (1976, p. 620)) and Wang (1990) showed that the confidence coefficient of this interval is at least $1-\kappa$.

Following the Tukey–Williams approach Thomas and Hultquist (1978) proposed a confidence interval for the between group variance in the unbalanced case where the distributions of the two involved quadratic forms in normal variables can only be approximated by multiples of a χ^2–distributed random variable, but again the expected values of the quadratic forms are parametric functions of the between group variance. The approximation to a χ^2–distribution is, however, not satisfactory if the ratio of between and within group variances is less than 0.25 and the design is rather unbalanced. To overcome this problem Burdick, Maqsood and Graybill (1986) considered a conservative confidence interval for the ratio of between and within group variance, which was used in Burdick and Eickman (1986) to construct a confidence interval for the between group variance based on the ideas of the Tukey–Williams method. In Burdick and Eickman (1986) a comparison of the confidence coefficients of the Thomas–Hultquist–interval and the Burdick–Eickman–interval are given by simulation studies. The results of the simulations studies indicated that the confidence coefficient is near $1-\kappa$ in most cases. If the approximation to a χ^2–

distribution in the Thomas–Hultquist approach is not so good, the resulting confidence interval can be very liberal, while in these situations the Burdick–Eickman–intervall can be very conservative.

Now we propose a confidence interval for the between group variance in the unbalanced design which is constructed from an exact confidence interval for the ratio of between and within group variance derived from Wald (1940), cf. also Searle, Casella, and McCulloch (1992, p. 78), Burdick and Graybill (1992, p. 186 f.), and an exact confidence interval of the error variance resp. an estimator of the error variance.

The structure of the paper is as follows: In section 2 a description of the unbalanced model and the properties of the mean sum of squares are presented. In section 3 the different approaches for constructing a confidence interval for the between group variance are described in detail. The section 4 explains the conducted simulation studies concerning the confidence coefficients of the three and contains the results of the simulation study. Finally, some conclusions are given.

Note that all mean sum of squares considered in the following are assumed to be positive, which is given with probability one.

2 The Model

We consider the unbalanced case of the one–way random effects model of ANOVA, i. e.

$$y_{ij} = \mu + a_i + e_{ij}\,, \qquad i = 1, \ldots, r,\ j = 1, \ldots, n_i > 1\,, \tag{1}$$

where y_{ij} denotes the observable variable, μ the fixed, but unknown grand mean, a_i the unobservable random effect with mean 0 and variance σ_a^2, and e_{ij} the error term with mean 0 and variance σ_e^2. We assume that the random variables $a_1, \ldots, a_r, e_{11}, \ldots e_{rn_r}$ are normally distributed and mutually stochastically independent. Furthermore, let $n = \sum_{i=1}^{r} n_i$ denote the number of the total observations.

In model (1) it holds that the mean sum of squares between the groups, i. e.

$$MS1 = \frac{1}{r-1} \sum_{i=1}^{r} n_i \left(\overline{y_{i.}} - \overline{y_{..}} \right)^2 \tag{2}$$

with $\overline{y_{i.}} = \sum_{j=1}^{n_i} y_{ij}/n_i$ and $\overline{y_{..}} = \sum_{i=1}^{r} \sum_{j=1}^{n_i} y_{ij}/n$, has the expected value

$$\mathrm{E}(MS1) = k\sigma_a^2 + \sigma_e^2\,, \qquad k = \frac{1}{r-1} \cdot \frac{n^2 - \sum_{i=1}^{r} n_i^2}{n}\,, \tag{3}$$

and

$$\frac{(r-1)MS1}{k\sigma_a^2 + \sigma_e^2} \sim \chi^2_{r-1}\,, \qquad \text{if } \sigma_a^2 = 0\,, \tag{4}$$

where χ^2_ν denotes a central chi–square distributed random variable with ν degrees of freedom.

The mean sum of squares within the groups, i. e.

$$MS2 = \frac{1}{n-r} \sum_{i=1}^{r} \sum_{j=1}^{n_i} (y_{ij} - \overline{y_{i.}})^2 \tag{5}$$

has expected value

$$\mathrm{E}(MS2) = \sigma_e^2 \tag{6}$$

and

$$(n-r)MS2/\sigma_e^2 \sim \chi^2_{n-r}\,. \tag{7}$$

According to (7), a $(1-\kappa)$–confidence interval for σ_e^2 is given by

$$\mathrm{CI}(\sigma_e^2) : \left[\frac{(n-r)MS2}{\chi^2_{n-r;\,1-\kappa/2}}\,;\, \frac{(n-r)MS2}{\chi^2_{n-r;\,\kappa/2}} \right]\,, \tag{8}$$

where $\chi^2_{\nu;\gamma}$ denotes the γ–quantile of a χ^2–distribution with ν degrees of freedom.

Due to (4) the approximation of the distribution of $MS1$ by a multiple of a χ^2–distribution is only satisfactory if the between group variance σ_a^2 is close to 0. Thus, the transfer of the confidence interval for σ_a^2 in the balanced case independently proposed by Tukey (1951) and Williams (1962) to the unbalanced case is not possible. In the next section we will therefore consider three different approaches of constructing a confidence interval for σ_a^2 in the unbalanced case.

3 Confidence intervals on σ_a^2

3.1 The Thomas–Hultquist confidence interval

Instead of *MS1* from (2) Thomas and Hultquist (1978) considered the sample variance of the group means given by

$$MS3 = \frac{1}{r-1}\sum_{i=1}^{r}\left(\overline{y_{i.}} - \frac{1}{r}\sum_{i=1}^{r}\overline{y_{i.}}\right)^2 . \tag{9}$$

They showed that it holds approximately

$$\frac{(r-1)MS3}{\sigma_a^2 + \sigma_e^2/\tilde{n}} \overset{\text{appr.}}{\sim} \chi^2_{r-1} , \tag{10}$$

where $\tilde{n}$ denotes the harmonic mean of the sample sizes of the r groups.

Furthermore, Thomas and Hultquist proved that $MS2$ and $MS3$ are stochastically independent, so that

$$\frac{\sigma_e^2}{\sigma_a^2 + \sigma_e^2/\tilde{n}} \cdot \frac{MS3}{MS2} \overset{\text{appr.}}{\sim} F_{r-1,\,n-r} , \tag{11}$$

where F_{ν_1,ν_2} denotes a F–distributed random variable with ν_1 and ν_2 degrees of freedom.

From (10) and (11) $(1-\kappa)$–confidence intervals for $\sigma_a^2+\sigma_e^2/\tilde{n}$ and σ_a^2/σ_e^2 can be constructed and adopting the ideas of constructing a confidence interval by Tukey and Williams to the present situation, Thomas and Hultquist proposed the following confidence interval for σ_a^2:

$$\mathrm{CI_{TH}}(\sigma_a^2) : \left[\frac{(r-1)}{\chi^2_{r-1;\,1-\kappa/2}}\left(MS3 - \frac{MS2}{\tilde{n}}F_{r-1,\,n-r;\,1-\kappa/2}\right) ; \right.$$
$$\left. \frac{(r-1)}{\chi^2_{r-1;\,\kappa/2}}\left(MS3 - \frac{MS2}{\tilde{n}}F_{r-1,\,n-r;\,\kappa/2}\right)\right] . \tag{12}$$

Due to Bonferroni's inequality the confidence coefficient of (12) is at least $(1-2\kappa)$, but one may hope that the actual confidence coefficient is near $(1-\kappa)$. However, in Thomas and Hultquist (1978) it is reported that the χ^2–approximation in (10) is not good for extremely unbalanced designs where the ratio $\eta = \sigma_a^2/\sigma_e^2$ is less than 0.25. Thus, in such situations the confidence interval (12) can be a liberal one, i. e. the confidence coefficient substantially lies below $(1-\kappa)$.

3.2 The Burdick–Eickman confidence interval

Burdick, Maqsood and Graybill (1986) suggested a confidence interval for the ratio $\eta = \sigma_a^2/\sigma_e^2$ which overcomes the problem with small ratios in the Thomas–Hultquist procedure and has a confidence coefficient of at least $1-\kappa$. This interval is given by

$$\mathrm{CI}(\eta):\left[\frac{MS3}{MS2}\cdot\frac{1}{F_{r-1,n-r;1-\kappa/2}}-\frac{1}{n_{\min}};\ \frac{MS3}{MS2}\cdot\frac{1}{F_{r-1,n-r;1-\kappa/2}}-\frac{1}{n_{\max}}\right] \tag{13}$$

with $n_{\min}=\min\{n_1,\dots,n_r\}$ and $n_{\max}=\max\{n_1,\dots,n_r\}$.

The difference between (13) and the confidence interval for $\eta=\sigma_a^2/\sigma_e^2$ in the Thomas–Hultquist procedure is that due to (11) Thomas and Hultquist subtract $1/\tilde{n}$ in both bounds instead of $1/n_{\min}$ and $1/n_{\max}$, respectively, in (13).

Using (13) and the confidence interval for $\sigma_a^2+\sigma_e^2/\tilde{n}$ due to (10), Burdick and Eickman (1986) investigated the confidence interval for σ_a^2 constructed by the Tukey-Williams method.

This interval is given by

$$\mathrm{CI}_{\mathrm{BE}}(\sigma_a^2):\left[\left(\frac{\tilde{n}L}{1+\tilde{n}L}\right)\cdot\frac{(r-1)MS3}{\chi^2_{r-1;1-\kappa/2}};\ \left(\frac{\tilde{n}U}{1+\tilde{n}U}\right)\cdot\frac{(r-1)MS3}{\chi^2_{r-1;1-\kappa/2}}\right], \tag{14}$$

with

$$L=\max\left\{0,\ \frac{MS3}{MS2}\cdot\frac{1}{F_{r-1,n-r;1-\kappa/2}}-\frac{1}{n_{\min}}\right\}$$

and

$$U=\max\left\{0,\ \frac{MS3}{MS2}\cdot\frac{1}{F_{r-1,n-r;\kappa/2}}-\frac{1}{n_{\max}}\right\}$$

3.3 Confidence interval based on Wald's confidence interval for η

Instead of approximative confidence intervals for η as in the Thomas–Hultquist and Burdick-Eickman approach we consider the exact confidence interval for η given in Wald (1940) to construct a confidence interval for σ_a^2.

Following Wald (1940) we observe that

$$\mathrm{Var}(\overline{y_{i.}}) = \sigma_a^2 + \sigma_e^2/n_i = \sigma_e^2/w_i \tag{15}$$

with $w_i = n_i/(1+\eta n_i)$, $i = 1, \ldots, r$.

Now, Wald considered the sum of squares

$$(r-1)MS4 = \sum_{i=1}^{r} w_i \left(\overline{y_{i.}} - \frac{\sum_{i=1}^{r} w_i \overline{y_{i.}}}{\sum_{i=1}^{r} w_i} \right)^2 \tag{16}$$

and proved that

$$(r-1)MS4/\sigma_e^2 \sim \chi^2_{r-1}\,.$$

Furthermore, $MS4$ and $MS2$ are stochastically independent so that

$$F_w(\eta) = \frac{MS4}{MS2} \sim F_{r-1,\,n-r}\,. \tag{17}$$

According to (17), an exact confidence interval for the ratio η can be constructed.

Wald showed that $(r-1)MS4$ is a strictly monotonously decreasing function in η, and so the bounds of the exact confidence interval are given as the solutions of the following two equations:

$$\begin{aligned} &\text{lower bound:} \quad F_w(\eta) = F_{r-1,\,n-r,\,1-\kappa/2} \\ &\text{upper bound:} \quad F_w(\eta) = F_{r-1,\,n-r,\,\kappa/2} \end{aligned} \tag{18}$$

Since $F_w(\eta)$ is a strictly monotonously decreasing function in η the solution of (18), if it exists, is unique. But due to the fact that η is nonnegative, $(r-1)MS4$ is bounded at $\eta = 0$, namely it holds

$$(r-1)MS4 \leq \sum_{i=1}^{r} n_i \left(\overline{y_{i.}} - \frac{\sum_{i=1}^{r} n_i \overline{y_{i.}}}{\sum_{n=1}^{r} n_i} \right)^2 . \tag{19}$$

Thus, a nonnegative solution of (18) may not exist. If such a solution of one of the equations in (18) does not exist, the corresponding bound in the confidence interval is set equal to zero. Note that the existence of a nonnegative solution in (18) only depends on the chosen κ.

Let us denote by η_L and η_U the solutions of the equations in (18), so we propose, using the confidence bounds from (8) for σ_e^2, the following confidence interval for σ_a^2

$$\mathrm{CI}(\sigma_a^2) : \left[\frac{(n-r)MS2}{\chi^2_{n-r;\,1-\kappa}} \cdot \eta_L \,;\, \frac{(n-r)MS2}{\chi^2_{n-r;\,\kappa}} \cdot \eta_U\right] , \tag{20}$$

which has a confidence coefficient of at least $(1-2\kappa)$ according to Bonferroni's inequality. But due to the fact that the confidence coefficient of $[\sigma_e^2 \cdot \eta_L, \sigma_e^2 \cdot \eta_U]$ is exactly $1-\kappa$, the resulting confidence interval (20) may be very conservative, i. e. the confidence coefficient is larger than $(1-\kappa)$. So, we also consider a confidence interval for σ_a^2 with the estimator $MS2$ for σ_e^2 instead of the bounds of the confidence interval for σ_e^2, i. e.

$$\widetilde{\mathrm{CI}}(\sigma_a^2) : [MS2 \cdot \eta_L \;;\; MS2 \cdot \eta_U] \;. \tag{21}$$

4 Simulation studies

In simulation studies we compare the confidence coefficients of the four different confidence intervals (12), (14), (20), and (21) for σ_a^2 in the unbalanced one–way random effects model. The simulations are conducted using SAS 6.12 under Windows NT. The means of the r groups, $\overline{y_{i.}}$, $i = 1, \ldots, r$, are independently generated using the SAS function RANNOR and the sum of squares within the groups, $(n-r) \cdot MS2$, is generated independently from $\overline{y_{i.}}$, $i = 1, \ldots, r$, using the SAS function RANGAM. During all simulations the error variance σ_e^2 is set equal to one, and for the variance between the groups, σ_a^2, we consider the values 0, 0.01, 0.05, 0.1, 0.25, 0.5, 0.75, 1, 2, 3, 4, 6, 8, and10. The different unbalanced designs, which we examined, are given in table 1, whereby eight of these patterns were also analysed by Burdick and Eickman (1986). For $r = 3$ and $r = 6$ we extend the analysis by considering some patterns which are not so extremely unbalanced as in Burdick and Eickman.

We consider two-sided confidence intervals with $\kappa = 0.1$ and $\kappa = 0.05$, respectively. All estimated confidence coefficients are based on 10,000 replications for every set of parameters. Based on normal approximation it means that a 95%–confidence interval for an estimated confidence coefficient $\hat{p} = 0.95$ is given by $[0.9456 \,;\, 0.9541]$ and for $\hat{p} = 0.9$ by $[0.8940 \,;\, 0.9057]$.

Table 1: Patterns of unbalanced designs used in simulations

Pattern	r	n_i
1*	3	5, 10, 15
2	3	10, 20, 30
3	3	5, 10, 100
4*	3	1, 1, 100
5*	3	2, 2, 100
6	6	5, 10, 15, 5, 10, 15
7	6	10, 20, 30, 10, 20, 30
8	6	5, 10, 15, 20, 25, 30
9*	6	1, 1, 1, 1, 1, 100
10*	6	2, 2, 2, 2, 2, 100
11*	10	1, 1, 4, 5, 6, 6, 8, 8, 10, 10
12*	10	2, 2, 4, 5, 6, 6, 8, 8, 10, 10
13*	10	3, 3, 4, 5, 6, 6, 8, 8, 10, 10

* These patterns were also considered in Burdick and Eickman (1986)

For solving the system of equations (18), we use the bisection method and choose as the precision of the solutions 10^{-14}. As starting values of the bisection method we use the ones proposed by Wald (1940), who showed that for the solutions of (18), say $\tilde{\eta} = \eta_L$ or η_U, it holds with $\tau = \kappa/2$ or $\tau = 1 - \kappa/2$

$$\frac{MS3}{MS2} \cdot \frac{1}{F_{r-1,n-r;\tau}} - \frac{1}{n_{\min}} \leq \tilde{\eta} \leq \frac{MS3}{MS2} \cdot \frac{1}{F_{r-1,n-r;\tau}} - \frac{1}{n_{\max}} . \tag{22}$$

Note that the lower bound for η_L and the upper bound for η_U coincide with the confidence interval for η considered by Burdick, Maqsood and Graybill (1986), cf. (13).

In table 2 the results of the simulation study are explicitly shown for all values of σ_a^2 with $\kappa = 0.1$ in pattern 5, where $\mathrm{CI}_{\mathrm{TH}}$ denotes the Thomas–Hultquist–interval from (12), $\mathrm{CI}_{\mathrm{BE}}$ the Burdick–Eickman–interval from (14), CI the interval from (20) and $\widetilde{\mathrm{CI}}$ from (21). We choose this pattern as an example to illustrate the characteristics we found in all simulation studies.

If $\sigma_a^2 = 0$ all confidence intervals are rather conservative. For small $\sigma_a^2 > 0$ the estimated

Table 2: Estimated confidence coefficients for the four different confidence intervals in pattern 5 with $\kappa = 0.1$

Pattern	σ_a^2	$\mathrm{CI}_{\mathrm{TH}}$	$\mathrm{CI}_{\mathrm{BE}}$	CI	$\widetilde{\mathrm{CI}}$
5	0	0.9377	0.9796	0.9465	0.9465
5	0.01	0.8874	0.9683	0.9026	0.9007
5	0.05	0.8890	0.9555	0.9070	0.8986
5	0.1	0.8912	0.9452	0.9141	0.9022
5	0.25	0.8961	0.9292	0.9227	0.9058
5	0.5	0.8946	0.9122	0.9244	0.9013
5	0.75	0.8984	0.9088	0.9280	0.9021
5	1	0.9035	0.9098	0.9352	0.9084
5	2	0.9000	0.9030	0.9363	0.9052
5	3	0.9084	0.9099	0.9414	0.9127
5	4	0.9049	0.9059	0.9404	0.9074
5	6	0.9028	0.9036	0.9422	0.9069
5	8	0.9000	0.9003	0.9394	0.9049
5	10	0.8969	0.8973	0.9380	0.9020

confidence coefficients of the Thomas–Hultquist interval lie below $1-\kappa$, but the difference to $1-\kappa$ in this pattern is not so severe. If σ_a^2 becomes larger, the estimated confidence coefficient lies near $1-\kappa$. The Burdick–Eickman–interval possesses for small $\sigma_a^2 > 0$ a high confidence coefficient, i. e. in these situations the interval is very conservative. The estimated confidence coefficient of this interval declines, if σ_a^2 becomes larger, and lies near $1-\kappa$ for large σ_a^2. The confidence interval CI, where the bounds of the $(1-\kappa)$–confidence interval for σ_e^2 are taken as estimates of the error variance, has a confidence coefficient near $1-\kappa$ for small $\sigma_a^2 > 0$, and the estimated confidence coefficient increases if σ_a^2 becomes larger. Thus, for large σ_a^2 this confidence interval may be rather conservative. For the confidence interval $\widetilde{\mathrm{CI}}$ with the mean squared error as the estimator of the error variance we get estimated confidence coefficients which lie near $1-\kappa$ for all $\sigma_a^2 > 0$.

If an increase or a decline of the estimated confidence coefficients is found, the estimated confidence coefficients attain a certain level near $\sigma_a^2 = 1$, so that for $\sigma_a^2 > 1$ there is little variation between the estimated confidence coefficients. Thus, for simplifying the presentation of all simulations we present in table 3 and 4 ranges of the estimated confidence coefficients for $0 < \sigma_a^2 \leq 1$ and $\sigma_a^2 > 1$ separately for all patterns with $\kappa = 0.1$, and $\kappa = 0.05$, respectively, where in the case $0 < \sigma_a^2 \leq 1$ we omit the values for $\sigma_a^2 = 0$, because as stated above the confidence intervals are rather conservative in this situation.

The most extreme result in table 3 and 4, respectively, is given in pattern 11. There, the Thomas–Hultquist–interval is very liberal for $0 < \sigma_a^2 \leq 1$, whereas the Burdick–Eickmann-interval produces estimated confidence coefficients greater than 0.99 in these situations. Generally speaking, the results just described for $\kappa = 0.1$ in pattern 5 are reflected in a similar way in all conducted simulations as well for $\kappa = 0.1$ as for $\kappa = 0.05$.

5 Conclusions

In our simulation studies we confirm the results of Burdick and Eickman (1986) that the Thomas–Hultquist–interval may be very liberal for small σ_a^2, i. e. the confidence coefficient considerably lies below $1 - \kappa$. In these situation the Burdick–Eickman–interval has a confidence coefficient which is always larger than $1 - \kappa$, but the interval can be very conservative. If σ_a^2 becomes larger, both intervals are very similar. The confidence interval CI deduced from Wald's confidence interval for the ratio η with the bounds of the confidence interval of the error variance as estimates for the error variance has always a confidence coefficient at least as great as $1 - \kappa$, but this interval can be very conservative for large σ_a^2. A good compromise for the whole range of σ_a^2 is the confidence interval $\widetilde{\mathrm{CI}}$ from (21), which has a confidence coefficient at least as great as $1 - \kappa$ for small σ_a^2, and for growing σ_a^2 the confidence interval only becomes moderately conservative.

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References

Boardman, T. J. (1974). Confidence intervals for variance components — A comparative Monte Carlo study. Biometrics 30, 251–262.

Burdick, R. K.; Eickman, J. (1986). Confidence intervals on the among group variance component in the unbalanced one–fold nested design. J. Statist. Comput. Simul. 26, 205–219.

Burdick, R. K.; Graybill, F. A. (1992). Confidence Intervals on Variance Components. Dekker, New York.

Burdick, R. K.; Maqsood, F.; Graybill, F. A. (1986). Confidence intervals on the intraclass correlation in the unbalanced one–way classification. Commun. Statist.–Theory Meth. 15, 3353–3378.

Graybill, F. A. (1976). Theory and Application of the Linear Model. Duxbury Press, North Scituate.

Searle, S. R.; Casella, G.; McCulloch, C. E. (1992). Variance components. Wiley, New York.

Thomas, J. D.; Hultquist, R. A. (1978). Interval estimation for the unbalanced case of the one–way random effects model. Ann. Statist. 6, 582–587.

Tukey, J. W. (1951). Components in regression. Biometrics 7, 33–69.

Wald, A. (1940). A note on the analysis of variance with unequal class frequencies. Ann. Math. Statist. 11, 96–100.

Wang, C. M. (1990). On the lower bound of confidence coefficients for a confidence interval on variance components. Biometrics 46, 187–192.

Williams, J. S. (1962). A confidence interval for variance components. Biometrika 49, 278–281.

Table 3: Ranges of estimated confidence coefficients for the four different confidence intervals (12), (14), (20), and (21) with $\kappa = 0.1$

Pattern	σ_a^2	CI_{TH}	CI_{BE}	CI	$\widetilde{CI}$
1	$<= 1$	0.8976 – 0.9056	0.9023 – 0.9530	0.9095 – 0.9604	0.9030 – 0.9184
	> 1	0.9002 – 0.9035	0.9002 – 0.9038	0.9609 – 0.9643	0.9140 – 0.9180
2	$<= 1$	0.8982 – 0.9048	0.8997 – 0.9492	0.9139 – 0.9483	0.9049 – 0.9097
	> 1	0.8963 – 0.9039	0.8963 – 0.9043	0.9460 – 0.9523	0.9033 – 0.9103
3	$<= 1$	0.8801 – 0.9046	0.9048 – 0.9707	0.9030 – 0.9372	0.8978 – 0.9073
	> 1	0.8978 – 0.9037	0.8979 – 0.9037	0.9346 – 0.9396	0.9025 – 0.9076
4	$<= 1$	0.8885 – 0.8968	0.9155 – 0.9714	0.9016 – 0.9263	0.8988 – 0.9063
	> 1	0.8971 – 0.9021	0.8999 – 0.9084	0.9333 – 0.9409	0.9020 – 0.9062
5	$<= 1$	0.8874 – 0.9035	0.9088 – 0.9683	0.9026 – 0.9352	0.8986 – 0.9084
	> 1	0.8969 – 0.9084	0.8973 – 0.9099	0.9363 – 0.9422	0.9020 – 0.9127
6	$<= 1$	0.8872 – 0.9065	0.8977 – 0.9713	0.9070 – 0.9650	0.8986 – 0.9207
	> 1	0.8997 – 0.9033	0.9003 – 0.9034	0.9675 – 0.9702	0.9142 – 0.9188
7	$<= 1$	0.8889 – 0.9048	0.8945 – 0.9661	0.9113 – 0.9555	0.9016 – 0.9110
	> 1	0.8967 – 0.9062	0.8967 – 0.9062	0.9519 – 0.9585	0.9050 – 0.9137
8	$<= 1$	0.8742 – 0.9065	0.9018 – 0.9797	0.9115 – 0.9570	0.9019 – 0.9152
	> 1	0.8958 – 0.9020	0.8966 – 0.9021	0.9525 – 0.9609	0.9061 – 0.9117
9	$<= 1$	0.8818 – 0.9004	0.9240 – 0.9727	0.9037 – 0.9398	0.8994 – 0.9092
	> 1	0.8987 – 0.9037	0.8994 – 0.9145	0.9488 – 0.9557	0.9067 – 0.9121
10	$<= 1$	0.8787 – 0.8987	0.9059 – 0.9711	0.9026 – 0.9427	0.9005 – 0.9087
	> 1	0.8983 – 0.9076	0.9000 – 0.9094	0.9491 – 0.9576	0.9060 – 0.9165
11	$<= 1$	0.8141 – 0.8965	0.9338 – 0.9951	0.9081 – 0.9773	0.9010 – 0.9269
	> 1	0.8999 – 0.9057	0.9010 – 0.9202	0.9790 – 0.9842	0.9205 – 0.9314
12	$<= 1$	0.8666 – 0.9068	0.9220 – 0.9926	0.9090 – 0.9779	0.9026 – 0.9281
	> 1	0.8990 – 0.9031	0.8992 – 0.9059	0.9813 – 0.9847	0.9245 – 0.9294
13	$<= 1$	0.8791 – 0.9062	0.9115 – 0.9874	0.9093 – 0.9807	0.9006 – 0.9280
	> 1	0.8945 – 0.9044	0.8946 – 0.9047	0.9799 – 0.9836	0.9215 – 0.9293

Table 4: Ranges of estimated confidence coefficients for the four different confidence intervals (12), (14), (20), and (21) with $\kappa = 0.05$

Pattern	σ_a^2	$\mathrm{CI}_{\mathrm{TH}}$	$\mathrm{CI}_{\mathrm{BE}}$	CI	$\widetilde{\mathrm{CI}}$
1	$<= 1$	0.9456 – 0.9541	0.9535 – 0.9772	0.9561 – 0.9841	0.9518 – 0.9625
	> 1	0.9484 – 0.9532	0.9484 – 0.9532	0.9841 – 0.9874	0.9597 – 0.9627
2	$<= 1$	0.9484 – 0.9527	0.9507 – 0.9760	0.9584 – 0.9818	0.9535 – 0.9571
	> 1	0.9483 – 0.9538	0.9484 – 0.9538	0.9784 – 0.9811	0.9537 – 0.9591
3	$<= 1$	0.9365 – 0.9524	0.9499 – 0.9871	0.9546 – 0.9711	0.9503 – 0.9550
	> 1	0.9478 – 0.9523	0.9478 – 0.9524	0.9717 – 0.9745	0.9506 – 0.9551
4	$<= 1$	0.9384 – 0.9485	0.9585 – 0.9874	0.9503 – 0.9673	0.9496 – 0.9544
	> 1	0.9454 – 0.9525	0.9457 – 0.9560	0.9710 – 0.9749	0.9491 – 0.9551
5	$<= 1$	0.9390 – 0.9527	0.9542 – 0.9853	0.9502 – 0.9707	0.9490 – 0.9554
	> 1	0.9492 – 0.9566	0.9492 – 0.9574	0.9728 – 0.9754	0.9516 – 0.9588
6	$<= 1$	0.9406 – 0.9540	0.9501 – 0.9849	0.9570 – 0.9882	0.9508 – 0.9624
	> 1	0.9489 – 0.9522	0.9493 – 0.9522	0.9882 – 0.9897	0.9598 – 0.9627
7	$<= 1$	0.9436 – 0.9548	0.9474 – 0.9839	0.9583 – 0.9840	0.9512 – 0.9604
	> 1	0.9479 – 0.9553	0.9479 – 0.9553	0.9813 – 0.9846	0.9531 – 0.9600
8	$<= 1$	0.9306 – 0.9528	0.9510 – 0.9906	0.9580 – 0.9825	0.9518 – 0.9575
	> 1	0.9447 – 0.9529	0.9447 – 0.9529	0.9812 – 0.9859	0.9519 – 0.9595
9	$<= 1$	0.9357 – 0.9496	0.9611 – 0.9876	0.9516 – 0.9733	0.9472 – 0.9574
	> 1	0.9496 – 0.9525	0.9505 – 0.9566	0.9781 – 0.9830	0.9546 – 0.9582
10	$<= 1$	0.9362 – 0.9494	0.9525 – 0.9858	0.9524 – 0.9761	0.9503 – 0.9569
	> 1	0.9473 – 0.9551	0.9485 – 0.9559	0.9783 – 0.9841	0.9526 – 0.9592
11	$<= 1$	0.8875 – 0.9463	0.9682 – 0.9971	0.9528 – 0.9912	0.9490 – 0.9682
	> 1	0.9454 – 0.9528	0.9491 – 0.9588	0.9925 – 0.9950	0.9639 – 0.9681
12	$<= 1$	0.9264 – 0.9546	0.9615 – 0.9968	0.9588 – 0.9932	0.9538 – 0.9686
	> 1	0.9500 – 0.9530	0.9504 – 0.9537	0.9929 – 0.9958	0.9658 – 0.9687
13	$<= 1$	0.9398 – 0.9528	0.9578 – 0.9950	0.9569 – 0.9940	0.9511 – 0.9681
	> 1	0.9483 – 0.9532	0.9484 – 0.9542	0.9928 – 0.9954	0.9647 – 0.9693