GUNESCH, Roland
Feldkirch

Use of metrics (distance functions) in teaching mathematics courses at the tertiary level

This article addresses some aspects of teaching metrics (distance functions) in university-level courses of mathematics. We will focus on mathematical, didactic, cognitive and institutional aspects.

The following arguments and didactical comments concerning the use of metrics in teaching mathematics courses represent the personal view of the author, based on experience with a large number of personally taught mathematics courses and on experience in mathematics research.

The notion of metric (distance function, metric space) appears naturally in the following tertiary-level courses: 1.) analysis of functions of one variable, 2.) analysis of functions of several variables, 3.) linear algebra, and 4.) geometry. (It also appears in graduate mathematics courses.) In the following sections, we treat the occurrence of the topic in these courses.

1.) Metrics in courses on analysis of functions of one variable

In courses on analysis, the concept of metric appears very often, although usually not by name or symbol, but implicitly in terms like $|x-y|$ (without calling it a metric). The triangle inequality (3) appears very prominently in analysis courses, and it is also referenced by name, whereas the other two axioms of the norm are usually not referenced by name. However, this creates another didactical challenge: In analysis courses, the name "triangle inequality" usually refers to the form (5) $|x+y| \leq |x|+|y|$, whereas the form (6) $|x-y| \leq |x-a|+|a-y|$, which is used very often in proofs and calculations, often is unknown to students (and even some instructors of recitation classes, as the author of this article experienced during his first year in university). Note that form (6) is just axiom (3) of a metric, so from the point of view of metric spaces, this is the *obvious* version of the triangle inequality. To overcome the difficulty of (6) not being known, in analysis proofs, it is customary to replace every single occurrence of (6) with the (longer) two-step calculation (7) $|x-y| = |x-a+a-y| \leq |x-a|+|a-y|$, which, from the point of view of the students, is acceptable but not didactically ideal; after all, how could anyone have guessed that the term $-a+a$ is required here? Thus, in the author's point of view, analysis courses would benefit from an early inclusion of the abstract definition of metric and proof of elementary properties of it, in particular the equivalence of equations (6) and (7).

In: L. Schick, M. Platz & A. Lambert (Hrsg.),
Beiträge zum Mathematikunterricht 2025.
58. Jahrestagung der Gesellschaft für Didaktik der Mathematik. WTM.
https://doi.org/10.37626/GA9783959873307.0

Note also that the usual proofs of the triangle inequality in analysis, while not very complicated, introduce some didactical challenges of their own (Gunesch, 2015).

2.) Metrics in courses on analysis of functions of several variables

In courses on *higher-dimensional* analysis, the concepts of abstract metric (and norm) are usually introduced early, since it becomes inevitable to talk about open sets and neighborhoods and error estimates vector-valued quantities, and the term $|x-y|$ (from one-dimensional analysis) has to be replaced by something higher-dimensional anyway. Compared to the usual topics in higher-dimensional analysis, a treatment of abstract metric spaces is quite usually easy. That does not mean that it appears early in the course: E.g., in the textbook (Barner & Flohr, 1988) the triangle inequality for norms already appears on page 10, but the triangle inequality for metrics (and the definition of metric) appear only on page 145 (just before proving the contraction mapping theorem, where it is required).

The definition of metric (space) is seemingly straightforward, with 3 simple axioms: (1) $\mathrm{d(x,y)=0} \Longleftrightarrow x = y$ (2) $d(x,y) = d(y,x)$ (3) $d(x,z) \leq d(x,y) + d(y,z)$ (with suitable all-quantors). Unfortunately, some textbooks and course notes replace (1) with the version (1b) $d(x,y) \geq 0 \text{ and } d(x,y) = 0 \Longleftrightarrow x = y$. This is redundant (because the statement (4) $d(x,y) \geq 0$ follows from (1), (2), (3)), but it is not mathematically wrong. Didactically, however, using version (1b) is problematic and hinders students' understanding. Firstly, it means that in all exercise problems, students have to do superfluous work. Secondly, it makes learning the definition more difficult, since the relevant part of the first axiom appears only in the second half of the sentence (after the word "and"). This can cause students to forget that in order to prove the equivalence, two directions are necessary. In the end, students have made an unnecessary extra calculation showing a condition that is true anyway, and have omitted another (required) calculation. Moreover, in practically all examples know to the author, this (unnecessary) calculation is longer than the (abstract) proof of statement (4). It would be interesting to measure the occurrence of such errors by students who were presented with axiom (1b) compared to axiom (1), and to explain the occurrence of such errors.

3.) Metrics in courses on linear algebra

When the notion of metric appears in courses on linear algebra, it is usually as a side-product of the definition of norm on vector spaces, since any norm induces a metric. On the other hand, the concept of norm is often treated as either a side topic or as special case of the scalar product (since each scalar product induces a norm).

Unfortunately, for the most prominent example, the Euclidean metric, it is not easy to show that it satisfies the triangle inequality (much harder than for the sum-norm or the maximum-norm); it requires the Cauchy-Schwarz inequality, which is difficult for students to guess and almost impossible for students to prove (textbook proofs of the Cauchy-Schwarz inequality are not very long, but usually require either very clever substitutions that cannot reasonably be guessed, or require very theoretical arguments which are not so obvious either, see for example the two proofs in Fischer & Springborn (2020).

4.) Metrics in courses on geometry (elementary geometry, differential geometry)

In courses on geometry (insofar as they are separate from courses on linear algebra), metric spaces may appear, but students' familiarity with them can hardly be a prerequisite. It is the observation of the author that students at the PH Vorarlberg, who take a mandatory geometry course after having completed courses on linear algebra, analysis, and higher-dimensional analysis, still struggle when asked (at the beginning of the course) to explain the triangle inequality for triangles in Euclidean space.

It is also noteworthy that, due to the simplicity of the axioms, metric geometry appears particularly suitable for inquiry-based learning; see Gunesch (2015, 2016) for that topic in general.

The notion of metrics appears prominent in courses on differential geometry, where Riemannian manifolds are studied, in particular curved (2-dimensional) surfaces in 3-space. It is interesting (and of considerable didactic importance) that the word "metric" is used in three contexts: the distance function inherited from the Euclidean space that the surface is embedded in; the "inner" distance obtained by the length of curves joining two points; and lastly, the scalar product is often denoted "Riemannian metric". See e.g. Bär (2010) or Burago, Burago & Ivanov (2001) as well as the following comments on the geometry of interior metric spaces.

Metrics as a topic also appear at graduate level mathematics courses. Metric geometry can actually be a one-semester course on its own (Burago, Burago & Ivanov, 2001; Gunesch, 2012).

Teaching geometry via metric structures instead of via differentiable structures offers several substantial didactic benefits. The traditional method of teaching geometry of shapes involves differential geometry and Riemannian geometry. Thus, there are substantial prerequisites, particular higher-dimensional analysis and linear algebra. If the notion of abstract manifold is used, prerequisites are at least some concepts from topology (Hausdorff space, countable base). Then, a tangent bundle (and scalar product) has to be introduced; however, the concept of smoothness of the resultant Riemannian metric is not so straightforward, since it involves the scalar product in different tangent spaces. Finally, one can introduce the length of differentiable curves by integrating tangent vectors. Much work is required to introduce the curvature tensor, which must then be reduced to some special numbers, such as sectional curvature, to obtain understandable results.

In comparison, in an interior metric space, it is possible to introduce length of curves and angles between curves without differentiability and tangent spaces (Burago, Burago & Ivanov, 2001). Moreover, it is possible to define geodesics without requiring a Levi-Civita connection. In fact, it is even possible to define upper and lower curvature bounds without having to define a curvature tensor. This creates the opportunity to give e.g. a proof of the Gauss-Bonnet theorem for students without a prerequisite of higher-dimensional analysis (Gunesch, 2012).

References

Barner, M., & Flohr, F. (1988): *Analysis II.* Walter de Gruyter.

Burago, D., Burago, Y., & Ivanov, S. (2001): *A Course in Metric Geometry*. American Mathematical Society, Graduate Studies in Mathematics.

Bär, Ch. (2010): *Elementare Differentialgeometrie*. Walter de Gruyter.

Fischer, G. & Springborn, B. (2020): *Lineare Algebra*, Grundkurs Mathematik. https://doi.org/10.1007/978-3-662-61645-1_1

Gunesch, R. (2012): Differential Geometry explained easily: A new teaching concept. In: M. Ludwig & M. Kleine (Eds.), *Beiträge zum Mathematikunterricht 2012*, WTM-Verlag, Münster, 321--324.

Gunesch, R. (2015): Inquiry-based learning in academic teaching compared to traditional teaching: an example. In F. Caluori, H. Linneweber-Lammerskitten & C. Streit (Hrsg.), *Beiträge zum Mathematikunterricht 2015*, WTM-Verlag, Münster, 336--339. http://dx.doi.org/10.17877/DE290R-16647

Gunesch, R. (2016): Forschendes Lernen als Zugang zu mathematisch anspruchsvollen Stellen in der Studierendenausbildung. In: Institut für Mathematik und Informatik Heidelberg (Hrsg.), *Beiträge zum Mathematikunterricht 2016*, WTM-Verlag, Münster, 345--348. http://dx.doi.org/10.17877/DE290R-17292