HADRON PROPERTIES FROM QCD SUM RULES

L.J. REINDERS, H. RUBINSTEIN and S. YAZAKI

CERN, Geneva, Switzerland

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Abstract:

We review the theoretical basis and the applications to hadronic physics of the QCD sum rules invented by Shifman, Vainshtein and Zakharov.

* Alexander von Humboldt fellow at the Physikalisches Institut, Nussallee 12, Bonn, Fed. Rep. Germany.
** Dept. of Physics, University of Stockholm, Stockholm, Sweden. On leave of absence from the Weizmann Institute, Rehovot, Israel.
*** Dept. of Physics, University of Tokyo, Tokyo, Japan.

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1. Introduction

Hadrons are, beyond reasonable doubt, bound states of quarks. It is far from clear however how to generate the observed spectrum and its properties, even if one accepts the idea of confinement. The reason for the complexity of the problem and the ensuing frustration is that asymptotic freedom and confinement imply that the theory must have a very complex infrared structure. It is therefore a great challenge to extract information on the spectrum from the rather simple Lagrangian of QCD.

The purpose of this report is to present a method which is based on the simplicity of the theory in the ultraviolet regime to extract information on the bound states of the system. In this method one cannot prove that the only possible states are colour singlets but once this assumption is made the wealth of predictions is rewarding. Use can be made of the phenomenology that has led to our present knowledge of hadrons (a) chiral symmetry and its spontaneous breakdown which gives a vacuum expectation value to $\bar{q}q$ and (b) duality which led to the string model. This embodies the concept of string tension or universal Regge slope and the rapid convergence in the saturation of the imaginary part of amplitudes by a few resonances. The combination of the properties of spontaneous symmetry breaking, duality and asymptotic freedom are the basis for the QCD sum rule method.

Following the original paper of Shifman, Vainshtein and Zakharov [1] a considerable amount of work has been done in extracting properties of hadrons using these methods. In this report we will give an extensive review of developments since the report of Novikov et al. [2]. We will restrict ourselves to the calculation of masses and couplings of resonances, though we will mention other subjects as well. The many papers using the QCD sum rule formalism to determine the masses of the light quarks have been reviewed recently by Gasser and Leutwyler [3] and will not be considered here unless particularly relevant. Other topics not included in this review are deep inelastic scattering and large transverse momentum scattering.

The idea of the QCD sum rule formalism is to approach the bound state problem in QCD from the asymptotic freedom side, i.e., to start at short distances and move to larger distances where confinement effects become important, asymptotic freedom starts to break down and resonances emerge as a reflection of the fact that quarks and gluons are permanently confined within hadrons. The breakdown of asymptotic freedom is signalled by the emergence of power corrections due to nonperturbative effects in the QCD vacuum. These are introduced via nonvanishing vacuum expectation values of quark and gluon condensate operators such as

$$\langle 0|\bar{q}q|0\rangle\,,\qquad \langle 0|G^a_{\mu\nu}G^a_{\mu\nu}|0\rangle\,,\tag{1.1}$$

where $q(x)$ is the quark field and $G^a_{\mu\nu}(x)$ is the gluon field tensor. In standard perturbation theory these matrix elements vanish after normal ordering.

A nonvanishing quark condensate has been known for a long time [4] and its significance for resonance physics can be seen from the following argument. Consider the vacuum polarizations induced by the vector ($\bar{q}\gamma_\mu q$) and axial vector ($\bar{q}\gamma_\mu\gamma_5 q$) currents. For massless quarks which should be a good approximation for u and d quarks there is no difference between the two in every order of perturbation theory. However the physical states for the two currents are completely different (ρ meson in the vector channel and π and A_1 in the axial vector channel) which is due to the spontaneous breaking of chiral symmetry signalled by a nonvanishing vacuum expectation value of $\bar{q}q$. On dimensional grounds it is clear that this has to lead to power corrections compared to the logarithmic behaviour of the perturbative contributions. It has been shown [1] that to produce the spectrum these corrections are more important than higher order α_s corrections.

The starting point of the approach is to write down the Wilson operator expansion [5] for the T-ordered product of two or more currents. The gluon and quark condensates appear as higher dimensional operators in this expansion. The coefficients of these operators contain the short distance part and can be calculated in terms of the Lagrangian parameters of the theory (α_s and the quark masses) in perturbation theory. Since the operator product expansion (OPE) has only been proven in perturbation theory its validity is by no means obvious when nonperturbative effects are included. Indeed there is a critical dimension, as shown by SVZ [1], at which nonperturbative effects cause the OPE to break down. A general discussion has been given by Novikov et al. in [6], and we will come back to this in the next section. Using dispersion relations we relate the n-point function to physical states and we will discuss methods (moments, Borel transforms) which ensure dominance of the lowest lying resonance. At this point one obtains an equation which relates the parameters of the QCD Lagrangian with hadronic masses and couplings. On the asymptotic freedom side of this equation the bare loop gives the main contribution and the nonperturbative terms give small but vital corrections. As we will discuss in detail later these methods have been applied with considerable success to two-point functions for equal mass quarks (charmonium, upsilonium) and light quark systems (mesons as well as baryons). In equal mass heavy quark systems it is only the gluon condensate operator which gives a contribution. Its vacuum expectation value which is a universal quantity will be treated phenomenologically and determined from the charmonium spectrum. Lattice determinations of this quantity agree and will be discussed as well. For pure light quark systems operators up to dimension six have to be taken into account. For the vacuum expectation values of operators involving light quark fields we will use the current algebra value of $\langle 0|\bar{q}q|0\rangle$.

For systems involving heavy and light quarks successful applications have only been made to open beauty systems. Contrary to pure light quark mesons where the quark condensate operator always appears multiplied by a light quark mass, in systems with one heavy and one light quark this operator is multiplied by the heavy quark mass and the corrections due to this operator become especially large. This has prevented us so far to perform reliable calculations for the masses of open charm states, but some applications to open beauty states have been made. The centre of mass of the orbital excitations is predicted to be higher than in potential models where the above mentioned term is absent.

For baryons the situation is again different. On dimensional grounds it can be shown that the $\bar{q}q$ operator now appears without a small quark mass and that this operator drives the baryon mass dynamics. A similar situation also exists in various three-point functions involving baryons as well as mesons. Since the Wilson coefficient of the $\bar{q}q$ operator involves less loop integrations than the bare loop or the coefficient of $G^a_{\mu\nu}G^a_{\mu\nu}$ it is comparatively easy to derive sum rules which involve only the chiral condensate. We will exploit this feature to study hadron couplings.

Although the total body of results on masses and couplings is very impressive the methods used so far are limited to the lowest lying level in each partial wave. The determination of radial excitations is most probably related to calculations with higher dimensional currents which include gluons. Even so the method of enhancing the lowest lying states by moments or Borel transforms is not always successful. In upsilonium, for example where the relative spacing is much smaller than in charmonium the method must be modified. In light quark and heavy–light quark systems the modification is simply to include a continuum contribution besides the resonance under consideration. This extra parameter s_0 is quite naturally determined by the criteria of stability and the spacing of levels as determined by the string tension.

The second approach to determine the spectrum of QCD is via Monte Carlo simulations on a lattice. This method, originally proposed by Wilson [7] has been put into practical calculations by a number of authors. Besides the original paper by Creutz [8] which showed that pure Yang–Mills theory has only one phase there has been a large amount of work in order to include fermions (for a recent review see [9]). It is probably the most promising method in the long run but at the present time its accuracy is limited by technical factors like the lattice size.

A completely different approach to spectroscopy is the nonrelativistic potential model approach. As pointed out by Appelquist and Politzer there is reason to believe that heavy quark systems become Coulombic. Indeed this method at the qualitative level has produced good results in a variety of problems. In the case of spectroscopy a naive potential model in which all elements of QCD known from one gluon exchange have been incorporated has been proposed by de Rujula, Georgi and Glashow [10], and developed further by Isgur and Karl [11] and others. For baryons composed of light quarks the agreement is surprisingly good though some terms have to be dropped arbitrarily. For mesons, light as well as heavy, the situation is not too encouraging. It does not appear to be possible to explain in a rather simple way the $J/\psi - \eta_c$ splitting. Nevertheless the concept of quarks with constituent masses which are about 300 MeV larger than the current quark masses seems a good approximation.

Though the potential model is useful as a zeroth-order approximation it cannot contain the whole story. More important, some of the parameters are not easily expressed as functions of the Lagrangian parameters. There is some work of Politzer along these lines [12], but it is not easy to see how, for instance, chiral symmetry breaking effects have been included in the one gluon exchange potential models. The nonperturbative effects which we include in the QCD sum rule formalism are nonlocal and it is therefore in principle not possible to describe these effects by a local potential.

The report is organized in the following way. In chapter 2 we discuss the general procedure, chapter 3 is devoted to the calculation of the Wilson coefficients of the perturbative and nonperturbative operators and contains all relevant technical details. In chapter 4 the applications for two-point functions of heavy and light quarks, including baryons are reviewed in detail. We also give a discussion of mesons and baryons with glue and glueballs. Chapter 5 treats the applications to three-point functions. Finally, we give our conclusions, outlook and summary of the results in chapter 6.

2. General procedure

2.1. Introduction

In this chapter we review the general procedure. Correlation functions are defined and their relation with resonance parameters is established. We discuss the operator product expansion applied to two-and three-point functions in QCD, which operators appear in general and which have to be taken into account for the various quark systems. We close this chapter with a discussion of the moment method and its limit, the Borel transform, which can be used to enhance the dominance of the lowest lying resonance in the sum rule.

2.2. Correlation functions

Consider a current of the general form $j_\Gamma(x) = \bar{q}_i \Gamma q_j$ where the indices i and j on the quark fields denote its flavour and Γ the tensor structure. Each current has definite J, P and C quantum numbers.

To get a current with definite isospin, one has to take the appropriate combination of quark flavours. The vacuum polarization induced by such a current is given by the two-point function

$$T_{\mu\nu\dots}\Pi^j(q^2) = \mathrm{i}\int \mathrm{d}x\, \mathrm{e}^{\mathrm{i}qx}\langle 0|\mathrm{T}(j_\Gamma(x) j_\Gamma(0))|0\rangle\,, \tag{2.1}$$

and is represented by the diagram in fig. 1, where the vertex Γ depends on the current, and the blob represents all possible diagrams which end in a quark–antiquark pair at the vertices. $\Pi^j(q^2)$ is a scalar function, $T_{\mu\nu\dots}$ a tensor depending on the current in question, and T on the right-hand side denotes the T-ordered product.

In the literature so far [1, 13–15] all possible currents with $J^{PC} = 0^{-+}$, 1^{--}, 0^{++}, 1^{++}, 1^{+-}, 2^{++} and 2^{-+} that couple to the observed physical meson states have been studied, i.e. the following set of currents:

$$\begin{aligned}
j_\mathrm{S} &= \bar{q}_i q_j\,, & J^{PC} &= 0^{++}\,,\\
j_\mathrm{P} &= \mathrm{i}\bar{q}_i\gamma_5 q_j\,, & J^{PC} &= 0^{-+}\,,\\
j_\mathrm{V} &= \bar{q}_i\gamma_\mu q_j\,, & J^{PC} &= 1^{--}\,,\\
j_\mathrm{A} &= \eta_{\mu\nu}\bar{q}_i\gamma_\nu\gamma_5 q_j\,, & J^{PC} &= 1^{++}\,,\\
j_{\mathrm{A}'} &= \bar{q}_i\partial_\mu\gamma_5 q_i\,, & J^{PC} &= 1^{+-}\,,\\
j_\mathrm{T} &= \mathrm{i}\bar{q}_i(\gamma_\mu\partial_\nu + \gamma_\nu\partial_\mu + \tfrac{2}{3}\eta_{\mu\nu}\not{\partial})q_i\,, & J^{PC} &= 2^{++}\,,\\
j_{\mathrm{T}'} &= \mathrm{i}\bar{q}_i(\gamma_\mu\gamma_5\partial_\nu + \gamma_\nu\gamma_5\partial_\mu + \tfrac{2}{3}\eta_{\mu\nu}\gamma_5\not{\partial})q_i\,, & J^{PC} &= 2^{-+}\,,
\end{aligned} \tag{2.2}$$

where $\eta_{\mu\nu} = q_\mu q_\nu/q^2 - g_{\mu\nu}$. The last three currents have not yet been studied for the flavour changing case. The $J^{PC} = 2^{-+}$ current has been constructed in [15]. The various currents differ by the current quark vertex Γ.

On grounds of analyticity $\Pi^j(q^2)$ is related to its imaginary part by a dispersion relation, with a number of subtractions depending on the current

$$\Pi^j(q^2) = \frac{(q^2)^n}{\pi}\int \frac{\operatorname{Im}\Pi^j(s)}{s^n(s-q^2)}\,\mathrm{d}s + \sum_{k=0}^{n-1} a_k (q^2)^k\,. \tag{2.3}$$

The unknown subtraction constants a_k can be removed by taking the appropriate number of derivatives of $\Pi^j(q^2)$. In turn the $\operatorname{Im}\Pi^j(s)$ is related to a cross section. In particular for the vector current $j_\mathrm{V}(x)$ we have

$$\operatorname{Im}\Pi^\mathrm{V}(s) = \frac{9}{64\pi^2\alpha^2}\, s\sigma(\mathrm{e}^+\mathrm{e}^- \to \text{hadrons})\,. \tag{2.4}$$

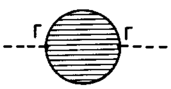


Fig. 1. Graphical representation of the two-point function (2.1). The dashed line indicates the current, and the blob stands for all possible diagrams which end in a quark–antiquark pair at the vertices.

By selecting a particular flavour, e.g. charm for $j_V(x) = \bar{c}\gamma_\mu c$ only mesons with open and hidden charm and $J^{PC} = 1^{--}$ appear in $\operatorname{Im} \Pi^V$, i.e. J/ψ, ψ', ψ'', ..., and continuum states above threshold ($D\bar{D}$ etc.). Similarly one can pick out states with other quantum numbers and/or quark content by choosing another current from (2.2). The reason why currents diagonal in flavour can be chosen will be given below. At this point, feeding hadronic states plus a continuum into $\operatorname{Im} \Pi^j(s)$ one obtains a representation of $\Pi^j(q^2)$ in terms of the parameters of the hadrons that correspond to the current $j(x)$.

Throughout we will use a narrow resonance approximation and write the imaginary part as a sum over δ functions, e.g. for the vector current of flavour q with charge e_q we have

$$\operatorname{Im} \Pi^V(s) = \frac{\pi}{e_q^2} \sum_{\text{res}} \frac{m_R^2}{g_R^2} \delta(s - m_R^2) + \frac{1}{4\pi}\left(1 + \frac{\alpha_s}{\pi}\right)\theta(s - s_0). \tag{2.5}$$

In this case g_R is related to the electronic width of the resonance but for most other currents the coupling has no direct physical significance.

The procedure for studying baryon resonances is completely identical. The currents are now constructed of three quark fields. For instance, for the nucleon and $\Delta(1232)$ ($J^P = \frac{3}{2}^+$) one can choose

$$\eta^N(x) = \varepsilon_{abc}(u^a(x) C\gamma^\mu u^b(x))\gamma_5\gamma_\mu d^c(x), \tag{2.6}$$

$$\eta_\mu^\Delta(x) = \varepsilon_{abc}(u^a(x) C\gamma_\mu u^b(x)) u^c(x), \tag{2.7}$$

C is the charge conjugation operator, and a, b and c are colour indices. The choice (2.6) for the nucleon current is not unique. It has been argued [16] that the coupling of the nucleon to other choices with the same quantum numbers should be small. We will come back to this point in section (4.6), where the baryon sum rules will be considered in detail.

Also the generalization to three-point functions is straightforward. In chapter 5 we will discuss applications to hadron couplings of Goldstone bosons, in particular trilinear meson couplings and baryon couplings to pions and kaons. In this case we consider correlation functions of three currents. For instance for the pion–nucleon interaction we have

$$A(p', p, q) = \int dx\, dy\, e^{ip'x - iqy} \langle 0|T(\eta_N(x) j_P(y) \bar{\eta}_N(0))|0\rangle. \tag{2.8}$$

In all these cases we will follow the same procedure as for two-point functions of mesonic currents and saturate each channel by resonances and a continuum contribution.

2.3. *The operator product expansion*

Following SVZ [1] we start at short distances, i.e. $Q^2 = -q^2$ large, and assume that in QCD the operator product expansion is valid for the T-ordered product of currents in (2.1) and (2.8). We will illustrate the procedure for the meson case (2.1):

$$i \int dx\, e^{iqx} T(j_\Gamma(x) j_\Gamma(0)) = C_I^\Gamma I + \sum_n C_n^\Gamma(q) O_n, \tag{2.9}$$

where I is the identity operator, C_I^Γ, C_n^Γ are the Wilson coefficients, and the O_n are local gauge invariant operators constructed from the quark and gluon fields. The c-number coefficients C_n^Γ obey renormalization group equations, depend on the parameters of the theory, and on the Lorentz indices and quantum numbers of $j_\Gamma(x)$ and O_n. The operators O_n in (2.9) are ordered by increasing dimension and the $C_n^\Gamma(q)$ fall off by corresponding powers of q^2. In the three-point function case the C_n^Γ are functions of the three external momenta p, p' and q. and fall off by appropriate powers of the squares of these momenta. Therefore, at short distances the operators with the lowest dimensions dominate, and give power corrections to the perturbative contributions stemming from the unit operator. Since we are interested in the vacuum expectation values we only have to consider spin zero operators. The complete set of operators with spin zero and dimension equal or less than six is [1]

$$
\begin{aligned}
&I \text{ (unit operator)} && d=0\,,\\
&O_m = m\bar{q}q\,, && d=4\,,\\
&O_G = G^a_{\mu\nu}G^a_{\mu\nu}\,, && d=4\,,\\
&O_\Gamma = \bar{q}\Gamma_1 q\bar{q}\Gamma_2 q\,, && d=6\,,\\
&O_\sigma = \tilde{m}\bar{q}\sigma_{\mu\nu}\frac{\lambda^a}{2}qG^a_{\mu\nu}\,, && d=6\,,\\
&O_f = f_{abc}G^a_{\mu\nu}G^b_{\nu\gamma}G^c_{\gamma\mu}\,, && d=6\,,
\end{aligned}
\tag{2.10}
$$

where m and $\tilde{m}$ are matrices in flavour space whose elements are proportional to quark masses, λ^a are the usual Gell–Mann SU(3) matrices, $\mathrm{Tr}(\lambda^a\lambda^b) = 2\delta^{ab}$, $\sigma_{\mu\nu} = \frac{1}{2}\mathrm{i}[\gamma_\mu, \gamma_\nu]$ and $G^a_{\mu\nu}$ is the gluon field tensor. Other operators can be reduced to those given in (2.10) by using the equations of motion.

The operators of dimension $d > 0$ give rise to $1/q^d$ power corrections. For most applications which have been considered so far higher dimensional operators are assumed to give negligible contributions. Explicit calculations of operators with dimension $d = 6$ and $d = 8$ in charmonium will be discussed in the next chapters.

In QED and other renormalizable quantum field theories one can give good arguments (within perturbation theory) for the validity of an expansion like (2.9). Its physical meaning is well known: the Wilson coefficients determine the short distance behaviour and the large distance part is contained in the matrix elements of the operators O_n. One must check that this separation is still meaningful in QCD.

It is important to note that the expansion (2.9) is an operator expansion. Therefore, the Wilson coefficients are independent of the process being discussed and can be calculated by sandwiching (2.9) between states that pick out a definite operator on the right-hand side.

For the polarization operators (2.1) and (2.8) we have to consider vacuum expectation matrix elements of O_n, which are per definition zero in perturbation theory, but in QCD nonperturbative effects (e.g. instantons) change the nature of the vacuum and induce nonvanishing vacuum expectation values for these operators. These matrix elements express the fact that at large distances the free particle propagators of quarks and gluons are modified by nonperturbative effects. This is by no means a new phenomenon. As mentioned in the Introduction, in the framework of current algebra the expectation value $\langle 0|\bar{q}q|0\rangle$ is known to be unequal to zero and responsible for the breakdown of chiral invariance, and the π–ρ–A_1 mass difference. These matrix elements contain all nonperturbative contributions, and consequently Feynman diagrammatic techniques can be used to calculate their Wilson coefficients. In the next section we will consider in detail which operators contribute for the various different quark systems and discuss the calculation of the Wilson coefficients. The vacuum values of the operators in (2.10) will be treated phenomenologically. It is reassuring that the values found by other methods (current algebra, lattice calculations) agree.

Because nonperturbative effects are included the validity of the OPE in QCD must be checked. As remarked earlier the standard derivations are all in the context of perturbation theory [17]. The extension of the OPE to QCD has been discussed by SVZ in [1] and [18] and by Novikov et al. [6]. Physically the problem is the following. Given the currents or composite operators we construct the correlation function (2.1) or (2.8). In a theory with a trivial vacuum one obtains to all orders in perturbation theory that only the identity operator in (2.9) survives when $x \to 0$. More precisely, the higher dimensional operators do not contribute to the connected Green functions after normal ordering. When the vacuum is more complicated, e.g. in the presence of condensates, how does the theory generalize?

A simple example to illustrate the problem is $\lambda\phi^4$ theory [19, 20]. In this case one allows the mass to become negative; the theory develops a $\langle\phi^2\rangle$ condensate and the naive perturbative vacuum is no longer the ground state. Renormalization, however, can be done in the symmetric or in the broken phase [21]. The Green functions may be computed in any of the two vacua.

In the broken phase, because of the condensate $\langle\phi^2\rangle$ extra terms will appear in the short distance expansion:

$$\langle\Omega_0|\phi(p)\phi(-p)|\Omega_0\rangle = C_I(p^2) + C_{\phi^2}(p^2)\langle\phi^2\rangle\,, \tag{2.11}$$

where Ω_0 stands for the perturbative vacuum. The Wilson coefficients are calculated in the perturbative vacuum and nonperturbative effects are simply taken into account by allowing $\langle\phi^2\rangle$ to be different from zero. This is the method proposed by SVZ [1]. For large p^2 it yields the same answer as calculating in the physical vacuum with the shifted Lagrangian and no condensate. The Lagrangian is

$$\mathscr{L} = \tfrac{1}{2}(\partial_\mu\phi)^2 - \frac{\lambda}{4!}\phi^4 - \frac{m^2}{2}\phi^2\,. \tag{2.12}$$

We are interested in $m^2 < 0$ in which case the vacuum is usually shifted. The new Lagrangian in terms of the shifted field $\rho = \phi - u$ reads

$$\mathscr{L} = \tfrac{1}{2}(\partial_\mu\rho)^2 - \tfrac{1}{2}\left(m^2 + \frac{\lambda u^2}{2}\right)\rho^2 - \frac{\lambda u}{3!}\rho^3 - \frac{\lambda}{4!}\rho^4\,, \tag{2.13}$$

where $u^2 = \langle\phi^2\rangle = -6m^2/\lambda$. Asymptotically ($p^2 \to \infty$) one obtains for the two-point function

$$\Gamma^{(2)}_{\Omega} = p^2 + m^2 + \frac{\lambda u^2}{2} + \frac{\lambda u^2}{32\pi^2}\ln\left(\frac{p^2}{m^2}\right) + \cdots\,. \tag{2.14}$$

It can now be verified [20] that calculating the Wilson coefficients $C_I(p^2)$ and $C_{\phi^2}(p^2)$ in (2.11), i.e. in the perturbative vacuum, the result (2.14) is reproduced.

This example of course has none of the complexities of QCD. Another theory studied along these lines is the non-Abelian, nonlinear sigma model [22]. In the large N limit and to leading order this model reproduces the SVZ expansion. David [22], claims that the gluon condensate is ill defined in the next order in $1/N$. This assertion is not correct since the intermediate renormalization is not taken into account in [22], and it can be shown [23] that by doing so there is no contradiction. However, a proof of the expansion is still not complete, even in this model, since one must prove that additive renormalization does not occur in any of the Green functions. In QCD itself, there only exists a calculation [24] in an external instanton field. There all terms arrange themselves with the appropriate coefficients. In particular all power corrections Q^{-2n} with $n>2$ must vanish. This non-trivial cancellation has been proven for $n=3$ by explicit calculation. The potential problems stemming from renormalization are not tested in this case.

There are two effects due to nonperturbative terms: first they induce nonvanishing vacuum matrix elements for the higher dimensional operators O_n and secondly they cause the breakdown of the OPE at some critical dimension d_{cr}. The Wilson expansion is valid as long as there is a clear distinction between short distances ($\propto 1/Q$) which determine the Wilson coefficients and large distances ($\propto 1/\mu$) which govern the matrix elements. Nonperturbative fluctuations of the quark and gluon fields which have a size independent of Q can be absorbed into the vacuum expectation values of the operators (2.10). Small size fluctuations ($\propto 1/Q$) go beyond the operator product expansion but the central point is that they only show up at some relatively high order in Q^{-2}.

The leading nonperturbative contribution at short distances is due to the one-instanton solution [25]. To get a rough estimate of the validity of the series one can use the dilute instanton gas approximation. In this approximation the expectation value of an operator like $G^a_{\mu\nu}G^a_{\mu\nu}$ can be estimated as

$$\langle 0|\frac{\alpha_s}{\pi}G^a_{\mu\nu}G^a_{\mu\nu}|0\rangle \cong 16\int_0^{\rho_c}\frac{\mathrm{d}\rho}{\rho^5}\,d(\rho)\,, \tag{2.15}$$

where $d(\rho)\propto \exp\{-2\pi/\alpha_s\}$ is the instanton density function (for small $\rho\, d(\rho)\propto\rho^{11}$) and ρ_c is a cutoff. The important point is that (2.15) does not diverge for small ρ. The first ultraviolet divergence appears (in a pure Yang–Mills SU(3) theory) when one attempts to calculate $\langle 0|G^6|0\rangle$, i.e. an operator of dimension 12. So beyond $d=12$ the OPE becomes invalid.

We will not use estimates like (2.15) to find the matrix elements. The one-instanton solution in the dilute gas approximation can only help to find the critical dimension from the convergence of the integrals at large and small ρ.

Having calculated the Wilson coefficients C^Γ_I and $C^\Gamma_n(q)$ we have an expression for the vacuum polarization $\Pi^j(q^2)$ or the three-point function in the deep Euclidean region in terms of the fundamental parameters of QCD and the matrix elements of O_n in the physical nonperturbative vacuum of QCD. Equating this expression with the physical representation discussed in (2.2), we have a relation between the parameters of the theory and hadron parameters. In one of the next sections we discuss methods to determine the properties of the lowest lying resonance in a particular channel.

2.4. Further analysis of the operator product expansion

Our main task will be to calculate the Wilson coefficients of the operators (2.10) for two- and three-point functions constructed of the meson currents (2.2) and/or baryon currents. Which operators play the main role depends on the type of quark system one considers.

Let us first consider heavy quark mesonic systems (charmonium, upsilon system) and assume one can neglect operators with dimension $d > 4$. We will see in chapter 4 under which conditions that is correct. Therefore, apart from the pure perturbative contributions contained in C_I^Γ we will only have the gluon condensate operator $G^a_{\mu\nu}G^a_{\mu\nu}$ and the two fermion operator $m\bar{q}q$. For heavy quarks like a c quark the latter will involve $\langle 0|\bar{c}c|0\rangle$ which can be put equal to zero. Consider the first-order α_s perturbative diagrams of fig. 2 for heavy quarks at $Q^2 = 0$ for the external momentum. The integrals which describe these diagrams are dominated by p^2, $k^2 \propto -m^2$ for the quark and gluon momenta, i.e., the quarks and gluons will be far off shell, and propagate only a short distance. Their propagators will not be affected by vacuum fluctuations and can be described by free particle propagators. The point $Q^2 = 0$ belongs to the asymptotic freedom region and standard perturbation theory can be applied.

To probe larger distances we have to move closer to the quark threshold $q^2 = -Q^2 = 4m^2$. This can also be done [1] by computing higher derivatives at $q^2 = 0$. These will involve integrals of the type

$$\int \frac{\mathrm{d}^4 p}{(p^2 + m^2)^n}\,, \qquad \int \frac{\mathrm{d}^4 p\, \mathrm{d}^4 k}{[(p+k)^2 + m^2]^n}\,, \tag{2.16}$$

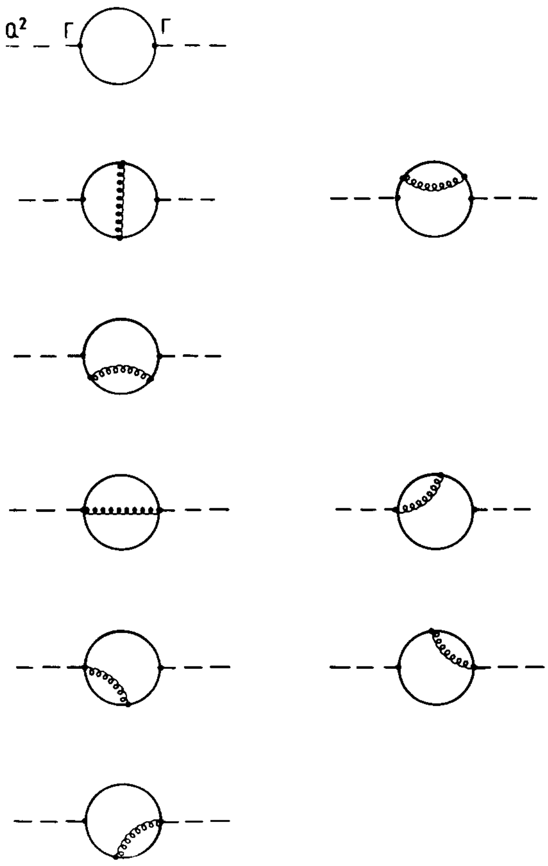


Fig. 2. Contributions to the vacuum polarization to first order in α_s. Curly lines depict gluons, continuous lines quarks, and dashed lines currents. Graphs with four-point vertices are only present for cases with derivative couplings.

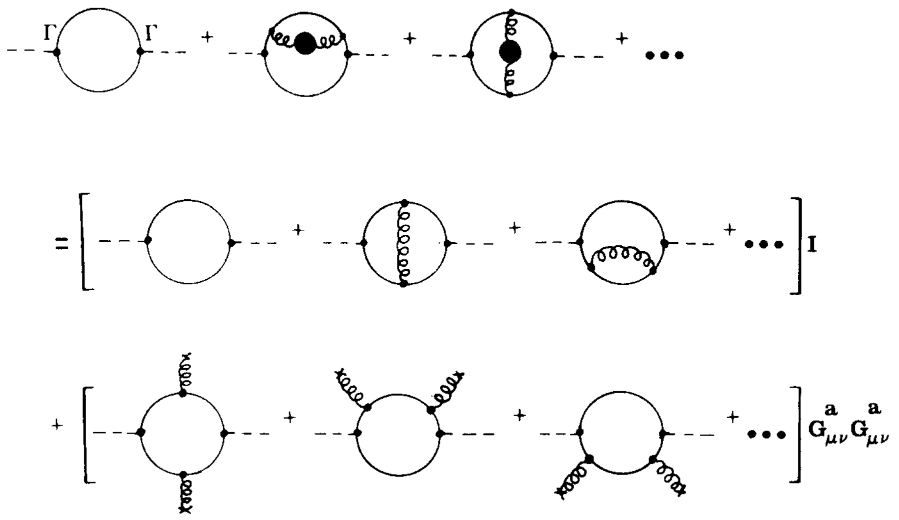
where all momenta are Euclidean. They are dominated by Minkovskian momenta p^2, $k^2 \propto -m^2/n$ and for high n both p^2 and k^2 tend to zero. Even at $p^2 = 0$ the heavy quark is highly virtual and its propagation is described by standard perturbation theory. Thus, for heavy quarks there will be no modification of the free particle propagator due to nonperturbative forces and $\langle 0|\bar{c}c|0\rangle = \langle 0|\bar{b}b|0\rangle \equiv 0$.

A more precise argument can be given in the framework of the heavy quark mass expansion [26] which enables one to write

$$\langle 0|m\bar{q}q|0\rangle \cong -\tfrac{1}{12}\langle 0|\frac{\alpha_s}{\pi} G^a_{\mu\nu}G^a_{\mu\nu}|0\rangle\,, \tag{2.17}$$

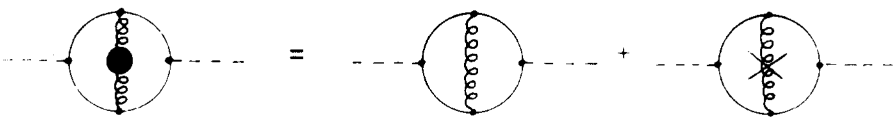
and the quark condensate contribution can be absorbed in the gluon condensate. Due to mixing of the quark and gluon condensate operators there is an extra contribution to the gluon condensate that exactly cancels (2.17) leaving us with a zero contribution to this order from a heavy quark condensate. We will see later that this mixing is specially important for systems with one heavy and one light quark since it cancels unwanted mass singularities.

On the other hand for k^2 tending to zero the gluon approaches its mass shell and modifications to the gluon propagator due to nonperturbative effects have to be taken into account. So to first order in α_s we can write for the vacuum polarization of heavy quarks:



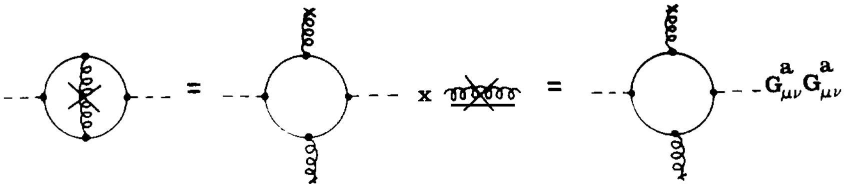
(2.18)

where the first set of diagrams on the right-hand side are the normal perturbative ones (contribute to C^Γ_I) and the second set forms the Wilson coefficient C^Γ_G. The blob indicates that the gluon propagator is modified by long distance effects, i.e.,



(2.19)

and in the OPE the last diagram factorizes into a short- and long-distance piece



(2.20)

So, this procedure tells us immediately which diagrams to calculate for the Wilson coefficient C_G. To see what form this extra contribution due to the modification of the gluon propagator at small k^2 takes, let us write down the loop integral of the diagrams in (2.18) which are of order g^2,

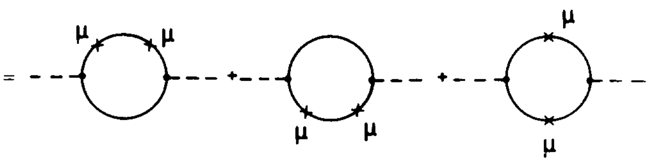
$$\Pi(q^2) = -\frac{\mathrm{i}}{(2\pi)^4} g^2 \int \mathrm{d}^4k\, D(k^2) C(q, k)\,. \tag{2.21}$$

Here, $D(k^2)$ stands for the full gluon propagator including nonperturbative pieces and in the Feynman gauge is given by

$$\langle 0|\mathrm{T}(A^a_\mu(x) A^b_\nu(0))|0\rangle = -\frac{\mathrm{i}\delta^{ab}}{(2\pi)^4} \int \mathrm{d}^4k\, \mathrm{e}^{-\mathrm{i}kx} D(k^2) g_{\mu\nu}\,, \tag{2.22}$$

while $C(q, k)$ represents the subdiagrams from which the gluon propagator is removed

$C(q, k) =$



(2.23)

The spirit of the operator product expansion is based on the observation that even at small k^2 $C(q, k)$ can be calculated in perturbation theory provided the external momentum q^2 is kept far enough from the threshold $4m^2$. Putting the perturbative form for all quark propagators into $C(q, k)$ and expanding in the small momentum k we get for $\Pi(q^2)$ the factorized expression

$$\Pi(q^2) = -\frac{\mathrm{i}}{(2\pi)^4} g^2 \int \mathrm{d}^4k\, D(k^2) \Big[C(q, 0) + k_\alpha \partial^\alpha_k C(q, k)|_{k=0} + k_\alpha k_\beta \frac{1}{2!} \partial^\alpha_k \partial^\beta_k C(q, k)|_{k=0} + \cdots \Big]_{\mathrm{pert}}\,. \tag{2.24}$$

The first two terms vanish because of gauge and Lorentz invariance, and we end up with the desired form in which long and short distance effects have been separated

$$\Pi(q^2) = \lim_{x\to 0} g^2 \sum_{n=0} \frac{(-1)^n}{(2n+2)!} \langle \mathrm{T}(G_{\mu\nu}(x) D_{\alpha_1} D_{\alpha_2} \ldots D_{\alpha_{2n}} G_{\mu\nu}(0))\rangle [\partial^{\alpha_1}_k \partial^{\alpha_2}_k \ldots \partial^{\alpha_{2n}}_k (\partial^2_k C(q, k))|_{k=0}]_{\mathrm{pert}}\,. \tag{2.25}$$

To include only the long-distance part into the matrix elements in (2.25) we have to subtract the perturbative contribution to the gluon propagator

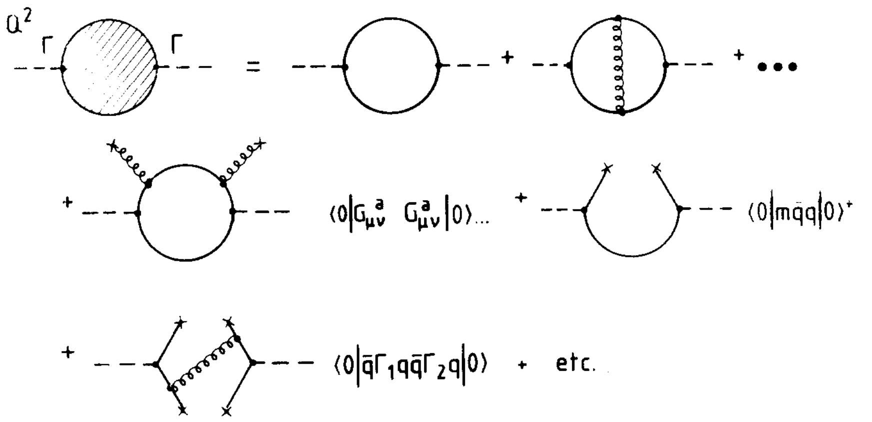
$$-\frac{\mathrm{i}}{(2\pi)^4}\int \mathrm{d}^4k\,[D(k^2)-D(k^2)_{\mathrm{pert}}]k^2\,. \tag{2.26}$$

With this definition of the gluon condensate we get for (2.18)

$$\Pi(q^2)=\Pi(q^2)_{\mathrm{pert}}+g^2\langle 0|G^a_{\mu\nu}G^a_{\mu\nu}|0\rangle\tfrac{1}{16}\partial_{k\alpha}\partial^\alpha_k C(q,k)|_{k=0}+\cdots, \tag{2.27}$$

and the expression for the diagrams in front of $G^a_{\mu\nu}G^a_{\mu\nu}$ is given by $\partial_{k\alpha}\partial^\alpha_k C(q,k)|_{k=0}$.

For systems containing light quarks the quark propagator will also be modified by nonperturbative vacuum fluctuations and quark condensate contributions have to be taken into account in the calculation of the vacuum polarization functions. Of course Q^2 cannot be taken zero in this case. Moreover, since these systems are governed by larger distances also the dimension $d=6$ operators in (2.10), in particular the four fermion operator O_Γ, play an important role. Diagrammatically we can present the vacuum polarization for light quarks in the form



(2.28)

Here Q^2 is large and all propagators in the diagrams which represent the Wilson coefficients carry large momenta while the momenta of the quark and gluon legs which disappear into the vacuum go to zero. For light–heavy systems like open charm and beauty dimension $d=6$ operators still give negligible contributions. For two-point functions of baryonic currents as well as for three-point functions the same arguments hold. Similar diagrammatic equations like (2.18) and (2.28) can be written down. In baryonic currents there will be an extra line originating in the Γ vertex and for three-point functions there will be three Γ vertices.

In various applications to baryon two-point functions and in particular to three-point functions we will use dimensional arguments to argue dominance of certain operators. For any of the currents (2.2) the polarization operator (2.1) has an even number of dimensions and all operators will contribute. For certain systems however the invariant function can have an odd number of dimensions. Since the Wilson coefficients can only be functions of Q^2, this implies that the operator $\bar{q}q$ will not be multiplied by a quark mass and will be greatly enhanced in the light quark case compared to even dimensional operators like $G^a_{\mu\nu}G^a_{\mu\nu}$ which will be multiplied by a small quark mass to match dimensions. For certain cases, for instance the pion–nucleon coupling constant which we will treat in detail later, this will significantly simplify the calculations.

In the next chapter we will discuss in detail the calculation of the Wilson coefficients for the two-point functions of meson and baryon currents. They will be functions of the external momentum Q^2, the quark masses and the coupling constant α_s, but contain no other free parameters. The vacuum expectation values of the operators O_n are also free parameters. They will be determined by various methods (phenomenologically, by PCAC, by instanton models, or by the vacuum saturation hypothesis) as we will discuss in detail in chapter 4.

To conclude this chapter we will now discuss the methods which have been developed to ensure that the lowest lying resonance dominates in a particular channel.

2.5. Moments and the Borel transform

So far, we have two expressions for the vacuum polarization operator: one in terms of physical resonance parameters as discussed in section (2.2) and the other a theoretical expression which is a function of q^2, α_s, the quark masses and the vacuum expectation values of the operators O_n. The theoretical expression has been calculated for large negative q^2 where asymptotic freedom prevails and perturbation theory can be used, and should be a good approximation to the physical $\Pi(q^2)$.

To be able to pick out the lowest lying resonance in a particular channel we define moments by taking derivatives of $\Pi(q^2)$, i.e. in the dispersion representation (2.3) we get

$$\begin{aligned} M^j_n(Q_0^2) &= \frac{1}{n!}\left(-\frac{\mathrm{d}}{\mathrm{d}Q^2}\right)^n \Pi^j(Q^2)|_{Q^2=Q_0^2}, \qquad (Q^2=-q^2) \\ &= \frac{1}{\pi}\int\limits_{4m_q^2}^{\infty} \frac{\mathrm{Im}\,\Pi(s)}{(s+Q_0^2)^{n+1}}\,\mathrm{d}s\,. \end{aligned} \tag{2.29}$$

Inserting the representation (2.5) for the imaginary part into (2.29) we can write

$$M^j_n(Q_0^2) = \frac{1}{e_q^2}\frac{m_{\mathrm{R}}^2}{g_{\mathrm{R}}^2}\frac{1}{(m_{\mathrm{R}}^2+Q_0^2)^{n+1}}[1+\delta^j_n(Q_0^2)]\,, \tag{2.30}$$

where m_{R} and g_{R} are the parameters of the lowest lying resonance and $\delta^j_n(Q_0^2)$ contains the contributions from higher resonances and the continuum. For high n, $\delta^j_n(Q_0^2)$ will go to zero because of the factors $[(m_{\mathrm{R}}^2+Q_0^2)/(m_{\mathrm{R'}}^2+Q_0^2)]^{n+1}$ it contains and because $m_{\mathrm{R'}} > m_{\mathrm{R}}$. So, from a certain n onwards $M^j_n(Q_0^2)$ will be practically equal to the contribution of the first resonance. This will be even more the case if we consider ratios of these moments

$$r^j_n(Q_0^2) = \frac{M^j_n(Q_0^2)}{M^j_{n-1}(Q_0^j)} = \frac{1}{m_{\mathrm{R}}^2+Q_0^2}\frac{1+\delta^j_n(Q_0^2)}{1+\delta^j_{n-1}(Q_0^2)}\,, \tag{2.31}$$

which immediately gives the mass m_{R} of the lowest resonance if we are at sufficiently high n where $\delta^j_n(Q_0^2) \cong \delta^j_{n-1}(Q_0^2)$.

For instance, for the charmed vector sector a large number of resonances is known and $M_n^V(Q_0^2)$ can be calculated to high accuracy. One can easily verify that the J/ψ alone gives about 50% of $M_1^V(Q_0^2=0)$ and already 90% of $M_4^V(Q_0^2=0)$. For $Q_0^2 \neq 0$, $\delta_n^j(Q_0^2)$ will converge less fast to zero and for very large Q_0^2 it will be difficult to extract the parameters of the lowest lying resonances from (2.30) or (2.31). Large (spacelike) Q_0^2 means moving away from the resonance region in the Q^2 plane up to a point from where it will be impossible to distinguish individual resonances. In principle this can be compensated by taking large n. Indeed, taking higher derivatives of $\Pi(q^2)$ means testing larger distances, i.e. moving towards the resonance region. The observation of SVZ is that for heavy quarks there is a region of Q_0^2 starting at about zero for which asymptotic freedom holds. In fact, all calculations for heavy quarks by SVZ [1, 27], have been performed at $Q_0^2=0$. As we have argued before in section (2.4) the high quark mass, compared to the QCD scale, ensures that even at $Q_0^2=0$ we are in the asymptotic freedom region.

Taking derivatives of the theoretical expressions we can write for the moments for heavy quarks (in which case only the gluon condensate gives a nonperturbative contribution)

$$M_n^j(\xi) = A^j(n)[1 + a_n(j;\xi)\alpha_s + b_n(j;\xi)\phi]\,, \tag{2.32}$$

where $\xi = Q_0^2/4m^2$ and

$$\phi = \frac{4\pi^2}{9}\,\frac{\langle 0|\dfrac{\alpha_s}{\pi}G^a_{\mu\nu}G^a_{\mu\nu}|0\rangle}{(4m^2)^2} \tag{2.33}$$

is the matrix element of the gluon condensate. $A^j(n)$ is the nth derivative of the bare loop contribution and the dimensionless coefficients $a_n(j;\xi)$ and $b_n(j;\xi)$ are the moments of the α_s contribution of the Wilson coefficients C_I^Γ and of C_G^Γ respectively, normalized with respect to the bare loop. Their calculation will be discussed in the next chapter. For $\xi=0$ they are tabulated in [28] and for $\xi \neq 0$ in [13]. The coefficients a_n and b_n grow with n (b_n like n^3) but decrease with ξ (or Q_0^2) which has to be chosen such that for a large range of n $a_n\alpha_s$, $b_n\phi \ll 1$ for first-order perturbation theory to make sense. We will see later in applications to the charmonium spectrum that for the calculations to be reliable and to minimize the contributions of higher dimensional operators one must choose Q_0^2 different from zero. For a certain value of ξ there will be a range of n values for which the experimental side of the moment eq. (2.30) (or the ratio (2.31)) is dominated by a single resonance, while the asymptotic freedom side (2.32) is still valid. For small n there occurs a breakdown as the effects of higher states in (2.30) should become noticeable, and at high n the expansion (2.32) does not hold when $b_n\phi$ becomes too large compared to 1. The stability region in n will change with ξ and one should study the stability of the moments for growing ξ. Further on we will apply this method to the charmonium and upsilon systems to extract the parameters of the lowest lying resonance in each channel.

A second method is a variation of the first in which ratios of moments like (2.31) for two different currents are considered. Since it is reasonable to believe that the parameters δ_n^j and $\delta_n^{j'}$ are very similar, in particular in the same partial wave, the corrections from higher resonances tend to cancel each other and we have

$$r_n^j/r_n^{j'} \cong (m_{R'}^2 + Q_0^2)/(m_R^2 + Q_0^2)\,. \tag{2.34}$$

This method is, unlike the first, incapable of giving the absolute normalization of the levels but should give the ratios of the levels with high accuracy. From dimensional considerations it is clear that $A^j(n)$ must be proportional to $(1/m^2)^n$; therefore r_n^j is proportional to $1/m^2$ and very sensitive to the quark mass, while the ratio (2.34) is independent of m.

Equation (2.32) is only true for equal mass heavy quark systems. For light-heavy systems the quark condensate will also give a contribution to (2.32) which again must be small compared to 1 for the expansion to be valid. For light quark systems more higher dimensional operators come in. The moment method can in principle also be used in this case using a large mass scale Q^2 where all corrections are small and taking derivatives with respect to Q^2. If Q^2 tends to infinity the number of derivatives which are calculable in a reliable way is also arbitrarily large and one can consider the limit [1]

$$Q^2 \to \infty\,, \qquad n \to \infty\,, \qquad Q^2/n \equiv M^2 \text{ fixed}\,. \tag{2.35}$$

In this way a new variable M^2 is introduced instead of Q^2. It corresponds to introducing the Borel transform of $\Pi^j(Q^2)$:

$$L_M \Pi^j(Q^2) = \lim_{\substack{Q^2,\, n\to\infty \\ Q^2/n = M^2}} \frac{1}{(n-1)!} (Q^2)^n \left(-\frac{\mathrm{d}}{\mathrm{d}Q^2}\right)^n \Pi^j(Q^2)\,, \tag{2.36}$$

which results in a Borel improvement of the series (2.9), as an operator of dimension d is suppressed by a factor $1/(\frac{1}{2}d-1)!$. Applying L_M to (2.3) we get

$$L_M \Pi^j(Q^2) = \frac{1}{\pi M^2} \int \exp(-s/M^2)\, \mathrm{Im}\, \Pi^j(s)\, \mathrm{d}s\,, \tag{2.37}$$

where all subtraction constants have disappeared and the weight function in the integral has been replaced by an exponential one. We note that the suppression of higher resonances is much less (for $M^2 \cong M_\mathrm{R}^2$) than for the moment method and (2.37) can only be used when the spacing between resonances is relatively large. For details of the properties of the Borel transform we refer to [1]. In chapter 4 we will apply this method to $L=0$ and $L=1$ light quark systems.

3. Calculation of the Wilson coefficients

3.1. Introduction

In this chapter we describe the calculation of the Wilson coefficients of the various operators in the OPE (2.9). Section (3.2) is devoted to the perturbative part of the polarization function $\Pi(q^2)$ for mesonic currents, i.e., to the Wilson coefficient of the identity operator in the OPE (to first order in α_s). The calculation will be described in some detail for the vector current ($J^{PC}=1^{--}$) with unequal mass quarks. This is a particular convenient example since the axial vector, scalar, and pseudoscalar cases can be obtained by a simple replacement. The description follows closely [13] where the equal mass results for all currents (2.2) (except the $J^{PC}=2^{-+}$ current) were obtained. Some of these were already obtained by SVZ in [1] and [27]. The unequal mass cases for currents without derivatives can be found in [29] and [30]. Other calculations of spectral functions for hadronic currents were reported in [31–33].

In [13] a number of checks have been performed for the equal mass cases, in particular the low-energy limit which confronts the results with those from nonrelativistic one gluon exchange, and the high-energy limit where a comparison is made with the solutions of the renormalization group equation for the current correlation functions at high energy. The results reported here agree with these tests.

Section (3.3) is devoted to the Wilson coefficients of the higher dimensional (nonperturbative) operators. First we discuss the coefficient functions of gluonic operators, with special emphasis on the fixed-point gauge technique. We will derive rules for treating the quark propagator in an external background gauge field. A detailed application to the Wilson coefficient of $G_{\mu\nu}G_{\mu\nu}$ for massive mesonic currents is made. Subsequently we discuss recent calculations of six- and eight-dimensional gluonic operators.

As explained in chapter 2 massive quark systems only get contributions from gluonic operators. Systems which contain light quarks have not only explicit quark condensate contributions, but the Wilson coefficient of the gluon condensate gets also a contribution from the light quark condensate. This is especially important in light–heavy systems to cancel mass divergences. These questions will be discussed in the second part of section (3.3).

Finally we will turn to pure light quark systems, including baryons. For calculations with baryonic currents (in general for massless quark systems) it is most convenient to work in the coordinate representation. The difference between mesonic and baryonic currents is due to the different physics of the two systems. As a consequence of chiral symmetry breaking, operators with quark fields appear without a quark mass in baryonic polarization functions. This makes it possible to calculate the coefficients starting with massless quarks whose propagators are simple in x-space. Throughout the calculations we will use the same conventions as Björken and Drell. In the Appendix we have collected all formulae for the gluonic contributions to equal mass heavy quark systems, as well as the full expressions for the polarization functions of light quark mesons and baryons, including mass corrections.

3.2. *Calculation of the perturbative part of $\Pi(q^2)$*

The diagrams which contribute to the perturbative part of the polarization function $\Pi(q^2)$ (to first-order in α_s) are given in fig. 2. The diagrams with contact four-point vertices are only present for currents with derivatives (like $J^{PC} = 1^{+-}$ and $J^{PC} = 2^{++}$). We have calculated the imaginary part of each diagram following Schwingers method [34] of on-mass-shell-renormalization.

To demonstrate the calculation in some detail, we take the flavour changing vector current $j_\mu = \bar{q}_1\gamma_\mu q_2$ as an example, where q_1 and q_2 have masses m_1 and m_2 respectively. From the result the expressions for the axial vector, scalar and pseudoscalar can then be obtained by the replacement $m_1 \to -m_1$. The imaginary part of the polarization function for this current can be written in the form

$$\operatorname{Im} \Pi_{\mu\nu}(q^2) = (q_\mu q_\nu - g_{\mu\nu}q^2)\operatorname{Im} \Pi^{(1)}(q^2) + q_\mu q_\nu \operatorname{Im} \Pi^{(0)}(q^2)\,, \tag{3.1}$$

where $\operatorname{Im} \Pi^{(1)}(q^2)$ is pure vector and $\operatorname{Im} \Pi^{(0)}(q^2)$ is related to the scalar current polarization function. In lowest order, i.e., for the bare loop diagram in fig. 2 the invariant functions $\operatorname{Im} \Pi^{(1)}(q^2)$ and $\operatorname{Im} \Pi^{(0)}(q^2)$

are given by

$$\mathrm{Im}\,\Pi^{(1)}(q^2) = \frac{N_c}{8\pi}\frac{\bar{q}^4}{3q^4}\left\{u(3-u^2) + \frac{(m_1-m_2)^2}{q^2}u^3\right\},$$

$$\mathrm{Im}\,\Pi^{(0)}(q^2) = \frac{N_c}{8\pi}\frac{(m_1-m_2)^2}{q^2}\frac{\bar{q}^4}{q^4}u^3, \tag{3.2}$$

where

$$u^2 = 1 - 4m_1m_2/\bar{q}^2 \quad \text{and} \quad \bar{q}^2 = q^2 - (m_1-m_2)^2. \tag{3.3}$$

The calculation of the first-order α_s correction to the imaginary part of the polarization function can be separated into two parts depending on the way of cutting the diagrams of fig. 2, namely the vertex correction (virtual gluon) and the real gluon emission part. The mass and wave function renormalizations are q^2 independent, and can be included in the redefinition of the quark mass and in the vertex correction as a normalization factor.

(1) *The vertex correction*

The flavour changing vector vertex is corrected as

$$\bar{q}_2\gamma_\mu q_1 \to \bar{q}_2[\gamma_\mu f_1(q^2) + \sigma_{\mu\nu}q^\nu f_2(q^2) + q_\mu f_3(q^2)]q_1, \tag{3.4}$$

where the form factors $f_i(q^2)$ are given by

$$f_1(q^2) = \frac{g^2}{8\pi^2}C_2(F)\left[-2 + \frac{1+2u^2}{2u}\ln\frac{1+u}{1-u} + \frac{1+u^2}{2u}\sum_{i=1,2}\ln\frac{1+u_i}{1-u_i}\ln\frac{u_i}{u} + \frac{1}{u}F(u)\right], \tag{3.5}$$

$$f_2(q^2) = \frac{g^2}{8\pi^2}C_2(F)\left[\frac{(m_1+m_2)}{q^2}\frac{1}{2u}\ln\frac{1+u}{1-u} - \frac{1}{4}\frac{(m_1-m_2)}{q^2}\ln\frac{m_1^2}{m_2^2}\right], \tag{3.6}$$

and

$$f_3(q^2) = -\frac{(m_1-m_2)}{q^2}\frac{g^2}{8\pi^2}C_2(F)\left[4 + \frac{1-4u^2}{2u}\ln\frac{1+u}{1-u} + 3\left\{\frac{1}{4}\left(\frac{m_1+m_2}{m_1-m_2}\right)\ln\frac{m_1^2}{m_2^2} - 1\right\}\right.$$
$$\left. -\frac{1}{2}\frac{(m_1^2-m_2^2)}{q^2}\ln\frac{m_1^2}{m_2^2} + \frac{(m_1-m_2)^2}{q^2}u\ln\frac{1+u}{1-u}\right]. \tag{3.7}$$

where

$$F(u) = (1+u^2)\left[\frac{\pi^2}{2} + \tfrac{1}{4}\sum_i\left(l\left(\frac{1+u_i}{2}\right) - l\left(\frac{1-u_i}{2}\right) - 4l(u_i) + l(u_i^2)\right.\right.$$
$$\left.\left. -\tfrac{1}{2}\ln^2\frac{1+u_i}{2} + \tfrac{1}{2}\ln^2\frac{1-u_i}{2}\right)\right] - u\ln\frac{\lambda^2}{m_1m_2}. \tag{3.8}$$

with u^2 and $\bar{q}^2$ defined in (3.3),

$$u_1 = \frac{\bar{q}^2}{q^2 + m_1^2 - m_2^2}\, u\,, \qquad u_2 = \frac{\bar{q}^2}{q^2 - m_1^2 + m_2^2}\, u\,,$$

and the Spence function $l(x) = -\int_0^x \mathrm{d}t\, t^{-1} \ln(1-t)$. In the form factor $f_1(q^2)$ we have included the contributions from the wave function renormalization to $\bar{q}_2$ and q_1

$$\bar{q}_2\gamma_\mu q_1 \to \bar{q}_2\gamma_\mu q_1 \sqrt{Z_{F_1}}\sqrt{Z_{F_2}} = \bar{q}_2\gamma_\mu q_1(1 + \tfrac{1}{2}\delta Z_{F_1} + \tfrac{1}{2}\delta Z_{F_2})\,. \tag{3.9}$$

The renormalization constants Z_{F_i} are calculated at the quark mass $\not{p} = m_i$ and given by

$$\delta Z_{F_i} = -\frac{g^2}{4\pi^2} C_2(F)\left[\tfrac{1}{2}\ln\frac{\Lambda}{m_i} + \tfrac{9}{8} + \ln\frac{\lambda}{m_i}\right]. \tag{3.10}$$

The ultraviolet divergence $\ln(\Lambda^2/m_1m_2)$ cancels in the vector vertex. The infrared divergence $\ln(\lambda^2/m_1m_2)$ (which will disappear when the real gluon emission is added) still remains at this stage.

In terms of the form factors (3.5), (3.6) and (3.7) the virtual gluon corrections to the invariant functions $\operatorname{Im}\Pi^{(1)}$ and $\operatorname{Im}\Pi^{(0)}$ can be written as

$$\operatorname{Im}\delta\Pi^{(1)}(q^2) = \frac{N_c}{4\pi}\left(\frac{\bar{q}^2}{q^2}\right)^2 u\left[f_1\left(1 - \frac{1}{3}\frac{\bar{q}^2}{q^2}u^2\right) - (m_1 + m_2)f_2\right], \tag{3.11}$$

and

$$\operatorname{Im}\delta\Pi^{(0)}(q^2) = \frac{N_c}{4\pi}\left(\frac{\bar{q}^2}{q^2}\right)^2 u^3\frac{(m_1 - m_2)^2}{q^2}\left[f_1 - \frac{q^2}{m_1 - m_2}f_3\right]. \tag{3.12}$$

(2) *The real gluon emission*

The relevant diagrams for one gluon emission are depicted in fig. 3. The amplitude for this process is

$$M_{\mu\nu} = \bar{u}(p_2)\left[gt^a\gamma_\nu\frac{1}{\not{p}_2 + \not{k} - m_2}\gamma_\mu + \gamma_\mu\frac{1}{-\not{p}_1 - \not{k} - m_1}gt^a\gamma_\nu\right]v(p_1)\,. \tag{3.13}$$

The contribution from one gluon emission to the imaginary part of the polarization function is expressed by

$$\operatorname{Im}\delta\Pi_{\mu\nu} = \tfrac{1}{2}(-g^{\rho\sigma})\left\langle\sum_{\text{spins}} M_{\mu\rho}M^{\dagger}_{\nu\sigma}\right\rangle, \tag{3.14}$$

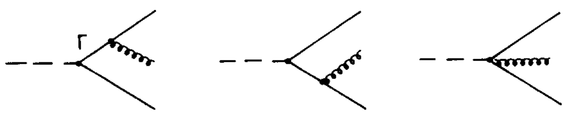


Fig. 3. Contributions to the one-gluon emission part of $\operatorname{Im}\Pi_{\text{pert}}$. They are obtained by cutting the relevant diagrams in fig. 2 in all possible ways.

where $\langle\cdots\rangle$ stands for the three-body phase space integral

$$\langle\cdots\rangle=\int\!\!\int\!\!\int \frac{\mathrm{d}^3k}{2\nu(2\pi)^3}\frac{\mathrm{d}^3p_1}{2E_1(2\pi)^3}\frac{\mathrm{d}^3p_2}{2E_2(2\pi)^3}(2\pi)^4\delta^{(4)}(q-k-p_1-p_2)\,. \tag{3.15}$$

After separating $\operatorname{Im}\delta\Pi_{\mu\nu}$ into the vector and scalar invariants we get

$$\operatorname{Im}\delta\Pi^{(1)}=\left\langle(-\underline{P}^2)\frac{\bar{q}^2}{q^2}\left(1-\frac{1}{3}\frac{\bar{q}^2}{q^2}u^2\right)+\frac{4}{3}\frac{1}{q^2}\frac{((p_1+p_2)\cdot k)^2}{(p_1\cdot k)(p_2\cdot k)}\left(1+\frac{(m_1-m_2)^2}{q^2}\right)-\frac{8}{3q^2}\right\rangle, \tag{3.16a}$$

and

$$\operatorname{Im}\delta\Pi^{(0)}=(m_1-m_2)^2/(q^2)^2\left\langle(-\underline{P}^2)\bar{q}^2u^2+2\frac{[(p_1+p_2)\cdot k]^2}{(p_1\cdot k)(p_2\cdot k)}\right\rangle. \tag{3.16b}$$

Here we have used the short-hand notation $\underline{P}$ for

$$\underline{P}=\left(\frac{p_1}{p_1\cdot k}-\frac{p_2}{p_2\cdot k}\right).$$

The three-body phase space integral can be performed along the same lines as in Schwinger's book [34], by putting a ficticious gluon mass (λ) in the infrared sensitive integral to get a consistent treatment of the infrared singularity with the one in the virtual gluon correction. Taking also the colour factor into account we obtain the following expression for the integrals:

$$\begin{aligned}\langle-\underline{P}^2\rangle=\frac{N_c}{4\pi}\frac{g^2}{8\pi^2}C_2(F)\frac{\bar{q}^2}{q^2}\Big[&(1+u^2)\Big\{\frac{\pi^2}{6}+\ln\frac{1+u}{1-u}\ln\frac{1+u}{2}+2l\Big(\frac{1-u}{1+u}\Big)+l\Big(\frac{1+u}{2}\Big)\\&-l\Big(\frac{1-u}{2}\Big)+\tfrac{1}{2}\sum_i\Big(-4l(u_i)+l(u_i^2)+l\Big(\frac{1+u_i}{2}\Big)-l\Big(\frac{1-u_i}{2}\Big)\Big)\Big\}\\&+u\Big\{4+3\ln\frac{1-u^2}{4u}-\ln u+\tfrac{1}{2}\sum_i\frac{1}{u_i}\ln\frac{1+u_i}{1-u_i}+2\ln\frac{q}{\bar{q}}\Big\}-F(u)\Big],\end{aligned} \tag{3.17}$$

$$\left\langle 2\frac{((p_1+p_2)\cdot k)^2}{(p_1\cdot k)(p_2\cdot k)}\right\rangle=\frac{N_c}{4\pi}\frac{g^2}{8\pi^2}C_2(F)q^2\left(\frac{\bar{q}^2}{q^2}\right)^2\left\{-\tfrac{3}{8}u(1+u^2)+\tfrac{1}{2}[u^2+\tfrac{3}{8}(1-u^2)^2]\ln\frac{1+u}{1-u}\right\}, \tag{3.18}$$

and

$$\begin{aligned}\langle 4\rangle=\frac{N_c}{4\pi}\frac{g^2}{8\pi^2}C_2(F)q^2\Big[&\left(\frac{\bar{q}^2}{q^2}\right)^2\left(u\frac{3-u^2}{4}-\tfrac{1}{8}(1-u^2)(3+u^2)\ln\frac{1+u}{1-u}\right)\\&+\frac{(m_1-m_2)^2}{q^2}\left(u\frac{\bar{q}^2}{q^2}-\left(1+\tfrac{1}{2}(1-u^2)\frac{\bar{q}^2}{q^2}\right)\ln\frac{1+u}{1-u}\right)+\frac{m_1^2-m_2^2}{q^2}\ln\frac{a+u}{a-u}\Big],\end{aligned} \tag{3.19}$$

where $a=(m_1+m_2)/(m_1-m_2)$.

Summing the vertex corrections (3.11) and (3.12) with the resulting expressions (3.16a) and (3.16b) for real gluon emission, we have for the first-order α_s perturbative corrections to the imaginary part of the polarization function

$$\begin{aligned}
\operatorname{Im}\delta\Pi^{(1)} = {} & \frac{N_c}{4\pi}\frac{g^2}{8\pi^2}C_2(F)\left(\frac{\bar{q}^2}{q^2}\right)^2\biggl[(1-\tfrac{1}{3}u^2)\left(\mathcal{B}(u,u_1,u_2)-\ln\frac{1+u}{1-u}\right) \\
& +\tfrac{5}{4}u-\tfrac{3}{4}u^3+(\tfrac{33}{24}+\tfrac{22}{24}u^2-\tfrac{7}{24}u^4)\ln\frac{1+u}{1-u}+\frac{1}{3}\frac{(m_1-m_2)^2}{q^2}\biggl\{u^2\biggl(\mathcal{B}(u,u_1,u_2) \\
& -\ln\frac{1+u}{1-u}\biggr)-\tfrac{3}{8}u+\tfrac{29}{8}u^3+(\tfrac{3}{16}+\tfrac{34}{16}u^2-\tfrac{13}{16}u^4)\ln\frac{1+u}{1-u}\biggr\} \\
& +\frac{1}{3}\frac{(m_1-m_2)^2}{\bar{q}^2}\left\{-2u\frac{q^2}{\bar{q}^2}-2u^3+\left(-2u^2(1-u^2)+\frac{q^2}{\bar{q}^2}(1-u^2)+2\left(\frac{q^2}{\bar{q}^2}\right)^2\right)\ln\frac{1+u}{1-u}\right\} \\
& +\frac{1}{3}\frac{m_1^2-m_2^2}{q^2}\left(\frac{3}{4}u\ln\frac{m_1^2}{m_2^2}-2\left(\frac{q^2}{\bar{q}^2}\right)^2\ln\frac{a+u}{a-u}\right)\biggr],
\end{aligned}\tag{3.20}$$

and for the scalar component

$$\begin{aligned}
\operatorname{Im}\delta\Pi^{(0)} = {} & \frac{(m_1-m_2)^2}{\bar{q}^2}\frac{N_c}{4\pi}\frac{g^2}{8\pi^2}C_2(F)\left(\frac{\bar{q}^2}{q^2}\right)^2\biggl[u^2\left(\mathcal{B}(u,u_1,u_2)-\ln\frac{1+u}{1-u}\right)-\tfrac{3}{8}u+\tfrac{45}{8}u^3 \\
& +(\tfrac{3}{16}+\tfrac{34}{16}u^2-\tfrac{13}{16}u^4)\ln\frac{1+u}{1-u}+\frac{(m_1-m_2)^2}{q^2}u^4\ln\frac{1+u}{1-u}+u^3\left(\frac{3}{4}\frac{m_1+m_2}{m_1-m_2}\ln\frac{m_1^2}{m_2^2}-3\right) \\
& -\frac{(m_1^2-m_2^2)}{2q^2}u^3\ln\frac{m_1^2}{m_2^2}\biggr],
\end{aligned}\tag{3.21}$$

where the function $\mathcal{B}(u, u_1, u_2)$ is common to all currents and given by

$$\begin{aligned}
\mathcal{B}(u,u_1,u_2) = {} & (1+u^2)\biggl\{\frac{\pi^2}{6}+\ln\frac{1+u}{1-u}\ln\frac{1+u}{2}+2l\left(\frac{1-u}{1+u}\right)+l\left(\frac{1+u}{2}\right)-l\left(\frac{1-u}{2}\right) \\
& +\tfrac{1}{2}\sum\left[l\left(\frac{1+u_i}{2}\right)-l\left(\frac{1-u_i}{2}\right)-4l(u_i)+l(u_i^2)\right]\biggr\}+3u\ln\frac{1-u^2}{4u}-u\ln u+2u\ln\frac{q}{\bar{q}} \\
& +\tfrac{1}{2}\sum_i\frac{u}{u_i}\ln\frac{1+u_i}{1-u_i}-\frac{1+u^2}{2}\sum_i\ln\frac{1+u_i}{1-u_i}\ln\frac{u}{u_i}.
\end{aligned}\tag{3.22}$$

As mentioned above, the expressions for the axial vector and the pseudoscalar unequal mass cases can be obtained by replacing $m_1\to -m_1$ (or $m_2\to -m_2$) in eqs. (3.20) and (3.21), respectively. Under this replacement ($m_1\to -m_1$) the variables change as $u\to 1/u$, $u_1\to u_1$, $u_2\to u_2$, and $\bar{q}^2\to\bar{q}^2u^2$, while the function $\mathcal{B}$ obeys the relation

$$u^2\mathcal{B}(1/u, u_1, u_2)=\mathcal{B}(u,u_1,u_2).$$

The leading behaviour at large q^2 of the axial current obtained in this way coincides with the vector current. In other words we have renormalized the axial form factor with respect to the vector one, namely,

$$F_A(q^2) = F_V(q^2) + \frac{1}{\pi}\int ds\, \frac{1}{s-q^2}[\mathrm{Im}\, F_A(s) - \mathrm{Im}\, F_V(s)]\,. \tag{3.23}$$

As for the pseudoscalar case, the current is defined by the divergence of the axial current $\partial^\mu j_{5\mu}$. These renormalization procedures differ from the ones used in [13] and [28] by a finite constant. There we have renormalized all form factors independently by putting $F_\Gamma(q^2 = 0) = 1$. For massive quarks (like m_c and m_b which are far from the chiral limit) there is no preferred renormalization procedure but to recover the correct massless limit the renormalization procedure adopted here is more convenient. Our results are in agreement with other calculations [33].

For the sake of completeness we list the results of the perturbative corrections in the equal mass case ($m_1 = m_2 = m$) for all S- and P- wave currents, including the $J^{PC} = 1^{+-}$ and $J^{PC} = 2^{++}$ currents. The last two contain derivative couplings and cannot be obtained from the expressions of the flavour changing current. We note that there do not exist expressions in the literature for the unequal mass cases of these currents. All expressions are the same as in [13], except for the polynomial terms in the axial vector case (u^3 term) and in the pseudoscalar case (u term) because of the change in the renormalization of the form factors at $q^2 = 0$.

In the limit $m_1 = m_2 = m$, $u_1 = u_2 = u$ and

$$\mathscr{B}(u, u, u) = A(u) + \ln\frac{1+u}{1-u}\,,$$

where $A(u)$ is defined as

$$A(u) = (1+u^2)\left[\frac{\pi^2}{6} + \ln\frac{1+u}{1-u}\ln\frac{1+u}{2} + 2l\left(\frac{1-u}{1+u}\right) + 2l\left(\frac{1+u}{2}\right) - 2l\left(\frac{1-u}{2}\right) - 4l(u) + l(u^2)\right]$$
$$+ 3u\ln\frac{1-u^2}{4u} - u\ln u\,. \tag{3.24}$$

For the pseudoscalar current we obtain

$$\mathrm{Im}\,\delta\Pi^{(P)} = \frac{\mathrm{Im}\,\Pi_0^{(P)}}{u}\frac{g^2}{4\pi^2}C_2(F)\left[A(u) + P^{(P)}(u)\ln\frac{1+u}{1-u} + Q^{(P)}(u)\right],$$

$$\mathrm{Im}\,\Pi_0^{(P)} = \frac{3}{8\pi}q^2 u\,, \qquad P^{(P)}(u) = \tfrac{19}{16} + \tfrac{2}{16}u^2 + \tfrac{3}{16}u^4\,, \qquad Q^{(P)}(u) = \tfrac{21}{8}u - \tfrac{3}{8}u^3\,. \tag{3.25}$$

For the vector current,

$$\operatorname{Im}\delta\Pi^{(\mathrm{V})}=\frac{\operatorname{Im}\Pi_0^{(\mathrm{V})}}{u(1-\frac{1}{3}u^2)}\frac{g^2}{4\pi^2}C_2(F)\left[(1-\tfrac{1}{3}u^2)A(u)+P^{(\mathrm{V})}(u)\ln\frac{1+u}{1-u}+Q^{(\mathrm{V})}(u)\right],$$

$$\operatorname{Im}\Pi_0^{(\mathrm{V})}=\frac{1}{8\pi}u(3-u^2),\qquad P^{(\mathrm{V})}(u)=\tfrac{33}{24}+\tfrac{22}{24}u^2-\tfrac{7}{24}u^4,\qquad Q^{(\mathrm{V})}(u)=\tfrac{5}{4}u-\tfrac{3}{4}u^3. \tag{3.26}$$

And for the P-wave currents ($J^{PC}=1^{+-},0^{++},1^{++}$),

$$\operatorname{Im}\delta\Pi^{(\Gamma)}=\frac{\operatorname{Im}\Pi_0^{(\Gamma)}}{u^3}\frac{g^2}{4\pi^2}C_2(F)\left[u^2A(u)+P^{(\Gamma)}(u)\ln\frac{1+u}{1-u}+Q^{(\Gamma)}(u)\right], \tag{3.27}$$

$\Gamma=(A';\,\partial_\mu\gamma_5,\,1^{+-})$

$$\operatorname{Im}\Pi_0^{(A')}=\frac{1}{8\pi}q^4u^3,\qquad P^{(A')}(u)=\tfrac{13}{16}+\tfrac{28}{16}u^2+\tfrac{17}{16}u^4-\tfrac{2}{16}u^6,\qquad Q^{(A')}(u)=-\tfrac{13}{8}u+\tfrac{47}{24}u^3+\tfrac{1}{4}u^5, \tag{3.28}$$

$\Gamma=(S;\,1,\,0^{++})$

$$\operatorname{Im}\Pi_0^{(S)}=\frac{3}{8\pi}q^2u^3,\qquad P^{(S)}(u)=\tfrac{3}{16}+\tfrac{34}{16}u^2-\tfrac{13}{16}u^4,\qquad Q^{(S)}(u)=-\tfrac{3}{8}u+\tfrac{45}{8}u^3, \tag{3.29}$$

$\Gamma=(A;\,\gamma_\mu\gamma_5,\,1^{++})$

$$\operatorname{Im}\Pi_0^{(A)}=\frac{1}{4\pi}q^2u^3,\qquad P^{(A)}(u)=\tfrac{21}{32}+\tfrac{59}{32}u^2+\tfrac{19}{32}u^4-\tfrac{3}{32}u^6,\qquad Q^{(A)}(u)=-\tfrac{21}{16}u+\tfrac{30}{16}u^3+\tfrac{3}{16}u^5. \tag{3.30}$$

Finally, the correction to the tensor current ($J^{PC}=2^{++}$) takes the form:

$$\operatorname{Im}\delta\Pi^{(\mathrm{T})}=\frac{3\operatorname{Im}\Pi_0^{(\mathrm{T})}}{u^3(1-\frac{2}{5}u^2)}\frac{g^2}{4\pi^2}C_2(F)\left[\tfrac{1}{3}u^2(1-\tfrac{2}{5}u^2)A(u)+P^{(\mathrm{T})}(u)\ln\frac{1+u}{1-u}+Q^{(\mathrm{T})}(u)\right], \tag{3.31}$$

$$\operatorname{Im}\Pi_0^{(\mathrm{T})}=\frac{1}{2\pi}q^4u^3(1-\tfrac{2}{5}u^2),$$
$$P^{(\mathrm{T})}(u)=\tfrac{1}{45}+\tfrac{313}{360}u^2+\tfrac{23}{96}u^4-\tfrac{409}{1440}u^6+\tfrac{31}{1440}u^8-\tfrac{1}{480}u^{10},$$
$$Q^{(\mathrm{T})}(u)=-\tfrac{2}{45}u-\tfrac{47}{540}u^3+\tfrac{229}{10800}u^5-\tfrac{1}{24}u^7+\tfrac{1}{240}u^9. \tag{3.32}$$

In phenomenological applications we have to calculate the moments of the polarization function according to (2.29), i.e. at $Q_0^2=0$,

$$M_n^J=(4m^2)^{-n}\frac{1}{\pi}\int_0^1\mathrm{d}u^2(1-u^2)^{n-1}\operatorname{Im}\Pi^J(u). \tag{3.33}$$

To calculate these integrals it is convenient to introduce the interpolation formula for the common part of the perturbative correction in each current, derived by Schwinger [34]. For the S-wave currents (0^{-+} and 1^{--}),

$$(1-\tfrac{1}{3}u^2)A(u)+P^{(V)}(u)\ln\frac{1+u}{1-u}+Q^{(V)}(u)=\pi u(1-\tfrac{1}{3}u^2)\left(\frac{\pi}{2u}-\tfrac{1}{4}(3+u)\left(\frac{\pi}{2}-\frac{3}{4\pi}\right)\right), \tag{3.34}$$

with $A(u)$, $P^{(V)}(u)$, and $Q^{(V)}(u)$ given by (3.24) and (3.26). For the P-wave currents (1^{+-}, 0^{++}, 1^{++}, and 2^{++})

$$u^2A(u)+P(u)\ln\frac{1+u}{1-u}+Q(u)=\pi u^3\left(\frac{\pi}{2u}-\frac{1+u}{2}\left(\frac{\pi}{2}-\frac{3}{\pi}\right)\right), \tag{3.35}$$

where $P(u)$ and $Q(u)$ are defined as

$$P(u)=\tfrac{5}{4}(1+u^2)^2-2 \quad \text{and} \quad Q(u)=\tfrac{3}{2}u^2(1+u^2)\,. \tag{3.35a}$$

For example, the interpolation formula for the scalar current becomes

$$\begin{aligned} u^2A(u)+P^{(S)}(u)\ln\frac{1+u}{1-u}+Q^{(S)}(u) &= \pi u^3\left(\frac{\pi}{2u}-\frac{1+u}{2}\left(\frac{\pi}{2}-\frac{3}{\pi}\right)\right) \\ &\quad +(P^{(S)}(u)-P(u))\ln\frac{1+u}{1-u}+Q^{(S)}(u)-Q(u)\,. \end{aligned} \tag{3.36}$$

From (3.29) we obtain for the last two terms

$$P^{(S)}(u)-P(u)=\tfrac{15}{16}-\tfrac{6}{16}u^2-\tfrac{33}{16}u^4\,, \qquad Q^{(S)}(u)-Q(u)=-\tfrac{15}{8}u+\tfrac{33}{8}u^3\,, \tag{3.37}$$

and (3.36) is now easy to handle in the integrals (3.33) (no Spence function left).

We will not repeat the checks on the equal mass results performed in [13]. The direct calculation of the flavour changing current makes it possible to obtain the expressions for other currents by changing the sign of one of the masses:

$$\Pi^{A}(q^2,m_1,m_2)=\Pi^{V}(q^2,\mp m_1,\pm m_2)\,, \qquad \Pi^{P}(q^2,m_1,m_2)=\Pi^{S}(q^2,\mp m_1,\pm m_2)\,, \tag{3.38}$$

and the result can be compared with the direct calculation starting from the currents with opposite chirality. The scalar (pseudoscalar) results can also be obtained from the divergence of the vector (axial vector) current $q^\mu q^\nu \Pi_{\mu\nu}$. A similar check is possible for the tensor current $(\gamma_\mu\partial_\nu)$ by projecting with $g^{\mu\nu}$ which gives the scalar polarization function. Finally, the expressions for any current should be an even function of (m_1-m_2) because of charge conjugation.

3.3. The calculation of the Wilson coefficients of higher dimensional operators

3.3.1. The plane wave method

First we will discuss the so-called plane wave method for calculating Wilson coefficients. As an example we consider the calculation of the coefficient C_G of the gluon field strength operator $O_G = G^a_{\mu\nu}G^a_{\mu\nu}$. The original calculations [1] and [13] have all been done with this method, and it is still the most convenient way for calculating Wilson coefficients of operators which contain quark fields.

Let us go back to the starting point of the operator product identity, which has in general the form (2.9). Because of gauge and Lorentz invariance one can construct two independent operators from $G^a_{\mu\nu}$ in the OPE

$$\mathrm{i}\int \mathrm{d}^4x\, \mathrm{e}^{\mathrm{i}qx}\, \mathrm{T}(j_\Gamma(x)j_\Gamma(0)) = C_I(q)I + C_G(q)G^a_{\mu\nu}G^a_{\mu\nu} + D_G(q)\{G^a_{\mu\lambda}q_\lambda G^a_{\mu\nu}q_\nu - \tfrac{1}{4}q^2 G^a_{\mu\nu}G^a_{\mu\nu}\} + \cdots . \tag{3.39}$$

The relevant term for the vacuum polarization is the operator $G^a_{\mu\nu}G^a_{\mu\nu}$, since the second term vanishes when sandwiched by vacuum states.

Since the operator product expansion is an operator identity, one can single out a particular operator by sandwiching by appropriately chosen states. The coefficient function reflects the short-distance behaviour of the product of currents and is independent of the state we sandwich with. For the present purpose, it is most convenient to take the matrix element between one gluon states (k, α) and (k, β) on both sides of (3.39). On the left-hand side we have the forward gluon scattering amplitude on a colour singlet current which can be decomposed into two invariant amplitudes

$$T_{\alpha\beta}(q, k) = \mathrm{i}\int \mathrm{d}^4x\, \mathrm{e}^{\mathrm{i}qx}\langle k, \alpha|\mathrm{T}(j_\Gamma(x)j_\Gamma(0))|k, \beta\rangle = L_{1\alpha\beta}C(q, k) + L_{2\alpha\beta}D(q, k)\,, \tag{3.40}$$

where the Lorentz tensors $L_{i\alpha\beta}$ have the form

$$L_{1\alpha\beta} = 4(k^2 g_{\alpha\beta} - k_\alpha k_\beta) = = \langle k, \alpha|G^a_{\mu\nu}(0)G^a_{\mu\nu}(0)|k, \beta\rangle\,, \tag{3.41a}$$

and

$$\begin{aligned} L_{2\alpha\beta} &= 2(k^2 q_\alpha q_\beta - k\cdot q q_\alpha k_\beta - k\cdot q q_\beta k_\alpha + (k\cdot q)^2 g_{\alpha\beta}) - q^2(k^2 g_{\alpha\beta} - k_\alpha k_\beta) \\ &= \langle k, \alpha|\{G^a_{\mu\nu}(0)G^a_{\mu\lambda}(0)q^\nu q^\lambda - \tfrac{1}{4}q^2 G^a_{\mu\nu}(0)G^a_{\mu\nu}(0)\}|k, \beta\rangle\,. \end{aligned} \tag{3.41b}$$

Comparing (3.40) with the right-hand side of (3.39) we find that the coefficient $C_G(q^2)$ is equal to the invariant amplitude $C(q, k)$ at $k_\mu = 0$,

$$C_G(q^2) = C(q, k)|_{k_\mu = 0}\,. \tag{3.42}$$

Thus, the procedure to extract $C(q, 0)$ from the scattering amplitude $T_{\alpha\beta}(q, k)$ amounts to the following: (i) write down the diagrams with two external gluons carrying a momentum k (i.e., the diagrams in front of $G^a_{\mu\nu}G^a_{\mu\nu}$ in (2.18)), (ii) expand the propagators in the gluon momenta k and keep terms that are bilinear in k, (iii) apply the projection $g^{\alpha\beta}$ and substitute $\frac{1}{4}g^{\rho\sigma}k^2$ for $k_\rho k_\sigma$. The first steps generate a lot of terms and make the calculations cumbersome. Moreover, if we want to go to higher dimensional gluonic operators we are faced with complicated projections (step (iii)) to extract a particular operator from all possible operators with the same dimension.

3.3.2. Fixed-point gauge technique

Especially for higher dimensional gluonic operators the plane wave method is too complicated. Recently considerable progress has been made by working in the so-called fixed-point gauge, originally discovered by Fock [35] and Schwinger [36] and rediscovered by a number of people [24, 37]. In this method one puts an external gauge field $A^a_\mu(x)$ into the Lagrangian with the gauge condition

$$(x-x_0)^\mu A^a_\mu(x) = 0\,, \tag{3.43}$$

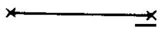
where x_0 is an arbitrary point in space which can be chosen to be the origin. Then, the external gauge field $A^a_\mu(x)$ can be expressed directly in terms of gauge covariant quantities

$$\begin{aligned} A_\mu(x) &= \int_0^1 \alpha\, \mathrm{d}\alpha\, G_{\nu\mu}(\alpha x) x^\nu \\ &= \tfrac{1}{2}x^\nu G_{\nu\mu}(0) + \tfrac{1}{3}x^\alpha x^\nu D_\alpha G_{\nu\mu}(0) + \tfrac{1}{8}x^\alpha x^\beta x^\nu D_\alpha D_\beta G_{\nu\mu}(0) + \cdots\,, \end{aligned} \tag{3.44}$$

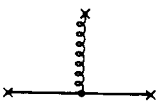
where $G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu$, $G_{\mu\nu} = G^a_{\mu\nu} t^a$, $D_\alpha = \partial_\alpha - \mathrm{i} g t^a A^a_\mu$, and $[t^a, t^b] = \mathrm{i} f^{abc} t^c$ $(t^a = \frac{1}{2}\lambda^a)$.

Gluonic power corrections can be calculated by considering the polarization function in this external background field. To calculate the $G^a_{\mu\nu}G^a_{\mu\nu}$ correction only the first term in the expansion (3.44) for A_μ has to be taken into account, for the $d=6$ correction also the next two, etc. The gauge-invariant structures $\cong G^2$, G^3, etc. emerge automatically and do not have to be constructed by hand as in the plane wave method. Because of the choice of the fixed point x_0 in (3.43) the fixed-point gauge violates translation invariance. In the final result for any gauge-invariant quantity however all translation noninvariant terms cancel.

Simple rules can now be formulated for the calculation of the coefficients [38] by calculating first the quark propagator in an external field $A_\nu(x)$ in the gauge (3.43). For the massive quark propagator one obtains (in the p-representation) expanding in the number of gluon legs attached to the quark line:

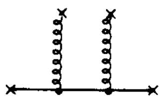


$$\doteq \frac{\mathrm{i}}{\not p - m} \text{ the free propagator;} \tag{3.45a}$$



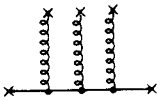
$$-\frac{\mathrm{i}}{4} g t^a G^a_{\kappa\lambda}(0) \frac{1}{(p^2-m^2)^2}\{\sigma_{\kappa\lambda}(\not p + m) + (\not p + m)\sigma_{\kappa\lambda}\} \tag{3.45b}$$

for one gluon line attached $(\sigma_{\kappa\lambda} = \frac{1}{2}\mathrm{i}[\gamma_\kappa, \gamma_\lambda])$;



$$-\frac{\mathrm{i}}{4} g^2 t^a t^b G^a_{\alpha\beta}(0) G^b_{\mu\nu}(0) \frac{(\not p + m)}{(p^2-m^2)^5}(f_{\alpha\beta\mu\nu} + f_{\alpha\mu\beta\nu} + f_{\alpha\mu\nu\beta})(\not p + m) \tag{3.45c}$$

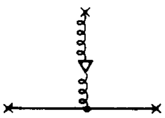
with two gluon lines attached. Here $f_{\alpha\beta\mu\nu} = \gamma_\alpha(\not{p} + m)\gamma_\beta(\not{p} + m)\gamma_\mu(\not{p} + m)\gamma_\nu(\not{p} + m)$.



$$\frac{\mathrm{i}}{48} g^3 f^{abc} G^a_{\gamma\delta} G^b_{\delta\varepsilon} G^c_{\varepsilon\gamma} \frac{1}{(p^2 - m^2)^6} (\not{p} + m)\{\not{p}(p^2 - 3m^2) + 2m(2p^2 - m^2)\}(\not{p} + m)\,. \tag{3.45d}$$

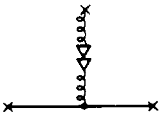
Here we have already averaged over the indices of the three gluon legs. It is quite straightforward but cumbersome to write down the general expression (i.e., before averaging) for any number of gluon legs in terms of $f_{\alpha\beta\ldots\mu\nu}$ defined in analogy to the $f_{\alpha\beta\mu\nu}$ in (3.45c). Each leg adds two indices to the f symbols and the number of indices is equal to the dimension of the operator.

The rules given so far only involve the first term in the expansion (3.44). To be able to calculate the coefficients of all possible gluonic operators with dimension up to 6 we also need the terms with one and two derivatives in (3.44). For instance,



$$\tfrac{1}{3}\mathrm{i} g t^a D_\alpha G^a_{\rho\mu}(0) \frac{(\not{p} + m)}{(p^2 - m^2)^4} (f_{\mu\rho\alpha} + f_{\mu\alpha\rho})(\not{p} + m) \tag{3.45e}$$

where $f_{\mu\rho\alpha} = \gamma_\mu(\not{p} + m)\gamma_\rho(\not{p} + m)\gamma_\alpha$.



$$-\tfrac{1}{8} g t^a (D_\alpha D_\beta + D_\beta D_\alpha) G^a_{\rho\mu}(0) \frac{(\not{p} + m)}{(p^2 - m^2)^5} (f_{\mu\rho\alpha\beta} + f_{\mu\alpha\beta\rho} + f_{\mu\alpha\rho\beta})(\not{p} + m) \tag{3.45f}$$

with $f_{\alpha\beta\mu\nu}$ as defined above. We will demonstrate the use of these rules later in an explicit calculation of the Wilson coefficients of $G^a_{\mu\nu} G^a_{\mu\nu}$ for massive quark systems.

For massless quarks (to be discussed at the end of this section) it is more convenient to work in the x-representation. The external field method can formally be extended to include fermion fields by adding the appropriate terms to the original Lagrangian

$$\Delta\mathscr{L} = \bar{q}(x)(\mathrm{i}\not{\partial} - m)\chi(x) + \bar{\chi}(x)(\mathrm{i}\not{\partial} - m)q(x) + g\bar{q}(x)\gamma^\mu t^a q(x) A_\mu(x)\,, \tag{3.46}$$

where χ and $\bar{\chi}$ are anticommuting external spinor fields and $A^a_\mu(x)$ is the external gauge field which satisfies (3.43) and (3.44). The quark propagator in the external fields can then be written as

$$S_{\mathrm{F}}(x; \chi, \bar{\chi}, G)^{ab} = \delta^{ab} s_0(x) x_\mu \gamma^\mu + \chi^a(x)\bar{\chi}^b(0) - \mathrm{i} g (t^k)^{ab} G^k_{\kappa\lambda}(0) s_1(x)[x_\mu \gamma^\mu \sigma_{\kappa\lambda} + \sigma_{\kappa\lambda} x_\mu \gamma^\mu] + \cdots\,, \tag{3.47}$$

where a, b refer to the colour indices. The first term is the free propagator with $s_0(x) = (\mathrm{i}/2\pi^2)(1/x^4)$. The second term comes from the external quark fields and is responsible for the $\bar{q}q$ condensate. The last term stands for the propagator with one gluon line attached ((3.45b)) with $s_1(x) = (1/32\pi^2)(1/x^2)$. The remarkable feature in this gauge is that (averaged over the indices of the gluon fields) there is no contribution from $(G^a_{\mu\nu})^2$ to the massless quark propagator as can be checked from (3.45c). Higher terms in the expansion (3.47) can be found in [39] (or derived from the rules given above), but they will not play a role in the calculations described in this report. For a beautiful review of the procedure of calculating in an external field in QCD see [110].

3.3.3. *The coefficients of gluonic operators in heavy quark systems*

We apply the fixed-point gauge technique to calculate the coefficient of $G^a_{\mu\nu}G^a_{\mu\nu}$ in the case of the flavour changing vector current $j_\mu(x) = \bar{q}_1(x)\gamma_\mu q_2(x)$. Up to terms necessary for our calculation the massive propagator takes the form

$$S_G^{aa'}(p, m) = \frac{\mathrm{i}}{\not{p} - m}\,\delta^{aa'} - \frac{\mathrm{i}}{4}\,g(t^c)^{aa'}G^c_{\kappa\lambda}\,\frac{1}{(p^2 - m^2)^2}\{\sigma_{\kappa\lambda}(\not{p} + m) + (\not{p} + m)\sigma_{\kappa\lambda}\}$$
$$+ \frac{\mathrm{i}}{12}\,g^2\delta^{aa'}G^a_{\alpha\beta}G^a_{\alpha\beta}\,m\,\frac{p^2 + m\not{p}}{(p^2 - m^2)^4}\,. \tag{3.48}$$

Putting the propagator S_G into the vector polarization and picking up all terms proportional to G^2 we find

$$\Pi^{G^2}_{\mu\nu}(q, m_1, m_2) = \left[\frac{\mathrm{i}}{(2\pi)^4}\int \mathrm{d}^4p\,\mathrm{Tr}(\gamma_\mu S_G(p, m_1)\gamma_\nu S_G(p - q, m_2))\right]_{G^2}$$
$$= \frac{1}{24}\left\langle\frac{\alpha_s}{\pi}G^2\right\rangle\left\{\int \mathrm{d}x\,x(1-x)\left[\left(\frac{3}{C} + \frac{x(1-x)}{C^2}\,q^2\right)g_{\mu\nu} + \frac{2x(1-x)}{C^2}\,q_\mu q_\nu\right]\right.$$
$$+ x^2\left[\left(-\frac{3m_1m_2}{C^2} + \frac{2m_1^3m_2}{C^3} + \frac{m_1^2}{C^2}\,x\left(1 + \frac{2}{C}\,x(1-x)q^2\right)\right)g_{\mu\nu} - \frac{4m_1^2}{C^3}\,x^2(1-x)q_\mu q_\nu\right]$$
$$+ (1-x)^2\left[\left(-\frac{3m_1m_2}{C^2} + \frac{2m_1m_2^3}{C^3} + \frac{m_2^2}{C^2}(1-x)\left(1 + \frac{2}{C}\,x(1-x)q^2\right)\right)g_{\mu\nu}\right.$$
$$\left.\left.- \frac{4m_2^2}{C^3}\,x(1-x)^2q_\mu q_\nu\right]\right\}, \tag{3.49}$$

where $C = xm_1^2 + (1-x)m_2^2 - x(1-x)q^2$. The three terms in the Feynman parameter integral (3.49) correspond to the diagrams (a), (b) and (c) in fig. 4. To obtain the first term we have used the substitution

$$\langle G^a_{\nu\mu}G^b_{\sigma\rho}\rangle = \tfrac{1}{96}\delta^{ab}(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho})\langle G^2\rangle\,, \tag{3.50}$$

where $\langle G^2\rangle$ stands for $\langle 0|G^a_{\mu\nu}G^a_{\mu\nu}|0\rangle$. After the Feynman integral we get the following analytic form for

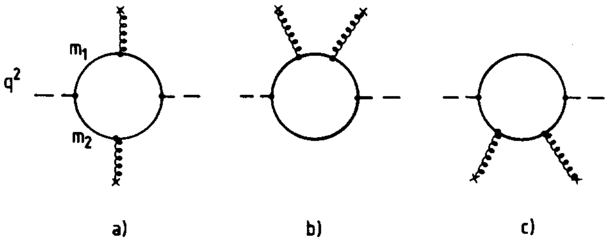


Fig. 4. Contributions to the Wilson coefficient of $G^a_{\mu\nu}G^a_{\mu\nu}$.

the coefficient function,

$$\Pi^{G^2}_{\mu\nu}(q, m_1, m_2) = \left\langle \frac{\alpha_s}{\pi} G^2 \right\rangle \left\{ (q_\mu q_\nu - q^2 g_{\mu\nu}) \left(C^{(V)}_G(u) - \frac{(m_1 - m_2)^2}{q^2} C^{(S)}_G(u) \right) + q_\mu q_\nu \frac{(m_1 - m_2)^2}{q^2} C^{(S)}_G(u) \right\} . \tag{3.51}$$

Here $C^{(V)}_G(u)$ and $C^{(S)}_G(u)$ are defined as

$$C^{(V)}_G(u) = \frac{1}{48} \frac{1}{\bar{q}^4} \left[\frac{3(1+u^2)(1-u^2)^2}{2u^5} \ln \frac{1+u}{1-u} - \frac{3u^4 - 2u^2 + 3}{u^4} \right], \tag{3.52}$$

and

$$C^{(S)}_G(u) = -\frac{1}{48} \frac{1}{\bar{q}^4} \left[\frac{3(3+u^2)(1-u^2)}{2u^3} \ln \frac{1+u}{1-u} - \frac{3u^4 + 4u^2 + 9}{u^2(1-u^2)} \right], \tag{3.53}$$

with the variables $u = (1 - 4m_1 m_2/\bar{q}^2)^{1/2}$ and $\bar{q}^2 = q^2 - (m_1 - m_2)^2$.

In the same way as for the perturbative calculation of section (3.2) the coefficients of the axial vector and the pseudoscalar current can be obtained by the replacement $m_1 \to -m_1$ ($u \to 1/u$ and $\bar{q}^2 \to \bar{q}^2 u^2$) in (3.52) and (3.53) respectively.

In the Appendix we have listed the coefficient functions of the G^2 operator for all S- and P-wave currents in the equal mass case. We will come back to the massless limit of these expressions when we discuss light quark meson systems. We note that the 1^{+-} and 2^{++} currents contain derivative couplings. The rules given above are not sufficient for these cases. No calculations using the fixed point gauge are known in the literature. The results quoted in the Appendix are from [13], where the plane wave method is used. An independent check of these results would be useful.

3.3.4. Higher dimensional operators

Recently the calculation of the coefficient functions of dimension six [40] and dimension eight [41] gluonic operators has been reported. As follows from the expansion (3.44) one can construct three different operators with dimension six

$$G^a_{\mu\nu} G^b_{\kappa\lambda} G^c_{\rho\sigma} , \tag{3.54a}$$

$$(D_\alpha G_{\mu\nu})^a (D_\beta G_{\kappa\lambda})^b , \tag{3.54b}$$

$$(D_\alpha D_\beta G_{\mu\nu})^a G^b_{\kappa\lambda} . \tag{3.54c}$$

Using the equation of motion

$$(D_\mu G_{\nu\mu})^a = g \sum \bar{\psi} \gamma_\nu t^a \psi = g j^a_\nu , \tag{3.55}$$

the Bianchi identities and commutation relations, the operators in (3.54) which contain derivatives can be expressed in (3.54a) and in $j^a_\mu j^a_\mu$ (j^a_μ defined in (3.55)). As a result there are two independent vacuum matrix elements:

$$O^6_1 = \langle 0|g^3 f^{abc} G^a_{\mu\nu} G^b_{\nu\lambda} G^c_{\lambda\mu}|0\rangle\,, \tag{3.56a}$$

$$O^6_2 = \langle 0|g^4 j^a_\mu j^a_\mu|0\rangle\,, \tag{3.56b}$$

where we have used the contraction

$$\begin{aligned}\langle 0|f^{abc} G^a_{\mu\nu} G^b_{\alpha\beta} G^c_{\rho\sigma}|0\rangle = \tfrac{1}{24}\langle 0|f^{abc} G^a_{\gamma\delta} G^b_{\delta\varepsilon} G^c_{\varepsilon\gamma}|0\rangle (g_{\mu\sigma}g_{\alpha\nu}g_{\beta\rho} + g_{\mu\beta}g_{\alpha\rho}g_{\sigma\nu} + g_{\alpha\sigma}g_{\mu\rho}g_{\nu\beta} + g_{\rho\nu}g_{\mu\alpha}g_{\beta\sigma} \\ - g_{\mu\beta}g_{\alpha\sigma}g_{\rho\nu} - g_{\mu\sigma}g_{\alpha\rho}g_{\nu\beta} - g_{\alpha\nu}g_{\mu\rho}g_{\beta\sigma} - g_{\beta\rho}g_{\mu\alpha}g_{\nu\sigma})\,.\end{aligned} \tag{3.57}$$

The number of operators grows rapidly with dimension. For $d = 8$ one finally ends up with the following set of seven independent matrix elements [41]:

$$\begin{aligned}&O^8_1 = \langle 0|g^4(d^{abc} G^a_{\mu\nu} G^b_{\alpha\beta})^2|0\rangle + \tfrac{2}{3}\langle 0|g^4(G^a_{\mu\nu} G^a_{\alpha\beta})^2|0\rangle\,, \qquad O^8_2 = \langle 0|g^4(f^{abc} G^a_{\mu\nu} G^b_{\alpha\beta})^2|0\rangle\,, \\ &O^8_3 = \langle 0|g^4(d^{abc} G^a_{\mu\alpha} G^b_{\alpha\nu})^2|0\rangle + \tfrac{2}{3}\langle 0|g^4(G^a_{\mu\alpha} G^a_{\alpha\nu})^2|0\rangle\,, \qquad O^8_4 = \langle 0|g^4(f^{abc} G^a_{\mu\alpha} G^b_{\alpha\nu})^2|0\rangle\,, \\ &O^8_5 = \langle 0|g^5 f^{abc} G^a_{\mu\nu} j^b_\mu j^c_\nu|0\rangle\,, \quad O^8_6 = \langle 0|g^3 f^{abc} G^a_{\mu\nu} G^b_{\nu\lambda}(D_\alpha D_\alpha G_{\lambda\mu})^c|0\rangle\,, \quad O^8_7 = \langle 0|g^4 j^a_\mu D_\alpha D_\alpha j^a_\mu|0\rangle\,.\end{aligned} \tag{3.58}$$

In the Appendix we have listed the coefficient functions of the operators (3.56) and (3.58) as calculated by Nikolaev and Radyushkin in the equal mass case for the scalar, pseudoscalar, vector and axial vector currents. No expressions exist as yet for the $J^{PC} = 2^{++}$ and 1^{+-} currents.

3.3.5. The coefficient of the G^2 operator in meson systems with light quarks

We will now apply the results obtained for massive quarks to systems which contain light quarks by taking the small quark mass limit. We discuss the coefficient for pure light quark systems, its connection with the operator $m\bar{q}q$ and the analytic properties of the coefficient in the massless limit, although for these systems the result can be most easily obtained in the coordinate representation starting from pure massless quarks, as we will see later.

Let us take the scalar current as an example. Starting with the current $j(x) = \bar{q}_1(x)q_2(x)$ applying the fixed-point gauge technique as in the case of the vector current we arrive at

$$C_{G_1}(q^2, m_1, m_2) = \frac{\alpha_s}{\pi}\left\{-\frac{1}{48}\frac{q^2}{\bar{q}^4}\left[\frac{3(3+u^2)(1-u^2)}{2u^3}\ln\frac{1+u}{1-u} - \frac{3(3-u^2)}{u^2}\right] + \frac{1}{12}\frac{1}{\bar{q}^2}\frac{(m_1-m_2)^2}{m_1 m_2}\right\}. \tag{3.59}$$

This form agrees with the scalar component of $\Pi^{G^2}_{\mu\nu}(q, m_1, m_2)$ ((3.51) and (3.53)) except for the constant $(\alpha_s/\pi)(1/12 m_1 m_2)$ (which of course does not play a role in the moments (2.32)). To apply this result to light quark systems by taking the small quark mass limit ($m_1, m_2 \to 0$), we have to take the contributions from the operator $m\bar{q}q$ into account. To understand the origin of this extra contribution, we have to go back to the starting point of the OPE (2.9). In order to single out the operator G^2 we sandwich with one gluon states. On the current product side (left-hand side) of the OPE, we get the matrix element

$$\mathrm{i}\int \mathrm{d}^4x\, \mathrm{e}^{\mathrm{i}qx}\langle k|\mathrm{T}(j(x)j(0))|k\rangle = C_{G_1}(q^2, m_1, m_2)\langle k|G^2|k\rangle\,, \tag{3.60}$$

with C_{G_1} given by (3.59). On the operator product side we have

$$\sum_n C_n\langle k|O_n|k\rangle = C_{m_1}(q^2)\langle k|m_1\bar{q}_1q_1|k\rangle + C_{m_2}\langle k|m_2\bar{q}_2q_2|k\rangle + C_G(q^2)\langle k|G^2|k\rangle + \cdots \tag{3.61}$$

where the coefficients C_{m_1} and C_{m_2} can be obtained from the Born diagrams (the bare loop diagram in fig. 2 with one quark line cut). Their explicit form will be given later when we study the analytic properties of these coefficients for vanishing quark masses.

Calculating the matrix element $\langle k|m\bar{q}q|k\rangle$,

$$\langle k|m\bar{q}q|k\rangle = -\frac{1}{12}\frac{\alpha_s}{\pi}\langle k|G^2|k\rangle + \cdots, \tag{3.62}$$

we see that it is of the same order in α_s as $C_G(q^2)$, since in lowest order the coefficient $C_m(q^2)$ is of zeroth order in α_s. Equating both sides of the OPE ((3.60) and (3.61)) we get the following expression for $C_G(q^2)$

$$C_G(q^2) = C_{G_1}(q^2, m_1, m_2) + C_{G_2}(q^2, m_1, m_2), \tag{3.63}$$

where C_{G_1} is given by (3.59) and

$$C_{G_2}(q^2, m_1, m_2) = \frac{\alpha_s}{12\pi}(C_{m_1}(q^2) + C_{m_2}(q^2)). \tag{3.64}$$

The physical interpretation of this mixing is the following [1]. At large q^2 two momentum regions contribute to the gluon condensate diagrams of fig. 4, $p^2 \cong q^2$ and $p^2 \cong m^2$. For small quark masses the latter region cannot be included in the coefficient C_G since some of the quark propagators go soft. This piece has to be subtracted and absorbed into the matrix element $\langle m\bar{q}q\rangle$.

Let us summarize the nonperturbative corrections from $\langle m\bar{q}q\rangle$ and $\langle G^2\rangle$ to the various systems.

(1) For pure light quarks systems the corrections can be written as

$$C_{m_1}\langle m_1\bar{q}_1q_1\rangle + C_{m_2}\langle m_2\bar{q}_2q_2\rangle + C_G\langle G^2\rangle. \tag{3.65}$$

(2) For systems which contain one light and one heavy quark ($m_2 \gg m_1$) we have

$$C_{m_1}\langle m_1\bar{q}_1q_1\rangle + \left(C_{G_1} + \frac{1}{12}\frac{\alpha_s}{\pi}C_{m_1}\right)\langle G^2\rangle + C_{m_2}\left\{\langle m_2\bar{q}_2q_2\rangle + \frac{1}{12}\frac{\alpha_s}{\pi}\langle G^2\rangle\right\}. \tag{3.66}$$

The last term in this expression vanishes when we apply the heavy quark mass expansion for the heavy quark condensate

$$\langle\bar{h}h\rangle = -\frac{1}{m_h}\frac{\alpha_s}{12\pi}\langle G^2\rangle + \cdots. \tag{3.67}$$

Further terms in the expansion (3.67) can be found in [42].

(3) For pure heavy quark systems the relevant expression is

$$C_{G_1}\langle G^2\rangle + C_{m_1}\left\{\langle m_1\bar{q}_1q_1\rangle + \frac{\alpha_s}{12\pi}\langle G^2\rangle\right\} + C_{m_2}\left\{\langle m_2\bar{q}_2q_2\rangle + \frac{\alpha_s}{12\pi}\langle G^2\rangle\right\} = C_{G_1}\langle G^2\rangle\,, \tag{3.68}$$

where again (3.67) has been used. This result is identical with the one calculated without taking account of $m\bar{q}q$ terms. Therefore, the heavy quark condensate has practically no effect on the polarization functions.

In order to study the small quark mass limit of the G^2 coefficient for pure light quark systems (both quarks massless) and for light–heavy quark systems (one quark massless), we have to derive the explicit form of C_{m_1} and C_{m_2}. C_{m_1} of $m_1\bar{q}q$ can be obtained by calculating the matrix element of the product of currents between one quark states $q_1(p)$ with $p^2 = m_1^2$. For the scalar current $j(x) = \bar{q}_1(x)q_2(x)$,

$$\mathrm{i}\int \mathrm{d}^4x\, \mathrm{e}^{\mathrm{i}qx}\langle p|\mathrm{T}(j(x)j^+(0))|p\rangle = -\frac{1}{m_1^2}\frac{p\cdot q + m_1^2 + m_1m_2}{(q+p)^2 - m_2^2}\, m_1\bar{u}(p)u(p)\,. \tag{3.69}$$

After averaging over the four-dimensional Euclidean angle, the coefficient $C_{m_1}(q^2)$ takes the form

$$C_{m_1}(q^2) = \frac{1}{2m_1^2q^2}\left[\frac{\bar{q}^4}{4m_1^2}u^2(1-u)^4 - (m_1+m_2)^2\right], \tag{3.70}$$

with u and $\bar{q}^2$ as before. The coefficient $C_{m_2}(q^2)$ can be obtained by interchanging m_1 and m_2.

We are now ready to study the analytic properties of C_G of (3.63). In the limit of vanishing quark masses C_{G_1} of (3.59) becomes

$$C_{G_1} \cong \frac{\alpha_s}{12\pi}\left(-\frac{1}{\bar{q}^2}\frac{(m_1-m_2)^2}{m_1m_2} + \frac{3}{2q^2}\right), \tag{3.71}$$

which is not analytic in m_1 and m_2. However, C_{G_2} of (3.64) contains a piece that precisely cancels the singularity

$$C_{G_2} \cong \frac{\alpha_s}{12\pi}\left(\frac{1}{\bar{q}^2}\frac{(m_1-m_2)^2}{m_1m_2} - \frac{3}{q^2}\right). \tag{3.72}$$

Therefore, the coefficient $C_G(q^2)$ is free from singularities at $m_1 = m_2 = 0$ and is equal to

$$\lim_{m_1,\, m_2\to 0} C_G = -\frac{\alpha_s}{8\pi}\frac{1}{q^2}\,. \tag{3.73}$$

This result can be obtained directly (and very easily) in the coordinate representation in the fixed-point gauge starting with massless quarks. In this case we do not have to take care of the operator $m\bar{q}q$ since it is absent in the OPE.

For light-heavy quark systems, the G^2 coefficient (the second term in (3.66)) is

$$C_{G_1}+\frac{\alpha_s}{12\pi}C_{m_1} \tag{3.74}$$

In the limit $m_1\to 0$ with m_2 finite C_{G_1} behaves as

$$C_{G_1}\cong\frac{\alpha_s}{12\pi}\left[\frac{m_2}{m_1}\frac{1}{(q^2-m_2^2)}-\frac{1}{2(q^2-m_2^2)}-\frac{m_2^2}{2(q^2-m_2^2)^2}\right], \tag{3.75}$$

while in the same limit the second term behaves as

$$\frac{\alpha_s}{12\pi}C_{m_1}\cong\frac{\alpha_s}{12\pi}\left[-\frac{m_2}{m_1}\frac{1}{(q^2-m_2^2)}-\frac{1}{2(q^2-m_2^2)}+\frac{m_2^2}{2(q^2-m_2^2)^2}\right]. \tag{3.76}$$

Therefore, the coefficient of the G^2 operator for the product of two scalar currents with $m_1=0$, $m_2\neq 0$ is given by

$$C_G=-\frac{\alpha_s}{12\pi}\frac{1}{q^2-m_2^2}. \tag{3.77}$$

Finally, we close this section with the following remark. For purely massless two-quark systems Dubovikov and Smilga [24] have proven that the coefficient of the G^3 operator is equal to zero. This has been confirmed by Hubschmid and Mallik [39] by explicit calculation for all quark bilinears.

3.3.6. *The coefficients of operators with quark fields in meson currents*

The operator $\bar{q}q$

The calculation of the coefficient of $m\bar{q}q$ has already been presented in the previous subsection. Here, we will discuss $m\bar{q}q$ in the context of chiral symmetry breaking. Indeed, in the QCD sum rules the role of $m\bar{q}q$ is rather special for the pseudoscalar current. To understand its role, let us consider the correlation function of the axial vector current $A_\mu(x)=\bar{u}(x)\gamma_\mu\gamma_5 d(x)$ and the pseudoscalar current $P(x)=\bar{d}(x)\mathrm{i}\gamma_5 u(x)$. The coefficient of $\bar{u}(0)u(0)$ calculated from the Born diagram (the bare loop diagram of fig. 2 with one quark line cut, in one vertex $\Gamma=\gamma_\mu\gamma_5$ and the other $\Gamma=\gamma_5$):

$$\begin{aligned}\mathrm{i}\int \mathrm{d}^4x\,\mathrm{e}^{\mathrm{i}qx}\langle p|\mathrm{T}(A_\mu(x)\bar{P}(0))|p\rangle &= -\bar{u}(p)\gamma_\mu\gamma_5\frac{1}{\not p+\not q-m}\mathrm{i}\gamma_5 u(p)\\ &=\mathrm{i}\frac{q_\mu}{q^2}\langle p|\bar{u}(0)u(0)|p\rangle\,. \end{aligned}\tag{3.78}$$

Here we have used the equation of motion and neglected higher orders in the quark mass. Therefore in the vacuum to vacuum correlation function including also the perturbative part we get

$$-\mathrm{i}q_\mu(m_\mathrm{u}+m_\mathrm{d})\frac{3}{8\pi^2}\ln\frac{\Lambda^2}{-q^2}+\mathrm{i}\frac{q_\mu}{q^2}(\langle\bar{u}(0)u(0)\rangle+\langle\bar{d}(0)d(0)\rangle)=\mathrm{i}\int \mathrm{d}^4x\,\mathrm{e}^{\mathrm{i}qx}\langle 0|\mathrm{T}(A_\mu(x)\bar{P}(0))|0\rangle\,. \tag{3.79}$$

One sees that in the chiral limit ($m_\mathrm{q}\to 0$) the $\bar{q}q$ condensate dominates, and balances in the sum rule with the pion pole term

$$-\mathrm{i}q_\mu\frac{f_\pi^2 m_\pi^2}{m_\mathrm{u}+m_\mathrm{d}}\frac{1}{q^2-m_\pi^2}, \tag{3.80}$$

which results in the well-known PCAC relation.

In chapter 5 we will exploit this relationship between $\langle\bar{q}q\rangle$ in the pseudoscalar current and the pion pole (Goldstone boson) to calculate couplings involving Goldstone bosons, e.g. $g_{\pi\mathrm{NN}}$, $g_{\omega\rho\pi}$.

The operators $m\bar{q}\sigma^{\mu\nu}t^a qG^a_{\mu\nu}$ *and* $(\bar{q}\gamma^\mu\lambda^a q)D^\nu G^a_{\mu\nu}$

For convenience we will consider the pseudoscalar polarization function $P(x)=\bar{u}(x)\mathrm{i}\gamma_5 d(x)$. To single out the operators of interest a state with one u quark and one gluon is chosen as the initial state and with one u quark as the final one. The current product side gives (the tree diagram of fig. 5a)

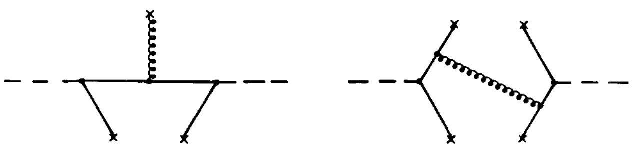
$$\begin{aligned}\mathrm{i}\int \mathrm{d}^4x\,\mathrm{e}^{\mathrm{i}qx}\langle p'|\mathrm{T}(P(x)\bar{P}(0))|p,k\rangle &= \bar{u}(p')\mathrm{i}\gamma_5\frac{\not{p}'+\not{q}+m}{(p'+q)^2-m^2}g\gamma^\mu t^a\frac{\not{p}+\not{q}+m}{(p+q)^2-m^2}\mathrm{i}\gamma_5 u(p)A^a_\mu(k)\\ &= -\frac{1}{2q^2}g\bar{u}(p')\gamma^\mu t^a u(p)A^a_\mu(k)+\frac{1}{2q^4}mg\bar{u}(p')[\gamma^\mu,\not{K}]t^a u(p)A^a_\mu(k)\\ &\quad+\frac{1}{3q^4}g\bar{u}(p')\gamma_\nu t^a u(p)(k^2g^{\nu\mu}-k^\nu k^\mu)A^a_\mu(k)\,,\end{aligned} \tag{3.81}$$

where $A^a_\mu(k)=\langle 0|A^a_\mu(0)|k\rangle$. To obtain this result we have used $(\not{p}-m)u(p)=0$, expanded the propagators in $2p\cdot q$ and $2p'\cdot q$ and projected on the Lorentz scalar by replacing

$$q_\alpha q_\beta\to\tfrac{1}{4}g_{\alpha\beta}q^2 \tag{3.82a}$$

and

$$q_\alpha q_\beta q_\gamma q_\delta\to\tfrac{1}{24}(g_{\alpha\beta}g_{\gamma\delta}+g_{\alpha\gamma}g_{\beta\delta}+g_{\alpha\delta}g_{\beta\gamma})q^4\,. \tag{3.82b}$$



a) b)

Fig. 5. Contributions to $m\bar{q}\sigma_{\mu\nu}t^a qG^a_{\mu\nu}$ (a) and to the four-fermion operator (b).

The first term in (3.81) gives the coefficient of the two-quark operator $\bar{u}(0)\,\mathrm{i}\not{D}u(0) = \bar{u}(0)(\mathrm{i}\not{\partial} + gt^a\not{A}^a)u(0)$. This operator does not appear in (2.10) since it reduces to $m\bar{u}(0)u(0)$ by using the equation of motion $\mathrm{i}\not{D}\psi = m\psi$. From the second and the third term we get the following contribution to the polarization operator involving the two operators in question:

$$-\frac{1}{2q^4}\langle g(m_u\bar{u}(0)\sigma^{\mu\nu}t^a u(0) + m_d\bar{d}(0)\sigma^{\mu\nu}t^a d(0))G^a_{\mu\nu}\rangle + \frac{1}{3q^4}\langle g(\bar{u}(0)\gamma_\mu t^a u(0) + \bar{d}(0)\gamma^\mu t^a d(0))D^\nu G^a_{\mu\nu}\rangle \tag{3.83}$$

where $\sigma^{\mu\nu} = \frac{1}{2}\mathrm{i}[\gamma^\mu, \gamma^\nu]$. Using the equation of motion $D^\nu G^a_{\mu\nu} = g\sum\bar{q}\gamma_\mu t^a q$ the vacuum expectation value of the second operator (3.83) can be replaced by the four-quark operator $g^2\langle\bar{u}\gamma^\mu t^a u\sum\bar{q}\gamma_\mu t^a q\rangle$.

Four-quark operators

To obtain the coefficients of four-quark operators we take the matrix element between two-quark states. For the example of the pseudoscalar current we choose the two quark states $u(p_1)d(p_2)$ and $d(p_{1'})u(p_{2'})$ which amounts to calculating diagrams with gluon exchange, like the one in fig. 5b:

$$\begin{aligned}\mathrm{i}\int \mathrm{d}^4x\, \mathrm{e}^{\mathrm{i}qx}\langle p_{1'}p_{2'}|\mathrm{T}(P(x)\bar{P}(0))|p_1p_2\rangle &= \frac{g^2}{q^6}\bar{d}(p_{1'})[\gamma^\mu, \not{q}]\gamma_5 t^a u(p_1)\bar{u}(p_{2'})[\gamma_\mu, \not{q}]\gamma_5 t^a d(p_2)\\ &= -\frac{g^2}{q^4}\langle p_{1'}p_{2'}|\bar{d}(0)\sigma^{\mu\nu}\gamma_5 t^a u(0)\bar{u}(0)\sigma_{\mu\nu}\gamma_5 t^a d(0)|p_1p_2\rangle\,.\end{aligned} \tag{3.84}$$

The vacuum expectation value of the resulting four-quark operators can be estimated in terms of $\langle\bar{q}q\rangle$ by the vacuum saturation hypothesis, to be discussed in section (4.3).

In this last section we have calculated examples of all operators which have so far been taken into account in mesonic correlation functions. Higher dimensional operators can be calculated in the same fashion by sandwiching with the appropriate states.

In the Appendix we list the full expressions of all polarization functions of light quark mesonic currents, including mass corrections for currents with strange quarks.

3.3.7. *The coefficient functions in massless quark systems*

In the pure massless case it is most convenient to work in the x-representation, especially for the G^2 coefficient and for gluonic coefficients in general. At the beginning of section (3.3) we have given the expression for the quark propagator ((3.47)) in external quark and gluon background fields. We will use this expression to calculate the G^2 coefficient of the massless scalar meson current and the nucleon current polarization functions as examples. The rules for calculating the various coefficients are:

(1) to substitute the quark propagator of (3.47) into all the quark lines in the Feynman diagrams,

(2) to expand the polarization function in products of the external fields and finally,

(3) to take the vacuum average of the products of the external fields such that these products have the same quantum numbers as the vacuum. For instance, the product $\chi^a_\alpha\bar{\chi}^b_\beta$ should be replaced by

$$\begin{aligned}\langle\chi^a_\alpha(x)\bar{\chi}^b_\beta(0)\rangle = -\tfrac{1}{12}\delta^{ab}\Big(\delta_{\alpha\beta} - \frac{\mathrm{i}m}{4}(x^\mu\gamma_\mu)_{\alpha\beta}\Big)\langle\bar{q}q\rangle - \frac{\delta^{ab}}{3\times 2^6}\Big(\delta_{\alpha\beta} - \frac{\mathrm{i}m}{6}(x^\mu\gamma_\mu)_{\alpha\beta}\Big)\\ \times\, x^2\Big\langle\bar{q}\sigma_{\mu\nu}\frac{\lambda^n}{2}gG^n_{\mu\nu}q\Big\rangle + \cdots\,,\end{aligned} \tag{3.85}$$

with m the quark mass. Some higher dimensional terms in (3.85) can be found in [43]. For the product $G^a_{\mu\nu}G^b_{\rho\sigma}$ one should use (3.50), etc.

If we want to calculate mass corrections and/or the contribution of $m\bar{q}q$ to the polarization operator we have to include the first-order mass correction to the free propagator

$$S_{\mathrm{F}}(x) = \frac{\mathrm{i}}{2\pi^2}\frac{x^\mu\gamma_\mu}{x^4} - \frac{m}{4\pi^2}\frac{1}{x^2} + \cdots . \tag{3.86}$$

Let us now apply our rules to the calculation of the scalar current polarization function. In lowest order the polarization function can be expanded as

$$\begin{aligned} \Pi(x;\chi,\bar{\chi},G) &= \langle \mathrm{T}(j(x)j(0))\rangle = -\mathrm{Tr}(S_{\mathrm{F}}(x;\chi,\bar{\chi},G)S_{\mathrm{F}}(-x;\chi,\bar{\chi},G)) \\ &= -\frac{3}{4\pi^2}\frac{1}{x^8}\mathrm{Tr}(x^\mu\gamma_\mu x^\nu\gamma_\nu) - \frac{1}{2\pi^2}\frac{m}{x^2}\langle\bar{q}q\rangle + s_0(x)\delta^{ab}\,\mathrm{Tr}(\chi^a(x)\bar{\chi}^b(0)x^\mu\gamma_\mu) \\ &\quad - s_0(x)\delta_{ab}\,\mathrm{Tr}(x^\mu\gamma_\mu\chi^a(-x)\bar{\chi}^b(0)) \\ &\quad + g^2\,\mathrm{Tr}(t^m t^n)G^m_{\kappa\lambda}(0)G^n_{\rho\psi}(0)\,\mathrm{Tr}([x^\mu\gamma_\mu\sigma^{\kappa\lambda} + \sigma^{\kappa\lambda}x^\nu\gamma_\nu][x^\mu\gamma_\mu\sigma^{\rho\psi} + \sigma^{\rho\psi}x^\nu\gamma_\nu]) . \end{aligned} \tag{3.87}$$

After the vacuum average we have for $\Pi(x;\chi,\bar{\chi},G)$ the compact form

$$\Pi(x;\chi,\bar{\chi},G) = -\frac{3}{\pi^4}\frac{1}{x^6} - \frac{3}{4\pi^2}\frac{m}{x^2}\langle\bar{q}q\rangle - \frac{1}{32\pi^2}\frac{1}{x^2}\left\langle\frac{\alpha_s}{\pi}G^2\right\rangle + \cdots . \tag{3.88}$$

One can easily verify that the coefficient of G^2 agrees with the small quark mass limit of the massive coefficient $C_G = C_{G_1} + C_{G_2}$.

In order to get the expression in the momentum representation one can use the formula [44]

$$\int \frac{\mathrm{d}^4x}{(x^2)^n}\,\mathrm{e}^{\mathrm{i}qx} = \frac{\mathrm{i}(-1)^n 2^{4-2n}\pi^2}{\Gamma(n-1)\Gamma(n)}(q^2)^{n-2}\ln(-q^2) , \qquad n \geq 2 , \tag{3.89a}$$

where polynomials in q^2 with divergent coefficients have been omitted. For $n = 1$ we have

$$\int \frac{\mathrm{d}^4x}{x^2}\,\mathrm{e}^{\mathrm{i}qx} = -\mathrm{i}\frac{4\pi^2}{q^2} . \tag{3.89b}$$

3.3.8. *The coefficients in baryonic currents*

Here we take the nucleon (proton) current as an example

$$\eta_{\mathrm{N}}(x) = \varepsilon_{abc}(u^a(x)^T C\gamma_\mu u^b(x))\gamma_\mu\gamma_5 d^c(x) . \tag{3.90}$$

In lowest order the polarization function can be written as

$$\Pi_{\mathrm{N}}(x;\chi,\bar{\chi},G) = -2\varepsilon_{abc}\varepsilon_{a'b'c'}\gamma_\mu\gamma_5 S_{\mathrm{F}}(x;\chi,\bar{\chi},G)^{cc'}\gamma_\nu\gamma_5\,\mathrm{Tr}(S_{\mathrm{F}}(x;\chi,\bar{\chi},G)^{aa'T}C\gamma_\mu S_{\mathrm{F}}(x;\chi,\bar{\chi},G)^{bb'}\gamma_\nu C) . \tag{3.91}$$

Applying the rules given above we get the result

$$\Pi_N(x;\chi,\bar{\chi},G)=\mathrm{i}\frac{24}{\pi^6}\frac{x^\mu\gamma_\mu}{x^{10}}+\frac{\mathrm{i}}{8\pi^4}\frac{x^\mu\gamma_\mu}{x^6}\left\langle\frac{\alpha_s}{\pi}G^2\right\rangle-\frac{\mathrm{i}}{3\pi^2}\frac{x^\mu\gamma_\mu}{x^4}\langle\bar{q}q\rangle^2-\frac{2}{\pi^4}\frac{1}{x^6}\langle\bar{q}q\rangle+\cdots. \tag{3.92}$$

This can be easily transformed into the momentum space representation by means of the formulas (3.89). The important difference between mesonic and baryonic currents is that the chiral symmetry breaking term $\langle\bar{q}q\rangle$ appears to set the scale of the baryonic polarization, whereas in the mesonic case its effect is softened by the associated quark mass in $\langle m\bar{q}q\rangle$.

In the Appendix we have given the complete expressions (including mass corrections) for the polarization functions of all $L=0$ octet and decuplet baryons.

We note that recently two calculations of the first-order α_s correction to the bare loop diagram and to the $\bar{q}q$ condensate for baryonic currents have been reported [45, 46]. The two calculations do not agree, but in both cases the corrections are rather large. Finally we note that for three-point functions the calculations can be performed in a similar way. We will discuss the relevant details in chapter 5.

4. Phenomenology of two-point functions

4.1. Introduction

The machinery described in the previous chapters can now be applied to physical systems. In this chapter we will systematically study meson and baryon systems. For each partial wave we choose a composite operator (current) with the proper quantum numbers (like the ones listed in (2.2), and (2.6) and (2.7)) and study the polarization function by one of the methods described in section 2.5.

In general, the two-point functions will have a well-defined flavour. Flavour mixing is suppressed; it occurs as a higher order correction in α_s. As a consequence $I=0$ and $I=1$ states are degenerate, in agreement with experiment ($S^*-\delta$, A_2-f, etc.); the U(1) problem in the pseudoscalar channel must be dealt with separately. We first consider mesons. Our analysis covers the following mesonic systems:

(1) heavy quark systems ($c\bar{c}$, $b\bar{b}$),
(2) light quark mesons with $L=0$ and $L=1$ without strange quarks (e.g. ρ, A_1),
(3) light quark mesons with strange quarks (ϕ, K^*),
(4) systems which contain light and heavy quarks (D, B),
(5) systems with gluons or $q\bar{q}$ pairs.

We then discuss baryon systems:

(1) the $L=0$ octet and decuplet baryons of light quarks,
(2) heavy baryons and baryons with gluons.

For all these systems sum rules are constructed to calculate the ground state and its coupling to the current whenever it has a physical meaning. Three-point functions are left for chapter 5. The physical assumptions originally proposed by SVZ [1] are simple. Based on the dual stringlike nature of hadrons one computes the polarization function in a region where asymptotic freedom holds. This representation is then analytically continued to larger distances by a moment or Borel transform method. The nonperturbative vacuum fluctuations force the quarks to bind. At this stage the expression is matched with a phenomenological form containing the resonances and a parametrization of the continuum states above threshold. Resonance parameters can then be extracted as a function of a few universal QCD parameters and the quark masses.

It is the dual string nature of the theory that provides for the fast convergence of the dispersion integral and the saturation by the lowest lying resonance. For light quark systems the location of the higher resonances can also be found from the slope of the linearly rising trajectories. We will assume that the spacing is given by $1/\alpha'^2$, where α' is the slope of the Regge trajectory. For heavy quarks the situation is more complicated (Coulombic gluon exchanges are not negligible), but fortunately there is only a weak dependence on the threshold position s_0.

The parameters that characterize the QCD side of the sum rule are the quark masses, α_s or Λ_{QCD}, and the condensates. The latter are the vacuum expectation values of the higher dimensional operators in the operator product expansion. They are gauge invariant, Lorentz invariant, and colour singlets, and the expectation values are universal parameters, i.e., independent of the quark or gluon system in question. It is reassuring that the number of parameters is rather small, since the operator product expansion series can be cut off at a low dimension. We will for instance see that under certain conditions and assumptions the contribution to heavy quark systems from the dimension six and eight operators in section (3.3d) can be neglected. All baryon and meson masses and their couplings depend therefore on a handful of parameters, and before analyzing the sum rules we discuss the value or at least the range of values of these parameters.

4.2. The operator matrix elements and the QCD parameters

1. *The gluon condensate* $\langle 0|(\beta(g)/g)G^a_{\mu\nu}G^a_{\mu\nu}|0\rangle$

This quantity drives resonance formation in heavy quark systems and is related to the trace of the energy momentum tensor [47]

$$\theta_{\mu\mu} = \frac{\beta(g)}{2g} G^a_{\mu\nu}G^a_{\mu\nu} + \sum_q m\bar{q}q\,. \tag{4.1}$$

Here $\beta(g)$ is the Callan Symanzik β function. Since $\theta_{\mu\mu}$ is renormalization group invariant, the same is true for $(\beta(g)/g)G^a_{\mu\nu}G^a_{\mu\nu}$ or to first order in α_s for $\alpha_s G^a_{\mu\nu}G^a_{\mu\nu}$. A vacuum expectation value of this quantity reflects the breakdown of conformal invariance. However, the absence of a quasi-conserved quantum number or an identifiable physical quantity related to this vacuum expectation value [22] creates a problem concerning its definition. There is neither a global symmetry (like chiral symmetry for the quark condensate $\bar{q}q$ or CP symmetry for the topological charge density related to $G^a_{\mu\nu}\tilde{G}^a_{\mu\nu}$ to make its definition unambiguous. As we will see below, the price to pay is that one must define a perturbative as well as a physical vacuum. We will discuss three ways for determining the value of the gluon condensate:

(a) as a parameter to be fitted in the various systems of heavy and light quarks, in particular in charmonium,

(b) by some theoretical model of the vacuum like the dilute instanton gas,

(c) by lattice simulations.

The first method assumes that the perturbation theory contributions are given by ordinary series of Feynman diagrams and that these are not affected by the renormalized vacuum matrix elements of the true theory, in other words that a strict separation of large and short distances is possible. The matrix elements are then treated as phenomenological parameters and fitted to reproduce the resonance spectrum. As we will see in the next section the charmonium spectrum is optimally suited for determining the universal value of the gluon condensate:

$$\langle 0|\frac{\alpha_s}{\pi} G^a_{\mu\nu} G^a_{\mu\nu}|0\rangle = (360 \pm 20\ \mathrm{MeV})^4 . \tag{4.2}$$

The second method attempts to calculate the matrix element from first principles. As a first approximation one can calculate its value induced by a classical instanton field, in the dilute gas approximation. One obtains

$$\langle 0|\frac{\alpha_s}{\pi} G^a_{\mu\nu} G^a_{\mu\nu}|0\rangle = 16 \int_0^{\rho_c} \frac{\mathrm{d}\rho}{\rho^5}\, d(\rho) , \tag{4.3}$$

where $d(\rho)$ is the instanton density. Unfortunately the large size instantons make the integration diverge, and one has to introduce a cutt-off ρ_c. The value of ρ_c is a key parameter for all instanton calculations and should be very well known to give a reliable estimate for matrix elements like (4.3).

The use of the free instanton gas approximation has been criticized in [48], where a more sophisticated model for the vacuum is proposed. Using (4.2) one can estimate the critical parameter ρ_c and calculate higher dimensional gluonic operators. The dilute gas approximation and Shuryak's model [48] differ by about a factor of two in the value of ρ_c, which leads to very large differences for the matrix elements of higher dimensional operators. We will come back to this later in this section when we discuss the matrix elements of $d = 6$ and $d = 8$ gluonic operators.

Finally we discuss the lattice calculations of the gluon condensate. The vacuum expectation value of $G^a_{\mu\nu} G^a_{\mu\nu}$ is infinite as a vacuum fluctuation as well as in perturbation theory. SVZ [1] suggest that the appropriate definition of the operator is to normal order it in the perturbative vacuum, and renormalize the perturbative series in the usual way. No further additive renormalizations are then needed. If present, they would spoil the relative weight of the different operators in the OPE and undermine the assumption that the Wilson coefficients both for the perturbative and nonperturbative operators can be calculated by series of Feynman diagrams. There are several examples (see chapter 2) which show that this prescription works, and so far no contradiction with this conjecture has been found. On the lattice the procedure goes as follows. Using the standard definition of the plaquette element with the Wilson action one defines [49, 50]

$$\phi = \left\langle \frac{\beta_{\mathrm{B}}(g_0)}{g_0} [\langle G^2_{\mathrm{B}}\rangle - \langle G^2_{\mathrm{B}}\rangle_{\mathrm{P}}] \right\rangle . \tag{4.4}$$

The subscript B means that bare quantities are considered. $\langle G^2_{\mathrm{B}}\rangle$ and $\langle G^2_{\mathrm{B}}\rangle_{\mathrm{P}}$ are the vacuum expectation values of $G^a_{\mu\nu} G^a_{\mu\nu}$ in the true vacuum and in the perturbative vacuum respectively. Nonperturbative contributions to ϕ are then supposed to be finite.

For the gauge group SU(N) (without any quarks) the connection between the Wilson plaquette energy W and G^2_{B} is given by

$$W = 1 - \frac{a^4}{48N} g_0^2 \langle G^2_{\mathrm{B}}\rangle + \mathrm{O}(a^5) \tag{4.5}$$

where N is the dimension of the gauge group, g_0 is the bare coupling constant, and a the lattice spacing. Combining (4.4) and (4.5) we get in the continuum limit

$$\phi = \lim_{a,\, g_0 \to 0} \frac{\beta_{\mathrm{B}}(g_0)}{g_0^3} \frac{48N}{a^4} [W_{\mathrm{P}} - W] . \tag{4.6}$$

The SVZ conjecture implies that ϕ is finite in the continuum limit. W_{P} is the perturbative part of W, which can be evaluated as a series in α_s (or $\beta \propto 1/g^2$). The terms in this expansion can be calculated by weak coupling perturbation theory on a finite lattice or determined from fits to Monte Carlo data at large β. To separate the perturbative and nonperturbative effects one has to go to small β, but not too small in order to have a reliable approximation in the continuum limit. At these intermediate values of β the value for ϕ can be found by comparing the Monte Carlo data with the behaviour for W predicted by the renormalization group

$$W = \sum_{n=0}^{\infty} \frac{d_n}{\beta^n} - A\beta^{2b_1/b_0^2} \mathrm{e}^{-\beta/2b_0} . \tag{4.7}$$

The sum on the right-hand side of (4.7) is W_{P}. The first few terms in the series are known, and the fit to the Monte Carlo data should be insensitive to the addition of further terms. In (4.7) b_0 and b_1 are the first two coefficients in the expansion of the Callan–Symanzik β function

$$b_0 = \tfrac{11}{3} N/16\pi^2 \quad \text{and} \quad b_1 = \tfrac{34}{3}(N/16\pi^2)^2 . \tag{4.8}$$

Combining the formulae given above, one finds

$$\langle 0| \frac{\alpha_s}{\pi} G^a_{\mu\nu} G^a_{\mu\nu} |0\rangle = -\frac{\phi}{4\pi^2 b_0} = A\beta^{2b_1/b_0^2} \mathrm{e}^{-\beta/2b_0} \frac{12N}{\pi^2 a^4} . \tag{4.9}$$

In the fit to the Monte Carlo data an exponential with the right slope (see (4.7)) is found after subtraction of the perturbative terms, and the gluon condensate can be measured. The resulting value for SU(3) agrees remarkably well with the phenomenological value (4.2), but given the uncertainties of Monte Carlo calculations these results have to be taken with some care. For more details see [49, 50].

2. *Higher dimensional gluonic operators*

The higher the dimension of the operator, the less known the value of its matrix element. For light quark $\bar{q}q$ mesons the Wilson coefficient of the six-dimensional operator $\langle 0|g^3 f_{abc} G^a_{\mu\nu} G^b_{\nu\rho} G^c_{\rho\mu}|0\rangle$ vanishes for all quark bilinears [24, 39], and this operator does not have to be considered. For heavy quark systems the contributions are suppressed by extra mass factors for dimensional reasons, but it is important to have an order of magnitude estimate for these matrix elements in order to check the convergence of the OPE series.

Lattice determinations of $\langle gfG^3\rangle$ have only been made for the gauge group SU(2). They yield [51]

$$\langle g^3 fGGG\rangle \cong +26 \left\langle \frac{\alpha_s}{\pi} G^2 \right\rangle^{3/2} . \tag{4.10}$$

In a similar way as $\langle(\alpha_s/\pi)G^2\rangle$ this quantity can also be estimated in instanton models. The dilute instanton gas approximation gives [1]

$$\langle g^3 fGGG\rangle = \frac{3\times 2^8\pi^2}{5}\int_0^{\rho_c}\frac{d\rho}{\rho^7}\,d(\rho) \cong \frac{48\pi^2}{5}\langle\rho_c^{-2}\rangle\left\langle\frac{\alpha_s}{\pi}G^2\right\rangle . \tag{4.11}$$

More sophisticated instanton models [52] give an order of magnitude larger than (4.10).

For dimension $d=8$ the seven independent matrix elements have been listed in (3.58). A rough approximation for the first five can be obtained using the hypothesis of vacuum saturation [1]. Although there are arguments based on the large N limit [53] that this approximation is less reliable for gluonic operators than for quark operators (see below), it should be a good order of magnitude estimate.

$$O_1^8 = \tfrac{65}{9}\pi^4\left(\left\langle\frac{\alpha_s}{\pi}G^2\right\rangle\right)^2 , \qquad O_2^8 = 5\pi^4\left(\left\langle\frac{\alpha_s}{\pi}G^2\right\rangle\right)^2 , \qquad O_3^8 = \tfrac{38}{9}\pi^4\left(\left\langle\frac{\alpha_s}{\pi}G^2\right\rangle\right)^2 ,$$

$$O_4^8 = \pi^4\left(\left\langle\frac{\alpha_s}{\pi}G^2\right\rangle\right)^2 , \qquad O_5^8 = -\tfrac{3}{4}g^4\langle\bar q q\rangle\left\langle g\bar q\sigma_{\mu\nu}\frac{\lambda^a}{2}G^a_{\mu\nu}q\right\rangle . \tag{4.12a}$$

The last two operators in (3.58) which contain derivatives can be reduced to other operators assuming scaling behaviour

$$O_6^8 = M^2\langle g^3 fGGG\rangle , \qquad O_7^8 = M^2\langle g^4 j^a_\mu j^a_\mu\rangle , \tag{4.12b}$$

where M^2 is a parameter which characterizes the average virtual momentum of the vacuum gluons and quarks. In [40] M^2 has been estimated to be $0.3\,\mathrm{GeV}^2$. For a recent and different estimate of the matrix elements O_1^8–O_5^8 see [54]. It involves an extension of the heavy quark mass expansion to higher dimensional operators, and leads to values of the same order of magnitude.

3. *Matrix elements of operators involving quarks*

(a) $\langle 0|g\bar q\sigma_{\mu\nu}(\lambda^a/2)G^a_{\mu\nu}q|0\rangle$

This matrix element which appears in O_5^8 in (4.12a) can be expressed in terms of $\langle\bar q q\rangle$ by the scaling hypothesis used for O_6^8 and O_7^8 in (4.12b) [40]. Using the fact that $G_{\mu\nu}\propto[D_\mu, D_\nu]$ and the equation of motion for massless quarks one finds

$$\langle 0|g\bar q\sigma_{\mu\nu}\frac{\lambda^a}{2}G^a_{\mu\nu}q|0\rangle = 2\langle 0|\bar q D_\mu D^\mu q|0\rangle = 2M^2\langle 0|\bar q q|0\rangle = m_0^2\langle 0|\bar q q|0\rangle . \tag{4.13}$$

Here, M^2 is the average virtual momentum of vacuum quarks which should be approximately equal to the average virtual momentum of gluons, $M^2\cong 0.3\,\mathrm{GeV}^2$ which roughly agrees with estimates from sum rules for baryons [43, 55] and from open charm states [56].

(b) *The quark condensates and light quark masses*

The light-quark condensate multiplied by the quark mass is a renormalization group invariant quantity. Its value is fixed by PCAC ($f_\pi = 133\,\mathrm{MeV}$):

$$(m_u + m_d)\langle 0|\bar{u}u + \bar{d}d|0\rangle = -m_\pi^2 f_\pi^2\,, \tag{4.14}$$

and a similar relation for $f_K^2 m_K^2$ involving $\langle m_s \bar{s}s\rangle$. Since m_u and m_d are not very well known the values of $\langle \bar{u}u\rangle$ and $\langle \bar{d}d\rangle$ are difficult to establish. From [3] we take (at a scale of 1 GeV)

$$\langle 0|\bar{u}u|0\rangle = \langle 0|\bar{d}d|0\rangle \equiv \langle 0|\bar{q}q|0\rangle = -(225 \pm 25\,\mathrm{MeV})^3\,, \tag{4.15}$$

and from (4.14) with $m = \frac{1}{2}(m_u + m_d)$

$$\langle 0|m\bar{q}q|0\rangle = -(100\,\mathrm{MeV})^4\,. \tag{4.16}$$

In QCD sum rule applications the masses of the u and the d quark are mostly set equal to zero. The difference between the u and d condensates can be calculated from chiral perturbation theory.

The parameters of the strange quark can actually be determined from the QCD sum rules for mesons containing strange quarks, in particular K* and ϕ (see section (4.4) and [57]):

$$\begin{aligned} &m_s(1\,\mathrm{GeV}) = 110 \pm 10\,\mathrm{MeV}\,, \\ &\langle 0|m_s\bar{s}s|0\rangle = -(210 \pm 5\,\mathrm{MeV})^4\,, \end{aligned} \tag{4.17}$$

which is a considerable improvement compared to the values quoted in [3]. The sum rules for baryons [16, 55] give a value for $\langle \bar{s}s\rangle$ at 1 GeV:

$$\langle \bar{s}s\rangle = (0.8 \pm 0.1)\langle \bar{u}u\rangle\,. \tag{4.18}$$

This value is of interest since it goes against the trend suggested by chiral perturbation theory (heavier quarks have larger condensates). The values (4.17) and (4.18) are all compatible with the PCAC relation for the K meson. The quark condensates of the heavy quarks do not play an important role in QCD sum rule calculations, as explained in detail in section (3.3e).

Finally we discuss lattice determinations of quark condensates. These calculations require Monte Carlo simulations with fermions [58, 59]. The physics comes down to the following. Since there is little trace in the spectrum of extra $\bar{q}q$ pairs in the wave functions (i.e., ordinary mesons are mainly $\bar{q}q$) the quenched approximation should be adequate. In Euclidean form the action reads

$$S_F = \int \mathrm{d}^D x \bar{\psi}(\not{D} + m)\psi\,, \tag{4.19}$$

where D_μ is the covariant derivative. One then defines the fermionic Green function in an external background field A_μ, with a probability distribution $d_\mu[A]$. Then

$$\langle \bar{\psi}(0)\psi(0)\rangle_A = \int \mathrm{d}\mu[A]\,\mathrm{Tr}[G(0,0|A)] = \frac{1}{V}\int \mathrm{d}^4x \langle x|\frac{1}{\not{D} + m}|x\rangle\,. \tag{4.20}$$

Introducing the spectral density $\rho(\mathrm{i}\lambda)$ for the operator $\not{D}$ one has

$$\langle\bar{\psi}(0)\psi(0)\rangle_A = \frac{1}{V}\int \mathrm{d}\lambda\, \rho(\mathrm{i}\lambda)\frac{1}{m-\mathrm{i}\lambda}\,. \tag{4.21}$$

Taking $m \to 0$ and recalling that in a vectorlike theory $\rho(\lambda) = -\rho(-\lambda)$ (both λ and $-\lambda$ are eigenvalues) one obtains

$$\lim_{m\to 0}\langle\bar{\psi}(0)\psi(0)\rangle_A = \frac{\pi}{V}\rho(0)\,. \tag{4.22}$$

Therefore the signal for chiral symmetry breaking is $\rho(0) \neq 0$. Lattice simulations of this quantity are possible [58]. The final result is a function of $\beta = 1/g^2$ and the QCD parameter Λ:

$$\langle\bar{\psi}\psi\rangle_{m=0} = 3.4(3\alpha_{\mathrm{B}})^{-4/11}\Lambda^3_{\mathrm{mom}}\,, \tag{4.23}$$

where $\alpha_{\mathrm{B}} = \dfrac{3}{2\pi}(6\beta - 2.75)$ and

$$\Lambda_{\mathrm{mom}} = \frac{\pi}{a}\frac{8\pi^2}{33}[(6\beta-2.75)]^{51/121}\exp\left[-\frac{4\pi^2}{33}(6\beta-2.75)\right],$$

a is the lattice spacing. For β about 1 this gives a value of about $(300\,\mathrm{MeV})^3$ for the condensate in reasonable agreement with PCAC.

(c) *Four-fermion operators*

The matrix elements of these operators are reduced to the square of $\langle\bar{q}q\rangle$, using Fierz transformations, saturating by the vacuum intermediate state and neglecting the contribution of all other states [1]. Arguments based on the large N_c expansion suggest that this approximation is good to within 10%, since corrections go like $1/N_c^2$ [53]. In the sum rules these terms are in general of the order of 10% of the leading contribution. The general formula can be found in [1]. For the various operators one finds

$$\begin{aligned}
&\langle 0|\bar{q}\sigma_{\mu\nu}\lambda^a q\bar{q}\sigma_{\mu\nu}\lambda^a q|0\rangle = -\tfrac{16}{3}\langle 0|\bar{q}q|0\rangle^2\,,\\
&\langle 0|\bar{q}\gamma_\mu\lambda^a q\bar{q}\gamma_\mu\lambda^a q|0\rangle = -\tfrac{16}{9}\langle 0|\bar{q}q|0\rangle^2\,,\\
&\langle 0|\bar{q}\gamma_5\lambda^a q\bar{q}\gamma_5\lambda^a q|0\rangle = -\tfrac{4}{9}\langle 0|\bar{q}q|0\rangle^2\,.
\end{aligned} \tag{4.24}$$

Similar combinations with $\sigma_{\mu\nu}\gamma_5$ and $\gamma_\mu\gamma_5$ give the opposite sign.

4. *The heavy quark masses*

The analysis of the moments for the charmonium and upsilon systems (using the abundant experimental information in the vector channel) gives a powerful means for determining the c and b quark masses. One obtains

$$m_c(p^2=-m_c^2)=1.26\pm 0.02\ \mathrm{GeV}\,, \tag{4.25}$$

$$m_b(p^2=-m_b^2)=4.23\pm 0.05\ \mathrm{GeV}\,. \tag{4.26}$$

The details of these determinations can be found in section (4.3).

5. *The strong coupling constant α_s*

Light quark physics is very insensitive to the precise value of α_s. This is not surprising since the Coulombic forces are small and the spectrum is generated by nonperturbative effects. As a consequence the particles lie on straight trajectories. In the Borel transformed sum rules this is reflected in the fact that in the equations for the resonance mass (see section (4.5)) the perturbative α_s correction (provided it is much smaller than 1) can be divided out and absorbed in the matrix elements of the non-perturbative operators. As we will see in section (4.3) charmonium is much better suited for determining α_s and yields

$$\alpha_s(4m_c^2)=0.20\pm 0.05\,, \tag{4.27}$$

which implies $\Lambda_{\mathrm{QCD}}=140\pm 40$ MeV to be interpreted as $\Lambda_{e^+e^-}$, the scheme in which the second order α_s correction for the vector channel ($e^+e^-\to J/\psi, \psi', \ldots$) is equal to zero. From (4.27) the value of α_s at any other scale can be calculated by the familiar renormalization group equation.

4.3. *Charmonium*

We discuss charmonium in detail for three reasons:

(1) It is simple theoretically: only gluonic power corrections contribute and under certain conditions only $G_{\mu\nu}G_{\mu\nu}$ has to be taken into account.

(2) There is detailed experimental information on the system.

(3) All relevant corrections have been calculated (up to dimension eight operators).

For the time being we will assume that the corrections from dimension six and eight operators can be neglected. Later in this section we will discuss their contribution in more detail. We also assume that the level spacings in all channels are approximately equal.

The calculation proceeds by the moment method described in section (2.5); the moments are defined by (2.28). The coefficients a_n and b_n can be found in the literature [1, 13, 28] or calculated from the expressions in the Appendix. The charmed quark mass m_c provides a natural large scale ($m_c \gg \Lambda_{\mathrm{QCD}}$). The original calculation of SVZ [1] used the moments at $\xi=0$ where $\xi=Q_0^2/4m_c^2$. This was done for convenience and was enough to predict the η_c at around 3.0 GeV (long before this particle was correctly identified) and to rule out the X(2.83) as the pseudoscalar state [27]. However at $Q^2=0$ the calculation is marginal, due to the large contribution of higher dimensional operators as we will see later. We emphasize that $\xi=0$ is not a reliable point for accurate mass and parameter determination. Therefore, as emphasized earlier [13], this parameter is very important. More precisely: though ξ is an unphysical parameter, it plays an important role in determining the validity region of the method. For the calculations to be valid the following conditions have to be satisfied: (a) asymptotic freedom must be valid; (b) the resonance chosen must dominate the sum rule; (c) the Wilson coefficients, which are functions of ξ, must always be small in the sense that the next order correction can be neglected. Figure 6 shows the role played by ξ. At $\xi = 0$ (i.e. $Q_0^2 = 0$) there is essentially no plateau where on the one hand the QCD expression is reliable ($n = 6$ is already outside the fiducial region) and where on the other hand the lowest lying resonance saturates the sum rule. If ξ is very large n has to be very large for the lowest lying resonance to dominate in which case higher order perturbative corrections cannot be neglected [2]. An example where conditions (b) and (c) are incompatible is open charm. The power correction proportional to $m_c\langle\bar{q}q\rangle$ (with $\langle\bar{q}q\rangle$ the light quark condensate) is so large that single resonance saturation becomes impossible when ξ is chosen large enough to make the correction small. The same figure shows that for $\xi = 1$ (S waves) there is a large plateau for determining the resonance mass. For P waves ξ has to be chosen larger: $\xi = 2.5$, in view of the fact that the coefficients b_n are roughly a factor three larger than for S waves.

The breakdown of the plateau at large and small n is understood. At large n perturbation theory breaks down: also the nonperturbative coefficients become of order 1 and therefore the calculation becomes invalid. In fact the coefficients grow as a power of n, a reflection of the fact that larger n implies larger distances. At small n higher resonances are relatively less suppressed. Because of the detailed experimental information available in the vector channel one can use these low moments to fix m_c [1]. On the theoretical side the nonperturbative contributions are unimportant for these moments.

Fortunately, for n between 6 and 10, where the plateau develops, the contribution of the continuum (given by the theta function in (2.5)) is less than 3%. This happy circumstance should be compared to the situation of light quark Borel transformed sum rules where the continuum contribution in the relevant region of M^2 is $\geq$30% (see section (4.5)). This is the reason why parameter determination with light quarks is sometimes less reliable.

The parameters α_s and m_c are ξ dependent. The running of α_s can be calculated from the asymptotic freedom formula:

$$\alpha_s(Q_0^2 + 4m_c^2) = \alpha_s(4m_c^2) \Big/ \left[1 + \frac{33-2n}{12\pi}\,\alpha_s(4m_c^2)\ln\frac{Q_0^2 + 4m_c^2}{4m_c^2}\right] \tag{4.28}$$

with n the number of flavours. The running of m_c can be absorbed in the choice of the renormalization

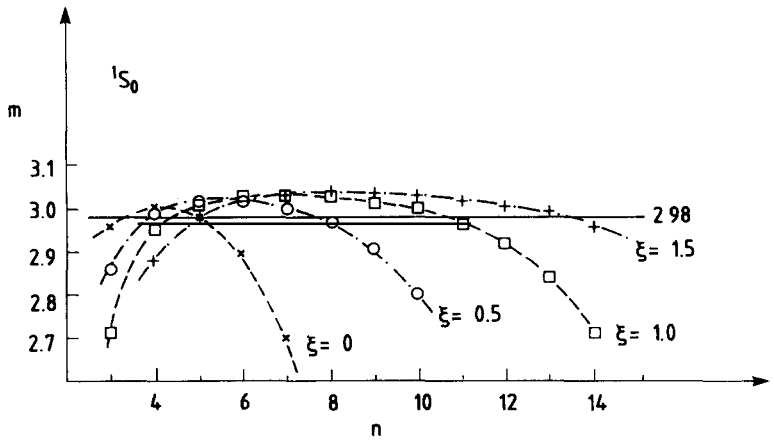


Fig. 6. Stability regions in n for the pseudoscalar S state of charmonium. The plot is for various values of ξ. Figure taken from [13].

point. In the calculations presented in the previous chapter the on-mass shell renormalization prescription of Schwinger has been used, i.e., the renormalization point is $p^2 = m_c^2$. Novikov et al. [2] already noticed that in order to minimize higher order corrections one should choose the renormalization point $p^2 = -m_c^2$ (in the Landau gauge at $Q_0^2 = 0$). When $\xi \neq 0$ one can either renormalize at $p^2 = -m_c^2$ or $p^2 = -(\xi + 1)m_c^2$. The two schemes give equivalent results. We prefer to work with the latter. In the Landau gauge $m_c(p^2 = -(\xi + 1)m_c^2)$ can be expressed in $m_c(p^2 = -m_c^2)$ by the following formula:

$$\frac{m_c(\xi)}{m_c} = 1 - \frac{\alpha_s}{\pi}\left\{\frac{2+\xi}{1+\xi}\ln(2+\xi) - 2\ln 2\right\}, \tag{4.29}$$

where $m_c = m_c(p^2 = -m_c^2)$. As a consequence the effective quark mass and coupling constant evolve to different values for S and P waves, which can be calculated from (4.28) and (4.29) in terms of $m_c(p^2 = -m_c^2)$ and $\alpha_s(4m_c^2)$.

We are now in the position to analyze the charmonium spectrum and to fit masses and parameters. As noted before, in the vector channel the low moments (or better the ratios (2.31)) can be used for a very accurate determination of m_c. This analysis of the ratios can be extended to $\xi \neq 0$. For a recent compilation of the data see [60]. The ratios are very sensitive to m_c (they behave as $1/m_c^2$) and the lowest ones are insensitive to changes in α_s and the gluon condensate. While at $\xi = 0$ essentially only r_3 and r_4 can be used, the number of suitable moments rapidly grows with ξ. We analyzed the ratios at $\xi = 0$, 1, and 2.5. Renormalizing the mass at $p^2 = -m_c^2$ we find

$$m_c(p^2 = -m_c^2) = 1.26 \pm 0.02\ \mathrm{GeV}. \tag{4.30}$$

Using the mass renormalization at $p^2 = -(\xi + 1)m_c^2$ confirms the validity of (4.29). In particular we obtain

$$m_c(p^2 = -(\xi + 1)m_c^2) = \begin{cases} 1.24 \pm 0{,}02 & \text{at} \quad \xi = 1, \\ 1.22 \pm 0.02 & \text{at} \quad \xi = 2.5. \end{cases} \tag{4.31}$$

We can now proceed to extract the parameters of the lowest lying resonance in each partial wave. To account more accurately for the continuum contributions we include the behaviour of the bare loop of each partial wave

$$\operatorname{Im} \Pi^j(s)_{\text{cont}} = \left(1 + \frac{\alpha_s}{\pi}\right) \operatorname{Im} \Pi_0^j(s)\theta(s - s_0). \tag{4.32}$$

The $\operatorname{Im} \Pi_0^j(s)$ are listed in chapter 3. This replaces the rather crude θ function behaviour of [13]. As usual we transfer the continuum contribution to the theoretical side of the sum rule.

Since the quark mass is now fixed, our parameters are α_s, ϕ, and s_0 (ϕ is defined in (2.33)). We have allowed the continuum threshold s_0 to be different for S and P waves although it could be argued that with the bare loop behaviour taken into account in (4.32) they are equal. Small variations in s_0 can be compensated by variations in m_c within the errors given in (4.31) and vice versa. We find

$$\begin{aligned} \sqrt{s_0} &= 4.0 \pm 0.2\ \mathrm{GeV} \quad \text{for} \quad \text{P waves}, \\ \sqrt{s_0} &= 3.8 \pm 0.2\ \mathrm{GeV} \quad \text{for} \quad \text{S waves}. \end{aligned} \tag{4.33}$$

These values are lower than in [13] since we have taken account of the kinematical tail in a proper way by including the bare loop behaviour in (4.32). We further observe that in order to obtain a $J/\psi-\eta_c$ splitting of at least 90 MeV (experimentally the splitting is 115 MeV) ϕ cannot be lower than 0.14×10^{-2}, but for ϕ larger than 0.20×10^{-2} it is difficult to accommodate the P waves. We find good P-wave solutions for $0.14\times10^{-2}\le\phi\le0.20\times10^{-2}$ and $0.13\le\alpha_s(\xi=2.5)\le0.20$, but the values of α_s and ϕ are correlated: high ϕ demands low α_s and vice versa. A good solution for both S and P waves is for instance

$$\begin{array}{ll} \text{S waves}(\xi=1) & \text{P waves}(\xi=2.5) \\ m_c = 1.23\,\text{GeV} & m_c = 1.21\,\text{GeV} \\ \phi = 0.18\times10^{-2} & \phi = 0.18\times10^{-2} \\ \alpha_s = 0.21 & \alpha_s = 0.17 \\ \sqrt{s_0} = 3.8\,\text{GeV} & \sqrt{s_0} = 3.8\,\text{GeV} \end{array} \tag{4.34}$$

but solutions can be found for the parameter values

$$\phi = (0.17\pm0.03)10^{-2}\,,\qquad \alpha_s(4m_c^2) = 0.20\pm0.05\,. \tag{4.35}$$

The first number implies $\langle(\alpha_s/\pi)G^a_{\mu\nu}G^a_{\mu\nu}\rangle = (360\pm20\,\text{MeV})^4$ and the second $\Lambda_{\text{QCD}} = 140\pm40$ MeV where Λ_{QCD} has been calculated using the second-order expression for α_s

$$\ln\left(\frac{Q^2}{\Lambda^2}\right) = \frac{16\pi^2}{\beta_0 g^2} - \left(\frac{\beta_1}{\beta_0^2}\right)\ln\left(\frac{16\pi^2+\beta_1/\beta_0 g^2}{\beta_0 g^2}\right);\qquad \alpha_s = g^2/4\pi\,.$$

The ratios r_n^j for the parameter set (4.34) are plotted in figs. 7 and 8. These give the following mass values for the levels:

$$\begin{array}{lll} {}^1\text{S}_0\text{: } 3.00\pm0.02\,\text{GeV}\,, & {}^3\text{S}_1\text{: } 3.09\pm0.02\,\text{GeV}\,, & {}^3\text{P}_0\text{: } 3.40\pm0.01\,\text{GeV}\,, \\ {}^1\text{P}_1\text{: } 3.51\pm0.01\,\text{GeV}\,, & {}^3\text{P}_1\text{: } 3.50\pm0.02\,\text{GeV}\,, & {}^3\text{P}_2\text{: } 3.57\pm0.02\,\text{GeV}\,. \end{array} \tag{4.36}$$

For the vector current the moments can be used directly to calculate the e^+e^- width of the J/ψ. We find

$$\Gamma^{J/\psi}_{e^+e^-} = 4.9\pm0.4\,\text{keV} \tag{4.37}$$

to be compared with the experimental value of 4.7 ± 0.6 keV [61]. This is an improvement over the original result of [13]. The agreement with the data is excellent. To resolve the remaining differences electromagnetic corrections have to be included. It has been suggested [62] that the tendency of the η_c to be systematically about 20 MeV too high might be due to mixing with a nearby (glueball?) state. The prediction for the ${}^1\text{P}_1$ agrees with preliminary ISR results [63].

Before closing this section we have to discuss the contribution from higher dimensional operators, calculated in [40, 41]. Including six and eight dimensional operators the expression for the nth moment

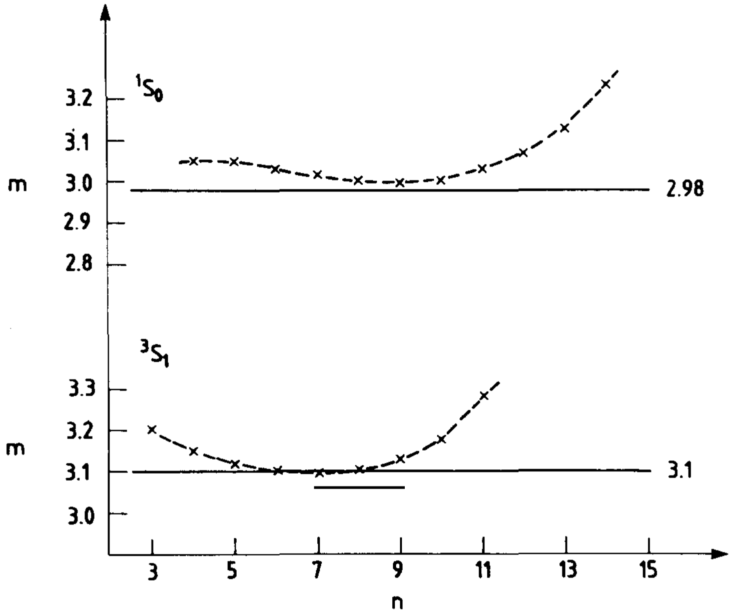


Fig. 7. Results for vector and pseudoscalar states of charmonium from the moments (2.31). All parameters as in (4.34). For comparison the experimental mass values have also been indicated.

at $Q^2 = 0(\xi = 0)$ can now be written as

$$M_n(Q^2=0) = A_n(Q^2=0)[1 + a_n(Q^2=0)\alpha_s + b_n(Q^2=0)\phi^{(2)} + c_n(Q^2=0)\phi^{(3)} + d_n(Q^2=0)\phi^{(4)}]\,, \tag{4.38}$$

where $\phi^{(2)} \equiv \phi$ and $\phi^{(3)}$ and $\phi^{(4)}$ are similar dimensionless quantities containing the matrix elements of six and eight dimensional operators. Even at $Q^2 = 0$ the terms $c_n^V\phi^{(3)}$ give negligible contributions for $n \leq 10$ and we will not consider them further here.

The dimension eight terms are of the form (in the vector channel)

$$d_n^V(Q^2=0)\phi^{(4)} = \frac{n(n+1)(n+2)(n+3)}{(2n+5)(2n+7)(2n+9)} \sum_{I=1}^{7} \left(\sum_{J=0}^{6} A(I,J) n^J \right) \phi_I^{(4)}\,, \tag{4.39}$$

where $\phi_I^{(4)} = \frac{4}{81}\langle O_I^8 \rangle/(4m^2)^4$. The operators O_I^8 are given in (3.58) and the matrix $A(I, J)$ in table 1. The matrix elements of the operators O_I^8 have been discussed in section (4.2). With these matrix elements Nikolaev and Radyushkin found that at $Q^2 = 0$ the contributions (4.39) to the moments (4.38) are large and grow rapidly with n. At $n = 6$ they are already so large (about 50% of the gluon condensate contribution) that they invalidate the original SVZ calculation [1]. This confirms that ξ must be chosen different from 0, in particular we have seen that for the vector channel $\xi = 1$. Shifting the expression (4.39) to $\xi \neq 0$ gives [64]

$$d_n^V(\xi)\phi^{(4)} = \frac{n(n+1)(n+2)(n+3)}{(2n+5)(2n+7)(2n+9)} (1+\xi)^{-4} \sum_{I=1}^{7} \left(\sum_{J=0}^{6} A(I,J) \frac{G(J,n)}{F(n,\frac{1}{2},n+\frac{5}{2};\rho)} \right) \phi_I^{(4)}\,, \tag{4.40a}$$

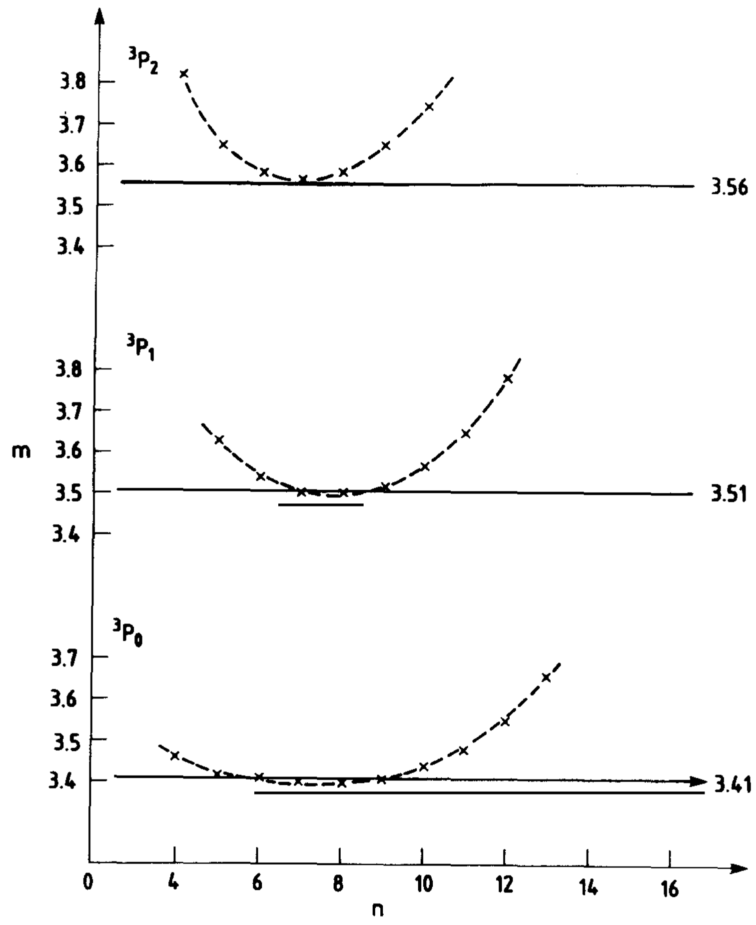


Fig. 8. Results for the P states of charmonium from the moments (2.31). All parameters as in (4.34). For comparison the experimental mass values have also been indicated.

where

$$G(J,n)=\frac{(n+1+J)!}{(n+1)!}F(n+4,\tfrac{7}{2}-J,n+\tfrac{5}{2};\rho)-\sum_{k=0}^{J-1}\alpha^J(k)G(k,n)\,, \tag{4.40b}$$

where $\alpha^J(0)=(J+1)!$, $\alpha^J(J-1)=(J+1)+\alpha^{J-1}(J-2)$ $(J\geq 2)$, and $\alpha^J(k)=(J+1)\alpha^{J-1}(k)+\alpha^{J-1}(k-1)$ $(k<J-1)$.

In fig. 9 we have plotted the $d=4$ and $d=8$ contributions to the moments $M_n^V(\xi)$ at $\xi=0$ and $\xi=1$. As can be seen, at $\xi=0$ the $d=8$ contribution is not small compared to the $d=4$ contribution. Already at $n=6$ it exceeds the uncertainty in the $d=4$ contribution which is about 20% in the determination of $\phi^{(2)}$. At $\xi=1$ the situation has changed drastically. Both contributions are suppressed but the $d=8$ contribution by a much larger factor than the $d=4$ one. At $n=6$ the $d=8$ terms are essentially zero and up to the moment $n=11$ their contribution is less than about 20% of the $d=4$ one. This confirms the correctness of the procedure followed in the beginning of this section for analyzing

Table 1
The values of the coefficients $A(I, J)$ in (4.39).

I \ J	0	1	2	3	4	5	6
1	−396	−1075/4	−811/24	38/3	29/8	1/4	0
2	−42	−257/56	3011/105	7891/480	2203/672	671/3360	−1/224
3	1638	15703/10	35209/60	1643/15	217/20	1/2	0
4	−510	6541/140	46569/140	12229/80	46519/1680	3443/1680	23/560
5	−2232	−161939/70	−35121/35	−27883/120	−8157/280	−1367/840	−3/280
6	0	−6344/35	−6686/35	−1171/15	−533/35	−289/210	−3/70
7	1080	14709/14	8797/21	10801/120	1937/168	247/280	9/280

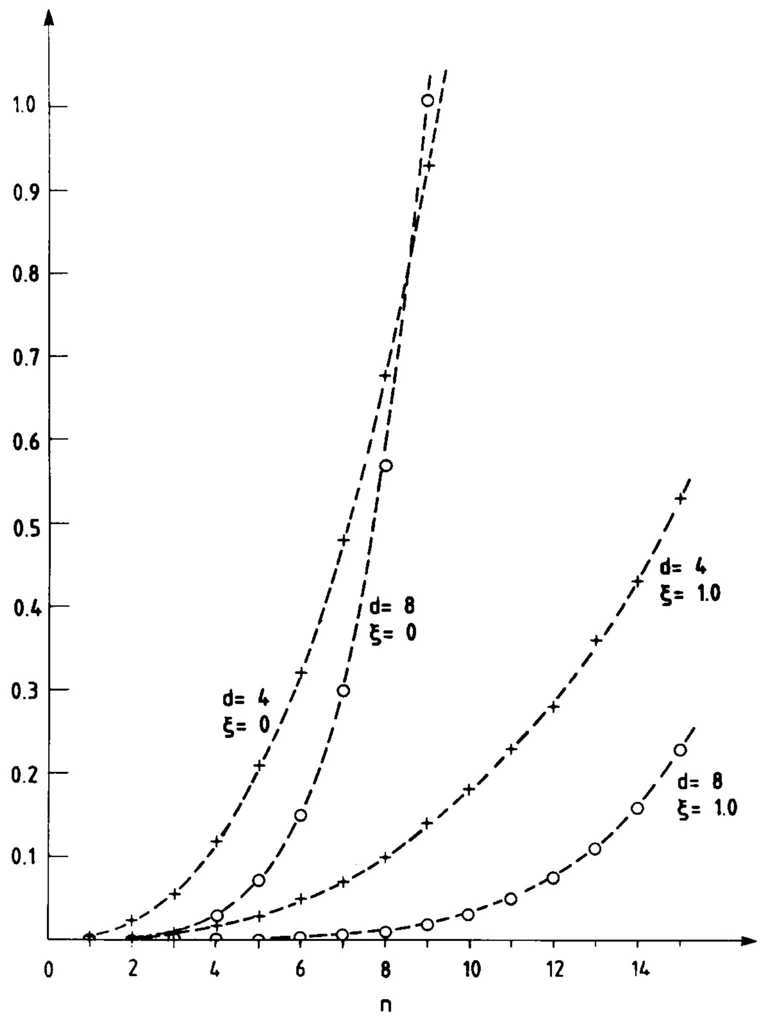


Fig. 9. Contributions of $d = 4$ and $d = 8$ operators to $M_n^V(\xi)$ for $\xi = 0$ and $\xi = 1$. The contributions are relative to the bare loop. The absolute value of the $d = 4$ contribution has been plotted. Figure taken from [64].

the charmonium system. Including dimension eight operators can only extend the stability to higher n. For a consistent treatment perturbative corrections of $O(\alpha_s^2)$ and first-order α_s corrections to the gluon condensate should also be taken into account since they are likely of the same order of magnitude.

4.4. The upsilon and higher quark systems

The transition towards higher quark masses renders the physics more complicated. The reason is that the system is governed by shorter distances and knowledge of higher order gluonic exchanges is necessary. The influence of the power corrections is much smaller. This is reflected in the fact that for the upsilon system the parameter ϕ defined by (2.33) will be a factor $(m_b/m_c)^4 \propto 100$ smaller. Naively one could say that it is sufficient to go to n large enough to feel the presence of the nonperturbative forces in our sum rules. This improves the saturation by the lowest lying resonance, but since the perturbative corrections also grow with n (roughly like $(\sqrt{n\alpha_s})^k$ for the kth order at $Q^2 = 0$) higher order corrections have to be included. A method to implement this programme exists [65] and we discuss it below.

Like in charmonium we can use the existing experimental information in the vector channel to determine the beauty quark mass m_b [66]. With $\alpha_s(4m_c^2)$ given by (4.35), (4.29) can be used to determine $\alpha_s(4m_b^2)$:

$$\alpha_s(4m_b^2) = 0.15 \pm 0.03\,. \tag{4.41}$$

Comparing the experimental and theoretical moments for $n = 2$ to 9 at $Q^2 = 0$ we find

$$m_b(p^2 = -m_b^2) = 4.23 \pm 0.05\ \mathrm{GeV}\,, \tag{4.42a}$$

which agrees with the value obtained in [13]. Using (4.29) we find for the on-shell mass

$$m_b(p^2 = +m_b^2) = 4.5 \pm 0.1\ \mathrm{GeV}\,. \tag{4.42b}$$

This rules out the on-shell value $m_b(p^2 = +m_b^2) = 4.80\ \mathrm{GeV}$ obtained by Voloshin [65].

In spite of the difficulties mentioned above the moment method can still be used at intermediate n to obtain an estimate of the splittings. For S waves the plateaus for the pseudoscalar and vector channels partly overlap at $\xi = 1$, and although the plateau levels are still going down with growing ξ, a reliable value for the ratio of the pseudoscalar and vector mass can be obtained from the ratios (2.34). The results are plotted in fig. 10 and give

$$m_\Upsilon - m_{\eta_b} \cong 60\ \mathrm{MeV}\,. \tag{4.43}$$

This estimate is independent of the mass of the quark. We notice that about 40 MeV of this splitting has a relativistic kinematic origin.

In the P-wave case no ξ-independent plateau can be achieved for low enough n to justify first-order perturbation theory, but also the stability regions for different J values occur at widely different values of n, which makes it impossible to use the ratios (2.34) and no reliable mass prediction can be made. Voloshin [65] has exploited an alternative method to study the upsilon system. As stated before, the

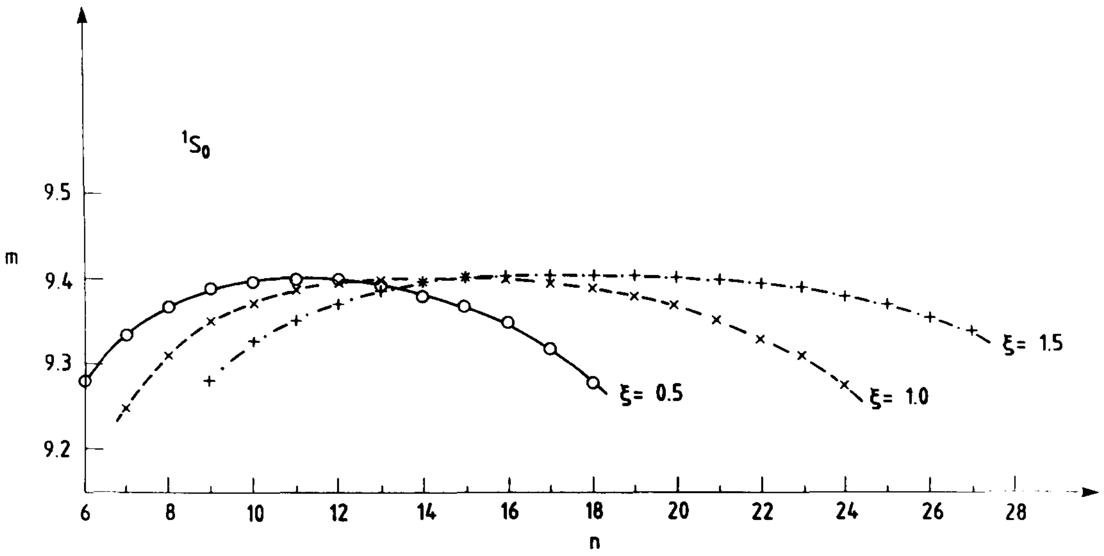


Fig. 10. Results for the pseudoscalar upsilon state from the ratios (2.34). The figure demonstrates that a stable ξ-independent plateau has been reached. Figure taken from [13].

nonperturbative contributions decrease with growing spacelike Q^2, and the power corrections are too small to drive resonance formation. One is forced to move to larger distances (closer to threshold) by taking Q^2 negative (timelike). However, this also increases the perturbative contributions and it is not sufficient anymore to take only first-order α_s corrections into account. This is expected since the system is much more "Coulombic" than charmonium. The leading contributions in terms of n (the moment) are the ladder type gluon exchanges and the kth order for $Q^2 = 0$ grows like $(\sqrt{n}\alpha_s)^k$ while the other contributions decrease with n. These graphs can be summed to all orders. Voloshin used a multipole expansion technique of the nonrelativistic Green function [67]. He also includes the first non-perturbative correction due to the gluon condensate and succeeds in summing all ladder exchanges with one gluon line cut. After Borel transforming the sum rule results in a splitting between the S and P levels in disagreement with experiment [68]. The calculation is unfortunately not accurate enough. As pointed out above the values of the parameters are not correct and this calculation cannot be considered as a test of QCD sum rules. In particular m_b is too high, the value of α_s is doubtful, the gluon condensate is not accurate and relativistic effects could be important.

Borel transformed sum rules have also been used in [69] for analyzing charmonium and upsilon systems, however without including higher order perturbative corrections.

The recent discovery of the top quark [70], if confirmed, offers the first possibility of studying heavy quarks in the dominant Coulombic regime. Voloshin and Leutwyler [71, 72] noticed that even in this Coulombic regime for large enough n (principal quantum number) the system is at large enough distances that the vacuum condensates become important again. Since these effects grow as n^6 different levels would have rapidly varying effects and a determination of the gluon condensate would be possible. The problem with this method is that the gluon condensate has only a meaning as a Euclidean operator. When used in the Schrödinger equation it is not Hermitian and although only used as a first-order perturbation the calculation is not rigorous.

4.5. Light quark meson systems

Since there is no heavy quark mass to set the scale as in the charmonium and upsilon systems, it is natural and convenient (although not compulsory) to use Borel transformed sum rules for light quark systems. Most $L=0$ and $L=1$ states have been studied [1, 14, 15]. All currents and the corresponding expressions for the Wilson coefficients have been collected in the Appendix. Here we discuss the results with emphasis on the physical aspects. For the technical details we refer the reader to the original papers.

1. *$L=0$ states*

ρ meson. To describe the method we discuss the ρ meson in detail. Using the polarization function given in (A.16) we obtain after Borel transforming and using the vacuum saturation hypothesis for the four-fermion operators:

$$\int \mathrm{e}^{-s/M^2}\, \mathrm{Im}\, \Pi(s)\, \mathrm{d}s = \frac{1}{8\pi} M^2 \Big[1 + \frac{\alpha_s(M)}{\pi} + \frac{8\pi^2}{M^4} \langle 0|m\bar{q}q|0\rangle + \frac{\pi^2}{3M^4} \langle 0| \frac{\alpha_s}{\pi} G^a_{\mu\nu} G^a_{\mu\nu}|0\rangle - \frac{448}{81} \frac{\pi^2 \alpha_s}{M^6} |\langle 0|\bar{q}q|0\rangle|^2 \Big]. \tag{4.44}$$

We now saturate $\mathrm{Im}\,\Pi(s)$ by one resonance (the ρ meson), plus a continuum with threshold s_0 in the form of a θ function (see (2.5)). The continuum contribution will be transferred to the right-hand side of (4.44). Taking the ratio of the resulting equation and its first derivative with respect to $1/M^2$ yields an equation for m_ρ^2. With the values for the matrix elements given in section (4.2), we get

$$m_\rho^2 = M^2 \Big[\Big(1 + \frac{\alpha_s}{\pi}\Big)\Big[1 - \Big(1 + \frac{s_0}{M^2}\Big) \mathrm{e}^{-s_0/M^2}\Big] - \frac{0.05}{M^4} + \frac{0.06}{M^6}\Big] \Big/ \Big[\Big(1 + \frac{\alpha_s}{\pi}\Big)\Big[1 - \mathrm{e}^{-s_0/M^2}\Big] + \frac{0.05}{M^4} - \frac{0.03}{M^6}\Big]. \tag{4.45}$$

Even at $M^2 = m_\rho^2 = 0.6\,\mathrm{GeV}^2$ the power corrections in (4.45) are relatively small (but not negligible).

The value of $s_0 \cong 1.5\,\mathrm{GeV}^2$ follows from the spacing in dual models. In this case it can also be inferred from the data in $\mathrm{e}^+\mathrm{e}^-$ annihilation. For this value of s_0 a stable mass prediction is obtained for a range of M^2. In fact the stability criterion itself also gives a good determination of s_0, but contrary to the charmonium situation the continuum contribution in (4.45) is not small.

To consider the situation in more detail we write for the prediction of the mass [1]:

$$m_\rho^2 = M^2 f_{\mathrm{cont}}(M^2) f_{\mathrm{th\,corr}}(M^2), \tag{4.46}$$

where $f_{\mathrm{th\,corr}}(M^2)$ is given by (4.45) without the continuum contributions ($s_0 = \infty$) and f_{cont} is the ratio of (4.45) and $f_{\mathrm{th\,corr}}$. These functions are plotted in fig. 11. Without power corrections $f_{\mathrm{th\,corr}} = 1$ and for $s_0 = \infty$, $f_{\mathrm{cont}} = 1$. The deviations from these values give a measure of the importance of the power corrections and the continuum. The arrows A and B in fig. 11 indicate the region in M^2 which is reliable from a theoretical point of view (higher order corrections are negligible) and where the resonance contribution is large. Curve 3 in fig. 11 shows that without power corrections there is no stability while curve 2 is the actual mass prediction.

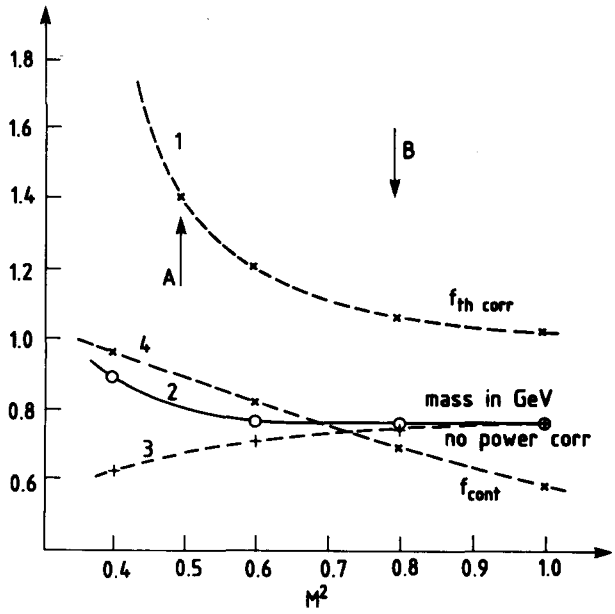


Fig. 11. The ρ meson mass with and without power corrections. The continuum threshold $s_0 = 1.5\,\mathrm{GeV}^2$. Also shown are the functions f_{cont} and $f_{\mathrm{th\,corr}}$ defined in the text. The region between the arrows A and B is considered to be reliable for determining the resonance parameters. Figure adopted from [1].

Without taking the ratio the sum rules can also be used to determine the ρ meson coupling. The final results for the ρ meson parameters are

$$g_\rho^2/4\pi \cong 2.42 \quad (\exp g_\rho^2/4\pi = 2.36 \pm 0.18)\,, \qquad m_\rho \cong 770\,\mathrm{MeV} \quad (\exp m_\rho = 769 \pm 3\,\mathrm{MeV})\,. \tag{4.47}$$

ω meson. To lowest order in α_s mesons with $I = 0$ and $I = 1$ in the same channel are degenerate. Therefore we have $m_\omega = m_\rho$. From studying $\rho\omega$ interference [73], one finds $m_{\mathrm{d}} - m_{\mathrm{u}}$ about 3 MeV and in particular a solution with vanishing u quark mass and m_{d} unequal zero seems to be ruled out.

ϕ meson. Here, mass corrections due to the strange quark mass have to be added. The main power correction is now given by the term $\langle m_s \bar{q} q \rangle$ which compared to the ρ meson results in a change of sign of the $1/M^4$ power correction. This is a pure octet term and the dynamical source of the Gell-Mann–Okubo formula for mesons. In addition there is the correction $6m_s^2/M^2$ to the bare loop which goes beyond octet. It amounts to about 10% of the dominant Gell-Mann–Okubo term but gives a significant contribution to the mass and coupling. It appears that for this sum rule to give the observed mass and its coupling requires the stringent values given by (4.17) for the strange mass and condensate [57]. For the resonance parameters we obtain

$$m_\phi = 1010 \pm 10 \quad (\exp 1019.6 \pm 0.1)\,, \qquad g_\phi^2/4\pi = 13.0 \pm 0.2 \quad (\exp 11.7 \pm 0.9)\,. \tag{4.48}$$

Higher quark masses tend to give an unacceptably high value for the coupling. Both the mass and the coupling depend on s_0, but both the stability of the sum rule and the spacing deduced from the string tension require $s_0 \cong 2\,\mathrm{GeV}^2$.

K^* meson. The K^* meson can be analyzed in the same fashion. The mass correction to the bare loop is about a factor four smaller than for the ϕ meson and the quark condensate a factor of two. The resulting value for the mass is in excellent agreement with experiment. Although the K^* is not very sensitive to m_s, the dependence on the condensate is very steep. For this case higher values of the condensate are favoured while the ϕ meson requires low values. The region of overlap determines the narrow range for $\langle m_s \bar{s}s\rangle$ given by (4.17). For more details see [57].

Pseudoscalar ($J^{PC} = 0^{-+}$) mesons. The pseudoscalar mesons (π, K) are too low in mass to be calculable by this method, but rough estimates of the couplings can be made [1]. As we will see below, the A_1 sum rules give indirect evidence for the existence of a massless or almost massless pseudoscalar state.

2. *$L = 1$ mesons*

P-wave states are in general a little more delicate. Several of the currents contain derivatives and the dispersion relations for the resulting polarization functions need more subtractions. Consequently the sum rules are more sensitive to the continuum contribution.

Scalar ($J^{PC} = 0^{++}$) states. Let us consider the sum rule generated by the $I = 1$ scalar current $j(x) = \frac{1}{2}(\bar{u}u - \bar{d}d)$. Using the expression given in the Appendix we obtain after Borel transforming

$$\int e^{-s/M^2} \operatorname{Im} \Pi(s)\, ds = \frac{3}{16\pi} M^4 \left[1 + \frac{11}{3} \frac{\alpha_s}{\pi} + \frac{8\pi^2}{M^4} \langle 0|m\bar{q}q|0\rangle + \frac{\pi^2}{3M^4} \langle 0| \frac{\alpha_s}{\pi} G^a_{\mu\nu} G^a_{\mu\nu} |0\rangle \right.$$
$$\left. - \frac{1408}{81} \frac{\pi^3 \alpha_s(M)}{M^6} \langle 0|\bar{q}q|0\rangle^2 \right]. \tag{4.49}$$

The sum rule is now proportional to M^4, due to the extra subtraction. It would be tempting to divide $\Pi(Q^2)$ by Q^2 and consider the resulting expression. In that case however the subtraction constant in (2.3) does not disappear under Borelization and would introduce an unknown $1/M^2$ power correction. The consequence is that in the second sum rule obtained by differentiating (4.49) with respect to $1/M^2$ the $1/M^4$ power corrections disappear. In principle one can take an arbitrary number of derivatives and the resulting sum rules are equivalent. However, in higher transforms the lower part of the spectral functions gets less enhanced and it will be increasingly difficult to extract the parameters of the lowest lying resonance. In fig. 12 we have plotted the resonance mass from the ratio of (4.49) and its derivative with respect to $1/M^2$ for some values of s_0. The results are fairly sensitive to the actual choice of s_0, which can be determined from the stability criterion of the sum rule. The result does not quite agree with the equal spacing argument. Not surprisingly the continuum contribution is larger than in the ρ-meson case. We find

$$m_\delta = 1.00 \pm 0.03\ \text{GeV} \quad (s_0 \cong 1.5\ \text{GeV}^2). \tag{4.50}$$

The analysis for the $I = 0$ current $j(x) = \frac{1}{2}(\bar{u}u + \bar{d}d)$ corresponding to the S*(980) leads to the same sum rule ($I = 0$ and $I = 1$ degeneracy).

To analyze the $(\bar{s}s)$ state corresponding to the $\varepsilon(1300)$ meson mass corrections due to the strange quark mass have to be added to the polarization function (see Appendix). We find

$$m_{\bar{s}s} \cong 1350\ \text{MeV} \quad (s_0 \cong 3\ \text{GeV}^2). \tag{4.51}$$

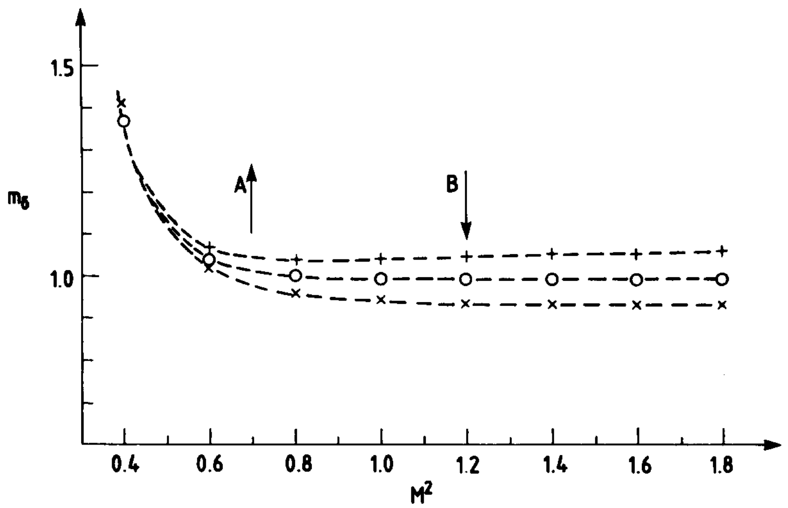


Fig. 12. The theoretical prediction for the mass of the δ meson from (4.49). The value of the continuum threshold s_0 for the various curves is: (+) $s_0 = 1.75\,\mathrm{GeV}^2$; (O) $s_0 = 1.5\,\mathrm{GeV}^2$; (×) $s_0 = 1.25\,\mathrm{GeV}^2$. Figure taken from [14].

The scalar states S* and δ are a little controversial for several reasons:

(a) Phenomenologically they have unusual decay modes. This has led to speculation that these states might be four-quark states [74].

(b) Theoretically they might have large instanton contributions. However, since we know that the $\bar{q}q$ states exist and we observe that the sum rules predict degenerate $I = 0$ and $I = 1$ states at about 1 GeV, we conjecture that the S* and the δ are the expected $\bar{q}q$ states.

Axial vector ($J^{PC} = 1^{++}$) *states.* For the A_1 meson, corresponding to the axial vector current $j_\mu(x) = \frac{1}{2}(\bar{u}\gamma_\mu\gamma_5 u - \bar{d}\gamma_\mu\gamma_5 d)$, we can write down two sum rules, one connected with the axial vector current and one with its divergence. In the latter case the pion also gives a contribution to the sum rule:

$$\operatorname{Im} \Pi_2(s) = \frac{\pi}{2} f_\pi^2 \delta(s) + \pi m_{A_1}^2 f_{A_1}^{-2} \delta(s - m_{A_1}^2) + \frac{1}{8\pi}\left(1 + \frac{\alpha_s}{\pi}\right)\theta(s - s_0)\,, \tag{4.52a}$$

while in the first sum rule we have

$$\operatorname{Im} \Pi_1(s) = \pi m_{A_1}^4 f_{A_1}^{-2} \delta(s - m_{A_1}^2) + \frac{1}{8\pi}\left(1 + \frac{\alpha_s}{\pi}\right) s\theta(s - s_0)\,. \tag{4.52b}$$

Here the constants f_{A_1} and f_π are defined in the usual way

$$\langle 0|\bar{u}\gamma_\mu\gamma_5 d|\pi\rangle = \mathrm{i}p_\mu f_\pi\,, \qquad \langle 0|\bar{u}\gamma_\mu\gamma_5 d|A_1\rangle = \sqrt{2} f_{A_1}^{-1} m_{A_1}^2 \varepsilon_\mu\,. \tag{4.53}$$

From (4.52) and from the actual expressions for the polarization functions it follows that up to the order we work one has $\operatorname{Im} \Pi_1(s) = s \operatorname{Im} \Pi_2(s)$ which implies that the continuum threshold s_0 is the same in (4.52a) and (4.52b). The behaviour of the two sum rules as a function of s_0 is different and it turns out that they only result in the same resonance mass for $s_0 \cong 1.75\,\mathrm{GeV}^2$, as can be seen in fig. 13a. This is about the position of the first radial recurrence of the pion, as expected. The experimental value $f_\pi = 133\,\mathrm{MeV}$ is used as input. For more details see [14]. We find for the mass (fig. 13a) and the coupling (fig. 13b):

$$m_{\mathrm{A}_1} = 1.15 \pm 0.04\,\mathrm{GeV} \quad (s_0 \cong 1.75\,\mathrm{GeV}^2)\,, \qquad 4\pi/f^2_{\mathrm{A}_1} \cong 0.15\text{–}0.18\,, \tag{4.54}$$

which agrees with one of the mass values usually quoted [61]. It does not appear to be possible to have the A_1 at around 1300 MeV as suggested nowadays. The value for the coupling is in good agreement with the experimental value quoted in [75].

The $I = 0$ partners of the A_1 are the D(1285) and the E(1420) mesons. Naively speaking the D meson should be degenerate with the A_1, but

(a) for this meson we have only one sum rule since the U(1) problem prevents us from treating the divergence of the $I = 0$ current in the same way,

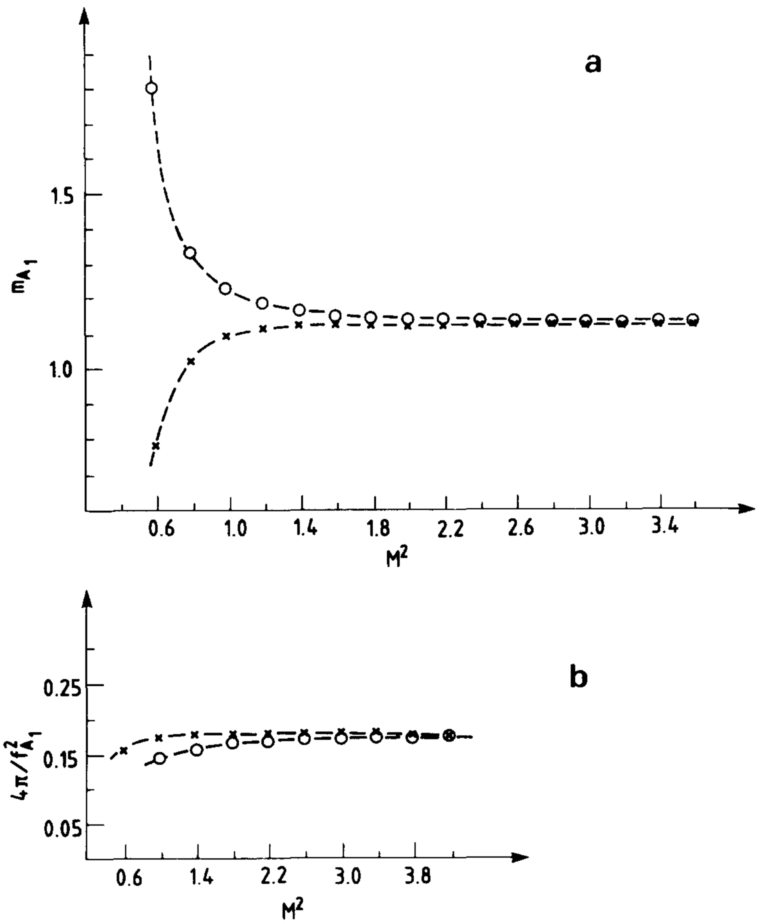


Fig. 13. The mass (a) and the coupling (b) from the two sum rules for the A_1 meson. The continuum threshold $s_0 = 1.75\,\mathrm{GeV}^2$. Figure taken from [14].

(b) after chiral breaking the threshold s_0 is very different since there is no $\rho\pi$ mode in this channel. We find

$$m_D \cong 1300\ \mathrm{MeV} \quad (s_0 \cong 2.6\ \mathrm{GeV}^2)\,, \tag{4.55}$$

but this is rather sensitive to the choice of s_0. The stability criterion of the sum rule alone would imply $m_D = 1200 \pm 100$ MeV. For the E meson we have to include the first-order (logarithmic) mass correction to the bare loop which is the same as in the scalar case. With the values of m_s and $\langle m_s \bar{s}s\rangle$ determined from the K* and ϕ mesons, we find a value for m_E which is about 50 MeV higher than the experimental value [57]

$$m_E = 1470^{+30}_{-10}\ \mathrm{MeV} \quad (s_0 = 3.3\ \mathrm{GeV}^2)\,. \tag{4.56}$$

The strange 1^+ states $Q_1(1280)$ and $Q_2(1400)$ have not been analyzed in the QCD sum rule approach.

The $J^{PC} = 1^{+-}$ states. Because of the derivative coupling ($j_\mu(x) = i\bar{q}\gamma_5 \overleftrightarrow{D}_\mu q$) we need a further subtraction in this case. The polarization function $\Pi(Q^2)$ is proportional to Q^4 (see Appendix). Consequently the power correction due to the gluon condensate is in first approximation independent of Q^2. We note that the quark condensate contributions for currents with derivative couplings are multiplied by an extra factor m^2/Q^2 and can safely be neglected for nonstrange quarks. In general, currents with derivative couplings have a $\ln Q^2$ term in the coefficient of the gluon condensate (even in lowest order) but this happens to have a coefficient zero in this case. As a consequence no power corrections of dimension four operators survive in the final Borel transformed sum rule. And the second sum rule obtained by differentiating with respect to $1/M^2$ does not contain any power corrections at all. It is therefore not surprising that it is not possible to determine the B meson mass in this way. The remaining power corrections are too small, and the continuum contribution dominates the sum rule completely.

The tensor ($J^{PC} = 2^{++}$) states. The analysis of these states goes along the same lines, using the polarization functions given in the Appendix. We note that in this case the gluon condensate coefficient proportional to $\ln Q^2$ does not vanish. The sum rules for the 2^{++} ($j_{\mu\nu}(x) = i\bar{q}\gamma_\mu \overleftrightarrow{D}_\nu q$) current have been analyzed in [14] and for the 2^{-+} ($j_{\mu\nu}(x) = i\bar{q}\gamma_5\gamma_\mu \overleftrightarrow{D}_\nu q$) in [15]. The M^6 power corrections for the 2^{-+} have the opposite sign compared to 2^{++}, which tends to make the 2^{-+} state ($A_3(1680)$ meson) heavier than the 2^{++} states (f(1270) and $A_2(1320)$). Because of the sign of the power corrections the stability criterion does not work for the f/A_2 meson to determine the threshold positions, which makes the predictive power rather small in this case. The results are

$$m_f = m_{A_2} \cong 1300\ \mathrm{MeV} \quad (s_0 \cong 2.65\ \mathrm{GeV}^2)\,, \qquad m_{A_3} \cong 1630\ \mathrm{MeV} \quad (s_0 \cong 3.5\ \mathrm{GeV}^2)\,. \tag{4.57}$$

Assuming f-meson dominance in the matrix element $\langle\pi|\theta_{\mu\nu}|\pi\rangle$ ($\theta_{\mu\nu}$ is the energy momentum tensor), similar to ρ meson dominance in $\langle\pi|j_\mu^{em}|\pi\rangle$ the coupling g_f of the f meson to its current can be expressed in terms of $g_{f\pi\pi}$. The resulting "experimental" value agrees remarkably well with the value $g_f \cong 0.040$ determined from the sum rules.

For the f′(1520), assuming it to be a pure $\bar{s}s$ state, strange quark mass corrections have to be included, including the quark condensate terms which although suppressed by an extra $(m_s/Q)^2$ factor are not negligible here. The resulting sum rule is rather complicated and gives

$$m_{f'} = 1540^{+40}_{-10}\ \mathrm{MeV} \quad (s_0 \cong 3.8\ \mathrm{GeV}^2)\,. \tag{4.58}$$

The analysis of the K*(1430) and L(1770) awaits the calculation of the unequal mass $J = 2$ polarization functions.

4.6. Systems with one light and one heavy quark

Light–heavy quark systems have been studied employing various QCD sum rule versions [56, 76–78]. In chapter 3 we have calculated in detail the two-point function for the unequal mass vector case from which the scalar, pseudoscalar, and axial vector polarization functions can be obtained. The result for the bare loop is given by (3.2) and the first-order α_s correction by (3.20) and (3.21). The nonperturbative corrections have been listed in the Appendix.

In a moment type sum rule we will have the following expression for the moment $M_n^J(\xi)$, instead of (2.32):

$$M_n^J(\xi) = A^J(n)[1 + a_n' + a_n\alpha_s + b_n\phi + c_n\phi_1]\,, \tag{4.59}$$

where a_n' is the first-order mass correction to the bare loop, ϕ is defined in (2.33) and

$$\phi_1 = -\frac{4\pi^2}{3}\frac{\langle 0|\bar{q}q|0\rangle}{m_2^3}\,, \tag{4.60}$$

where m_2 is the heavy quark mass and $\langle 0|\bar{q}q|0\rangle$ the light quark condensate. Contributions from higher dimensional operators have also been calculated [56, 78], but can be neglected. The role of the quark condensate (4.60) is very important in these systems due to the fact that the light quark condensate appears without being multiplied by the light quark mass. This induces a very large splitting between opposite parity states.

For open charm states $m_2 = m_c \cong 1.27\,\text{GeV}$ and $\phi_1 \cong 0.1$. It can easily be verified that $c_n\phi_1 \cong n^3\phi_1$ and even for low n much too large for the approximation to make sense. The method of moving to large spacelike Q^2 employed in the charmonium system does not decrease the coefficients fast enough (compared to the increase of the contributions of higher states) to make reliable calculations possible.

Borel transformed sum rules seem to be better suited for the open charm system. A very crude version, in the limit of the heavy quark mass going to infinity, has been employed by Shuryak [77]. A more careful analysis has been performed by Aliev and Eletsky [78]. It still does not appear to be possible to calculate the masses of the open charm states because of the large continuum contributions, but using the experimental value for m_D as input, the leptonic decay constant f_D can be determined. Using the definition (4.53) this gives

$$f_D \cong 200\,\text{MeV}\,, \tag{4.61}$$

which agrees well with the value found by Shuryak.

The difficulty with the moment method mentioned above is not present in the open beauty case [76]. Here we have $m_2 = m_b \cong 4.26\,\text{GeV}^2$ (normalized at $p^2 = -m_b^2$) and therefore $\phi_1 \cong 3 \times 10^{-3}$. Choosing an appropriate spacelike Q^2 ($Q^2 = 3m_b^2$) a good window of moments can be obtained, where all corrections are not too large for the approximations to make sense. All parameters are now fixed, m_b and ϕ are known from the analysis of the upsilon and charmonium systems and ϕ_1 from current algebra. The only free parameter is the continuum threshold s_0. For the nonstrange pseudoscalar channel s_0 is chosen to reproduce the open beauty state [61]. The values of s_0 for the other channels follow then quite naturally. The results are presented in table 6. The mass values are uncertain by 50–100 MeV due to the dependence on the continuum threshold. Therefore, we cannot give reliable values for the vector-pseudoscalar and axial-scalar mass differences. We predict about 700 MeV splitting between the nonstrange S- and P-wave levels. Without a quark condensate term this splitting would go down to about 350 MeV. We note that the potential model gives 500 MeV for the open charm system [79].

The result for the coupling f_B in [76] differs by a factor two from the value obtained in [78]. There are two reasons for this discrepancy:

(1) the rough θ-function model adopted for the continuum in [76] overestimates the coupling, and

(2) the too large value for the quark mass m_b used in [78] leads to an underestimate.

We have redone the analysis of [78] with the proper value (4.42b) for the on-shell beauty quark mass. This gives the value

$$f_B = 190 \pm 30\,\mathrm{MeV}\,. \tag{4.62}$$

We expect that the values of the other couplings in [76] also have to be reduced by about 30%. The resulting couplings have also been listed in table 6.

4.7. Baryons

To study these systems one constructs a composite operator made of three quarks with the appropriate quantum numbers corresponding to the baryon in question. Equation (2.6) gives a possible choice for the nucleon ($J^P = \frac{1}{2}^+$)

$$\eta_N(x) = \varepsilon_{abc}(u^a(x)C\gamma_\mu u^b(x))\gamma_5\gamma_\mu d^c(x)\,, \tag{4.63}$$

where C is the charge conjugation matrix, $C^2 = -1$, and the latin indices refer to colour. Since two quarks are always in an antisymmetric representation of SU(3) colour, they must be in a symmetric isospin state which apart from (4.63) can also be achieved by a $\sigma_{\mu\nu}$ coupling

$$\eta'_N(x) = \varepsilon_{abc}(u^a(x)C\sigma_{\mu\nu}u^b(x))\gamma_5\sigma_{\mu\nu}d^c(x)\,. \tag{4.64}$$

Restricting ourselves to currents without derivatives, (4.63) and (4.64) are the only possible nucleon currents.

Both these currents or any linear combination should couple to the nucleon. We will follow Ioffe [16] and work with (4.63). The reason for this choice is that the two-point function of (4.64) does not get any contribution from the lowest dimensional chiral symmetry breaking operators. Therefore, the resonance is expected to couple weakly to this current. This has been confirmed by the analysis of [80]. Using an arbitrary linear combination of (4.63) and (4.64), it is found that the contribution of (4.64) is small (about 2%).

A complete set of currents for the $L = 0$ octet and decuplet baryons is given by

$$\begin{aligned}
\eta_N &= \varepsilon_{abc}(u^a(x)C\gamma_\mu u^b(x))\gamma_5\gamma_\mu d^c(x)\,,\\
\eta_\Sigma &= \varepsilon_{abc}(u^a(x)C\gamma_\mu u^b(x))\gamma_5\gamma_\mu s^c(x)\,,\\
\eta_\Lambda &= \sqrt{\tfrac{2}{3}}\,\varepsilon_{abc}\{(u^a(x)C\gamma_\mu s^b(x))\gamma_5\gamma_\mu d^c(x) - (d^a(x)C\gamma_\mu s^b(x))\gamma_5\gamma_\mu u^c(x)\}\,,\\
\eta_\Xi &= -\,\varepsilon_{abc}(s^a(x)C\gamma_\mu s^b(x))\gamma_5\gamma_\mu u^c(x)\,,
\end{aligned} \tag{4.65}$$

for the octet and

$$
\begin{aligned}
&\eta^{\mu}_{\Delta} = \varepsilon_{abc}(u^a(x)C\gamma_\mu u^b(x))u^c(x)\,,\\
&\eta^{\mu}_{\Sigma^*} = \sqrt{\tfrac{1}{3}}\,\varepsilon_{abc}\{2(u^a(x)C\gamma_\mu s^b(x))u^c(x) + (u^a(x)C\gamma_\mu u^b(x))s^c(x)\}\,,\\
&\eta^{\mu}_{\Xi^*} = \sqrt{\tfrac{1}{3}}\,\varepsilon_{abc}\{2(s^a(x)C\gamma_\mu u^b(x))s^c(x) + (s^a(x)C\gamma_\mu s^b(x))u^c(x)\}\,,\\
&\eta^{\mu}_{\Omega} = \varepsilon_{abc}(s^a(x)C\gamma_\mu s^b(x))s^c(x)\,,
\end{aligned}
\tag{4.66}
$$

for the decuplet.

To illustrate the baryon case we will consider the nucleon case in some detail. It was observed by Ioffe [16] and Chung et al. [81] that the dominant power corrections come from chiral symmetry breaking operators. The two-point function for the current (4.63) has two invariant functions

$$
\mathrm{i}\int \mathrm{d}^4x\, \mathrm{e}^{\mathrm{i}qx}\langle 0|\mathrm{T}(\eta_{\mathrm{N}}(x)\bar\eta_{\mathrm{N}}(0))|0\rangle = \Pi_1(q^2) + \not{q}\,\Pi_2(q^2)\,. \tag{4.67}
$$

Counting dimensions one easily finds that $\Pi_1(q^2)$ has an odd number of dimensions while $\Pi_2(q^2)$ is even because of the factor $\not{q}$. This implies that in performing the operator product expansion the Wilson coefficients of the even dimensional operators (I, $G^a_{\mu\nu}G^a_{\mu\nu}$, $\bar q\Gamma q\bar q\Gamma q$) in $\Pi_1(q^2)$ will be proportional to the small quark mass m_q while the operator $\bar qq$ will appear without m_q and gives the dominant contribution to $\Pi_1(q^2)$, the more so since the contribution of the five-dimensional operator $\bar q\sigma_{\mu\nu}G_{\mu\nu}q$ turns out to vanish for the baryon octet [38]. All operators contribute to $\Pi_2(q^2)$ but in the limit of massless quarks there will be no contribution from the quark condensate.

Therefore the diagrams of fig. 14a contribute to $\Pi_2(q^2)$ and those of fig. 14b to $\Pi_1(q^2)$. In chapter 3 we have explained that these diagrams can most easily be calculated in coordinate space, and the expression for each of the diagrams in fig. 14 can be read off from (3.92). Transforming to momentum space by means of formula (3.89) we find

$$
\begin{aligned}
\Pi_1(q^2) &= -\frac{1}{4\pi^2}\langle\bar qq\rangle q^2\ln(-q^2) + \cdots\,,\\
\Pi_2(q^2) &= +\frac{1}{64\pi^2}q^4\ln(-q^2) + \frac{1}{32\pi^2}\left\langle\frac{\alpha_s}{\pi}G^2\right\rangle\ln(-q^2) + \frac{2\langle\bar qq\rangle^2}{3q^2} + \cdots\,.
\end{aligned}
\tag{4.68}
$$

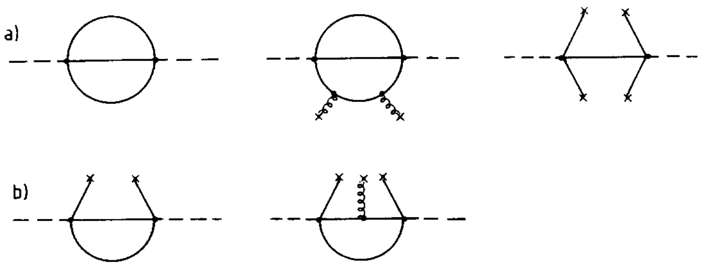


Fig. 14. Diagrams to be calculated for the Wilson coefficients of baryonic currents. The figures (a) give the contributions to $\Pi_2(q^2)$ and (b) to $\Pi_1(q^2)$ as defined by (4.67).

The gluon condensate contribution has first been calculated in [38] using the fixed-point gauge. The dots in (4.68) stand for contributions from higher dimensional operators, some of which have been calculated in [43], and α_s corrections [45, 46].

In the Appendix we have listed the expressions for the polarization functions of all $L = 0$ octet and decuplet baryons, including operators up to dimension $d = 6$ and mass corrections. On the phenomenological side of the sum rule we take the nucleon state into account with coupling λ_N to the current, therefore

$$\Pi(q) = \lambda_N^2 \frac{\not{q} + M_N}{q^2 - M_N^2} + \text{continuum}, \tag{4.69}$$

where the coefficient of $\not{q}$ gives $\Pi_2(q^2)$ and the M_N piece $\Pi_1(q^2)$. Taking the Borel transform of (4.68) and (4.69), we arrive at two independent sum rules (no continuum contribution has yet been introduced)

$$M^6 + bM^2 + \tfrac{4}{3}a^2 = 2(2\pi)^4 \lambda_N^2 \exp(-M_N^2/M^2), \tag{4.70a}$$

$$2aM^4 = 2(2\pi)^4 M_N \lambda_N^2 \exp(-M_N^2/M^2), \tag{4.70b}$$

where $b = \pi^2\langle(\alpha_s/\pi)G^2\rangle \cong 0.17\ \text{GeV}^4$ and $a = -(2\pi)^2\langle\bar{q}q\rangle \cong 0.5\ \text{GeV}^3$. In first approximation we neglect the power corrections in (4.70a) and take the ratio with (4.70b). At $M^2 \cong M_N^2$ this results in [16]

$$M_N = \{-2(2\pi)^2\langle 0|\bar{q}q|0\rangle\}^{1/3} \cong 1\ \text{GeV}, \tag{4.71a}$$

$$\lambda_N^2 = M_N^6 e/2(2\pi)^4 \cong 10^{-3}\ \text{GeV}^6. \tag{4.71b}$$

This rough calculation contains interesting physics. Not only is the estimate (4.71a) astonishingly accurate, it also shows the dependence of the baryon mass on the quark condensate ($M_N \to 0$ for $\langle\bar{q}q\rangle \to 0$). The coupling λ_N^2 of (4.71b) measures the probability for the three quarks in the proton to be at the same point, and appears as a factor in the proton decay amplitude. The results (4.71) can be improved by considering the full expressions and by including the (unfortunately very large) continuum contributions. Solving for the nucleon mass one obtains

$$M_N(M^2) = \frac{2aM^4[1 - \exp(-s_0/M^2)(s_0/M^2 + 1)]}{M^6[1 - \exp(-s_0/M^2)(s_0^2/2M^4 + s_0/M^2 + 1)] + \tfrac{4}{3}a^2 + bM^2}. \tag{4.72a}$$

The resulting expressions for the other members of the octet are (the continuum expressions are not explicitly given)

$$M_\Lambda(M^2) = \frac{-2mM^6 + 2a(3-\gamma)M^4 + \tfrac{8}{3}a^2m(2-\gamma)}{3M^6 + 2am(1-3\gamma)M^2 + 3bM^2 + \tfrac{4}{3}a^2(3+4\gamma)}, \tag{4.72b}$$

$$M_\Sigma(M^2) = \frac{2mM^6 + 2a(1+\gamma)M^4 + \tfrac{8}{3}a^2m}{M^6 - 2am(1+\gamma)M^2 + bM^2 + \tfrac{4}{3}a^2}, \tag{4.72c}$$

$$M_\Xi(M^2) = \frac{2aM^4 + 4a^2m(1+\gamma)}{M^6 + bM^2 + \tfrac{4}{3}a^2(1+\gamma)^2}, \tag{4.72d}$$

where $\gamma = \langle 0|\bar{s}s|0\rangle/\langle 0|\bar{u}u|0\rangle - 1$ and $m \equiv m_s$. A check of these formulae is given by keeping the octet pieces of the symmetry breaking and verifying that these satisfy the Gell-Mann Okubo formula.

For the decuplet states the tensor structure is more complicated since we deal with $J = \frac{3}{2}$ states. The two invariant functions proportional to $g_{\mu\nu}$ and $\not{q} g_{\mu\nu}$ will receive contributions from $J = \frac{3}{2}$ states only (at least for the Δ). On the phenomenological side we use the Rarita-Schwinger formalism for the propagator. The resulting formulae are

$$M_\Delta = \frac{\frac{20}{3}aM^4(1 - \frac{1}{2}m_0^2/M^2)}{M^6 + \frac{20}{3}a^2 - \frac{25}{18}bM^2}, \tag{4.73a}$$

$$M_{\Sigma^*}(M^2) = \frac{\frac{20}{3}aM^4(1 + \frac{1}{3}\gamma)(1 - \frac{1}{2}m_0^2/M^2) + \frac{5}{2}mM^6 + \frac{10}{3}a^2m}{M^6 + 5am(1 - \frac{1}{3}\gamma)M^2 - \frac{25}{18}bM^2 + \frac{20}{3}a^2(1 + \frac{2}{3}\gamma)}, \tag{4.73b}$$

$$M_{\Xi^*}(M^2) = \frac{\frac{20}{3}aM^4(1 + \frac{2}{3}\gamma)(1 - \frac{1}{2}m_0^2/M^2) + 5mM^6 + \frac{20}{3}a^2m(1 + \gamma)}{M^6 + 10am(1 + \frac{1}{3}\gamma)M^2 - \frac{25}{18}bM^2 + \frac{20}{3}a^2(1 + \gamma)(1 + \frac{1}{3}\gamma)}, \tag{4.73c}$$

$$M_\Omega(M^2) = \frac{\frac{20}{3}aM^4(1 + \gamma)(1 - \frac{1}{2}m_0^2/M^2) + \frac{15}{2}mM^6 + 10a^2m(1 + \gamma)^2}{M^6 + 15am(1 + \gamma)M^2 - \frac{25}{18}bM^2 + \frac{20}{3}a^2(1 + \gamma)^2}. \tag{4.73d}$$

The simplicity of the structure of these formulae reflects the fact that for the decuplet all quark pairs are in a spin one state. The parameter m_0^2 is connected with the operator $\bar{q}\sigma_{\mu\nu}G_{\mu\nu}q$ and defined by (4.13), $m_0^2 \cong 0.5$–$1.0\ \mathrm{GeV}^2$.

Even more sophisticated formulae including higher dimensional operators, anomalous dimensions, and different thresholds for the continuum contributions to the bare loop and quark condensate are given in [43]. Reference [80] contains second-order corrections in the strange quark mass. Recent calculations [45, 46] of the first order α_s corrections to the bare loop and quark condensate for the nucleon and delta suggest that these are far greater than contributions from higher dimensional operators, anomalous dimensions, etc., and it does not seem to be sensible to include the latter before the α_s corrections. We note however that this cannot be done before the discrepancies between [45] and [46] have been resolved.

A reasonable fit to the experimental masses for acceptable values of the parameters can be obtained from formulae (4.72) and (4.73). However, the continuum threshold that should fix the stability of the sum rule comes out higher than expected from the data. The five dimensional operator $\bar{q}\sigma_{\mu\nu}G_{\mu\nu}q$ (i.e., the parameter m_0^2) is essential in order to obtain the correct splitting between the octet and the decuplet. An independent determination (for instance on a lattice) of this parameter, which resembles a colour magnetic moment, is of interest.

In the more sophisticated approach of [43] better agreement with the data can be obtained at the expense of introducing further parameters. In this work information on the negative parity baryons in the 70 multiplet is also obtained in qualitative agreement with experiment.

As a final point we note that irrespective of the details the formulae (4.72) show a striking behaviour as a function of γ. As can be seen from fig. 15 the proper ordering of the octet masses requires

$$\gamma = \frac{\langle 0|\bar{s}s|0\rangle}{\langle 0|\bar{u}u|0\rangle} - 1 = -0.17 \pm 0.05\,. \tag{4.74}$$

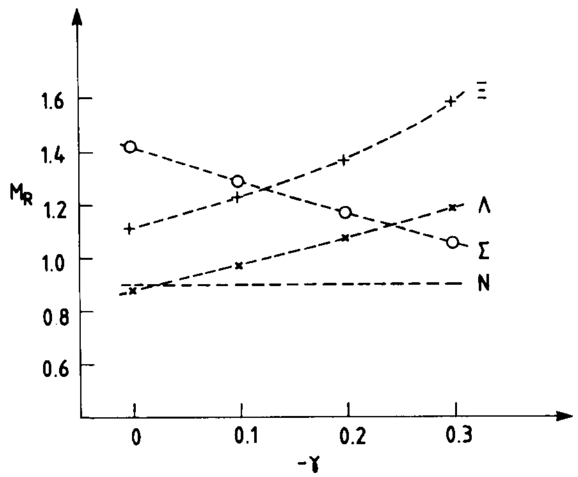


Fig. 15. Plot of the octet masses as a function of γ. Figure taken from [55].

This result is insensitive to large changes in other parameters. The value agrees with [43] and with the result obtained from mesons [57]. An independent determination would be useful.

The results for baryons are quite significant. There are however some limitations:

(1) due to the density-of-states single baryon saturation is hard to achieve, and an accuracy of better than 10% in the values for the masses seems beyond reach;

(2) higher dimensional operators are important, which introduces new parameters;

(3) perturbative contributions may be large.

4.8. Glueballs, hybrids, heavy baryons, and radial excitations

In this section we describe some work on more controversial states. Paradoxically, these are of utmost importance since they check novel and specific aspects of QCD.

1. *Glueballs*

The problem of glueballs is central to QCD. Unfortunately there is no guiding experimental evidence on these states. The more complete papers on this question are by Novikov et al. [82, 83]. In these papers several general theoretical issues are studied. In particular it is established that:

(a) Following Landau–Yang's theorem two gluon vector states do not exist. As a consequence it is speculated that these states are composed of at least three gluons and therefore should be heavier. Although this argument is appealing it does not rule out collective states to which these composite operators might not couple.

(b) In [83] the following low energy theorems are proven:

$$\mathrm{i}\int \mathrm{d}x\langle 0|\mathrm{T}\Big\{\frac{3\alpha_s}{4\pi}\,G^2(x),\frac{3\alpha_s}{4\pi}\,G^2(0)\Big\}|0\rangle \cong \frac{18}{b}\Big\langle\frac{\alpha_s}{\pi}\,G^2\Big\rangle + \mathrm{O}(m_q)\,, \tag{4.75}$$

$$\mathrm{i}\int \mathrm{d}x\langle 0|\mathrm{T}\Big\{\sum_{\mathrm{u,d,s}} m_{\mathrm{q}}\bar{q}(x)q(x),\frac{\alpha_s}{\pi}\,G^2(0)\Big\}|0\rangle \cong \frac{24}{b}\sum_{\mathrm{u,d,s}}\langle m_{\mathrm{q}}\bar{q}q\rangle + \mathrm{O}(m_{\mathrm{q}}^2)\,, \tag{4.76}$$

where $b = \frac{1}{3}(11N_c - 2N_f)$ and G^2 stands for $G^a_{\mu\nu}G^a_{\mu\nu}$. The first theorem can be used to calculate the leading power corrections to the scalar glueball two-point function generated by the current

$$j_s(x) = \frac{3\alpha_s}{4\pi} G^a_{\mu\nu}(x) G^a_{\mu\nu}(x) . \tag{4.77}$$

Using standard Borel transform methods and saturating with one resonance σ with mass m_σ plus a continuum one obtains the following sum rule

$$\frac{\pi^2 \langle 0|\alpha_s G^2|\sigma\rangle}{2m_\sigma^2 M^4} \exp(-m_\sigma^2/M^2) + \frac{1}{M^4} \int_{s_0}^{\infty} \exp(-s/M^2) s \alpha_s^2(s)\, \mathrm{d}s$$
$$= \alpha_s^2(M^2)\left[1 + \frac{1}{M^4}\left\langle \frac{\alpha_s}{\pi} G^2 \right\rangle \left(-\frac{2\pi^3}{\alpha_s(M^2)} + \frac{16\pi^4}{b\alpha_s^2(M^2)} \right) + \mathrm{O}(M^{-6})\right], \tag{4.78}$$

where the second piece of the power correction is a reflection of the low energy theorem (4.75). It can be seen (using $\alpha_s = 0.3$ at 1 GeV and the standard value for the gluon condensate) that for $M^2 < 4\ \mathrm{GeV}^2$ the bare loop does not dominate in (4.78) and therefore the usual duality which we saw for instance in the case of the ρ meson, does not hold. This is a problem that consistently plagues the states beyond the standard quark model, and is essentially due to the fact that the leading power corrections come from diagrams which have one loop integration less than the bare loop. This is not the case for two-point functions of currents which consist only of quark fields.

Although a sum rule like (4.78) points to a high mass for the scalar glueball, it can be argued qualitatively [82–84] that the mass value for scalar gluonium should be about 1.4 GeV, while pseudo-scalar gluonium is expected at about 2.2 GeV. The shift of the latter is due to the fact that the η' occupies the duality interval in that channel. Tensor gluonium comes out at about 2 GeV, which is relatively low for this high spin state. Unfortunately these predictions are relatively soft and therefore they do not constitute serious tests of the theory.

2. *Hybrids*

These states stand a better chance to be discovered since, some of them at least cannot mix with conventional states. Typical examples are $J^{PC} = 1^{-+}$ and 0^{--}. These states for any type of quarks require at least one additional gluon to form a $\bar{q}qG$ state.

Several calculations exist for light quarks [85–87] and although these differ somewhat, the prediction for the $I = 1$, $J^{PC} = 1^{-+}$ state is at about 1.5 GeV, while the 0^{--} comes out much higher. Unfortunately, the results are not free of problems: due to the sign of the gluon condensate contribution there is no stability in M^2 for the 1^{-+} state, which may indicate that this system does not bind. There is no experimental evidence for any of these states.

3. *Heavy baryons*

Shuryak [77] has made an effort to calculate these states in the limit of the heavy quark mass going to infinity. Although his results are qualitatively interesting, due to the aforementioned problems with baryons they are not very reliable. In particular the density of states and therefore the importance of the continuum is a difficult problem.

4. *Radial excitations*

The methods described in this report are very poor in determining radial excitations. The moments as well as the Borel transform are devised to ensure the dominance of the lowest lying state. For heavy quarks all higher resonances are consequently pushed into the first few moments and the resolution is very poor. We believe that the recurrences should couple strongly to operators of higher dimensions (with gluons). Some qualitative calculations for baryons with $\bar{q}qG$ currents indeed reproduce the Roper resonance but systematic calculations along these lines do not exist.

5. Three-point functions

5.1. Introduction

Although in principle there is relatively little new in going from two- to three-point functions, in practice there are many interesting problems. The types of questions which we will discuss in this chapter are:

(1) decays of hadrons into two γ (e.g. $\eta_c \to 2\gamma$);
(2) couplings of two hadrons to an on-shell γ (e.g. $J/\psi \to \eta_c\gamma$)
(3) couplings of three hadrons. As we will see below this is particularly simple and instructive when one of the hadrons is a Goldstone boson and the other two have similar masses (e.g. $g_{\omega\rho\pi}$, $g_{\pi NN}$);
(4) form factors.

In general the procedure is similar to the two-point function case and often information on the latter is required to cancel the couplings to the current. Consider for instance the transition $J/\psi \to \eta_c\gamma$. The phenomenological side of the sum rule for this transition contains, apart from the three particle coupling, the couplings of the J/ψ and η_c to their respective currents. A double moment analysis is performed in the vector and the pseudoscalar channels. From the two-point function analysis it is known at which moments the J/ψ and η_c are dominant and these results can be used to cancel the couplings to the currents.

Analyses along these lines were already performed by Novikov et al. [2] who realized that the decay of heavy quarkonium into gluons and/or photons is dominated by the bare loop diagram. Radiative transitions in charmonium but always within the bare loop approximation have been discussed in [88]. In the next sections we discuss these decays in detail including perturbative and nonperturbative corrections. As a second application we will discuss couplings of Goldstone bosons to baryons ($g_{\pi NN}$, $g_{\Delta\pi N}$) and mesons ($g_{\omega\rho\pi}$) [89] based on the observation that the $\bar{q}q$ condensate contributions dominate in the chiral limit and balance with the pion contribution on the phenomenological side of the sum rule (see chapter (3.3f)). Finally we will discuss conventional sum rule approaches to form factors and couplings.

5.2. Decays of heavy quark systems: $\eta_c \to 2\gamma$ and other radiative decays

We consider the three-point function

$$A_{\mu\nu}(q, q_1, q_2) = e^2 Q_c^2 \int d^4x\, d^4y \exp[-i(qx + q_2 y)]\langle 0|T(j_\mu(0) j_5(x) j_\nu(y))|0\rangle$$
$$= 3\alpha Q_c^2 \varepsilon_{\mu\nu\alpha\beta} q_1^\alpha q_2^\beta A(q^2, q_1^2, q_2^2)\,, \qquad q = q_1 + q_2\,. \tag{5.1}$$

The currents in (5.1) are the heavy quark vector ($\bar{c}\gamma_\mu c$) and pseudoscalar ($\mathrm{i}\bar{c}\gamma_5 c$) currents, and Q_c is the charge of the quark. First we will consider the case when the external momenta q_1^2 and q_2^2 of the vector currents are equal to zero. In this way, (5.1) describes the decay of heavy quark pseudoscalar mesons into two (real) photons, e.g. $\eta_c \to 2\gamma$. This case is the simplest for several reasons:

(1) the heavy quark mass ensures asymptotic freedom for the quark even at $q^2 = 0$ for the pseudoscalar current;

(2) masslessness of the physical photons simplifies the calculations considerably;

(3) the dominant nonperturbative effects will be due to the gluon condensate for the same reasons as in the case of the two-point functions of heavy quark currents.

The diagrams to be calculated are given in fig. 16. Diagram (a) is the bare loop contribution which can be easily calculated and gives [2, 88] in terms of a dispersion integral

$$A_0(q^2) = \frac{1}{\pi} \int_{4m^2}^{\infty} \mathrm{d}s \, \frac{2m}{s(s-q^2)} \ln \frac{1+u}{1-u} , \tag{5.2}$$

where $u^2 = 1 - 4m^2/s$ with m the quark mass. The other six diagrams in fig. 16 give the first-order α_s correction to the amplitude. The calculation of these diagrams is complicated and can be done most easily by considering the imaginary part in the q^2 channel. The dominant contribution is the Coulombic piece ($\cong 1/u$) (diagram (b)). We have only taken the Coulombic contribution plus the next order ($\cong u^0$) into account and neglected all higher order corrections in u. In this case we get

$$\mathrm{Im}\, A_1(s) = \frac{2m}{s} \frac{4}{3} \frac{\alpha_s}{\pi} (2u) \left[\frac{\pi^2}{2u} - 2 + 2C \right] , \tag{5.3}$$

where $C \cong -0.3$ is the sum of all contributions of zeroth order in u not contained in the Coulombic piece.

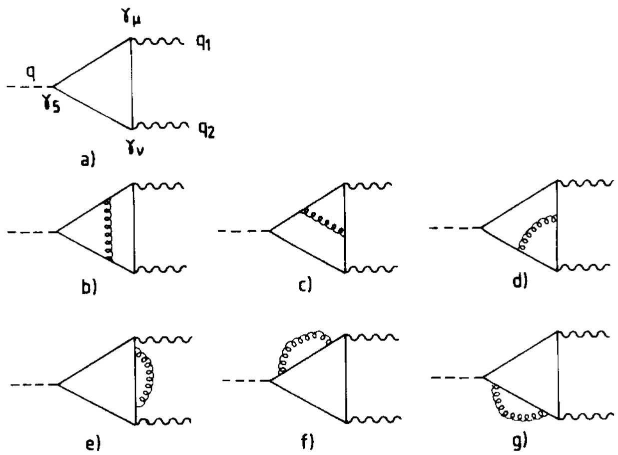


Fig. 16. Contributions to the three-point function to first order in α_s. Wavy lines depict vector currents, dashed lines pseudoscalar currents, solid lines quarks, and curly lines gluons.

The nonperturbative gluon condensate contributions in the operator product expansion of the T-ordered product in (5.1) are obtained from the diagrams in fig. 16 by cutting the gluon line as explained in chapter 2. Since only one loop integration is involved the calculations although complicated can be performed analytically. We get for the Wilson coefficient

$$C_G(q^2) = \frac{(u^2-1)}{8\pi u^4}\left[\frac{33u^6 - 13u^4 - u^2 - 3}{2u}\ln\frac{1+u}{1-u} - 33u^4 + 2u^2 + 3\right], \tag{5.4}$$

where $u^2 = 1 - 4m^2/q^2$. This expression calculated in [90] has been confirmed by various authors [91–93]. The contribution to $A(q^2)$ is given by

$$C_G(q^2)\phi = C_G(q^2)\frac{4\pi^2}{9}\langle 0|\frac{\alpha_s}{\pi}G^a_{\mu\nu}G^a_{\mu\nu}|0\rangle/(4m^2)^2\,. \tag{5.5}$$

The values of ϕ and m_c are given in chapter 4.

To apply these formulae to the decay $\eta_c \to 2\gamma$ we follow the same procedure as in the two-point function case and define moments by differentiating $A(Q^2)$ at $Q^2 = -q^2 = 0$. This leads to the following moment equation

$$M_n = \frac{2}{\pi}\frac{1}{m^{2n+1}}\frac{n!n!}{(2n+2)!}[1 + a_n^{(3)}(\mathrm{P})\alpha_s + b_n^{(3)}(\mathrm{P})\phi]\,, \tag{5.6}$$

with

$$a_n^{(3)}(\mathrm{P}) = \tfrac{4}{3}\left[\pi\frac{(2n+1)!}{n!n!2^{2n+1}} + \frac{4}{\pi}\frac{n+1}{2n+3}(-1+C) - \frac{3(2n+1)\ln 2}{2\pi}\right], \tag{5.7}$$

and

$$b_n^{(3)}(\mathrm{P}) = -\frac{n+1}{2n+3}n(n^2+n+2)\,. \tag{5.8}$$

The last term in $a_n^{(3)}$ results from the mass renormalization at the point $p^2 = -m^2$, consequently m in (5.6) is $m(p^2 = -m^2)$. It can be seen from (5.8) that the nonperturbative contributions b_n grow very fast with n. As in the two-point function case, increasing n means moving closer to the resonance region in the q^2 (pseudoscalar) channel.

To construct the phenomenological side of the sum rule we adopt a narrow resonance approximation in this channel and include two resonances

$$\mathrm{Im}\,A(q^2) = \pi ghM_{\eta_c}\delta(s - M^2_{\eta_c}) + \pi g'h'M_{\eta'_c}\delta(s - M^2_{\eta'_c})\,, \tag{5.9}$$

where $g(g')$ are the couplings of the $\eta_c(\eta'_c)$ to the pseudoscalar currents ($\langle 0|\mathrm{i}\bar{c}\gamma^5 c|P\rangle = M^2_{\mathrm{P}}g$) and $h(h')$ determines the $\eta_c \to 2\gamma(\eta'_c \to 2\gamma)$ width (see fig. 17)

$$\Gamma(\eta_c \to 2\gamma) = (3Q_c^2\alpha)^2h^2M_{\eta_c}/64\pi\,. \tag{5.10}$$

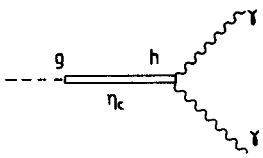


Fig. 17. Graphical representation of the sum rule for $\eta_c \to 2\gamma$. g is the coupling of the meson to the pseudoscalar current and h the coupling to the photons.

From (5.9) we can calculate phenomenological expressions for the moments M_n. Equating with (5.6) expresses the product of the couplings as a function of α_s, ϕ, and the quark mass m.

$$gh + g'h'\left(\frac{M_{\eta_c}}{M_{\eta_c'}}\right)^{2n+1} = \frac{2}{\pi}\left(\frac{M_{\eta_c}}{m}\right)^{2n+1} \frac{n!n!}{(2n+2)!}[1 + a_n^{(3)}(\mathrm{P})\alpha_s + b_n^{(3)}(\mathrm{P})\phi] . \tag{5.11}$$

To eliminate the couplings g we return to the two-point function case (2.32), which gave a similar moment equation for the couplings g^2. Saturating the imaginary part with two resonances we get at $Q_0^2 = 0$

$$g^2 + g'^2\left(\frac{M_{\eta_c}}{M_{\eta_c'}}\right)^{2k} = \frac{3}{8\pi^2}\left(\frac{M_{\eta_c}}{m}\right)^{2k} \frac{k!(k-1)!}{(2k+1)!}[1 + a_k^{\mathrm{P}}\alpha_s + b_k^{\mathrm{P}}\phi] , \tag{5.12}$$

with a_n^{P} and b_n^{P} given in table 1 of [28].

Experimentally both M_{η_c} and $M_{\eta_c'}$ are known [94] and one can easily estimate the contamination of the η_c' resonance in the sum rules (5.11) and (5.12) as a function of n. In analogy to the vector meson case where the coupling to the current is directly related to the e^+e^- width we expect that $g'^2 \cong \frac{1}{3}g^2$, and it can be easily verified that for $n \geq 4$ the contamination of the η_c' in (5.11) and (5.12) is less than 10%. From the discussion in chapter 4 we know that at $Q^2 = 0$ moments four and five are suitable for determining the coupling and/or mass of the resonance. The contamination by higher resonances is small at these moments while at the same time the perturbative and nonperturbative contributions are small compared to the bare loop. If we want to use higher moments we have to evaluate the moments (5.8) at spacelike $Q^2 \neq 0$. Neglecting the higher resonances and after some algebra we obtain

$$h^2 = \frac{32}{3}\left(\frac{M_{\eta_c}}{m}\right)^{2n} \frac{n!n!}{(2n+2)!}\frac{2n+3}{n+1}[1 + (2a_n^{(3)}(\mathrm{P}) - a_{n+1}^{\mathrm{P}})\alpha_s + (2b_n^{(3)}(\mathrm{P}) - b_{n+1}^{\mathrm{P}})\phi] . \tag{5.13}$$

We have chosen $k = n + 1$ in (5.12) since as can be seen in table 2 the coefficients a_n and b_n are similar in the two cases. Substituting $n = 4, 5$ in (5.13) we find ($\alpha_s = 0.2$)

$$h^2 = 3.3 \pm 0.3 ,$$

and for the $\eta_c \to 2\gamma$ width using (5.10)

$$\Gamma(\eta_c \to 2\gamma) = (4.6 \pm 0.4)\,\mathrm{keV} , \tag{5.14}$$

Table 2
Values of the coefficients a_n^{P}, b_n^{P}, $a_n^{(3)}(\mathrm{P})$, and $b_n^{(3)}(\mathrm{P})$ for the decay $\eta_c \to 2\gamma$

n	a_n^{P}	b_n^{P}	$a_n^{(3)}(\mathrm{P})$	$b_n^{(3)}(\mathrm{P})$
1	0.28	2.4	0.94	−1.6
2	0.74	3.4	0.79	−6.9
3	0.75	0.0	0.52	−18.7
4	0.57	−10.9	0.19	−40.0
5	0.28	−32.3	−0.19	−73.8
6	−0.07	−67.2	−0.61	−123.2
7	−0.47	−118.6	−1.06	−191.1
8	−0.90	−189.5	−1.55	−280.4

which is less than one would naively expect from the ratio

$$\frac{\Gamma(\eta_c \to 2\gamma)}{\Gamma(J/\psi \to e^+e^-)} = \tfrac{4}{3}\,.$$

In this ratio the wave functions of the η_c and J/ψ at the origin have been assumed to be equal and α_s corrections have been neglected. It is known [95] that for some of these decays large corrections should exist in view of the fact that the experimental value for $J/\psi \to \eta_c\gamma$ is much less than one would expect from these simple considerations. We will come back to $J/\psi \to \eta_c\gamma$ later. Our result (5.14) for $\eta_c \to 2\gamma$ indicates that the wave functions at the origin for the J/ψ and η_c may differ by 40%. We find from (5.14)

$$|R_{1S}(0)|^2 = 0.33\ \mathrm{GeV}^3\,, \tag{5.15a}$$

while from $J/\psi \to e^+e^-$ one has

$$|R_{3S}(0)|^2 = 0.55\ \mathrm{GeV}^3\,. \tag{5.15b}$$

The value (5.14) is about 10% higher than in our original paper [90], due to the new values of m_c and α_s. Due to the high power of m in (5.13) the result is very sensitive to the quark mass. The $\eta_c \to 2\gamma$ widths given in [91–93] are higher than (5.14). In these papers the perturbative corrections are not included.

Two photon decays of the P and D states of charmonium can be studied in a similar way. The pseudoscalar current in (5.1) has to be replaced by the appropriate P or D wave current. In [2] these decays have been studied in the bare loop approximation.

We have obtained the following expression for the Wilson coefficient C_G for the decay $0^{++} \to 2\gamma$

$$C_G(0^{++} \to 2\gamma) = \frac{u^2-1}{8\pi u^2}\left[\frac{9-5u^2-9u^4+21u^6}{2u}\ln\frac{1+u}{1-u} - 9 + 2u^2 - 21u^4\right]. \tag{5.16}$$

The amplitude for this decay is

$$A_{\mu\nu}(q, q_1, q_2) = 3\alpha Q_c^2[q_{2\mu}q_{1\nu} - (q_1\cdot q_2)g_{\mu\nu}]A(q^2)\,. \tag{5.17}$$

For the perturbative part of $A(q^2)$ we have (taking only the Coulombic piece into account in the first order α_s correction)

$$\operatorname{Im} A(q^2) = \frac{2m}{s} u^2 \left[\ln \frac{1+u}{1-u} + \frac{4}{3} \frac{\alpha_s}{\pi} (2u) \left(\frac{\pi^2}{2u} - 2 \right) \right], \tag{5.18}$$

which leads to the following moment equation:

$$M_n = \frac{4}{\pi} \frac{1}{(m)^{2n+1}} \frac{n!n!}{(2n+4)!} (3n+4)[1 + a_n^{(3)}(\mathrm{S})\alpha_s + b_n^{(3)}(\mathrm{S})\phi], \tag{5.19}$$

with

$$a_n^{(3)}(\mathrm{S}) = \frac{4}{3} \left[\pi \frac{(2n+3)!2^{-2n-2}}{n!(n+1)!(3n+4)} + \frac{4}{\pi} \frac{(n+1)(n+2)}{(2n+5)(3n+4)} - \frac{3(2n+1)\ln 2}{2\pi} \right], \tag{5.19a}$$

$$b_n^{(3)}(\mathrm{S}) = -(n+1)(n+2) \frac{9n^3 + 43n^2 + 64n + 24}{(2n+5)(3n+4)}. \tag{5.19b}$$

As in the two-point function case we find that the nonperturbative contribution for the P states has an extra factor of 3 compared to the S states. From the numerical values in table 3 it can be seen that we can use moments 2, 3 and 4. Saturating $\operatorname{Im} A(s)$ by a single resonance

$$\operatorname{Im} A(s) = \pi g h_+ M_{\chi_0}^3 \delta(s - M_{\chi_0}^2), \tag{5.20}$$

and using the two-point function results ([28] and chapter 2) to eliminate the coupling g we find

$$h_+^2 = \frac{64}{9} \left(\frac{M_{\chi_0}}{m} \right)^{4n+2-2k} \left[\frac{n!n!}{(2n+4)!} (3n+4) \right]^2 \frac{(2k+3)!}{(k-1)!(k+1)!}$$
$$\times [1 + (2a_n^{(3)}(\mathrm{S}) - a_k^{\mathrm{S}})\alpha_s + (2b_n^{(3)}(\mathrm{S}) - b_k^{\mathrm{S}})\phi]. \tag{5.21}$$

Table 3
Values of the coefficients a_n^{S}, b_n^{S}, $a_n^{(3)}(\mathrm{S})$, and $b_n^{(3)}(\mathrm{S})$ for the decay $\chi_0 \to 2\gamma$

n	a_n^{S}	b_n^{S}	$a_n^{(3)}(\mathrm{S})$	$b_n^{(3)}(\mathrm{S})$
1	0.52	−8.6	1.13	−17.1
2	0.92	−34.7	0.77	−52.8
3	0.79	−87.3	0.32	−118.3
4	0.46	−175.4	−0.18	−222.7
5	0.04	−308.0	−0.72	−374.9
6	−0.43	−494.1	−1.29	−584.0
7	−0.96	−742.7	−1.89	−858.8
8	−1.51	−1062.9	−2.50	−1208.6

The numerical values of the coefficients a_n and b_n are given in table 3. For $n = 3$ and $k = 4$ we find

$$h_+^2 = 2.30 \pm 0.3\,, \tag{5.22}$$

and consequently

$$\Gamma(\chi_0 \to 2\gamma) = (3Q_c^2\alpha)^2 h_+^2 M_{\chi_0}/64\pi = 3.7 \pm 0.5\,\mathrm{keV}\,. \tag{5.23}$$

Experimentally no value has been given so far for this decay. No attempts have yet been made to calculate other two photon decays by this method.

For q_1^2 or q_2^2 unequal zero (5.1) can be used to describe the decay $J/\psi \to \eta_c\gamma$, and similarly radiative decays of the charmonium P states into $J/\psi\gamma$. In this case the sum rules are applied simultaneously in the pseudoscalar (η_c) channel and the massive vector (J/ψ) channel. In the bare loop approximation these decays have been studied in [88]. A refinement of these results is greatly hampered by the complexity of the perturbative calculations. The diagrams of fig. 16 have to be calculated for the momenta of two of the external legs unequal zero. No expressions have so far been given in the literature. The generalization of the nonperturbative calculations is relatively easy, and in the following we will only take these into account, assuming that the perturbative corrections are not too large. The magnitude of the nonperturbative corrections then determines which moments can be used for determining the width. Analyses along these lines are performed in [96] and [97].

The width of the decay $J/\psi \to \eta_c\gamma$, is given by

$$\Gamma(J/\psi \to \eta_c\gamma) = \frac{\alpha Q_c^2}{24}|A(J/\psi \to \eta_c\gamma)|^2 \frac{m_\psi^3}{m_{\eta_c}^2}\left(1 - \frac{m_{\eta_c}^2}{m_\psi^2}\right)^3 . \tag{5.24}$$

The amplitude for the three-point function (5.1) with two heavy quark currents and one electromagnetic current now reads

$$A_{\mu\nu}(q, q_1, q_2) = 3\frac{Q_c e}{4\pi}\varepsilon_{\mu\nu\alpha\beta}q_1^\alpha q_2^\beta A(q^2, q_1^2, q_2^2)\,. \tag{5.25}$$

Calculating double moments by differentiating $A(q^2, q_1^2, q_2^2)$ with respect to q^2 and q_1^2 we find

$$M_{ij} = \frac{2}{\pi}\frac{1}{(m)^{2k+1}}\frac{\Gamma(k+1)\Gamma(k+1)}{\Gamma(2k+3)}[1 + b_{ij}\phi]\,. \tag{5.26}$$

where $k = i + j$, ϕ is defined by (2.33), and the coefficient b_{ij} reads

$$b_{ij} = -\frac{k+1}{2k+3}[k(k^2 + 3k + 4) + 2(i - j)(k + 1) + 4ij]\,. \tag{5.27}$$

The subscript j refers to the jth moment in the pseudoscalar channel and i to the ith moment in the vector channel. For $i = 0$ (5.27) reduces to (5.8). From (5.27) one easily finds that the power corrections in (5.26) grow rapidly with i and j, and are less than about 30% of the bare loop contribution for $k=(i+j)\leq 6$. On the phenomenological side we have for $A(q^2, q_1^2, q_2^2)$

$$A(q^2, q_1^2, q_2^2) = \frac{4\pi}{3Q_c} g \frac{m_{\eta_c}}{(q^2 - m_{\eta_c}^2)} \frac{m_\psi^2}{g_\psi} \frac{A(J/\psi \to \eta_c\gamma)}{(q_1^2 - m_\psi^2)} + \cdots, \tag{5.28}$$

where g is the usual pseudoscalar decay constant defined as in (5.9) and g_ψ is directly related to the e^+e^- width of J/ψ,

$$\Gamma(J/\psi \to e^+e^-) = (4\pi\alpha^2/3g_\psi^2)m_\psi. \tag{5.29}$$

Taking moments of (5.28) and combining with (5.26) we find

$$A(J/\psi \to \eta_c\gamma) = \frac{g_\psi}{2\pi} \frac{(m_{\eta_c})^{2j+1}}{g} (m_\psi)^{2i} M_{ij}. \tag{5.30}$$

The two-point function results can now be used to cancel the couplings g and g_ψ. Substituting the numerical values for the various parameters for $i, j = 2, 3$, we find

$$A(J/\psi \to \eta_c\gamma) \cong 4.$$

The error in this number due to contamination by higher resonance contributions is estimated to be about 50%. From (5.24) we then finally obtain

$$\Gamma(J/\psi \to \eta_c\gamma) = 2.7\,\mathrm{keV} \pm 50\%, \tag{5.31}$$

which is a factor 2–3 higher than the experimental value [98]. The value (5.31) is an upper limit because of the higher resonance contributions in (5.28).*

5.3. The pion nucleon coupling constant $g_{\pi NN}$

In this and the following two sections we will consider hadron couplings to Goldstone bosons [89] (in particular pions). These couplings can be calculated in two ways: via two-point functions and via three-point functions. In both cases the special nature of Goldstone bosons is exploited to select certain operators in the operator product expansion for the product of currents, similar to the derivation of the PCAC relation in chapter 3. It may be worthwhile to emphasize at this point that when q^2 (i.e., the momentum flowing through the Goldstone boson current) is large, like in the form factor case the situation is completely different.

Consider the three-point function constructed of two baryon currents $\eta_B(x)$ and the pseudoscalar meson current $j_5(x)$

$$A(p, p', q) = \int \mathrm{d}x \int \mathrm{d}y \langle 0|\mathrm{T}(\eta_B(x) j_5(y) \bar{\eta}_B(0))|0\rangle \exp(\mathrm{i}p'x - \mathrm{i}qy). \tag{5.32}$$

* In a recent paper [111] Beilin and Radyushkin have also calculated the first order α_s correction to $J/\psi \to \eta_c\gamma$. This does not change the value (5.31), but reduces the error to about 20%. Therefore, the theoretical prediction for $\Gamma(J/\psi \to \eta_c\gamma)$ seems to be definitely larger than 2 keV, in clear disagreement with the present experimental data.

For the π^0 we choose the current

$$j_5(x) = \bar{u}(x)\,\mathrm{i}\gamma_5 u(x) - \bar{d}(x)\,\mathrm{i}\gamma_5 d(x)\,. \tag{5.33}$$

Phenomenologically the pion–baryon coupling can be represented by the diagram in fig. 18, λ_B is the coupling of the baryon to its current and g_P is the coupling of the meson to its current. For the neutral pion we have

$$g_\mathrm{P} = \frac{f_\pi}{\sqrt{2}}\frac{m_\pi^2}{m_\mathrm{q}}, \tag{5.34}$$

with $f_\pi = 133\,\mathrm{MeV}$ the pion decay constant and m_q the quark mass. We assume as usual that each channel is saturated by a single resonance. For the transition $\mathrm{B}\to\pi\mathrm{B}$ with B a $J=\frac{1}{2}$ baryon we then get for the phenomenological side of $A(p,p',q)$ (with $p^2 = p'^2$)

$$A(p,p',q) = \lambda_\mathrm{B}^2 \frac{M_\mathrm{B}}{(p^2 - M_\mathrm{B}^2)^2}(\not{q}\,\mathrm{i}\gamma_5) g_{\pi\mathrm{BB}} \frac{1}{q^2 - m_\pi^2}\frac{f_\pi}{\sqrt{2}}\frac{m_\pi^2}{m_\mathrm{q}}, \tag{5.35}$$

which is just the product of the two fermion propagators, the pion propagator and the various couplings. Equation (5.35) results from the effective Lagrangean $\mathscr{L}(\pi\mathrm{BB}) = g_{\pi\mathrm{BB}}\bar{B}\,\mathrm{i}\gamma_5(\tau\cdot\pi)B$. For baryons with higher spin, e.g., the $\Delta(1232)$ a similar structure is obtained multiplied by the appropriate projection operator (see section (5.4)). Due to the pion there will always be a $1/q^2$ pole on the phenomenological side (neglecting the pion mass).

Since the operator product expansion is only valid in the deep Euclidean region we have to take $p^2 = -Q^2$ with Q^2 large and spacelike in (5.35). To get rid of all possible subtraction constants and improve the saturation by the lowest lying baryon resonance we will apply the Borel transform (2.36) to $A(p,p',q)$ with respect to Q^2. This gives for (5.35)

$$\lambda_\mathrm{B}^2 \frac{\exp(-M_\mathrm{B}^2/M^2)}{M^4} M_\mathrm{B}(\not{q}\,\mathrm{i}\gamma_5) g_{\pi\mathrm{BB}} \frac{1}{q^2 - m_\pi^2}\frac{f_\pi}{\sqrt{2}}\frac{m_\pi^2}{m_\mathrm{q}}, \tag{5.36}$$

where M^2 is the new mass scale connected with Q^2 via the Borel transform.

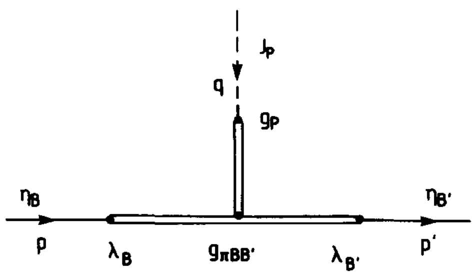


Fig. 18. Diagrammatic representation of the sum rule for vertex functions. η_B and $\eta_{\mathrm{B}'}$ are baryon currents, J_P is the pseudoscalar meson current. λ_B and $\lambda_{\mathrm{B}'}$ are the couplings of the lowest-lying baryons to the currents, g_P is the coupling of the pseudoscalar meson to the current, and $g_{\pi\mathrm{BB}'}$ is the three-point coupling.

To determine the pion–baryon coupling constant theoretically we will follow our observation in deriving the PCAC relation in section (3.3) and identify the leading terms in the operator product expansion which have a $1/q^2$ term with (5.35), i.e., we will determine the Wilson coefficients which have a $1/q^2$ term for the lowest dimensional operators in the OPE. Perturbative diagrams like the ones given in fig. 19 behave logarithmically. Next we have the diagrams corresponding to the Wilson coefficients of the three-dimensional operator $\bar{q}q$ (with q the light quark field) which (in lowest order in α_s) can be obtained by cutting one quark line of fig. 19a. This gives diagrams like in fig. 20. It can easily be seen that only the diagrams of fig. 20a and 20b have a $1/q^2$ term.

The Wilson coefficient of the four-dimensional operator $G^a_{\mu\nu}G^a_{\mu\nu}$ is obtained from all first-order perturbative diagrams in fig. 19b by cutting the gluon line. Some of these diagrams will have a $1/q^2$ term but all diagrams belonging to this operator can be neglected for the following reason. Equation (5.35) and (5.36) are proportional to $\not{q}$, so the theoretical expression for $A(p, p', q)$ will also be proportional to $\not{q}$. From (5.32) it can be seen that the total number of dimensions of $A(p, p', q)$ is even; therefore taking into account the overall $\not{q}$ factor the Wilson coefficients of all even dimensional operators will be proportional to the small mass m_q, while the operator $\bar{q}q$ does not have this m_q and its contribution will be greatly enhanced compared to the other operators. This is similar to the $\bar{q}q$ dominance in the baryon two-point function case. This implies that up to dimension four we only have to take into account quark condensate contributions and that we can neglect all perturbative and gluon condensate contributions.

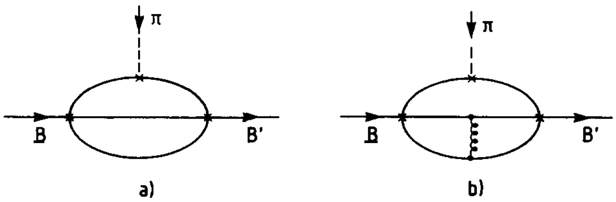


Fig. 19. The correlation function of two baryon currents and one meson current. Figure (a) is the bare loop contribution and fig. (b) a first-order α_s correction diagram.

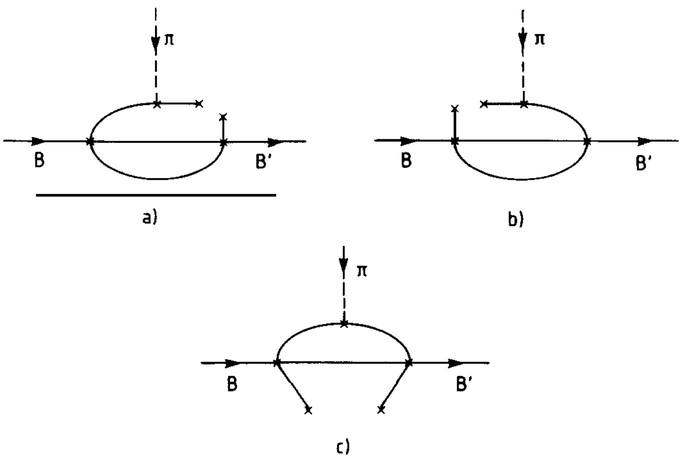


Fig. 20. Diagrams that contribute to the Wilson coefficient of the operator $\bar{q}q$. Only diagrams (a) and (b) have a $1/q^2$ term with q the pion momentum.

To actually calculate the diagrams of figs. 20a and 20b we again make use of the fact that the Wilson coefficients in the OPE are independent of the state and we can sandwich the product of currents by two single quark states to select the quark condensate Wilson coefficients. Let us apply this to the calculation of the pion–nucleon coupling constant $g_{\pi NN}$

$$A(p, p', q) = C_{\rm d}\langle 0|\bar{d}d|0\rangle + C_{\rm u}\langle 0|\bar{u}u|0\rangle + \cdots . \tag{5.37}$$

To select the coefficient $C_{\rm d}$ we sandwich by single d-quark states with momentum p_3 and colour c, c':

$$C_{\rm q}\delta_{c'c}\bar{d}(p_3)d(p_3) = \int\int {\rm d}x\, {\rm d}y\, \exp({\rm i}p'x - {\rm i}qy)\langle p_3^c|\eta_{\rm N}(x)J_{\rm P}(y)\bar{\eta}_{\rm N}(0)|p_3^{c'}\rangle , \tag{5.38}$$

where the nucleon current $\eta_{\rm N}(x)$ is given by (2.6). The calculation of the right-hand side of this expression is cumbersome but straightforward and amounts to calculating the diagrams of fig. 20 by ordinary Feynman diagrammatic techniques. We find (in the limit $p_3 \to 0$)

$$C_{\rm d} = -\frac{1}{3}\frac{1}{(2\pi)^2}\frac{1}{q^2}\left[\{(p'^2 - p^2 - q^2)\not{p} + q^2\not{q}\}\ln\frac{\Lambda^2}{-p^2} + \{p'^2\not{q} + (p'^2 - p^2 + q^2)\not{p}'\}\ln\frac{\Lambda^2}{-p'^2}\right]({\rm i}\gamma_5) . \tag{5.39}$$

The logarithmic factors result from the loop integration with Λ the ultraviolet cutoff. Taking the limit $p'^2 \to p^2$ and collecting only the $1/q^2$ terms we get

$$C_{\rm d} = -\frac{2}{3}\frac{1}{(2\pi)^2}\frac{\not{q}}{q^2}({\rm i}\gamma_5)p^2\ln\frac{\Lambda^2}{-p^2} . \tag{5.40}$$

Similarly we find for $C_{\rm u}$ in the same limit

$$C_{\rm u} = -\frac{10}{3}\frac{1}{(2\pi)^2}\frac{\not{q}}{q^2}({\rm i}\gamma_5)p^2\ln\frac{\Lambda^2}{-p^2} . \tag{5.41}$$

And the total result with $\langle 0|\bar{u}u|0\rangle = \langle 0|\bar{d}d|0\rangle = \langle 0|\bar{q}q|0\rangle$ is

$$C_{\rm u}\langle 0|\bar{u}u|0\rangle + C_{\rm d}\langle 0|\bar{d}d|0\rangle = -\frac{1}{\pi^2}\frac{\not{q}}{q^2}({\rm i}\gamma_5)\left(p^2\ln\frac{\Lambda^2}{-p^2}\right)\langle 0|\bar{q}q|0\rangle . \tag{5.42}$$

To be able to compare with the phenomenological side given by (5.36) with B = N we take $p^2 = -Q^2$ and apply the Borel transform with respect to Q^2. Notice that we have assumed that the Borel singularities in the limit of equal baryon momenta are given by (5.40) and (5.41). Equating with (5.36) we find

$$\lambda_{\rm N}^2\frac{\exp(-M_{\rm N}^2/M^2)}{M^4}M_{\rm N}g_{\pi NN}\frac{f_\pi}{\sqrt{2}}m_\pi^2 = \frac{1}{(2\pi)^2}M^2(-4m_{\rm q}\langle 0|\bar{q}q|0\rangle) . \tag{5.43}$$

Using the PCAC relation $f_\pi^2 m_\pi^2 = -4m_q\langle 0|\bar{q}q|0\rangle$ and rearranging the terms we obtain for the pion–nucleon coupling constant

$$g_{\pi NN} = \frac{\sqrt{2}}{(2\pi)^2} f_\pi \frac{M^6}{M_N} \frac{1}{\lambda_N^2} \exp(M_N^2/M^2) . \tag{5.44}$$

The only unknown on the right-hand side of this relation is the coupling of the nucleon to the current, which we can take from the baryon analysis in section (4.6). Using (4.70a) we get the final result for $g_{\pi NN}$

$$g_{\pi NN} = 2(2\pi)^2\sqrt{2}\frac{f_\pi}{M_N}\left(1 + \frac{4}{3}\frac{a^2}{M^6} + \frac{b}{M^4}\right)^{-1} , \tag{5.45}$$

where $a = -(2\pi)^2\langle 0|\bar{q}q|0\rangle$, and $b = \pi^2\langle 0|(\alpha_s/\pi)G^a_{\mu\nu}G^a_{\mu\nu}|0\rangle$. In first approximation, i.e., neglecting the power corrections ($a = b = 0$), (5.45) is completely independent of M. Even with the power corrections the dependence in the region $M \cong M_N$ is rather weak. For M^2 very small the power corrections blow up and relation (5.45) is no longer valid. For M^2 very large the nucleon will not dominate the dispersion integral and higher excitations and continuum contributions would have to be taken into account. For $M^2 \cong M_N^2$ the power corrections are still manageable ($\cong$30%) and the nucleon gives the dominant contribution to the dispersion integral. Using the canonical values for the parameters a and b and substituting $M = 1$ GeV into (5.45) we get the prediction

$$g_{\pi NN} \cong 12.5 , \tag{5.46}$$

which compares very favourably with the experimental result $g_{\pi NN} = 13.5$ [99]. In fact the uncertainties in the quark and gluon condensate values are such (see section 4.6) that the prediction (5.46) has an error of about 20%.

We note that the relation (5.45) has a completely different structure (even in zeroth order) than the Goldberger–Treiman relation:

$$g_{\pi NN} = \sqrt{2}M_N/f_\pi , \tag{5.47}$$

which yields a high value for $g_{\pi NN}$ because M_N is relatively large compared to f_π while in (5.45) the large numerical coefficient ensures that $g_{\pi NN}$ is large. We also note that the Goldberger–Treiman relation is derived by soft pion techniques, i.e., q^2 is very small ($\cong m_\pi^2$) while in our case q^2 is large to ensure that the operator product expansion can be applied.

Instead of (4.70a) we can also use (4.70b) to eliminate the coupling λ_N in (5.43). Combining the resulting equation

$$g_{\pi NN} = 4m_q M^2\sqrt{2}/f_\pi m_\pi^2 \tag{5.48}$$

with (5.45) we find an estimate for the light quark mass $m_q = \frac{1}{2}(m_u + m_d)$,

$$m_q(M^2) = \frac{(2\pi)^2 f_\pi^2 m_\pi^2}{2M^2 M_N} \cong 7\,\text{MeV} \quad \text{at} \quad M^2 \cong 1\,\text{GeV} , \tag{5.49}$$

which is the same as found from QCD sum rules for the axial divergence [3].

Let us now consider the calculation of the pion–nucleon coupling constant via the two-point function. In this case we use the fact that $\langle 0|\bar{q}\gamma_5 q|\pi\rangle \neq 0$ while for all other operators this matrix element vanishes. Therefore, sandwiching the two-point function of the two baryon currents by the vacuum and a one-pion state will select the operator $\bar{q}\gamma_5 q$. For completeness we give here the total expression for the quark condensate part (to zeroth order in α_s) of the operator product expansion for the correlation function of two nucleon currents

$$\begin{aligned} \mathrm{i}\int \mathrm{d}^4x\, \mathrm{e}^{\mathrm{i}px}\,\mathrm{T}(\eta_\mathrm{N}(x)\bar{\eta}_\mathrm{N}(0)) &= \frac{-1}{(2\pi)^2}\ln\frac{\Lambda^2}{-p^2}\,[\tfrac{1}{3}(\not{p}\gamma^\alpha\not{p} + 8p^2\gamma^\alpha)\bar{u}(0)\gamma_\alpha u(0) \\ &+ \tfrac{1}{3}(\not{p}\gamma^\alpha\gamma_5\not{p} - 4p^2\gamma^\alpha\gamma_5)\bar{u}(0)\gamma_\alpha\gamma_5 u(0) + p^2\bar{d}(0)d(0) + p^2(\mathrm{i}\gamma_5)\bar{d}(0)\,\mathrm{i}\gamma_5 d(0) \\ &+ \tfrac{1}{3}(\not{p}\gamma^\alpha\not{p} + 2p^2\gamma^\alpha)\bar{d}(0)\gamma_\alpha d(0) + \tfrac{1}{3}(\not{p}\gamma^\alpha\gamma_5\not{p} - 2p^2\gamma^\alpha\gamma_5)\bar{d}(0)\gamma_\alpha\gamma_5 d(0) \\ &- \tfrac{1}{6}\not{p}\sigma^{\alpha\beta}\not{p}\,\bar{d}(0)\sigma_{\alpha\beta}d(0)]\,, \end{aligned} \tag{5.50}$$

and we get

$$\langle 0|\,\mathrm{i}\int \mathrm{d}^4x\,\mathrm{e}^{\mathrm{i}px}\,\mathrm{T}(\eta_\mathrm{N}(x)\bar{\eta}_\mathrm{N}(0))|\pi^0(q\to 0)\rangle = -\frac{p^2}{(2\pi)^2}\ln\frac{\Lambda^2}{-p^2}(\mathrm{i}\gamma_5)\langle 0|\bar{d}(0)\,\mathrm{i}\gamma_5 d(0)|\pi^0\rangle\,. \tag{5.51}$$

We note that the Wilson coefficient in this case is identical to the p-dependent part of the Wilson coefficient for the three-point function case ((5.42)). This justifies a posteriori our procedure of taking $p'^2 = p^2 = -Q^2$ (i.e., taking the soft pion limit in the residue of the pion pole) in the three-point function case. Using

$$\tfrac{1}{2}[\bar{u}\,\mathrm{i}\gamma_5 u - \bar{d}\,\mathrm{i}\gamma_5 d] = \frac{1}{\sqrt{2}}\frac{f_\pi m_\pi^2}{m_\mathrm{u}+m_\mathrm{d}}\,\pi^{(0)} \tag{5.52}$$

for the right-hand side of (5.51), the effective pion–nucleon coupling $\mathscr{L} = g_{\pi\mathrm{NN}}\bar{N}\,\mathrm{i}\gamma_5(\tau\cdot\pi)N$ and saturation by the nucleon for the left-hand side, we obtain after Borel transforming with respect to $Q^2 = -p^2$

$$g_{\pi\mathrm{NN}}\frac{\lambda_\mathrm{N}^2}{M^2}\exp(-M_\mathrm{N}^2/M^2) = \frac{1}{(2\pi)^2}M^2\frac{1}{\sqrt{2}}\frac{f_\pi m_\pi^2}{m_\mathrm{u}+m_\mathrm{d}}\,. \tag{5.53}$$

Eliminating the nucleon-current coupling by (4.70b) we arrive at the Goldberger–Treiman relation (5.47) (independent of M^2).

5.4. The $\pi^+ p\to\Delta^{++}$ transition

As a second example we consider the $\pi^+ p\to\Delta^{++}$ transition. In this case we are interested in the three-point function

$$A_\mu(p,p',q) = \int \mathrm{d}x\,\mathrm{d}y\,\exp(\mathrm{i}p'x - \mathrm{i}qy)\mathrm{T}(\eta_\Delta^\mu(x)J_\pi(y)\bar{\eta}_\mathrm{N}(0))\,, \tag{5.54}$$

where the Δ^{++} current is given by (2.7) and the proton current by (2.6). For the π^+ current we have

$$J_{\pi^+}(x) = \bar{u}(x)\,\mathrm{i}\gamma_5 d(x)\,. \tag{5.55}$$

Again we can sandwich (5.54) by single quark states to find the Wilson coefficients C^{u}_μ and C^{d}_μ of the quark condensate operators $\bar{u}u$ and $\bar{d}d$. The results are

$$C^{\mathrm{d}}_\mu = \tfrac{4}{3}\mathrm{i}[g_{\mu\nu}\not{p} - \tfrac{3}{8}\gamma_\mu\gamma_\nu\not{p} + \tfrac{3}{8}(\gamma_\mu p_\nu - \gamma_\nu p_\mu) - \tfrac{1}{8}(\gamma_\mu p_\nu + \gamma_\nu p_\mu)]\not{p}\,\frac{q^\nu}{q^2}\frac{1}{(2\pi)^2}\ln\frac{\Lambda^2}{-p^2}\,, \tag{5.56}$$

and the same for C^{u}_μ with p replaced by p'.

On the phenomenological side of the sum rule we again have a diagram like fig. 18. One of the baryons is the nucleon, the other one the Δ. The $\Delta\pi\mathrm{N}$ coupling is usually taken to be

$$\mathscr{L}_{\Delta\pi\mathrm{N}} = g_{\Delta\pi\mathrm{N}}\bar{N}(x)\Delta_\mu(x)\partial^\mu\pi(x)\,, \tag{5.57}$$

and using the Rarita–Schwinger expression for a spin $\frac{3}{2}$ projection operator we find

$$\begin{aligned}&\mathrm{i}\lambda_\Delta\lambda_\mathrm{N}\frac{f_\pi m_\pi^2}{2m_\mathrm{q}}g_{\Delta\pi\mathrm{N}}\frac{1}{p'^2 - M_\Delta^2}\frac{1}{p^2 - M_\mathrm{N}^2}\frac{1}{q^2 - m_\pi^2}\left[g_{\mu\nu} - \tfrac{1}{3}\gamma_\mu\gamma_\nu - \frac{2p'_\mu p'_\nu}{3M_\Delta^2} + \frac{\gamma_\mu p'_\nu - \gamma_\nu p'_\mu}{3M_\Delta}\right]\\ &\quad\times(\not{p}' + M_\Delta)q^\nu(\not{p} + M_\mathrm{N})\,.\end{aligned} \tag{5.58}$$

On the theoretical side we have the sum of C^{u}_μ and C^{d}_μ multiplied by $\langle 0|\bar{q}q|0\rangle$. We now proceed as before by taking the Borel transform with respect to Q^2 $(p^2 = p'^2 = -Q^2)$ and identify the $g_{\mu\nu}$ coefficients on both sides which leads to the equation (with $M_\mathrm{B} = \frac{1}{2}(M_\mathrm{N} + M_\Delta)$)

$$\lambda_\Delta\lambda_\mathrm{N}\frac{f_\pi m_\pi^2}{2m_\mathrm{q}}g_{\Delta\pi\mathrm{N}}\frac{M^2 - 2M_\mathrm{B}^2}{M^4}\exp(-M_\mathrm{B}^2/M^2) = \frac{8}{3}\frac{1}{(2\pi)^2}M^2\langle 0|\bar{q}q|0\rangle\,. \tag{5.59}$$

Using the PCAC relation and rearranging the terms we get

$$g_{\Delta\pi\mathrm{N}} = \frac{4}{3}\frac{1}{(2\pi)^2}\frac{M^6}{\lambda_\Delta\lambda_\mathrm{N}}\frac{f_\pi}{2M_\mathrm{B}^2 - M^2}\exp(M_\mathrm{B}^2/M^2)\,. \tag{5.60}$$

Using (4.70a) and the analogous equation for λ_Δ [16]

$$M^6 + \tfrac{20}{3}a^2 - \tfrac{25}{18}bM^2 = 5(2\pi)^4\lambda_\Delta^2\exp(-M_\Delta^2/M^2)\,, \tag{5.61}$$

we finally get

$$g_{\Delta\pi\mathrm{N}} = \tfrac{4}{3}\sqrt{10}(2\pi)^2\frac{f_\pi}{2M_\mathrm{B}^2 - M^2}\left(1 + \frac{20}{3}\frac{a^2}{M^6} - \frac{25}{18}\frac{b}{M^4}\right)^{-1/2}\left(1 + \frac{4}{3}\frac{a^2}{M^6} + \frac{b}{M^4}\right)^{-1/2}, \tag{5.62}$$

or in first approximation using (5.40) and $M^2 \cong M_B^2$

$$g_{\Delta\pi N} = \tfrac{2}{3}\sqrt{5}\,\frac{M_N}{M_B^2}\, g_{\pi NN} \cong 15\ \mathrm{GeV}^{-1}\,. \tag{5.63}$$

The formula for the decay width

$$\Gamma(\Delta^{++} \to \pi^+ p) = \frac{1}{6}\,\frac{g_{\Delta\pi p}^2}{4\pi}\,\frac{(m_\Delta + m_N)^2 - m_\pi^2}{m_\Delta^2}\, p^3\,, \tag{5.64}$$

gives for $g_{\Delta\pi N}$

$$g_{\Delta\pi N}^2/4\pi \cong 20\ \mathrm{GeV}^{-2}\,, \tag{5.65}$$

in excellent agreement with (5.63).

We can also derive formula (5.62) by considering the correlation function of a nucleon and a Δ current, and selecting the $\bar{d}\gamma_\mu\gamma_5 u$ piece in the OPE. We find

$$\begin{aligned} \mathrm{i}\int \mathrm{d}^4x\, \mathrm{e}^{\mathrm{i}px}\, \mathrm{T}(\eta_\Delta^\mu(x)\bar{\eta}_N(0)) = \frac{-1}{(2\pi)^2}\ln\frac{\Lambda^2}{-p^2}\,\frac{4}{3}\,&[g_{\mu\nu}\not{p} + \tfrac{3}{8}(\gamma_\mu p_\nu - \gamma_\nu p_\mu) - \tfrac{1}{8}(\gamma_\mu p_\nu + \gamma_\nu p_\mu) \\ &- \tfrac{3}{8}\gamma_\mu\gamma_\nu\not{p}]\not{p}\,\bar{d}(0)\gamma_\nu\gamma_5 u(0) + \cdots\,. \end{aligned} \tag{5.66}$$

We use the effective Lagrangian (5.57) in the left-hand side of (5.66) which gives

$$\lambda_\Delta\lambda_N g_{\Delta\pi N}\,\frac{1}{p^2 - M_\Delta^2}\,\frac{1}{p^2 - M_N^2}\,[g_{\mu\nu} - \tfrac{1}{3}\gamma_\mu\gamma_\nu + \cdots](\not{p} + M_\Delta)q^\nu(\not{p} + M_N)\,. \tag{5.67}$$

For the right-hand side we use

$$\bar{d}(0)\gamma_\mu\gamma_5 u(0) = -f_\pi \partial_\mu \pi^+\,, \tag{5.68}$$

and obtain

$$\frac{1}{(2\pi)^2}\ln\frac{\Lambda^2}{-p^2}\, f_\pi \tfrac{4}{3}[g_{\mu\nu} - \tfrac{3}{8}\gamma_\mu\gamma_\nu + \cdots]p^2 q^\nu\,, \tag{5.69}$$

after sandwiching between the vacuum and a π^+ state. Taking the Borel transform and equating the $g_{\mu\nu}$ pieces on both sides we get the same result as (5.62). The examples given here can be generalized to include the couplings to Goldstone bosons of all baryons in the $L = 0$ octet and decuplet, in particular the F/D ratio $\alpha = D/(F + D)$ can be estimated. In the SU(3) limit this gives [89]

$$\alpha = \tfrac{7}{12}\,, \tag{5.70}$$

in good agreement with the value $\alpha \cong 0.6$ from the ratio of the experimental values for the $\Lambda\Sigma\pi$ and $NN\pi$ couplings [99].

5.5. The $g_{\omega\rho\pi}$ coupling

Finally we give an example involving only mesons. As before it can easily be seen from dimensional considerations that quark condensate contributions dominate the trilinear meson couplings. The possible diagrams are the same as fig. 20 with one quark line less. The calculation of these diagrams is extremely simple as they do not contain any loop integrations. Only figs. 20a and 20b contain a $1/q^2$ pole due to the pion. The result for the invariant amplitude is

$$\frac{\langle 0|\bar{q}q|0\rangle}{q^2}\left(\frac{1}{p^2}+\frac{1}{p'^2}\right). \tag{5.71}$$

On the phenomenological side we have

$$\frac{g_{\omega\rho\pi}}{(q^2-m_\pi^2)(p^2-m_\rho^2)(p'^2-m_\omega^2)}\,\frac{\sqrt{2}f_\pi m_\pi^2}{2m_q}\,\frac{m_\omega^2 m_\rho^2}{g_\omega g_\rho}. \tag{5.72}$$

Identifying the $1/q^2$ pole in (5.71) and (5.72), taking $p^2=p'^2=-Q^2$ and applying the Borel transform on both sides with respect to Q^2 we get (with $m_\omega^2=m_\rho^2$)

$$g_{\omega\rho\pi}\frac{\exp(-m_\rho^2/M^2)}{M^4}\,\frac{\sqrt{2}f_\pi m_\pi^2}{2m_q}\,\frac{m_\omega^2 m_\rho^2}{g_\omega g_\rho}=-2\frac{\langle 0|\bar{q}q|0\rangle}{M^2}. \tag{5.73}$$

In the same way as the couplings of the baryons to the currents we can use the two-point function results of chapter 4 to eliminate g_ω and g_ρ.

$$\frac{12\pi^2 m_\rho^2}{g_\rho^2}\exp(-m_\rho^2/M^2)=\tfrac{3}{2}M^2[1+\alpha_s/\pi+\text{higher corrections}]. \tag{5.74}$$

Explicit expressions of the higher corrections are given by (4.44). At $M^2\cong m_\rho^2$ the sum of all corrections amounts to $\cong 10\%$.

Substituting (5.74) into (5.73), using PCAC and rearranging the terms we get

$$g_{\omega\rho\pi}=\sqrt{2}(2\pi)^2 f_\pi/m_\rho^2, \tag{5.75}$$

which is numerically very close to the current algebra result

$$g_{\omega\rho\pi}=2g_\rho/m_\rho=2/f_\pi. \tag{5.76}$$

The relation between our result and (5.76) can be seen more clearly if we use (5.73) directly at $M^2\cong m_\rho^2$. Using $f_\pi=m_\rho/g_\rho=m_\omega/g_\omega$ and PCAC we get

$$g_{\omega\rho\pi}=(2g_\rho/m_\rho)(e/2\sqrt{2}), \tag{5.77}$$

which compares very well with (5.76).

Other trilinear couplings can, at least in principle, be treated in the same way. However, the method only works well when the two mesons which couple to the pion have approximately equal masses, since that provides a natural scale for the Borel variable M^2.

5.6. Form factors

Another interesting application of the QCD sum rule method is the study of the high q^2 behaviour of three-point functions. Here the presence of nonperturbative effects can be felt up to the highest known momentum transfer. In particular, calculations that include only perturbative contributions [100] fail by one order of magnitude to explain the size of the experimental data.

We consider the pion form factor in detail. We use the standard definition of the three-point function

$$A_{\mu\alpha\beta}(p_1, p_2) = -\int \exp(-\mathrm{i}p_1x + \mathrm{i}p_2y)\langle 0|\mathrm{T}\{j^\dagger_{\alpha 5}(x) j_\mu(0) j_{\beta 5}(y)\}|0\rangle\, \mathrm{d}^4x\, \mathrm{d}^4y\,. \tag{5.78}$$

Here, the electromagnetic current $j_\mu(x)$ is sandwiched between the two axial currents $j_{\alpha 5}(x)$ and $j_{\beta 5}(x)$ which project nontrivially on a one pion state $|P\rangle$ with a well defined strength given by the well-known formula

$$\langle 0|j_{\alpha 5}(0)|P\rangle = \mathrm{i}f_\pi P_\alpha\,. \tag{5.79}$$

To avoid instanton contributions it is convenient to use the axial and not the pseudoscalar current [101].

The amplitude can be expanded into a number of invariant amplitudes which obey (with subtractions) dispersion relations. We can calculate the invariant amplitude in the deep Euclidean region because of asymptotic freedom. Using a double dispersion relation, the function under scrutiny reads, after Borel transforming with respect to the pion variables p_1^2 and p_2^2

$$A(M_1^2, M_2^2, q^2) = \frac{1}{\pi}\int_0^\infty \frac{\mathrm{d}s_1\, \mathrm{d}s_2}{M_1^2 M_2^2}\rho(s_1, s_2, q^2) \exp\left\{-\frac{s_1}{M_1^2} - \frac{s_2}{M_2^2}\right\}, \tag{5.80}$$

where the dynamics is in the function $\rho(s_1, s_2, q^2)$. One then isolates the Lorentz structure corresponding to the pion form factor

$$\langle 0|j_{\beta 5}|p_2\rangle\langle p_2|j_\mu|p_1\rangle\langle p_1|j^\dagger_{\alpha 5}|0\rangle \cong f_\pi^2 F_\pi(Q^2) p_1^\alpha p_2^\beta (p_1^\mu + p_2^\mu)\,. \tag{5.81}$$

We know the general procedure to calculate the perturbative and nonperturbative (gluon and chiral condensate) contributions to the function in the limit when the quark masses are taken to vanish. Isolating the pion pole and taking the expression for $M_1^2 = M_2^2 = M^2$, the form factor after some algebra reads [101, 102]

$$f_\pi^2 F_\pi(Q^2) = \frac{1}{\pi^2}\int_0^{s_0} \mathrm{d}s_1 \int_0^{s_0} \mathrm{d}s_2 \rho(s_1, s_2, Q^2) \exp\left(-\frac{s_1 + s_2}{M^2}\right)$$
$$+ \frac{\alpha_s\langle G^2\rangle}{12\pi M^2} + \frac{208\pi\alpha_s\langle\bar{q}q\rangle^2}{81M^4}\left(1 + \frac{2}{13}\frac{Q^2}{M^2}\right) + \mathrm{O}\left(\frac{1}{M^6}\right). \tag{5.82}$$

It is easy to convince oneself that in the relevant region the power terms are by far more important than the perturbative corrections [101, 102, 103]. In first approximation $\rho(s_1, s_2, Q^2)$ is given by the free quark spectral function and reads

$$\rho(s_1, s_2, Q^2) = \frac{3Q^4}{4}\left\{\left(\frac{d}{dQ^2}\right)^2 + \frac{Q^2}{3}\left(\frac{d}{dQ^2}\right)^3\right\}\frac{1}{[(s_1+s_2+Q^2)^2 - 4s_1s_2]^{1/2}}. \tag{5.83}$$

In [101] the authors produce an approximate formula that is accurate theoretically to about 10% and fits the data very well, as can be seen from fig. 21. As usual the formula can only be used when the power corrections are smaller than about 30%, otherwise the expansion breaks down. The limit here is about 6 GeV^2. In the limit Q^2 very large one recuperates the simple $1/Q^2$ behaviour as demanded by general arguments of compositeness [104, 105]. It is interesting that the dynamical aspects of QCD determine the form factor behaviour in the intermediate region mainly through power corrections. Ioffe and Smilga have considered several other transition moments and the static limit of the nucleon and octet magnetic moments [106]. The agreement is qualitatively good but whether it solves the presumed 10% discrepancy of the nonrelativistic quark model is still an open question. Applications to axial form factors and G_V/G_A are also available [107].

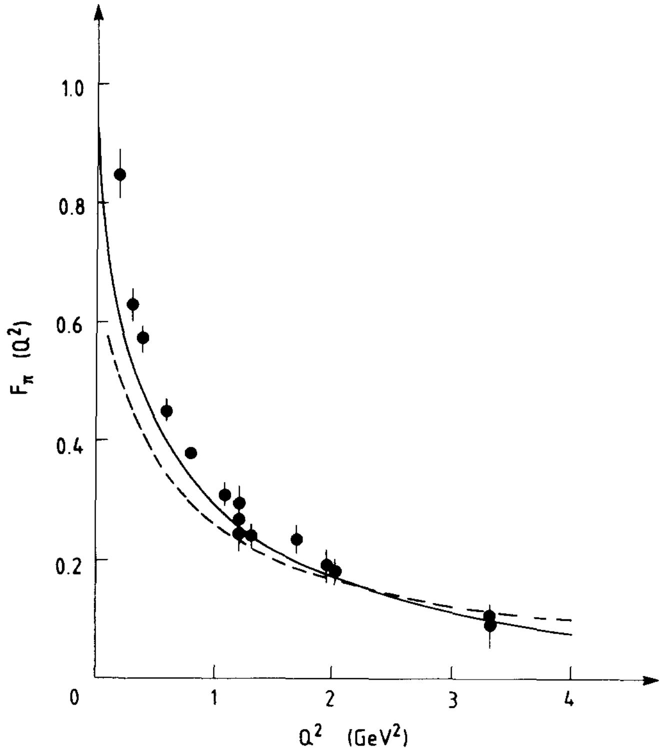


Fig. 21. Plot of the pion form factor $F_\pi(Q^2)$. The form factor is plotted as a function of Q^2, and compared to the experimental data. Figure taken from [101].

6. Conclusions

In this report we have reviewed the method of Shifman, Vainshtein and Zacharov [1] for studying a variety of problems in QCD. We have tried to give a clear and coherent view of the matter which we hope will be as convincing to you as it is to us.

The main subjects we dealt with include (a) an analysis of the status of the theoretical basis, (b) a discussion of all calculations we think are well established, (c) a discussion of recent developments. Inevitably our bias has put emphasis on some subjects and neglected others. Many papers have appeared in the five years of existence of the subject, and not all could be given proper treatment. For more specific information on certain subjects see also the reviews [108, 109] and various conference reports.

QCD sum rules provide a systematic way for studying strong interaction properties in a framework that only uses QCD parameters. In this it differs from many QCD like theories like bag and potential models that often introduce ad hoc quantities that cannot be related to the QCD Lagrangian. However, it is not a fundamental theory either since it does not allow to study questions like confinement. Nevertheless it has very good predictive power in terms of the quark masses, a few condensates (one to three depending on the problem), and $\mathscr{L}_{\mathrm{QCD}}$. All mesons and baryons belonging to the 35 and 56 multiplets of SU(6) with $L = 0$ and 1 can be calculated. This is true for most light, heavy and light–heavy systems. In some cases the couplings can be calculated as well. One gains understanding on symmetry breaking and on parameters like the quark masses, and condensates.

All these predictions require substantial technical effort. In the relevant sections we have discussed the present status of these calculations and in particular technical innovations like the calculational scheme in the fixed-point gauge. These calculational developments may have applications elsewhere.

In chapter 4 we discussed the predictions for the best known meson and baryon systems and took advantage of the feedback provided by the data to check on parameters and procedures like the subtraction of the continuum, the influence of neglected states and other issues. We find that the conventional states are indeed very well understood by the method. Applications to strong interaction couplings of Goldstone bosons and radiative decays of quarkonium are also discussed. Although approximations are needed nonperturbative effects are definitely present in agreement with the data. There are also interesting results on wave functions and decay constants like f_{B}.

Even so, there is some frustration in the fact that the method is unable to yield the simple pattern of rising Regge trajectories of the dual models that is seen experimentally. The Regge behaviour that so neatly puts together all positive and negative angular momenta cannot yet be retrieved by this method. This is one major challenge left open.

There are also technical difficulties concerning open charm states. No window is available in the parameter space to determine the masses of these objects. Open beauty states are accessible but the small splittings and the high density of states makes it impossible to make very detailed predictions. There are also technical difficulties in the calculations of some electromagnetic transitions.

Heavier quarks like beauty are also not completely understood. Upsilonium is on the boundary of the Coulombic and power corrections regime. Moreover the splittings are so small that a relativistic treatment is required. These calculations are not yet complete. Early claims about level spacings in these systems should not be taken seriously.

To complete our summary of the conventional states we have to emphasize the baryon mass formula of Ioffe. However, a detailed understanding of baryons is still lacking. In particular, the perturbative corrections are still under discussion. The first results of these difficult calculations do not agree with

Table 4
Results for light quark mesons with $L = 0$

State	J^{PC}	Mass Exp	Mass Theor	Coupling Exp	Coupling Theor	Remarks
π		140	–	$f_\pi = 133$	125	Masses too low; direct
K		495	–	$f_K \cong f_\pi$		instantons; f_π, f_K well
	0^{-+}					computed.
η		550	–			
η'		920	–			
				$g^2/4\pi$		Masses and couplings
ω		780	770	2.4	2.4	calculated with ≅10%
ρ		770	770	2.4	2.4	acc. ρ–ω interference
	1^{--}					has also been obtained.
K*		890	890	1.39	1.46	Relevant parameters are
ϕ		1020	1010	12.0	13.0	$\langle G^2\rangle$, $\langle\bar{q}q\rangle$, α_s, s_0,
						$m_q \equiv 0$, $I = 0$, 1 degeneracy,
						SU(3) breaking o.k., m_s
						not well fixed.

each other. It seems probable that an accurate determination of the baryon properties will still take some time. In total there is a large number of predictions, many verified, that support the theory. In light–heavy systems there are novel contributions from chiral symmetry breaking terms which await confirmation.

We have summarized a sample of the results in tables 4, 5, 6, and 7. Baryons have not been included

Table 5
Results for light quark mesons with $L = 1$

J^{PC}	State	Mass Exp	Mass Theor	Coupling Exp	Coupling Theor	Remarks
2^{++}	A_2	1320	1300	Couplings not directly		$I = 1$, 0 degeneracy
	f	1270	1300	useful; $g_f = 0.04$ cal-		m_s low ≅120 MeV
	K**	1430	–	culated with further		Large $1/M^4$ term gives bound
	f'	1520	1540	assumption, agrees		on $\langle G^2\rangle$
				with exp.		
1^{++}	A_1	1200	1150	$4\pi/f^2_{A_1} = 0.15$	0.16	Two sum rules for A_1;
	D	1285	1300			$m_s(1\ \text{GeV}) = 110 \pm 10$ MeV
	E	1420	1470			$\langle m_s\bar{s}s\rangle = -(0.20 \pm 0.01)10^{-2}\ \text{GeV}^4$
	Q_1	1270	–			for D only one sum rule, since
	Q_2	1414	–			divergence of axial current
						has U(1) problem in this
						channel
0^{++}	δ	980	1000			δ, S* assumed to be pure $\bar{q}q$,
	S*	980	1000			no instanton contributions in-
	ε	1300	1350			cluded
1^{+-}	B	1240	?			No calculation possible; power
						corrections vanish at one-loop
						level
2^{-}	A_3	1680	1630			

Table 6
Results for heavy quark mesons with $L = 0$ and $L = 1$

J^{PC}	State	Mass (GeV) Exp	Theor	Coupling	Remarks
Charmonium					
1^{--}	J/ψ	3.10	3.09 ± 0.02	Only for J/ψ	$m_c(p^2 = -m_c^2) = 1.26$ GeV
0^{-+}	η_c	2.98	3.00 ± 0.02	exp: $\Gamma_{e^+e^-} = 4.7 \pm 0.6$ keV	very accurate indep. of s_0.
0^{++}	χ_0	3.42	3.40 ± 0.01	theor: $\Gamma_{e^+e^-} \cong 4.9$ keV	Gluon condensate same as
1^{++}	χ_1	3.51	3.50 ± 0.02		in light quark case
2^{++}	χ_2	3.56	3.57 ± 0.02		$\left\langle \frac{\alpha_s}{\pi} G^2 \right\rangle \cong (360 \pm 20\ \text{MeV})^4$.
1^{+-}		3.51	3.51 ± 0.01		Same parameters fit P waves
Bottonium					
					$m_b(p^2 = -m_b^2) \cong 4.23$ GeV
1^{--}	Υ	9.46	$m_\Upsilon - m_{\eta_b} \cong 60$ MeV		moment method fails, no single
0^{--}	η_B	?			resonance saturation
Open bottom					
0^{-+}	$(\bar{u}b)$	5.27	5.31	$f_p \cong 190 \pm 30$ MeV	Continuum very important;
1^{--}			5.38	$g_V^2/4\pi \cong 16$	splittings cannot be resolved.
0^{++}			6.13	$f_S \cong 270$ MeV	S–P splitting large because
1^{++}			6.17	$g_A^2/4\pi \cong 10$	of $m_Q\langle \bar{q}q \rangle$
0^{-+}	$(\bar{s}b)$		5.42	$f_P \cong 210$ MeV	
1^{--}			5.46	$g_V^2/4\pi \cong 12$	
0^{++}			6.29	$f_S \cong 270$ MeV	
1^{++}			6.34	$g_A^2/4\pi \cong 9$	

since we feel that the calculations need improvement. Some missing states cannot be calculated (B meson, for example) or have not yet been computed (K**).

The reader will have noticed that we have refrained from comparing the theory with potential models. We believe that potential models give a qualitative understanding and are useful in many instances. However, they are QCD-like theories and at different stages require assumptions beyond QCD. Spin forces are an example. There is no easy way, if any, which relates the two approaches.

Table 7
Couplings $g^2/4\pi$ of Goldstone bosons to hadrons

	Exp	Theory	
πNN	14.5	12.5	all within 20% of exp value
$\pi\Sigma\Sigma$	13 ± 2	10	$\alpha = D/(F + D) = \frac{7}{12}$
η_8NN	$\cong 4.5$	6.4	(compare $\alpha(\text{SU}(6)) = \frac{2}{3}$)
KΣN	$\cong 1$	1	
πNΔ	$\cong 15\ \text{GeV}^{-2}$	$18\ \text{GeV}^{-2}$	
$\omega\rho\pi$	$\cong 16\ \text{GeV}^{-1}$	$13\ \text{GeV}^{-2}$	

The theory has also been used for calculating glueballs hybrid states and for composite models of quarks and leptons. These are interesting subjects but not yet settled. In fact it is quite remarkable that the QCD degrees of freedom due to gluons are so difficult to pin down. The detection of a few of these states will be very important to see how our stringlike saturation assumptions work in these cases. It is not impossible that the method fails if it is used naively since the duality between physical states and quarks and gluons may be different.

There are many questions being studied and the method might also prove useful for other problems. In spite of the achievements sofar, there are still important problems like establishing the values of the parameters (condensates) from first principles and fully understanding the theoretical basis. Nevertheless the QCD sum rule method remains one of the most powerful methods for extracting information from the deceivingly simple Lagrangian of QCD.

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Appendix

In this Appendix we list the following expressions:

(1) The coefficients of $G^a_{\mu\nu}G^a_{\mu\nu}$ in the equal mass case of mesons with $J^{PC} = 0^{-+}, 1^{--}, 1^{++}, 0^{++}, 1^{+-}$, and 2^{++}.

(2) The coefficients of the 6 and 8 dimensional gluonic operators as given in [40] and [41] for the equal mass case of mesons with $J^{PC} = 0^{-+}$, 1^{--}, 1^{++}, and 0^{++}.

(3) The quark condensate and gluon condensate contributions to mesons consisting of unequal mass quarks.

(4) The expressions to the polarization functions of light quark mesons.

(5) The expressions for the polarization functions of all octet and decuplet $L = 0$ baryons.

The expressions for the currents with the various J^{PC} quantum numbers are given by (2.2).

(1) The coefficients C^J_G of the operator $(\alpha_s/\pi)G^a_{\mu\nu}G^a_{\mu\nu}$ for mesons with equal mass quarks of mass m.

$$J^{PC} = 1^{--}: \quad C^{V}_G(u) = \frac{1}{48Q^4}\left[\frac{3(1+u^2)(1-u^2)^2}{2u^5}\ln\frac{1+u}{1-u} - \frac{3u^4-2u^2+3}{u^4}\right], \tag{A.1a}$$

$$J^{PC} = 0^{-+}: \quad C^{P}_G(u) = \frac{-1}{48Q^2}\left[\frac{3(3u^2+1)(1-u^2)}{2u^5}\ln\frac{1+u}{1-u} - \frac{9u^4+4u^2+3}{u^4(1-u^2)}\right], \tag{A.1b}$$

$$J^{PC} = 0^{++}: \quad C^{S}_G(u) = \frac{1}{48Q^2}\left[\frac{3(3+u^2)(1-u^2)}{2u^3}\ln\frac{1+u}{1-u} - \frac{9+4u^2+3u^4}{u^2(1-u^2)}\right], \tag{A.1c}$$

$$J^{PC} = 1^{+-}: \quad C^{A'}_G(u) = \tfrac{1}{3}C^{S}_G(u), \tag{A.1d}$$

$$J^{PC}=1^{++}:\quad C_G^A(u)=\frac{1}{48Q^4}\left[-\frac{(1-u^2)^2}{2u^3}\ln\frac{1+u}{1-u}+\frac{1+u^2}{u^2}\right], \tag{A.1e}$$

$$J^{PC}=2^{++}:\quad C_G^T(u)=-\frac{1}{18Q^4}\left[\frac{1}{4u^3}(9-36u^2+123u^4-160u^6)\ln\frac{1+u}{1-u}\right.$$
$$\left.+\frac{1}{6u^2(1-u^2)}(-27-290u^2+689u^4-480u^6)\right]. \tag{A.1f}$$

As usual $u^2=1-4m^2/q^2$.

For calculating the moments at $Q^2=0$ [28] or $Q^2\neq 0$ [13] it is more convenient to express the coefficients in terms of the integrals [40]

$$J_N(Q^2/m^2)=\int_0^1 [1+x(1-x)Q^2/m^2]^{-N}\,dx. \tag{A.2}$$

The very simple expressions read

$$C_G^V=\frac{1}{12Q^4}(-1+3J_2-2J_3),\qquad C_G^P=\frac{1}{24Q^2}(5+6J_1-15J_2+J_3),$$

$$C_G^S=\frac{1}{8Q^2}(-1-2J_1+3J_2),\qquad C_G^{A'}=\tfrac{1}{3}C_G^S,\qquad C_G^A=\frac{1}{4Q^4}(1-J_2). \tag{A.3}$$

(2) The coefficients of the 6 and 8 dimensional operators (3.56) and (3.58) (from [41]) in terms of the integrals (A.2); $\xi=Q^2/m^2$

$J^{PC}=1^{--}$:

$$O_1^6:\ \frac{1}{72\pi^2Q^6}\left(\tfrac{2}{15}+4J_2-\tfrac{31}{3}J_3+\tfrac{43}{5}J_4-\tfrac{12}{5}J_5-\tfrac{1}{10}\xi\right), \tag{A.4}$$

$$O_2^6:\ \frac{1}{36\pi^2Q^6}\left(\tfrac{41}{45}+\tfrac{2}{3}J_1-J_2-\tfrac{4}{9}J_3-\tfrac{26}{15}J_4+\tfrac{8}{5}J_5+\tfrac{1}{3}\xi J_1-\tfrac{3}{5}\xi\right).$$

$$O_1^8:\ \frac{1}{432\pi^2Q^8}\left(\tfrac{85}{12}+15J_1-\tfrac{57}{2}J_2-\tfrac{301}{3}J_3+\tfrac{871}{4}J_4-141J_5+30J_6-\tfrac{1}{2}\xi J_1+\xi-\tfrac{3}{4}\xi^2\right),$$

$$O_2^8:\ \frac{1}{432\pi^2Q^8}\left(\tfrac{13}{56}-\tfrac{19}{2}J_1+\tfrac{427}{4}J_2-\tfrac{575}{2}J_3+\tfrac{13553}{40}J_4-\tfrac{13669}{70}J_5+\tfrac{1391}{28}J_6-\tfrac{45}{14}J_7-\tfrac{3}{4}\xi J_1+\tfrac{3}{35}\xi-\tfrac{9}{140}\xi^2\right)$$

$$O_3^8:\ \frac{1}{432\pi^2Q^8}\left(-\tfrac{661}{30}-60J_1+294J_2-\tfrac{1172}{3}J_3+\tfrac{3143}{10}J_4-\tfrac{978}{5}J_5+60J_6+2\xi J_1-\tfrac{14}{5}\xi+\tfrac{21}{10}\xi^2\right)$$

$$O_4^8:\ \frac{1}{432\pi^2Q^8}\left(\tfrac{1145}{84}+33J_1-\tfrac{99}{2}J_2-\tfrac{983}{3}J_3+\tfrac{13027}{20}J_4-\tfrac{12591}{35}J_5+\tfrac{131}{14}J_6+\tfrac{207}{7}J_7+\tfrac{1}{2}\xi J_1+\tfrac{104}{35}\xi-\tfrac{27}{14}\xi^2\right), \tag{A.5}$$

$$O_5^8:\ \frac{1}{432\pi^2Q^8}\left(-\tfrac{663}{70}+22J_1-269J_2+722J_3-\tfrac{8893}{10}J_4+\tfrac{19762}{35}J_5-\tfrac{935}{7}J_6-\tfrac{54}{7}J_7+3\xi J_1+\tfrac{248}{35}\xi-\tfrac{102}{35}\xi^2\right),$$

$$O_6^8:\ \frac{1}{432\pi^2Q^8}\left(\tfrac{8}{35}+8J_1-88J_2+234J_3-\tfrac{1066}{5}J_4+\tfrac{284}{35}J_5+\tfrac{572}{7}J_6-\tfrac{216}{7}J_7-\tfrac{23}{35}\xi+\tfrac{12}{35}\xi^2\right),$$

$$O_7^8:\ \frac{1}{432\pi^2Q^8}\left(\tfrac{881}{210}-2J_1+7J_2+\tfrac{142}{3}J_3-\tfrac{1377}{10}J_4+\tfrac{4666}{35}J_5-\tfrac{555}{7}J_6+\tfrac{162}{7}J_7-\xi J_1-\tfrac{128}{35}\xi+\tfrac{54}{35}\xi^2\right).$$

$J^{PC}=0^{-+}$:

$$O_1^6:\ \frac{1}{144\pi^2Q^4}\left(\tfrac{14}{5}+9J_1-48J_2+62J_3-\tfrac{153}{5}J_4+\tfrac{24}{5}J_5+\tfrac{1}{10}\xi\right),$$

$$O_2^6:\ \frac{1}{72\pi^2Q^4}\left(\tfrac{67}{15}+8J_1-25J_2+\tfrac{22}{3}J_3+\tfrac{42}{5}J_4-\tfrac{16}{5}J_5-\xi J_1+\tfrac{3}{5}\xi\right), \tag{A.6}$$

$$O_1^8:\ \frac{1}{432\pi^2Q^6}\left(\tfrac{35}{8}+\tfrac{93}{2}J_1-\tfrac{1533}{4}J_2+782J_3-\tfrac{5325}{8}J_4+246J_5-30J_6+\tfrac{3}{4}\xi J_1+\tfrac{3}{2}\xi+\tfrac{3}{8}\xi^2\right),$$

$$O_2^8:\ \frac{1}{432\pi^2Q^6}\left(\tfrac{53}{112}+15J_1-\tfrac{1239}{8}J_2+403J_3-\tfrac{37323}{80}J_4+\tfrac{18507}{70}J_5+\tfrac{3621}{56}J_6+\tfrac{45}{14}J_7+\tfrac{3}{8}\xi J_1+\tfrac{27}{70}\xi+\tfrac{9}{280}\xi^2\right),$$

$$O_3^8:\ \frac{1}{432\pi^2Q^6}\left(\tfrac{29}{4}-24J_1+237J_2-401J_3+\tfrac{2463}{20}J_4+\tfrac{588}{5}J_5-60J_6-3\xi J_1-\tfrac{12}{5}\xi-\tfrac{21}{20}\xi^2\right),$$

$$O_4^8:\ \frac{1}{432\pi^2Q^6}\left(\tfrac{1355}{56}+183J_1-\tfrac{5721}{4}J_2+2789J_3-\tfrac{17613}{8}J_4+\tfrac{4407}{7}J_5+\tfrac{999}{28}J_6-\tfrac{207}{7}J_7+\tfrac{3}{4}\xi J_1+\tfrac{39}{7}\xi+\tfrac{27}{28}\xi^2\right), \tag{A.7}$$

$$O_5^8:\ \frac{1}{432\pi^2Q^6}\left(\tfrac{995}{28}+18J_1+\tfrac{513}{2}J_2-1112J_3+\tfrac{28263}{20}J_4-\tfrac{25974}{35}J_5+\tfrac{1725}{14}J_6+\tfrac{54}{7}J_7-\tfrac{15}{2}\xi J_1-\tfrac{144}{35}\xi+\tfrac{51}{35}\xi^2\right),$$

$$O_6^8:\ \frac{1}{432\pi^2Q^6}\left(-\tfrac{27}{7}-33J_1+237J_2-\tfrac{717}{2}J_3+\tfrac{147}{2}J_4+\tfrac{1464}{7}J_5-\tfrac{1086}{7}J_6+\tfrac{216}{7}J_7-\tfrac{45}{28}\xi-\tfrac{6}{35}\xi^2\right),$$

$$O_7^8:\ \frac{1}{432\pi^2Q^6}\left(-\tfrac{801}{28}-54J_1+\tfrac{441}{2}J_2-246J_3+\tfrac{855}{4}J_4-\tfrac{1350}{7}J_5+\tfrac{1545}{14}J_6-\tfrac{162}{7}J_7+\tfrac{9}{2}\xi J_1+\tfrac{15}{7}\xi-\tfrac{27}{35}\xi^2\right).$$

$J^{PC}=0^{++}$:

$$O_1^6:\ \frac{1}{144\pi^2Q^4}\left(-\tfrac{14}{5}-9J_1+42J_2-44J_3+\tfrac{69}{5}J_4-\tfrac{3}{10}\xi\right), \tag{A.8}$$

$$O_2^6:\ \frac{1}{72\pi^2Q^4}\left(\tfrac{13}{15}-2J_1+9J_2+\tfrac{4}{3}J_3-\tfrac{46}{5}J_4-\xi J_1-\tfrac{9}{5}\xi\right),$$

$$O_1^8:\ \frac{1}{432\pi^2Q^6}\left(-\tfrac{133}{8}-\tfrac{135}{2}J_1+\tfrac{1611}{4}J_2-577J_3+\tfrac{2427}{8}J_4-45J_5+\tfrac{3}{4}\xi J_1-\tfrac{3}{2}\xi-\tfrac{9}{8}\xi^2\right),$$

$$O_2^8\colon \frac{1}{432\pi^2Q^6}\left(-\tfrac{2903}{560}-\tfrac{45}{2}J_1+\tfrac{1233}{8}J_2-\tfrac{565}{2}J_3+\tfrac{18453}{80}J_4-\tfrac{3033}{35}J_5+\tfrac{675}{56}J_6+\tfrac{3}{8}\xi J_1-\tfrac{27}{70}\xi-\tfrac{27}{280}\xi^2\right),$$

$$O_3^8\colon \frac{1}{432\pi^2Q^6}\left(\tfrac{673}{20}+108J_1-531J_2+517J_3-\tfrac{753}{20}J_4-90J_5-3\xi J_1+\tfrac{12}{5}\xi+\tfrac{63}{20}\xi^2\right),$$

$$O_4^8\colon \frac{1}{432\pi^2Q^6}\left(-\tfrac{2773}{56}-234J_1+\tfrac{5823}{4}J_2-2086J_3+\tfrac{36543}{40}J_4+\tfrac{3888}{35}J_5-\tfrac{3105}{28}J_6+\tfrac{3}{4}\xi J_1-\tfrac{237}{35}\xi-\tfrac{81}{28}\xi^2\right), \tag{A.9}$$

$$O_5^8\colon \frac{1}{432\pi^2Q^6}\left(\tfrac{5359}{140}+60J_1-\tfrac{663}{2}J_2+706J_3-\tfrac{14841}{20}J_4+\tfrac{8412}{35}J_5+\tfrac{405}{14}J_6-\tfrac{15}{2}\xi J_1-\tfrac{192}{35}\xi-\tfrac{153}{35}\xi^2\right),$$

$$O_6^8\colon \frac{1}{432\pi^2Q^6}\left(\tfrac{312}{35}+45J_1-243J_2+\tfrac{345}{2}J_3+\tfrac{2619}{10}J_4-\tfrac{12636}{35}J_5+\tfrac{810}{7}J_6+\tfrac{309}{140}\xi+\tfrac{18}{35}\xi^2\right),$$

$$O_7^8\colon \frac{1}{432\pi^2Q^6}\left(-\tfrac{2469}{140}-12J_1-\tfrac{15}{2}J_2+96J_3-\tfrac{3261}{20}J_4+\tfrac{6684}{35}J_5-\tfrac{1215}{14}J_6+\tfrac{9}{2}\xi J_1+\tfrac{93}{85}\xi+\tfrac{81}{35}\xi^2\right).$$

$J^{PC}=1^{++}$:

$$O_1^6\colon \frac{1}{72\pi^2Q^6}\left(-\tfrac{1}{15}-5J_2+\tfrac{29}{3}J_3-\tfrac{23}{5}J_4+\tfrac{1}{10}\xi\right), \tag{A.10}$$

$$O_2^6\colon \frac{1}{108\pi^2Q^6}\left(-\tfrac{13}{15}+2J_1-9J_2-\tfrac{4}{3}J_3+\tfrac{46}{5}J_4+\xi J_1+\tfrac{9}{5}\xi\right),$$

$$O_1^8\colon \frac{1}{432\pi^2Q^8}\left(-\tfrac{59}{12}-12J_1+\tfrac{69}{2}J_2+\tfrac{68}{3}J_3-\tfrac{281}{4}J_4+30J_5-\tfrac{1}{2}\xi J_1-\tfrac{1}{2}\xi+\tfrac{3}{4}\xi^2\right),$$

$$O_2^8\colon \frac{1}{432\pi^2Q^8}\left(\tfrac{1649}{280}+17J_1-\tfrac{413}{4}J_2+175J_3-\tfrac{5639}{40}J_4+\tfrac{1903}{35}J_5-\tfrac{225}{28}J_6-\tfrac{3}{4}\xi J_1-\tfrac{3}{70}\xi+\tfrac{9}{140}\xi^2\right),$$

$$O_3^8\colon \frac{1}{432\pi^2Q^8}\left(\tfrac{347}{30}+48J_1-282J_2+\tfrac{1240}{3}J_3-\tfrac{2509}{10}J_4+60J_5+2\xi J_1+\tfrac{7}{5}\xi-\tfrac{21}{10}\xi^2\right),$$

$$O_4^8\colon \frac{1}{432\pi^2Q^8}\left(-\tfrac{8339}{420}-26J_1-\tfrac{47}{2}J_2+\tfrac{922}{3}J_3-\tfrac{5293}{20}J_4-\tfrac{1654}{35}J_5+\tfrac{1035}{14}J_6+\tfrac{1}{2}\xi J_1-\tfrac{31}{35}\xi+\tfrac{27}{14}\xi^2\right), \tag{A.11}$$

$$O_5^8\colon \frac{1}{432\pi^2Q^8}\left(-\tfrac{937}{70}-28J_1+207J_2-500J_3+\tfrac{5131}{10}J_4-\tfrac{1116}{7}J_5-\tfrac{135}{7}J_6+3\xi J_1+\tfrac{44}{35}\xi+\tfrac{102}{35}\xi^2\right),$$

$$O_6^8\colon \frac{1}{432\pi^2Q^8}\left(-\tfrac{33}{35}-16J_1+118J_2-162J_3-\tfrac{193}{5}J_4+\tfrac{6184}{35}J_5-\tfrac{540}{7}J_6+\tfrac{1}{35}\xi-\tfrac{12}{35}\xi^2\right),$$

$$O_7^8\colon \frac{1}{432\pi^2Q^8}\left(-\tfrac{85}{42}-4J_1+35J_2-\tfrac{212}{3}J_3+\tfrac{219}{2}J_4-\tfrac{908}{7}J_5+\tfrac{405}{7}J_6-\xi J_1-\tfrac{4}{7}\xi-\tfrac{54}{35}\xi^2\right).$$

(3) The coefficients C_{m_1} of $m_1\bar{q}_1q_1$ and C_G of $(\alpha_s/\pi)G^a_{\mu\nu}G^a_{\mu\nu}$ in the polarization functions of mesons consisting of quarks with masses m_1 and m_2 [29, 30] (apart from q^2-independent terms); m_1 is

considered to be light and m_2 heavy, so for $\langle m_2\bar{q}_2q_2\rangle$ we can use the heavy quark mass expansion (see chapter 3).

$J^{PC}=1^{--}$:

$$C_{m_1}=\left(1-\frac{\bar{q}^2}{q^2}\frac{u^2}{3}\right)\left(\frac{\bar{q}^4}{8m_1^4q^4}(1-u^2)-\frac{(m_1-m_2)^2}{2m_1^2q^4}\right)+\frac{(m_1+m_2)}{3m_1q^4}\,, \tag{A.12}$$

$$C_G=\frac{1}{48}\frac{1}{\bar{q}^4}\left[\frac{3(1-u^2)^2(1+u^2)}{u^4}\frac{1}{2u}\ln\frac{1+u}{1-u}-\frac{3u^4-2u^2+3}{u^4}\right.$$
$$\left.+\frac{(m_1-m_2)^2}{q^2}\left(\frac{3(1-u^2)(3+u^2)}{u^2}\frac{1}{2u}\ln\frac{1+u}{1-u}-\frac{3u^4+4u^2+9}{u^2(1-u^2)}\right)\right]+\frac{1}{12}C_{m_1}\,, \tag{A.13}$$

$\bar{q}^2$, and u^2 are defined by (3.3). The axial vector case $J^{PC}=1^{++}$ can be obtained from (A.12) and (A.13) by changing $m_1\to -m_1$ i.e. $u\to 1/u$ and $\bar{q}^2\to\bar{q}^2u^2$.

$J^{PC}=0^{-+}$:

$$C_{m_1}=(8m_1^4q^2)^{-1}(\bar{q}^2)^2(1-u)^2-(2m_1^2q^2)^{-1}(m_1-m_2)^2\,, \tag{A.14}$$

$$C_{G_1}=\frac{\alpha_s}{48\pi}\frac{1}{4m_1m_2}\frac{q^2}{\bar{q}^2}\left[\frac{3(3u^2+1)(1-u^2)^2}{u^4}\frac{1}{2u}\ln\frac{1+u}{1-u}-\frac{9u^4+4u^2+3}{u^4}\right]+\tfrac{1}{12}C_{m_1}\,. \tag{A.15}$$

The coefficients for the scalar case can be found from (A.14) and (A.15) by replacing $m_1\to -m_1$.

(4) The expressions for the polarization functions of light quark mesons [1, 14, 15] (including mass corrections) for the currents $j_\Gamma=\bar{q}\Gamma q$ of (2.2)

$J^{PC}=1^{--}$:

$$(q_\mu q_\nu-q^2g_{\mu\nu})\left\{-\frac{1}{4\pi^2}\left(1+\frac{\alpha_s}{\pi}\right)\ln\frac{Q^2}{\mu^2}+\frac{6m^2}{Q^2}+\frac{2}{Q^4}m\bar{q}q+\frac{\alpha_s}{12\pi Q^4}G^a_{\mu\nu}G^a_{\mu\nu}\right.$$
$$\left.-\frac{2\pi\alpha_s}{Q^6}(\bar{q}\gamma_\mu\gamma_5\lambda^aq)(\bar{q}\gamma_\mu\gamma_5\lambda^aq)-\frac{4\pi\alpha_s}{9Q^6}(\bar{q}\gamma_\mu\lambda^aq)(\bar{q}\gamma_\mu\lambda^aq)\right\}. \tag{A.16}$$

$J^{PC}=0^{-+}$:

$$\frac{3}{8\pi^2}\left(1+\frac{11}{3}\frac{\alpha_s}{\pi}\right)Q^2\ln\frac{Q^2}{\mu^2}-\frac{1}{Q^2}m\bar{q}q+\frac{\alpha_s}{8\pi Q^2}G^a_{\mu\nu}G^a_{\mu\nu}+\frac{\pi\alpha_s}{Q^4}(\bar{q}\sigma_{\mu\nu}\gamma_5\lambda^aq)(\bar{q}\sigma_{\mu\nu}\gamma_5\lambda^aq)$$
$$+\frac{2\pi\alpha_s}{3Q^4}(\bar{q}\gamma_\mu\lambda^aq)\sum_{q=u,d,s,\ldots}\bar{q}\gamma_\mu\lambda^aq\,. \tag{A.17}$$

$J^{PC}=0^{++}$:

$$\frac{3}{8\pi^2}\left(1+\frac{11}{3}\frac{\alpha_s}{\pi}\right)Q^2\ln\frac{Q^2}{\mu^2}+\frac{3}{Q^2}m\bar{q}q+\frac{\alpha_s}{8\pi Q^2}G^a_{\mu\nu}G^a_{\mu\nu}+\frac{mg_s}{2Q^4}(\bar{q}\sigma_{\mu\nu}\lambda^aq)G^a_{\mu\nu}$$
$$+\frac{\pi\alpha_s}{Q^4}(\bar{q}\sigma_{\mu\nu}\lambda^aq)(\bar{q}\sigma_{\mu\nu}\lambda^aq)+\frac{2\pi\alpha_s}{3Q^4}(\bar{q}\gamma_\mu\lambda^aq)\sum_{q=u,d,s,\ldots}\bar{q}\gamma_\mu\lambda^aq\,. \tag{A.18}$$

$J^{PC} = 1^{++}$:

$$(q_\mu q_\nu - q^2 g_{\mu\nu})\left\{-\frac{1}{4\pi^2}\left(1+\frac{\alpha_s}{\pi}\right)\ln\frac{Q^2}{\mu^2} - \frac{2}{Q^4} m\bar{q}q + \frac{\alpha_s}{12\pi Q^4} G^a_{\mu\nu}G^a_{\mu\nu} - \frac{2\pi\alpha_s}{Q^6}(\bar{q}\gamma_\mu\lambda^a q)(\bar{q}\gamma_\mu\lambda^a q)\right.$$

$$\left. - \frac{4\pi\alpha_s}{9Q^6}(\bar{q}\gamma_\mu\lambda^a q) \sum_{q=u,d,s,\ldots} \bar{q}\gamma_\mu\lambda^a q\right\}. \tag{A.19}$$

$J^{PC} = 1^{+-}$:

$$(q_\mu q_\nu - q^2 g_{\mu\nu})\left\{\frac{1}{8\pi^2}\left(1+\frac{\alpha_s}{\pi}\right)Q^2\ln\frac{Q^2}{\mu^2} - \frac{\alpha_s}{24\pi Q^2} G^a_{\mu\nu}G^a_{\mu\nu} + \frac{mg_s}{6Q^4}(\bar{q}\sigma_{\mu\nu}\lambda^a q)G^a_{\mu\nu}\right.$$

$$\left. + \frac{4\pi\alpha_s}{Q^4}(\bar{q}\gamma_5\lambda^a q)(\bar{q}\gamma_5\lambda^a q) - \frac{4\pi\alpha_s}{9Q^4}(\bar{q}\gamma_\mu\lambda^a q) \sum_{q=u,d,s,\ldots} \bar{q}\sigma_{\mu\nu}\lambda^a q\right\}. \tag{A.20}$$

$J^{PC} = 2^{++}$:

$$P_{\mu\nu\rho\sigma}\left\{-\frac{3}{10\pi^2}\left(1-\frac{\alpha_s}{\pi}\right)Q^4\ln\frac{Q^2}{\mu^2} + \text{constant} + \frac{8\alpha_s}{9\pi}\ln\frac{Q^2}{\mu^2} G^a_{\mu\nu}G^a_{\mu\nu} - \frac{4\pi\alpha_s}{Q^2}(\bar{q}\gamma_\mu\lambda^a q)(\bar{q}\gamma_\mu\lambda^a q)\right.$$

$$\left. - \frac{4mg_s}{3Q^2}(\bar{q}\sigma_{\mu\nu}\lambda^a q)G^a_{\mu\nu}\right\},$$

$$P_{\mu\nu\rho\sigma} = \tfrac{1}{2}(\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho} - \tfrac{2}{3}\eta_{\mu\nu}\eta_{\rho\sigma}), \qquad \eta_{\mu\nu} = q_\mu q_\nu/q^2 - g_{\mu\nu}. \tag{A.21}$$

The polarization function for the $J^{PC} = 2^{-+}$ current is identical to (A.21) except for the four fermion operator which should be replaced by

$$-\frac{4\pi\alpha_s}{Q^2}(\bar{q}\gamma_5\gamma_\mu\lambda^a q)(\bar{q}\gamma_5\gamma_\mu\lambda^a q).$$

For studying states with one or two strange quarks one has to add mass corrections to (A.16)–(A.21). For most currents it is sufficient to consider mass corrections to the bare loop only. In addition, of course, the terms in (A.16)–(A.21) which contain a mass have to be changed accordingly as well as the quark condensates. The mass corrections to the bare loop for the invariant functions are

$J^{PC} = 1^{--}$:

$$\bar{s}s:\quad \Delta\Pi(q) = -\frac{3}{2\pi^2}\frac{m_s^2}{Q^2}, \tag{A.16a}$$

$$\bar{s}u:\quad \Delta\Pi(q) = -\frac{3}{8\pi^2}\frac{m_s^2}{Q^2}\ln Q^2. \tag{A.16b}$$

For the following cases we just give the equal mass results

$$J^{PC} = 0^{-+}:\quad \Delta\Pi(q) = \frac{3}{4\pi^2} m_s^2 \ln Q^2, \tag{A.17a}$$

$$J^{PC} = 0^{++}: \quad \Delta\Pi(q) = \frac{9}{4\pi^2} m_s^2 \ln Q^2 , \tag{A.18a}$$

$$J^{PC} = 1^{++}: \quad \Delta\Pi(q) = \frac{3}{4\pi^2} m_s^2 \ln Q^2 . \tag{A.19a}$$

In the 2^{++} case we have to add to the invariant function in the $\bar{s}s$ case

$$\Delta\Pi(q) = -\frac{1}{2\pi^2} m_s^2 Q^2 \ln \frac{Q^2}{m_s^2} + \frac{4m_s^3}{Q^2} \bar{s}s + \left(\tfrac{29}{6} \ln \frac{Q^2}{m_s^2} - \tfrac{153}{18}\right) \frac{m_s^2}{Q^2} \frac{\alpha_s}{\pi} G^a_{\mu\nu} G^a_{\mu\nu} . \tag{A.21a}$$

(5) The polarization functions of all octet and decuplet $L = 0$ baryons up to operators of dimension $d = 6$ and to first order in the strange quark mass m; $\langle\bar{q}q\rangle = \langle\bar{u}u\rangle = \langle\bar{d}d\rangle$. For the octet states we have

$$\Pi(q) = \mathrm{i} \int \mathrm{d}^4x\, \mathrm{e}^{\mathrm{i}qx} \langle 0|\mathrm{T}(\eta(x), \bar{\eta}(0))|0\rangle = \Pi_1(q^2) + \not{q}\,\Pi_2(q^2) , \tag{A.22}$$

with $\Pi_1(^2)$ and $\Pi_2(q^2)$ as follows:

$$\text{N}: \quad \Pi_1(q^2) = -\frac{1}{4\pi^2} \langle\bar{q}q\rangle q^2 \ln(-q^2) , \tag{A.23}$$

$$\Pi_2(q^2) = \frac{1}{64\pi^4} q^4 \ln(-q^2) + \frac{1}{32\pi^2} \left\langle \frac{\alpha_s}{\pi} G^2 \right\rangle \ln(-q^2) + \frac{2\langle\bar{q}q\rangle^2}{3q^2} .$$

$$\Lambda: \quad \Pi_1(q^2) = -\frac{1}{12\pi^2} (4\langle\bar{q}q\rangle - \langle\bar{s}s\rangle) q^2 \ln(-q^2) - \frac{m}{96\pi^4} q^4 \ln(-q^2) + \frac{4m}{9q^2} (3\langle\bar{q}q\rangle^2 - \langle\bar{s}s\rangle\langle\bar{q}q\rangle) , \tag{A.24}$$

$$\Pi_2(q^2) = \frac{1}{64\pi^4} q^4 \ln(-q^2) - \frac{m}{12\pi^2} (4\langle\bar{q}q\rangle - 3\langle\bar{s}s\rangle) \ln(-q^2) + \frac{1}{32\pi^2} \left\langle \frac{\alpha_s}{\pi} G^2 \right\rangle \ln(-q^2)$$
$$+ \frac{2}{9q^2} (4\langle\bar{s}s\rangle\langle\bar{q}q\rangle - \langle\bar{q}q\rangle^2) .$$

$$\Sigma: \quad \Pi_1(q^2) = -\frac{1}{4\pi^2} \langle\bar{s}s\rangle q^2 \ln(-q^2) + \frac{m}{32\pi^2} q^4 \ln(-q^2) + \frac{4m}{3q^2} \langle\bar{q}q\rangle^2 ,$$

$$\Pi_2(q^2) = \frac{1}{64\pi^4} q^4 \ln(-q^2) + \frac{m}{4\pi^2} \langle\bar{s}s\rangle \ln(-q^2) + \frac{1}{32\pi^2} \left\langle \frac{\alpha_s}{\pi} G^2 \right\rangle \ln(-q^2) + \frac{2\langle\bar{q}q\rangle^2}{3q^2} . \tag{A.25}$$

$$\Xi: \quad \Pi_1(q^2) = -\frac{1}{4\pi^2} \langle\bar{q}q\rangle q^2 \ln(-q^2) + \frac{2m}{q^2} \langle\bar{q}q\rangle\langle\bar{s}s\rangle , \tag{A.26}$$

$$\Pi_2(q^2) = \frac{1}{64\pi^4} q^4 \ln(-q^2) + \frac{1}{32\pi^2} \left\langle \frac{\alpha_s}{\pi} G^2 \right\rangle \ln(-q^2) + \frac{2\langle\bar{s}s\rangle^2}{3q^2} .$$

For the decuplet states the tensor structure is more complicated, but we are only interested in the invariant functions proportional to $g_{\mu\nu}$ and $g_{\mu\nu}\not{q}$,

$$\Pi_{\mu\nu}(q) = \mathrm{i}\int \mathrm{d}^4x\, \mathrm{e}^{\mathrm{i}qx}\langle 0|\mathrm{T}(\eta_\mu(x), \bar{\eta}_\nu(0))|0\rangle = g_{\mu\nu}\Pi_1(q^2) + g_{\mu\nu}\not{q}\,\Pi_2(q^2) + \cdots, \tag{A.27}$$

with $\Pi_1(q^2)$ and $\Pi_2(q^2)$ as follows:

$$\Delta:\quad \Pi_1(q^2) = -\frac{1}{3\pi^2}\langle\bar{q}q\rangle q^2\ln(-q^2) + \frac{1}{6\pi^2}\left\langle g\bar{q}\sigma_{\mu\nu}\frac{\lambda^a}{2}G^a_{\mu\nu}q\right\rangle\ln(-q^2), \tag{A.28}$$

$$\Pi_2(q^2) = \frac{1}{10}\frac{1}{(2\pi)^4}q^4\ln(-q^2) - \frac{5}{9}\frac{1}{32\pi^2}\left\langle\frac{\alpha_s}{\pi}G^2\right\rangle\ln(-q^2) + \frac{4\langle\bar{q}q\rangle^2}{3q^2}.$$

$$\Sigma^*:\quad \Pi_1(q^2) = -\frac{1}{9\pi^2}(2\langle\bar{q}q\rangle + \langle\bar{s}s\rangle)q^2\ln(-q^2) + \frac{1}{6\pi^2}\left\langle g\bar{q}\sigma_{\mu\nu}\frac{\lambda^a}{2}G^a_{\mu\nu}q\right\rangle\ln(-q^2) + \frac{m}{64\pi^4}q^4\ln(-q^2) + \frac{2m}{3q^2}\langle\bar{q}q\rangle^2, \tag{A.29}$$

$$\Pi_2(q^2) = \frac{1}{10}\frac{1}{(2\pi)^4}q^4\ln(-q^2) - \frac{5}{9}\frac{1}{32\pi^2}\left\langle\frac{\alpha_s}{\pi}G^2\right\rangle\ln(-q^2) - \frac{m}{12\pi^2}(4\langle\bar{q}q\rangle - \langle\bar{s}s\rangle)\ln(-q^2) + \frac{4}{9}\frac{\langle\bar{q}q\rangle^2 + 2\langle\bar{s}s\rangle\langle\bar{q}q\rangle}{q^2}.$$

$$\Xi^*:\quad \Pi_1(q^2) = -\frac{1}{9\pi^2}(\langle\bar{q}q\rangle + 2\langle\bar{s}s\rangle)q^2\ln(-q^2) + \frac{1}{6\pi^2}\left\langle g\bar{q}\sigma_{\mu\nu}\frac{\lambda^a}{2}G^a_{\mu\nu}q\right\rangle\ln(-q^2) + \frac{m}{32\pi^4}q^4\ln(-q^2) + \frac{4m}{3q^2}\langle\bar{q}q\rangle\langle\bar{s}s\rangle, \tag{A.30}$$

$$\Pi_2(q^2) = \frac{1}{10}\frac{1}{(2\pi)^4}q^4\ln(-q^2) - \frac{5}{9}\frac{1}{32\pi^2}\left\langle\frac{\alpha_s}{\pi}G^2\right\rangle\ln(-q^2) - \frac{m}{6\pi^2}(2\langle\bar{q}q\rangle + \langle\bar{s}s\rangle)\ln(-q^2) + \frac{4}{9}\frac{\langle\bar{s}s\rangle^2 + 2\langle\bar{s}s\rangle\langle\bar{q}q\rangle}{q^2}.$$

$$\Omega:\quad \Pi_1(q^2) = -\frac{1}{3\pi^2}\langle\bar{s}s\rangle q^2\ln(-q^2) + \frac{1}{6\pi^2}\left\langle g\bar{q}\sigma_{\mu\nu}\frac{\lambda^a}{2}G^a_{\mu\nu}q\right\rangle\ln(-q^2) + \frac{3m}{64\pi^4}q^4\ln(-q^2) + \frac{2m}{q^2}\langle\bar{s}s\rangle^2, \tag{A.31}$$

$$\Pi_2(q^2) = \frac{1}{10}\frac{1}{(2\pi)^4}q^4\ln(-q^2) - \frac{5}{9}\frac{1}{32\pi^2}\left\langle\frac{\alpha_s}{\pi}G^2\right\rangle\ln(-q^2) - \frac{3m}{4\pi^2}2\langle\bar{s}s\rangle\ln(-q^2) + \frac{4}{3}\frac{\langle\bar{s}s\rangle^2}{q^2}.$$

Corrections of order m^2 to the octet as well as the decuplet formulas can be found in [76], while [43] contains contributions from higher dimensional operators.

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