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Improved Bounds for the Slope and Curvature of $\bar{B} \to D^{(*)}\ell\,\bar{\nu}$ Form Factors

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Abstract

We derive a theoretically allowed domain for the slope $\widehat{\varrho}^2$ and curvature $\widehat{c}$ of the physical form factor appearing in the decay $\bar{B} \to D^*\ell\,\bar{\nu}$. Using heavy-quark symmetry, we relate this function to a particular $\bar{B} \to D$ form factor free of ground-state B_c poles below the threshold for BD production, for which almost model-independent constraints are derived from QCD using unitarity and analyticity. Our results are of interest for the extraction of $|V_{cb}|$ from the recoil spectrum in exclusive semileptonic B decays. Making conservative estimates of the theoretical uncertainties, we find (up to $1/m_Q$ corrections) $\widehat{\varrho}^2 < 1.11$ and $\widehat{c} \simeq 0.66\widehat{\varrho}^2 - 0.11$. We also derive the corresponding bounds for the form factor in the decay $\bar{B} \to D\,\ell\,\bar{\nu}$.

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1 Introduction

One of the most accurate methods of extracting the parameter $|V_{cb}|$ of the Cabibbo-Kobayashi-Maskawa matrix is based on the study of the exclusive semileptonic decay $\bar{B} \to D^* \ell\, \bar{\nu}$. Heavy-quark symmetry (for a review see Ref. [1]) can be employed to eliminate, to a large extent, hadronic uncertainties from the theoretical description of this process [2]–[5]. The analysis consists in measuring the recoil spectrum, i.e. the distribution in the kinematic variable

$$w = v_B \cdot v_{D^*} = \frac{E_{D^*}}{m_{D^*}} = \frac{m_B^2 + m_{D^*}^2 - q^2}{2 m_B m_{D^*}}\,, \tag{1}$$

which is the product of the four-velocities of the mesons. Here E_{D^*} denotes the recoil energy of the D^* meson in the parent rest frame, and $q^2 = (p_B - p_{D^*})^2$ is the momentum transfer squared. The differential decay rate is given by [5]

$$\begin{aligned}\frac{\mathrm{d}\Gamma(\bar{B} \to D^* \ell\, \bar{\nu})}{\mathrm{d}w} &= \frac{G_F^2}{48\pi^3}\, (m_B - m_{D^*})^2\, m_{D^*}^3 \sqrt{w^2 - 1}\, (w+1)^2 \\ &\times \left[1 + \frac{4w}{w+1}\, \frac{m_B^2 - 2w\, m_B m_{D^*} + m_{D^*}^2}{(m_B - m_{D^*})^2} \right] |V_{cb}|^2\, \mathcal{F}^2(w)\,, \end{aligned} \tag{2}$$

where $\mathcal{F}(w)$ is the (suitably defined) hadronic form factor for the decay. Apart from symmetry-breaking corrections that can be calculated using the heavy-quark effective theory [6], $\mathcal{F}(w)$ coincides with the universal Isgur-Wise function $\xi(w)$ [3, 4], which describes the long-distance physics associated with the light degrees of freedom in the heavy mesons and is normalized to unity at the zero-recoil point $w = 1$. As a consequence, the normalization of the physical form factor is determined up to small perturbative corrections and power corrections of order $(\Lambda_{\mathrm{QCD}}/m_Q)^2$ [7], where we use m_Q generically for m_b or m_c. Detailed calculations of these corrections lead to $\mathcal{F}(1) = 0.91 \pm 0.03$ [5], [8]–[11]. Therefore, an accurate determination of $|V_{cb}|$ can be obtained by extrapolating the differential decay rate to $w = 1$.

This analysis has been performed by several experimental groups [12]–[15]. The existing data are compatible with a linear dependence of the form factor $\mathcal{F}(w)$ on w, with possible corrections induced by a non-zero curvature. In general, one may define

$$\mathcal{F}(w) = \mathcal{F}(1) \left\{ 1 - \widehat{\varrho}^2\, (w-1) + \widehat{c}\, (w-1)^2 + O[(w-1)^3] \right\}, \tag{3}$$

where $\widehat{\varrho}$ and $\widehat{c}$ are referred to as the charge radius and the convexity, respectively. In order to guide the extrapolation to zero recoil, it is of interest to gain theoretical information about these parameters. In the present paper, we readdress this problem and derive bounds for $\widehat{\varrho}^2$ and $\widehat{c}$ that are much stronger than those obtained in previous analyses [16]–[19].

A model-independent method of constraining the q^2 dependence of form factors using analyticity properties of QCD spectral functions and unitarity was proposed some time ago in Refs. [20]–[23]. More recently, the same method was applied to the elastic form factor of the B meson [16]–[18], which is related by heavy-quark symmetry to the Isgur-Wise function. The resulting bounds prove to be rather weak, however, due to the presence of the Υ poles below the $B\bar{B}$ threshold. The lack of information about the residues of these poles, related to the unknown $\Upsilon B\bar{B}$ couplings, reduce considerably the constraining power of the method. The technique of treating, in an optimal way, poles with unknown residues is to multiply the form factor by specific functions in the complex q^2 plane (the so-called Blaschke factors) with zeros at the positions of the poles, but with unit modulus along the physical cut. Since these functions have modulus less than unity below the cut (and in particular at the zero-recoil point), they weaken the bounds on the form factor. Hence, the more poles there are below threshold, the weaker these bounds are. In a recent paper, Boyd et al. have applied this approach to the form factors of interest for $\bar{B}\to D^{(*)}\,\ell\,\bar{\nu}$ decays, i.e. to the matrix elements of flavour-changing heavy-quark currents [19].[1] However, once again the constraining power of the method is strongly reduced because of the presence of possible $(\bar{b}c)$ bound states below the threshold for $BD^{(*)}$ production. Up to now, pseudoscalar B_c and vector B_c^* mesons have not been observed experimentally; they are predicted, however, by potential models [26]–[28]. The uncertainty in these model calculations weakens the derived constraints even further.

Instead of relying on model-dependent predictions about the properties of $B_c^{(*)}$ mesons, we adopt a different strategy: we identify a specific $\bar{B}\to D$ form factor, which does not receive contributions from the ground-state B_c poles. By applying the method of Refs. [16]–[23], we derive strong model-independent constraints on the slope and curvature of this form factor. Heavy-quark symmetry is then used to relate the form factor to the function $\mathcal{F}(w)$ describing $\bar{B}\to D^*\ell\,\bar{\nu}$ decays, and to translate the constraints into bounds for the charge radius $\widehat{\varrho}$ and the convexity $\widehat{c}$. These relations receive symmetry-breaking corrections, which can however be estimated. They turn out to weaken the bounds only softly, so that our constraints are stronger than those obtained in previous analyses. Thus, our results should be used in future determinations of $|V_{cb}|$. At the end, we briefly consider the corresponding bounds on the form factor in the decay $\bar{B}\to D\,\ell\,\bar{\nu}$.

2 Derivation of the bounds

We consider the flavour-changing vector current $V^\mu=\bar{c}\,\gamma^\mu b$ and write its matrix element between $\bar{B}$- and D-meson states in terms of hadronic form factors $F_0(q^2)$

[1] Applications to the heavy-to-light transitions $\bar{B}\to\pi,\rho\,\ell\,\bar{\nu}$ have been considered in Refs. [24, 25].

and $F_1(q^2)$ defined as $(q = p - p')$

$$\langle D(p')|V^\mu|\bar{B}(p)\rangle = \left[(p+p')^\mu - \frac{m_B^2 - m_D^2}{q^2}\, q^\mu\right] F_1(q^2) + \frac{m_B^2 - m_D^2}{q^2}\, q^\mu\, F_0(q^2)\,. \quad (4)$$

The form factors are real analytic functions in the complex q^2 plane, cut along the real axis from the branch point $t_0 = (m_B + m_D)^2$ to infinity. Below the threshold t_0, poles can appear at $q^2 = m^2_{B_c^*}$, and also branch points due to non-resonant states. Their contribution to the form factors can be obtained using crossing symmetry and the unitarity relation

$$2\,\mathrm{Im}\,\langle\, 0\,|V^\mu|\bar{B}(p)\,\bar{D}(-p')\rangle = \sum\!\!\!\!\!\!\int_{\Gamma} \mathrm{d}\rho_\Gamma\,(2\pi)^4\,\delta^{(4)}(p_\Gamma - q)\,\langle\, 0\,|V^\mu|\Gamma\rangle\,\langle\Gamma|\bar{B}(p)\,\bar{D}(-p')\rangle\,, \quad (5)$$

where the sum is over all possible hadron states Γ with the appropriate flavour quantum numbers. The form factor $F_1(q^2)$ receives pole contributions from B_c^* vector mesons.[2] The scalar form factor $F_0(q^2)$, however, does not couple to ground-state B_c or B_c^* mesons. We note that $F_0(q^2)$ can, in principle, receive contributions from scalar B_c states; however, these are expected to be broad resonances above the branch point due to two-particle intermediate states of the form $(B_c^{(*)} + h)$, where h is a light hadron. We shall discuss these sub-threshold singularities below. Assuming for the moment that their effect is negligible, the form factor $F_0(q^2)$ can be considered a real analytic function in the complex q^2 plane except for the cut from t_0 to infinity.

Consider then the vacuum polarization tensor

$$\begin{aligned}\Pi^{\mu\nu}(q) &= i\int \mathrm{d}^4x\, e^{iq\cdot x}\,\langle\, 0\,|\,T\{V^\mu(x), V^{\dagger\nu}(0)\}|\,0\,\rangle \\ &= (q^\mu q^\nu - g^{\mu\nu}q^2)\,\Pi(q^2) + g^{\mu\nu}D(q^2)\,.\end{aligned} \quad (6)$$

The invariant amplitudes $\Pi(q^2)$ and $D(q^2)$ satisfy once-subtracted dispersion relations, so it is convenient to consider their first derivatives with respect to q^2. Applying a unitarity relation similar to (5) to the corresponding spectral functions, we obtain the positivity conditions $(t = q^2)$

$$\mathrm{Im}\,\Pi(t + i\epsilon) \geq 0\,, \qquad \mathrm{Im}\,D(t + i\epsilon) \geq 0\,; \qquad t \geq t_0\,. \quad (7)$$

Therefore, if we retain in the unitarity sum only the contribution of the two-particle state $|\bar{B}\,\bar{D}\rangle$, we obtain rigorous lower bounds on the spectral functions. Being interested in the form factor $F_0(t)$, we retain the inequality for the longitudinal amplitude $D(t)$, which reads

$$\mathrm{Im}\,D(t + i\epsilon) \geq \frac{n_f}{16\pi t^2}\, t_0\, t_1 \sqrt{(t - t_0)(t - t_1)}\,|F_0(t)|^2\,, \quad (8)$$

[2] Similarly, the form factors parametrizing the matrix elements for $\bar{B} \to D^*$ transitions receive pole contributions from ground-state pseudoscalar (B_c) or vector (B_c^*) mesons.

where

$$t_0 = (m_B + m_D)^2\,, \qquad t_1 = (m_B - m_D)^2\,, \tag{9}$$

and n_f is the number of light flavours. The factor of n_f appears since SU(n_f) light-flavour multiplets of heavy-meson states contribute with the same weight to the unitarity sum. Below we will take $n_f = 2.5$ to account for the breaking of flavour symmetry by the mass of the strange quark. Using the above inequality in the dispersion relation

$$D'(q^2) = \frac{1}{\pi}\int\limits_{t_0}^{\infty}\mathrm{d}t\,\frac{\mathrm{Im}\,D(t+i\epsilon)}{(t-q^2)^2}\,, \tag{10}$$

we obtain the inequality

$$\frac{n_f}{16\pi^2}\,\frac{t_0\,t_1}{D'(q^2)}\int\limits_{t_0}^{\infty}\mathrm{d}t\,\frac{\sqrt{(t-t_0)(t-t_1)}}{t^2(t-q^2)^2}\,|F_0(t)|^2 \le 1\,. \tag{11}$$

On the other hand, if $q^2 \ll (m_b + m_c)^2$, the quantity $D'(q^2)$ can be calculated using the operator product expansion. The leading-order expression is

$$D'(q^2) = \frac{N_c}{4\pi^2}\int\limits_0^1\mathrm{d}x\,x(1-x)\,\frac{x m_b^2 + (1-x)m_c^2 - m_b m_c}{x m_b^2 + (1-x)m_c^2 - x(1-x)q^2}\,, \tag{12}$$

where m_b and m_c are the heavy-quark masses. The choice of the value of q^2 will be discussed below. We note that, for $q^2 = 0$, one obtains the simple result

$$D'(0) = \frac{N_c}{24\pi^2}\left\{\frac{(1-4r+r^2)(1+r+r^2)}{(1-r^2)^2} - \frac{12r^3\ln r}{(1-r^2)^3}\right\}, \tag{13}$$

where $r = m_c/m_b$. Expressions for the order-α_s corrections to the polarization functions in the case of unequal quark masses have been derived in Refs. [29, 30]. The leading non-perturbative contribution is proportional to the gluon condensate and, for dimensional reasons, suppressed by four powers of a large mass scale. As in the case of QCD sum rules for the charmonium and bottomonium systems [31], this contribution is very small. Below, we shall include possible effects of all these corrections in a very conservative way.

It is convenient to perform the conformal mapping

$$z = \frac{\sqrt{t_0 - t} - \sqrt{t_0 - t_1}}{\sqrt{t_0 - t} + \sqrt{t_0 - t_1}}\,, \tag{14}$$

which transforms the cut t-plane onto the interior of the unit disc $|z| < 1$, such that the point t_1 is mapped onto the origin $z = 0$, while the branch point t_0 is mapped onto $z = -1$. Applying standard techniques in the theory of analytic functions [32], we write the inequality (11) as

$$\frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\theta \, |\phi(e^{i\theta})|^2 \, |\tilde{F}(e^{i\theta})|^2 \leq 1 \,, \tag{15}$$

where $\tilde{F}(z) \equiv F_0(t(z))$. The function $\phi(z)$ is analytic and without zeros inside the unit disc, and its modulus squared on the boundary of the unit disc is equal to the weight function appearing in (11) multiplied by the Jacobian of the conformal transformation. In general

$$\ln \phi(z) = \frac{1}{4\pi}\int_0^{2\pi} \mathrm{d}\theta \, \frac{e^{i\theta}+z}{e^{i\theta}-z} \, \ln |\phi(e^{i\theta})|^2 \,. \tag{16}$$

In the present case, a straightforward calculation gives [23]:

$$\phi(z) = \phi(0) \, \frac{(1+z)(1-z)^{3/2}}{(1-d_1 z)^2(1-d_2 z)^2} \,, \tag{17}$$

where

$$\begin{aligned} \phi(0) &= \sqrt{\frac{n_f}{2\pi D'(q^2)}} \, \frac{\sqrt{t_0\, t_1}}{t_0 - t_1} \, \frac{(1-d_1)^2(1-d_2)^2}{16} \,, \\ d_1 &= \frac{\sqrt{t_0} - \sqrt{t_0 - t_1}}{\sqrt{t_0} + \sqrt{t_0 - t_1}} \,, \\ d_2 &= \frac{\sqrt{t_0 - q^2} - \sqrt{t_0 - t_1}}{\sqrt{t_0 - q^2} + \sqrt{t_0 - t_1}} \,. \end{aligned} \tag{18}$$

The inequality (15) is a condition for the norm on the Hilbert space H^2 of analytic functions, which implies rigorous constraints for the values of the function $\tilde{F}(z)$ and its derivatives at interior points [32]. We shall transform these constraints into bounds on the slope and curvature of the function

$$f_0(w) = f_0(1) \left\{ 1 - \varrho_0^2\,(w-1) + c_0\,(w-1)^2 + O[(w-1)^3] \right\}, \tag{19}$$

which is related to the original form factor $F_0(q^2)$ by

$$F_0(q^2) = \frac{m_B + m_D}{2\sqrt{m_B m_D}} \left[1 - \frac{q^2}{(m_B+m_D)^2} \right] f_0(w(q^2)) \,. \tag{20}$$

The relation between w and q^2 is given by (1) with m_D^* replaced by m_D. The definition (20) is such that in the heavy-quark limit the function $f_0(w)$ coincides with the Isgur-Wise form factor [33]. The zero-recoil point $w = 1$ corresponds to

$q^2 = t_1$, i.e. to $z = 0$. The parameters $f_0(1)$, ϱ_0^2 and c_0 are related to the function $\tilde{F}(z)$ and its derivatives with respect to z through

$$\begin{aligned} \tilde{F}(0) &= \beta\, f_0(1)\,, \\ \frac{\tilde{F}'(0)}{\tilde{F}(0)} &= 4 - 8\varrho_0^2\,, \\ \frac{\tilde{F}''(0)}{\tilde{F}(0)} &= 128 c_0 - 96\varrho_0^2 + 16\,, \end{aligned} \tag{21}$$

where

$$\beta = \frac{2\sqrt{m_B m_D}}{m_B + m_D} \simeq 0.879\,. \tag{22}$$

Using well-known results in the interpolation theory for the Hilbert space H^2 [32], we obtain from (15) the inequality

$$[(\phi\,\tilde{F})(0)]^2 + [(\phi\,\tilde{F})'(0)]^2 + \frac{1}{4}\,[(\phi\,\tilde{F})''(0)]^2 < 1\,. \tag{23}$$

Written in terms of ϱ_0^2 and c_0, this becomes the equation for an ellipse:

$$(\varrho_0^2 - \rho^2)^2 + 64\left[(c_0 - k) - \chi\,(\varrho_0^2 - \rho^2)\right]^2 < K^2\,, \tag{24}$$

where

$$\begin{aligned} \rho^2 &= \frac{7}{16} + \frac{d_1 + d_2}{4}\,, \\ \chi &= \frac{11}{16} + \frac{d_1 + d_2}{4}\,, \\ k &= \frac{115}{512} + \frac{11}{64}\,(d_1 + d_2) + \frac{d_1^2 + 4 d_1 d_2 + d_2^2}{64}\,, \\ K &= \frac{1}{8}\sqrt{\frac{1}{[\phi(0)\,\beta\,f_0(1)]^2} - 1}\,. \end{aligned} \tag{25}$$

From (24), one can derive strict upper and lower bounds on the charge radius and the convexity of the form factor $f_0(w)$, which read

$$\begin{aligned} \rho^2 - K < \varrho_0^2 < \rho^2 + K\,, \\ k - \sqrt{\tfrac{1}{64} + \chi^2}\,K < c_0 < k + \sqrt{\tfrac{1}{64} + \chi^2}\,K\,. \end{aligned} \tag{26}$$

Note that the large numerical factor 64 in front of the second term in (24) implies a strong correlation between the slope parameter ϱ_0^2 and the curvature c_0 (since $K^2 \ll 1$, see below). In other words, the resulting ellipse in the ϱ_0^2–c_0 plane is almost degenerate to a line, and to a good approximation

$$c_0 \simeq \chi\varrho_0^2 + (k - \chi\rho^2)\,. \tag{27}$$

It is remarkable that the parameters in this relation only depend on the meson masses and q^2; the dynamical information encoded in $\phi(0)$ and $f_0(1)$ does not enter here (once these parameters are such that $K^2 \ll 64$).

Let us comment, at this point, on an obvious extension of our approach. Instead of (23), one could write a more general inequality involving higher derivatives of the product $\phi\,\tilde{F}$. This would further constrain the form factor near zero recoil. We refrain from presenting such an extension because of two approximations inherent in our treatment: first, sub-threshold singularities give an increasingly larger contribution to higher derivatives of $\phi\,\tilde{F}$ (see eq. (34) below); secondly, the relations among the form factors implied by heavy-quark symmetry, which will be used later to derive from our results bounds on the form factors of interest in semileptonic B decays, are expected to break down for higher derivatives of the form factors.

For the numerical evaluation of the bounds, we first consider the case where $q^2 = 0$, so that

$$d_1 = d_2 = \left(\frac{\sqrt{m_B} - \sqrt{m_D}}{\sqrt{m_B} + \sqrt{m_D}}\right)^2 \simeq 0.065\,. \tag{28}$$

This leads to $\rho^2 \simeq 0.470$, $\chi \simeq 0.720$ and $k \simeq 0.247$. The only model-dependent quantity is the "radius" K, which depends on the product $\phi(0)\,f_0(1)$. Using $r = m_c/m_b = 0.29 \pm 0.03$ for the ratio of the heavy-quark masses, as well as $n_f = 2.5$, we find $\phi(0) = 0.282 \pm 0.018$. As discussed above, QCD corrections are expected to modify this result in a moderate way. Moreover, in the heavy-quark limit we have (including short-distance corrections) $f_0(1) \simeq 1.02$ [1], and corrections to this result are of order $(\Lambda_{\rm QCD}/m_Q)^2$ and should not exceed 10%. Thus, we believe it is conservative to assume that the total uncertainty in the product $\phi(0)\,f_0(1)$ is less than 30%. As our goal is to derive bounds, we take the smallest possible value, i.e. $\phi(0)\,f_0(1) = 0.20$, which leads to the largest value of K and thus to the largest allowed domain in the ϱ_0^2–c_0 plane. With this choice, we obtain $K \simeq 0.70$. According to (26), the allowed intervals for the slope and curvature are then given by

$$-\,0.23 < \varrho_0^2 < 1.17\,, \qquad -0.26 < c_0 < 0.76\,, \tag{29}$$

and the approximate relation (27) between the two parameters reads

$$c_0 \simeq 0.72\varrho_0^2 - 0.09\,. \tag{30}$$

The corresponding narrow ellipse is shown by the solid curve in Fig. 1. The allowed region can be reduced further by considering values $q^2 > 0$; however, the QCD calculation is reliable only if $q^2 \ll (m_b + m_c)^2$. As an example, we show by the dashed and dotted curves the two ellipses obtained for $q^2 = 10$ and 20 GeV^2. The gain that can be achieved in this way is rather small, and to be conservative we shall take $q^2 = 0$ from now on.

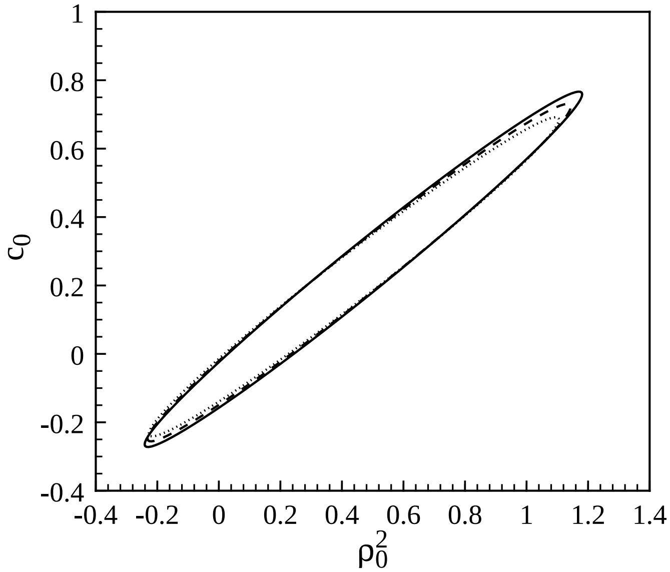

Figure 1: Allowed regions in the ϱ_0^2–c_0 plane for different values of q^2.

3 Sub-threshold singularities

A last question that has to be discussed is the effect of possible sub-threshold singularities. These are due to states of the form $(B_c^{(*)} + h)$, where h is a light hadron, or to scalar B_c resonances, which contribute to the unitarity sum in (5). Because these scalar mesons are predicted to have masses that lie above the branch point due to two-particle intermediate states [26]–[28], we shall treat them as resonances in these two-particle channels. The lowest intermediate states in the unitarity relation for the form factor $F_0(t)$ are then given by $(B_c^{(*)} + \pi)$. However, these contributions are suppressed by isospin symmetry, so the lowest states which contribute significantly are $(B_c^{(*)} + \eta, \omega)$. Consequently, the sub-threshold cut is rather short. We parametrize its contribution to the discontinuity of the form factor by (a similar parametrization was adopted in Ref. [19]):

$$\mathrm{Im}\, F_0(t + i\epsilon) = \frac{C}{\sqrt{t_0}} \sqrt{t - t_+}\,, \quad t_+ \leq t \leq t_0\,, \tag{31}$$

where $t_+ = (m_{B_c} + m_\eta)^2$, and C is a dimensionless quantity expected to be of order unity. We estimated this quantity in a model where the form factors and amplitudes in the unitarity integral in (5) are saturated by the nearest resonances, and find that C is a slowly varying quantity in the interval $t_+ < t < t_0$, which takes values in the range 0.5–3.[3]

Once an approximation of $\mathrm{Im}\, F_0$ is adopted, the effect of sub-threshold singularities can be treated in an exact way [34]. In this case, the function $\tilde{F}(z)$

[3]We note that the contribution of the sub-threshold cut is suppressed by the OZI rule, since the couplings $B_c B_c^* h$ appearing in $\mathrm{Im}\, F_0$ are described by "hair-pin diagrams".

obtained after the conformal mapping (14) is no longer analytic. Yet, the formalism presented above can be applied to the new function

$$g(z) = \phi(z)\tilde{F}(z) - \frac{1}{\pi}\int_{-1}^{z_+} \mathrm{d}x\, \frac{\phi(x)\,\mathrm{Im}\,\tilde{F}(x)}{x-z}\,, \tag{32}$$

where $z_+ \equiv z(t_+)$ is the position of the lowest sub-threshold branch point. We have used the fact that the function $\phi(z)$ is real analytic inside the unit disc. By definition, the function $g(z)$ is analytic for $|z| < 1$, since the discontinuity of the product $\phi(z)\tilde{F}(z)$ is compensated by the subtraction term. Substituting (32) into the inequality (15), we obtain

$$\frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\theta \left| g(e^{i\theta}) + \frac{1}{\pi}\int_{-1}^{z_+} \mathrm{d}x\, \frac{\phi(x)\,\mathrm{Im}\,\tilde{F}(x)}{x-e^{i\theta}}\right|^2 \leq 1\,. \tag{33}$$

By performing a Fourier expansion of the integrand, which contains now both positive- and negative-frequency Fourier coefficients, and taking into account the orthogonality of these coefficients, we obtain, after a straightforward calculation, the following inequality instead of (23):

$$\begin{aligned} &\left[\phi(0)\tilde{F}(0) - \delta_0\right]^2 + \left[(\phi\,\tilde{F})'(0) - \delta_1\right]^2 + \frac{1}{4}\left[(\phi\,\tilde{F})''(0) - \delta_2\right]^2 \\ &< 1 - \frac{1}{\pi^2}\int_{-1}^{z_+} \mathrm{d}x \int_{-1}^{z_+} \mathrm{d}y\, \frac{\phi(x)\phi(y)\,\mathrm{Im}\,\tilde{F}(x)\,\mathrm{Im}\,\tilde{F}(y)}{1-xy} \equiv 1 - \delta_{xy}\,, \end{aligned} \tag{34}$$

where

$$\delta_n = \frac{n!}{\pi}\int_{-1}^{z_+} \mathrm{d}x\, \frac{\phi(x)\,\mathrm{Im}\,\tilde{F}(x)}{x^{n+1}}\,. \tag{35}$$

This inequality shows the corrections induced by the additional branch point below the threshold t_0.

The net effect is that the parameters describing the shape of the ellipse in (24) are modified. Instead of ρ^2, k and K in (25), we now obtain new parameters $\rho^2 + \delta\rho^2$, $k + \delta k$ and $\sqrt{K^2 + \delta K^2}$, where

$$\begin{aligned} \delta\rho^2 &= -\frac{\delta_1}{8N}\,, \\ \delta k &= -\left(\frac{11}{128} + \frac{d_1 + d_2}{32}\right)\frac{\delta_1}{N} + \frac{\delta_2}{128N}\,, \\ \delta K^2 &= \frac{1}{64}\left[\frac{2\delta_0}{N} - \frac{\delta_0^2 + \delta_{xy}}{N^2}\right], \end{aligned} \tag{36}$$

and $N = \phi(0)\,\beta\,f_0(1)$. Numerically, we find that

$$\begin{aligned} \delta\rho^2 &\simeq -4.0 \times 10^{-3}\,C\,, \\ \delta k &\simeq -2.1 \times 10^{-3}\,C\,, \\ \delta K^2 &\simeq -0.7 \times 10^{-3}\,C - 10^{-5}C^2\,. \end{aligned} \tag{37}$$

For C of order unity, these corrections have a negligible effect on the bounds. Note, in particular, that the parameter χ remains unchanged, and that the combination $(k - \chi\rho^2)$ changes by only $0.8 \times 10^{-3}\,C$. Thus, the approximate linear relation (27) is essentially unaffected by sub-threshold singularities.

4 Bounds on $\bar{B} \to D^{(*)}\ell\,\bar{\nu}$ form factors

Our final task is to use heavy-quark symmetry to translate the bounds and constraints derived above into bounds for the slope and curvature of the form factor $\mathcal{F}(w)$, which describes the recoil spectrum in the decay $\bar{B} \to D^*\ell\,\bar{\nu}$. Our definition of the parameters ϱ_0^2 and c_0 was such that they agree, in the heavy-quark limit, with the slope $\widehat{\varrho}^2$ and the curvature $\widehat{c}$ of the physical form factor $\mathcal{F}(w)$. Differences are induced, however, by corrections that break the heavy-quark symmetry. In general, there are perturbative corrections of order $\alpha_s(m_Q)$, and power corrections of order $\Lambda_{\rm QCD}/m_Q$. A calculation of the latter requires non-perturbative techniques such as lattice field theory; this is beyond the scope of this paper. Here we shall include the short-distance corrections, which can be calculated in a model-independent way using perturbation theory.

To derive the relations between the slope and curvature parameters of different form factors we use the results of Refs. [35, 36, 1], where the relations between heavy-meson form factors have been calculated including one-loop QCD corrections. Expanding the corresponding lengthy expressions around the zero-recoil point $w = 1$, we find ($r = m_c/m_b$)

$$\widehat{\varrho}^2 = \varrho_0^2 - \frac{4\alpha_s}{9\pi}\Big[1 - 4\psi(r)\Big] + O(\Lambda_{\rm QCD}/m_Q) \tag{38}$$

for the relation between the slope parameters, and

$$\begin{aligned}\widehat{c} = c_0 &- \frac{2\alpha_s}{27\pi}\left[\frac{7 - 2r + 7r^2}{(1-r)^2} + \frac{1 + 42r + r^2}{(1-r)^2}\,\psi(r)\right] \\ &- \frac{4\alpha_s}{9\pi}\,\varrho_0^2\Big[1 - 4\psi(r)\Big] + O(\Lambda_{\rm QCD}/m_Q)\end{aligned} \tag{39}$$

for the relation between the curvatures, where

$$\psi(r) = \frac{r}{(1-r)^2}\left(\frac{1+r}{1-r}\,\ln r + 2\right). \tag{40}$$

To evaluate these expressions, we use $r = 0.29$ and $\alpha_s = \alpha_s(\sqrt{m_b m_c}) = 0.26$, leading to

$$\widehat{\varrho}^2 \simeq \varrho_0^2 - 0.06\,, \qquad \widehat{c} \simeq c_0 - 0.06 - 0.06\varrho_0^2\,. \tag{41}$$

The corresponding ellipse is shown by the dark-shaded area in Fig. 2. Note that the boundary of the allowed region is uncertain by an amount of order $\Lambda_{\rm QCD}/m_Q$.

Given this fact, we can safely replace the ellipse with the approximate linear relation between the curvature and the slope parameter analogous to (27). This relation is the central result of our analysis:

$$\widehat{c} \simeq 0.66\widehat{\varrho}^2 - 0.11 + O(\Lambda_{\rm QCD}/m_Q)\,. \tag{42}$$

The average experimental value of the slope parameter, as extracted from a linear fit to the recoil spectrum, is $\widehat{\varrho}^2 = 0.82\pm0.09$ [12]–[15],[37]. Allowing for a positive curvature of the form factor, the true slope parameter may be somewhat larger; however, the experimental value is already close to the upper bound for $\widehat{\varrho}^2$ in Fig. 2. Using this value, we predict that

$$\widehat{c} \gtrsim 0.43 \pm 0.06 + O(\Lambda_{\rm QCD}/m_Q)\,. \tag{43}$$

We thus expect a moderate positive curvature of the form factor $\mathcal{F}(w)$.

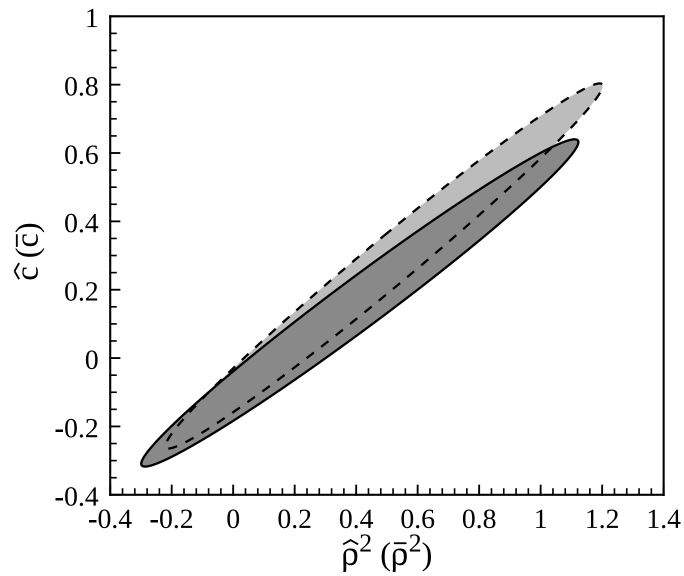

Figure 2: Allowed regions for the slope and curvature of the form factors $\mathcal{F}(w)$ (dark-shaded area) and $\mathcal{G}(w)$ (light-shaded area), which describe $\bar{B} \to D^{(*)}\ell\,\bar{\nu}$ decays.

Let us, for completeness, also consider the semileptonic decay $\bar{B} \to D\,\ell\,\bar{\nu}$. The differential decay rate for this process is given by [5]

$$\frac{\mathrm{d}\Gamma(\bar{B}\to D\,\ell\,\bar{\nu})}{\mathrm{d}w} = \frac{G_F^2}{48\pi^3}\,(m_B+m_D)^2\,m_D^3\,(w^2-1)^{3/2}\,|V_{cb}|^2\,\mathcal{G}^2(w)\,, \tag{44}$$

where $w = v_B\cdot v_D$, and the hadronic form factor $\mathcal{G}(w)$ obeys an expansion similar to (3):

$$\mathcal{G}(w) = \mathcal{G}(1)\left\{1 - \bar{\varrho}^2\,(w-1) + \bar{c}\,(w-1)^2 + O[(w-1)^3]\right\}. \tag{45}$$

In this case, the normalization at zero recoil is known only up to first-order power corrections [33]. An explicit calculation using the QCD sum-rule approach leads to $\mathcal{G}(1)/\mathcal{F}(1) = 1.08 \pm 0.06$ [38], which can be combined with $\mathcal{F}(1) = 0.91 \pm 0.03$ to give $\mathcal{G}(1) = 0.98 \pm 0.07$. Thus, in principle, a measurement of the recoil spectrum in $\bar{B} \to D\,\ell\,\bar{\nu}$ decay provides an independent determination of $|V_{cb}|$ with a theoretical accuracy not much less than in the decay $\bar{B} \to D^*\ell\,\bar{\nu}$. For the slope and curvature of the form factor $\mathcal{G}(w)$, we obtain the relations

$$\begin{aligned}
\bar{\varrho}^2 &= \varrho_0^2 + \frac{4\alpha_s}{9\pi}\Big[1 + 3\psi(r)\Big] + O(\Lambda_{\rm QCD}/m_Q)\,, \\
\bar{c} &= c_0 + \frac{4\alpha_s}{45\pi}\left[\frac{2 - 9r + 2r^2}{(1-r)^2} - \frac{30r}{(1-r)^2}\,\psi(r)\right] \\
&\quad + \frac{4\alpha_s}{9\pi}\,\varrho_0^2\Big[1 + 3\psi(r)\Big] + O(\Lambda_{\rm QCD}/m_Q)\,.
\end{aligned} \tag{46}$$

Numerically, this gives $\bar{\varrho}^2 \simeq \varrho_0^2 + 0.02$ and $\bar{c} \simeq c_0 + 0.01 + 0.02\varrho_0^2$. The corresponding ellipse is shown by the light-shaded area in Fig. 2. The approximate linear relation is

$$\bar{c} \simeq 0.74\bar{\varrho}^2 - 0.09 + O(\Lambda_{\rm QCD}/m_Q)\,. \tag{47}$$

5 Conclusion

Using analyticity, unitarity and heavy-quark symmetry, we have derived from QCD conservative bounds on the slope and curvature of the form factors describing the semileptonic decays $\bar{B} \to D^{(*)}\ell\,\bar{\nu}$. The allowed regions for these parameters are displayed in Fig. 2. Our method employs heavy-quark symmetry in such a way as to avoid the problem of sub-threshold poles due to ground-state B_c mesons. Thus, our bounds are stronger than the ones obtained in previous analyses. In particular, we find that the curvature $\widehat{c}$ and the slope $\widehat{\varrho}^2$ of the form factor $\mathcal{F}(w)$ governing the decay $\bar{B} \to D^*\ell\,\bar{\nu}$ are related by $\widehat{c} \simeq 0.66\,\widehat{\varrho}^2 - 0.11$, and that $\widehat{\varrho}^2 < 1.11$. We propose to use the first of these relations in future determinations of $|V_{cb}|$ from the recoil spectrum in this decay. It allows the linear fit of this spectrum to be extended to a quadratic one without introducing a new parameter.

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