

Semileptonic B and Λ_b Decays and Local Duality in QCD

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The inclusive and exclusive semileptonic decay distributions for $b \rightarrow c$ decay are computed in the Shifman-Voloshin limit. The inclusive decay distributions (computed using an operator product expansion) depend on quark masses, and the exclusive decay distributions depend on hadron masses. Nevertheless, we show explicitly how the first two terms in the $1/m$ expansion match between the inclusive and exclusive decays. Agreement between the inclusive and exclusive decay rates requires a minimum smearing region of size Λ_{QCD} before local duality holds in QCD. The α_s corrections to the inclusive and exclusive decay rates are also shown to agree to order $(\log m)/m^2$. The α_s/m^2 corrections are used to obtain the α_s correction to Bjorken's inequality on the slope of the Isgur-Wise function.

I. INTRODUCTION

The semileptonic decay of hadrons containing b quarks allows one to measure the V_{ub} and V_{cb} elements of the quark mixing matrix. Reliable model independent values for these matrix elements can only be obtained if one can accurately calculate the hadronic decay distributions (or decay rates) from QCD. Two approaches to semileptonic $b \rightarrow c$ decay are to study exclusive decay modes such as $B \rightarrow De^- \nu_e$, $B \rightarrow D^* e^- \nu_e$, or $\Lambda_b \rightarrow \Lambda_c e^- \nu_e$, or inclusive decay modes such as $B \rightarrow X_{u,c} e^- \nu_e$. The inclusive and exclusive modes have been analyzed using heavy quark effective theory (HQET) [1,2].

The exclusive decay rates can be obtained by computing decay form factors using HQET, and then integrating over the allowed phase space. The final answer typically depends on the hadron masses, and on hadron matrix elements of various quark and gluon operators.

The inclusive decay rate can be obtained using an operator product expansion (OPE) to write the square of the decay amplitude as an expansion in a series of local operators. The matrix elements of this series between hadron states then gives the decay rate as an expansion in $1/m_b$. The Wilson coefficients are functions of kinematic variables q^2 and $v \cdot q$. The OPE can be justified for values of $v \cdot q$ which are far from the physical region; the utility of the expansion comes from relating a contour integral in the unphysical region of the complex $v \cdot q$ plane to an integral over the physical region [3]. Unfortunately, the contour integral always has a segment close to the physical region, where the OPE may not be valid. Nevertheless, it is expected that the OPE computation of the differential distributions (including non-perturbative corrections) is valid, provided the results are smeared over an energy of order Λ_{QCD} . The idea that the parton calculation agrees with the full QCD answer for *smeared* inclusive distributions is known as local duality.

It is important to test the assumptions used in the OPE based calculation of inclusive decay rates. One way to do this is to compare the inclusive decay rate to the exclusive decay rate summed over all allowed channels; they must be equal by definition. This equality will arise nontrivially, because the OPE gives the inclusive rate in terms of quark masses, rather than hadron masses. Higher dimension operators in the OPE must enter in precisely the correct way to compensate for the mismatch. A demonstration that this occurs alleviates many of the concerns about the validity of local duality in QCD.

In this paper, we demonstrate by explicit computation that the inclusive decay rates computed by an OPE or by summing over exclusive rates are equal for hadrons containing a heavy quark in the Shifman-Voloshin (SV) limit [2] $m_b, m_c \gg \delta m = m_b - m_c \gg \Lambda_{\text{QCD}}$. The equality will be shown to hold to two orders in the $1/m$ expansion, and to first order in α_s , for which explicit calculations exist in HQET. In the SV limit, there are $1/m$ corrections to the inclusive and exclusive decay rates. We discuss the origin of these $1/m$ corrections, and show how they match between the exclusive and inclusive decays. The differential decay distributions for the inclusive and exclusive decay rates are also shown to be equal, provided they are smeared over a region of size Λ_{QCD} . This is the expected size of the minimum smearing region required before local duality holds in QCD. We discuss the variables that should be used when comparing inclusive and exclusive decay distributions. Finally, by studying the $1/m^2$ corrections, we

obtain inequalities on the slope of the Isgur-Wise function. We also comment on hadronic matrix element inequalities obtained earlier in the literature. The α_s/m^2 corrections to the decay width are used to obtain the α_s corrections to Bjorken's bound on the slope of the Isgur-Wise function at zero recoil.

The kinematics of semileptonic B decay are reviewed in Sec. II. The inclusive decay rate for $b \rightarrow c$ is computed in Sec. III; the exclusive decays $B \rightarrow D$ and $B \rightarrow D^*$ are computed in Sec. IV, and are shown to agree to two orders in the $1/m$ expansion in the SV limit. The electron spectrum and differential decay distributions for the inclusive and exclusive decays are shown to agree to two orders in $1/m$ in the SV limit in Sec. V. The α_s corrections to the decay width and decay tensors are studied in Sec. VI. The α_s correction to Bjorken's bound is derived in Sec. VII. We also study inequalities on hadronic matrix elements in this section. The results of earlier sections are extended to Λ_b decays in Sec. VIII.

II. KINEMATICS

In this section, we review some well-known results for the kinematics of three-body decays which will be used extensively in the rest of this paper. We will consider the decay of a hadron H_b containing a b -quark. The formulæ can be applied to the decay of a b -quark by replacing the hadron mass M_{H_b} by the quark mass, m_b .

Consider the decay $H_b \rightarrow Xe^- \nu_e$, where X is some hadronic state with invariant mass M_X .¹ The momenta of H_b , X , e and ν_e will be denoted by p_{H_b} , p_X , p_e and p_ν respectively, and $q = p_{H_b} - p_X = p_e + p_\nu$ is the momentum transferred from the hadron system to the leptons by the virtual W boson. The velocity four-vector v is defined by $p_{H_b} = M_{H_b}v$, so that $v = (1, 0, 0, 0)$ defines the rest frame of the decay hadron. The hadronic matrix elements for $H_b \rightarrow Xe^- \nu_e$ can only depend on the variables q^2 , $q^0 = q \cdot v$, and the masses M_{H_b} and M_X . A quantity of experimental interest is the electron energy E_e in the rest frame of the decaying hadron.

The momentum transfer to the hadronic system varies between $0 \leq q^2 \leq q_{\max}^2$, where

$$q_{\max}^2 = (M_{H_b} - M_X)^2. \quad (2.1)$$

At $q^2 = q_{\max}^2$ the final state hadronic system X is at rest in the rest frame of the decaying H_b hadron. This kinematic point is known as the zero-recoil point. The mass-shell condition $p_X^2 = (p_{H_b} - q)^2 = M_X^2$ relates q^2 and $q \cdot v$,

$$q \cdot v = \frac{M_{H_b}^2 - M_X^2 + q^2}{2M_{H_b}}. \quad (2.2)$$

As q^2 ranges from 0 to q_{\max}^2 for a fixed value of M_X , $q \cdot v$ ranges from $(M_{H_b}^2 - M_X^2)/2M_{H_b}$ to $M_{H_b} - M_X$. The relation between q^2 and $q \cdot v$ is plotted in Fig. 1 for different values of the final state hadronic mass M_X . The boundaries of the allowed region are (including all allowed values of M_X)

$$q^2 = (q \cdot v)^2, \quad (2.3a)$$

$$q^2 = 0, \quad 0 \leq q \cdot v \leq \frac{M_{H_b}^2 - M_{\min}^2}{2M_{H_b}} \quad (2.3b)$$

$$q \cdot v = \frac{M_{H_b}^2 - M_{\min}^2 + q^2}{2M_{H_b}}, \quad 0 \leq q^2 \leq (M_{H_b} - M_{\min})^2. \quad (2.3c)$$

The upper edge of the allowed region, $q^2 = (v \cdot q)^2$, corresponds to the zero-recoil point for different hadronic states. The minimum hadronic mass M_{\min} is the pion mass for $b \rightarrow u$ decays, and is the D meson mass for $b \rightarrow c$ decays.

In the q^2 - E_e plane, E_e varies from 0 to $(M_{H_b}^2 - M_X^2)/2M_{H_b}$, and for a given value of E_e , the q^2 range is

$$0 \leq q^2 \leq 2E_e M_{H_b} - \frac{2E_e M_X^2}{M_{H_b} - 2E_e}. \quad (2.4)$$

The maximum allowed value for q^2 , Eq. (2.1), corresponds to an electron energy of $(M_{H_b} - M_X)/2$. The allowed region in the q^2 - E_e plane is plotted in Fig. 2. For a given hadronic mass M_X , the allowed region is the interior of one of the curves.

¹ X can be a multi-particle state. The electron and neutrino masses will be neglected for simplicity.

In the SV limit, $M_{H_b}, M_X \gg M_{H_b} - M_X \gg \Lambda_{\text{QCD}}$, the kinematically allowed regions for the variables for a fixed value of M_X are:

$$0 \leq q^2 \leq (\delta M_X)^2, \quad (2.5)$$

$$\delta M_X - \frac{(\delta M_X)^2}{2M_{H_b}} \leq q \cdot v \leq \delta M_X, \quad (2.6)$$

$$0 \leq E_e \leq \delta M_X - \frac{(\delta M_X)^2}{2M_{H_b}} \approx \delta M_X, \quad (2.7)$$

where $\delta M_X \equiv M_{H_b} - M_X$. The phase space volume for three-body decay, which is proportional to $\int dq^2 dE_e$, is of order $(\delta M_X)^3$ in the SV limit.

III. INCLUSIVE DECAYS

Semileptonic $b \rightarrow c$ decay is due to the weak hamiltonian density

$$\begin{aligned} H_W &= -V_{cb} \frac{4G_F}{\sqrt{2}} \bar{c}\gamma^\mu P_L b \bar{e}\gamma_\mu P_L \nu_e \\ &= -V_{cb} \frac{4G_F}{\sqrt{2}} J_h^\mu J_{\ell\mu}, \end{aligned} \quad (3.1)$$

where P_L is the left handed projection operator $(1 - \gamma_5)/2$, and J_h^μ and J_ℓ^μ are the hadronic and leptonic currents, respectively. The inclusive differential decay rate for a hadron H_b containing a b -quark to decay semileptonically, $H_b \rightarrow X_{u,c} e \bar{\nu}_e$ is determined by the hadronic tensor

$$\begin{aligned} W^{\mu\nu} &= (2\pi)^3 \sum_X \delta^4(p_{H_b} - q - p_X) \times \\ &\langle H_b(v, s) | J_h^{\mu\dagger} | X \rangle \langle X | J_h^\nu | H_b(v, s) \rangle. \end{aligned} \quad (3.2)$$

The hadron state $|H_b(v, s)\rangle$ is normalized to v^0 instead of to the usual relativistic normalization of $2E$, as this is more convenient for the heavy quark expansion. $W^{\mu\nu}$ can be expanded in terms of five form factors if one spin-averages over the initial state,

$$\begin{aligned} W^{\mu\nu} &= -g^{\mu\nu} W_1 + v^\mu v^\nu W_2 - i\epsilon^{\mu\nu\alpha\beta} v_\alpha q_\beta W_3 \\ &+ q^\mu q^\nu W_4 + (q^\mu v^\nu + q^\nu v^\mu) W_5. \end{aligned} \quad (3.3)$$

W_1 and W_2 have mass dimension -1 , W_3 and W_5 have mass dimension -2 , and W_4 has mass dimension -3 . The form factors are functions of the invariants q^2 and $q \cdot v$, and will also depend on the initial hadron H_b and the final quark mass m_c . The spin averaged differential semileptonic decay rate is

$$\begin{aligned} \frac{d\Gamma}{dq^2 dE_e dE_\nu} &= \frac{|V_{cb}|^2 G_F^2}{2\pi^3} \left[W_1 q^2 \right. \\ &\left. + W_2 \left(2E_e E_\nu - \frac{1}{2} q^2 \right) + W_3 q^2 (E_e - E_\nu) \right], \end{aligned} \quad (3.4)$$

where E_e and E_ν are the electron and neutrino energies in the H_b rest frame, q^2 is the invariant mass of the lepton pair, and the kinematic variables are to be integrated over the region $q^2 \leq 4E_e E_\nu$. The terms proportional to q^μ or q^ν in Eq. (3.2) do not contribute to the decay rate if one neglects the electron mass.

The invariant tensors W_i can be written as

$$W_i = -\frac{1}{\pi} \text{Im} T_i, \quad (3.5)$$

where T_i are defined via the hadronic matrix element of the time-ordered product of the two currents,

$$\begin{aligned}
T^{\mu\nu} &= -i \int d^4x \, e^{-iq \cdot x} \langle H_b | T \left(J_h^{\mu\dagger}(x) J_h^\nu(0) \right) | H_b \rangle \\
&= -g^{\mu\nu} T_1 + v^\mu v^\nu T_2 - i \epsilon^{\mu\nu\alpha\beta} v_\alpha q_\beta T_3 + q^\mu q^\nu T_4 \\
&\quad + (q^\mu v^\nu + q^\nu v^\mu) T_5.
\end{aligned} \tag{3.6}$$

The tensors T_i have been computed to order $1/m^2$ [4–7]. We will compute the inclusive and exclusive rates not only for the case of physical interest, in which the weak currents are left-handed, but also for more general currents. For this reason, it is useful to break T_i into the pieces that arise from the time ordered product of two vector currents, T_i^{VV} , two axial currents, T_i^{AA} , and one vector and axial current T_i^{AV} , which have been computed separately [7]. The explicit expressions for the tensors are given in Appendix A for completeness. The $1/m^2$ corrections are written in terms of two hadronic matrix elements,

$$G = Z_b \langle H_b(v) | \bar{b}_v \frac{g\sigma^{\mu\nu} G_{\mu\nu}}{4} b_v | H_b(v) \rangle, \tag{3.7}$$

$$K = - \langle H_b(v) | \bar{b}_v \frac{(iD)^2}{2} b_v | H_b(v) \rangle, \tag{3.8}$$

which give the energy of the heavy quark in the hadron due to the color magnetic moment interaction and kinetic energy, respectively. Z_b is a renormalization factor equal to unity at a scale $\mu = m_b$. In the $m_b \rightarrow \infty$ limit, K and G are finite, and of order Λ_{QCD}^2 . The matrix elements K and G differ from the dimensionless matrix elements K_b and G_b of Ref. [6] by a factor of m_b^2 , $K = m_b^2 K_b$ and $G = m_b^2 G_b$.

The total inclusive decay rate for the case of $V - A$ decay is given by using Eqs. (A1), and has been obtained previously [6–8],

$$\begin{aligned}
\Gamma^L(B \rightarrow X_c) &= \frac{G_F^2 m_b^5}{192\pi^3} |V_{cb}|^2 \left[(1 - 8r + 8r^3 - r^4 - 12r^2 \log r) + \frac{G}{m_b^2} (3 - 8r + 24r^2 - 24r^3 + 5r^4 + 12r^2 \log r) \right. \\
&\quad \left. + \frac{K}{m_b^2} (-1 + 8r - 8r^3 + r^4 + 12r^2 \log r) \right],
\end{aligned} \tag{3.9}$$

where $r = m_c^2/m_b^2$. Expanding the inclusive decay rate in the SV limit for $b \rightarrow c$ decay gives

$$\begin{aligned}
\Gamma^L(B \rightarrow X_c) &= \frac{G_F^2}{192\pi^3} |V_{cb}|^2 \left[\frac{64}{5} (\delta m)^5 - \frac{96}{5} \frac{(\delta m)^6}{m_b} + 64 \frac{(\delta m)^4 G}{m_b} \right. \\
&\quad \left. + \frac{32}{35m_b^2} (\delta m)^5 (9(\delta m)^2 - 154G - 14K + \dots) \right] + \dots,
\end{aligned} \tag{3.10}$$

where $\delta m = m_b - m_c$ is the difference of quark masses. Terms of order $1/m_b^3$ have been neglected. In addition, we have neglected $1/m_b^2$ terms of order $\Lambda_{\text{QCD}}^3/\delta m m_b^2$. Such terms can arise in the SV limit on expanding the $1/m_b^3$ corrections to semileptonic decay, which have not been computed.

The leading order term in the decay rate has changed from $G_F^2 m_b^5/192\pi^3$ to $G_F^2 (\delta m)^5/15\pi^3$. The phase space volume is proportional to $(\delta m)^3$ in the SV limit; the remaining $(\delta m)^2$ arises from factors such as q^2 in Eq. (3.4). In the limit $m_b \rightarrow \infty$ with δm fixed, the first term approaches a constant. The second and third terms in Eq. (3.10) are $1/m_b$ corrections to the decay rate, since δm and G are both fixed in the SV limit as $m_b \rightarrow \infty$. There are no $1/m_b$ corrections to the tensors T_i , since the only possible operator that can occur in HQET at order $1/m$, $\bar{b}(v \cdot D)b$ has vanishing matrix elements by the equations of motion [3]. Nevertheless, in the SV limit, one still obtains a $1/m$ correction to the inclusive rate. The G operator contribution to T_i gives a term proportional to $(\delta m)^4$ in the total rate, rather than $(\delta m)^5$, which enhances the correction by one power of m_b in the SV limit. It is interesting that the $1/m_b$ correction depends only on G , and not on K . We will return to this point in the section on exclusive decays.

One can also compute the total inclusive rate for a purely vector current “weak decay” using T_i^{VV} . The resulting computation is straightforward, and leads to the expression

$$\Gamma^V(B \rightarrow X_c) = \frac{G_F^2 m_b^5}{96\pi^3} |V_{cb}|^2 \left[1 - 2\sqrt{r} - 8r - 18r^{3/2} + 18r^{5/2} + 8r^3 + 2r^{7/2} - r^4 - 12(r^{3/2} + r^2 + r^{5/2}) \log r \right]$$

$$\begin{aligned}
& +G \left(3 + \frac{26\sqrt{r}}{3} - 8r - 18r^{3/2} + 24r^2 + 18r^{5/2} - 24r^3 - \frac{26r^{7/2}}{3} + 5r^4 + \left(8\sqrt{r} - 12r^{3/2} + 12r^2 + 12r^{5/2} \right) \log r \right) \\
& +K \left(-1 + 2\sqrt{r} + 8r + 18r^{3/2} - 18r^{5/2} - 8r^3 - 2r^{7/2} + r^4 + 12 \left(r^{3/2} + r^2 + r^{5/2} \right) \log r \right) \Bigg]. \tag{3.11}
\end{aligned}$$

Expanding this in the SV limit gives

$$\begin{aligned}
\Gamma^V(B \rightarrow X_c) &= \frac{G_F^2}{192\pi^3} |V_{cb}|^2 \left[\frac{64}{5} (\delta m)^5 - \frac{96}{5} \frac{(\delta m)^6}{m_b} \right. \\
&+ \frac{32}{35m_b^2} (\delta m)^5 \left(12(\delta m)^2 - 70G - 14K + \dots \right) \Bigg] \\
&+ \dots \tag{3.12}
\end{aligned}$$

The structure is similar to the expansion of the left-handed current in Eq. (3.10), except that the $1/m_b$ correction is independent of any hadronic matrix elements. We will return to Eqs. (3.10) and (3.12) after we have computed the exclusive decay rates.

IV. EXCLUSIVE DECAYS

In the SV limit for B meson decay via $b \rightarrow c$ transitions, the dominant hadronic final states are the D and D^* . All other states contribute to the decay rate only at order $1/m_b^2$ because of heavy quark symmetry. At zero recoil, the leading order terms in the $1/m$ expansion of the vector and axial currents are generators of the heavy quark symmetry in the effective theory, so acting on the B meson, they can only produce a linear combination of the D and D^* . The amplitude to produce excited states must either be of order $1/m$, or vanish as $v \cdot v' \rightarrow 1$. In either case, the decay rate has at least a $1/m^2$ suppression factor.

An interesting example of this suppression is for the decays $B \rightarrow D\pi e\nu$ and $B \rightarrow D^*\pi e\nu$. Chiral perturbation theory can be used to compute the amplitude for these decays when the pion momentum is small compared with the chiral symmetry breaking scale Λ_χ . One might expect that the amplitude for producing the additional pion is proportional to \mathbf{p}_π/f_π , and so is unsuppressed when $\mathbf{p}_\pi \sim f_\pi$. However, an explicit computation [9] shows that there is a cancellation in the sum of the pion emission from the initial meson, final meson, and vertex, and that the total amplitude at zero recoil is proportional to $M_{B^*} - M_B$ or $M_{D^*} - M_D$, and is of order $1/m$.

The $B \rightarrow D$ and $B \rightarrow D^*$ matrix elements of the vector and axial currents can be parameterized in terms of six form factors,

$$\langle D(p') | V^\mu | \bar{B}(p) \rangle = (p + p')^\mu f_+ + (p - p')^\mu f_-, \tag{4.1a}$$

$$\langle D^*(p') | V^\mu | \bar{B}(p) \rangle = ig\epsilon^{\mu\nu\alpha\beta} \epsilon_\nu^* p'_\alpha p_\beta, \tag{4.1b}$$

$$\begin{aligned}
\langle D^*(p') | A^\mu | \bar{B}(p) \rangle &= f_0 \epsilon^{*\mu} + \\
&[(p + p')^\mu a_+ + (p - p')^\mu a_-] p \cdot \epsilon^*, \tag{4.1c}
\end{aligned}$$

where f_\pm , f_0 , a_\pm and g are functions of $w = v \cdot v'$ (or equivalently, of q^2). The states in Eq. (4.1) have the usual relativistic normalization of $2E$.

The contribution of the D and D^* final states to the hadronic tensors W_i can be computed by squaring the matrix elements Eq. (4.1), and averaging over the D^* polarizations. The result for a left-handed current $L = \frac{1}{2}(V - A)$ is

$$8M_B W_1 = 0, \tag{4.2}$$

$$8M_B W_2 = 4f_+^2 M_B^2 \delta(\Delta_D), \tag{4.3}$$

$$8M_B W_3 = 0, \tag{4.4}$$

$$8M_B W_4 = (f_+ - f_-)^2 \delta(\Delta_D), \tag{4.5}$$

$$8M_B W_5 = 2f_+ (f_- - f_+) M_B \delta(\Delta_D), \tag{4.6}$$

from D final states, and

$$8M_B W_1 = [g^2 (p \cdot q^2 - M_B^2 q^2) + f_0^2] \delta(\Delta_{D^*}), \quad (4.7)$$

$$8M_B W_2 = \left[-q^2 g^2 + \frac{f_0^2}{M_{D^*}^2} + 4a_+^2 \left(-M_B^2 + \frac{(M_B^2 - p \cdot q)^2}{M_{D^*}^2} \right) + f_0 a_+ \left(-4 + 4 \frac{M_B^2}{M_{D^*}^2} - 4 \frac{p \cdot q}{M_{D^*}^2} \right) \right] M_B^2 \delta(\Delta_{D^*}), \quad (4.8)$$

$$8M_B W_3 = 2g f_0 M_B \delta(\Delta_{D^*}), \quad (4.9)$$

$$8M_B W_4 = \left[-g^2 M_B^2 + \frac{f_0^2}{M_{D^*}^2} + (a_+ - a_-)^2 \left(-M_B^2 + \frac{(M_B^2 - p \cdot q)^2}{M_{D^*}^2} \right) + 2f_0 (a_+ - a_-) \frac{M_B^2 - p \cdot q}{M_{D^*}^2} \right] \delta(\Delta_{D^*}), \quad (4.10)$$

$$8M_B W_5 = \left[p \cdot q g^2 - \frac{f_0^2}{M_{D^*}^2} + 2a_+^2 \left(M_B^2 - \frac{(M_B^2 - p \cdot q)^2}{M_{D^*}^2} \right) + f_0 a_+ \left(1 - 3 \frac{(M_B^2 - p \cdot q)}{M_{D^*}^2} \right) + f_0 a_- \left(\frac{(M_B^2 - p \cdot q)}{M_{D^*}^2} - 1 \right) + 2a_+ a_- \left(-M_B^2 + \frac{(M_B^2 - p \cdot q)^2}{M_{D^*}^2} \right) \right] M_B \delta(\Delta_{D^*}) \quad (4.11)$$

from D^* final states. Δ_{D, D^*} are defined by

$$\Delta_D = (p_B - q)^2 - M_D^2, \quad (4.12)$$

$$\Delta_{D^*} = (p_B - q)^2 - M_{D^*}^2. \quad (4.13)$$

Note that all the form factors in Eqs. (4.2)–(4.11) depend implicitly on q^2 . The W 's for a vector current or an axial current can be obtained trivially from Eqs. (4.2)–(4.11) by setting either the axial or vector form factors to zero, and multiplying by four, since $L^2 = (V - A)^2/4$.

The decay rate for $B \rightarrow D$ and $B \rightarrow D^*$ is obtained by integrating W_i (or equivalently, the square of the matrix elements in Eq. (4.1)) over the allowed kinematic region using Eq. (3.4).

$$w - 1 = v \cdot v' - 1 = \frac{(M_B - M_D)^2 - q^2}{2M_B M_D}, \\ = \frac{q_{\max}^2 - q^2}{2M_B M_D}, \quad (4.14)$$

so that in the SV limit $w - 1$ is of order $1/M^2$ over the entire kinematic region for the decay process. Thus we can expand the form factors in a series in $w - 1$,

$$f(w) = f(1) + (w - 1) \frac{df}{dw}(1) + \dots \quad (4.15)$$

where f denotes any of the six form factors in Eq. (4.1).

For left-handed currents, the decay rate for $B \rightarrow D$ is

$$\Gamma^L(B \rightarrow D) = \frac{G_F^2}{192\pi^3} |V_{cb}|^2 \left[\frac{16}{5} (\delta M)^5 f_+^2 - \frac{24}{5} \frac{(\delta M)^6}{M_B} f_+^2 + \frac{8}{35M_B^2} (\delta M)^7 f_+ \left(10 \frac{df_+}{dw} + 9f_+ \right) + \dots \right], \quad (4.16)$$

and for $B \rightarrow D^*$ is

$$\Gamma^L(B \rightarrow D^*) = \frac{G_F^2}{192\pi^3} |V_{cb}|^2 \left[\frac{12}{5} (\delta M^*)^5 \frac{f_0^2}{M_B^2} + \frac{2}{5} \frac{(\delta M^*)^6}{M_B} f_0 \left(8a_+ - \frac{f_0}{M_B^2} \right) + \frac{2}{35M_B^2} (\delta M^*)^7 \left(40M_B^2 a_+^2 - 8a_+ f_0 + 22 \frac{df_0}{dw} \frac{f_0}{M_B^2} + 7 \frac{f_0^2}{M_B^2} + 8M_B^2 g^2 \right) + \dots \right], \quad (4.17)$$

where

$$\delta M = M_B - M_D, \quad (4.18)$$

$$\delta M^* = M_B - M_{D^*}, \quad (4.19)$$

are hadron mass differences, and all form factors are evaluated at $w = 1$. The neglected terms are of order $1/M_B^3$. Equations (4.16) and (4.17) are important results from this section, and will be used repeatedly in the rest of this paper. The decay rates for a purely vector current weak decay are obtained by multiplying Eqs. (4.16-4.17) by four, and setting the axial form factors to zero,

$$\Gamma^V(B \rightarrow D) = \frac{G_F^2}{192\pi^3} |V_{cb}|^2 \left[\frac{64}{5} (\delta M)^5 f_+^2 - \frac{96}{5} \frac{(\delta M)^6}{M_B} f_+^2 + \frac{32}{35M_B^2} (\delta M)^7 f_+ \left(10 \frac{df_+}{dw} + 9f_+ \right) + \dots \right], \quad (4.20)$$

$$\Gamma^V(B \rightarrow D^*) = \frac{G_F^2}{192\pi^3} |V_{cb}|^2 \left[\frac{64}{35} (\delta M^*)^7 g^2 + \dots \right]. \quad (4.21)$$

The hadron masses can be computed in terms of quark masses in HQET to order $1/m^2$,

$$M_B = m_b + \bar{\Lambda} + \frac{K}{m_b} + \frac{G}{m_b} + \frac{K'}{m_b^2} + \frac{G'}{m_b^2} + \dots, \quad (4.22)$$

$$M_D = m_c + \bar{\Lambda} + \frac{K}{m_c} + \frac{G}{m_c} + \frac{K'}{m_c^2} + \frac{G'}{m_c^2} + \dots, \quad (4.23)$$

$$M_{B^*} = m_b + \bar{\Lambda} + \frac{K}{m_b} - \frac{G}{3m_b} + \frac{K'}{m_b^2} - \frac{G'}{3m_b^2} + \dots, \quad (4.24)$$

$$M_{D^*} = m_c + \bar{\Lambda} + \frac{K}{m_c} - \frac{G}{3m_c} + \frac{K'}{m_c^2} - \frac{G'}{3m_c^2} + \dots, \quad (4.25)$$

where K and G are defined in Eqs. (3.7-3.8). K' and G' are the spin-independent and spin-dependent splittings at order $1/m^3$. The hadron mass difference δM is²

$$\begin{aligned} \delta M &= M_B - M_D = m_b - m_c + (K + G) \left(\frac{1}{m_b} - \frac{1}{m_c} \right) \\ &= \delta m \left[1 - \frac{K + G}{m_b^2} \right] + \mathcal{O} \left(\frac{1}{m_b^3} \right), \end{aligned} \quad (4.26)$$

in the SV limit. Note that the hadron mass difference δM is equal to the quark mass difference δm to order $1/m^2$. Similarly,

$$\begin{aligned} \delta M^* &= M_{B^*} - M_{D^*} \\ &= m_b - m_c + (K + G) \left(\frac{1}{m_b} - \frac{1}{m_c} \right) + \frac{4}{3} \frac{G}{m_c} \\ &= \delta m \left[1 + \frac{4G}{3m_b \delta m} - \frac{K + G}{m_b^2} + \frac{4G'}{3m_b^2 \delta m} \right] + \mathcal{O} \left(\frac{1}{m_b^3} \right), \end{aligned} \quad (4.27)$$

in the SV limit. There is a $1/m$ term in the relation between hadron and quark mass differences for the D^* , which arises because of the $D^* - D$ mass difference, and is proportional to the matrix element G . This $1/m$ term is required to reproduce the correct $1/m$ correction to the exclusive rate in Eq. (3.10), and explains why the $1/m$ correction in that formula depended only on the matrix element G , and not on K .

To compare with the exclusive rate, we also need the form factors at zero recoil in an expansion in $1/m_b$ [10],

$$f_+(1) = \frac{M_B + M_D}{2\sqrt{M_B M_D}} \left[1 + f_+^{(2)} \frac{1}{m_b^2} + \dots \right], \quad (4.28a)$$

$$f_0(1) = 2\sqrt{M_B M_{D^*}} \left[1 + f_0^{(2)} \frac{1}{m_b^2} + \dots \right], \quad (4.28b)$$

²G for B and D mesons differ by anomalous scaling between $\mu = m_b$ and $\mu = m_c$. This affects δM at order $\alpha_s G \delta m / m_b^2$, and is neglected here.

$$a_+(1) = -\frac{1}{2}\sqrt{\frac{1}{M_B M_{D^*}}} \left[1 + a_+^{(1)} \frac{1}{m_b} + \dots \right], \quad (4.28c)$$

$$g(1) = \sqrt{\frac{1}{M_B M_{D^*}}} [1 + \dots]. \quad (4.28d)$$

The slopes of f_0 and f_+ at zero recoil $df_0/dw(1)$, $df_+/dw(1)$ are simply related to the slope $-\rho^2$ of the Isgur Wise function $\xi(w) = \xi(1) - \rho^2(w-1) + \mathcal{O}(w-1)^2$ by

$$\begin{aligned} \frac{df_0}{dw}(1) &= \sqrt{M_B M_{D^*}} (1 - 2\rho^2) + \mathcal{O}(\delta m), \\ \frac{df_+}{dw}(1) &= -\rho^2 + \mathcal{O}(\delta m). \end{aligned} \quad (4.29)$$

Substituting Eqs. (4.26), (4.27), (4.28) and (4.29) into Eqs. (4.16) and (4.17), and adding, gives the total rate

$$\begin{aligned} \Gamma^L(B \rightarrow D + D^*) &= \frac{G_F^2}{192\pi^3} |V_{cb}|^2 \left[\frac{64}{5} (\delta M)^5 - \frac{96}{5} \frac{(\delta M)^6}{M_B} + 64 \frac{G (\delta M)^4}{M_B} \right. \\ &\quad \left. + \frac{8}{35M_B^2} (\delta M)^5 \left(44 (\delta M)^2 - 32\rho^2 (\delta M)^2 - 14a_+^{(1)} \delta M + 28f_+^{(2)} + 84f_0^{(2)} - 224G + 280 \frac{G'}{\delta M} + 280 \frac{G\bar{\Lambda}}{\delta M} + \frac{2240}{3} \frac{G^2}{(\delta M)^2} \right) \right] \\ &\quad + \dots \end{aligned} \quad (4.30)$$

Similarly, the total decay rate for vector current weak decay is given by

$$\Gamma^V(B \rightarrow D + D^*) = \frac{G_F^2}{192\pi^3} |V_{cb}|^2 \left[\frac{64}{5} (\delta M)^5 - \frac{96}{5} \frac{(\delta M)^6}{M_B} + \frac{32}{35M_B^2} (\delta M)^5 \left(\frac{29}{2} (\delta M)^2 - 10\rho^2 (\delta M)^2 + 28f_+^{(2)} \right) \right] + \dots \quad (4.31)$$

It is useful to rewrite the inclusive decay rates Eqs. (3.10) and (3.12) using the hadron mass difference δM , rather than the quark mass difference δm . The expressions are

$$\begin{aligned} \Gamma^L(B \rightarrow X_c) &= \frac{G_F^2}{192\pi^3} |V_{cb}|^2 \left[\frac{64}{5} (\delta M)^5 - \frac{96}{5} \frac{(\delta M)^6}{M_B} + 64 \frac{(\delta M)^4 G}{M_B} \right. \\ &\quad \left. + \frac{32}{35M_B^2} (\delta M)^5 \left(9 (\delta M)^2 - 21\bar{\Lambda} (\delta M) - 84G + 56K + \frac{70G\bar{\Lambda}}{\delta M} + \dots \right) \right] + \dots, \end{aligned} \quad (4.32)$$

$$\Gamma^V(B \rightarrow X_c) = \frac{G_F^2}{192\pi^3} |V_{cb}|^2 \left[\frac{64}{5} (\delta M)^5 - \frac{96}{5} \frac{(\delta M)^6}{M_B} + \frac{32}{35M_B^2} (\delta M)^5 \left(12 (\delta M)^2 - 21\bar{\Lambda} \delta M + 56K + \dots \right) \right] + \dots, \quad (4.33)$$

The exclusive decay rates Eqs. (4.30) and (4.31) agree with the inclusive decay rates Eqs. (4.32) and (4.33) to two orders in the $1/m$ expansion. The leading order term in the hadronic decay rate is proportional to $(\delta M)^5$ or $(\delta M^*)^5$. This functional dependence is essential for the exclusive and inclusive decay rates to match. The parameter $\bar{\Lambda}$ cancels in δM and δM^* , so $\bar{\Lambda}$ does not enter the relation between the inclusive and exclusive rates. The $1/m$ corrections to the rates also match between the inclusive and exclusive decays. The $1/m$ matching is non-trivial in the case of $V-A$ decay, where it depends on the matrix element G . The G dependence arises from the $1/m$ term in δM^* in Eq. (4.17), and implicitly through the M_{D^*} dependence in the form factors Eq. (4.28). Agreement between the inclusive and exclusive decays at order $1/m$ strongly suggests that agreement will persist to all orders in the $1/m$ expansion including non-perturbative corrections.

At order $1/m^2$, the analysis is more complicated because excited states contribute to the exclusive decays. Nevertheless, one can still obtain a useful inequality, since the B partial decay width to excited states is positive, so that

$$\Gamma^L(B \rightarrow X_c) \geq \Gamma^L(B \rightarrow D + D^*). \quad (4.34)$$

This inequality implies that

$$\frac{32}{35} \left(9 (\delta M)^2 - 21 \bar{\Lambda} (\delta M) - 84G + 56K + \frac{70G\bar{\Lambda}}{\delta M} \right) \geq \frac{8}{35} \left(44 (\delta M)^2 - 32\rho^2 (\delta M)^2 - 14a_+^{(1)} \delta M + 28f_+^{(2)} + 84f_0^{(2)} - 224G + 280 \frac{G'}{\delta M} + 280 \frac{G\bar{\Lambda}}{\delta M} + \frac{2240}{3} \frac{G^2}{(\delta M)^2} \right). \quad (4.35)$$

The parameters G , K , $\bar{\Lambda}$, $a_+^{(1)}$ and $f_+^{(2)}$ are all of order Λ_{QCD}^2 . Since $\delta M \gg \Lambda_{\text{QCD}}$, one can neglect those terms to obtain the Bjorken inequality on the slope of the meson Isgur-Wise function [11]

$$\rho^2 > \frac{1}{4}. \quad (4.36)$$

We will obtain the α_s corrections to the Bjorken inequality using the α_s corrections to the inclusive and exclusive decay rates in Sec. VI.

V. DIFFERENTIAL DISTRIBUTIONS

In this section we will show that all the differential decay distributions for semileptonic $B \rightarrow X_c$ and $B \rightarrow D + D^*$ decay agree to two orders in the $1/m_b$ expansion, provided they are smeared over a region of size Λ_{QCD} . We will first demonstrate this for the electron spectrum, and then generalize the result to the tensors T_i . The equality of the tensors T_i means that all moments of the inclusive and exclusive distributions agree to order $1/m^2$. The comparison of T_i for inclusive and exclusive decays sheds some light on how local duality arises in QCD. It also specifies which variables should be used when comparing the inclusive and exclusive decay distributions. We will only give the equations for the physically interesting case of $V - A$ interactions; it is straightforward to obtain the corresponding results for a vector current decay.

A. Electron Spectrum

The electron spectrum in semileptonic b decay can be computed using the tensors T_i , and Eq. (3.4). Expanding the known results [6–8] in the SV limit gives

$$\begin{aligned} \frac{d\Gamma}{dE_e} = \frac{2G_F^2}{\pi^3} |V_{cb}|^2 E_e^2 (E_e - \delta M) & \left[(E_e - \delta M) + \right. \\ & \left. \frac{1}{M_B} \left(2(\delta M)^2 - 5E_e \delta M + 4E_e^2 - 2G \right) \right]. \end{aligned} \quad (5.1)$$

The endpoint of the inclusive electron spectrum is obtained using quark masses,

$$E_{\text{end},q} = \frac{m_b^2 - m_c^2}{2m_b}. \quad (5.2)$$

The electron spectrum from $B \rightarrow D$ decay is

$$\begin{aligned} \frac{d\Gamma}{dE_e} = \frac{G_F^2}{2\pi^3} |V_{cb}|^2 E_e^2 (E_e - \delta M) & \left[(E_e - \delta M) + \right. \\ & \left. \frac{1}{M_B} \left((\delta M)^2 - 2E_e \delta M + 2E_e^2 \right) \right], \end{aligned} \quad (5.3)$$

and from $B \rightarrow D^*$ decay is

$$\begin{aligned} \frac{d\Gamma}{dE_e} = \frac{G_F^2}{2\pi^3} |V_{cb}|^2 E_e^2 (E_e - \delta M^*) & \left[3(E_e - \delta M^*) + \right. \\ & \left. \frac{1}{M_B} \left(7(\delta M^*)^2 - 18E_e \delta M^* + 14E_e^2 \right) \right]. \end{aligned} \quad (5.4)$$

The endpoints of the electron spectrum for $B \rightarrow D$ and $B \rightarrow D^*$ semileptonic decay are

$$E_{\text{end},D} = \frac{M_B^2 - M_D^2}{2M_B}, \quad E_{\text{end},D^*} = \frac{M_B^2 - M_{D^*}^2}{2M_B}, \quad (5.5)$$

respectively.

The electron spectrum from the inclusive decays, Eq. (5.1) agrees with the sum of the electron spectrum from $B \rightarrow D$ and $B \rightarrow D^*$ from Eq. (5.3) and Eq. (5.4). The G term arises when δM^* is rewritten in terms of δM using Eq. (4.27). It is important to note that the spectra agree when written in terms of a measurable quantity, such as the electron energy E_e . The spectra would not agree at order $1/m$ if written in terms of dimensionless variables, such as $y_{\text{incl}} = 2E_e/m_b$ and $y_{\text{excl}} = 2E_e/m_B$, because there is a $1/m$ correction proportional to $\bar{\Lambda}$ in m_b/M_B .

The endpoint of the inclusive electron spectrum agrees with the endpoint of the $B \rightarrow D$ spectrum to two orders in $1/m$,

$$E_{\text{end},D} = E_{\text{end},q} \left[1 + \mathcal{O}\left(\frac{1}{m_b^2}\right) \right], \quad (5.6)$$

but the endpoint of the $B \rightarrow D^*$ spectrum differs at order $1/m$,

$$E_{\text{end},D^*} = E_{\text{end},q} \left[1 + \frac{4G}{3m_b\delta m} + \mathcal{O}\left(\frac{1}{M_B^2}\right) \right] \quad (5.7)$$

However, the leading term in the $B \rightarrow D^*$ spectrum vanishes quadratically at its endpoint, and the first correction vanishes linearly. Thus, the difference between the quark and hadron endpoints does not enter the expression for the electron spectrum until the $1/m^2$ correction, and the two spectra agree point-by-point to order $1/m$, without any smearing.

B. The tensors T_i

We will now compare the differential decay distributions for $B \rightarrow X_c$ and $B \rightarrow D + D^*$ decays. All the information in the differential decay distributions for arbitrary decays is contained in the hadronic tensor $W_{\mu\nu}$ for VV , AV and AA currents. We will compare these for inclusive and exclusive decays, and show that they agree to two orders in $1/m$. The tensor $W_{\mu\nu}$ can be expanded into Lorentz invariant functions W_i defined in Eq. (3.3). Since q is of order δm in the SV limit, W_1 and W_2 must match to two orders in $1/m$; W_3 and W_4 must match to first order in $1/m$; W_4 does not have to match. The inclusive and exclusive decay tensors only match when suitably smeared. This is obvious, because W_i for the exclusive decays contains only δ -functions, whereas W_i for the inclusive decay contains δ -functions, as well as their first and second derivatives. The gradients of δ functions are an approximation to a δ function of shifted argument, $\delta(x+a) = \delta(x) + a\delta'(x) + \dots$. The presence of singular functions such as the delta function and its derivatives requires that the inclusive and exclusive decay tensors be smeared, so that the correction terms $\delta'(x)$ are “smaller” than the leading order term $\delta(x)$.

In comparing the inclusive with exclusive decays, it is convenient to use the spectral representation to define the time ordered product $T_{\mu\nu}$ for exclusive decays, and to compute the tensors T_i , rather than their imaginary parts W_i . The tensors T_i can be obtained trivially from W_i by the replacement

$$\delta(\Delta_D) \rightarrow \frac{1}{\Delta_D}, \quad (5.8)$$

$$\delta(\Delta_{D^*}) \rightarrow \frac{1}{\Delta_{D^*}}, \quad (5.9)$$

$$(5.10)$$

in Eqs. (4.2)–(4.11), where Δ_{D,D^*} are defined in Eq. (4.12).

The tensors T_i for inclusive decay are given in Eqs. (A1). We can expand them in the SV limit, treating $\Delta_q = m_b^2 - m_c^2 - 2m_b q \cdot v + q^2$ as typically of order $m_b \Lambda_{\text{QCD}}$. The physical region corresponds to $\Delta_q = 0$. Assuming $\Delta_q \sim m_b \Lambda_{\text{QCD}}$ means that one is working in the complex $q \cdot v$ plane at least a distance Λ_{QCD} away from the physical cut (See Fig. 3). This implies that one can only compute physical quantities smeared over an energy range of order Λ_{QCD} [12]. Expanding Eqs. (A1) gives

$$T_1^{VV} = \frac{\delta m - q \cdot v}{\Delta_q} + \mathcal{O}\left(\frac{\Lambda}{m_b^2}\right), \quad (5.11)$$

$$T_2^{VV} = 2\frac{m_b}{\Delta_q} + \mathcal{O}\left(\frac{\Lambda}{m_b^2}\right), \quad (5.12)$$

$$T_3^{VV} = 0, \quad (5.13)$$

$$T_4^{VV} = \mathcal{O}\left(\frac{1}{m_b^3}\right), \quad (5.14)$$

$$T_5^{VV} = -\frac{1}{\Delta_q} + \mathcal{O}\left(\frac{1}{m_b^2}\right), \quad (5.15)$$

$$T_1^{AA} = 2\frac{m_b}{\Delta_q} - \left(\frac{\delta m + q \cdot v}{\Delta_q} + \frac{16}{3\Delta_q^2}m_b G\right) + \mathcal{O}\left(\frac{\Lambda}{m_b^2}\right), \quad (5.16)$$

$$T_2^{AA} = 2\frac{m_b}{\Delta_q} - \frac{16}{3\Delta_q^2}m_b G + \mathcal{O}\left(\frac{\Lambda}{m_b^2}\right), \quad (5.17)$$

$$T_3^{AA} = 0, \quad (5.18)$$

$$T_4^{AA} = \mathcal{O}\left(\frac{1}{m_b^3}\right), \quad (5.19)$$

$$T_5^{AA} = -\frac{1}{\Delta_q} + \mathcal{O}\left(\frac{1}{m_b^2}\right), \quad (5.20)$$

$$T_{i \neq 3}^{AV} = 0, \quad (5.21)$$

$$T_3^{AV} = -\frac{1}{\Delta_q} + \mathcal{O}\left(\frac{1}{m_b^2}\right), \quad (5.22)$$

where $\Lambda \sim \Lambda_{\text{QCD}}$ or $\sim \delta m$. The corresponding expansions of T_i for $B \rightarrow D$ are

$$T_1^{VV} = 0, \quad (5.23)$$

$$T_2^{VV} = \frac{2M_B}{\Delta_D} + \mathcal{O}\left(\frac{\Lambda}{M_B^2}\right), \quad (5.24)$$

$$T_3^{VV} = 0, \quad (5.25)$$

$$T_4^{VV} = \mathcal{O}\left(\frac{1}{\Lambda M_B^2}\right), \quad (5.26)$$

$$T_5^{VV} = -\frac{1}{\Delta_D} + \mathcal{O}\left(\frac{1}{M_B^2}\right), \quad (5.27)$$

$$T_i^{AA} = 0, \quad (5.28)$$

$$T_i^{AV} = 0, \quad (5.29)$$

and for $B \rightarrow D^*$ are

$$T_1^{AA} = 2\frac{M_B - \delta M^*}{\Delta_{D^*}} + \mathcal{O}\left(\frac{\Lambda}{M_B^2}\right), \quad (5.30)$$

$$T_2^{AA} = \frac{2M_B + q \cdot v - \delta M^*}{\Delta_{D^*}} + \mathcal{O}\left(\frac{\Lambda}{M_B^2}\right), \quad (5.31)$$

$$T_3^{AA} = 0, \quad (5.32)$$

$$T_4^{AA} = \mathcal{O}\left(\frac{1}{\Lambda M_B^2}\right), \quad (5.33)$$

$$T_5^{AA} = -\frac{1}{\Delta_{D^*}} + \mathcal{O}\left(\frac{1}{M_B^2}\right), \quad (5.34)$$

$$T_{i \neq 3}^{AV} = 0, \quad (5.35)$$

$$T_3^{AV} = -\frac{1}{\Delta_{D^*}} + \mathcal{O}\left(\frac{1}{M_B^2}\right), \quad (5.36)$$

and T_i^{VV} are all zero to this order. In deriving these equations, we have assumed that Δ_D and Δ_{D^*} are of order $M_B\Lambda_{\text{QCD}}$.

Using the relation between the D^* and D masses Eq. (4.27), one can write

$$\Delta_{D^*} = \Delta_D + \frac{8}{3}G \quad (5.37)$$

so that

$$\frac{1}{\Delta_{D^*}} = \frac{1}{\Delta_D} - \frac{8}{3} \frac{G}{\Delta_D^2} + \dots \quad (5.38)$$

The expansion is allowed since we are assuming that Δ_D is of order $M_B\Lambda_{\text{QCD}}$. We also need the relation

$$\frac{m_b}{\Delta_q} = \frac{M_B}{\Delta_D} \left[1 + \mathcal{O}\left(\frac{1}{m_b^2}\right) \right]. \quad (5.39)$$

One can now compare T_i for inclusive and exclusive decays. Most of the $T_{1,2}$'s agree to order $1/m^2$ and $T_{3,5}$'s agree to order $1/m$. The only subtlety is that each of the tensors T_1^{VV} , and $T_{1,2}^{AA}$ for inclusive decay is equal to the corresponding tensor for exclusive decay plus the term

$$\frac{\delta m - q \cdot v}{\Delta_q}. \quad (5.40)$$

This term can be rewritten as

$$\frac{1}{2m_b} + \frac{(\delta m)^2 - q^2}{2m_b\Delta_q} \quad (5.41)$$

using the identity $\Delta_q = 2m_b(\delta m - q \cdot v) + q^2 - (\delta m)^2$. The second term is of order $1/m_b^2$, and is of the same order as terms we have neglected. The first term is of order $1/m$, and must be retained. *However, it is analytic, and has no imaginary part.* Thus $W_{\mu\nu}$, which is the imaginary part of $T_{\mu\nu}$ and describes the decay distributions, agrees between the inclusive and exclusive decays at first two orders in the $1/m$ expansion.

The above calculation shows that the first two terms of all the decay distributions must match between the inclusive and exclusive decays. It also shows that the distributions match when considered as a function of $q \cdot v$ and q^2 . They would not agree at order $1/m$ if written in terms of $p \cdot q$ and q^2 , since m_b and M_B differ at order $1/m$. We also required $\Delta_{q,D} \sim M_B\Lambda_{\text{QCD}}$ and the expansion Eq. (5.39) to obtain an equality between the inclusive and exclusive decay distributions. This means that one must smear over a region of order Λ_{QCD} before duality between the parton and hadron calculations will hold. In particular, the smearing region must be much larger than the D^*-D mass difference of 140 MeV, which is of order Λ_{QCD}^2/m . Finally, note that $T^{\mu\nu}$ for inclusive and exclusive decays can differ by polynomial terms with no imaginary parts. This corresponds to having possible subtraction constants in dispersion relations for W_i .

VI. α_s CORRECTIONS

The order $\alpha_s(m_b)$ corrections to the total weak decay rate for b quarks has been computed [13]. The α_s corrections that depend on $\alpha_s(m_b)$ should match between the inclusive and exclusive decay rates. We will see that this is indeed true, by explicit calculation. The agreement between the inclusive and exclusive decays is very interesting, because the first three terms have the form $(\delta m)^5$, $(\delta m)^6$, and $(\delta m)^6 \log(\delta m)$. All three terms must match, since the excited states first appear at order $(\delta m)^7$. We study the differential decay distributions for inclusive and exclusive decays, and show how they agree to order α_s . We then use the positivity of the partial width into excited states to obtain the α_s correction to Bjorken's inequality.

A. The Decay Width

The total weak decay rate for b quark decay can be written as

$$\Gamma = \frac{G_F^2 m_b^5}{192\pi^3} |V_{cb}|^2 \left[R \left(\frac{m_c^2}{m_b^2} \right) + \frac{\alpha_s}{\pi} A_0 \left(\frac{m_c}{m_b} \right) \right] \quad (6.1)$$

where ($r = m_c^2/m_b^2$)

$$R(r) = 1 - 8r + 8r^3 - r^4 - 12r^2 \log r \quad (6.2)$$

is the leading order term in Eq. (3.9). The function A_0 is given explicitly in [14]. Expanding A_0 and R in the SV limit gives

$$A_0 = -\frac{64}{5} (\delta m)^5 + \frac{96}{5} \frac{(\delta m)^6}{m_b} + \left(-\frac{632608}{33075} + \frac{2048}{315} \log \left(\frac{2\delta m}{m_b} \right) \right) \frac{(\delta m)^7}{m_b^2}, \quad (6.3)$$

$$R = \frac{64}{5} (\delta m)^5 - \frac{96}{5} \frac{(\delta m)^6}{m_b} + \frac{288}{35} \frac{(\delta m)^7}{m_b^2}. \quad (6.4)$$

The terms in the expansion Eq. (6.3) of A_0 must be obtained from the α_s corrections to the exclusive form factors, using the same mass renormalization scheme (in this case, on-shell renormalization). In HQET, the α_s corrections in the SV limit can be computed by matching from QCD to the effective theory with heavy b and c quarks at the scale m_b , which keeps the entire functional dependence on m_c/m_b , followed by renormalization group scaling from m_b to some hadronic scale μ of order Λ_{QCD} . The Isgur-Wise function is the matrix element of HQET operators renormalized at the scale μ , and does not contain any large logarithms. The form factors to order α_s are given in Eq. (B1).

The total B decay rate at order $(\delta m)^5$ and $(\delta m)^6$ is obtained by using the form factors Eq. (B1) and substituting in Eq. (4.16,4.17) ignoring for the moment, the scale factor S . The result is

$$\begin{aligned} \Gamma^L = \frac{G_F^2}{192\pi^3} |V_{cb}|^2 & \left[\left(1 - \frac{\alpha_s}{\pi} \right) \left(\frac{64}{5} (\delta M)^5 - \frac{96}{5} \frac{(\delta M)^6}{M_B} \right) + 64 \left(1 - \frac{4\alpha_s}{3\pi} \right) \frac{(\delta M)^4 G}{M_B} \right. \\ & \left. + \frac{1}{M_B^2} (\delta M)^7 \left(\frac{352}{35} - \frac{256}{35} \rho^2 - \frac{\alpha_s}{\pi} \left(\frac{304}{315} - \frac{704}{105} \rho^2 \right) + \dots \right) \right] + \dots, \end{aligned} \quad (6.5)$$

where we have only retained terms of order $(\delta M)^7$ in the $1/m^2$ corrections. The result Eq. (6.5) is precisely what is required for the first two terms of the inclusive and exclusive decay rates to match at order α_s . The terms of order $(\delta M)^5$ and $(\delta M)^6$ agree. The agreement between the $G(\delta M)^4$ terms can not be checked, since the α_s corrections to the inclusive decays including $1/m$ corrections have not been computed.

The $(\delta m)^7 \log(\delta m)$ term arises from renormalization group scaling between m_b and some renormalization point μ which is of order Λ_{QCD} . The anomalous dimension vanishes as $w \rightarrow 1$, which is why the log first appears multiplying the $(\delta m)^7$ term. All the form factors are multiplied by an overall scaling factor [15],

$$S = \left(\frac{\alpha_s(m_b)}{\alpha_s(\mu)} \right)^{\gamma(w)/2b_0} \quad (6.6)$$

where $\gamma(w)$ is the velocity dependent anomalous dimension,

$$\gamma(w) = \frac{16}{3} \left[w \frac{\log(w + \sqrt{w^2 - 1})}{\sqrt{w^2 - 1}} - 1 \right], \quad (6.7)$$

and b_0 is the first term in the QCD β -function. Expanding Eq. (6.6) and Eq. (6.7) in α_s gives

$$\begin{aligned} S &= 1 + \gamma(w) \frac{\alpha_s}{4\pi} \log \frac{\mu}{m_b} + \mathcal{O}(\alpha_s)^2 \\ &= 1 + \frac{8}{9} \frac{\alpha_s}{\pi} (w - 1) \log \frac{\mu}{m_b} + \mathcal{O}(w - 1)^2. \end{aligned} \quad (6.8)$$

The net effect of S is to change the slope of the form factors at zero recoil by a large logarithm. Having extracted this large log, we refrain from denoting the scale at which α_s should be evaluated, because the difference is of order α_s^2 . The contribution of the renormalization group scale factor S to the total rate can be obtained from Eq. (4.30) by the substitution

$$\rho^2 \rightarrow \rho^2 - \frac{8\alpha_s}{9\pi} \log \frac{\mu}{m_b}, \quad (6.9)$$

which gives a contribution to the total rate of

$$\frac{G_F^2}{192\pi^3} |V_{cb}|^2 \frac{2048}{315} \frac{\alpha_s}{\pi} \frac{(\delta m)^7}{m_b^2} \log \frac{\mu}{m_b}. \quad (6.10)$$

The $\log m_b$ dependence matches that in Eq. (6.3). The remaining terms in Eq. (6.3) are of order $(\delta m)^7$, or of order $(\delta m)^7 \log \delta m/\mu$. These will be discussed in the next subsection.

The order α_s corrections to the vector current decay have not been calculated previously. The exclusive decay calculation gives a prediction for the α_s corrections to the inclusive decay rate. Since the exclusive decay rate to order $(\delta m)^7$ is given entirely by $B \rightarrow D$, and $f_+(1)$ gets a renormalization group scaling correction, but no finite α_s correction, one obtains

$$\begin{aligned} \Gamma^V = \frac{G_F^2}{192\pi^3} |V_{cb}|^2 & \left[\frac{64}{5} (\delta m)^5 - \frac{96}{5} \frac{(\delta m)^6}{m_b} \right. \\ & \left. + \frac{512}{63} \frac{\alpha_s}{\pi} \frac{(\delta m)^7}{m_b^2} \log \frac{2\delta m}{m_b} + \mathcal{O}(\delta m)^7 \right]. \end{aligned} \quad (6.11)$$

We have verified the first two terms by a direct calculation of the inclusive rate.

B. The Differential Distributions

The differential distributions for the exclusive and inclusive decays agree to order α_s/m^2 . The α_s corrections to the exclusive decay tensors are simply obtained by using the expressions Eq. (B1) for the α_s corrections to the form factors, multiplying by the scale factor Eq. (6.6) and substituting in the full expressions Eq. (4.2–4.11).

The α_s corrections to the inclusive tensors arise from virtual gluon corrections, and from gluon bremsstrahlung. The virtual and real contributions are separately infrared divergent, but their sum is finite. The virtual graph contributions to W_i are proportional to $\delta(\Delta_q)$, since the final state is still a single quark. Thus the residue of the pole in Fig. 3 is modified at order α_s . The bremsstrahlung graphs have a gluon and quark in the final state, so they have a two particle cut for $0 \leq q \cdot v \leq \delta m$, as show in Fig. 3. For the exclusive and inclusive decay tensors to match, it must be that the two particle cut is suppressed by $(\delta m/m_b)^2$ relative to the α_s corrections to the residue at the pole, since the exclusive D and D^* states produce only a pole contribution, and the excited states which produce a cut in the exclusive decay tensors are of order $(\delta m/m_b)^2$. This can be verified explicitly. Falk, Luke and Savage have calculated the α_s corrections to the hadronic mass distribution in Ref. [14]. Expanding their Eq. (3.8) in the SV limit gives (for $s \neq m_c^2$)

$$\begin{aligned} \frac{d\Gamma}{ds} = \frac{G_F^2}{192\pi^3} |V_{cb}|^2 \frac{\alpha_s}{\pi} \frac{1}{s - m_c^2} \frac{8(m_b^2 - s)^5}{315m_b^9} & \left[9(m_b^2 - s)^2 \right. \\ & \left. - 35m_b \delta m (m_b^2 - s) + 42m_b^2 \delta m^2 \right], \end{aligned} \quad (6.12)$$

where s is the invariant mass of the final hadronic system. We have taken $s - m_b^2$ as of order $m_b \delta m$ and retained the leading term in δm , but kept the explicit singularity at m_c^2 since we intend to study the behavior about that singularity. Eq. (6.12) shows immediately that the two-particle cut's contribution to $d\Gamma/ds$ is of order $(\delta m)^6$, and to Γ is naively of order $(\delta m)^7$. However, there is a divergence as $s \rightarrow m_c^2$, so that the total width due to the bremsstrahlung graphs has a logarithmic contribution,

$$\frac{G_F^2}{192\pi^3} |V_{cb}|^2 \frac{\alpha_s}{\pi} \frac{2048}{315} \frac{(\delta m)^7}{m_b^2} \log \frac{m_b^2 - m_c^2}{\mu^2} \quad (6.13)$$

on integrating over the allowed region $m_c^2 \leq s \leq m_b^2$, where μ is an infrared regulator. The infrared divergence is canceled by the virtual graphs, which have a logarithmic contribution of the form $-(2048/315)(\delta m)^7 \log m_b^2/\mu^2$. The sum of the real and virtual graphs gives the infrared finite contribution $(2048/315)(\delta m)^7 \log(2\delta m/m_b)$ in Eq. (6.3). On the exclusive side, there are no infrared divergences because of confinement effects. $B \rightarrow D, D^*$ decays give a contribution of order $-(2048/315)(\delta M)^7 \log M_B^2/\Lambda^2$ to the decay width, and the excited states give a contribution of order $(2048/315)(\delta M)^7 \log 2M_B\delta M/\Lambda^2$, where Λ is some hadronic mass scale of order Λ_{QCD} .

We have shown that the only corrections to the inclusive tensors W_i at order $(\delta m)^5$ and $(\delta m)^6$ are due to the virtual graphs, and are proportional to $\delta(\Delta_q)$. Following the arguments of sec. V, all that remains is to show that the coefficients of the delta functions agree for the inclusive and exclusive decays. The α_s corrections to the tensors are not known explicitly, but we do know that their integrals over phase space agree. We expect that the tensors themselves will agree to this order as well.

VII. $1/M^2$ CORRECTIONS AND SUM RULES

We have demonstrated consistency between the inclusive and exclusive decay rates at order $1/m$. Excited states first contribute to the decay rates at order $1/m^2$. Since the contribution of the excited states to the decay rates are positive, one can get inequalities by comparing the $1/m^2$ corrections to the inclusive and exclusive decay widths.

A. An Inequality on the Slope of the Isgur-Wise Function

The inclusive $V - A$ decay width Eqs (4.32) must be greater than the sum of the $B \rightarrow D$ and $B \rightarrow D^*$ widths given in Eq. (4.30). The leading order term and the first correction cancel, so one obtains an inequality on the $1/m^2$ terms. In the SV limit, the $1/m^2$ terms can be organized as an expansion in powers of $\Lambda_{\text{QCD}}/\delta M$, since we are assuming that $m_b, m_c \gg \delta M \gg \Lambda_{\text{QCD}}$. Keeping only the leading order terms in the inequality (including α_s corrections), gives

$$\rho^2(\mu) > \frac{1}{4} + \frac{\alpha_s}{\pi} \left(\frac{2563}{945} - \frac{8}{9} \log \frac{2\delta m}{\mu} \right). \quad (7.1)$$

This is the Bjorken bound [11], $\rho^2 > 1/4$, including α_s corrections.

Choosing $\mu = 2\delta m$ in Eq. (7.1) gives the inequality

$$\rho^2(\mu) > \frac{1}{4} + \frac{\alpha_s}{\pi} \frac{2563}{945}, \quad (7.2)$$

which holds provided that μ is large compared with Λ_{QCD} , so that Λ_{QCD}/μ terms can be neglected. For example, if $\alpha_s = 0.22$, its value at m_b , the bound is

$$\rho^2(\mu) > 0.44. \quad (7.3)$$

The bound Eq. (7.2) has no heavy quark mass dependence, since $\rho^2(\mu)$ is a parameter of the effective theory. The bound was derived in the SV limit, but since ρ^2 is independent of m_b and m_c , it is valid for any numerical values of the heavy quark masses, as long as $m_b, m_c \gg \Lambda_{\text{QCD}}$.

The dependence of $\rho^2(\mu)$ for large values of μ is governed by perturbation theory, and is known to be $\rho^2(\mu) = \rho^2(\mu_0) + 8\alpha_s/(9\pi) \log \mu/\mu_0$. Since $\rho^2(\mu)$ increases with μ the most restrictive bound on ρ^2 is obtained by choosing μ in Eq. (7.2) to be as small as possible, subject to the condition that $\Lambda_{\text{QCD}}/\mu \ll 1$.

The quantity $\rho^2(\mu)$ is not directly measurable, but it can be related to the slope of the decay form factors, which can be measured. It is convenient to define $\mathcal{F}(w) = (2/(1+w))(f_0(w)/f_0(1))$, which is equal to the Isgur-Wise function $\xi(w)$ when $1/m$ and α_s corrections are neglected. Including the α_s corrections computed in Ref. [16,17] gives

$$\frac{d\mathcal{F}}{dw}(1) \approx -\rho^2 - 0.01, \quad (7.4)$$

where we have used $\mu = m_c$ and $m_c/m_b = 0.3$ to evaluate the correction. Thus the slope of \mathcal{F} at $w = 1$ is bounded above by -0.45 .

The ALEPH collaboration [18] quotes a slope of -0.39 ± 0.21 , while the CLEO collaboration [19] quotes a slope of -0.84 ± 0.13 . Both used linear fits; a QCD derived parametrization brings the ALEPH slope into closer agreement with the above CLEO slope [20]. All are consistent with the bound (7.3).

The inequality Eq. (7.1) depends on δm . The analysis of the previous section indicates that the effect of the excited states is to replace δm in the logarithm by a scale of order Λ_{QCD} . This would further strengthen the bound on the slope of the form factor at zero recoil.

B. Inequalities on Hadronic Matrix Elements

The inequality,

$$K + G > 0 \quad (7.5)$$

has been obtained in Ref. [21] by studying W_{ii} for vector current decay. Let us here consider the integral of W_i^i (summed on i) over the allowed kinematic region for semileptonic decay. This can be thought of as the decay rate for a B hadron in a modified theory, where W_i^i replaces the tensor combination in Eq. (3.4). The positivity conditions of a “real” decay must still hold, since W_i^i is the sum of the absolute value squared of hadronic matrix elements. The advantage of working with W_i^i is that the matrix element K appears in the decay rate at leading order in $1/m$. The reason is that the D and D^* rates vanish at zero recoil, and so are suppressed over the entire kinematic region in the SV limit.

The inclusive rate is given straightforwardly by integrating Eqs. (A1) using Eq. (3.4) with the square brackets replaced by $3W_1 + (v \cdot q^2 - q^2)W_4$,

$$\int W_i^i = \frac{G_F^2}{192\pi^3} \frac{16(\delta m)^3}{5m_b^2} [9(\delta m)^2 + 40K + \mathcal{O}(1/m_b^3)]. \quad (7.6)$$

The exclusive rate is computed from Eqs. (4.2)–(4.11),

$$\int W_i^i = \frac{G_F^2}{192\pi^3} \frac{(\delta M)^5}{M_B^2} \frac{144}{5} + exc + \mathcal{O}(1/M_B^3), \quad (7.7)$$

where exc denotes the contribution from excited states, and is positive. Comparing the two rates leads to the inequality

$$K > 0. \quad (7.8)$$

Though the derivation of Eq. (7.8) depends on the SV limit, it is valid even away from this limit, since K is defined in the effective theory and is independent of the quark masses. A closely related sum rule has been analyzed under different assumptions in Ref. [21]. Our relation Eq. (7.8) agrees with Eq. (121) of that reference when $m_b = m_c$. The inequality Eq. (7.5) was obtained in Ref. [21] by considering $b \rightarrow c$ decays, and then taking the limit $m_c \gg m_b$. We obviously can not take this limit here, since we are considering a physical decay.

There is an important correction to Eq. (7.8) which invalidates the above derivation of the inequality. As pointed out by Falk, Luke and Savage, there are α_s corrections which can not be neglected in the derivation of Eq. (7.8) [22]. These corrections are of order $\alpha_s (\delta m)^2$, which are parametrically much larger than K , which is of order Λ_{QCD}^2 . It is straightforward to check that the sign of the corrections is such that any value of K is allowed.

VIII. $\Lambda_B \rightarrow \Lambda_C$ DECAY

The entire analysis of $B \rightarrow D, D^*$ decay presented so far can be extended to $\Lambda_b \rightarrow \Lambda_c$ decays. We will not present the details of all the computations here, since the results are very similar to those for B meson decay.

The form factors for $\Lambda_b \rightarrow \Lambda_c$ decay are defined by [23]

$$\begin{aligned} \langle \Lambda_c, v' | V^\mu | \Lambda_b, v \rangle &= \bar{u}(v') (F_1 \gamma^\mu + F_2 v^\mu + F_3 v'^\mu) u(v), \\ \langle \Lambda_c, v' | A^\mu | \Lambda_b, v \rangle &= \bar{u}(v') (G_1 \gamma^\mu + G_2 v^\mu + G_3 v'^\mu) \gamma_5 u(v), \end{aligned}$$

where the states are normalized to v^0 . The form factors for $\Lambda_b \rightarrow \Lambda_c$ decay near zero recoil can be computed using HQET. The expansion of the form factors around $w = 1$ is given in Eq. (C4).

The decay tensors W_i for $\Lambda_b \rightarrow \Lambda_c$ are computed by squaring the matrix element and summing over the polarizations of the final state Λ_c . The results for the spin-independent decay tensors are summarized in Eq. (C1). Decay tensors S_i for inclusive polarized Λ_b decay were given in Ref. [6]. The decay tensors for $\Lambda_b \rightarrow \Lambda_c$ and $\Lambda_b \rightarrow X_c$ agree in such a way that the differential decay rates are equal to two orders in $1/m$ for the spin-dependent and spin-independent

terms, i.e. $W_{1,2}$ and $S_{6,8}$ agree to two orders in $1/m$, while $W_{3,5}$ and $S_{1,2,7,9}$ agree to first order in $1/m$. It is straightforward to verify this result, and the details are omitted here.

The total decay rate for $\Lambda_b \rightarrow \Lambda_c$ decay for a left handed current is

$$\begin{aligned} \Gamma^L(\Lambda_b \rightarrow \Lambda_c) = & \frac{G_F^2}{192\pi^3} |V_{cb}|^2 \left[\frac{16}{5} (\delta M)^5 \left[(F_1 + F_2 + F_3)^2 + 3G_1^2 \right] - \frac{8}{5} \frac{(\delta M)^6}{M_{\Lambda_b}} [(F_1 + F_2 + F_3)(3F_1 + 5F_2 + F_3) \right. \\ & + G_1(9G_1 + 2G_2 + 2G_3)] + \frac{4}{35} \frac{(\delta M)^7}{M_{\Lambda_b}^2} \left[20(F_1 + F_2 + F_3) \left(\frac{dF_1}{dw} + \frac{dF_2}{dw} + \frac{dF_3}{dw} \right) + 44G_1 \frac{dG_1}{dw} + 24F_1^2 + 78F_1F_2 \right. \\ & \left. \left. + 65F_2^2 + 22F_1F_3 + 46F_2F_3 + 9F_3^2 + 48G_1^2 + 42G_1G_2 + 5G_2^2 + 14G_1G_3 + 10G_2G_3 + 5G_3^2 \right] \right] + \dots, \end{aligned} \quad (8.1)$$

where $\delta M = M_{\Lambda_b} - M_{\Lambda_c}$ and all the form factors are evaluated at zero recoil.

The rate for $\Lambda_b \rightarrow \Lambda_c$ decay including α_s corrections is obtained by using the form factors Eq. (C4), and substituting into Eq. (8.1),

$$\begin{aligned} \Gamma^L(\Lambda_c) = & \frac{G_F^2}{192\pi^3} |V_{cb}|^2 \left[\left(1 - \frac{\alpha_s}{\pi}\right) \left(\frac{64}{5} (\delta M)^5 - \frac{96}{5} \frac{(\delta M)^6}{M_{\Lambda_b}} \right) \right. \\ & \left. + \frac{1}{M_{\Lambda_b}^2} (\delta M)^7 \left(\frac{288}{35} - \frac{256}{35} \rho^2 + \frac{\alpha_s}{\pi} \left(\frac{32}{45} + \frac{704}{105} \rho^2 \right) \right) + \frac{2048}{315} \log \frac{m_b}{\mu} \right] + \dots, \end{aligned} \quad (8.2)$$

There are no G terms in the Λ_b decay rate because the light degrees of freedom have spin zero, so matrix elements of the $\bar{b}\sigma^{\mu\nu}G_{\mu\nu}b$ operator vanish in the Λ_b . Also note that there are no $\bar{\Lambda}$ terms in Eq. (8.2), even though the individual form factors depend on $\bar{\Lambda}$ through $\epsilon_{b,c}$.

Comparing Eq. (8.2) with the inclusive decay rates Eq. (6.1) gives the bound on the slope of the Isgur-Wise function for baryons,

$$\rho^2(\mu) > \frac{\alpha_s}{\pi} \left(\frac{2563}{945} - \frac{8}{9} \log \frac{2\delta m}{\mu} \right), \quad (8.3)$$

which gives the α_s correction to the lowest order result $\rho^2 \geq 0$ [24]. An analysis similar to that for mesons gives

$$\rho^2(\mu) > \frac{\alpha_s}{\pi} \frac{2563}{945}, \quad (8.4)$$

valid for μ large enough that $\Lambda_{QCD}/\mu \ll 1$. For $\alpha_s = 0.22$, the bound is $\rho^2 > 0.19$. The bound for the physical form factor G_1 is that the logarithmic slope is bounded above by -0.20 , using the relation between the Isgur-Wise function and the form factors given in [16,17], $\mu = m_c$, and $m_c/m_b = 0.3$.

IX. CONCLUSIONS

We have shown by explicit computation that the decay widths for $B \rightarrow X_c l \bar{\nu}$ and $B \rightarrow (D + D^*) l \bar{\nu}$ decays agree to two orders in the $1/m$ expansion in the SV limit, including α_s radiative corrections. The agreement is non-trivial because the $1/m$ term depends on the non-perturbative matrix element G , and because the parton decay at order α_s involves the two-body hadronic final state cg . The matrix element G from the operator product expansion enters in precisely the way necessary to compensate for the difference between quark and meson kinematics. The two-body charm-gluon cut enters at $1/m^2$, as necessary to match the continuum contribution from multi-hadron states. The differential rates also agree to two orders in $1/m$. The electron spectra agree to two orders in $1/m$ without any smearing over E_e . The double differential rates agree as well, provided one smears over a kinematic region at least as large as Λ_{QCD} . These results show that (smeared) local duality holds in our example to two orders in $1/m$. This is a non-trivial check on the validity of local duality, since non-perturbative corrections come in at first order in $1/m$.

The partial width for B decays to excited states is positive, so one obtains an inequality on the $1/m^2$ corrections to the decay widths which gives the α_s corrections to Bjorken's inequality on ρ^2 . A conservative estimate of the appropriate hadronic scale leads to an upper bound on the slope of a measurable decay form factor of -0.45 . A similar analysis leads to the conclusion that the kinetic energy operator of the heavy quark has positive matrix

element, $K > 0$, if one neglects α_s corrections. Unlike the case of the Bjorken bound, the α_s terms are not merely a correction to the leading order result and invalidate the derivation of the $K > 0$ bound.

We have also shown that the differential decay distributions and total widths for polarized and unpolarized $\Lambda_b \rightarrow X_c l \bar{\nu}$ and $\Lambda_b \rightarrow \Lambda_c l \bar{\nu}$ agree to two orders in $1/m$, in the SV limit. At order $1/m^2$ we find α_s corrections to Bjorken's inequality on the slope of the baryon form factor. The logarithmic slope of the form factor G_1 has an upper bound of -0.20 .

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APPENDIX A: HADRONIC TENSORS FOR INCLUSIVE DECAY

The tensors T_i for semileptonic B decay are summarized here. The results are from Ref. [7], and have been converted to the notation and sign conventions used in Ref. [6]. The tensors for unpolarized Λ_b decay are obtained by setting $G = 0$, and replacing K in Eq. (3.8) by the corresponding matrix element in the Λ_b .

$$T_1^{VV} = \frac{1}{\Delta_q} \left[(m_b - m_c - q \cdot v) + (G + K) \frac{1}{m_b} \left(\frac{1}{3} + \frac{m_c}{m_b} \right) \right] \quad (\text{A1a})$$

$$+ \frac{2}{\Delta_q^2 m_b} \left[-\frac{1}{3} G [(4m_b - 3q \cdot v)(m_b - m_c - q \cdot v) + 2(q \cdot v^2 - q^2)] + K \left[q \cdot v (m_b - m_c - q \cdot v) - \frac{2}{3} (q \cdot v^2 - q^2) \right] \right] \quad (\text{A1b})$$

$$+ \frac{8}{3\Delta_q^3} K (m_b - m_c - q \cdot v) (q \cdot v^2 - q^2), \quad (\text{A1c})$$

$$T_2^{VV} = \frac{1}{\Delta_q} \left[2m_b + \frac{10}{3m_b} (G + K) \right] + \frac{4}{3\Delta_q^2} [-2G(m_b - m_c) + 5Gq \cdot v + 7Kq \cdot v] + \frac{16}{3\Delta_q^3} m_b K (q \cdot v^2 - q^2), \quad (\text{A1d})$$

$$T_4^{VV} = \frac{8}{3m_b \Delta_q^2} (K + G), \quad (\text{A1e})$$

$$T_5^{VV} = -\frac{1}{\Delta_q} - \frac{2}{3\Delta_q^2} \left[5 \frac{q \cdot v}{m_b} (G + K) + 4K \right] - \frac{8}{3\Delta_q^3} K (q \cdot v^2 - q^2), \quad (\text{A1f})$$

$$T_3^{AV} = -\frac{1}{\Delta_q} - \left[-4G + \frac{10}{3} (K + G) \frac{q \cdot v}{m_b} \right] \frac{1}{\Delta_q^2} - \frac{8}{3\Delta_q^3} K (q \cdot v^2 - q^2), \quad (\text{A1g})$$

$$T_3^{VV} = T_1^{AV} = T_2^{AV} = T_4^{AV} = T_5^{AV} = 0, \quad (\text{A1h})$$

and T_i^{AA} is obtained from T_i^{VV} by changing the sign of m_c . Here $\Delta_q = m_b^2 - m_c^2 - 2m_b q \cdot v + q^2$.

APPENDIX B: $B \rightarrow D, D^*$ FORM FACTORS

The expansions of the $B \rightarrow D$ and $B \rightarrow D^*$ form factors around $w = 1$ (using the results in [16,17]) are summarized here. The deviation $w - 1$ is treated as order $(\delta m)^2$.

$$f_+ = \frac{M_B + M_D}{2\sqrt{M_B M_D}} \left[1 + \frac{5\alpha_s}{18\pi} \left(\frac{\delta m}{m_b} \right)^2 + f_+^{(2)} \frac{1}{m_b^2} + \mathcal{O}(\delta m)^3 \right] S(w) \xi(w), \quad (\text{B1a})$$

$$f_0 = 2\sqrt{M_B M_{D^*}} \left[\frac{1+w}{2} + f_0^{(2)} \frac{1}{m_b^2} + \frac{\alpha_s}{\pi} \left(-\frac{2}{3} + \frac{2}{9}(w-1) + \frac{1}{6} \left(\frac{\delta m}{m_b} \right)^2 \right) + \mathcal{O}(\delta m)^3 \right] S(w) \xi(w), \quad (\text{B1b})$$

$$a_+ = -\frac{1}{2} \sqrt{\frac{1}{M_B M_{D^*}}} \left[1 + a_+^{(1)} \frac{1}{m_b} + \frac{\alpha_s}{\pi} \left(-\frac{2}{3} + \frac{1}{9} \frac{\delta m}{m_b} \right) + \mathcal{O}(\delta m)^2 \right] S(w) \xi(w), \quad (\text{B1c})$$

$$g = \sqrt{\frac{1}{M_B M_{D^*}}} \left[1 + \frac{2\alpha_s}{3\pi} + \mathcal{O}(\delta m)^2 \right] S(w) \xi(w). \quad (\text{B1d})$$

The Isgur-Wise function $\xi(w)$ can be expanded about $w = 1$,

$$\xi(w) = 1 - \rho^2 (w - 1) + \dots, \quad (\text{B2})$$

which defines the slope parameter ρ^2 . The renormalization group scaling factor $S(w)$ is

$$S = \left(\frac{\alpha_s(m_b)}{\alpha_s(\mu)} \right)^{\gamma(w)/2b_0} \quad (\text{B3})$$

where $\gamma(w)$ is the velocity dependent anomalous dimension,

$$\gamma(w) = \frac{16}{3} \left[w \frac{\log(w + \sqrt{w^2 - 1})}{\sqrt{w^2 - 1}} - 1 \right], \quad (\text{B4})$$

and b_0 is the first term in the QCD β -function. Expanding $\gamma(w)$ around $w = 1$ gives

$$\gamma(w) = \frac{32}{9} (w - 1) + \dots \quad (\text{B5})$$

APPENDIX C: Λ_B FORM FACTORS

The form factors for $\Lambda_b \rightarrow \Lambda_c$ decay for a left-handed current are

$$\frac{4}{M_{\Lambda_c}} W_1 = X_1 \delta(\Delta_{\Lambda_c}), \quad (C1a)$$

$$\frac{4}{M_{\Lambda_c}} W_2 = \left[X_2 + 2 \frac{M_{\Lambda_b}}{M_{\Lambda_c}} X_5 + X_4 \left(\frac{M_{\Lambda_b}}{M_{\Lambda_c}} \right)^2 \right] \delta(\Delta_{\Lambda_c}), \quad (C1b)$$

$$\frac{4}{M_{\Lambda_c}} W_3 = \frac{X_3}{M_{\Lambda_c}} \delta(\Delta_{\Lambda_b}), \quad (C1c)$$

$$\frac{4}{M_{\Lambda_c}} W_4 = \frac{X_4}{M_{\Lambda_c}^2} \delta(\Delta_{\Lambda_c}), \quad (C1d)$$

$$\frac{4}{M_{\Lambda_c}} W_5 = - \left[\frac{X_5}{M_{\Lambda_c}} + \frac{M_{\Lambda_b}}{M_{\Lambda_c}^2} X_4 \right] \delta(\Delta_{\Lambda_c}), \quad (C1e)$$

where X_i are defined by

$$X_1 = (w-1) F_1^2 + (w+1) G_1^2, \quad (C2a)$$

$$X_2 = (w+1) F_2^2 + 2F_1 F_2 + (w-1) G_2^2 + 2G_1 G_2, \quad (C2b)$$

$$X_3 = 2F_1 G_1, \quad (C2c)$$

$$X_4 = (w+1) F_3^2 + 2F_1 F_3 + (w-1) G_3^2 - 2G_1 G_3, \quad (C2d)$$

$$X_5 = (w+1) F_2 F_3 + F_1 (F_1 + F_2 + F_3) + (w-1) G_2 G_3 + G_1 (G_1 + G_3 - G_2), \quad (C2e)$$

and

$$\Delta_{\Lambda_c} = (p_{\Lambda_b} - q)^2 - M_{\Lambda_c}^2. \quad (C3)$$

The form factors for more general currents can be trivially obtained from the above by rescaling the axial or vector current form factors.

The expansion of the Λ_b form factors around $w = 1$ at order α_s is (using the results in [16,17])

$$F_1 = \left[\left(1 + \frac{2\alpha_s}{3\pi} \right) (1 + \epsilon_b + \epsilon_c) + \frac{\alpha_s}{\pi} \left(\frac{w-1}{9} + \frac{(\delta m)^2}{6m_b^2} \right) + \mathcal{O}(\delta m)^3 \right] S(w) \zeta(w) \quad (C4a)$$

$$F_2 = \left[-\frac{\alpha_s}{3\pi} (1 + \epsilon_b + \epsilon_c) - \epsilon_c - \frac{\alpha_s}{9\pi} (1 + \epsilon_b + \epsilon_c) \frac{\delta m}{m_b} + \frac{\alpha_s}{9\pi} \left(w-1 - \frac{(\delta m)^2}{2m_b^2} \right) + \mathcal{O}(\delta m)^3 \right] S(w) \zeta(w) \quad (C4b)$$

$$F_3 = \left[-\frac{\alpha_s}{3\pi} (1 + \epsilon_b + \epsilon_c) - \epsilon_b + \frac{\alpha_s}{9\pi} (1 + \epsilon_b + \epsilon_c) \frac{\delta m}{m_b} + \frac{\alpha_s}{9\pi} \left(w-1 + \frac{(\delta m)^2}{2m_b^2} \right) + \mathcal{O}(\delta m)^3 \right] S(w) \zeta(w) \quad (C4c)$$

$$G_1 = \left[1 - \frac{2\alpha_s}{3\pi} + \frac{\alpha_s}{\pi} \left(\frac{5(w-1)}{9} + \frac{(\delta m)^2}{6m_b^2} \right) + \mathcal{O}(\delta m)^3 \right] S(w) \zeta(w), \quad (C4d)$$

$$G_2 = \left[-\frac{\alpha_s}{\pi} (1 + 2\epsilon_b) \left(\frac{7}{9} + \frac{\delta m}{3m_b} \right) - \epsilon_c \left(1 + \frac{8\alpha_s}{9\pi} \right) + \frac{\alpha_s}{90\pi} \left(22(w-1) - 17 \frac{(\delta m)^2}{m_b^2} \right) + \mathcal{O}(\delta m)^3 \right] S(w) \zeta(w), \quad (C4e)$$

$$G_3 = \left[\frac{\alpha_s}{\pi} (1 + 2\epsilon_c) \left(\frac{7}{9} - \frac{\delta m}{3m_b} \right) + \epsilon_b \left(1 + \frac{8\alpha_s}{9\pi} \right) - \frac{\alpha_s}{90\pi} \left(22(w-1) + 13 \frac{(\delta m)^2}{m_b^2} \right) + \mathcal{O}(\delta m)^3 \right] S(w) \zeta(w). \quad (C4f)$$

Here

$$\epsilon_{b,c} = \frac{\bar{\Lambda}}{2m_{b,c}}, \quad (C5)$$

and the scale factor $S(w)$ is the same as for meson form factors. In the above expansion, we have omitted terms that arise from the $1/m_b^2$ operators in HQET, analogous to the $f_+^{(2)}/m_b^2$ terms in the meson form factors. Such terms do not affect the inequality on ρ^2 . Note that $\bar{\Lambda}$ and $\rho^2 = d\zeta/dw(1)$ for baryons are different from $\bar{\Lambda}$ and $\rho^2 = d\xi/dw(1)$ for mesons, even though they are denoted by the same symbol.

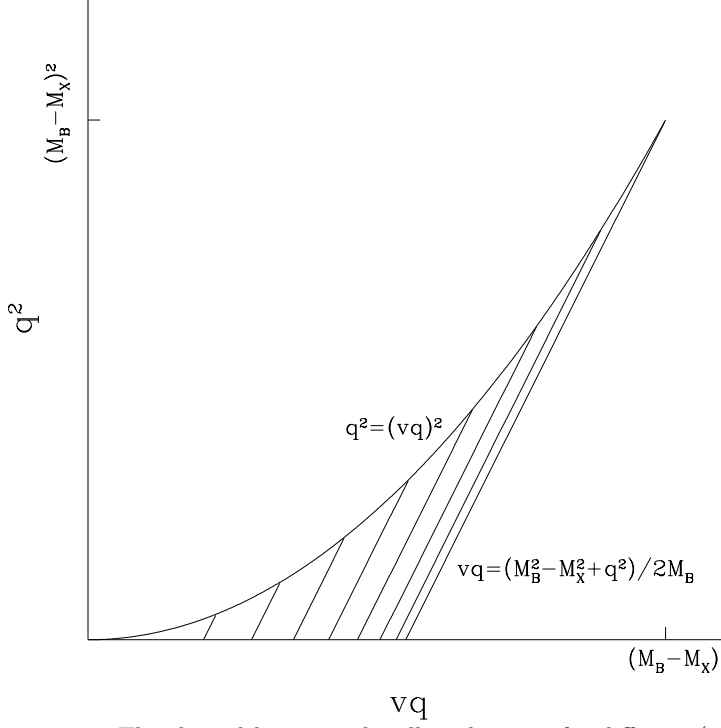


FIG. 1. Plot of q^2 against $q \cdot v$. The slanted lines are the allowed curves for different (equally spaced) values of M_X . The upper edge of the allowed region is the zero recoil point for different hadronic states X . The right-hand edge of the allowed region is from the lightest allowed state X .

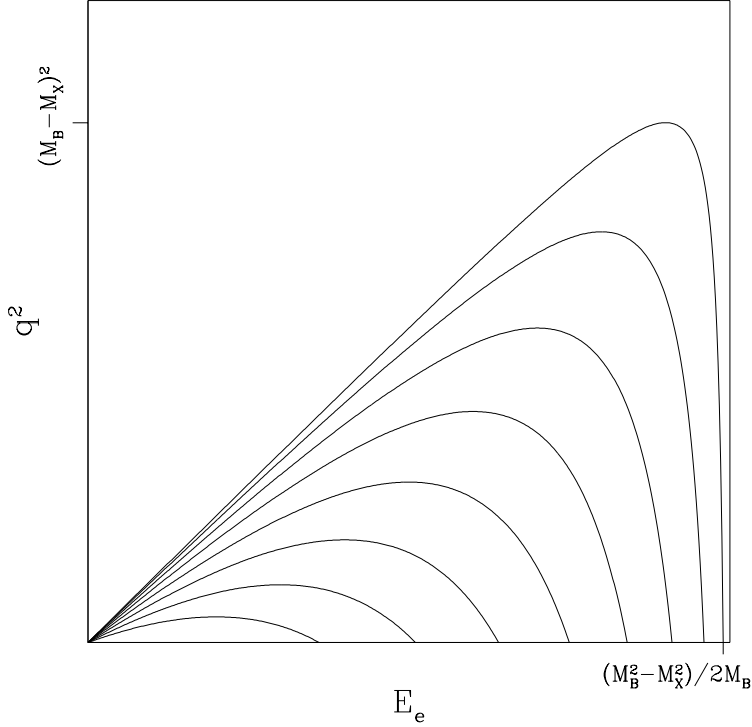


FIG. 2. Plot of q^2 against the electron energy E_e . The allowed region for different (equally spaced) values of the final state mass M_X is the interior of one of the curves. The total allowed region is the interior of the largest curve, which corresponds to the lowest allowed mass M_X . The zero recoil point for a given M_X is at the maximum allowed value of q^2 .

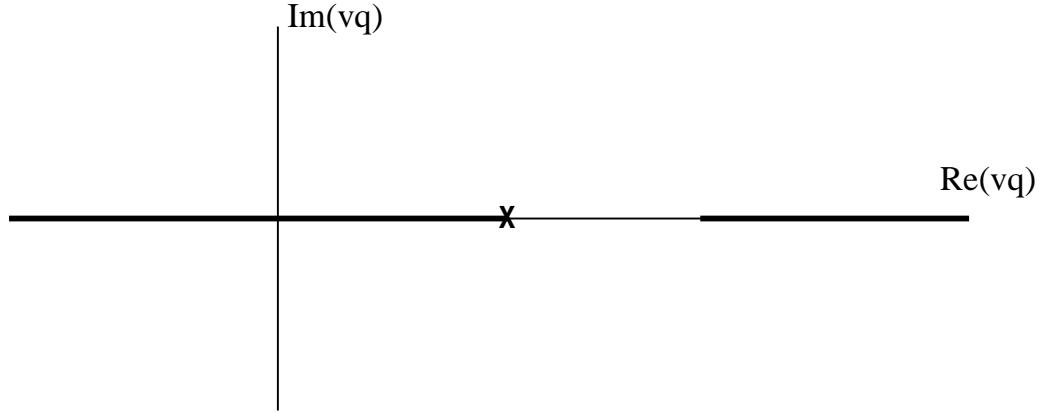


FIG. 3. Analytic structure of T_i in the complex $v \cdot q$ plane. The physical cut for semileptonic decay is along the real axis for $q \cdot v \geq 0$. The free quark decay tensors have a pole marked by a \times . The gluon bremsstrahlung graphs produce a cut to the left of the pole. The left hand cut for $q \cdot v < 0$, and the right hand cut correspond to crossed processes, and are not relevant for semileptonic $b \rightarrow c$ decay.