Subleading form factors at order $1/m_Q$ in terms of leading quantities using the non-forward amplitude in HQET

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Abstract

We consider the non-forward amplitude within the Heavy Quark Effective Theory. We show that one can obtain new information on the subleading corrections in $1/m_Q$. We illustrate the method by deriving new simple relations between the *functions* $\xi_3(w)$ and $\overline{\Lambda}\xi(w)$ and the sums $\sum_n \Delta E_j^{(n)} \tau_j^{(n)}(1)\tau_j^{(n)}(w)$ ($j = \frac{1}{2}, \frac{3}{2}$), that involve leading quantities, namely the Isgur-Wise functions $\tau_j^{(n)}(w)$ and the level spacings $\Delta E_j^{(n)}$. Our results follow because the non-forward amplitude $B(v_i) \to D^{(n)}(v') \to B(v_f)$ depends on three variables $(w_i, w_f, w_{if}) = (v_i \cdot v', v_f \cdot v', v_i \cdot v_f)$ independent in a certain domain, and we consider the zero recoil frontier $(w, 1, w)$ where only a finite number of J^P states contribute $\left(\frac{1}{2}^+, \frac{3}{2}^+\right)$. These sum rules reduce to known results at $w = 1$, for $\overline{\Lambda}$ obtained by Voloshin, and for $\xi_3(1)$ obtained by Le Yaouanc et al. and by Uraltsev, and generalizes them to all values of w. We discuss phenomenological applications of these results, in particular the check of Bakamjian-Thomas quarks models and the comparison with the QCD Sum Rules approach.

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1 Introduction.

In the leading order of the heavy quark expansion of QCD, Bjorken sum rule (SR) [1, 2] relates the slope of the elastic Isgur-Wise (IW) function $\xi(w)$, to the IW functions of the transition between the ground state $j^P = \frac{1}{2}^-$ and the $j^P = \frac{1}{2}^+, \frac{3}{2}^+$ excited states, $\tau_{1/2}^{(n)}(w)$, $\tau_{3/2}^{(n)}(w)$ at zero recoil $w = 1$ (n is a radial quantum number). This SR leads to the lower bound $-\xi'(1) = \rho^2 \geq \frac{1}{4}$. A new SR was formulated by Uraltsev in the heavy quark limit [3], involving also $\tau_{1/2}^{(n)}(w)$, $\tau_{3/2}^{(n)}(w)$, that implies, combined with Bjorken SR, the much stronger lower bound $\rho^2 \geq \frac{3}{4}$. A basic ingredient in deriving this bound is the consideration of the non-forward amplitude $B(v_i) \to D^{(n)}(v') \to B(v_f)$, allowing for general v_i, v_f, v' and where B is a ground state meson. In refs. [4, 5, 6] we have developed, in the heavy quark limit of QCD, a manifestly covariant formalism within the Operator Product Expansion (OPE), using the matrix representation for the whole tower of heavy meson states [7]. We did recover Uraltsev SR plus a general class of SR that allow to bound also higher derivatives of the IW function. In particular, we found two bounds for the curvature $\xi''(1) = \sigma^2$ in terms of ρ^2, that imply $\sigma^2 \geq \frac{15}{16}$.

The object of the present paper is to extend the formalism to IW functions at subleading order in $1/m_Q$.

The general SR obtained from the OPE can be written in the compact way [4]

$$L_{Hadrons}(w_i, w_f, w_{if}) = R_{OPE}(w_i, w_f, w_{if}) \tag{1}$$

where the l.h.s. is the sum over the intermediate D states, while the r.h.s. is the OPE counterpart. Using the trace formalism [8], this expression writes, in the heavy quark limit [4] :

$$\sum_{D=P,V} \sum_n \mathrm{Tr}\left[\overline{B}_f(v_f)\Gamma_f D^{(n)}(v')\right] \ \mathrm{Tr}\left[\overline{D}(v')\Gamma_i B_i(v_i)\right] \xi^{(n)}(w_i)\xi^{(n)}(w_f) + \text{Excited states}$$
$$= -2\xi(w_{if}) \ \mathrm{Tr}\left[\overline{B}_f(v_f)\Gamma_f P'_+ \Gamma_i B_i(v_i)\right] \tag{2}$$

where

$$w_i = v_i \cdot v' \qquad w_f = v_f \cdot v' \qquad w_{if} = v_i \cdot v_f \tag{3}$$

and

$$P'_+ = \frac{1 + \not{v}'}{2} \tag{4}$$

is the positive energy projector on the intermediate c quark, and we assume that the IW functions are real. The ground state B meson can be either a pseudoscalar or a vector, the doublet $\frac{1}{2}^-$. The heavy quark currents considered in the previous expression are

$$\overline{h}_{v'}\Gamma_i h_{v_i} \qquad\qquad \overline{h}_{v_f}\Gamma_f h_{v'} \tag{5}$$

$B(v)$, $D(v)$ are the 4×4 matrices representing the B, D states [8], and $\overline{B} = \gamma^0 B^+ \gamma^0$ denotes the Dirac conjugate matrix. The domain for the variables (w_i, w_f, w_{if}) is [4] :

$$\begin{gathered} w_i \geq 1 \qquad\qquad w_f \geq 1 \\ w_i w_f - \sqrt{(w_i^2-1)(w_f^2-1)} \leq w_{if} \leq w_i w_f + \sqrt{(w_i^2-1)(w_f^2-1)}\ . \end{gathered} \tag{6}$$

We will now consider $1/m_Q$ corrections to the heavy quark limit SR (2).

The paper is organized as follows. In Section 2 we set the problem of obtaining sum rules involving subleading quantities in $1/m_Q$ within the OPE. In Section 3 we make explicit the formalism of the corrections in $1/m_b$, the b-quark being the external quark, using the formalisms of Falk and Neubert [9] and of Leibovich et al. [10] to parametrize the $\frac{1}{2}^- \to \frac{1}{2}^-$ and $\frac{1}{2}^- \to \frac{1}{2}^+, \frac{3}{2}^+$ form factors, that we extend to the $\frac{1}{2}^- \to \frac{3}{2}^-$ transitions. In Sections 4 and 5 we write down the SR obtained respectively if the initial and final meson is a pseudoscalar B or a vector B^*. We factorize polynomials in the variables (w_i, w_f, w_{if}) that allow to obtain simple results going to the interesting frontier $(w, 1, w)$ of the domain (6). We generalize our results to any $\frac{1}{2}^- \to j^P$ transition. From the obtained SR we get enough information to write our fundamental results for the subleading quantities $\overline{\Lambda}\xi(w)$ and $\xi_3(w)$ in terms of leading quantities in Section 6. In Section 7 we use as input the results of the Bakamjian-Thomas class of quark models – that satisfy all the necessary properties in the heavy quark limit – to obtain phenomenologically useful results, that appear to be consistent, in our quite different approach, with the QCD Sum Rules. Finally, in Section 8 we conclude and set up the program that remains to be pursued. In Appendix A we demonstrate the identity between two subleading parameters defined by Falk and Neubert [9] and by Uraltsev [3]. In Appendix B we discuss the experimental situation of the leading P-wave IW functions $\tau_{1/2}^{(0)}(w)$ and $\tau_{3/2}^{(0)}(w)$.

2 The Operator Product Expansion and the corrections at first order in $1/\mathbf{m_Q}$.

Our starting point [11] is the T-product

$$T_{fi}(q) = i\int d^4x\ e^{-iq\cdot x}\ \ < B(p_f)|T[J_f(0)J_i(x)]|B(p_i) > \tag{7}$$

where $J_f(x)$, $J_i(y)$ are the currents (the convenient notation for the subindices i, f will appear clear below) :

$$J_f(x) = \overline{b}(x)\Gamma_f c(x) \qquad\qquad J_i(y) = \overline{c}(y)\Gamma_i b(y) \tag{8}$$

and p_i is in general different from p_f.

Inserting in this expression hadronic intermediate states, $x^0 < 0$ receives contributions from the direct channel with hadrons with a single heavy quark c, while $x^0 > 0$ receives contributions from hadrons with $b\overline{c}b$ quarks, the Z diagrams :

$$\begin{aligned} T_{fi}(q) = &\sum_{X_c}(2\pi)^3\ \delta^3(\mathbf{q}+\mathbf{p}_i-\mathbf{p}_{X_c})\frac{< B_f|J_f(0)|X_c >< X_c|J_i(0)|B_i >}{q^0+E_i-E_{X_c}+i\varepsilon} \\ &-\sum_{X_{\overline{c}bb}}(2\pi)^3\ \delta^3(\mathbf{q}-\mathbf{p}_f+\mathbf{p}_{X_{\overline{c}bb}})\frac{< B_f|J_i(0)|X_{\overline{c}bb} >< X_{\overline{c}bb}|J_f(0)|B_i >}{q^0-E_f+E_{X_{\overline{c}bb}}-i\varepsilon}\ . \end{aligned} \tag{9}$$

We will consider the following limit

$$m_c \gg m_b \gg \Lambda_{QCD}\ . \tag{10}$$

The difference between the two energy denominators is large

$$q^0 - E_f + E_{X_{\overline{c}bb}} - \left(q^0 + E_i - E_{X_c}\right) \sim 2m_c\ . \tag{11}$$

Therefore, we can in this limit neglect the second term, and we will consider the imaginary part of the direct diagram, the first term in (9), the piece proportional to

$$\delta\left(q^0 + E_i - E_{X_c}\right)\ . \tag{12}$$

One can see this point otherwise. The two cuts corresponding to the two terms in (9) are widely separated, and one can isolate the imaginary part of the first term by a suitable integration contour in the q^0 complex plane. Notice that one can choose q^0 such that there is a left-hand cut, even in the conditions (10). This means that

q^0 is of the order of m_c and $m_c - q^0$ is fixed, of the order m_b. Our conditions are, in short, as follows :

$$\Lambda_{QCD} \ll m_b \sim m_c - q^0 \ll q^0 \sim m_c \ , \tag{13}$$

consistent with (12). To summarize, we are considering the heavy quark limit for the c quark, but we allow for a large finite mass for the b quark.

Unlike the case of the forward amplitude ($p_i = p_f$), the imaginary part of the direct diagram in (9) will not be related to a positive definite absorptive part, because we are in the more general case of the non-forward amplitude. However, we are allowed to consider this imaginary part.

In the conditions (13), or choosing the suitable integration contour [12] [13], we can write therefore, integrating over q^0

$$T_{fi}^{abs}(\mathbf{q}) \cong \sum_{X_c} (2\pi)^3 \ \delta^3 \left(\mathbf{q} + \mathbf{p}_i - \mathbf{q}_{X_c}\right) \ < B_f|J_f(0)|X_c > < X_c|J_i(0)|B_i > \ . \tag{14}$$

Finally, integrating over $\mathbf{q}_{X_c}$ and defining $v' = \frac{q+p_i}{m_c}$ one gets

$$T_{fi}^{abs} \cong \sum_{D_n} < B_f(v_f)|J_f(0)|D_n(v') > < D_n(v')|J_i(0)|B_i(v_i) > \tag{15}$$

where we have denoted by $D_n(v')$ the charmed intermediate states.

The T-product matrix element $T_{fi}(q)$ (7) is given, alternatively, in terms of quarks and gluons, by the expression

$$T_{fi}(q) = -\int d^4x \ e^{-iq\cdot x} \ < B(p_f)|\overline{b}(0)\Gamma_f S_c(0,x)\Gamma_i b(x)|B(p_i) > \tag{16}$$

where $S_c(0,x)$ is the c quark propagator in the background of the soft gluon field [14].

Since we are considering the absorptive part in the c heavy quark limit of the direct graph in (9), this quantity can be then identified with (16) where $S_c(x,0)$ is replaced by the following expression [15]

$$S_c(0,x) \to e^{im_c v'\cdot x} \ \Phi_{v'}[0,x] D_{v'}(x) \tag{17}$$

where $D_{v'}(x)$ is the *cut* free propagator of a heavy quark

$$D_{v'}(x) = P'_+ \int \frac{d^4k}{(2\pi)^4} \delta(k\cdot v') e^{ik\cdot x} = P'_+ \int_{-\infty}^{\infty} \frac{dt}{2\pi} \delta^4(x - v't) \tag{18}$$

with the positive energy projector defined by

$$P'_+ = \frac{1}{2}(1 + \not{v}') \ . \tag{19}$$

The eikonal phase $\Phi_{v'}[0, x]$ in (17) corresponds to the propagation of the c quark from the point $x = v't$ to the point 0, that is given by

$$\Phi_{v'}[0, v't] = P \exp\left(-i \int_0^t ds\ v' \cdot A(v's)\right) \ . \tag{20}$$

This quantity takes care of the dynamics of the soft gluons in HQET along the classical path $x = v't$.

Inserting (17)-(19) into (16) we obtain

$$\begin{aligned} T_{fi}^{abs}(q) = \int d^4x\ e^{-i(q-m_c v')\cdot x} \int_{-\infty}^{\infty} \frac{dt}{2\pi} \delta^4(x - v't) \\ < B(p_f)|\overline{b}(0)\Gamma_f P'_+ \Phi_{v'}[0, x]\Gamma_i b(x)|B(p_i) > \ + O(1/m_c) \ . \end{aligned} \tag{21}$$

Integrating over x in (21) and making explicit (20),

$$\begin{aligned} &T_{fi}^{abs}(q) = \int_{-\infty}^{\infty} \frac{dt}{2\pi}\ e^{-i(q-m_c v')\cdot v't} \\ &< B(p_f)|\overline{b}(0)\Gamma_f P'_+ P \exp\left(-i \int_0^t ds\ v' \cdot A(v's)\right) \Gamma_i b(v't)|B(p_i) > \\ &+ O(1/m_c) \ . \end{aligned} \tag{22}$$

Performing first the integration over q^0 one obtains simply $\delta(v'^0 t)$, that forces $t = 0$, and making the trivial integration over t one obtains finally the OPE matrix element that must be identified with (15) :

$$T_{fi}^{abs} \cong < B(p_f)|\overline{b}(0)\Gamma_f \frac{1 + \not{v}'}{2v'^0} \Gamma_i b(0)|B(p_i) > \ + O(1/m_c) \ . \tag{23}$$

Therefore, we end up with the sum rule

$$\begin{aligned} &\sum_{D_n} < B_f(v_f)|J_f(0)|D_n(v') > < D_n(v')|J_i(0)|B_i(v_i) > \\ &= < B(v_f)|\overline{b}(0)\Gamma_f \frac{1 + \not{v}'}{2v'^0} \Gamma_i b(0)|B(v_i) > \ + O(1/m_c) \end{aligned} \tag{24}$$

that is valid for *all powers* of an expansion in $1/m_b$, but only to leading order in $1/m_c$.

On the other hand, making use of the HQET equations of motion, the field $b(x)$ in (24) can be decomposed into upper and lower components as follows [16]

$$b(x) = e^{-im_b v\cdot x} \left(1 + \frac{1}{2m_b + iv\cdot \overrightarrow{D}}\ i\overrightarrow{\not{D}} \right) h_v(x) \tag{25}$$

where the second term corresponds to the lower components and can be expanded in a series in powers of D_μ/m_b, and v is an arbitrary four-velocity.

Taking into account the normalization of the states in the trace formalism and the sign convention for the matrix elements, we recover, in the heavy quark limit (neglecting the term in $1/2m_b$), the master formula (2) obtained in ref. [4].

Including the first order in $1/m_b$, the sum rule reads

$$\begin{aligned}
&\sum_{D_n} < B_f(v_f)|J_f(0)|D_n(v')> < D_n(v')|J_i(0)|B_i(v_i)> \\
&= < B(p_f)|\overline{h}_{v_f}(0)\Gamma_f \frac{1+\not{v}'}{2v'^0}\Gamma_i h_{v_i}(0)|B(p_i)> \\
&+\frac{1}{2m_b}\ < B(p_f)|\overline{h}_{v_f}(0)\Big[(-i\overleftarrow{\not{D}})\Gamma_f \frac{1+\not{v}'}{2v'^0}\Gamma_i + \Gamma_f \frac{1+\not{v}'}{2v'^0}\Gamma_i(i\overrightarrow{\not{D}})\Big]h_{v_i}(0)|B(p_i)> \\
&+ O(1/m_c) + O(1/m_b^2)\ .
\end{aligned} \tag{26}$$

Therefore, in the OPE side we have, besides the leading dimension 3 operator

$$O^{(3)} = \overline{h}_{v_f}\Gamma_f P'_+\Gamma_i h_{v_i} \tag{27}$$

the dimension 4 operator

$$O^{(4)} = \overline{h}_{v_f}\left[(-i\overleftarrow{\not{D}})\Gamma_f P'_+\Gamma_i + \Gamma_f P'_+\Gamma_i(i\overrightarrow{\not{D}})\right] h_{v_i}\ . \tag{28}$$

In the SR we have to compute the l.h.s. including terms of order $1/2m_b$. These terms have been parametrized by Falk and Neubert for the $\frac{1}{2}^-$ doublet and by Leibovich et al. for the transitions between the ground state $\frac{1}{2}^-$ and the $\frac{1}{2}^+$, $\frac{3}{2}^+$ excited states. These $1/m_b$ corrections are of two classes : perturbations of the current, and perturbations of the Lagrangian (kinetic and magnetic). This will be our guideline to compute the l.h.s. of the SR, although we will consider the whole tower of excited states.

A remark is in order here, that was already made in ref. [11]. Had we taken higher moments of the form $\int dq^0(q^0)^n T_{fi}^{abs}(q^0)$ $(n>0)$, instead of the lowest one

$n = 0$, the integration over q^0 that leads to the simple sum rules (24) or (26) would involve higher dimension operators, giving a whole tower of sum rules [17], [15], even in the *leading* heavy quark limit. Our point of view in this paper is different. We consider the lowest moment $n = 0$, while we expand in powers of $1/m_b$, keeping the first order in this parameter.

Concerning the OPE side in (26), the dimension 4 operator $O^{(4)}$ (28) is nothing else but the $1/m_b$ perturbation of the heavy current $O^{(3)} = \overline{h}_{v_f} \Gamma_f P'_+ \Gamma_i h_{v_i}$ since this operator, containing the Dirac matrix $\Gamma_f P'_+ \Gamma_i$ between heavy quark fields, can be considered as a heavy quark current. Indeed, following Falk and Neubert, the $1/m_b$ perturbation of any heavy quark current $\overline{h}_{vf} \Gamma h_{vi}$ is given by

$$\overline{h}_{v_f} \left(- \frac{i\overleftarrow{\not{D}}}{2m_b} \right) \Gamma h_{v_i} + \overline{h}_{v_f} \Gamma \left(\frac{i\overrightarrow{\not{D}}}{2m_b} \right) h_{v_i} \ . \qquad (29)$$

However, this perturbation of the current does not exhaust all perturbations in $1/m_b$. Indeed, we need also to compute the perturbation of the initial and final wave functions $|B_i(v_i)>$, $|B_f(v_f)>$ due to the kinetic and magnetic perturbations of the Lagrangian. This can be done easily following also the prescriptions of Falk and Neubert to compute these corrections in $1/m_b$ for the leading matrix element $< B_f(v_f)|\overline{h}_{vf} \Gamma_f P'_+ \Gamma_i h_{vi}|B_i(v_i)>$, as we will see below.

3 Setting the formalism for the calculation of the corrections in $1/\mathbf{m_b}$.

Considering B or B^* initial and final mesons, we can perturb the SR (2) by $1/m_c$ and $1/m_b$ terms. The perturbation of the r.h.s. is parametrized by six new subleading IW functions concerning the ground state $\frac{1}{2}^-$, denoted by $L_i(w)$ ($i = 1, \cdots 6$), in the notation of Falk and Neubert [9].

As for the l.h.s., considering for the moment as intermediate D states the multiplets $\frac{1}{2}^-$, $\frac{1}{2}^+$, $\frac{3}{2}^+$, we have three types of matrix elements

$$\begin{aligned} &< D \left(\tfrac{1}{2}^-\right) (v')|\overline{c}\Gamma b|B(v)> \\ &< D \left(\tfrac{1}{2}^+\right) (v')|\overline{c}\Gamma b|B(v)> \\ &< D \left(\tfrac{3}{2}^+\right) (v')|\overline{c}\Gamma b|B(v)> \ . \end{aligned} \qquad (30)$$

The corrections in $1/m_b$ or $1/m_c$ to the first matrix element are given by the same ground state subleading IW functions $L_i(w)$ $(i = 1, \cdots 6)$, while the $O(1/m_b)$ and $O(1/m_c)$ corrections to the matrix elements $B \to D\left(\frac{1}{2}^+\right)$, $D\left(\frac{3}{2}^+\right)$ have been carefully studied by Leibovich, Ligeti, Steward and Wise [10], and result in a number of new subleading IW functions. All these corrections are of two types, perturbations of the heavy quark current, and perturbations of the Lagrangian.

Moreover, since, as pointed out by Leibovich et al. [10, Section VI], the states $D\left(\frac{3}{2}^-, 1^-\right)$ contribute also to zero recoil at order $1/m_Q$, we will consider the contribution of the matrix elements

$$< D\left(\tfrac{3}{2}^-, J^-\right)(v')|\overline{c}\Gamma b|B(v) > \qquad (J = 1, 2)\ . \tag{31}$$

We will show that these contributions do not spoil the simple result presented below, that can be expressed only in terms of the leading IW functions $\tau_{1/2}(w)$ and $\tau_{3/2}(w)$. We argue also that higher j^P intermediate states do not contribute.

Let us again underline that we will not take into account radiative hard gluon corrections, as computed in [12] for Bjorken SR, in [3] for Uraltsev SR and in [13] for our SR concerning the curvature of the IW function [6].

We begin with the general SR in the heavy quark limit (2) and perturb the heavy quark limit matrix elements with $1/m_c$ and $1/m_b$ corrections. The general expression could then be written, making explicit the leading and the $1/m_c$ and $1/m_b$ parts :

$$\begin{aligned}
&G_0(w_i, w_f, w_{if}) + E_0(w_i, w_f, w_{if}) + \frac{1}{2m_b}\left[G_b(w_i, w_f, w_{if}) + E_b(w_i, w_f, w_{if})\right] \\
&+\frac{1}{2m_c}\left[G_c(w_i, w_f, w_{if}) + E_c(w_i, w_f, w_{if})\right] \\
&= R_0(w_i, w_f, w_{if}) + \frac{1}{2m_b}R_b(w_i, w_f, w_{if}) + \frac{1}{2m_c}R_c(w_i, w_f, w_{if})
\end{aligned} \tag{32}$$

where the subindex 0 means the heavy quark limit, while the subindex b or c correspond to the subleading corrections in $1/m_b$ or $1/m_c$, and G or E mean, respectively, ground state or excited state contributions.

In the heavy quark limit, one has

$$G_0(w_i, w_f, w_{if}) + E_0(w_i, w_f, w_{if}) = R_0(w_i, w_f, w_{if}) \tag{33}$$

that leads to equation (2) and to the results quoted above [1]-[6].

In expression (32) we can vary m_b and m_c as independent parameters and obtain new SR for the subleading quantities.

To obtain information on the $1/m_b$ corrections, it is relatively simple to proceed as follows. We will assume the formal limit of Section 2 :

$$m_c \gg m_b \gg \Lambda_{QCD} \tag{34}$$

and perturb both sides of the SR (33) by $1/m_b$ terms. This heuristic procedure gives the same results as the method demonstrated in Section 2.

In this limit, since the parameter $1/m_b$ can be varied at will, one obtains the relation

$$G_b(w_i, w_f, w_{if}) + E_b(w_i, w_f, w_{if}) = R_b(w_i, w_f, w_{if}) \ . \tag{35}$$

One can compute $G_b(w_i, w_f, w_{if})$ and $E_b(w_i, w_f, w_{if})$ using respectively the formalism of Falk and Neubert [9] and the one of Leibovich et al. [10], and obtain SR for the different subleading IW functions $L_i(w)$ $(i = 1, \cdots 6)$.

Of course, one can obtain SR by taking the opposite limit $m_b \gg m_c$, that must be consistent with the preceding ones. In ref. [11] we did adopt the Shifman-Voloshin limit [18] $m_b, m_c \gg m_b - m_c \gg \Lambda_{QCD}$ for the forward amplitude.

To be explicit, let us define these functions from the current matrix elements, following the notation of Falk and Neubert [9] :

$$\begin{aligned}
&< D(v')|\overline{Q}'\Gamma Q|B(v) > \cong -\xi(w)\ \mathrm{Tr}\left[\overline{D}(v')\Gamma B(v)\right] \\
&-\frac{1}{2m_b}\ \mathrm{Tr}\left\{\overline{D}(v')\Gamma\left[P_+ L_+(v,v') + P_- L_-(v,v')\right]\right\}
\end{aligned} \tag{36}$$

in the formal limit $m_c \gg m_b$ (34) that we adopt here.

The 4×4 matrices write, respectively, for pseudoscalar and vector mesons :

$$\begin{aligned}
M(v) &= P_+(v)(-\gamma_5) \\
M(v) &= P_+(v)\ \not\varepsilon_v
\end{aligned} \tag{37}$$

while the subleading $1/m_b$ functions are for pseudoscalar and vector mesons :

$$P_+(v)L_+(v,v') + P_-(v)L_-(v,v') = \left[L_1(w)P_+(v) + L_4(w)P_-(v)\right](-\gamma_5) \tag{38}$$

$$P_+(v)L_+(v,v') + P_-(v)L_-(v,v') =$$

$$P_+(v)\left[\not{\varepsilon}_v L_2(w) + (\varepsilon_v \cdot v')\, L_3(w)\right] + P_-(v)\left[\not{\varepsilon}_v L_5(w) + (\varepsilon_v \cdot v')\, L_6(w)\right] \quad (39)$$

where $w = v \cdot v'$.

The matrix elements to excited states write [10]

$$\begin{aligned}
&< D\left(\tfrac{3}{2}^+\right)(v')|\bar{c}\Gamma b|B(v)> \cong \sqrt{3}\ \tau_{3/2}(w)\ \mathrm{Tr}\left[v_\sigma \overline{D}^\sigma(v')\Gamma B(v)\right] \\
&+ \frac{1}{2m_b}\Big\{\mathrm{Tr}\left[S^{(b)}_{\sigma\lambda}\overline{D}^\sigma(v')\Gamma\gamma^\lambda B(v)\right] + \eta^{(b)}_{ke}\ \mathrm{Tr}\left[v_\sigma \overline{D}^\sigma(v')\Gamma B(v)\right] \\
&+ \mathrm{Tr}\left[R^{(b)}_{\sigma\alpha\beta}\overline{D}^\sigma(v')\Gamma P_+(v) i\sigma^{\alpha\beta}B(v)\right]\Big\}
\end{aligned}$$

$$\begin{aligned}
&< D\left(\tfrac{1}{2}^+\right)(v')|\bar{c}\Gamma b|B(v)> \cong 2\tau_{1/2}(w)\ \mathrm{Tr}\left[\overline{D}(v')\Gamma B(v)\right] \\
&+ \frac{1}{2m_b}\Big\{\mathrm{Tr}\left[S^{(b)}_{\lambda}\overline{D}(v')\Gamma\gamma^\lambda B(v)\right] + \chi^{(b)}_{ke}\ \mathrm{Tr}\left[\overline{D}(v')\Gamma B(v)\right] \\
&+ \mathrm{Tr}\left[R^{(b)}_{\alpha\beta}\overline{D}(v')\Gamma P_+(v) i\sigma^{\alpha\beta}B(v)\right]\Big\} \qquad (40)
\end{aligned}$$

where

$$\begin{aligned}
D^\sigma_{2^+}(v') &= P_+(v')\varepsilon^{\sigma\nu}_{v'}\gamma_\nu \\
D^\sigma_{1^+}(v') &= -\sqrt{\tfrac{3}{2}}\ P_+(v')\varepsilon^\nu_{v'}\gamma_5\left[g^\sigma_\nu - \frac{1}{3}\gamma_\nu\left(\gamma^\sigma - v'^\sigma\right)\right] \\
D_{1^+}(v') &= P_+(v')\varepsilon^\nu_{v'}\gamma_5\gamma_\nu \\
D_{0^+}(v') &= P_+(v')\ . \qquad (41)
\end{aligned}$$

The notations $S^{(b)}_{\sigma\lambda}$, $S^{(b)}_{\lambda}$ denote the perturbations to the current, and $\eta^{(b)}_{ke}$, $\chi^{(b)}_{ke}$ and $R^{(b)}_{\sigma\alpha\beta}$ and $R^{(b)}_{\alpha\beta}$ denote respectively the kinetic and the magnetic perturbations to the Lagrangian. In the preceding relations (40) $B(v)$ can be a pseudoscalar or a vector.

Expanded in terms of Lorentz covariant factors and subleading IW functions, these tensor quantities read [10] :

$$\begin{aligned}
S^{(b)}_{\sigma\lambda} &= v_\sigma\left[\tau^{(b)}_1(w)v_\lambda + \tau^{(b)}_2(w)v'_\lambda + \tau^{(b)}_3(w)\gamma_\lambda\right] + \tau^{(b)}_4(w)g_{\sigma\lambda} \\
S^{(b)}_{\lambda} &= \zeta^{(b)}_1(w)v_\lambda + \zeta^{(b)}_2(w)v'_\lambda + \zeta^{(b)}_3(w)\gamma_\lambda \\
R^{(b)}_{\sigma\alpha\beta} &= \eta^{(b)}_1(w)v_\sigma\gamma_\alpha\gamma_\beta + \eta^{(b)}_2(w)v_\sigma v'_\alpha\gamma_\beta + \eta^{(b)}_3(w)g_{\sigma\alpha}v'_\beta \\
R^{(b)}_{\alpha\beta} &= \chi^{(b)}_1(w)\gamma_\alpha\gamma_\beta + \chi^{(b)}_2(w)v'_\alpha\gamma_\beta\ . \qquad (42)
\end{aligned}$$

The IW functions relevant to the current perturbation are not independent, due to the equations of motion :

$$\tau^{(b)}_1(w) + w\tau^{(b)}_2(w) - \tau^{(b)}_3(w) + \tau^{(b)}_4(w) = 0$$

$$\zeta_1^{(b)}(w) + w\zeta_2^{(b)}(w) - \zeta_3^{(b)}(w) = 0 \tag{43}$$

and, at zero recoil, one has

$$\begin{aligned} \tau_4^{(b)}(1) &= \sqrt{3}\ \Delta E_{3/2}\ \tau_{3/2}(1) \\ \zeta_3^{(b)}(1) &= -\Delta E_{1/2}\ \tau_{1/2}(1) \end{aligned} \tag{44}$$

where a radial quantum number n is implicit and $\Delta E_{3/2}$, $\Delta E_{1/2}$ are the mass differences between the excited states and the ground state.

Since, as pointed out above, we will also consider the intermediate states $D\left(\frac{3}{2}^-, 1^-\right)$, let us give the relevant formulae, parallel to (40)-(44)

$$\begin{aligned} &< D\left(\tfrac{3}{2}^-\right)(v')|\overline{c}\Gamma b|B(v) > \cong \sqrt{3}\ \sigma_{3/2}(w)\ \mathrm{Tr}\left[v_\sigma \overline{D}^\sigma(v')\Gamma B(v)\right] \\ &+ \frac{1}{2m_b}\Big\{\mathrm{Tr}\left[T_{\sigma\lambda}^{(b)}\overline{D}^\sigma(v')\Gamma\gamma^\lambda B(v)\right] + \rho_{ke}^{(b)}\ \mathrm{Tr}\left[v_\sigma \overline{D}^\sigma(v')\Gamma B(v)\right] \\ &+ \mathrm{Tr}\left[V_{\sigma\alpha\beta}^{(b)}\overline{D}^\sigma(v')\Gamma P_+(v) i\sigma^{\alpha\beta}B(v)\right]\Big\} \\ &D_{1^-}^\sigma(v') = D_{1^+}^\sigma(v')(-\gamma_5) \\ &T_{\sigma\lambda}^{(b)} = v_\sigma\left[\sigma_1^{(b)}(w)v_\lambda + \sigma_2^{(b)}(w)v'_\lambda + \sigma_3^{(b)}(w)\gamma_\lambda\right] + \sigma_4^{(b)}(w)g_{\sigma\lambda} \\ &V_{\sigma\alpha\beta}^{(b)} = \rho_1^{(b)}(w)v_\sigma\gamma_\alpha\gamma_\beta + \rho_2^{(b)}(w)v_\sigma v'_\alpha\gamma_\beta + \rho_3^{(b)}(w)g_{\sigma\alpha}v'_\beta \\ &\sigma_1^{(b)}(w) + w\sigma_2^{(b)}(w) - \sigma_3^{(b)}(w) + \sigma_4^{(b)}(w) = 0 \\ &\sigma_4^{(b)}(1) = \sqrt{3}\ \Delta E\left(\tfrac{3}{2}^-\right)\sigma_{3/2}(1)\ . \end{aligned} \tag{45}$$

At zero recoil, Luke's theorem [19] imposes

$$L_1(1) = L_2(1) = 0 \tag{46}$$

while it can be shown that $L_4(1)$, $L_5(1)$, $L_6(1)$ are not linearly independent [9], and are related to two quantities, namely

$$\overline{\Lambda} = m_B - m_b = m_D - m_c \tag{47}$$

and the quantity called $\xi_3(1)$ by Falk and Neubert or $\overline{\Sigma}$ by Uraltsev [3] :

$$\begin{aligned} L_4(1) &= -\overline{\Lambda} + 2\xi_3(1) \\ L_5(1) &= -\overline{\Lambda} \\ L_6(1) &= -\overline{\Lambda} - \xi_3(1)\ . \end{aligned} \tag{48}$$

We demonstrate the identity $\xi_3(1) \equiv \overline{\Sigma}$ in Appendix A. Considering the forward amplitude, i.e. taking $w_{if} = 1$, two SR can be obtained for subleading corrections at zero recoil, as we will see below :

$$\overline{\Lambda} = 2\sum_n \Delta E_{1/2}^{(n)} |\tau_{1/2}^{(n)}(1)|^2 + 4\sum_n \Delta E_{3/2}^{(n)} |\tau_{3/2}^{(n)}(1)|^2$$
$$\xi_3(1) \equiv \overline{\Sigma} = 2\sum_n \Delta E_{3/2}^{(n)} |\tau_{3/2}^{(n)}(1)|^2 - 2\sum_n \Delta E_{1/2}^{(n)} |\tau_{1/2}^{(n)}(1)|^2 \tag{49}$$

where $\Delta E_j^{(n)}$ and $\tau_j^{(n)}(1)$ $(j = \frac{1}{2}, \frac{3}{2})$ are the corresponding level spacings and transition IW functions between the ground state $\frac{1}{2}^-$ and the P-wave states $\frac{1}{2}^+$ and $\frac{3}{2}^+$. The first SR is Voloshin SR [20], and the second one was discovered by A. Le Yaouanc et al. [11] and by Uraltsev [3]. We have adopted the notation of Isgur and Wise for the transition IW functions [2].

4 B Meson sum rule.

We take as initial and final states the ground state pseudoscalar meson at different four-velocities $B(v_i)$ and $B(v_f)$ and, as in [3]-[6], the axial currents aligned along the corresponding four-velocities, $\Gamma_i = \not{v}_i\gamma_5$ and $\Gamma_f = \not{v}_f\gamma_5$. Then, the subleading SR (35) writes :

$$\begin{aligned}
&\sum_n \Big\{ (w_i w_f - w_{if}) \xi^{(n)}(w_i) \left[L_1^{(n)}(w_f) + L_4^{(n)}(w_f) \right] \\
&+ (1 - w_i) 2\tau_{1/2}^{(n)}(w_i) F_{1/2}^{(n)}(w_f) \\
&+ \left[(w_i w_f - w_{if})^2 - \frac{1}{3}(w_i^2 - 1)(w_f^2 - 1) \right] \sqrt{3}\ \tau_{3/2}^{(n)}(w_i)\ F_{3/2}^{(n)}(w_f) \\
&+ (w_i w_f - w_{if})\sqrt{3}\ \sigma_{3/2}^{(n)}(w_i)\ G_{3/2}^{(n)}(w_f) + \text{Higher } j^P \text{ states} + (i \leftrightarrow f) \Big\} \\
&= -\left[2L_1(w_{if})(1 + w_{if} - w_i - w_f) - 2L_4(w_{if})(1 - w_{if}) \right]
\end{aligned} \tag{50}$$

The first, second, third and fourth term in the l.h.s. of (50) correspond to the $\left(\frac{1}{2}^-, 1^-\right)$, $\left(\frac{1}{2}^+, 0^+\right)$, $\left(\frac{3}{2}^+, 2^+\right)$ and $\left(\frac{3}{2}^-, 1^-\right)$ intermediate D states. No other states j^P with $j \leq \frac{3}{2}$ appear in the l.h.s. because of the number of γ_5 matrices involved in the traces over Dirac matrices.

In equation (50) we have made explicit the subleading $\frac{1}{2}^- \to \frac{1}{2}^-$ elastic functions $L_j^{(n)}$ $(j = 1, 4)$ and we have factorized, when possible, for the contributions $\frac{1}{2}^- \to \frac{1}{2}^+, \frac{3}{2}^+$ and $\frac{3}{2}^-$, polynomials in (w_i, w_f, w_{if}) that vanish at the frontier $(w, 1, w)$ of the domain (6). The IW functions $\tau_{1/2}^{(n)}(w)$, $\tau_{3/2}^{(n)}(w)$ and $\sigma_{3/2}^{(n)}(w)$ do not vanish at $w = 1$. Similarly, the complicated functions $F_{1/2}^{(n)}(w)$, $F_{3/2}^{(n)}(w)$ and $G_{3/2}^{(n)}(w)$ are given in terms of the form factors defined in Section 3 and do not vanish in general for $w = 1$.

It is not necessary to give all the explicit expressions of these functions since we are interested in the frontier of the domain $(w_i, w_f, w_{if}) = (w, 1, w)$ and, as we will see below, only $F_{1/2}^{(n)}(1)$ will contribute.

There are two crucial features in expression (50). First, the appearance of the subleading functions $L_1(w_{if})$, $L_4(w_{if})$ in the r.h.s., since we consider the whole allowed domain for the variables (w_i, w_f, w_{if}). Second, the polynomials in (w_i, w_f, w_{if}), that result from the sum over the spin $J = 1, 2$ polarizations [4] :

$$\sum_\lambda \varepsilon_{v'}^{(\lambda)*\mu} \ \varepsilon_{v'}^{(\lambda)\nu} \ v_{f\mu} v_{i\nu} = w_i w_f - w_{if}$$
$$\sum_\lambda \varepsilon_{v'}^{(\lambda)*\mu\nu} \ \varepsilon_{v'}^{(\lambda)\rho\sigma} \ v_{f\mu} v_{f\nu} v_{i\rho} v_{i\sigma} = (w_i w_f - w_{if})^2 - \frac{1}{3}(w_i^2 - 1)(w_f^2 - 1) \quad (51)$$

where λ runs over the $2J + 1$ polarizations.

These polynomials will imply the vanishing of the corresponding contributions at $(w_i, w_f, w_{if}) = (w, 1, w)$. This will occur also for higher j^P intermediate states, because one obtains, in all generality [4], for the projector on the polarization tensor of a particle of integer spin J, contracted with v_i and v_f four-velocities :

$$T_{v'}^{\nu_1 \cdots \nu_J, \mu_1 \cdots \mu_J} = \sum_\lambda \varepsilon_{v'}^{(\lambda)*\nu_1 \cdots \nu_J} \ \varepsilon_{v'}^{(\lambda)\mu_1 \cdots \mu_J} \quad (52)$$

$$v_{f\nu_1} \cdots v_{f\nu_J} \ T_{v'}^{\nu_1 \cdots \nu_J, \mu_1 \cdots \mu_J} \ v_{i\mu_1} \cdots v_{i\mu_J} =$$
$$\sum_{k=0}^{J/2} (-1)^k \frac{(J!)^2}{(2J)!} \ \frac{(2J-2k)!}{k!(J-k)!(J-2k)!} (w_i^2 - 1)^k (w_f^2 - 1)^k (w_i w_f - w_{if})^{J-2k} \quad (53)$$

that vanishes for $J > 0$ at $(w_i, w_f, w_{if}) = (w, 1, w)$.

Therefore, at the frontier

$$(w_i, w_f, w_{if}) \to (w, 1, w) \quad (54)$$

the SR (50) will write, very simply, dividing by a factor $(w - 1)$

$$L_4(w) = \sum_n \tau_{1/2}^{(n)}(w) F_{1/2}^{(n)}(1) \ . \quad (55)$$

We only need the functions $F_{1/2}^{(n)}(w)$ for $w=1$. The calculation gives, for all w,

$$\begin{aligned} F_{1/2}^{(n)}(w) &= (1-w)\Big[\zeta_1^{(b)(n)}(w)+(1+2w)\zeta_2^{(b)(n)}(w)+\chi_{kin}^{(b)(n)}(w)+6\chi_1^{(b)(n)}(w) \\ &\quad -2(1+w)\chi_2^{(b)(n)}(w)\Big]+2(1+2w)\zeta_3^{(b)(n)}(w) \end{aligned} \tag{56}$$

and for $w=1$,

$$F_{1/2}^{(n)}(1)=6\zeta_3^{(b)(n)}(1) \tag{57}$$

and from the relation (44) we obtain finally

$$L_4(w)=-6\sum_n \Delta E_{1/2}^{(n)}\ \tau_{1/2}^{(n)}(1)\ \tau_{1/2}^{(n)}(w)\ . \tag{58}$$

5 B* meson sum rule.

We take as initial and final states the ground state vector meson at different four-velocities $B^{*(\lambda_i)}(v_i)$ and $B^{*(\lambda_f)}(v_f)$ and we adopt the particular case of the B^* polarizations (see appendix A of ref. [4]) :

$$\varepsilon_i=\frac{v_f-w_{if}v_i}{\sqrt{w_{if}^2-1}} \qquad\qquad \varepsilon_f=\frac{v_i-w_{if}v_f}{\sqrt{w_{if}^2-1}} \tag{59}$$

that satisfy $\varepsilon_i\cdot v_i=\varepsilon_f\cdot v_f=0$ and $\varepsilon_i^2=\varepsilon_f^2=-1$. With the definitions (59) one has $\varepsilon_i\cdot\varepsilon_f=w_{if}$, but we can change one global sign in (59) to make $\varepsilon_i\cdot\varepsilon_f=-w_{if}$ and therefore $\varepsilon_i\cdot\varepsilon_f\to -1$ when $v_i\to v_f$. The sum rules, being linear in ε_i and in ε_f, do not depend on this overall sign.

Then, performing the relevant traces, the subleading SR (35) writes

$$\begin{aligned} &\sum_n \Big\{ \left(\varepsilon_f^*\cdot v'\right)(\varepsilon_i\cdot v')\,\xi^{(n)}(w_i)\left[L_2^{(n)}(w_f)-L_5^{(n)}(w_f)\right. \\ &\left.-(1-w_f)L_3^{(n)}(w_f)+(1+w_f)L_6^{(n)}(w_f)\right] \\ &+\ 2\tau_{1/2}^{(n)}(w_i)\left[\left(\varepsilon_f^*\cdot\varepsilon_i\right)K_1^{(n)}(w_f,w_i)+(\varepsilon_i\cdot v_f)\left(\varepsilon_f^*\cdot v'\right)K_2^{(n)}(w_f,w_i)\right. \\ &\left.+\left(\varepsilon_f^*\cdot v_i\right)(\varepsilon_i\cdot v')\,K_3^{(n)}(w_f)+(\varepsilon_i\cdot v')\left(\varepsilon_f^*\cdot v'\right)K_4^{(n)}(w_f,w_i,w_{if})\right] \\ &+\ \sqrt{3}\ \tau_{3/2}(w_i)\left[\left(\varepsilon_f^*\cdot\varepsilon_i\right)S_1^{(n)}(w_f,w_i)+\left(\varepsilon_f^*\cdot v'\right)(\varepsilon_i\cdot v_f)\,S_2^{(n)}(w_f,w_i)\right. \\ &\left.+\left(\varepsilon_f^*\cdot v_i\right)(\varepsilon_i\cdot v')\,S_3^{(n)}(w_f,w_i)+(\varepsilon_i\cdot v')\left(\varepsilon_f^*\cdot v'\right)S_4^{(n)}(w_f,w_i,w_{if})\right]+(i\leftrightarrow f)\Big\} \\ &+\ \sqrt{3}\ \tau_{3/2}^{(n)}(w_i)T^{(n)}(w_f)\Big[(w_{if}-w_iw_f)Tr\left[\gamma^\mu \not{v}'\not{v}_f\not{\varepsilon}_f^*\gamma_5\right]Tr\left[\gamma_\mu\not{v}'\not{v}_i\not{\varepsilon}_i\gamma_5\right] \\ &+\ Tr\left[\not{v}_f\not{v}'\not{v}_i\not{\varepsilon}_i\gamma_5\right]Tr\left[\not{v}_i\not{v}'\not{v}_f\not{\varepsilon}_f^*\gamma_5\right]\Big] \end{aligned}$$

$$
\begin{aligned}
&+ \sqrt{3}\sigma_{3/2}^{(n)}(w_i)U^{(n)}(w_f, w_i)\ Tr\left[\gamma^\mu \not{v}' \not{v}_f \not{\varepsilon}_f^* \gamma_5\right]\ Tr\left[\gamma_\mu \not{v}' \not{v}_i \not{\varepsilon}_i \gamma_5\right] \\
&+ \sqrt{3}\sigma_{3/2}^{(n)}(w_i)\left[v_{f\mu}\varepsilon_{f\nu}^* T_{v'}^{\mu\nu,\rho\sigma} v_{i\rho}\varepsilon_{i\sigma} V_1^{(n)}(w_f, w_i)\right. \\
&+ v_{f\mu}\varepsilon_{f\nu}^* T_{v'}^{\mu\nu,\rho\sigma} v_{i\rho} v_{i\sigma}(\varepsilon_i \cdot v') V_2^{(n)}(w_f, w_i) \\
&+ v_{f\mu} v_{f\nu} T_{v'}^{\mu\nu,\rho\sigma} v_{i\rho}\varepsilon_{i\sigma}(\varepsilon_f^* \cdot v') V_3^{(n)}(w_f, w_i) \\
&\left.+ v_{f\mu} v_{f\nu} T_{v'}^{\mu\nu,\rho\sigma} v_{i\rho} v_{i\sigma}(\varepsilon_f^* \cdot v')(\varepsilon_i \cdot v') V_4^{(n)}(w_f, w_i)\right] \\
&+ \text{Higher } j^P \text{ intermediate states} + \ (i \leftrightarrow f)\Big\} \\
&= -\Big\{ -2\left[\left(\varepsilon_f^* \cdot \varepsilon_i\right)(1 - w_i - w_f + w_{if}) + \left(\varepsilon_f^* \cdot v'\right)(\varepsilon_i \cdot v_f)\right. \\
&\left.+ \left(\varepsilon_f^* \cdot v_i\right)(\varepsilon_i \cdot v') - \left(\varepsilon_f^* \cdot v_i\right)(\varepsilon_i \cdot v_f)\right] L_2(w_{if}) \\
&- \left[(\varepsilon_i \cdot v_f)\left(\varepsilon_f^* \cdot v'\right)(w_{if} - 1) + \left(\varepsilon_f^* \cdot v_i\right)(\varepsilon_i \cdot v')(w_{if} - 1)\right. \\
&\left.- (\varepsilon_i \cdot v_f)\left(\varepsilon_f^* \cdot v_i\right)(w_i + w_f - 2)\right] L_3(w_{if}) \\
&+ 2\left[\left(\varepsilon_f^* \cdot \varepsilon_i\right)(1 - w_{if}) + \left(\varepsilon_f^* \cdot v_i\right)(\varepsilon_i \cdot v_f)\right] L_5(w_{if}) \\
&+ \left[-(\varepsilon_i \cdot v_f)\left(\varepsilon_f^* \cdot v'\right)(w_{if} + 1) - \left(\varepsilon_f^* \cdot v_i\right)(\varepsilon_i \cdot v')(w_{if} + 1)\right. \\
&\left.+ (\varepsilon_i \cdot v_f)\left(\varepsilon_f^* \cdot v_i\right)(w_i + w_f - 2)\right] L_6(w_{if})\Big\}
\end{aligned}
\tag{60}
$$

In the l.h.s. of equation (60), the first term corresponds to the intermediate states $\left(\frac{1}{2}^-, 0^-\right) + \left(\frac{1}{2}^-, 1^-\right)$ (both spins contribute due to the fact that one has more four-vectors and a γ_5 in the traces than in the pseudoscalar case), the second, third and fourth terms correspond to the contributions $\left(\frac{1}{2}^+, 1^+\right)$, $\left(\frac{3}{2}^+, 1^+\right)$ and $\left(\frac{3}{2}^+, 2^+\right)$ and the fifth and sixth terms correspond to the contributions $\left(\frac{3}{2}^-, 1^-\right)$ and $\left(\frac{3}{2}^-, 2^-\right)$. We have made explicit the subleading $\frac{1}{2}^- \to \frac{1}{2}^-$ functions $L_j^{(n)}$ $(j = 2, 3, 5, 6)$ and we have kept the explicit dependence on the initial and final polarizations ε_i, ε_f. This allows to factorize, when possible, for the contributions $\frac{1}{2}^- \to \frac{1}{2}^+$, $\frac{3}{2}^+$ and $\frac{3}{2}^-$, polynomials in (w_i, w_f, w_{if}) that vanish at the frontier $(w, 1, w)$. The IW functions $\tau_{1/2}^{(n)}(w)$, $\tau_{3/2}^{(n)}(w)$ and $\sigma_{3/2}^{(n)}(w)$ that do not vanish in general for $w = 1$, and the complicated functions $K_j^{(n)}$ $(j = 1, \cdots 4)$, $S_j^{(n)}$ $(j = 1, \cdots 4)$, $T^{(n)}$, $U^{(n)}$ and $V_j^{(n)}$ $(j = 1, \cdots 4)$ can be computed in terms of the form factors defined in Section 3.

The tensor $T_{v'}^{\mu\nu,\rho\sigma}$, that appears in the l.h.s. of eq. (60), is given in terms of the $J = 2$ polarization tensors by the expression

$$
T_{v'}^{\mu\nu,\rho\sigma} = \sum_\lambda \varepsilon_{v'}^{(\lambda)\mu\nu}\ \varepsilon_{v'}^{(\lambda)\rho\sigma} \tag{61}
$$

In order to see clearly which terms survive at the frontier $(w_i, w_f, w_{if}) = (w, 1, w)$, it is not necessary to go to the details that we have given in Section 4 for the sake of clarity. It is enough to realize that the limit

$$(w_i, w_f, w_{if}) \to (w, 1, w) \tag{62}$$

corresponds to the limit

$$v_f \to v' \quad , \qquad v_i \to v \tag{63}$$

($v \cdot v' = w$) and make use of the orthogonality conditions between the intermediate states polarization tensors and v'. Explicit calculations confirm this simple argument.

In the limit (63), we have

$$\begin{aligned}
&\varepsilon_f^* \cdot \varepsilon_i \to w \\
&\varepsilon_f^* \cdot v' \to 0 \\
&\varepsilon_i \cdot v' \to -\sqrt{w^2-1} \\
&\varepsilon_f^* \cdot v_i \to -\sqrt{w^2-1} \\
&\varepsilon_i \cdot v_f \to -\sqrt{w^2-1} \\
&Tr\left[\gamma^\mu \not{v}' \not{v}_f \not{\varepsilon}_f^* \gamma_5\right] \to 0 \\
&Tr\left[\not{v}_f \not{v}' \not{v}_i \not{\varepsilon}_i \gamma_5\right], Tr\left[\not{v}_i \not{v}' \not{v}_f \not{\varepsilon}_f^* \gamma_5\right] \to 0 \\
&T_{v'}^{\mu\nu,\rho\sigma} \; v_{f\mu} \varepsilon_{f\nu}^* v_{i\rho} \varepsilon_{i\sigma} \to 0 \\
&T_{v'}^{\mu\nu,\rho\sigma} \; v_{f\mu} v_{f\nu} v_{i\rho} \varepsilon_{i\sigma} \to 0 \\
&T_{v'}^{\mu\nu,\rho\sigma} \; v_{f\mu} \varepsilon_{f\nu}^* v_{i\rho} v_{i\sigma} \to 0 \\
&T_{v'}^{\mu\nu,\rho\sigma} \; v_{f\mu} v_{f\nu} v_{i\rho} v_{i\sigma} \to 0 \; .
\end{aligned} \tag{64}$$

The last limits follow from the orthogonality condition $T^{\mu\nu,\rho\sigma} v'_\rho = T^{\mu\nu,\rho\sigma} v'_\mu = 0$. Notice that the "Higher j^P intermediate states" contributions in equation (60), similarly to the four last expressions (64) and to the general expression (53), due to the symmetry in $(\nu_1, \nu_2, \cdots \nu_J)$ and in $(\mu_1, \mu_2, \cdots \mu_J)$ and the linearity in ε_i and ε_f^*, will be proportional to the following quantities

$$\begin{aligned}
&T_{v'}^{\nu_1\nu_2\cdots\nu_J,\mu_1\mu_2\cdots\mu_J} \; v_{f\nu_1} v_{f\nu_2} \cdots \varepsilon_{f\nu_J}^* \; v_{i\mu_1} v_{i\mu_2} \cdots v_{i\mu_{J-1}} \varepsilon_{i\mu_J} \\
&T_{v'}^{\nu_1\nu_2\cdots\nu_J,\mu_1\mu_2\cdots\mu_J} \; v_{f\nu_1} v_{f\nu_2} \cdots v_{f\nu_J} \; v_{i\mu_1} v_{i\mu_2} \cdots v_{i\mu_{J-1}} \varepsilon_{i\mu_J} \\
&T_{v'}^{\nu_1\nu_2\cdots\nu_J,\mu_1\mu_2\cdots\mu_J} \; v_{f\nu_1} v_{f\nu_2} \cdots \varepsilon_{f\nu_J}^* \; v_{i\mu_1} v_{i\mu_2} \cdots v_{i\mu_{J-1}} v_{i\mu_J} \\
&T_{v'}^{\nu_1\nu_2\cdots\nu_J,\mu_1\mu_2\cdots\mu_J} \; v_{f\nu_1} v_{f\nu_2} \cdots v_{f\nu_J} \; v_{i\mu_1} v_{i\mu_2} \cdots v_{i\mu_{J-1}} v_{i\mu_J} \; .
\end{aligned} \tag{65}$$

The tensor $T_{v'}^{\nu_1\nu_2\cdots\nu_J,\mu_1\mu_2\cdots\mu_J}$ is given, in terms of the polarization tensor of an intermediate state of spin J, by expression (52), and the last quantity in (65) is given by the polynomial in (w_i, w_f, w_{if}) (53).

By the same argument as before, due to the orthogonality conditions

$$T_{v'}^{\nu_1\nu_2\cdots\nu_J,\mu_1\mu_2\cdots\mu_J}\ v'_{\mu_k} = T_{v'}^{\nu_1\nu_2\cdots\nu_J,\mu_1\mu_2\cdots\mu_J}\ v'_{\nu_k} = 0 \qquad (k = 1\cdots J) \tag{66}$$

all the quantities (65) go to 0 in the limit $v_f \to v'$ (63).

Therefore, in the limit $(w_i, w_f, w_{if}) \to (w, 1, w)$, the SR (60), taking into account its symmetry in $i \leftrightarrow f$ and the *asymmetry* in $i \leftrightarrow f$ of the limit (63), becomes the much simpler expression :

$$\begin{aligned}
&\sum_n \Big\{2\tau_{1/2}^{(n)}(w)\left[wK_1^{(n)}(1,w) + (w^2-1)K_3^{(n)}(1)\right] \\
&+ 2\tau_{1/2}^{(n)}(1)\left[wK_1^{(n)}(w,1) + (w^2-1)K_2^{(n)}(w,1)\right]\Big\} \\
&+ \Big\{\sqrt{3}\ \tau_{3/2}^{(n)}(w)\left[wS_1^{(n)}(1,w) + (w^2-1)S_3^{(n)}(1,w)\right] \\
&+ \sqrt{3}\ \tau_{3/2}^{(n)}(w)\left[wS_1^{(n)}(w,1) + (w^2-1)S_2^{(n)}(w,1)\right]\Big\} \\
&= -2(w-1)L_5(w) + 2(w^2-1)L_6(w)\ .
\end{aligned} \tag{67}$$

Therefore, we have only to compute the functions $K_1^{(n)}(w_i, w_f)$, $K_2^{(n)}(w_i, w_f)$, $K_3^{(n)}(w_i)$, $S_1^{(n)}(w_i, w_f)$, $S_2^{(n)}(w_i, w_f)$ and $S_3^{(n)}(w_i, w_f)$. The explicit calculation gives, for general (w_i, w_f),

$$\begin{aligned}
&K_1^{(n)}(w_i, w_f) = -(w_i-1)(w_f-1)\Big[\zeta_1^{(b)(n)}(w_i) + (2w_i+1)\zeta_2^{(b)(n)}(w_i) \\
&+\chi_{kin}^{(b)(n)}(w_i) - 2\chi_1^{(b)(n)}(w_i)\Big] - 2(1-w_f)w_i\zeta_3^{(b)(n)}(w_i) \\
&K_2^{(n)}(w_i, w_f) = -(1-w_f)\Big[\zeta_1^{(b)(n)}(w_i) + (2w_i+1)\zeta_2^{(b)(n)}(w_i) - 2\zeta_3^{(b)(n)}(w_i) \\
&+\chi_{kin}^{(b)(n)}(w_i) - 2\chi_1^{(b)(n)}(w_i) + 2\chi_2^{(b)(n)}(w_i)\Big] \\
&K_3^{(n)}(w_i) = -(1-w_i)\Big[\zeta_1^{(b)(n)}(w_i) + (2w_i+1)\zeta_2^{(b)(n)}(w_i) \\
&+\chi_{kin}^{(b)(n)}(w_i) - 2\chi_1^{(b)(n)}(w_i)\Big] - 2w_i\zeta_3^{(b)(n)}(w_i)
\end{aligned} \tag{68}$$

and

$$\begin{aligned}
S_1^{(n)}(w_i, w_f) &= -\frac{1}{6}(w_f^2-1)G_1^{(n)}(w_i) \\
S_2^{(n)}(w_i, w_f) &= -\frac{1}{6}(w_f^2-1)G_2^{(n)}(w_i) \\
S_3^{(n)}(w_i, w_f) &= -\frac{1}{6}(2-w_f)G_1^{(n)}(w_i)
\end{aligned} \tag{69}$$

where in the preceding equations

$$G_1^{(n)}(w_i) = (w_i^2-1)\Big[\tau_1^{(b)(n)}(w_i) + (2w_i+1)\tau_2^{(b)(n)}(w_i) - 2\tau_3^{(b)(n)}(w_i)$$
$$+\chi_{kin}^{(b)(n)}(w_i) - 2\eta_1^{(b)(n)}(w_i) - 3\eta_3^{(b)(n)}(w_i)\Big] + 2(w_i+1)^2\tau_4^{(b)(n)}(w_i)$$

$$G_2^{(n)}(w_i) = (2-w_i)\tau_1^{(b)(n)}(w_i) + (2w_i+1)(2-w_i)\tau_2^{(b)(n)}(w_i) + 2w_i\tau_3^{(b)(n)}(w_i)$$
$$-2w_i\tau_4^{(b)(n)}(w_i) + (2-w_i)\chi_{kin}^{(b)(n)}(w_i) - 2(2-w_i)\eta_1^{(b)(n)}(w_i)$$
$$+4(w_i-1)\eta_2^{(b)(n)}(w_i) + 3(w_i-2)\eta_3^{(b)(n)}(w_i) \qquad (70)$$

and we get, for the quantities needed in equation (67),

$$\begin{aligned}
K_1^{(n)}(1,w) &= -2(1-w)\zeta_3^{(b)(n)}(1)\\
K_3^{(n)}(1) &= -2\zeta_3^{(b)(n)}(1)\\
K_1^{(n)}(w,1) &= K_2^{(n)}(w,1) = 0\\
S_1^{(n)}(1,w) &= -\frac{4}{3}(w^2-1)\tau_4^{(b)(n)}(1)\\
S_3^{(n)}(1,w) &= -\frac{4}{3}(2-w)\tau_4^{(b)(n)}(1)\\
S_1^{(n)}(w,1) &= S_2^{(n)}(w,1) = 0
\end{aligned} \qquad (71)$$

that gives, dividing the SR by the factor $2(w-1)$,

$$-L_5(w) + (w+1)L_6(w)$$
$$= -\sum_n \left[2\tau_{1/2}^{(n)}(w)\ \zeta_3^{(b)(n)}(1) + \frac{4}{3}(w+1)\sqrt{3}\ \tau_{3/2}^{(n)}(w)\ \tau_4^{(b)(n)}(1)\right] \qquad (72)$$

and using (44) one gets finally :

$$-L_5(w) + (w+1)L_6(w)$$
$$= 2\sum_n \Delta E_{1/2}^{(n)}\ \tau_{1/2}^{(n)}(w)\ \tau_{1/2}^{(n)}(1) - 4(w+1)\sum_n \Delta E_{3/2}^{(n)}\ \tau_{3/2}^{(n)}(w)\ \tau_{3/2}^{(n)}(1)\ . \qquad (73)$$

6 Basic results.

Let us recall the two sum rules that we have obtained in the two preceding sections :

$$L_4(w) = -6\sum_n \Delta E_{1/2}^{(n)}\ \tau_{1/2}^{(n)}(1)\ \tau_{1/2}^{(n)}(w) \qquad (74)$$

$$
\begin{aligned}
- L_5(w) + (w+1)L_6(w) &= 2\sum_n \Delta E_{1/2}^{(n)} \ \tau_{1/2}^{(n)}(1) \ \tau_{1/2}^{(n)}(w) \\
&\quad - 4(w+1)\sum_n \Delta E_{3/2}^{(n)} \ \tau_{1/2}^{(n)}(1) \ \tau_{3/2}^{(n)}(w) \ . \qquad (75)
\end{aligned}
$$

Due to the equations of motion, the functions $L_i(w)$ $(i = 4, 5, 6)$ are not independent and, as shown in ref. [9], are given in terms of the elastic IW function $\xi(w)$, a subleading function $\xi_3(w)$ and the $\overline{\Lambda}$ parameter $(\overline{\Lambda} = m_B - m_b)$:

$$
\begin{aligned}
L_4(w) &= -\overline{\Lambda}\xi(w) + 2\xi_3(w) \\
L_5(w) &= -\overline{\Lambda}\xi(w) \\
L_6(w) &= -\frac{2}{w+1}\left[\overline{\Lambda}\xi(w) + \xi_3(w)\right] \ . \qquad (76)
\end{aligned}
$$

Therefore, from (74)-(76) we obtain the interesting relations, *valid for all* w :

$$
\begin{aligned}
\overline{\Lambda}\xi(w) &= 2(w+1)\sum_n \Delta E_{3/2}^{(n)} \ \tau_{3/2}^{(n)}(1) \ \tau_{3/2}^{(n)}(w) \\
&\quad + 2\sum_n \Delta E_{1/2}^{(n)} \ \tau_{1/2}^{(n)}(1) \ \tau_{1/2}^{(n)}(w) \qquad (77)
\end{aligned}
$$

$$
\begin{aligned}
\xi_3(w) &= (w+1)\sum_n \Delta E_{3/2}^{(n)} \ \tau_{3/2}^{(n)}(1) \ \tau_{3/2}^{(n)}(w) \\
&\quad - 2\sum_n \Delta E_{1/2}^{(n)} \ \tau_{1/2}^{(n)}(1) \ \tau_{1/2}^{(n)}(w) \ . \qquad (78)
\end{aligned}
$$

These remarkably simple relations are the basic results of the present paper. They reduce to the known results (49) for $w = 1$. It is important to notice that both the *subleading* quantities $\overline{\Lambda}\xi(w)$ and $\xi_3(w)$ can be expressed in terms of *leading* quantities, namely the IW functions $\tau_j^{(n)}(w)$ and the level spacings $\Delta E_j^{(n)}$ $(j = \frac{1}{2}, \frac{3}{2})$.

Very remarkably, equation (77) shows that the leading IW function $\xi(w)$ appears constrained to be a combination of the *averages*

$$
\frac{1}{\overline{\Lambda}}\sum_n \Delta E_j^{(n)} \tau_j^{(n)}(1) \tau_j^{(n)}(w) \qquad (j = \tfrac{1}{2}, \tfrac{3}{2}) \qquad (79)
$$

or, conversely, the fundamental constant $\overline{\Lambda}$ is given by the ratio of *functions* :

$$
\begin{aligned}
\overline{\Lambda} &= \frac{1}{\xi(w)}\left[2(w+1)\sum_n \Delta E_{3/2}^{(n)} \ \tau_{3/2}^{(n)}(1) \ \tau_{3/2}^{(n)}(w) \right. \\
&\quad \left. + 2\sum_n \Delta E_{1/2}^{(n)} \ \tau_{1/2}^{(n)}(1) \ \tau_{1/2}^{(n)}(w)\right] \ . \qquad (80)
\end{aligned}
$$

It is worth to underline that all other subleading IW functions except $\zeta_3^{(b)}(1)$ cannot contribute to the l.h.s. of the SR (50) because the polynomials (51) vanish at the frontier of the domain $(w_i, w_f, w_{if}) = (w, 1, w)$. In the case of the B^* SR, equation (60), for the same reason only $\zeta_3^{(b)}(1)$ and $\tau_4^{(b)}(1)$ survive in the l.h.s. at $(w_i, w_f, w_{if}) = (w, 1, w)$.

7 Phenomenological discussion.

To illustrate our results for the subleading form factors, we will concentrate on some functions that play a role in the analysis of $B \to D(D^*)\ell\nu$, and about which we can get information from the relations obtained in the preceding Section on $\overline{\Lambda}\xi(w)$ and $\xi_3(w)$. Of particular interest are the functions

$$L_4(w) \qquad \text{and} \qquad \eta(w) = \frac{\xi_3(w)}{\overline{\Lambda}\xi(w)} \tag{81}$$

and their values and derivatives at zero recoil.

The function $L_4(w)$ appears at first order in $1/m_Q$ in the differential semi-leptonic rate of $B \to D\ell\nu$ [9]. This subleading IW function is specially important, but can be expressed, from (76), in terms of $\xi(w)$, $\overline{\Lambda}$ and the commonly used function $\eta(w)$ (see for example [13] and references therein) :

$$L_4(w) = -\overline{\Lambda}\xi(w)\,[1 - 2\eta(w)] \tag{82}$$

7.1 Check of the Bakamjian-Thomas quark models.

We would like first to test whether the results found in the class of relativistic quark models of the Bakamjian-Thomas type [21] are consistent with the sum rules found in this paper, in particular the w dependence. This class of models yield covariant form factors in the heavy quark limit that satisfy Isgur-Wise scaling, and Bjorken and Uraltsev SR. It is a class of models in the sense that one can choose the dynamics in the hadron rest frame, and then compute the corresponding Isgur-Wise functions with the boosted wave functions.

The dynamics at rest that describes in the most accurate way the $Q\overline{Q}$, $Q\overline{q}$ and $q\overline{q}$ spectra (where Q and q denote respectively heavy and light quarks) is the phenomenological Hamiltonian set up by Godfrey and Isgur [22], containing a confining piece, a short distance piece with asymptotic freedom, plus spin-dependent interactions.

7.1.1 Hypothesis of saturation by n = 0 states.

Using this model within the Bakamjian and Thomas scheme, one finds, for the IW functions of the $n = 0$ states (n denoting the radial quantum number) [23] :

$$\begin{aligned}
\xi(w) &= \left(\frac{2}{w+1}\right)^{2\rho^2} \\
\tau_{3/2}^{(0)}(w) &= \tau_{3/2}^{(0)}(1)\left(\frac{2}{w+1}\right)^{2\sigma_{3/2}^2} \\
\tau_{1/2}^{(0)}(w) &= \tau_{1/2}^{(0)}(1)\left(\frac{2}{w+1}\right)^{2\sigma_{1/2}^2}
\end{aligned} \tag{83}$$

with

$$\rho^2 = 1.02 \tag{84}$$

for the slope of the elastic IW function, and, for the transition IW functions to the lowest P-wave states

$$\begin{aligned}
\tau_{3/2}^{(0)}(1) &= 0.5394 \qquad & \sigma_{3/2}^2 &= 1.50 \\
\tau_{1/2}^{(0)}(1) &= 0.2248 \qquad & \sigma_{1/2}^2 &= 0.83\ .
\end{aligned} \tag{85}$$

It has been shown that these $n = 0$ transition IW functions dominate the Bjorken [1], [2] and Uraltsev SR [3], that read, respectively

$$\begin{aligned}
\rho^2 &= \frac{1}{4} + \sum_n |\tau_{1/2}^{(n)}(1)|^2 + 2\sum_n |\tau_{3/2}^{(n)}(1)|^2 \\
\\
\sum_n |\tau_{3/2}^{(n)}(1)|^2 &- \sum_n |\tau_{1/2}^{(n)}(1)|^2 = \frac{1}{4}\ .
\end{aligned} \tag{86}$$

Keeping only the $n = 0$ states and the numbers quoted above we get

$$\begin{aligned}
&\frac{1}{4} + |\tau_{1/2}^{(0)}(1)|^2 + 2|\tau_{3/2}^{(0)}(1)|^2 = 0.882 \\
&|\tau_{3/2}^{(0)}(1)|^2 - |\tau_{1/2}^{(0)}(1)|^2 = 0.240
\end{aligned} \tag{87}$$

to be compared, respectively, with $\rho^2 = 1.02$ (84) and with 1/4 in the r.h.s. of the second equation (86). The $n = 0$ states give a dominant contribution, and saturate Bjorken SR at the 10 % level. Uraltsev SR is even more accurate in this approximation.

Let us first test equation (80), saturating it with the $n = 0$ states :

$$\overline{\Lambda} \cong \frac{1}{\xi(w)} \left[2(w+1)\Delta E_{3/2}^{(0)} \ \tau_{3/2}^{(0)}(1) \ \tau_{3/2}^{(0)}(w) + \ 2\Delta E_{1/2}^{(0)} \ \tau_{1/2}^{(0)}(1) \ \tau_{1/2}^{(0)}(w) \right] \ . \tag{88}$$

Inserting the phenomenological IW functions (83) and the values for $\Delta E_j^{(0)}$ ($j = \frac{1}{2}, \frac{3}{2}$) obtained in the same BT scheme [24] :

$$\Delta E_{3/2}^{(0)} \cong \Delta E_{1/2}^{(0)} = 0.406 \text{ GeV} \tag{89}$$

we get indeed a value of $\overline{\Lambda}$ that is quite stable in the whole physical region of $B \to D^* \ell \nu$, $1 \leq w \leq 1.5$. We find

$$\overline{\Lambda} = 0.513 \pm 0.015 \ . \tag{90}$$

The stability of the result for $\overline{\Lambda}$ is quite remarkable, and results essentially from the function $f(w) = (w+1)\tau_{3/2}^{(0)}(w)$ in the r.h.s., that has a slope $f'(1) = -1$, very close to the slope of $\xi(w)$, $\xi'(1) = -1.02$.

On the other hand, one gets, from (77), (78), (81) and the $n = 0$ approximation :

$$\eta(1) = 0.380 \qquad \eta'(1) = -0.006 \qquad \eta''(1) = 0.0003 \ . \tag{91}$$

For the moment, let us notice that the accuracy of the $n = 0$ approximation depends strongly on the considered quantity, at least at $w = 1$. We have seen that in Bjorken and Uraltsev SR the $n = 0$ states dominate, but the precision is quite different in both cases. As for the subleading quantities, defining

$$R = \frac{\sum\limits_n \Delta E_{1/2}^{(n)} |\tau_{1/2}^{(n)}(1)|^2}{\sum\limits_n \Delta E_{3/2}^{(n)} |\tau_{3/2}^{(n)}(1)|^2} \tag{92}$$

one gets in BT models, keeping only $n = 0$

$$R_{BT} = 0.174 \ . \tag{93}$$

7.1.2 Excited states ($\mathbf{n \neq 0}$) contribution.

Let us now discuss how this can be modified by $n \neq 0$ states. Although the results obtained keeping only the $n = 0$ states are encouraging, one must address the question of the $n \neq 0$ excited states contribution to the SR. This question can have a clear cut answer within the BT scheme, but asks for further numerical calculations to compute $\Delta E_j^{(n)}$ and the w-dependent form factors $\tau_j^{(n)}(w)$ $(j = \frac{1}{2}, \frac{3}{2})$, and will be done in a near future.

At $w = 1$, we can for the moment state that a sum including higher $n \neq 0$ states leads, for the quantity (92), to the value [11]

$$R_{BT} \cong 0.24 \tag{94}$$

giving

$$\eta(1) \cong 0.34 \ , \tag{95}$$

that differs from (91) by 10 %. Therefore, it is of importance to compute the contributions of $n \neq 0$ states for the different quantities, in particular to $\overline{\Lambda}$ of which (90) is only a lower limit, due to the positivity of the different contributions. It is also worth to check whether the inclusion of the $n \neq 0$ states in (80) yields indeed a constant.

In practice, one is anyway confronted to sums truncated to a definite n_{max}. It is of importance to notice that the dependence of the sum on n_{max} requires the consideration of the radiative corrections. On the one hand, one approach proposes to identify the renormalization scale μ with $\Delta E^{(n_{max})}$ [25]. On the other hand, another point of view [12], [13] distinguishes between the cut in n, given by a scale Δ such that $\Delta E^{(n_{max})} \cong \Delta$ and the renormalization point μ, although eventually both scales can be chosen to be proportional. The discussion of the contribution of the higher n states is not simple and cannot be done without including the radiative corrections.

7.2 Comparison with the QCD Sum Rules approach.

Although more precise calculations remain to be done in the BT model, let us qualitatively compare with other approaches.

The approach used up to now to obtain information on the subleading functions has been the QCD Sum Rules (QCDSR) approach [26] (for a review, see [27]). Moreover, radiative corrections to the subleading $1/m_Q$ corrections have also been computed within this scheme in these works. For a recent discussion of the subleading IW functions and their radiative corrections see [28] and [13].

We must first notice that the results obtained from the SR of the present paper and those of QCDSR are quite different in spirit. In our approach we have used the BT quark model to compute the r.h.s. of the SR (77) and (78), while in the QCDSR approach one computes directly the l.h.s. of these equations.

The subleading IW functions are non-perturbative quantities, and their calculation within the QCDSR approach is to some extend model-dependent because it is subject to a number of approximations.

Hence the interest of having information on these non-perturbative quantities within the present method of Bjorken-like SR. We have considered here only a limited number of quantities, namely the subleading corrections of the current perturbation type : $L_4(w)$, $L_5(w)$ and $L_6(w)$ in the notation of Falk and Neubert [9]. Moreover, the radiative corrections to these quantities within the present approach have not been computed. We must compare the values obtained to the ones of the QCDSR without including radiative corrections.

Without including QCD corrections, the QCDSR method gives the values [27]

$$\begin{aligned} &\overline{\Lambda} = 0.50 \\ &\eta(1) = \frac{1}{3} \\ &\eta'(1) \cong 0 \end{aligned} \tag{96}$$

and sets [13]

$$\eta''(1) = 0 \tag{97}$$

that are qualitatively consistent with our results (91), (95).

Notice that, as already pointed out in [11], the QCDSR algebraic value $\eta(1) = \frac{1}{3}$ would correspond to the value

$$R_{QCDSR} = \frac{1}{4} \tag{98}$$

that is very close to the value (94) including $n \neq 0$ states [29], [30].

As we realize in this comparison with the results of QCDSR, we have only computed in our approach a part of the subleading non-perturbative corrections, namely the $1/m_Q$ perturbations to the current. This is a part of a larger program that should include the subleading quantities related to the perturbations of the Lagrangian, namely $L_1(w)$, $L_2(w)$ and $L_3(w)$ or, in the more usual phenomenological notation, $\chi_1(w)$, $\chi_2(w)$ and $\chi_3(w)$ (see, for example [13] and [28]).

8 Conclusion and outlook.

In this paper we have shown that the consideration of the non-forward amplitude leads to powerful results for the subleading form factors at order $1/m_Q$, at least in the case of the functions that correspond to perturbations of the heavy quark current, $L_4(w)$, $L_5(w)$ and $L_6(w)$, or equivalently $\overline{\Lambda}\xi(w)$ and $\xi_3(w)$, where $\xi(w)$ is the leading elastic IW function and $\overline{\Lambda} = m_B - m_b$.

The parameters $1/m_b$ and $1/m_c$ are independent, and we have in this paper studied the sum rules coming from the terms in $1/m_b$, i.e. taking $m_c \to \infty$. The method in this case appears somewhat cumbersome but straightforward. The final results are very simple. We have considered in the SR intermediate states with orbital angular momenta of the light quark $\ell = 0, 1$ and 2. The consequences that we draw from these states made explicit have been easily generalized to all ℓ.

Within the framework of the OPE and the non-forward amplitude, the sum rules of the type $B(v_i) \to D^{(n)}(v') \to B(v_f)$ that depend on the three variables $(w_i, w_f, w_{if}) = (v_i \cdot v', v_f \cdot v', v_i \cdot v_f)$, allow to write $\overline{\Lambda}\xi(w)$ and $\xi_3(w)$ in terms of leading quantities, namely the transition IW functions $\tau_{1/2}^{(n)}(w)$, $\tau_{3/2}^{(n)}(w)$ and the corresponding level spacings $\Delta E_{1/2}^{(n)}$, $\Delta E_{3/2}^{(n)}$. This has been possible by taking the limit to the frontier of the domain $(w_i, w_f, w_{if}) = (w, 1, w)$. Then, most of the excited intermediate states that contribute to the hadronic side of the SR vanish, because zero recoil is chosen on one side, namely $w_f = 1$, and only the P-wave IW functions $\tau_{1/2}^{(n)}(w)$, $\tau_{3/2}^{(n)}(w)$ survive. As a result, the fundamental quantity of HQET $\overline{\Lambda}$ appears to be a *ratio of leading functions* and $\xi_3(w)$ is also given in terms of leading functions.

To proceed further phenomenologically, we have used as an Ansatz for these functions the results of the Bakamjian-Thomas quark model, that gives covariant form factors in the heavy quark limit, satisfies IW scaling and also Bjorken and Uraltsev sum rules. One obtains in this way for a very wide range of w the expected constancy for $\overline{\Lambda}$, with a numerical value of the order of 0.5. This value is in agreement with the QCDSR approach. We find also numerical agreement between our approach for the ratio of functions $\eta(w) = \xi_3(w)/\overline{\Lambda}\xi(w)$ and the QCDSR results. It must be emphasized that the confirmation of the results of the QCDSR approach by our rigorous method (with the phenomenological input of the BT model) is quite encouraging, since both methods are very different in spirit.

The program to study the subleading form factors presented in this paper should be pursued in several directions. First, one should also compute the – in principle independent – sum rules of the $1/m_c$ type (c being the intermediate quark), that should be consistent with the ones of the $1/m_b$ type computed here. We safely conjecture that no new SR for the form factors $\overline{\Lambda}\xi(w)$ and $\xi_3(w)$ will be found. Second, one should compute within the BT scheme the $n \neq 0$ contributions to the r.h.s. of the SR (77) and (78). Thirdly, one should study the other subleading quantities in the same spirit, namely the form factors that come from perturbations of the Lagrangian, i.e. $L_1(w)$, $L_2(w)$, $L_3(w)$, that remain rather uncertain in the QCDSR approach. However, it is not clear that *usable* relations could be obtained in this case, in terms of computed form factors in the BT class of relativistic quark models. But this direction should be pursued. Finally, one should follow and discuss the experimental situation for the $n = 0$ P-wave IW functions $\tau_{1/2}^{(0)}(w)$, $\tau_{3/2}^{(0)}(w)$. As shown in Appendix B, the observed values of $\tau_{1/2}^{(0)}(1)$ do not seem to fit at present Uraltsev sum rule and the predictions of the BT quark models. However, the $\frac{1}{2}^+$ states are very wide, and new semileptonic and non-leptonic data are necessary to finally settle the question of the values for $\tau_{1/2}^{(0)}(w)$.

Appendix A.

We give here a proof of the identity between the subleading quantities $\overline{\Sigma}$ defined by Uraltsev [3] and $\xi_3(1)$ defined by Falk and Neubert [9]

$$\xi_3(1) = \overline{\Sigma} \ . \tag{A.1}$$

Uraltsev expands the matrix element between two B^* mesons for small velocity transfer $\vec{u}$ (formula (14) of the first ref. [3]),

$$< B^*(\varepsilon', \vec{u})|\overline{Q} i D_j Q(0)|B^*(\varepsilon, \vec{0}) >$$

$$= -\frac{\overline{\Lambda}}{2} u_j(\varepsilon'^* \cdot \varepsilon) + \frac{\overline{\Sigma}}{2}\left[(\vec{\varepsilon}\,'^* \cdot \vec{u})\varepsilon_j - \varepsilon_j'^*(\vec{\varepsilon}\cdot\vec{u})\right] + O(\vec{u}^{\,2}) \tag{A.2}$$

where Q is the heavy quark field.

On the other hand, using the formula of Falk and Neubert (3.4) [9], that is valid for all Γ^α

$$< B^*(\varepsilon', u')|\overline{Q}\Gamma^\alpha i D_\alpha Q(0)|B^*(\varepsilon, u) >= -Tr\left[\xi_\alpha(u, u')\overline{\mathcal{B}}^*(\varepsilon', u')\Gamma^\alpha \mathcal{B}^*(\varepsilon, u)\right] \tag{A.3}$$

where

$$\xi_\alpha(u, u') = \xi_+(w)(u+u')_\alpha + \xi_-(w)(u-u')_\alpha - \xi_3(w)\gamma_\alpha \tag{A.4}$$

one can write, in a covariant way

$$< B^*(\varepsilon', u)|\overline{Q} i D_j Q(0)|B^*(\varepsilon, v) >= -Tr\left[\xi_j(u, v)\overline{\mathcal{B}}^*(\varepsilon', u)\mathcal{B}^*(\varepsilon, v)\right] \ . \tag{A.5}$$

Computing in terms of the $\xi_j(u, v)$ the matrix element (A.2) one has

$$< B^*(\varepsilon', \vec{u})|\overline{Q} i D_j Q(0)|B^*(\varepsilon, \vec{0}) > \ = -Tr\left[\xi_j(u, v)\overline{\mathcal{B}}^*(\varepsilon', u)\mathcal{B}^*(\varepsilon, v)\right] \tag{A.6}$$

with

$$u = \left(\sqrt{1+\vec{u}^2}, \vec{u}\right) \qquad v = (1, \vec{0}) \ . \tag{A.7}$$

Therefore

$$< B^*(\varepsilon', \vec{u})|\overline{Q} i D_j Q(0)|B^*(\varepsilon, \vec{0}) > \ = -Tr\left[\xi_j(u, v)\not{\varepsilon}'^* P_+(u) P_+(v)\not{\varepsilon}\right]$$

$$= -Tr\left[\left(\xi_+(w)(u+v)_j + \xi_-(w)(u-v)_j - \xi_3(w)\gamma_j\right)\not{\varepsilon}'^* P_+(u) P_+(v)\not{\varepsilon}\right] \tag{A.8}$$

where $w = \sqrt{1+\vec{u}^2}$.

Taking into account that *under the trace* (A.8) [9]

$$2\xi_+(1) + \xi_3(1) = 0 \tag{A.9}$$

and the relation

$$\xi_-(1) = -\frac{\overline{\Lambda}}{2} \tag{A.10}$$

one obtains, after some algebra, expanding in powers of $\vec{u}$,

$$< B^*(\varepsilon', \vec{u})|\overline{Q} i D_j Q(0)|B^*(\varepsilon, \vec{0}) >$$

$$= -\frac{\overline{\Lambda}}{2} u_j(\varepsilon'^* \cdot \varepsilon) + \frac{\xi_3(1)}{2}\left[(\vec{\varepsilon}\,'^* \cdot \vec{u})\varepsilon_j - \varepsilon_j'^*(\vec{\varepsilon}\cdot\vec{u})\right] + O(\vec{u}^2) \tag{A.11}$$

that compared with (A.2) demonstrates the identity (A.1).

Appendix B. Comment on the experimental situation for $\tau_{1/2}(\mathbf{w})$, $\tau_{3/2}(\mathbf{w})$.

The aim of the present paper has been a theoretical one. However, a brief comment on the experimental situation for the states $D^{**}(j^P, J^P)$ and the corresponding IW functions $\tau_{1/2}(w)$, $\tau_{3/2}(w)$ is in order.

In the framework of the Bakamjian-Thomas quark model, we have given the prediction of the shape of the functions $\tau_{1/2}^{(n)}(w)$, $\tau_{3/2}^{(n)}(w)$ for the $n = 0$ states. We have seen that these $n = 0$ IW functions almost saturate the Bjorken and Uraltsev SR, and give quite reasonable results for the subleading quantities $\overline{\Lambda}\xi(w)$ and $\xi_3(w)$.

However, for the $\frac{1}{2}^+$ states, the present experimental situation seems at odds with these expectations.

The more complete experimental analysis, from Belle, reports four excited states D^{**} above the ground state in non-leptonic decays $B^- \to D^{**0}\pi^-$ [31], two broad and two narrow, that complete the expected number of $n = 0$ states with parity $P = +$. The wide states should correspond to the (j^P, J^P) states $\left(\frac{1}{2}^+, 0^+\right)$, $\left(\frac{1}{2}^+, 1^+\right)$, and the narrow to $\left(\frac{3}{2}^+, 1^+\right)$, $\left(\frac{3}{2}^+, 2^+\right)$. In the following, we denote these states D_J^j. Indeed, the strong decays proceed through $D_0^{1/2} \to D\pi$, $D_1^{1/2} \to D^*\pi$ (S-wave),

$D_1^{3/2} \to D^*\pi$, $D_2^{3/2} \to D\pi$, $D^*\pi$ (D-wave). The product of branching ratios $B(B^- \to D^{**0}\pi^-) \times B(D^{**0} \to D^{(*)0}\pi^-)$ have been measured. Assuming factorization of π^- emission, assuming also that the states 1^+ are unmixed, and using a simple quark model for elementary pion emission for the decays $B \to D^{**}\pi$ to estimate needed spin counting coefficients, one finds qualitatively, as we will see below, a magnitude

$$|\tau_{1/2}^{(0)}(w_0)| \sim |\tau_{3/2}^{(0)}(w_0)| \tag{B.1}$$

where $w_0 \cong \frac{m_B^2+m_{D^{**}}^2}{2m_B m_{D^{**}}}$ is the value of w for $q^2 = m_\pi^2 \cong 0$. This situation is supported by older experiments measuring $B \to D^{**}\pi$ or $B \to D^{**}\ell\nu$ [32]. If the identification of these states is the correct one, the experimental situation for the states $\frac{1}{2}^+$ is at odds with the expectation of the Bakamjian-Thomas quark model. However, an approximate saturation of Bjorken and Uraltsev SR by the $n = 0$ states is not excluded within 1σ.

However, before concluding about these states, new experimental confirmation is needed, mostly for the *very wide* $\frac{1}{2}^+$ states. On the other hand, on the phenomenological side, one should take into account $1/m_c$ and $1/m_b$ corrections in the decays $B(B \to D^{**}\pi)$ [10], and moreover a sensible theoretical scheme for the decays $D^{**} \to D^{(*)}\pi$ is needed, including $1/m_c$ corrections. New data on the semileptonic decays $B \to D^{**}\ell\nu$, that would allow to extract directly the functions $\tau_{1/2}^{(0)}(w)$, $\tau_{3/2}^{(0)}(w)$ would also be very welcome. A needed detailed analysis to extract $\tau_{1/2}^{(0)}(w)$, $\tau_{3/2}^{(0)}(w)$ from present data is beyond the scope of the present paper. Our aim below is to have only a qualitative estimation.

The Belle data on the candidates to the four P-wave states D^{**} are the following :

$D^{**}\left(\frac{3}{2}^+, 2^+\right)$

$$\begin{gathered}
M_2^{3/2} = (2460.7 \pm 2.1 \pm 3.1) \text{ MeV} \\
\Gamma_2^{3/2} = (46.4 \pm 4.4 \pm 3.1) \text{ MeV} \\
B(B^- \to D_2^{3/2\ 0}\pi^-) \times B(D_2^{3/2\ 0} \to D^+\pi^-) = (3.5 \pm 0.3 \pm 0.5) \times 10^{-4} \\
B(B^- \to D_2^{3/2\ 0}\pi^-) \times B(D_2^{3/2\ 0} \to D^{*+}\pi^-) = (2.0 \pm 0.3 \pm 0.5) \times 10^{-4}
\end{gathered} \tag{B.2}$$

$D^{**}\left(\frac{3}{2}^+, 1^+\right)$

$$M_1^{3/2} = (2423.9 \pm 1.7 \pm 0.2) \text{ MeV}$$

$$\Gamma_1^{3/2} = (26.7 \pm 3.1 \pm 2.2)\ \text{MeV}$$

$$B(B^- \to D_1^{3/2\ 0}\pi^-) \times B(D_1^{3/2\ 0} \to D^{*+}\pi^-) = (6.2 \pm 0.5 \pm 1.1) \times 10^{-4} \tag{B.3}$$

$D^{**}\left(\frac{1}{2}^+, 0^+\right)$

$$M_0^{1/2} = (2290 \pm 22 \pm 20)\ \text{MeV}$$

$$\Gamma_0^{1/2} = (305 \pm 30 \pm 25)\ \text{MeV}$$

$$B(B^- \to D_0^{1/2\ 0}\pi^-) \times B(D_0^{1/2\ 0} \to D^{+}\pi^-) = (5.5 \pm 0.5 \pm 0.8) \times 10^{-4} \tag{B.4}$$

$D^{**}\left(\frac{1}{2}^+, 1^+\right)$

$$M_1^{1/2} = (2400 \pm 30 \pm 20)\ \text{MeV}$$

$$\Gamma_1^{1/2} = (380 \pm 100 \pm 100)\ \text{MeV}$$

$$B(B^- \to D_1^{1/2\ 0}\pi^-) \times B(D_1^{1/2\ 0} \to D^{*+}\pi^-) = (4.1 \pm 0.5 \pm 0.8) \times 10^{-4}\ . \tag{B.5}$$

Assuming that these states decay essentially into two-body modes, i.e. $B(D_2^{3/2} \to (D + D^*)\pi)$, $B(D_1^{3/2} \to D^*\pi)$, $B(D_0^{1/2} \to D\pi)$, $B(D_1^{1/2} \to D^*\pi)$, the following branching ratios are given by a Clebsch-Gordan coefficient

$$B(D_1^{3/2\ 0} \to D^{*+}\pi^-) = B(D_0^{1/2\ 0} \to D^{+}\pi^-) = B(D_1^{1/2\ 0} \to D^{*+}\pi^-) = \frac{2}{3}\ . \tag{B.6}$$

To estimate $B(D_2^{3/2\ 0} \to D^{+}\pi^-)$ and $B(D_2^{3/2\ 0} \to D^{*+}\pi^-)$, we use the spin counting of the non-relativistic quark model.

In the limit of neglecting spin-dependent perturbations in the spectrum, i.e. assuming the pairs (D, D^*), $(D_2^{3/2}, D_1^{3/2})$ and $(D_1^{1/2}, D_0^{1/2})$ to be degenerate and that all the D_J^j are degenerate, simple angular momentum calculations give, for the *total* widths :

$$\Gamma(D_2^{3/2}) = \Gamma(D_1^{3/2}) \qquad\qquad \Gamma(D_0^{1/2}) = \Gamma(D_1^{1/2}) \tag{B.7}$$

$$\Gamma(D_2^{3/2} \to D^*\pi) = \frac{3}{2}\Gamma(D_2^{3/2} \to D\pi)\ . \tag{B.8}$$

This last relation gives the needed spin counting coefficient.

It is easy to obtain this factor by realizing that to have the D wave (1 denoting the quark emitting a pion and taking Oz along the pion momentum) one needs the operator $(\sigma_1^z k_\pi)\exp(iz_1 k_\pi) \to i k_\pi^2 \sigma_1^z z_1$. We have then, for the non-vanishing amplitudes

$$M(D_2^{3/2} \to D\pi) = <1\ 0, 1\ 0|2\ 0> <0\ 0|\sigma_1^z|1\ 0> <0\ 0|\mathcal{Y}_1^z|1\ 0>$$

$$M(D_2^{3/2(\pm 1)} \to D^{*(\pm 1)}\pi) = <1\ 0, 1\ \pm 1|2\ \pm 1> <1\ \pm 1|\sigma_1^z|1\ \pm 1> <0\ 0|\mathcal{Y}_1^z|1\ 0> \tag{B.9}$$

that gives

$$M(D_2^{3/2(\pm 1)} \to D^{*(\pm 1)}\pi) = \pm\frac{\sqrt{3}}{2}\ M(D_2^{3/2} \to D\pi) \tag{B.10}$$

and hence (B.8).

We now take into account the actual masses. Since both $D_2^{3/2} \to D\pi$ and $D_2^{3/2} \to D^*\pi$ proceed through the D-wave, we will have

$$\frac{\Gamma(D_2^{3/2} \to D^*\pi)}{\Gamma(D_2^{3/2} \to D\pi)} = \frac{3}{2}\ \frac{p^{*5}}{p^5} \cong 0.40 \tag{B.11}$$

in an obvious notation. Therefore, we obtain the branching ratios

$$B(D_2^{3/2\ 0} \to D^+\pi^-) \cong 0.48 \qquad B(D_2^{3/2\ 0} \to D^{*+}\pi^-) \cong 0.19\ . \tag{B.12}$$

From these BR we find, adding the errors in quadrature

$$\begin{aligned} B(B^- \to D_2^{3/2\ 0}\pi^-) &= (7.3 \pm 1.2) \times 10^{-4} \quad (\text{from } D_2^{3/2\ 0} \to D^+\pi^-) \\ B(B^- \to D_2^{3/2\ 0}\pi^-) &= (10.5 \pm 3.1) \times 10^{-4} \quad (\text{from } D_2^{3/2\ 0} \to D^{*+}\pi^-)\ . \end{aligned} \tag{B.13}$$

We realize that the value for $B(B^- \to D_2^{3/2\ 0}\pi^-)$ differs if one obtains it from $D_2^{3/2\ 0} \to D^+\pi^-$ or from $D_2^{3/2\ 0} \to D^{*+}\pi^-$, although they agree within 1σ. Using (B.5) for the other modes, and taking into account the uncertainty from both results (B.13) one finds

$$\begin{aligned} B(B^- \to D_2^{3/2\ 0}\pi^-) &= (9.8 \pm 3.8) \times 10^{-4} \\ B(B^- \to D_1^{3/2\ 0}\pi^-) &= (9.3 \pm 1.7) \times 10^{-4} \\ B(B^- \to D_1^{1/2\ 0}\pi^-) &= (6.1 \pm 1.4) \times 10^{-4} \\ B(B^- \to D_0^{1/2\ 0}\pi^-) &= (8.2 \pm 1.4) \times 10^{-4} \end{aligned} \tag{B.14}$$

The decays $B^- \to D^{**0}\pi^-$ proceed through two different diagrams : a color-allowed diagram with π^- emission, and a color-suppressed diagram with D^{**0} emission. We will now assume that the π^- emission diagram dominates, although this hypothesis could be incorrect, as we argue at the end of this appendix. However, assuming factorization of π^- emission and that the states 1^+ are unmixed, we find for the decay rates, from [2] :

$$\Gamma = \frac{G_F^2}{16\pi} \, |V_{cb}|^2 \, f_\pi^2 \, \frac{p}{m_B^2} \overline{|M(B \to D^{**}\pi)|^2} \, . \tag{B.15}$$

$$\begin{aligned}
\overline{|M(B \to D_2^{3/2}\pi)|^2} &= 2m_{D^{**}} m_B (m_B + m_{D^{**}})^2 (w_0^2 - 1)^2 |\tau_{3/2}(w_0)|^2 \\
\overline{|M(B \to D_1^{3/2}\pi)|^2} &= 2m_{D^{**}} m_B (m_B - m_{D^{**}})^2 (w_0 + 1)^2 (w_0^2 - 1) |\tau_{3/2}(w_0)|^2 \\
\overline{|M(B \to D_1^{1/2}\pi)|^2} &= 4m_{D^{**}} m_B (m_B - m_{D^{**}})^2 (w_0^2 - 1) |\tau_{1/2}(w_0)|^2 \\
\overline{|M(B \to D_0^{1/2}\pi)|^2} &= 4m_{D^{**}} m_B (m_B + m_{D^{**}})^2 (w_0 - 1)^2 |\tau_{1/2}(w_0)|^2
\end{aligned} \tag{B.16}$$

with

$$w_0 \cong \frac{m_B^2 + m_{D^{**}}^2}{2m_B m_{D^{**}}} \qquad\qquad p \cong \frac{m_B^2 - m_{D^{**}}^2}{2m_B} \tag{B.17}$$

the subindex 0 denoting the value of w for $q^2 = m_\pi^2 \cong 0$, and $m_{D^{**}}$ the mass of the corresponding $D(j^P, J^P)$ state.

It is interesting to notice that the rates (B.15), (B.16) are given by the expressions

$$\Gamma(B \to D_2^{3/2}\pi) = \Gamma(B \to D_1^{3/2}\pi) = \frac{G_F^2}{16\pi} |V_{cb}|^2 m_B^3 \, f_\pi^2 \frac{(1-r)^5(1+r)^7}{16r^3} \left| \tau_{3/2}\left(\frac{1+r^2}{2r}\right)\right|^2 \tag{B.18}$$

$$\Gamma(B \to D_1^{1/2}\pi) = \Gamma(B \to D_0^{1/2}\pi) = \frac{G_F^2}{16\pi} |V_{cb}|^2 m_B^3 \, f_\pi^2 \frac{(1-r)^5(1+r)^3}{2r} \left| \tau_{1/2}\left(\frac{1+r^2}{2r}\right)\right|^2 \tag{B.19}$$

where $r = \frac{m_{D(3/2)}}{m_B}$ and $r = \frac{m_{D(1/2)}}{m_B}$ respectively in the first and the second relations. The equalities $\Gamma(B \to D_2^{3/2}\pi) = \Gamma(B \to D_1^{3/2}\pi)$ and $\Gamma(B \to D_1^{1/2}\pi) = \Gamma(B \to D_0^{1/2}\pi)$ follow from heavy quark symmetry, since there is a single helicity amplitude in all decays.

Using the central values for the masses, but taking into account the errors in (B.14), we find respectively for the different modes, roughly :

$$\begin{array}{ll} B \to D_2^{3/2}\pi & |\tau_{3/2}(1.30)| = 0.29 \pm 0.06 \\ B \to D_1^{3/2}\pi & |\tau_{3/2}(1.32)| = 0.27 \pm 0.04 \\ B \to D_1^{1/2}\pi & |\tau_{1/2}(1.33)| = 0.35 \pm 0.04 \\ B \to D_0^{1/2}\pi & |\tau_{1/2}(1.37)| = 0.39 \pm 0.06 \end{array} \tag{B.20}$$

Within 1σ there is consistency between the different determinations of $|\tau_{3/2}(w_0)|$ and $|\tau_{1/2}(w_0)|$, but errors increase considering both determinations. We conclude safely that we will have the numbers

$$|\tau_{3/2}(1.31)| = 0.28 \pm 0.06$$

$$|\tau_{1/2}(1.35)| = 0.38 \pm 0.07 \tag{B.21}$$

Extrapolating now with the Bakamjian-Thomas form factors (83), (85) for $\tau_{3/2}(w)$ and $\tau_{1/2}(w)$, we get

$$\begin{aligned} |\tau_{3/2}(1)| &= 0.44 \pm 0.10 \\ |\tau_{1/2}(1)| &= 0.49 \pm 0.10 \end{aligned} \tag{B.22}$$

to be compared with the values in the BT model $|\tau_{3/2}(1)|^{BT} = 0.54$, $|\tau_{1/2}(1)|^{BT} = 0.22$. We find agreement for $|\tau_{3/2}(1)|$ within errors, but $|\tau_{1/2}(1)|$ is much too large compared with the BT model.

Saturating Bjorken and Uraltsev SR with the $n = 0$ states, we get a contribution to the Bjorken and Uraltsev SR that lies, within 1σ, in the following range, keeping only the $n = 0$ states :

$$\rho^2 \cong \frac{1}{4} + |\tau_{1/2}(1)|^2 + 2|\tau_{3/2}(1)|^2 = 0.90 \pm 0.28$$

$$|\tau_{3/2}(1)|^2 - |\tau_{1/2}(1)|^2 = -0.04 \pm 0.18\ . \tag{B.23}$$

Therefore, within 1σ, low values for ρ^2 are not excluded but the second value is too small compared to the r.h.s. $\frac{1}{4}$ of Uraltsev SR (86).

We must keep in mind however that the estimation (B.22), that relies on the simple hypothesis of the dominance of π^- emission could be incorrect owing to two facts. First, in many decay modes the color-suppressed diagrams are empirically not so suppressed. Second, the diagram of D^{**0} emission is not computable on the ground of first principles in the BBNS QCD factorization scheme [33], since the emitted meson is composed of heavy-light quarks. Therefore, one must keep in mind that new data on semileptonic decays $B \to D^{**}\ell\nu$ (where the statistics is much smaller than in non-leptonic decays), that directly measure the functions $\tau_{1/2}(w)$ and $\tau_{3/2}(w)$, are necessary to settle the question of the magnitude of $\tau_{1/2}(1)$, $\tau_{3/2}(1)$ and their comparison with Bjorken and Uraltsev SR and with the predictions of Bakamjian-Thomas models.

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