HEAVY-QUARK SYMMETRY

Matthias NEUBERT

Stanford Linear Accelerator Center, Stanford University, Stanford, CA 94309, USA

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Matthias Neubert [1]

Stanford Linear Accelerator Center, Stanford University, Stanford, CA 94309, USA

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[1] Present address: Theory Division, CERN, CH-1211 Geneva 23, Switzerland.

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Abstract

We review the current status of heavy-quark symmetry and its applications to weak decays of hadrons containing a single heavy quark. After an introduction to the underlying physical ideas, we discuss in detail the formalism of the heavy-quark effective theory, including a comprehensive treatment of symmetry breaking corrections. We then illustrate some nonperturbative approaches, which aim at a dynamical, QCD-based calculation of the universal form factors of the effective theory. The main focus is on results obtained using QCD sum rules. Finally, we perform an essentially model-independent analysis of semileptonic B meson decays in the context of the heavy-quark effective theory.

0. Preface

A scan through the Review of Particle Properties gives an impression of the great variety of phenomena caused by the weak interactions. Attempts to understand this rich phenomenology have led to much progress in particle physics. Today, the standard model of the strong and electroweak interactions provides a most successful description of the physics currently accessible with particle accelerators. In spite of its success, however, many open questions remain. In particular, a large number of undetermined parameters are associated with the flavor sector of the theory. These parameters are the quark and lepton masses, as well as the four angles of the Cabibbo-Kobayashi-Maskawa matrix, which describes the mixing of the mass eigenstates of the quarks under the weak interactions. Quite obviously, weak decays offer the most direct way to determine these mixing angles and to test the flavor sector of the standard model. But they are also ideally suited for a study of that part of strong interaction physics which is least understood: the nonperturbative long-distance forces, which are responsible for the confinement of quarks and gluons into hadrons. Indeed, these two aspects cannot be separated from each other. An understanding of the connection between quark and hadron properties is a prerequisite for a precise determination of the parameters of the standard model.

In many instances this puts important limitations on the amount of information that can be deduced from weak interaction experiments at low energies. The theoretical description of hadron properties often relies on very naive bound state models with no direct connection to the underlying theory of quantum chromodynamics (QCD). Indeed, at low energies this theory has proved so intractable to analytical approaches that reliable predictions can only be made based on symmetries. A well known example is chiral symmetry, which arises since the current masses of the light quarks are small compared to the intrinsic mass scale of the strong interactions given, say, by the mass of the proton. It is then useful to consider, as a first approximation, the limiting case of n_f massless quarks, in which the theory has an $SU_L(n_f) \times SU_R(n_f)$ chiral symmetry, which is spontaneously broken to $SU_V(n_f)$. Associated with this is a set of massless Goldstone bosons. This symmetry pattern persists in the presence of small mass terms for the quarks. The Goldstone bosons, which also acquire small masses due to these perturbations, can be identified with the light pseudoscalar mesons. The low energy theorems of current algebra predict relations between the scattering and decay amplitudes for processes involving a different number of these particles, and chiral perturbation theory provides the modern framework for a systematic analysis of the symmetry breaking corrections. This is quite a powerful concept, which allows predictions from first principles in an energy regime where the standard perturbative approach to quantum field theory, i.e., an expansion in powers of the coupling constant, breaks down. A nice illustration is provided by the calculation of the matrix element of the flavor-changing vector current between a kaon and a pion state. The Ademollo-Gatto theorem states that, at zero momentum transfer, the corresponding form factor is normalized up to corrections which are of second order in the symmetry breaking parameter $m_s - m_u$. Chiral perturbation theory can be used to calculate the leading corrections in an expansion in the light quark masses in a model independent way. From such an analysis, the form factor can be predicted with an accuracy of 1%. This makes it possible to obtain a very precise determination of the element V_{us} of the Cabibbo-Kobayashi-Maskawa matrix from an analysis of the semileptonic decay $K \to \pi \ell \bar{\nu}$.

Recently, it has become clear that a similar situation arises in the opposite limit of very large quark masses. When the Compton wave length $1/m_Q$ of a heavy quark bound inside a hadron is much smaller than a typical hadronic distance of about 1 fm, the heavy-quark mass is unimportant for the low energy properties of the state. The strong interactions of such a heavy quark with light quarks and gluons can be described by an effective theory, which is invariant under changes of the flavor and spin of the heavy quark. This heavy-quark symmetry leads to similar predictions than chiral symmetry. In the limit $1/m_Q \to 0$, relations between decay amplitudes for processes involving different heavy quarks arise, and matrix elements of the flavor-changing weak currents become normalized at the point of zero velocity transfer. There is even an analog of the Ademollo–Gatto theorem: In certain cases the leading symmetry breaking corrections are of second order in $1/m_Q$. The existence of an exact symmetry limit of the theory increases the prospects for a precise determination of the element V_{cb} of the quark mixing matrix. Ultimately, it can help to promote the description of weak decays of hadrons containing a heavy quark from the level of naive quark models to a theory of strong interactions. The hope is to start from the model-independent relations that are valid in the limit of infinite heavy-quark masses, and to include the leading symmetry breaking corrections in a systematic expansion in $1/m_Q$. The theoretical framework for such an analysis is provided by the so-called heavy-quark effective theory.

In this review, we present the current status of heavy-quark symmetry and of the heavy-quark effective theory, with emphasis on the exclusive weak decays of hadrons containing a single heavy quark. For these processes the theory has already been developed to an extent which is no less elaborate than, e.g., the modern formulation of chiral perturbation theory. In particular, the symmetry breaking corrections have been analyzed in great detail, and many quantitative predictions of sometimes remarkable accuracy can be made. This review can be divided into four parts: In the first two chapters we present the physical picture and the ideas behind heavy-quark symmetry in an intuitive, introductory way. Similar introductions have been given by Georgi [199], Grinstein [200], Isgur and Wise [201], and Mannel [202]. These notes were very helpful in preparing the material for these chapters. The second part (chapters 3 and 4) is devoted to a detailed and comprehensive description of the formalism of heavy-quark effective theory, and of the state of the art in the analysis of symmetry breaking corrections. In particular, we derive the complete expressions for meson and baryon weak decay amplitudes to order $1/m_Q$ in the heavy-quark expansion, and to next-to-leading order in QCD perturbation theory. The discussion of these topics is necessarily more advanced and, with the exception of the introductory sections 3.1–3.4 and 4.1–4.3, addresses the experts in the field. The third part consists of chapter 5, in which we discuss nonperturbative techniques which aim at a dynamical, QCD-based calculation of hadronic matrix elements. We give a general introduction to the QCD sum rule approach of Shifman, Vainshtein, and Zakharov, which might help readers not familiar with this subject to get an appreciation of the physical motivations of the method. In the subsequent sections we present many interesting results that have been obtained recently by applying the sum rule technique to the heavy-quark effective theory. In the last part (chapter 6) we come back to the phenomenological applications of heavy-quark symmetry. We present a comprehensive analysis of exclusive semileptonic B meson decays in the context of the new theoretical framework provided by the heavy-quark effective theory. The virtue of this approach is that model-independent aspects of the analysis can be clearly separated from model-dependent ones. Ultimately, this may help to change the perspective in heavy-quark phenomenology from a comparison of the data with models to a language which deals with certain types of corrections in a well-defined expansion in QCD. This would be a significant development.

1. Heavy-quark symmetry

1.1. The physical picture

There are several reasons why the strong interactions of systems containing heavy quarks are easier to understand than those of systems containing only light quarks. The first is asymptotic freedom, the fact that the effective coupling constant of QCD becomes weak in processes with large momentum transfer, corresponding to interactions at short distance scales [1,2]. This is a unique feature of nonabelian gauge theories [3]. The familiar phenomenon of screening of charges at large distances can be overcompensated by an antiscreening due to the selfcouplings of the nonabelian gauge fields. This can lead to an effective charge which increases at large distances, and decreases at small distances. In QCD, the antiscreening mechanism dominates (as long as the number n_f of quark flavors does not exceed sixteen). The effective coupling constant

$$\alpha_s(Q^2) = \frac{g_{\rm eff}^2(Q^2)}{4\pi} = \frac{12\pi}{(33-2n_f)\ln(Q^2/\Lambda_{\rm QCD}^2)} \tag{1.1}$$

decreases at large Q^2, i.e., the strong interactions become weak at short distances. At large distances (small Q^2), on the other hand, the coupling becomes strong, leading to nonperturbative phenomena such as the confinement of quarks and gluons on a length scale $R_{\rm had} \sim 1/\Lambda_{\rm QCD} \sim 1$ fm, which determines the typical size of hadrons.[2] Roughly speaking, $\Lambda_{\rm QCD} \sim 0.2$ GeV is the energy scale that separates the regions of large and small coupling constant. When the mass of a quark Q is much larger than this scale, $m_Q \gg \Lambda_{\rm QCD}$, Q is called a heavy quark. The quarks of the standard model fall naturally into two classes: u, d and s are light quarks, whereas c, b and t are heavy quarks.[3] For heavy quarks, the effective coupling constant $\alpha_s(m_Q)$ is small, implying that on length scales comparable to the Compton wavelength $\lambda_Q \sim 1/m_Q$ the strong interactions are perturbative and much like the electromagnetic interactions. In fact, the quarkonium systems ($\bar{Q}Q$), which have a size of order $\lambda_Q/\alpha_s(m_Q) \ll R_{\rm had}$, are very much hydrogen-like. After the discovery of asymptotic freedom, their properties could be predicted [4] before the observation of charmonium, and later of bottomonium states.

For systems composed of a heavy quark and other light constituents, things are more complicated. The size of such systems is determined by $R_{\rm had}$, and the typical momenta exchanged between the heavy and light constituents are of order $\Lambda_{\rm QCD}$. The heavy quark is surrounded by a most complicated, strongly interacting cloud of light quarks, antiquarks, and gluons. This cloud is sometimes referred to as the "brown muck", a term invented by Isgur to emphasize that the properties of such systems cannot be calculated from first principles (at least not in a perturbative way). In this case, it is the fact that

$$m_Q \gg \Lambda_{\rm QCD} \qquad \Rightarrow \qquad \lambda_Q \ll R_{\rm had}, \tag{1.2}$$

that the Compton wavelength of the heavy quark is much smaller than the size of the hadron, which leads to simplifications. To resolve the quantum numbers of the heavy quark would require a hard

[2] We use units where $\hbar = c = 1$. The relation between length and energy scales is $0.2\ \mathrm{GeV} \approx 1\ \mathrm{fm}^{-1}$.

[3] Ironically, the top quark is of no relevance to our discussion, since it is too heavy to form hadronic bound states before it decays into b+W.

probe with $Q^2 \gtrsim m_Q^2$. The soft gluons which couple to the "brown muck" can only resolve distances much larger than λ_Q. Therefore, the light degrees of freedom are blind to the flavor (mass) and spin orientation of the heavy quark. They only experience its color field, which extends over large distances because of confinement. In the rest frame of the heavy quark, it is in fact only the electric color field that is important; relativistic effects such as color magnetism vanish as $m_Q \to \infty$. Since the heavy-quark spin participates in interactions only through such relativistic effects, it decouples. That the heavy-quark mass becomes irrelevant can be seen as follows: As $m_Q \to \infty$, the heavy quark and the hadron that contains it have the same velocity. In the hadron's rest frame, the heavy quark is at rest, too. The wave function of the "brown muck" follows from a solution of the field equations of QCD subject to the boundary condition of a static triplet source of color at the location of the heavy quark. This boundary condition is independent of m_Q, and so is the solution for the configuration of the light degrees of freedom.

It follows that, in the $m_Q \to \infty$ limit, hadronic systems which differ only in the flavor or spin quantum numbers of the heavy quark have the same configuration of light degrees of freedom. Although this observation still does not allow one to calculate what this configuration is, it provides relations between the properties of such particles as the heavy mesons B, D, B* and D*, or the heavy baryons Λ_b and Λ_c (to the extent that corrections to the infinite quark-mass limit are small in these systems). Isgur and Wise realized that these relations result from new symmetries of the effective strong interactions of heavy quarks at low energies [5]. The configuration of light degrees of freedom in a hadron containing a single heavy quark $Q(v,s)$ with spin s and velocity v does not change if this quark is replaced by another heavy quark $Q'(v,s')$ with different flavor or spin, but with the same velocity. Both heavy quarks lead to the same static color field. It is not necessary that the heavy quarks Q and Q′ have similar masses. What is important is that their masses are large compared to Λ_{QCD}. For N_h heavy-quark flavors there is thus an $SU(2N_h)$ spin–flavor symmetry group. These symmetries are in close correspondence to very familiar properties of atoms: The flavor symmetry is analogous to the fact that different isotopes have the same chemistry, since to good approximation the wave function of the electrons is independent of the mass of the nucleus. The electrons only see the total nuclear charge. The spin symmetry is analogous to the fact that the hyperfine levels in atoms are nearly degenerate. The nuclear spin decouples in the limit $m_e/m_N \to 0$.

The heavy-quark symmetry is an approximate symmetry, and corrections arise since the quark masses are not infinite. These corrections are of order Λ_{QCD}/m_Q. The condition $m_Q \gg \Lambda_{QCD}$ is necessary and sufficient for a system containing a heavy quark to be close to the symmetry limit. In many respects, heavy-quark symmetry is complementary to chiral symmetry, which arises in the opposite limit of small quark masses, $m_q \ll \Lambda_{QCD}$. There is an important distinction, however. Whereas chiral symmetry is a symmetry of the QCD Lagrangian in the limit of vanishing quark masses, heavy-quark symmetry is not a symmetry of the Lagrangian (not even an approximate one), but rather a symmetry of an effective theory which is a good approximation to QCD in a certain kinematic region. It is realized only in systems in which a heavy quark interacts predominantly by the exchange of soft gluons. In such systems the heavy quark is almost on-shell; its momentum fluctuates around the mass shell by an amount of order Λ_{QCD}. The corresponding changes in the velocity of the heavy quark vanish as $\Lambda_{QCD}/m_Q \to 0$. The velocity becomes a conserved quantity and is no longer a dynamical degree of freedom [6].[4]

Results derived based on heavy-quark symmetry are model-independent consequences of QCD in a well defined limit. The symmetry breaking corrections can, at least in principle, be studied in a systematic way. To this end, it is however necessary to recast the QCD Lagrangian for heavy quarks,

$$\mathcal{L}_Q = \bar{Q}\,(\mathrm{i}\not{D} - m_Q)\,Q\,, \tag{1.3}$$

into a form suitable for taking the limit $m_Q \to \infty$.

1.2. An effective theory

Inside a hadronic bound state, a heavy quark Q interacting with light constituents exchanges momenta much smaller than its mass m_Q. To good approximation the heavy quark moves with the hadron's velocity v. Being almost on-shell, the heavy quark's momentum P_Q is close to the "kinetic" momentum $m_Q v$ resulting from the hadron's motion:

$$P_Q^\mu = m_Q v^\mu + k^\mu\,. \tag{1.4}$$

Here v is the four-velocity of the hadron. It satisfies $v^2 = 1$. The "residual" momentum k is of order Λ_{QCD}. It results from the interactions of the heavy quark with light degrees of freedom and is thus of dynamical origin. In order to describe the properties of such a system in the case of a very heavy quark, it is appropriate to consider the limit $m_Q \to \infty$ with v and k kept fixed [6]. In this limit the Feynman rules of QCD simplify [7]. In particular, the heavy-quark propagator becomes[5]

$$\frac{\mathrm{i}}{\not{P}_Q - m_Q + \mathrm{i}\epsilon} = \frac{\mathrm{i}}{v\cdot k + \mathrm{i}\epsilon}\,\frac{\not{v}+1}{2} + \mathrm{O}(k/m_Q) \to \frac{\mathrm{i}}{v\cdot k + \mathrm{i}\epsilon}\,P_+\,, \tag{1.5}$$

where P_+ is a positive-energy projection operator. It is defined by

$$P_\pm = (1 \pm \not{v})/2\,, \qquad P_\pm^2 = P_\pm\,, \quad P_\pm P_\mp = 0\,. \tag{1.6}$$

Since $P_+\gamma^\mu P_+ = P_+ v^\mu P_+$, the coupling of a heavy quark to gluons can be simplified, too. To leading order in $1/m_Q$ the vertex $\mathrm{i}g T_a \gamma^\mu$ can be replaced by

$$\mathrm{i}g\,T_a\,v^\mu\,, \tag{1.7}$$

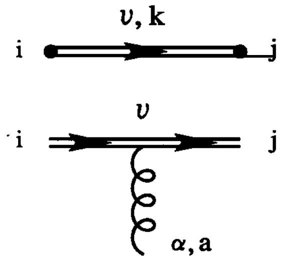
where g is the QCD coupling constant, and T_a are the generators of color SU(3), normalized according to $\mathrm{Tr}(T_a T_b) = \frac{1}{2}\delta_{ab}$. The effective Feynman rules (1.5) and (1.7) are depicted in Fig. 1.1. It has become standard to represent the effective heavy-quark propagator by a double line to indicate that it is different from the propagator in QCD. According to the decomposition (1.4), the propagator is labeled by a velocity and a residual momentum. The velocity is conserved by the strong interactions.

These Feynman rules follow from the Lagrangian

$$\mathcal{L}_{\mathrm{eff}} = \bar{h}_v\,\mathrm{i}v\cdot D\,h_v = \bar{h}_v\left(\mathrm{i}v^\mu\,\partial_\mu + g\,T_a\,v^\mu A_\mu^a\right)h_v\,, \tag{1.8}$$

[4] In systems containing more than one heavy quark, the heavy-quark velocities are not conserved. In quarkonium states, for instance, the typical momenta are of order $\alpha_s(m_Q)m_Q$, corresponding to changes in the velocities of order $\alpha_s(m_Q)$. The spin–flavor symmetry does not apply in this case.

[5] In the rest frame, this so-called "static" propagator was introduced by Eichten and Feinberg [8].



$$= \frac{i}{v\cdot k}\,\frac{1+\slashed{v}}{2}\,\delta_{ji}$$

$$= ig\,(T_a)_{ji}\,v^\alpha$$

Fig. 1.1. Feynman rules of the heavy-quark effective theory (i, j and a are color indices).

where D is the gauge covariant derivative. The effective heavy-quark field $h_v(x)$ is related to the original field $Q(x)$ by

$$Q(x) \approx \exp(-im_Q v\cdot x)\, h_v(x)\,, \tag{1.9}$$

and is furthermore subject to the on-shell constraint

$$\slashed{v}\, h_v = h_v = P_+\, h_v\,. \tag{1.10}$$

Because of this condition, h_v is effectively a two-component field. The phase factor in (1.9) removes the "kinetic" piece $m_Q v$ from the heavy-quark momentum, so that in momentum space a derivative acting on h_v produces the residual momentum k. $\mathcal{L}_{\rm eff}$ is the Lagrangian of the so-called heavy-quark effective theory. In the rest frame, where $v^\mu = (1, \mathbf{0})$, it was introduced by Eichten and Hill [9]. The covariant version presented here is due to Georgi [6].

The effective Lagrangian is only an approximation to the corresponding part of the QCD Lagrangian given in (1.3), since the heavy quark field Q also contains a component H_v satisfying $\slashed{v}H_v = -H_v$. In the rest frame, h_v has only upper components, whereas H_v has only lower components. The field H_v thus corresponds to the "small" components of the full spinor Q. It can be eliminated by means of the equations of motion, leading to corrections of order $1/m_Q$ to the effective Lagrangian. It is important to note, however, that (1.8) is not a nonrelativistic approximation. The velocity v is arbitrary; nothing prevents its spatial components from being of order unity. For a single heavy quark, it is of course possible to transform into the rest frame, in which $\mathcal{L}_{\rm eff}$ reduces to the leading term in a nonrelativistic expansion [10]. But the possibility of v being arbitrary becomes important when one studies processes involving heavy quarks moving at different velocities, such as a heavy-quark decay mediated by a weak current. In chapter 2, we will derive the effective Lagrangian from QCD in such a way that corrections arising at order $1/m_Q$ (or higher) can be systematically included. For the moment, let us take $\mathcal{L}_{\rm eff}$ as it stands and study its global symmetries [6].

Because of (1.9) and (1.10), the effective field h_v annihilates a heavy quark Q with velocity v, but does not create an antiquark. Correspondingly, the Feynman propagator in the effective theory as given in (1.5) has only a single pole (as compared to the two poles of the full propagator). Similarly, $\bar{h}_v$ creates a heavy quark with velocity v, but does not annihilate an antiquark. Thus in the effective theory the number of heavy quarks is conserved; pair creation is not present. If one wants to describe processes with external states containing heavy antiquarks, one has to start from a different effective Lagrangian (see chapter 2).

Since there are no Dirac matrices present in the effective Lagrangian, interactions of the heavy quark with gluons leave its spin unchanged. Associated with this is an SU(2) symmetry group under which $\mathcal{L}_{\rm eff}$ is invariant. The action of this symmetry on the heavy-quark fields becomes most transparent in the rest frame, where the generators S^i of spin SU(2) can be chosen

$$S^i = \frac{1}{2}\begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix} = \frac{1}{2}\gamma_5\gamma^0\gamma^i\,, \tag{1.11}$$

where σ^i are the Pauli matrices, and we use the Dirac representation for the γ-matrices:

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \qquad \gamma_5 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}. \tag{1.12}$$

Here I denotes the 2×2 unit matrix. In a general frame, one defines a set of three orthonormal vectors e^i orthogonal to v, and takes the generators of the spin symmetry as

$$S^i = \tfrac{1}{2}\gamma_5\, \slashed{v}\, \slashed{e}^i\,. \tag{1.13}$$

So defined, the matrices S^i satisfy the commutation relations of SU(2) and commute with $\slashed{v}$:

$$[S^i, S^j] = \mathrm{i}\epsilon^{ijk}S^k, \qquad [\slashed{v}, S^i] = 0\,. \tag{1.14}$$

An infinitesimal SU(2) transformation

$$h_v \to (1 + \mathrm{i}\boldsymbol{\epsilon}\cdot\boldsymbol{S})\, h_v \tag{1.15}$$

leaves the Lagrangian invariant,

$$\delta\mathcal{L}_{\rm eff} = \bar h_v\, [\mathrm{i}v\cdot D, \mathrm{i}\boldsymbol{\epsilon}\cdot\boldsymbol{S}\,]\, h_v = 0\,, \tag{1.16}$$

and preserves the on-shell condition (1.10):

$$\slashed{v}\,(1 + \mathrm{i}\boldsymbol{\epsilon}\cdot\boldsymbol{S})\, h_v = (1 + \mathrm{i}\boldsymbol{\epsilon}\cdot\boldsymbol{S})\,\slashed{v}\, h_v = (1 + \mathrm{i}\boldsymbol{\epsilon}\cdot\boldsymbol{S})\, h_v\,. \tag{1.17}$$

Finally, the mass of the heavy quark does not appear in the effective Lagrangian. When there are $N_{\rm h}$ heavy quarks moving at the same velocity, one can simply extend (1.8) by writing

$$\mathcal{L}_{\rm eff} = \sum_{i=1}^{N_{\rm h}} \bar h_v^i\, \mathrm{i}v\cdot D\, h_v^i\,. \tag{1.18}$$

This Lagrangian is clearly invariant under rotations in flavor space. When combined with the spin symmetry, the symmetry group becomes promoted to SU($2N_{\rm h}$). This is the heavy-quark spin–flavor symmetry [5,6]. Its physical content is that, in the limit $m_Q \gg \Lambda_{\rm QCD}$, the strong interactions of a heavy quark become independent of its mass and spin.

When one wants to describe processes with heavy quarks of different velocity, one simply adds corresponding terms to $\mathcal{L}_{\rm eff}$. For instance, the appropriate effective Lagrangian for the decay of a heavy quark $Q(v)$ into another heavy quark $Q'(v')$ would be

$$\mathcal{L}_{\rm eff} = \bar h_v\, \mathrm{i}v\cdot D\, h_v + \bar h'_{v'}\, \mathrm{i}v'\cdot D\, h'_{v'}\,. \tag{1.19}$$

It has to be supplemented by a source term mediating the transition.

As written above, the effective Lagrangian is not invariant under Lorentz transformations, which change the velocity v. This is not a problem, however. One has to keep in mind that heavy-quark symmetry is not a symmetry of the QCD Lagrangian, but rather a symmetry of the S-matrix in a kinematic region where a heavy quark interacts with light degrees of freedom by the exchange of soft gluons, which leave its velocity unchanged. Under a Lorentz boost Λ, the color field of the heavy quark is boosted, and the appropriate effective Lagrangian is obtained by replacing v in (1.18) by the boosted velocity Λv. Georgi has proposed an effective Lagrangian which contains heavy quarks of all possible flavors and velocities [6]:

$$\mathcal{L}_{\rm eff} = \sum_{i=1}^{N_{\rm h}} \sum_{v} \bar h_v^i \, {\rm i} v\cdot D \, h_v^i \,. \tag{1.20}$$

This Lagrangian has a symmetry group $[\mathrm{SU}(2N_{\rm h})]^{N_v}$, where N_v counts the (infinite) number of velocity intervals. Although there is nothing wrong with this formulation, the notation might suggest an even larger symmetry group $\mathrm{SU}(2N_{\rm h}N_v)$, corresponding to rotations of heavy quarks with different velocity into each other. But this is not a symmetry because of the QCD interactions. The color fields emitted by heavy quarks moving at different velocities look different to the light degrees of freedom. The fact that heavy quarks with different velocities are in no way related to each other is sometimes referred to as "velocity superselection rule" [6]. For practical purposes these subtleties are irrelevant (see, however, the discussion in Ref. [11]). Whatever formulation of $\mathcal{L}_{\rm eff}$ one prefers, the fact is that only heavy quarks moving at the same velocity are related by an $\mathrm{SU}(2N_{\rm h})$ spin–flavor symmetry. For each heavy quark in a given process, it suffices to write down an effective Lagrangian as given in (1.8).

1.3. Spectroscopic implications

The spin–flavor symmetry leads to many interesting relations between the properties of hadrons containing a heavy quark. The most direct consequences concern the spectroscopy of such states [12]. In the $m_{\rm Q} \to \infty$ limit, the spin of the heavy quark and the total angular momentum j of the light degrees of freedom are separately conserved by the strong interactions. Because of heavy-quark symmetry, the dynamics is independent of the spin and mass of the heavy quark. Hadronic states can thus be classified by the quantum numbers (flavor, spin, parity etc.) of the light degrees of freedom. The spin symmetry predicts that, for fixed $j \neq 0$, there is a doublet of degenerate states with total spin $J = j \pm \frac{1}{2}$. The flavor symmetry relates the properties of states with different heavy-quark flavor.

Consider, as an example, the ground-state mesons containing a heavy quark. In this case the light degrees of freedom have the quantum numbers of a light antiquark, and the degenerate states are the pseudoscalar ($J = 0$) and vector ($J = 1$) mesons. In the charm and bottom systems, one knows experimentally

$$m_{\rm B^*} - m_{\rm B} \approx 46~\mathrm{MeV}, \qquad m_{\rm D^*} - m_{\rm D} \approx 142~\mathrm{MeV}, \qquad m_{\rm D_s^*} - m_{\rm D_s} \approx 142~\mathrm{MeV}. \tag{1.21}$$

These mass splittings are in fact reasonably small. To be more specific, at order $1/m_{\rm Q}$ one expects hyperfine corrections to resolve the degeneracy, for instance $m_{\rm B^*} - m_{\rm B} \propto 1/m_{\rm b}$. This leads to the refined prediction $m_{\rm B^*}^2 - m_{\rm B}^2 \approx m_{\rm D^*}^2 - m_{\rm D}^2 \approx \mathrm{const}$. The data are compatible with this:

$$m_{\rm B^*}^2 - m_{\rm B}^2 \approx 0.49~\mathrm{GeV}^2, \qquad m_{\rm D^*}^2 - m_{\rm D}^2 \approx 0.55~\mathrm{GeV}^2. \tag{1.22}$$

The spin symmetry also predicts that $m^2_{B_S^*} - m^2_{B_S} \approx m^2_{D_S^*} - m^2_{D_S} \approx$ const., but this constant could in principle be different from that for nonstrange mesons, since the flavor quantum numbers of the light degrees of freedom are different in both cases. Experimentally, however, $m^2_{D_S^*} - m^2_{D_S} \approx m^2_{D^*} - m^2_{D}$, indicating that to first approximation hyperfine corrections are independent of the flavor of the "brown muck".

One can also study excited meson states, in which the light constituents carry orbital angular momentum. It is tempting to interpret $D_1(2420)$ with $J^P = 1^+$ and $D_2(2460)$ with $J^P = 2^+$ as the spin doublet corresponding to $j = \frac{3}{2}$. In fact, the small mass difference $m_{D_2^*} - m_{D_1} \approx 35$ MeV supports this assertion. One then expects

$$m^2_{B_2^*} - m^2_{B_1} \approx m^2_{D_2^*} - m^2_{D_1} \approx 0.17\ \mathrm{GeV}^2 \tag{1.23}$$

for the corresponding states in the bottom system. The fact that this mass splitting is smaller than for the ground-state mesons in (1.22) is not unexpected. For instance, in the nonrelativistic constituent quark model the light antiquark in these excited mesons is in a p-wave state, and its wave function at the location of the heavy quark vanishes. Hence, in this model hyperfine corrections are strongly suppressed.

A typical prediction of the flavor symmetry is that the "excitation energies" for states with different quantum numbers of the light degrees of freedom are approximately the same in the charm and bottom systems. For instance, one expects

$$\begin{aligned} &m_{B_S} - m_B \approx m_{D_S} - m_D \approx 100\ \mathrm{MeV}, \qquad m_{B_1} - m_B \approx m_{D_1} - m_D \approx 557\ \mathrm{MeV}, \\ &m_{B_2^*} - m_B \approx m_{D_2^*} - m_D \approx 593\ \mathrm{MeV}, \end{aligned} \tag{1.24}$$

Recently, the first relation has been confirmed very nicely by the discovery of the B_S meson by the ALEPH collaboration at LEP [13]. The observed mass, $m_{B_S} = 5.369 \pm 0.006$ GeV, corresponds to an excitation energy $m_{B_S} - m_B = 90 \pm 6$ MeV.

There are many more applications of heavy-quark symmetry to the strong interactions of hadrons containing a heavy quark. For instance, there exist relations between the decay amplitudes describing the emission of light particles such as π, ρ, or $\pi\pi$ off a heavy meson or baryon [12].

1.4. Weak decay form factors

Of particular interest are the relations between the weak decay form factors of heavy mesons, which parameterize hadronic matrix elements of currents between two meson states containing a heavy quark. These relations have been derived by Isgur and Wise [5], generalizing ideas developed by Nussinov and Wetzel [14], and by Voloshin and Shifman [15,16]. For the purpose of this discussion it is appropriate to work with a mass-independent normalization of meson states,

$$\langle M(p')|M(p)\rangle = 2(p^0/m_M)(2\pi)^3\delta^3(\boldsymbol{p} - \boldsymbol{p}'), \tag{1.25}$$

instead of the more conventional, relativistic normalization

$$\langle \widetilde{M}(p')|\widetilde{M}(p)\rangle = 2p^0(2\pi)^3\,\delta^3(\boldsymbol{p} - \boldsymbol{p}'). \tag{1.26}$$

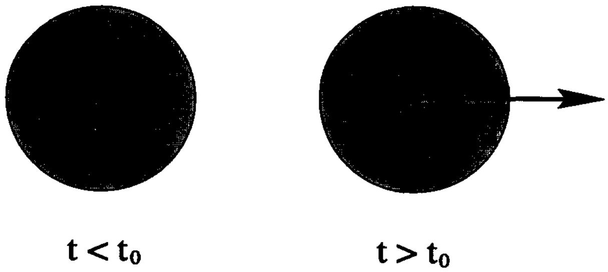


Fig. 1.2. The action of an external heavy-quark current, as seen by the light degrees of freedom in the initial state.

In the first case, $p^0/m_M = v^0$ depends only on the meson velocity. As we shall see, it is more natural for heavy-quark systems to use velocity instead of momentum variables. We will thus write $|M(v)\rangle$ instead of $|M(p)\rangle$. The relation to the conventionally normalized states is

$$|M(v)\rangle = m_M^{-1/2}|\widetilde{M}(p)\rangle\,. \tag{1.27}$$

In the $m_Q \to \infty$ limit, the states $|M(v)\rangle$ are completely characterized by the configuration of the light degrees of freedom. The virtue of working with a mass-independent normalization is that the shape and normalization of the wave function are independent of the heavy-quark mass. We will identify these states with the eigenstates of the effective Lagrangian $\mathcal{L}_{\rm eff}$ in (1.8) supplemented by the standard QCD Lagrangian for the light quarks and gluons.

Consider now the elastic scattering of a pseudoscalar meson, $P(v) \to P(v')$, induced by an external vector current coupled to the heavy quark contained in P. Before the action of the current, the light degrees of freedom in the initial state orbit around the heavy quark, which acts as a source of color moving with the meson's velocity v. On average, this is also the velocity of the "brown muck". The action of the current is to instantaneously (at $t = t_0$) replace the color source by one moving at velocity v', as indicated in Fig. 1.2. If $v = v'$, nothing really happens. The light degrees of freedom do not even realize that there was a current acting on the heavy quark. If the velocities are different, however, the "brown muck" suddenly finds itself interacting with a moving (relative to its rest frame) color source. Soft gluons have to be exchanged in order to rearrange the light degrees of freedom and build the final state meson moving at velocity v'. This rearrangement leads to a form factor suppression, which reflects the fact that as the velocities become more and more different, the probability for an elastic transition decreases. The important observation is that, in the $m_Q \to \infty$ limit, the form factor can only depend on the Lorentz boost $\gamma = v \cdot v'$ that connects the rest frames of the initial and final state mesons. That the form factor is independent of the heavy-quark mass can also be seen from the following intuitive argument: The light constituents in the initial and final state carry momenta which are typically of order $\Lambda_{\rm QCD} v$ and $\Lambda_{\rm QCD} v'$, respectively. Thus, they suffer a typical momentum transfer $q^2 \sim \Lambda_{\rm QCD}^2(v \cdot v' - 1)$, which is in fact independent of m_Q.

The result of this discussion is that in the effective theory, which provides the appropriate framework to consider the limit $m_Q \to \infty$ with the quark velocities kept fixed, the hadronic matrix element describing the scattering process can be written as [5]

$$\langle P(v')\,|\,\bar{h}_{v'}\gamma^\mu h_v\,|P(v)\rangle = \xi(v \cdot v')\,(v + v')^\mu\,, \tag{1.28}$$

with a form factor $\xi(v\cdot v')$ that does not depend on m_Q. Since the matrix element is invariant under complex conjugation combined with an interchange of v and v', the function $\xi(v\cdot v')$ must be real. That there is no term proportional to $(v-v')^\mu$ can be seen by contracting the matrix element with $(v-v')_\mu$, and using $\not{v}h_v = h_v$ and $\bar{h}_{v'}\not{v}' = \bar{h}_{v'}$.

We can now use the flavor symmetry to replace the heavy quark Q in one of the meson states by a heavy quark Q′ of different flavor, thereby turning P into another pseudoscalar meson P′. At the same time, the current becomes a flavor-changing vector current. In the $m_Q\to\infty$ limit, this is a symmetry transformation under which the effective Lagrangian is invariant. Hence the matrix element

$$\langle P'(v')|\,\bar{h}'_{v'}\gamma^\mu h_v\,|P(v)\rangle = \xi(v\cdot v')\,(v+v')^\mu \tag{1.29}$$

is still determined by the same function $\xi(v\cdot v')$. This universal form factor is called the Isgur–Wise function, after the discoverers of this relation [5].

For equal velocities, the vector current $J^\mu = \bar{h}'_v\gamma^\mu h_v = \bar{h}'_v v^\mu h_v$ is conserved in the effective theory irrespective of the flavor of the heavy quarks, since

$$\mathrm{i}\partial_\mu J^\mu = \bar{h}'_v\,\mathrm{i}v\cdot D\,h_v + \bar{h}'_v\,\mathrm{i}v\cdot\overleftarrow{D}\,h_v = 0 \tag{1.30}$$

by the equation of motion which follows from the effective Lagrangian (1.8). The conserved charges

$$N_{Q'Q} = \int \mathrm{d}^3x\, J^0(x) = \int \mathrm{d}^3x\, h'^\dagger_v(x)\,h_v(x) \tag{1.31}$$

are generators of the flavor symmetry. The diagonal generators simply count the number of heavy quarks, whereas the off-diagonal ones replace a heavy quark by another: $N_{QQ}|P(v)\rangle = |P(v)\rangle$ and $N_{Q'Q}|P(v)\rangle = |P'(v)\rangle$. It follows that

$$\langle P'(v)|\,N_{Q'Q}\,|P(v)\rangle = \langle P(v)|P(v)\rangle = 2v^0\,(2\pi)^3\,\delta^3(\mathbf{0})\,, \tag{1.32}$$

and from a comparison with (1.29) one concludes that the Isgur–Wise function is normalized at the point of equal velocities:

$$\xi(1) = 1\,. \tag{1.33}$$

This can easily be understood in terms of the physical picture discussed above: When there is no velocity change, the light degrees of freedom see the same color field and are in an identical configuration before and after the action of the current. There is no form factor suppression. The Isgur–Wise function can in fact be regarded as being the "overlap" of identical "brown muck" configurations boosted relative to each other by $\gamma = v\cdot v'$. Since

$$E_{\mathrm{recoil}} = m_{P'}\,(v\cdot v' - 1) \tag{1.34}$$

is the recoil energy of the daughter meson P′ in the rest frame of the parent meson P, the point $v\cdot v' = 1$ is referred to as the zero recoil limit.

It is instructive to rewrite the above results in terms of a more familiar parameterization of the hadronic matrix elements in (1.28) and (1.29), which uses the relativistic normalization of states in (1.26), and form factors $F_i(q^2)$ which are functions of the invariant momentum transfer q^2. To be specific, let us consider the case of B and D mesons,[6] and make the comparison using the parameterization [17]

$$\langle \widetilde{B}(p')|\bar{b}\gamma^\mu b|\widetilde{B}(p)\rangle = F_{\rm el}(q^2)\,(p+p')^\mu\,,$$

$$\langle \widetilde{D}(p')|\bar{c}\gamma^\mu b|\widetilde{B}(p)\rangle = F_1(q^2)\left((p+p')^\mu - \frac{m_{\rm B}^2 - m_{\rm D}^2}{q^2}\,q^\mu\right) + F_0(q^2)\,\frac{m_{\rm B}^2 - m_{\rm D}^2}{q^2}\,q^\mu\,, \tag{1.35}$$

where $q = p - p'$. The form factors $F_0(q^2)$ and $F_1(q^2)$ are subject to the constraint $F_0(0) = F_1(0)$, which eliminates the spurious pole at $q^2 = 0$. From the comparison of (1.35) with (1.28), one obtains for the elastic form factor

$$\xi(v\cdot v') = \lim_{m_{\rm b}\to\infty} F_{\rm el}(q^2)\,, \tag{1.36}$$

$$v\cdot v' = 1 + (-q^2/2m_{\rm B}^2)\,. \tag{1.37}$$

For the flavor-changing decay, comparison with (1.29) gives [18] (we use m generically for $m_{\rm b}$ and $m_{\rm c}$)

$$\xi(v\cdot v') = \lim_{m\to\infty} R F_1(q^2) = \lim_{m\to\infty} R\left(1 - \frac{q^2}{(m_{\rm B}+m_{\rm D})^2}\right)^{-1} F_0(q^2)\,, \tag{1.38}$$

$$R = \frac{2\sqrt{m_{\rm B} m_{\rm D}}}{m_{\rm B}+m_{\rm D}} \approx 0.88\,, \qquad v\cdot v' = \frac{m_{\rm B}^2 + m_{\rm D}^2 - q^2}{2m_{\rm B} m_{\rm D}}\,. \tag{1.39}$$

The infinite mass limits have to be taken such that the velocity transfer $v\cdot v'$ stays finite.

Although the above relations are nothing but a transfer of conventions, they show that the heavy-quark flavor symmetry is rather peculiar in that it relates form factors at different momentum transfer, but same velocity transfer. The elastic form factor $F_{\rm el}(q^2)$ at spacelike momentum transfer ($q^2 < 0$) is related to the weak decay form factors $F_1(q^2)$ and $F_0(q^2)$ at timelike momentum transfer ($q^2 > 0$). Whereas by current conservation the elastic form factor is normalized at $q^2 = 0$, the weak form factors are normalized at maximum momentum transfer $q^2_{\rm max} = (m_{\rm B} - m_{\rm D})^2$. In both cases, of course, the normalization point corresponds to the zero recoil limit $v\cdot v' = 1$.

Since the Fock state of a heavy meson can only be independent of the heavy-quark mass on length scales greater than $\lambda_{\rm Q} = 1/m_{\rm Q}$, the relations (1.36) and (1.38) must break down when these states are probed at distances smaller than that. This happens when the recoil energy of the light degrees of freedom in the parent rest frame becomes of order of the heavy-quark mass. Therefore, the heavy-quark symmetry relations are only valid as long as $(v\cdot v' - 1)$ is much smaller than $m_{\rm Q}/\Lambda_{\rm QCD}$ [5]. Fortunately, this requirement is well satisfied for the phenomenologically interesting semileptonic $\mathrm{B}\to \mathrm{D}\ell\bar{\nu}$ transition, for which

$$(v\cdot v' - 1)_{\rm max} = \frac{(m_{\rm B} - m_{\rm D})^2}{2m_{\rm B} m_{\rm D}} \approx 0.6\,. \tag{1.40}$$

The heavy-quark spin symmetry leads to additional relations among weak decay form factors. It can be used to relate matrix elements involving vector mesons to those involving pseudoscalar mesons.

[6] In contrast to the standard nomenclature, we shall denote mesons containing a bottom quark by B, not $\bar{\rm B}$.

In the effective theory, a vector meson with longitudinal polarization ϵ_3 is related to a pseudoscalar meson by

$$|V(v,\epsilon_3)\rangle = 2S_Q^3|P(v)\rangle, \tag{1.41}$$

where S_Q^3 is a hermitean operator which acts on the spin of the heavy quark Q. It follows that

$$\begin{aligned}&\langle V'(v',\epsilon_3)|\bar{h}'_{v'}\Gamma h_v|P(v)\rangle\\&\quad = \langle P'(v')|2[S_{Q'}^3,\bar{h}'_{v'}\Gamma h_v]|P(v)\rangle = \langle P'(v')|\bar{h}'_{v'}(2S^3\Gamma)h_v|P(v)\rangle,\end{aligned} \tag{1.42}$$

where Γ can be an arbitrary combination of Dirac matrices, and S^3 is a matrix representation of the operator $S_{Q'}^3$ as given in (1.13). It is easiest to evaluate this expression in the rest frame of the final state meson, where

$$v'^\mu = (1,0,0,0), \qquad \epsilon_3^\mu = (0,0,0,1), \qquad S^3 = \tfrac{1}{2}\gamma_5\gamma^0\gamma^3. \tag{1.43}$$

It is then straightforward to obtain the following commutation relations for the components of the weak current $(V-A)^\mu = \bar{h}'_{v'}\gamma^\mu(1-\gamma_5)h_v$

$$\begin{aligned}&2[S_{Q'}^3, V^0 - A^0] = A^3 - V^3, \qquad 2[S_{Q'}^3, V^3 - A^3] = A^0 - V^0,\\&2[S_{Q'}^3, V^1 - A^1] = \mathrm{i}(A^2 - V^2), \qquad 2[S_{Q'}^3, V^2 - A^2] = -\mathrm{i}(A^1 - V^1).\end{aligned} \tag{1.44}$$

Using (1.42) and (1.44), one can relate the matrix element of the weak current between a pseudoscalar and a vector meson to the matrix element of the vector current between two pseudoscalar mesons given in (1.29). The result can be written in the covariant form [5]

$$\begin{aligned}&\langle V'(v',\epsilon)|\bar{h}'_{v'}\gamma^\mu(1-\gamma_5)h_v|P(v)\rangle\\&\quad = \mathrm{i}\epsilon^{\mu\nu\alpha\beta}\epsilon^*_\nu v'_\alpha v_\beta \xi(v\cdot v') - [\epsilon^{*\mu}(v\cdot v'+1) - v'^\mu \epsilon^*\cdot v]\xi(v\cdot v'),\end{aligned} \tag{1.45}$$

where $\epsilon^{0123} = -\epsilon_{0123} = -1$. Once again, the matrix element is completely described in terms of the universal Isgur–Wise form factor.

For the semileptonic process $B \to D^*\ell\bar{\nu}$, let us study the implications of this prediction in a more conventional form factor basis defined by [17]

$$\begin{aligned}&\langle \widetilde{D}^*(p',\epsilon)|\bar{c}\gamma^\mu(1-\gamma_5)b|\widetilde{B}(p)\rangle\\&\quad = \frac{2\mathrm{i}\epsilon^{\mu\nu\alpha\beta}}{m_B + m_{D^*}}\epsilon^*_\nu p'_\alpha p_\beta V(q^2)\\&\qquad - \left((m_B + m_{D^*})\epsilon^{*\mu}A_1(q^2) - \frac{\epsilon^*\cdot q}{m_B + m_{D^*}}(p+p')^\mu A_2(q^2) - 2m_{D^*}\frac{\epsilon^*\cdot q}{q^2}q^\mu A_3(q^2)\right)\\&\qquad - 2m_{D^*}\frac{\epsilon^*\cdot q}{q^2}q^\mu A_0(q^2).\end{aligned} \tag{1.46}$$

The form factor $A_3(q^2)$ is given by the linear combination

$$A_3(q^2) = \frac{m_B + m_{D^*}}{2m_{D^*}} A_1(q^2) - \frac{m_B - m_{D^*}}{2m_{D^*}} A_2(q^2), \tag{1.47}$$

and is furthermore subject to the constraint $A_0(0) = A_3(0)$, so that there is no pole at $q^2 = 0$. Comparing (1.46) with (1.45), one finds that [18]

$$\begin{aligned}\xi(v\cdot v') &= \lim_{m\to\infty} R^* V(q^2) = \lim_{m\to\infty} R^* A_0(q^2) = \lim_{m\to\infty} R^* A_2(q^2) \\ &= \lim_{m\to\infty} R^* \left(1 - \frac{q^2}{(m_B + m_{D^*})^2}\right)^{-1} A_1(q^2).\end{aligned} \tag{1.48}$$

$R^* \approx 0.89$ and the relation between $v\cdot v'$ and q^2 are the same as in (1.39), but with m_D replaced by m_{D^*}. (Note that $m_D = m_{D^*}$ in the $m_c \to \infty$ limit.)

One could go on and study matrix elements between two vector mesons, or matrix elements of currents with a different Dirac structure. The physical picture is always the same: The hadronic dynamics results from the interaction of the "brown muck" with the color field of the heavy quark, and the boost of this field due to the action of an external current leads to a form factor suppression, which is described by the Isgur–Wise function. In the infinite quark-mass limit, this single, normalized function suffices to describe any meson matrix element of the type $\langle M'|\bar{Q}'\Gamma Q|M\rangle$.

Eqs. (1.38) and (1.48) summarize the relations imposed by heavy-quark symmetry on the weak decay form factors describing the semileptonic decay processes $B \to D\ell\bar{\nu}$ and $B \to D^*\ell\bar{\nu}$. These relations are model independent consequences of QCD in the limit where $m_b, m_c \gg \Lambda_{QCD}$. They can be used to test the consistency of models. Since $R = R^*$ in the infinite quark-mass limit, the QCD prediction is that the hadronic form factors F_1, V, A_0 and A_2 become identical, i.e., they have the same normalization and q^2-dependence. The same is true for the form factors F_0 and A_1. However, they differ from the former ones by a kinematic factor $[1 - q^2/(m_B + m_{D^{(*)}})^2]$. In the infinite mass limit this has the form of a pole term, since $m_b + m_c$ is the pole mass associated with the flavor-changing current [18]. It turns out that none of the quark models proposed in the literature is fully consistent with these relations. Let us briefly demonstrate this for three of the most popular models. Since in the limit of vanishing lepton mass F_0 and A_0 do not contribute to semileptonic decay rates, we shall focus on the remaining form factors F_1, V, A_1, and A_2. To the extent that the bottom and charm quarks are sufficiently heavy, the QCD prediction is

$$F_1(q^2) \approx V(q^2) \approx A_2(q^2) \approx \left(1 - \frac{q^2}{(m_B + m_{D^*})^2}\right)^{-1} A_1(q^2). \tag{1.49}$$

At maximum momentum transfer, corresponding to zero recoil, the first three form factors should have values close to $R^{-1} \approx R^{*-1} \approx 1.13$, whereas $A_1(q^2_{max})$ should approach $R^* \approx 0.89$. In the nonrelativistic constituent quark model of Isgur, Scora, Grinstein, and Wise (ISGW), all form factors have the same q^2-dependence, namely [19]

$$h\exp[-0.03(q^2_{max} - q^2)], \tag{1.50}$$

where h is computed from an overlap integral of nonrelativistic meson wave functions. The numerical values are $h = 1.13$, 1.19, 1.06, and 0.94 for F_1, V, A_2, and A_1, respectively. The normalizations at q^2_{max} are reasonably close to the asymptotic QCD prediction, but the q^2-dependence of A_1 relative to the other form factors is incompatible with heavy-quark symmetry. This becomes apparent in Fig. 1.3, where we show the four quantities in (1.49) as a function of q^2. A similar situation is encountered with the relativistic wave function model of Bauer, Stech, and Wirbel (BSW) [17]. Again all form factors are assumed to have the same q^2-dependence, given by the pole ansatz

$$h(1 - q^2/m_{\rm pole}^2)^{-n}, \tag{1.51}$$

with $n = 1$. In this model, the normalization at $q^2 = 0$ is computed from overlap integrals of relativistic light-cone wave functions. Numerically, h = 0.69, 0.71, 0.69, and 0.65 for F_1, V, A_2, and A_1. The pole masses are assumed to be $m_{\rm pole} = 6.34$ GeV for F_1 and V, and 6.73 GeV for A_1 and A_2. Again, the normalizations at maximum momentum transfer turn out to be close to the asymptotic prediction (see Fig. 1.3), but the relative q^2-dependence of A_1 is inconsistent with (1.49). Körner and Schuler (KS) adopt the same pole ansatz as shown above, with universal parameters $h = 0.7$ and $m_{\rm pole} = 6.34$ GeV for all form factors [20]. However, they adjust the coefficient n to be consistent with the QCD power counting rules for asymptotically large values of q^2 [21]. This gives $n = 1$ for F_1 and A_1, and $n = 2$ for V and A_2. In this model the relative q^2-dependence of F_1 is not in accordance with (1.49).[7] This little exercise clearly demonstrates the importance and usefulness of heavy-quark symmetry as an exact limiting case. It also elucidates the potential uncertainties inherent in models, which often rely on ad hoc assumptions. On the other hand, one may say that if these models have any physical significance they indicate that deviations from the heavy quark symmetry limit are moderate. This provides support for an approach where one starts from the limit of an exact symmetry, and adds on symmetry breaking corrections in a systematic way.

Many of the physical ideas presented in this chapter are not new to the discussion of heavy-quark effective theory. They have emerged from the work of many researchers over more than a decade. Prior to the modern formulation of the heavy-quark effective Lagrangian, these ideas have been incorporated to some extent into models describing hadronic bound states, such as the nonrelativistic constituent quark model [19,22–24]. The concept of a new flavor symmetry for hadrons containing a heavy quark was introduced as early as 1980 by Shuryak [25], who later studied many properties of heavy mesons and baryons in the context of QCD sum rules [26]. The infinite quark-mass limit in QCD was used by Eichten and Feinberg to study the heavy-quark–antiquark potential [8]. They introduced the static propagator in position space. The utility of the static approximation, and of the closely related nonrelativistic approximation [10], for computations in lattice gauge theory was discussed by Eichten [27], and by Lepage and Thacker [28]. Close, Gounaris, and Paschalis were the first to realize the approximate normalization of the weak decay form factors at zero recoil, and the relations between the $B \to D$ and $B \to D^*$ semileptonic decay rates [29]. A clear, model-independent formulation of the physical ideas of the spin–flavor symmetry was developed in the important works by Nussinov and Wetzel [14], and Voloshin and Shifman [15,16]. These papers contain much of the present knowledge about the weak decay form factors close to the endpoint region, in particular the model-independent normalization at zero recoil following from the heavy-quark flavor symmetry. Voloshin and Shifman even anticipated the vanishing of certain $1/m_Q$ corrections to the normalization at zero recoil [16], a result which is nowadays referred to as Luke's theorem [30]. They were also the first to understand the renormalization of currents containing heavy-quark fields, which leads to corrections that depend logarithmically on the heavy-quark mass [15]. Such corrections were also

[7] The region of applicability of the power counting rules corresponds to values $v \cdot v' \sim m_Q^2/\Lambda_{\rm QCD}^2$, which is outside of the range of validity of heavy-quark symmetry. There is no contradiction between the two limits.

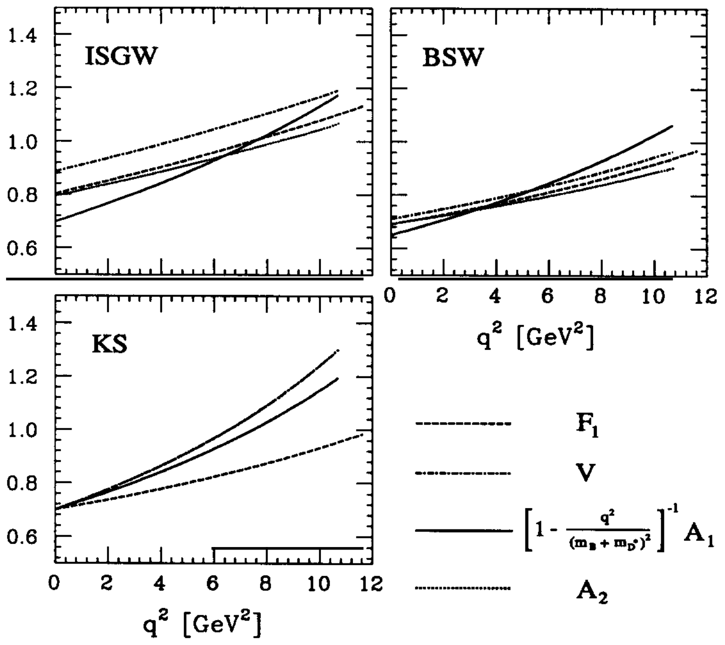


Fig. 1.3. Predictions for the weak decay form factors F_1, V, A_1, and A_2 over the kinematic range in q^2 accessible in semileptonic B decays, according to the quark models of ISGW, BSW, and KS. A_1 is multiplied by a kinematic factor as shown in (1.49). In the KS model the form factors V and A_2 are identical.

investigated by Politzer and Wise [31]. Another cornerstone were the two famous papers by Isgur and Wise [5], in which they pointed out that the simplifications arising in heavy-quark systems are consequences of the low energy effective Lagrangian that describes the strong interactions of heavy quarks with light degrees of freedom. The relations between decay form factors, which were previously derived mainly by intuition or in the framework of particular models, emerged as model-independent predictions of QCD in a well defined limit. They also realized the special role played by the heavy-quark velocities, and showed that the symmetry relations are not only valid near zero recoil, but in fact over the entire kinematic range in the velocity transfer variable $v \cdot v'$. A little later, Eichten and Hill constructed the effective Lagrangian for a heavy quark (in its rest frame) to leading and subleading order in the $1/m_Q$ expansion [9,32]. Grinstein established the validity of this effective theory at leading order in $1/m_Q$, but to all orders in perturbation theory [33]. Finally, Georgi reformulated the effective Lagrangian in a covariant way, which is necessary for studying processes involving heavy quarks with different velocities [6]. The establishment of this effective field theory for heavy quarks opened the way to more detailed investigations of renormalization group effects, to a systematic analysis of power corrections to the infinite quark-mass limit, and to many phenomenological applications of heavy quark symmetry.

The purpose of this review is to present the current status of these developments. The following chapters deal with a more rigorous derivation of the results presented above, as well as with a systematic discussion of the corrections to the exact symmetry limit. The presentation will often be more technical, but this should not obscure the simplicity of the underlying ideas of heavy-quark symmetry. In chapter 6, we will return to phenomenology and discuss in detail the semileptonic decays $B \to D^{(*)}\ell\bar{\nu}$ within the new theoretical framework provided by the heavy-quark effective theory.

2. Heavy-quark effective theory

2.1. Effective field theories

Effective field theories are an important tool in theoretical physics. The reason is simple: For the understanding of a physical process it is usually counterproductive to consider it in the context of a "theory of everything" (even if this existed). It is better to use a level of description that is most adequate to the problem at hand. In other words, one takes into account those aspects of the "full theory" which are important, and ignores others which are irrelevant. So are Newton's laws, Maxwell's equations, and the laws of thermodynamics sufficient to account for most of the phenomena of our everyday life, whereas the more refined descriptions of quantum mechanics and relativity are necessary to understand the physics at smaller distances and larger energies. For the energies presently accessible with particle accelerators, the language of local quantum field theories, in form of the standard model of the strong and electroweak interactions, has proved to provide a most adequate level of description. We are well aware that none of these concepts truly is the "theory of everything". But nevertheless, in their range of applicability they can all be used to make calculations of sometimes incredible accuracy.

In particle physics, it is often the case that the effects of a very heavy particle become irrelevant at low energies. It is then useful to construct a low energy effective theory in which this heavy particle no longer appears. Eventually, this effective theory will be easier to deal with than the full theory. A familiar example is Fermi's theory of the weak interactions. For the description of weak decays of hadrons one can safely approximate the weak interactions by point-like four-fermion couplings governed by a dimensionful coupling constant G_F. Only at energies much larger than the masses of hadrons can one resolve the structure of the intermediate vector bosons W and Z. This example is also instructive in that it shows that it is usually the low energy effective theory which is known first. Only as one proceeds to higher energies its limitations become apparent. In fortunate circumstances this can lead to the discovery of a new effective theory (in this case the standard model of electroweak interactions), which provides an adequate level of description for higher energies.

Technically, the process of removing the degrees of freedom of a heavy particle involves the following steps [34-36]: One first identifies the heavy particle fields and "integrates them out" in the generating functional of the Green functions of the theory. This is possible since at low energies the heavy particle does not appear as an external state. However, although the action of the full theory is usually a local one, what results after this first step is a nonlocal effective action. The nonlocality is related to the fact that in the full theory the heavy particle (with mass M) can appear in virtual processes and propagate over a short but finite distance $\Delta x \sim 1/M$. Thus a second step is required to get to a local effective Lagrangian: The nonlocal effective action is rewritten as an infinite series of local terms in an operator product expansion [37,38]. Roughly speaking, this corresponds to an expansion in $1/M$. It is in this step that the short- and long-distance physics is disentangled. The long-distance physics corresponds to interactions at low energies and is the same in the full and the effective theory. But short-distance effects arising from quantum corrections involving large virtual momenta are not reproduced in the effective theory up to this point. In a third step, they have to be added in a perturbative way using renormalization group techniques. This procedure is called matching. It leads to renormalizations of the coefficients of the local operators in the effective Lagrangian. An example is the effective Lagrangian for nonleptonic weak decays, in which radiative corrections from hard gluons with virtual momenta between m_W and some renormalization scale μ give rise to Wilson coefficients, which renormalize the local four-fermion interactions [39–41].

The heavy-quark effective theory (HQET) is constructed to provide a simplified description of processes where a heavy quark interacts with light degrees of freedom by the exchange of soft gluons. Clearly, m_Q is the high energy scale in this case, and Λ_{QCD} is the scale of the hadronic physics one is interested in. However, a subtlety arises since one wants to describe the properties and decays of hadrons which contain a heavy quark. Hence it is not possible to remove the heavy quark completely from the effective theory. What is possible, however, is to integrate out the "small components" in the full heavy-quark spinor, which describe fluctuations around the mass shell.

2.2. Derivation of the effective Lagrangian

The starting point in the construction of the low energy effective theory is the observation that a very heavy quark bound inside a hadron moves with essentially the hadron's velocity v and is almost on-shell. Its momentum can be written as $P_Q = m_Q v + k$, where k is much smaller than $m_Q v$. Interactions of the heavy quark with light degrees of freedom change this residual momentum by an amount of order $\Delta k \sim \Lambda_{QCD}$, but the corresponding changes in the heavy-quark velocity vanish as $\Lambda_{QCD}/m_Q \to 0$. For this situation it is appropriate to introduce "large" and "small" component fields h_v and H_v by

$$h_v(x) = \exp(im_Q v \cdot x)\, P_+\, Q(x), \qquad H_v(x) = \exp(im_Q v \cdot x)\, P_-\, Q(x), \tag{2.1}$$

so that

$$Q(x) = \exp(-im_Q v \cdot x)\, [h_v(x) + H_v(x)]. \tag{2.2}$$

The heavy-quark mass m_Q is defined as "the mass in the Lagrangian" to make the manipulations below work. A precise definition of this parameter will be given in section 2.6. Because of the projection operators $P_\pm = \frac{1}{2}(1 \pm \not{v})$, the new fields satisfy $\not{v}\, h_v = h_v$ and $\not{v}\, H_v = -H_v$. In the rest frame, h_v corresponds to the upper two components of Q, while H_v corresponds to the lower ones. Whereas h_v annihilates a heavy quark with velocity v, H_v creates a heavy antiquark with velocity v. If the heavy quark was on-shell, the field H_v would be absent. If one wants to describe a situation where a hadron contains a heavy antiquark, the sign of the velocity (momentum) has to be reversed. In this case, one would introduce large and small component fields by

$$h_v^-(x) = \exp(-im_Q v \cdot x)\, P_-\, Q(x), \qquad H_v^-(x) = \exp(-im_Q v \cdot x)\, P_+\, Q(x), \tag{2.3}$$

so that

$$Q(x) = \exp(im_Q v \cdot x)\, [h_v^-(x) + H_v^-(x)]. \tag{2.4}$$

The effective Lagrangian for this case is simply obtained by replacing $v \to -v$ and $h_v \to h_v^-$ in the equations presented below.

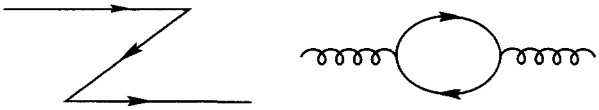


Fig. 2.1. Virtual fluctuations involving pair creation of heavy quarks. Time flows to the right.

In terms of the new fields, the QCD Lagrangian for heavy quarks given in (1.3) takes the following form:

$$\mathcal{L}_Q = \bar{h}_v \mathrm{i} v \cdot D h_v - \bar{H}_v (\mathrm{i} v \cdot D + 2m_Q) H_v + \bar{h}_v \, \mathrm{i} \not{D}_\perp H_v + \bar{H}_v \, \mathrm{i} \not{D}_\perp h_v , \tag{2.5}$$

where

$$D_\perp^\mu = D^\mu - v^\mu v \cdot D \tag{2.6}$$

is orthogonal to the heavy-quark velocity, $v \cdot D_\perp = 0$. In the rest frame, $D_\perp^\mu = (0, \boldsymbol{D})$ contains just the space components of the covariant derivative. From (2.5) it is apparent that h_v describes massless degrees of freedom, whereas H_v corresponds to fluctuations with twice the heavy-quark mass. These are the heavy degrees of freedom which will be eliminated. The fields are mixed by the presence of the third and fourth terms, which describe pair creation or annihilation of heavy quarks and antiquarks. As shown in the first diagram in Fig. 2.1, in a virtual process a heavy quark propagating forward in time can turn into a virtual antiquark propagating backward in time, and then turn back into a quark. The energy of the intermediate quantum state $hh\bar{H}$ is larger than the energy of the incoming heavy quark by at least $2m_Q$. Because of this large energy, the virtual quantum fluctuation can only propagate over a short distance $\Delta x \sim 1/m_Q$. On hadronic scales set by $R_{\mathrm{had}} = 1/\Lambda_{\mathrm{QCD}}$, the process essentially looks like a local interaction of the form

$$\bar{h}_v \mathrm{i} \not{D}_\perp (1/2m_Q) \mathrm{i} \not{D}_\perp h_v , \tag{2.7}$$

where we have simply replaced the propagator for H_v by $\mathrm{i}/2m_Q$. A more correct treatment is to integrate out the small component field H_v, thereby deriving a nonlocal effective action for the large component field h_v, which can then be expanded in terms of local operators. Before doing this, let us mention a second type of virtual corrections which involve pair creation, namely heavy-quark loops. An example is depicted in the second diagram in Fig. 2.1. Another one would be the triangle graph with two external gluons and an axial vector current, which gives rise to the chiral anomaly. Such heavy-quark loops cannot be described in terms of the effective fields h_v and H_v, since the quark velocities in the loop are not conserved, and are in no way related to hadron velocities. However, these short-distance processes are proportional to the small coupling constant $\alpha_s(m_Q)$ and can be calculated in perturbation theory. They lead to corrections which are added onto the low energy effective theory in the matching procedure. This will be discussed in detail in chapter 3.

On a classical level, the heavy degrees of freedom represented by H_v can be eliminated using the equations of motion of QCD. Substituting (2.2) into $(\mathrm{i}\not{D} - m_Q)\, Q = 0$ gives

$$\mathrm{i}\not{D}\, h_v + (\mathrm{i}\not{D} - 2m_Q)\, H_v = 0 , \tag{2.8}$$

and multiplying this by $P_\pm$ one derives the two equations

$$-\mathrm{i}v\cdot Dh_v = \mathrm{i}\not{D}_\perp H_v, \qquad (\mathrm{i}v\cdot D + 2m_\mathrm{Q})H_v = \mathrm{i}\not{D}_\perp h_v. \tag{2.9}$$

The second can be solved to give

$$H_v = (\mathrm{i}v\cdot D + 2m_\mathrm{Q} - \mathrm{i}\epsilon)^{-1}\mathrm{i}\not{D}_\perp h_v, \tag{2.10}$$

which shows that the small component field H_v is indeed of order $1/m_\mathrm{Q}$. One can now insert this solution into the first equation to obtain the equation of motion for h_v. It is easy to see that this equation follows from the effective Lagrangian

$$\mathcal{L}_\mathrm{eff} = \bar{h}_v\mathrm{i}v\cdot Dh_v + \bar{h}_v\mathrm{i}\not{D}_\perp(\mathrm{i}v\cdot D + 2m_\mathrm{Q} - \mathrm{i}\epsilon)^{-1}\mathrm{i}\not{D}_\perp h_v, \tag{2.11}$$

which is the correct generalization of (1.8) for large but finite heavy-quark mass. This is the Lagrangian of the heavy-quark effective theory (HQET). Clearly, the second term precisely corresponds to the first class of virtual processes depicted in Fig. 2.1.

Mannel, Roberts, and Ryzak have derived this Lagrangian in a more elegant way by manipulating the generating functional for QCD Green's functions containing heavy-quark fields [42]. They start from the field redefinition (2.2) and couple the large component fields h_v to external sources ρ_v. Green's functions with an arbitrary number of h_v fields can be constructed by taking derivatives with respect to ρ_v. No sources are needed for the heavy degrees of freedom represented by H_v. The functional integral over these fields is Gaussian and can be performed explicitly, leading to the nonlocal effective action

$$S_\mathrm{eff} = \int \mathrm{d}^4x\,\mathcal{L}_\mathrm{eff} - \mathrm{i}\ln\Delta, \tag{2.12}$$

with $\mathcal{L}_\mathrm{eff}$ as given in (2.11). The appearance of the logarithm of the determinant

$$\Delta = \exp[\tfrac{1}{2}\,\mathrm{Tr}\ln(\mathrm{i}v\cdot D + 2m_\mathrm{Q} - \mathrm{i}\epsilon)] \tag{2.13}$$

is a quantum effect not present in the classical derivation given above. However, in this case the determinant can be regulated in a gauge invariant way, and by choosing the axial gauge $v\cdot A = 0$ it is seen that $\ln\Delta$ is just an irrelevant constant [42,43].

Because of the phase factor in (2.2), the x-dependence of the effective heavy-quark field is weak, i.e., the Fourier transform of h_v contains only the small residual momenta k. Derivatives acting on h_v produce powers of k, which is much smaller than m_Q. Hence the nonlocal effective Lagrangian (2.11) allows for a derivative expansion in $\mathrm{i}D/m_\mathrm{Q}$. Taking into account that h_v contains a P_+ projection operator, and using the identity

$$P_+\mathrm{i}\not{D}_\perp\mathrm{i}\not{D}_\perp P_+ = P_+[(\mathrm{i}D_\perp)^2 + \tfrac{1}{2}g\sigma_{\alpha\beta}G^{\alpha\beta}]P_+, \tag{2.14}$$

where

$$[\mathrm{i}D^\alpha, \mathrm{i}D^\beta] = \mathrm{i}gG^{\alpha\beta} = \mathrm{i}gT_aG_a^{\alpha\beta} \tag{2.15}$$

is the gluon field strength tensor, one finds that [32,44]

$$\mathcal{L}_\mathrm{eff} = \bar{h}_v\mathrm{i}v\cdot Dh_v + \frac{1}{2m_\mathrm{Q}}\bar{h}_v(\mathrm{i}D_\perp)^2h_v + \frac{g}{4m_\mathrm{Q}}\bar{h}_v\sigma_{\alpha\beta}G^{\alpha\beta}h_v + \mathrm{O}(1/m_\mathrm{Q}^2). \tag{2.16}$$

The leading term coincides with the effective Lagrangian derived in chapter 1. The new operators arising at order $1/m_Q$ are easiest to identify in the rest frame:

$$O_{\rm kin} = \frac{1}{2m_Q}\bar h_v (iD_\perp)^2 h_v \to -\frac{1}{2m_Q}\bar h_v (i\boldsymbol{D})^2 h_v \tag{2.17}$$

is just the gauge covariant extension of the kinetic energy arising from the off-shell residual motion of the heavy quark. The second operator is the nonabelian analog of the Pauli term, which describes the interaction of the heavy-quark spin with the gluon field:

$$O_{\rm mag} = \frac{g}{4m_Q}\bar h_v \sigma_{\alpha\beta} G^{\alpha\beta} h_v \to -\frac{g}{m_Q}\bar h_v \boldsymbol{S}\cdot \boldsymbol{B}_c h_v . \tag{2.18}$$

Here $\boldsymbol{S}$ is the spin operator defined in (1.11), and $B_c^i = -\frac{1}{2}\epsilon^{ijk}G^{jk}$ are the components of the color-magnetic gluon field. This chromo-magnetic "hyperfine" interaction is a relativistic effect, which scales like $1/m_Q$. This is the origin of the heavy-quark spin symmetry.

2.3. The $1/m_Q$ expansion

In the absence of radiative corrections, eq. (2.16) defines the operator product expansion of the Lagrangian of HQET as a series of local, higher dimension operators multiplied by powers of $1/m_Q$. The explicit expression for H_v in (2.10) can be used to derive a similar expansion for the full heavy-quark field $Q(x)$:

$$\begin{aligned} Q(x) &= \exp(-im_Q v\cdot x)\left(1 + \frac{1}{(iv\cdot D + 2m_Q - i\epsilon)} i\not{D}_\perp\right) h_v(x) \\ &= \exp(-im_Q v\cdot x)\,(1 + i\not{D}_\perp/2m_Q + \cdots) h_v(x) . \end{aligned} \tag{2.19}$$

This relation contains the recipe how to construct (at tree level) any operator in HQET that contains one or more heavy-quark fields. For instance, the vector current $V^\mu = \bar q\gamma^\mu Q$ composed of a heavy and a light quark is represented as

$$V^\mu(x) = \exp(-im_Q v\cdot x)\,\bar q(x)\gamma^\mu(1 + i\not{D}_\perp/2m_Q + \cdots) h_v(x) . \tag{2.20}$$

Matrix elements of this current can be parameterized by hadronic form factors, and the purpose of using an effective theory is to make the m_Q-dependence of these form factors explicit.

Consider, as an example, the matrix element of $V^\mu(0)$ between a heavy meson $M(v)$ and the vacuum:[8]

$$\langle 0|V^\mu|M(v)\rangle = \langle 0|\bar q\gamma^\mu h_v|M(v)\rangle + (1/2m_Q)\langle 0|\bar q\gamma^\mu i\not{D}_\perp h_v|M(v)\rangle + \cdots . \tag{2.21}$$

It would be nice if the matrix elements on the right-hand side of this equation were independent of m_Q (once the $1/m_Q$ prefactor has been removed). Then the second term would give the leading power correction to the first one. However, the equation of motion for h_v derived from (2.16) contains $1/m_Q$ corrections, too. This leads to an m_Q-dependence of any hadronic matrix element of operators

[8] These matrix elements define meson decay constants, which are hadronic parameters of primary importance (see section 4.6).

containing such fields. Another way to say this is that the eigenstates of the effective Lagrangian $\mathcal{L}_{\rm eff}$ (supplemented by the standard QCD Lagrangian for the light quarks and gluons) depend, at higher order in $1/m_Q$, on the heavy-quark mass. This is no surprise, since the effective Lagrangian is equivalent to the Lagrangian of the full theory.

It is better to organize things in a slightly different way, by working with the eigenstates of only the leading term in (2.16),

$$\mathcal{L}_{\rm HQET} \equiv \bar{h}_v \mathrm{i} v\cdot D h_v , \tag{2.22}$$

and treating the higher dimension operators perturbatively as power corrections [30,45,46]:

$$\mathcal{L}_{\rm power} = \frac{1}{2m_Q}\mathcal{L}_1 + \frac{1}{4m_Q^2}\mathcal{L}_2 + \cdots . \tag{2.23}$$

Then the equation of motion satisfied by h_v is exactly

$$\mathrm{i}v\cdot D h_v = 0, \tag{2.24}$$

and the states of the effective theory are truly independent of m_Q. However, these states are now different from the states of the full theory. For instance, instead of (2.21) we should write

$$\begin{aligned}\langle 0|V^\mu|M(v)\rangle_{\rm QCD} &\\ = \langle 0|\bar{q}\gamma^\mu h_v|M(v)\rangle_{\rm HQET} &+ \frac{1}{2m_Q}\langle 0|\bar{q}\gamma^\mu \mathrm{i}\not{D}_\perp h_v|M(v)\rangle_{\rm HQET} \\ &+ \frac{1}{2m_Q}\langle 0|\mathrm{i}\int \mathrm{d}y\, T\{\bar{q}\gamma^\mu h_v(0), \mathcal{L}_1(y)\}|M(v)\rangle_{\rm HQET} + \mathrm{O}(1/m_Q^2) .\end{aligned} \tag{2.25}$$

The matrix elements on the right-hand side are now independent of the heavy-quark mass. The mass dependence of the states $|M(v)\rangle_{\rm QCD}$ is reflected in HQET by the appearance of the third term, which arises from an insertion of the first-order power corrections in (2.23) into the leading-order matrix element of the current. This insertion can be thought of as being a correction to the wave function of the heavy meson. From now on we will omit the subscript on the states. We will instead use the symbol "$\cong$" and write

$$V^\mu(0) \cong \bar{q}\gamma^\mu h_v + (1/2m_Q)\bar{q}\gamma^\mu \mathrm{i}\not{D}_\perp h_v + \frac{1}{2m_Q} i\int \mathrm{d}y\, T\{\bar{q}\gamma^\mu h_v(0), \mathcal{L}_1(y)\} + \cdots , \tag{2.26}$$

to indicate that the operators on the two sides of this equation have to be evaluated between different states.

As long as one does not go beyond the order $1/m_Q$ in the heavy quark expansion, the equation of motion (2.24) can be used to reduce the set of operators in HQET [47–49]. In particular, one may replace $\mathrm{i}D_\perp^\mu h_v$ by $\mathrm{i}D^\mu h_v$ in the first-order power corrections to the effective Lagrangian. This yields

$$\mathcal{L}_1 = \bar{h}_v(\mathrm{i}D)^2 h_v + \tfrac{1}{2} g\bar{h}_v \sigma_{\alpha\beta} G^{\alpha\beta} h_v . \tag{2.27}$$

At higher orders in the $1/m_Q$ expansion, some care has to be used when applying the equation of motion inside time-ordered products, since in that case the derivative $v\cdot D$ can act on the time-ordering symbol, leading to local contact terms. Since in this review we are mainly interested in first-order power corrections, we shall not discuss these subtleties further. The interested reader is referred to the literature [50–52].

2.4. Hadron masses in the effective theory

As discussed above, the hadron states used in matrix elements in the effective theory are different from the physical states of the full theory. Consider, as an example, the ground-state pseudoscalar and vector mesons P and V. In HQET these states are strictly degenerate, corresponding to the leading order in the $1/m_{\rm Q}$ expansion. Due to the QCD interactions, their common mass $M_{\rm M}$ differs from the heavy-quark mass by a constant amount $\bar\Lambda$ of order $\Lambda_{\rm QCD}$:

$$\bar\Lambda = M_{\rm M} - m_{\rm Q}. \tag{2.28}$$

Note that $\bar\Lambda$ is independent of the heavy-quark flavor. Since the phase factor in (2.2) effectively removes the mass of the heavy quark from the states, it is this difference which determines the x-dependence of matrix elements in the effective theory [30]. For the purposes of this review it suffices to consider the matrix elements of either "heavy–light" operators $O_{\rm hl}$, which transform a heavy into a light quark, or "heavy–heavy" operators $O_{\rm hh}$, which transform a heavy quark into another one of, in general, different flavor and velocity. Examples of the first type are the effective current operators on the right-hand side of (2.26), whereas examples of heavy–heavy currents have been considered in section 1.4. From translational invariance it follows that in HQET

$$\begin{aligned} \langle X(p')|O_{\rm hl}(x)|M(v)\rangle &= \exp[\mathrm{i}(p' - \bar\Lambda v)\cdot x]\,\langle X(p')|O_{\rm hl}(0)|M(v)\rangle, \\ \langle M'(v')|O_{\rm hh}(x)|M(v)\rangle &= \exp[\mathrm{i}\bar\Lambda(v' - v)\cdot x]\,\langle M'(v')|O_{\rm hh}(0)|M(v)\rangle, \end{aligned} \tag{2.29}$$

where $X(p')$ is some light final state with total momentum p'. The parameter $\bar\Lambda$ determines the "effective mass" of meson states in HQET. It is natural to associate this with the mass of the light degrees of freedom. This assertion can actually be put in the form of an operator definition of $\bar\Lambda$: Let $\bar q\Gamma h_v$ be an interpolating current with the quantum numbers of the heavy meson (for instance $\Gamma = \gamma_5$ for $M = P$, and $\Gamma = \gamma^\mu$ for $M = V$). Then a covariant and gauge invariant definition of the "mass" M_ℓ of the light degrees of freedom is [53]

$$M_\ell \equiv \frac{\langle 0|\bar q\,\mathrm{i}v\cdot\overleftarrow{D}\,\Gamma h_v|M(v)\rangle}{\langle 0|\bar q\Gamma h_v|M(v)\rangle}. \tag{2.30}$$

Using the equation of motion (2.24), as well as the x-dependence of matrix elements as given above, it is immediate to find that

$$M_\ell = \frac{\mathrm{i}v\cdot\partial\langle 0|\bar q\Gamma h_v|M(v)\rangle}{\langle 0|\bar q\Gamma h_v|M(v)\rangle} = \bar\Lambda. \tag{2.31}$$

Although we have derived these results for meson states, it goes without saying that they are equally valid for heavy baryons. Of course, the value of $\bar\Lambda$ is different in the two cases. It will turn out that this mass parameter plays a crucial role in the description of the leading power corrections to heavy meson and heavy baryon decay form factors. It is the "brown muck" that the heavy quark is recoiling against, and the ratio $\varepsilon_Q = \bar{\Lambda}/2m_Q$ determines the canonical size of deviations from the infinite quark-mass limit [30,45,54].

Let us come back to the meson masses for a moment. The physical masses of pseudoscalar and vector mesons are, of course, not exactly degenerate. They differ from the common mass of the states in HQET by amounts of order $1/m_Q$. These differences can be accounted for in the effective theory. The physical masses m_P and m_V obey an expansion of the form

$$m_M - m_Q = \bar{\Lambda} + \Delta m_M^2/2m_Q + \cdots, \tag{2.32}$$

where Δm_M^2 arises from the presence of the first-order power corrections in (2.23). In the meson rest frame, one has

$$\langle M(v)|M(v)\rangle \Delta m_M^2 = -\langle M(v)| \int d^3x\, \mathcal{L}_1(x)|M(v)\rangle. \tag{2.33}$$

With the normalization of states as given in (1.25), this is equivalent to [46]

$$\Delta m_M^2 = -\tfrac{1}{2}\langle M(v)|\mathcal{L}_1(0)|M(v)\rangle \equiv -\lambda_1 - d_M\lambda_2. \tag{2.34}$$

The parameter λ_1 parameterizes the mass shift due to the kinetic operator, whereas λ_2 describes the effect of the chromo-magnetic interaction. Using the tensor methods described in section 4.1, it is easy to show that $d_P = 3$ for a pseudoscalar meson, and $d_V = -1$ for a vector meson, such that

$$\lambda_1 = -\tfrac{1}{4}\left(\Delta m_P^2 + 3\Delta m_V^2\right) \tag{2.35}$$

is the spin-averaged mass shift, while λ_2 determines the vector-pseudoscalar mass splitting:

$$m_V^2 - m_P^2 \approx \Delta m_V^2 - \Delta m_P^2 = 4\lambda_2. \tag{2.36}$$

2.5. Reparameterization invariance

The construction of HQET starts from the decomposition of the heavy quark momentum into a "large" and a "small" piece: $P_Q = m_Q v + k$. The effective theory essentially provides an expansion in k/m_Q. In order for it to be consistent, it is necessary that the residual momentum k does not scale with m_Q, i.e., stay finite and of order Λ_{QCD} as $m_Q \to \infty$. Beyond the leading order in the $1/m_Q$ expansion, there are certain ambiguities in how one defines the decomposition of the heavy-quark momentum. But the predictions of HQET must, of course, be unambiguous. This requirement leads to "hidden" symmetries, which relate the coefficients of different operators in the $1/m_Q$ expansion.

One source of ambiguity is related to the definition of the heavy quark velocity. Until now, we have always adopted the natural choice of identifying it with the velocity v of the hadron. But nothing prevents us from using some different quark velocity $w = v + q/m_Q$, where q is of order Λ_{QCD} and subject to the constraint $2v\cdot q + q^2/m_Q = 0$, so that $w^2 = v^2 = 1$. In fact,

$$P_Q^\mu = m_Q v^\mu + k^\mu = m_Q w^\mu + k'^\mu, \tag{2.37}$$

with $k' = k - q$, provides equivalent parameterizations of the heavy quark momentum [11]. The effective theories constructed by using v or w as the heavy-quark velocity must give the same results, i.e., up to terms that vanish by the equation of motion, the effective Lagrangian must be invariant under the reparameterization in (2.37). Physically, this just means that it is the heavy quark momentum which is a well defined quantity. The way one splits it into a large and a small piece is irrelevant.

Luke and Manohar have investigated the implications of this simple statement in detail [55]. Roughly speaking, they find that for the effective Lagrangian to be invariant under reparameterizations of the momentum it is necessary that the velocity and the covariant derivative always appear in the combination

$$\mathcal{V}^\mu = v^\mu + \mathrm{i}D^\mu/m_\mathrm{Q}, \tag{2.38}$$

which is the gauge covariant extension of the operator $P_\mathrm{Q}/m_\mathrm{Q}$. Things are slightly more subtle, however, because the heavy-quark spinor fields themselves transform nontrivially under a reparameterization. A consistent transformation law for the transition from v to w is

$$h_w = \exp(\mathrm{i}q\cdot x)\, \Lambda^{-1}(\hat{\mathcal{V}}, w)\, \Lambda(\hat{\mathcal{V}}, v)\, h_v, \tag{2.39}$$

where $\Lambda(\hat{\mathcal{V}}, v)$ is a spinor representation of the Lorentz boost which transforms v into the invariant "velocity" $\hat{\mathcal{V}} = \mathcal{V}/|\mathcal{V}|$.[9] The phase factor comes from the definition of the heavy-quark fields in HQET. Using that

$$\exp(\mathrm{i}q\cdot x)\, \mathcal{V}^\mu = \left(v^\mu + \frac{\mathrm{i}D^\mu}{m_\mathrm{Q}} + \frac{q^\mu}{m_\mathrm{Q}}\right) \exp(\mathrm{i}q\cdot x) = \mathcal{W}^\mu \exp(\mathrm{i}q\cdot x), \tag{2.40}$$

one finds that the field $\tilde{h}_v \equiv \Lambda(\hat{\mathcal{V}}, v)\, h_v$ transforms homogeneously under reparameterizations [55]:

$$\tilde{h}_w = \exp(\mathrm{i}q\cdot x)\, \tilde{h}_v, \qquad \mathcal{W}^\mu \tilde{h}_w = \exp(\mathrm{i}q\cdot x)\, \mathcal{V}^\mu \tilde{h}_v. \tag{2.41}$$

The rest is straightforward. The most general reparameterization invariant effective Lagrangian must be build from $\mathcal{V}$ and $\tilde{h}_v$. The same is true for the operator product expansion of any external current in the effective theory. Using the explicit form of the Lorentz boost $\Lambda(\hat{\mathcal{V}}, v)$, one can show that to order $1/m_\mathrm{Q}$

$$\tilde{h}_v = (1 + \mathrm{i}\not{D}/2m_\mathrm{Q} + \cdots)\, h_v = \tfrac{1}{2}(1 + \hat{\not{\mathcal{V}}})\, h_v + \cdots. \tag{2.42}$$

This is nothing but the projection operator corresponding to the "true" velocity $\hat{\mathcal{V}}$.

Reparameterization invariance is a powerful concept in that it relates the coefficients of operators appearing at different order in the $1/m_\mathrm{Q}$ expansion. For instance, the leading and subleading terms in the effective Lagrangian (2.16) can be cast into the explicitly invariant form

$$\mathcal{L}_\mathrm{eff} = \tfrac{1}{2} m_\mathrm{Q} \bar{\tilde{h}}_v \{ (\mathcal{V}^2 - 1) - \tfrac{1}{2}\mathrm{i}\sigma_{\alpha\beta}[\mathcal{V}^\alpha, \mathcal{V}^\beta] + \cdots \} \tilde{h}_v, \tag{2.43}$$

where the ellipsis represents a term that vanishes by the equation of motion. Whereas the chromo-magnetic operator is reparameterization invariant by itself, the kinetic operator must be combined with the leading term in the effective Lagrangian into an invariant combination. For this to be still the case in the presence of quantum corrections, it is necessary that these two operators acquire the same renormalization factor [55].

[9] A proper definition of $\hat{\mathcal{V}}$ requires an operator ordering prescription. However, since $\mathcal{V}^2 = 1 + \mathrm{O}(1/m_\mathrm{Q}^2)$ by the equation of motion, this subtlety is irrelevant at order $1/m_\mathrm{Q}$.

Similarly statements apply for the expansion of currents in HQET. Consider again the heavy-light current from (2.20). At order $1/m_Q$, it can be written in the reparameterization invariant form

$$V^\mu(0) = \bar{q}\gamma^\mu \tilde{h}_v. \qquad (2.44)$$

As before, this form must be preserved by renormalization. This will be discussed in detail in chapter 3.

2.6. The residual mass term

Another ambiguity in the construction of HQET is related to the choice of the heavy-quark mass m_Q. Since quarks cannot appear as physical states, but are always confined into hadrons, there is no natural way to define their mass on-shell. Although one can ignore confinement effects and define an "on-shell" mass as the pole of the renormalized propagator to a given order in perturbation theory, such a concept becomes meaningless beyond the perturbative expansion. Alternative definitions of quark masses use unphysical subtraction schemes such as MS or $\overline{\rm MS}$. One could also work with some ad hoc definition, for instance taking m_Q to be the mass of the lightest hadron that contains the heavy quark, or this mass minus some fixed amount. This freedom poses the question which of the parameters and predictions of HQET are unambiguous. Consider, as an example, the mass parameter $\bar{\Lambda}$. Its definition in (2.28) clearly depends on the choice of m_Q. On the other hand, the quantity M_ℓ defined in (2.30) has no obvious relation to the heavy quark mass. How then can these two parameters be equal?

To resolve this puzzle, Falk, Neubert and Luke introduced the notion of a residual mass δm of the heavy quark in the effective theory [53]. The idea is similar to that of reparameterization invariance. The decomposition $P_Q = m_Q v + k$ can be rearranged to read

$$P_Q^\mu = (m_Q - \delta m)v^\mu + (k^\mu + \delta m\, v^\mu) \equiv m_Q' v^\mu + k'^\mu\,, \qquad (2.45)$$

which is a legitimate decomposition into a large and a small momentum as long as δm is of order $\Lambda_{\rm QCD}$. The effective theory constructed by using m_Q' in the field redefinition (2.2) must lead to the same results as the effective theory obtained by using m_Q. Let us denote the heavy-quark fields in the two theories with and without a residual mass term by h_v and $h_v^{\delta m}$, respectively. They simply differ by a phase, so that

$$h_v = \exp(i\delta m\, v\cdot x)\, h_v^{\delta m}, \qquad {\rm i}D^\mu h_v = \exp({\rm i}\delta m\, v\cdot x)\, ({\rm i}D^\mu - \delta m\, v^\mu)\, h_v^{\delta m}. \qquad (2.46)$$

Consider now changes in the definition of the heavy-quark mass m_Q', corresponding to changes in δm. From (2.45) it follows that

$$\partial m_Q'/\partial\delta m = -1, \qquad \partial k'^\mu/\partial\delta m = v^\mu. \qquad (2.47)$$

Substituting the covariant derivative for k', we conclude that the combinations

$$m_Q^* \equiv m_Q' + \delta m, \qquad {\rm i}\mathcal{D}^\mu \equiv {\rm i}D^\mu - \delta m\, v^\mu \qquad (2.48)$$

are invariant under redefinitions of δm. Notice that the generalized derivative ${\rm i}\mathcal{D}$ is precisely what appears in (2.46). The argument is now similar to that of the previous section. For nonvanishing δm, the parameters m_Q' and δm and the covariant derivative must always appear in the combinations m_Q^*

and $i\mathcal{D}$. This ensures that matrix elements calculated in the effective theory are independent of the choice of δm, i.e., of the definition of the heavy-quark mass. One can combine this requirement with reparameterization invariance by replacing $\mathcal{V}$ in the expressions of the previous section by

$$\mathcal{V}^\mu = v^\mu + i\mathcal{D}^\mu/m_Q^* . \tag{2.49}$$

For $\delta m \neq 0$, the effective Lagrangian of HQET must be generalized to [53]

$$\mathcal{L}_{\rm HQET} = \bar{h}_v^{\delta m} iv\cdot\mathcal{D}h_v^{\delta m} = \bar{h}_v^{\delta m} iv\cdot D h_v^{\delta m} - \delta m\,\bar{h}_v^{\delta m} h_v^{\delta m} . \tag{2.50}$$

Due to an incomplete cancellation of the full theory mass by the field redefinition, the heavy quark has a residual mass of order $\Lambda_{\rm QCD}$ in the effective theory. Such a term is not forbidden by heavy-quark symmetry. Since the predictions of the effective theory are independent of the value of δm, however, it is a legitimate choice to set $\delta m = 0$ in (2.50). This choice has been adopted implicitly in basically all works on HQET in continuum field theory. [10] In this case the generalized derivative reduces to the usual covariant derivative, and the heavy-quark mass m_Q coincides with the invariant mass m_Q^*.

The above discussion shows that in HQET there emerges a natural way to define a heavy-quark mass [56]: If the effective Lagrangian is constructed using the invariant mass m_Q^*, it does not contain a residual mass term. In a way, this provides a nonperturbative generalization of the concept of a pole mass. In fact, in perturbation theory m_Q^* is the pole mass, since in the absence of a residual mass the renormalized heavy-quark propagator has a pole at $k = 0$, corresponding to $P_Q^2 = m_Q^{*2}$.

In a theory with nonvanishing δm, the equation of motion is $iv\cdot Dh_v^{\delta m} = \delta m\, h_v^{\delta m}$. Hence, there appears an additional contribution $-\delta m$ on the right-hand side of (2.31), such that

$$M_\ell = (M_M - m_Q') - \delta m = M_M - m_Q^* \equiv \bar{\Lambda} . \tag{2.51}$$

This resolves the puzzle mentioned above: The quantity M_ℓ defined in (2.30) is independent of the choice of the heavy quark mass m_Q'; it only depends on the invariant combination m_Q^*. When defined by the above equation, $\bar{\Lambda}$ becomes an invariant parameter, too. But one can also consider (2.51) as a definition of m_Q^*, since the quantity M_ℓ is independently defined in terms of the ratio of matrix elements in (2.30). This definition becomes useful for very heavy quarks, since the difference between the physical meson mass m_M and the effective theory mass M_M vanishes as $1/m_Q$.

Since the discussion in this section is subtle, let us restate it this way: In perturbation theory, one can define m_Q as the position of the pole in the renormalized heavy-quark propagator. Alternatively, one may define a heavy-quark mass m_Q^* from the phase of the field h_v in HQET, by requiring that $iv\cdot Dh_v = 0$. In perturbation theory, these two definitions are equivalent. However, the second one is more general and does not rely on a perturbative expansion. With this definition, $\bar{\Lambda} = m_M - m_Q^*$ becomes a meaningful low energy parameter. When one chooses another definition, one must calculate the residual mass δm and include the perturbations induced by $\delta m \neq 0$. These are collected in a compact way in this section.

From now on we shall follow the standard procedure and set $\delta m = 0$ and $m_Q \equiv m_Q^*$, keeping in mind that by means of (2.51) the parameters m_Q and $\bar{\Lambda}$ are unambiguously defined.

[10] We note, however, that if the theory is regulated by a dimensionful cutoff, such as in lattice gauge theory, a residual mass term will be induced by radiative corrections even if it is not present at tree level.

3. Renormalization

3.1. Matching

In the previous chapter we have discussed the first two steps in the construction of HQET. Integrating out the small components in the heavy quark spinor fields, a nonlocal effective action was derived which allowed for a naive expansion in powers of $1/m_Q$. A similar expansion could be written down for any external current. The effective Lagrangian and the effective currents derived that way correctly reproduce the long-distance physics of the full theory. They cannot describe the short-distance physics correctly, however. The reason is obvious: The heavy quark participates in strong interactions through its coupling to gluons. These gluons can be soft or hard, i.e., their virtual momenta can be small, of order of the confinement scale, or large, of order of the heavy-quark mass. But hard gluons can resolve the nonlocality of the propagator of the small component fields H_v. Their effects are not taken into account in the naive operator product expansion, which was used in the derivation of the effective Lagrangian in (2.16) and the effective vector current in (2.26). So far, the effective theory provides an appropriate description only at scales $\mu \ll m_Q$.

In this chapter we will discuss the systematic treatment of short-distance corrections. A new feature of such corrections is that through the running coupling constant they induce a logarithmic dependence on the heavy-quark mass [15], whereas so far the dependence on m_Q was always powerlike. The important observation is that $\alpha_s(m_Q)$ is small, so that these effects can be calculated in perturbation theory. Consider, as an example, matrix elements of the vector current $V^\mu = \bar{Q}\gamma^\mu Q$. In QCD this current is conserved and needs no renormalization [57]. Its matrix elements are free of ultraviolet divergences. Still, these matrix elements can have logarithmic dependence on m_Q from the exchange of hard gluons with virtual momenta comparable to the heavy-quark mass. If one goes over to the effective theory by taking the limit $m_Q \to \infty$, these logarithms diverge. Consequently, the vector current in the effective theory does require a renormalization [31]. Its matrix elements depend on an arbitrary renormalization scale μ, which separates the regions of short- and long-distance physics. If μ is chosen such that $\Lambda_{\rm QCD} \ll \mu \ll m_Q$, the effective coupling constant in the region between μ and m_Q is small, and perturbation theory can be used to compute the short-distance corrections. These corrections have to be added to the matrix elements of the effective theory, which only contain the long-distance physics below the scale μ. Schematically, then, the relation between matrix elements in the full and in the effective theory is

$$\langle V^\mu \rangle_{\rm QCD} = C_0(\mu)\, \langle V_0(\mu)\rangle_{\rm HQET} + (C_1(\mu)/2m_Q)\, \langle V_1(\mu)\rangle_{\rm HQET} + \cdots . \tag{3.1}$$

The short-distance, or Wilson coefficients $C_i(\mu)$ are defined by this relation. Order by order in perturbation theory they can be computed from a comparison of the matrix elements in both theories. Since the effective theory is constructed to reproduce correctly the low energy behavior of the full theory, this "matching" procedure is independent of any long-distance physics, such as infrared singularities, nonperturbative effects, the nature of the external states used in the matrix elements, physical cuts, etc. Only at high energies do the two theories differ, and these differences are corrected for by the short-distance coefficients.

The calculation of the coefficient functions in perturbation theory uses the powerful methods of the renormalization group. It is in principle straightforward, yet in practice rather tedious. Much of the recent work on heavy-quark symmetry has been devoted to this subject. We will therefore discuss it in detail, starting with some general remarks on renormalization, composite operators, and the renormalization group.

3.2. *The renormalized Lagrangian*

In quantum field theory, the parameters and fields of the Lagrangian have no direct physical significance. Green functions of these "bare" quantities diverge at higher orders in perturbation theory. It is necessary to renormalize the "bare" parameters and fields before they can be related to observable quantities. In an intermediate step the theory has to be regularized. The most convenient regularization scheme in QCD is dimensional regularization [58–60], in which the dimension of space-time is analytically continued to $D = 4 - 2\epsilon$, with ϵ being infinitesimal. Loop integrals which are logarithmically divergent in four dimensions become finite for $\epsilon > 0$. The Green functions of bare fields in the regularized theory are finite, but they diverge in the limit $\epsilon \to 0$. Throughout this review we shall use the so-called naive dimensional regularization scheme with anticommuting γ_5 [61]. This is convenient since it will allow us to treat vector and axial vector currents on the same footing. From the fact that the action $S = \int \mathrm{d}^D x\, \mathcal{L}(x)$ is dimensionless, one can derive the mass dimensions of the fields and parameters of the theory. For QCD, one finds in particular that the "bare" gauge coupling g^{bare} is no longer dimensionless if $D \neq 4$:

$$\dim[\, g^{\text{bare}}\,] = (4 - D)/2 = \epsilon\,. \tag{3.2}$$

In a renormalizable theory it is possible to rewrite the Lagrangian in terms of renormalized quantities in such a way that Green's functions of the renormalized fields remain finite as $\epsilon \to 0$. For QCD one defines [11]

$$\begin{aligned} &Q^{\text{bare}} = Z_Q^{1/2}\, Q^{\text{ren}}, \qquad q^{\text{bare}} = Z_q^{1/2}\, q^{\text{ren}}, \qquad A^{\text{bare}} = Z_A^{1/2} A^{\text{ren}}, \\ &g^{\text{bare}} = \mu^\epsilon Z_g\, g^{\text{ren}}, \qquad m_Q^{\text{bare}} = Z_m\, m_Q^{\text{ren}}, \end{aligned} \tag{3.3}$$

where Q and q refer to heavy and light quarks, respectively, and μ is an arbitrary mass scale introduced to render the renormalized coupling constant dimensionless. Similarly, in HQET one defines a renormalized heavy-quark field by

$$h_v^{\text{bare}} = Z_h^{1/2}\, h_v^{\text{ren}}\,. \tag{3.4}$$

From now on the superscripts "ren" will be omitted.

Both for many formal considerations and for actual calculations it is advantageous to use the background field method, which is a technique for quantizing gauge theories preserving explicit gauge invariance [62–65]. In this method the renormalization of the coupling constant is related to that of the gauge fields by

$$Z_g = Z_A^{-1/2}\,, \tag{3.5}$$

so that the combination $g^{\text{bare}} A^{\text{bare}} = \mu^\epsilon g A$ is not renormalized. This guarantees that the form of the covariant derivative and the gluon field strength tensor is preserved during renormalization, i.e.

$$\mathrm{i}D^\mu = \mathrm{i}\partial^\mu + \mu^\epsilon g\, T_a A_a^\mu, \qquad G_a^{\mu\nu} = \partial^\mu A_a^\nu - \partial^\nu A_a^\mu + \mu^\epsilon g\, f_{abc}\, A_b^\mu A_c^\nu, \tag{3.6}$$

[11] For simplicity, we do not consider the renormalization of the gauge parameter and of the ghost fields.

where $G_a^{\mu\nu} = Z_A^{-1/2}(G_a^{\mu\nu})^{\rm bare}$. The only modification of the Feynman rules is the replacement of $g^{\rm bare}$ by $\mu^\epsilon g$. The background field method also greatly simplifies the renormalization of composite operators (see section 3.3). It will be adopted throughout this work.

Written in terms of the renormalized quantities, the QCD Lagrangian for a heavy and a light quark interacting with gauge fields becomes

$$\begin{aligned}\mathcal{L}_{\rm QCD} &= -\tfrac{1}{4} Z_A G_{\mu\nu,a} G_a^{\mu\nu} + Z_Q \bar{Q}\,(i\not{D} - Z_m m_Q)\,Q + Z_q\,\bar{q}\,i\not{D} q \\ &= -\tfrac{1}{4} G_{\mu\nu,a} G_a^{\mu\nu} + \bar{Q}\,(i\not{D} - m_Q)\,Q + \bar{q}\,i\not{D} q + \text{counterterms}. \end{aligned} \tag{3.7}$$

The generalization to more flavors is obvious. Similarly, the effective Lagrangian of HQET becomes

$$\mathcal{L}_{\rm HQET} = Z_h \bar{h}_v iv\cdot D h_v = \bar{h}_v\, iv\cdot D\, h_v + \text{counterterms}. \tag{3.8}$$

If the effective theory is regulated by a dimensionful regulator (as in lattice gauge theory, for instance), one should in principle also allow for a mass term $\delta m\,\bar{h}_v h_v$, even if no mass term is present in the bare Lagrangian. As discussed in section 2.6, however, such a residual mass can be removed by a field redefinition [53]. The renormalized Lagrangian as written above is thus a perfectly legitimate choice, but it assumes a particular definition of the heavy-quark mass.

The renormalization factors Z_i have to be constructed in such a way that Green's functions of renormalized fields stay finite in the limit $\epsilon \to 0$. But this condition does not determine the Z_i completely. To define them in an unambiguous way, one has to choose a subtraction scheme. The regularization method together with a subtraction prescription define the renormalization scheme. In QCD one faces the problem that quarks and gluons cannot be observed as free particles due to confinement. Unlike in QED, there is no direct way to relate their properties (such as the quark masses or the gauge coupling) to observable quantities. In the absence of a natural physical definition of the renormalized parameters, one usually chooses a subtraction prescription which is most convenient. A widely used scheme is the so-called minimal subtraction (MS), in which the Z_i factors are defined to subtract the $1/\epsilon$ poles in the bare quantities [59]. In this scheme, and using the background field method, the renormalization of the gauge fields to two-loop order is accomplished by the gauge independent factor

$$Z_A = 1 + \frac{\beta_0}{\epsilon}\frac{\alpha_s}{4\pi} + \frac{\beta_1}{2\epsilon}\left(\frac{\alpha_s}{4\pi}\right)^2, \tag{3.9}$$

where [12] [1,2,66–68]

$$\beta_0 = 11 - \tfrac{2}{3} n_f, \qquad \beta_1 = 102 - \tfrac{38}{3} n_f, \tag{3.10}$$

and n_f is the number of quark flavors. Because of (3.5) this also determines the renormalization of the coupling constant.

Of particular importance is the scale dependence of the renormalized coupling constant. From (3.3) one finds that

$$\mu {\rm d}g/{\rm d}\mu = -\epsilon\, g - g\, Z_g^{-1}\,\mu {\rm d}Z_g/{\rm d}\mu \equiv \beta(g,\epsilon). \tag{3.11}$$

[12] Throughout this work we evaluate the QCD coefficients for $N_c = 3$ colors, and only display the dependence on the number of flavors explicitly.

In the limit $\epsilon \to 0$ this reduces to the usual QCD β-function $\beta(g)$. In the MS scheme a very simple relation between $\beta(g)$ and Z_g can be derived [69]. Since Z_g depends on μ only implicitly through the μ-dependence of g, one can use the chain rule to obtain

$$\beta(g,\epsilon) = -\epsilon\, g - g\,\beta(g,\epsilon)\,\partial \ln Z_g/\partial g. \tag{3.12}$$

Now Z_g is a sum of poles,

$$Z_g = 1 + \sum_{k=1}^{\infty} \frac{1}{\epsilon^k}\, Z_{g,k}(g)\,, \tag{3.13}$$

with coefficients $Z_{g,k}(g)$ that have a perturbative expansion in the renormalized coupling g. On the other hand, since QCD is a renormalizable theory the β-function must be finite in the limit $\epsilon \to 0$, hence $\beta(g,\epsilon) = \beta(g) + \sum_{k=1}^{\infty} \epsilon^k \beta^{(k)}(g)$. But from (3.12) it follows that for $k > 1$ the coefficients vanish, and

$$\beta(g,\epsilon) = \beta(g) - \epsilon\, g\,. \tag{3.14}$$

Furthermore, the coefficients $Z_{g,k}$ must conspire such that all poles cancel in (3.12). From this constraint one can derive important relations between the $Z_{g,k}$ [69]. The only finite contribution to $\beta(g)$ arises when the ϵg term in (3.14) hits the $1/\epsilon$ pole in Z_g. This yields the useful relation

$$\beta(g) = g^2\, \partial Z_{g,1}/\partial g, \tag{3.15}$$

which is true to all orders in perturbation theory. Using the background field relation (3.5) and the two-loop result for Z_A in (3.9), one sees that the first two coefficients in the perturbative expansion of the β-function,

$$\beta(g) = -\beta_0\, \frac{g^3}{16\pi^2} - \beta_1\, \frac{g^5}{(16\pi^2)^2} + \cdots, \tag{3.16}$$

are precisely the ones given in (3.10).

Another useful subtraction scheme is modified minimal subtraction ($\overline{\rm MS}$) [73], in which one takes into account that in dimensional regularization the $1/\epsilon$ poles always appear in the combination

$$1/\hat\epsilon \equiv 1/\epsilon - \gamma_{\rm E} + \ln 4\pi, \tag{3.17}$$

and defines the Z_i to subtract the $1/\hat\epsilon$ poles from the divergent quantities. This scheme is particularly useful for one-loop calculations, for which it is trivially related to the MS scheme. The coefficients β_0 and β_1 are the same in both schemes. They are actually the same in all renormalization schemes in which the renormalized coupling constant is gauge independent [70–72].

Integrating (3.11) in the limit $\epsilon \to 0$ one obtains the running coupling constant to two-loop accuracy:

$$\alpha_s(\mu) = \frac{g^2}{4\pi^2} = \frac{4\pi}{\beta_0\, \ln(\mu^2/\Lambda^2)} \left(1 - \frac{\beta_1}{\beta_0^2}\, \frac{\ln\ln(\mu^2/\Lambda^2)}{\ln(\mu^2/\Lambda^2)} \right). \tag{3.18}$$

The QCD scale parameter Λ is scheme dependent. It has become standard to evaluate the running coupling constant in the $\overline{\rm MS}$ scheme, $\alpha_s(\mu) \equiv \alpha_s^{\overline{\rm MS}}(\mu)$, and to denote the corresponding value of the scale parameter by $\Lambda_{\overline{\rm MS}}$. In other schemes this parameter is different, for instance [73]

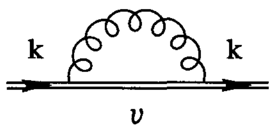


Fig. 3.1. Self-energy $-i\Sigma(v\cdot k)$ of a heavy quark in HQET. The velocity v is conserved by the strong interactions.

$$\Lambda_{\rm MS} = e^{\delta}\,\Lambda_{\overline{\rm MS}}; \qquad \delta = \tfrac{1}{2}\,(\gamma_E - \ln 4\pi)\,. \tag{3.19}$$

This scheme dependence becomes important at next-to-leading order in perturbation theory:

$$\alpha_s^{\rm MS}(\mu) = \alpha_s(\mu)\,[\,1 + \beta_0\,\delta\,\alpha_s(\mu)/2\pi + \cdots]\,, \tag{3.20}$$

i.e., only by going beyond the leading order is one sensitive to the precise definition of α_s.

The above discussion of the effective coupling constant ignores quark mass effects. It assumes the idealized situation where quarks with masses larger than μ decouple [74] and are neglected completely. Then n_f is the number of light quarks with mass below μ, and the coefficients of the β-function change by discrete amounts as one crosses quark thresholds. The QCD scale parameter $\Lambda_{\overline{\rm MS}}$ is adjusted accordingly, such that $\alpha_s(\mu)$ becomes a continuous function of μ.

Let us now discuss the renormalization of the quark fields. In the MS scheme the wave function renormalization factors are gauge dependent. In Feynman gauge, the expressions arising at one-loop order are

$$Z_Q = Z_q = 1 - \alpha_s/3\pi\epsilon\,, \qquad Z_h = 1 + 2\alpha_s/3\pi\epsilon\,. \tag{3.21}$$

In the full theory, heavy and light quark fields are renormalized in the same way, since dimensional regularization with minimal subtraction provides a mass-independent renormalization scheme. In the effective theory, the calculation of Z_h from the self-energy diagram shown in Fig. 3.1 was performed by Politzer and Wise [31]. The wave function renormalization factors are also known at two-loop order. The QCD result was derived in Ref. [75], and the calculation in the effective theory has been performed in Refs. [76–78].

Although the wave function renormalization factors given in (3.21) render the quark self-energies finite, the MS scheme is not the most convenient scheme for matching calculations. The reason is the following: As mentioned in section 3.1, the short-distance coefficients are independent of the external states. Their calculation can be performed using whatever states are most convenient, for instance on-shell quarks. It is then better to perform an on-shell wave function renormalization, such that the renormalized propagator has a pole with unit residue on the mass shell. This prescription absorbs entirely the contribution of self-energy diagrams into the field renormalization. It is then sufficient to consider only one-particle irreducible diagrams in the matching calculation. In any other scheme, there would be finite self-energy contributions after renormalization, which could contribute to the matching.

Since on-shell quark states are unphysical, the corresponding wave function renormalization factors are infrared divergent. As long as no nonabelian vertices are encountered, the infrared singularities can be regulated by associating a fictitious mass λ with the virtual gluons, taking the limit $\lambda \to 0$ at the end of the calculation. The on-shell renormalization of light and heavy quarks in QCD is completely analogous to that of the electron in QED. In the effective theory, Z_h is related to the heavy-quark self-energy by:

$$Z_h^{-1} = 1 - \frac{\partial \Sigma(v \cdot k)}{\partial (v \cdot k)}\bigg|_{v\cdot k=0}, \tag{3.22}$$

where k is the residual off-shell momentum. At one-loop order, the on-shell wave function renormalization factors in Feynman gauge are [9,79]

$$Z_Q = 1 - \frac{\alpha_s}{3\pi}\left(\frac{1}{\hat\epsilon} - \ln\frac{m_Q^2}{\mu^2} - 2\ln\frac{m_Q^2}{\lambda^2} + 4\right),$$

$$Z_q = 1 - \frac{\alpha_s}{3\pi}\left(\frac{1}{\hat\epsilon} + \ln\frac{\mu^2}{\lambda^2} - \frac{1}{2}\right), \qquad Z_h = 1 + \frac{2\alpha_s}{3\pi}\left(\frac{1}{\hat\epsilon} + \ln\frac{\mu^2}{\lambda^2}\right), \tag{3.23}$$

with $1/\hat\epsilon$ as defined in (3.17). The pole terms are, of course, the same as in the MS scheme. Notice that the μ-dependent term in Z_h can be obtained by setting $m_Q = \mu$ in Z_Q, and that the infrared singular pieces in Z_h and Z_Q are the same. What is relevant to matching calculations is the difference $Z_Q - Z_h$, in which the regulator λ disappears.

3.3. Composite operators, short-distance expansion, and the renormalization group

Similar to the fields and coupling constants, any composite operator built from quark and gluon fields may require a renormalization beyond that of its component fields. Such operators can be divided into three classes: Gauge invariant operators that do not vanish by the equations of motion (class I), gauge invariant operators that vanish by the equations of motion (class II), and operators which are not gauge invariant (class III). In general, operators with the same dimension and quantum numbers mix under renormalization. The advantage of the background field technique is, however, that a class I operator cannot mix with class III operators, so that only gauge invariant operators need to be considered [80]. Furthermore, class II operators are irrelevant since their matrix elements vanish by the equations of motion. It it thus sufficient to consider class I operators only.

For a set $\{O_i\}$ of n class I operators that mix under renormalization, one defines an $n \times n$ matrix of renormalization factors Z_{ij} by (summation over j is understood)

$$O_i^{\text{bare}} = Z_{ij}\, O_j\,, \tag{3.24}$$

such that the matrix elements of the renormalized operators O_j remain finite as $\epsilon \to 0$. Since the bare operators are composed of bare fields, this definition of Z_{ij} is independent of the renormalization of the component fields. In contrast to the bare operators, the renormalized operators depend on the subtraction scale via the μ-dependence of Z_{ij}:

$$\mu \mathrm{d}O_i/\mathrm{d}\mu = (\mu\, \mathrm{d}Z_{ij}^{-1}/\mathrm{d}\mu)\, O_j^{\text{bare}} = -\gamma_{ik}\, O_k, \tag{3.25}$$

where

$$\gamma_{ik} = Z_{ij}^{-1}\, \mu\, \mathrm{d}Z_{jk}/\mathrm{d}\mu \tag{3.26}$$

are called the anomalous dimensions. It is convenient to introduce a compact matrix notation in which $\boldsymbol{O}$ is the vector of renormalized operators, $\hat{Z}$ is the matrix of renormalization factors, and $\hat{\gamma}$ denotes the anomalous dimension matrix. Then

$$\hat{\gamma} = \hat{Z}^{-1}\mu\,\mathrm{d}\hat{Z}/\mathrm{d}\mu, \tag{3.27}$$

and the running of the renormalized operators is controlled by the renormalization group equation

$$(\mu\,\mathrm{d}/\mathrm{d}\mu + \hat{\gamma})\boldsymbol{O} = 0. \tag{3.28}$$

In the MS scheme the matrix $\hat{Z}$ obeys an expansion

$$\hat{Z} = \hat{1} + \sum_{k=1}^{\infty} \frac{1}{\epsilon^k}\,\hat{Z}_k(g), \tag{3.29}$$

and by requiring that the anomalous dimension matrix be finite as $\epsilon \to 0$, one finds in analogy to (3.15) that $\hat{\gamma}$ can be computed in terms of the coefficient of the $1/\epsilon$ pole:

$$\hat{\gamma}(g) = -g\,\partial\hat{Z}_1/\partial g. \tag{3.30}$$

The same relation holds in the $\overline{\mathrm{MS}}$ scheme.

In HQET, one is mainly interested in composite operators appearing in the operator product expansion of a weak current J in terms of local operators of the effective theory. In the full theory, neither vector nor axial vector currents need to be renormalized. But the effective current operators in HQET do require renormalization. Let us for simplicity assume that there is a single large mass scale m in the problem. Then at leading order in $1/m$ the short-distance expansion of the QCD operator J in the effective theory reads

$$J_{\mathrm{QCD}} \cong C_i(\mu)\,J_i(\mu) + \mathrm{O}(1/m) = C_i(\mu)\,Z_{ij}^{-1}(\mu)\,J_j^{\mathrm{bare}} + \mathrm{O}(1/m)\,, \tag{3.31}$$

where we have indicated the μ-dependence of the renormalized operators. A similar expression appears at each successive order in $1/m$. This gives the correct generalization of (3.1) in the case of operator mixing. In general, a complete set of operators with the same quantum numbers appears on the right-hand side. Order by order in $1/m$, the μ-dependence of the Wilson coefficients has to cancel against that of the renormalized operators.

At the high energy scale $\mu = m$, the coefficients have a perturbative expansion in $\alpha_s(m)$:

$$C_i(m) = C_{i,0} + C_{i,1}\,\alpha_s(m)/4\pi + \cdots. \tag{3.32}$$

For scales below m, expressions for $C_i(\mu)$ to order α_s can be derived from a comparison of one-loop matrix elements of the operators in (3.31) in the full and in the effective theory. For $\mu \ll m$ such a treatment is unsatisfactory, however. The scale in the running coupling constant cannot be fixed from a one-loop calculation, but $\alpha_s(\mu)$ and $\alpha_s(m)$ can differ substantially. One would like to resolve the scale-ambiguity problem by going beyond the leading order in perturbation theory. Furthermore, the perturbative expansion of the Wilson coefficients is known to contain large logarithms of the type $[\beta_0\,\alpha_s \ln(m/\mu)]^n$, which one should sum to all orders. Both goals can be achieved by using the renormalization group [73,81]. From (3.28) and the fact that the product $C_i(\mu)\,J_i(\mu)$ must be μ-

independent, one can derive the renormalization group equation satisfied by the coefficient functions. It reads

$$(\mu\,\mathrm{d}/\mathrm{d}\mu - \hat{\gamma}^{\mathrm{t}})\boldsymbol{C}(\mu) = 0, \tag{3.33}$$

where $\hat{\gamma}^{\mathrm{t}}$ denotes the transposed anomalous dimension matrix, and we have collected the coefficients into a vector $\boldsymbol{C}(\mu)$. In general, the Wilson coefficients can depend on μ both explicitly or implicitly through the running coupling constant $g(\mu)$. Using [13]

$$\mu\,\mathrm{d}/\mathrm{d}\mu = \mu\partial/\partial\mu + \beta(g)\,\partial/\partial g(\mu), \tag{3.34}$$

it is straightforward to obtain a formal solution of the renormalization group equation. It reads

$$\boldsymbol{C}(\mu) = \hat{U}(\mu, m)\,\boldsymbol{C}(m), \tag{3.35}$$

with the evolution matrix [73,82]

$$\hat{U}(\mu, m) = T_g \exp\left(\int\limits_{g(m)}^{g(\mu)} \mathrm{d}g'\,\frac{\hat{\gamma}^{\mathrm{t}}(g')}{\beta(g')} \right). \tag{3.36}$$

Here T_g means an ordering in the coupling constant such that the couplings increase from right to left (for $\mu < m$). This is necessary since, in general, the anomalous dimension matrices at different values of g do not commute: $[\hat{\gamma}(g_1), \hat{\gamma}(g_2)] \neq 0$. Eq. (3.36) can be solved perturbatively by expanding the β-function [cf. (3.16)] and the anomalous dimension matrix in powers of the renormalized coupling constant

$$\hat{\gamma}(g) = \hat{\gamma}_0\, g^2/16\pi^2 + \hat{\gamma}_1 (g^2/16\pi^2)^2 + \cdots. \tag{3.37}$$

Consider first the important case of only a single coefficient function, or equivalently, when there is no operator mixing. Then the matrix $\hat{\gamma}$ reduces to a number, and the evolution is described by a function $U(\mu, m)$, for which the perturbative solution of (3.36) yields

$$U(\mu, m) = \left(\frac{\alpha_s(m)}{\alpha_s(\mu)}\right)^a \left(1 + \frac{\alpha_s(m) - \alpha_s(\mu)}{4\pi} S + \cdots\right), \tag{3.38}$$

where

$$a = \gamma_0/2\beta_0, \qquad S = \gamma_1/2\beta_0 - \gamma_0\beta_1/2\beta_0^2. \tag{3.39}$$

The terms shown explicitly in (3.38) correspond to the so-called next-to-leading order in renormalization group improved perturbation theory. In this approximation the running coupling constant has two-loop accuracy, and the leading and subleading logarithms

$$[\alpha_s \ln(m/\mu)]^n, \qquad \alpha_s[\alpha_s \ln(m/\mu)]^n \tag{3.40}$$

are summed correctly to all orders. To achieve this, however, it is necessary to calculate the two-loop coefficient γ_1 of the anomalous dimension. When γ_1 is not known, it is only possible to evaluate the evolution function in the so-called leading logarithmic approximation, in which

$$U_0(\mu, m) = [\alpha_s(m)/\alpha_s(\mu)]^a. \tag{3.41}$$

[13] Recall that in HQET the heavy-quark masses are fixed, μ-independent parameters.

This still sums the leading logarithms to all orders, but does not contain the nonlogarithmic terms of order α_s.

When (3.38) is combined with the initial condition (3.32) for the Wilson coefficients at the high energy scale $\mu = m$, one obtains the next-to-leading order result

$$C(\mu) = \left(1 - \frac{\alpha_s(\mu)}{4\pi} S\right) U_0(\mu, m) \left(C_0 + \frac{\alpha_s(m)}{4\pi} (C_0 S + C_1)\right). \tag{3.42}$$

In this equation the terms involving the high energy coupling constant $\alpha_s(m)$ are scheme independent [73,81]. $U_0(\mu, m)$ involves only the one-loop coefficients γ_0 and β_0 and is scheme independent by itself. For the coefficient $C_0 S + C_1$ in parentheses things are more subtle, however. The one-loop matching coefficient C_1, the two-loop anomalous dimension γ_1, and the QCD scale parameter Λ in the running coupling constant are scheme dependent. But they conspire to give $\alpha_s(m)$ a scheme independent coefficient. Let us demonstrate this for the MS and $\overline{\mathrm{MS}}$ schemes, for which γ_1 and hence S in (3.39) are the same. Before the subtraction of the ultraviolet divergent pole, the expression for $C(\mu)$ obtained from one-loop matching is of the form

$$C(\mu) = C_0 \left[1 + \frac{\gamma_0}{2} \frac{\alpha_s}{4\pi} \left(\frac{1}{\hat{\epsilon}} - \ln(m^2/\mu^2)\right)\right] + c_1 \frac{\alpha_s}{4\pi}, \tag{3.43}$$

which is clearly a solution of the renormalization group equation (3.33). Comparing this to (3.32) one finds that $C_1^{\overline{\mathrm{MS}}} = c_1$ and $C_1^{\mathrm{MS}} = c_1 - \gamma_0\, \delta\, C_0$, where δ is given in (3.19). But the relation (3.20) between the coupling constants in both schemes is such that it precisely compensates this difference in (3.42), since

$$[\alpha_s^{\mathrm{MS}}(m)]^a = [\alpha_s(m)]^a [1 + \gamma_0\, \delta \alpha_s(m)/4\pi + \cdots]. \tag{3.44}$$

In other schemes (such as the 't Hooft-Veltman scheme for γ_5 [58]), the two-loop anomalous dimension will in general be different from that in the MS and $\overline{\mathrm{MS}}$ schemes, but it is still true that the coefficient of $\alpha_s(m)$ in the next-to-leading order solution is scheme independent. On the contrary, the coefficient S of $\alpha_s(\mu)$ in (3.42) does depend on the renormalization procedure, but this is not a surprise. Only when the μ-dependent terms in $C(\mu)$ are combined with the μ-dependent matrix elements of the renormalized operators in the effective theory one can expect to obtain a renormalization-group invariant result. It will therefore be useful to factorize the solution (3.42) in the form $C(\mu) \equiv \widehat{C}(m)\, K(\mu)$, where $\widehat{C}(m)$ is renormalization-group invariant and contains all dependence on the large mass scale m, whereas $K(\mu)$ is scheme dependent but independent of m,

$$\widehat{C}(m) = [\alpha_s(m)]^a \left(C_0 + \frac{\alpha_s(m)}{4\pi} (C_0\, S + C_1)\right),$$

$$K(\mu) = [\alpha_s(\mu)]^{-a} \left(1 - \frac{\alpha_s(\mu)}{4\pi} S\right). \tag{3.45}$$

In the case of operator mixing, the solution of the renormalization group equation is more complicated [73,82]. At next-to-leading order, eq. (3.42) can be generalized in the form

$$\hat{U}(\mu, m) = \left(1 - \frac{\alpha_s(\mu)}{4\pi} \hat{S}\right) \hat{U}_0(\mu, m) \left(1 + \frac{\alpha_s(m)}{4\pi} \hat{S}\right), \tag{3.46}$$

where

$$\hat{U}_0(\mu, m) = \exp\left[\frac{\hat{\gamma}_0^{\mathrm{t}}}{2\beta_0} \ln\left(\frac{\alpha_s(m)}{\alpha_s(\mu)}\right)\right] \tag{3.47}$$

is the evolution matrix in leading logarithmic approximation, and $\hat{S}$ contains the next-to-leading corrections to this evolution. This matrix can be shown to satisfy the algebraic equation

$$\hat{S} + \frac{1}{2\beta_0}\,[\hat{\gamma}_0^{\mathrm{t}}, \hat{S}] = \frac{1}{2\beta_0}\,\hat{\gamma}_1^{\mathrm{t}} - \frac{\beta_1}{2\beta_0^2}\,\hat{\gamma}_0^{\mathrm{t}}\,. \tag{3.48}$$

In the case without mixing the commutator vanishes, and the result for $\hat{S}$ reduces to (3.39). To proceed further, suppose there exists a matrix $\hat{V}$ which diagonalizes the one-loop anomalous dimension matrix:

$$\hat{V}^{-1}\hat{\gamma}_0^{\mathrm{t}}\,\hat{V} = \hat{\gamma}_0^{\mathrm{diag}}\,. \tag{3.49}$$

Collect the diagonal components $\gamma_{0,i}$ into a vector $\boldsymbol{\gamma}_0$. Then the leading-log evolution matrix is

$$\hat{U}_0(\mu, m) = \hat{V}\left[\left(\frac{\alpha_s(m)}{\alpha_s(\mu)}\right)^{\boldsymbol{a}}\right]_{\mathrm{diag}} \hat{V}^{-1}, \qquad \boldsymbol{a} = \frac{\boldsymbol{\gamma}_0}{2\beta_0}, \tag{3.50}$$

and furthermore $\hat{S} = \hat{V}\,\hat{T}\,\hat{V}^{-1}$, where the elements of $\hat{T}$ are given by

$$T_{ij} = \frac{(\hat{V}^{-1}\hat{\gamma}_1^{\mathrm{t}}\,\hat{V})_{ij}}{2\beta_0 + \gamma_{0,i} - \gamma_{0,j}} - \delta_{ij}\,\frac{\beta_1\,\gamma_{0,i}}{2\beta_0^2}\,. \tag{3.51}$$

3.4. Heavy–light currents

After this general review of the formalism, let us now follow in detail the matching and renormalization procedure for the case of the vector and axial vector currents built out of a heavy and a light quark:

$$V^\mu = \bar{q}\,\gamma^\mu Q\,, \quad A^\mu = \bar{q}\,\gamma^\mu\gamma_5 Q\,. \tag{3.52}$$

For simplicity the light quark is considered massless, $m_q = 0$. The $1/m_Q$ expansion of the vector current has been given at tree level in (2.26). Radiative corrections modify this result. The effective current operators present at tree level are renormalized, and additional operators are induced. Since in HQET the heavy-quark velocity v is not a dynamical degree of freedom, the effective current operators can explicitly depend on it. The most general short-distance expansion of the vector current in the effective theory contains two operators of dimension three (for simplicity we evaluate the current at $x = 0$):

$$V^\mu \cong \sum_{i=1,2} C_i(\mu)\,J_i + O(1/m_Q)\,. \tag{3.53}$$

The renormalized operators J_i can be written as

$$J_i = Z_{ij}^{-1}\,J_j^{\mathrm{bare}} = Z_{ij}^{-1}\,Z_h^{1/2}\,Z_q^{1/2}\,\bar{q}\,\Gamma_j\,h_v\,, \tag{3.54}$$

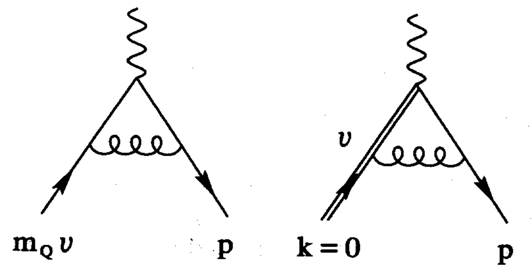


Fig. 3.2. One-loop corrections to the matrix elements of the vector current in QCD and in the effective theory. The wavy line represents the current. The external momenta are on-shell.

$$\Gamma_1 = \gamma^\mu, \qquad \Gamma_2 = v^\mu. \tag{3.55}$$

At tree level the coefficients are $C_1 = 1$ and $C_2 = 0$, and one recovers (2.26). The advantage of working with a regularization scheme with anticommuting γ_5 is that, to all orders in $1/m_Q$, the operator product expansion of the axial vector current is simply obtained from that of the vector current by replacing $\bar{q} \to -\bar{q}\gamma_5$ [53]. The Wilson coefficients remain unchanged. The reason is that in any loop diagram the γ_5 from the current can be moved outside next to the light quark spinor. For $m_q = 0$, this operation always leads to a minus sign. Hence we will only consider the case of the vector current in detail.

Explicit expressions for $C_i(\mu)$ at order α_s are obtained from the comparison of the one-loop matrix elements of the currents in the full and in the effective theory. As discussed in section 3.1, it is legitimate to perform the matching calculation with on-shell quark states. Then the matrix elements can be written $\langle V^\mu\rangle = \bar{u}_q \Gamma^\mu u_Q$, where the heavy-quark spinor satisfies $\not{v} u_Q = u_Q$. At tree level the vertex function in both theories is simply $\Gamma^\mu = \gamma^\mu$. The one-particle irreducible one-loop diagrams are shown in Fig. 3.2. In Feynman gauge, the vertex correction in QCD contributes

$$\frac{\alpha_s}{3\pi}\left(\frac{1}{\hat{\epsilon}} + \ln(\mu^2/\lambda^2) - 1\right)\gamma^\mu + \frac{2\alpha_s}{3\pi}\, v^\mu . \tag{3.56}$$

This has to be supplemented by the tree level result and the wave function renormalization of the external lines, which add $Z_Q^{1/2} Z_q^{1/2} \gamma^\mu$. The complete one-loop vertex function is

$$\Gamma^\mu_{\rm QCD} = \left(1 + \frac{\alpha_s}{2\pi}[\ln(m_Q^2/\lambda^2) - \tfrac{11}{6}]\right)\gamma^\mu + \frac{2\alpha_s}{3\pi}\, v^\mu. \tag{3.57}$$

As required by the nonrenormalization theorem for partially conserved currents, this result is gauge independent and ultraviolet finite [57]. In the limit of degenerate quark masses the vector is conserved. The space integral over its time component is the generator of a flavor symmetry. It cannot be renormalized. When the symmetry is softly broken by the presence of mass splittings, this is only relevant for small loop momenta but does not affect the ultraviolet region. Thus, there can only be a finite renormalization. Notice, however, that (3.57) does contain an infrared singularity, because the matrix element was calculated using unphysical states.

In the effective theory, the vertex correction to the matrix elements of any of the two bare operators $J_i^{\rm bare} = \bar{q}^{\rm bare}\, \Gamma_i\, h_v^{\rm bare}$ is

$$(\alpha_s/3\pi)[(1/\hat{\epsilon}) + \ln(\mu^2/\lambda^2) + 1]\Gamma_i, \tag{3.58}$$

and adding to this the renormalized tree graph contribution $Z_h^{1/2} Z_q^{1/2} \Gamma_i$ one obtains

$$\{1 + (\alpha_s/2\pi)[(1/\hat{\epsilon}) + \ln(\mu^2/\lambda^2) + \tfrac{5}{6}]\}\Gamma_i, \tag{3.59}$$

It is generally true that the dimension-three operators are renormalized multiplicatively and irrespectively of their Dirac structure [15]. Since there is no approximate flavor symmetry relating light and heavy quarks in the effective theory, it is not unexpected that the matrix elements of the bare currents are ultraviolet divergent. In the $\overline{\text{MS}}$ scheme, the currents are renormalized by a diagonal matrix $\hat{Z}$ with components

$$Z_{11} = Z_{22} = 1 + \alpha_s/2\pi\hat{\epsilon}, \qquad Z_{12} = Z_{21} = 0. \tag{3.60}$$

From (3.53) it then follows that the renormalized one-loop vertex function is

$$\Gamma^\mu_{\text{HQET}} = \{1 + (\alpha_s/2\pi)[\ln(\mu^2/\lambda^2) + \tfrac{5}{6}]\}[C_1(\mu)\,\gamma^\mu + C_2(\mu)\,v^\mu]. \tag{3.61}$$

Notice that the λ-dependence is the same as in (3.57). The short-distance coefficients $C_i(\mu)$, which follow from a comparison of the two results, are independent of the infrared regulator:

$$C_1(\mu) = 1 + (\alpha_s/\pi)[\ln(m_Q/\mu) - \tfrac{4}{3}], \qquad C_2(\mu) = 2\alpha_s/3\pi. \tag{3.62}$$

As argued in section 3.1, the matching procedure ensures by construction that the Wilson coefficients are independent of the infrared regularization scheme. Nevertheless, it is worthwhile to check this by repeating the calculation of $C_i(\mu)$ in a different scheme. One can, for instance, regulate the infrared singularities by keeping the external quarks off-shell, which leads to the same result [76]. Here we shall take the opportunity to introduce another scheme, which turns out to be extremely economic for matching calculations. In consists in using dimensional regularization for both the ultraviolet and the infrared singularities of Green's functions [32,83]. This results in the great simplification that loop integrals which depend on no mass scale other than μ vanish. For instance, in this scheme only heavy-quark fields are renormalized in the full theory; massless quarks are not:

$$Z_Q = 1 - (\alpha_s/\pi)[(1/\hat{\epsilon}) - \ln(m_Q^2/\mu^2) + \tfrac{4}{3}], \tag{3.63}$$

but $Z_q = 1$. Using this scheme, one readily obtains for the QCD vertex function

$$\Gamma^\mu_{\text{QCD}} = \{1 - (\alpha_s/2\pi)[(1/\hat{\epsilon}) - \ln(m_Q^2/\mu^2) + \tfrac{8}{3}]\}\gamma^\mu + (2\alpha_s/3\pi)v^\mu. \tag{3.64}$$

The $1/\hat{\epsilon}$ pole now comes from an infrared singularity. For the same reason that there was no renormalization of the light-quark field in QCD, all loop diagrams in the effective theory vanish in dimensional regularization. This is why this scheme is so superb for matching calculations. The vertex function in HQET is simply given by the tree level result

$$\Gamma^\mu_{\text{HQET}} = C_1(\mu)\,\gamma^\mu + C_2(\mu)\,v^\mu\,. \tag{3.65}$$

From the comparison of the two vertex functions after the subtraction of the $1/\hat{\epsilon}$ pole in (3.64), we indeed recover our previous result (3.62).

What remains to be done is the renormalization group improvement of the one-loop calculation. Since there is no operator mixing, the solution of the renormalization group equation proceeds as in

(3.38). The anomalous dimension matrix is proportional to the unit matrix, and from (3.30) and (3.60) one obtains the one-loop coefficient

$$\gamma_0^{\rm hl} = -4. \tag{3.66}$$

This is the so-called hybrid anomalous dimension of heavy–light currents derived by Voloshin and Shifman [15]. In the context of the effective theory, it has been calculated by Politzer and Wise [31]. For the next-to-leading order solution of the renormalization group equation one also needs the scheme dependent two-loop coefficient $\gamma_1^{\rm hl}$. In dimensional regularization with anticommuting γ_5 this coefficient is independent of the Dirac structure of the current. It is the same in the MS and $\overline{\rm MS}$ schemes. The two-loop calculation has been performed by Ji and Musolf [76], and by Broadhurst and Grozin [77]. They obtain

$$\gamma_1^{\rm hl} = -\tfrac{254}{9} - \tfrac{56}{27}\pi^2 + \tfrac{20}{9}n_{\rm f}. \tag{3.67}$$

The final result for the Wilson coefficients can be written in the factorized form

$$C_i(m_{\rm Q}) \equiv \widehat{C}_i(m_{\rm Q})\, K_{\rm hl}(\mu)\,, \tag{3.68}$$

where the functions $\widehat{C}_i(m_{\rm Q})$ are renormalization-group invariant and contain all dependence on the heavy-quark mass. All μ-dependence is contained in the function $K_{\rm hl}(\mu)$, which is universal for all heavy–light current operators. Taking into account the initial values of the coefficients at the matching scale $\mu = m_{\rm Q}$, one finds from (3.45)

$$\begin{aligned}
\widehat{C}_1(m_{\rm Q}) &= [\alpha_s(m_{\rm Q})]^{-2/\beta_0}\{1 + [\alpha_s(m_{\rm Q})/\pi]\,Z_{\rm hl}\},\\
\widehat{C}_2(m_{\rm Q}) &= [\alpha_s(m_{\rm Q})]^{-2/\beta_0}\,2\alpha_s(m_{\rm Q})/3\pi,\\
K_{\rm hl}(\mu) &= [\alpha_s(\mu)]^{2/\beta_0}\{1 - [\alpha_s(\mu)/4\pi]\,S\}.
\end{aligned} \tag{3.69}$$

The coefficient [76]

$$Z_{\rm hl} = \tfrac{1}{4}S - \tfrac{4}{3} = 3\,\frac{153 - 19n_{\rm f}}{(33 - 2n_{\rm f})^2} - \frac{381 + 28\pi^2 - 30n_{\rm f}}{36(33 - 2n_{\rm f})} - \frac{4}{3} \tag{3.70}$$

is independent of the renormalization scheme, but S is not.

3.5. Leading-power corrections to the effective Lagrangian

We have seen in chapter 2 that at order $1/m_{\rm Q}$ there appear dimension-four operators in the effective Lagrangian which explicitly violate the spin–flavor symmetry. The leading corrections have been given in (2.27). They involve the kinetic operator $O_{\rm kin}$ and the chromo-magnetic operator $O_{\rm mag}$. Just as the current operators discussed in the previous section, these are composite operators which, in general, have to be renormalized. But as discussed in section 2.5, reparameterization invariance requires that the kinetic operator have the same coefficient as the leading term in the effective Lagrangian, so that $Z_{\rm kin} = 1$ to all orders in perturbation theory.[14] Then only the chromo-magnetic operator needs to be renormalized. As usual we define

[14] In its original form (2.17), the kinetic operator also contains a class II operator, which vanishes by the equation of motion. We shall not discuss its renormalization since it is irrelevant.

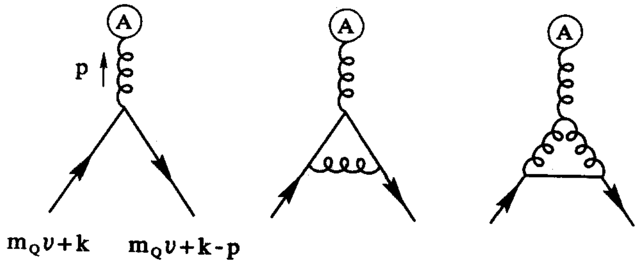


Fig. 3.3. Diagrams for the calculation of the heavy-quark-gluon vertex function in QCD. The background field is denoted by A.

$$O_{\rm mag} = Z_{\rm mag}^{-1} O_{\rm mag}^{\rm bare} = Z_{\rm mag}^{-1}\, Z_h (g/4m_Q)\, \bar h_v \sigma_{\alpha\beta} G^{\alpha\beta} h_v. \tag{3.71}$$

The aim is to calculate the short-distance coefficient which multiplies the renormalized operator in

$$\frac{1}{2m_Q}\mathcal{L}_1 = \frac{1}{2m_Q}\bar h_v (iD)^2 h_v + C_{\rm mag}(\mu)\frac{g}{4m_Q}\bar h_v \sigma_{\alpha\beta}\, G^{\alpha\beta}\, h_v\,, \tag{3.72}$$

which replaces the tree level expression in (2.27). $C_{\rm mag}(\mu)$ can be obtained from a calculation of the Green function of two heavy quarks and a background gluon field, to one-loop order in the full and in the effective theory. The diagrams in QCD are shown in Fig. 3.3. The momentum assignments are such that p is the outgoing momentum of the background field, and k and $(k-p)$ are the residual momenta of the heavy quarks. To order $1/m_Q$, it is sufficient to keep terms linear in k or p. The external quarks can be taken on-shell, in which case $v\cdot k = v\cdot p = 0$. A subtlety which has to be taken into account is that, according to (2.19), the QCD spinor $u_Q(P_Q, s)$ is related to the spinor $u_h(v,s)$ of the effective theory by

$$u_Q(P_Q, s) = (1 + \not{k}/2m_Q + \cdots) u_h(v,s), \quad P_Q = m_Q v + k. \tag{3.73}$$

In the matching calculation one has to use the same spinors in both theories. We thus define a vertex function Γ^μ by writing the amplitude as $i\mu^\epsilon g A_{\mu,a}(p)\, \bar u_h \Gamma^\mu T_a u_h$, so that at tree level in QCD

$$\Gamma^\mu_{\rm QCD,0} = \left(1 + \frac{\not{k}-\not{p}}{2m_Q}\right)\gamma_\mu \left(1 + \frac{\not{k}}{2m_Q}\right) + \cdots = v^\mu + \frac{(2k-p)^\mu}{2m_Q} + \frac{[\gamma^\mu, \not{p}]}{4m_Q} + \cdots \tag{3.74}$$

Here the ellipses represent terms of higher order in k or p, and we have used that between the heavy-quark spinors γ^μ can be replaced by v^μ.

The contributions to the vertex function arising at one-loop order are shown in Fig. 3.3. The last diagram involves the nonabelian three-gluon vertex. Its infrared divergence cannot be regulated by the introduction of a gluon mass. Following Eichten and Hill, we present the calculation using dimensional regularization for both the ultraviolet and the infrared singularities [32]. One finds that in the $\overline{\rm MS}$ scheme the one-loop contribution to the QCD vertex function is

$$\Gamma^\mu_{\rm QCD,1} = -\frac{[\gamma^\mu, \not{p}]}{4m_Q}\frac{3\alpha_s}{2\pi}[\ln(m_Q/\mu) - \tfrac{13}{9}] + \cdots \tag{3.75}$$

As emphasized in the previous section, this scheme has the great advantage that all loop integrals in HQET vanish. Hence, in the effective theory the vertex function is simply given by the tree level contributions from the operators in the effective Lagrangian,

$$\Gamma^\mu_{\rm HQET} = v^\mu + \frac{(2k-p)^\mu}{2m_Q} + C_{\rm mag}(\mu)\,\frac{[\gamma^\mu, \not p]}{4m_Q} + \cdots \tag{3.76}$$

The second term, which comes from the kinetic operator in (3.72), is in fact the same as in the full theory, as required by reparameterization invariance. For the coefficient of the chromo-magnetic operator, one obtains from a comparison of (3.74)–(3.76) the one-loop result

$$C_{\rm mag}(\mu) = 1 - (3\alpha_s/2\pi)\,[\ln(m_Q/\mu) - \tfrac{13}{9}]. \tag{3.77}$$

From the fact that this must satisfy the renormalization group equation (3.33), it follows that the one-loop coefficient of the anomalous dimension is [44]

$$\gamma_0^{\rm mag} = 6. \tag{3.78}$$

Unfortunately, the two-loop coefficient $\gamma_1^{\rm mag}$ is not yet known. This means that in the next-to-leading order solution of the renormalization group equation,

$$C_{\rm mag}(\mu) = \left(\frac{\alpha_s(m_Q)}{\alpha_s(\mu)}\right)^{3/\beta_0} \left(1 + \frac{13}{6}\frac{\alpha_s(m_Q)}{\pi} + \frac{\alpha_s(m_Q) - \alpha_s(\mu)}{4\pi} S_{\rm mag}\right), \tag{3.79}$$

the coefficient $S_{\rm mag}$ is unknown. One can then either work with the leading logarithmic approximation, or with the hybrid form

$$C_{\rm mag}(\mu) = [\alpha_s(m_Q)/\alpha_s(\mu)]^{3/\beta_0}(1 + \tfrac{13}{6}\alpha_s/\pi), \tag{3.80}$$

in which the scale in the next-to-leading correction is not determined (except for $\mu = m_Q$).

3.6. Power corrections to heavy–light currents

Let us now come back to the discussion of heavy–light currents and discuss their short-distance expansion at next-to-leading order in $1/m_Q$. Then there appears in (3.53) a set of dimension-four operators, which we shall denote by O_j and T_k:

$$V^\mu \cong \sum_{i=1,2} C_i(\mu)\,J_i + \frac{1}{2m_Q}\sum_j B_j(\mu)\,O_j + \frac{1}{2m_Q}\sum_k A_k(\mu)\,T_k + O(1/m_Q^2)\,. \tag{3.81}$$

In the limit $m_q = 0$, there are six independent local class I operators of dimension four in the effective theory. They can be chosen as [53,54,84]

$$\begin{aligned} O_1 &= \bar q\,\gamma^\mu\, i\not D h_v, & O_4 &= \bar q\,(-iv\cdot \overleftarrow{D})\,\gamma^\mu h_v\,,\\ O_2 &= \bar q\,v^\mu\, i\not D\, h_v\,, & O_5 &= \bar q\,(-iv\cdot \overleftarrow{D})\,v^\mu h_v\,,\\ O_3 &= \bar q\, iD^\mu h_v\,, & O_6 &= \bar q\,(-i\overleftarrow{D^\mu})\,h_v\,. \end{aligned} \tag{3.82}$$

In addition to these local operators, there are power corrections resulting from a combination of one of the leading-order currents J_i with a term of order $1/m_Q$ from the effective Lagrangian. The corresponding nonlocal operators T_k are

$$\frac{T_1}{2m_Q} = \mathrm{i}\int \mathrm{d}y\, T\{J_1(0), O_{\rm kin}(y)\}, \qquad \frac{T_2}{2m_Q} = \mathrm{i}\int \mathrm{d}y\, T\{J_2(0), O_{\rm kin}(y)\},$$

$$\frac{T_3}{2m_Q} = \mathrm{i}\int \mathrm{d}y\, T\{J_1(0), O_{\rm mag}(y)\}, \qquad \frac{T_4}{2m_Q} = \mathrm{i}\int \mathrm{d}y\, T\{J_2(0), O_{\rm mag}(y)\}, \tag{3.83}$$

with $O_{\rm kin}$ and $O_{\rm mag}$ as given in (2.17) and (2.18). Only O_1, T_1, and T_3 appeared in the tree level expansion of the current in (2.26). But since the operators O_j and T_k have the same dimension and quantum numbers, they all mix under renormalization. This mixing is described by a 10×10 anomalous dimension matrix $\hat{\gamma}$.

Several of the Wilson coefficients in (3.81) can be deduced without a detailed calculation [83]. The local operators O_j cannot mix into the nonlocal ones. The coefficients of the time-ordered products are thus simply the products of the coefficients of their local component operators, i.e.:

$$A_1(\mu) = C_1(\mu), \qquad A_2(\mu) = C_2(\mu),$$

$$A_3(\mu) = C_1(\mu) C_{\rm mag}(\mu), \qquad A_4(\mu) = C_2(\mu)\, C_{\rm mag}(\mu). \tag{3.84}$$

Furthermore, the operators O_1 to O_3, which contain a covariant derivative acting on the heavy-quark field, are not reparameterization invariant by themselves. According to the discussion in section 2.5, they have to be combined with the dimension-three operators J_i into a reparameterization invariant form. The corresponding extension of the operators J_i is unique and reads

$$J_1 \overset{\rm RPI}{\rightarrow} \bar{q}\gamma^\mu \tilde{h}_v = \bar{q}\gamma^\mu (1 + \mathrm{i}\not{D}/2m_Q) h_v + \cdots,$$

$$J_2 \overset{\rm RPI}{\rightarrow} \bar{q}\mathcal{V}^\mu \tilde{h}_v = \bar{q}(v^\mu + \mathrm{i}D^\mu/m_Q)(1 + \mathrm{i}\not{D}/2m_Q) h_v + \cdots. \tag{3.85}$$

This implies

$$B_1(\mu) = C_1(\mu), \qquad B_2(\mu) = \tfrac{1}{2} B_3(\mu) = C_2(\mu). \tag{3.86}$$

What remains to be determined, then, are the coefficients $B_4(\mu)$ to $B_6(\mu)$. Their calculation at one-loop order proceeds similar to that described in the previous section. It is again extremely economic to use dimensional regularization, so that all loop integrals in the effective theory vanish. At order $1/m_Q$, the current matrix element in the full theory is obtained from the first diagram shown in Fig. 3.2 by assigning momenta $P_Q = m_Q v + k$ and $p_q = p$ to the external quarks and keeping terms linear in k and p. The matching calculation has been performed in Ref. [83]. For the coefficients B_1 to B_3 one indeed finds the relations given in (3.86). The one-loop expressions for the remaining coefficient functions are (in the $\overline{\rm MS}$ scheme)

$$B_4(\mu) = (4\alpha_s/3\pi)\,[3\ln(m_Q/\mu) - 1],$$

$$B_5(\mu) = -(4\alpha_s/3\pi)\,[2\ln(m_Q/\mu) - 3],$$

$$B_6(\mu) = -(4\alpha_s/3\pi)\,[\ln(m_Q/\mu) - 1]. \tag{3.87}$$

To go beyond this result is tedious, since it requires a calculation of the 10×10 anomalous dimension matrix $\hat{\gamma}$. To some extent the structure of this matrix is determined, however. Since the operators O_1, O_2, O_3 and T_k renormalize multiplicatively, they can mix into O_4, O_5, and O_6, but not vice versa. Therefore, the anomalous dimension matrix must be of the form

$$\hat{\gamma} = \begin{pmatrix} \gamma^{\rm hl} I & \hat{\gamma}_A & 0 \\ 0 & \hat{\gamma}_B & 0 \\ 0 & \hat{\gamma}_C & \hat{\gamma}_D \end{pmatrix}, \tag{3.88}$$

where $\gamma^{\rm hl}$ is the anomalous dimension of the dimension three operators J_i, I denotes the 3×3 unit matrix, and

$$\hat{\gamma}_D = \mathrm{diag}\Big(\gamma^{\rm hl}, \gamma^{\rm hl}, \gamma^{\rm hl} + \gamma^{\rm mag}, \gamma^{\rm hl} + \gamma^{\rm mag}\Big)\,. \tag{3.89}$$

Furthermore, the 3×3 submatrix $\hat{\gamma}_B$ can be constructed by noting that the equation of motion $\mathrm{i}v\cdot Dh_v = 0$ can be used to rewrite

$$O_4 = -\mathrm{i}v \cdot \partial J_1 \qquad O_5 = -\mathrm{i}v \cdot \partial J_2, \qquad O_6 - O_3 = -\mathrm{i}\partial^\mu(\bar{q}h_v)\,. \tag{3.90}$$

The total derivatives of the currents on the right-hand side renormalize multiplicatively and in the same way as the dimension three operators J_i. The additional power of the external momentum carried by the current does not affect the divergences of loop diagrams. It follows that

$$(\hat{\gamma}_B)_{ij} = \gamma^{\rm hl}\,\delta_{ij} + \delta_{i3}\,(\hat{\gamma}_A)_{3j}\,. \tag{3.91}$$

For the complete next-to-leading order solution of the renormalization group equation the submatrices $\hat{\gamma}_A$ and $\hat{\gamma}_C$, as well as the anomalous dimension $\gamma^{\rm mag}$ of the chromo-magnetic operator, would have to be calculated at two-loop order. This has not yet been done. So far, only the one-loop matrices are known from calculations by Falk and Grinstein [84], and Neubert [83]. They are

$$\hat{\gamma}_A = \frac{\alpha_s}{3\pi}\begin{pmatrix} -2 & -4 & 6 \\ 0 & 0 & 0 \\ -1 & -2 & 3 \end{pmatrix}, \qquad \hat{\gamma}_C = \frac{\alpha_s}{3\pi}\begin{pmatrix} -8 & 0 & 0 \\ 0 & -8 & 0 \\ -2 & 12 & -2 \\ 0 & 8 & 0 \end{pmatrix}. \tag{3.92}$$

This information is sufficient to obtain the leading-log solution of the renormalization group equation (3.33). One way to proceed is to rewrite this equation as an inhomogeneous differential equation for the unknown coefficients B_4, B_5, and B_6 [84]. In obvious notation, this yields

$$(\mu\,\mathrm{d}/\mathrm{d}\mu - \hat{\gamma}_B^{\rm t})\boldsymbol{B}_{4-6}(\mu) = \hat{\gamma}_A^{\rm t}\,\boldsymbol{B}_{1-3}(\mu) + \hat{\gamma}_C^{\rm t}\,\boldsymbol{A}(\mu)\,. \tag{3.93}$$

The coefficient functions on the right-hand side are known. An alternative approach, which we shall discuss here, is to proceed following the general procedure outlined in section 3.3. This has the advantage that it can be straightforwardly generalized to next-to-leading order, once the two-loop anomalous dimensions are known. The leading-log evolution matrix in (3.47) is trivial to obtain when the one-loop anomalous dimension matrix $\hat{\gamma}_0$ can be diagonalized. This is, however, not possible in the present case. In such a situation, one constructs the matrix $\hat{W}$ which brings $\hat{\gamma}_0$ into Jordan form [83]:

$$\hat{W}^{-1}\hat{\gamma}_0^{\rm t}\hat{W} = \hat{\gamma}_J\,. \tag{3.94}$$

This is convenient enough for an exponentiation in closed form. The evolution matrix is then given by

$$\hat{U}_0(\mu, m) = \hat{W}\, \exp(t\,\hat{\gamma}_J)\, \hat{W}^{-1}, \tag{3.95}$$

$$t = (1/2\beta_0)\, \ln[\alpha_s(m_Q)/\alpha_s(\mu)]. \tag{3.96}$$

According to (3.35), the Wilson coefficients are obtained by acting with $\hat{U}_0(\mu, m_Q)$ on the vector containing the initial values of the coefficients at the matching scale $\mu = m_Q$. In leading logarithmic approximation it is sufficient to work with the tree level expressions, i.e., $B_1(m_Q) = A_1(m_Q) = A_3(m_Q) = 1$, and the other coefficients vanish. The result is

$$\begin{aligned} B_4(\mu) &= \tfrac{34}{27}\mathrm{e}^{-4t} - \tfrac{4}{27}\mathrm{e}^{2t} - \tfrac{10}{9} - \tfrac{32}{3}t\,\mathrm{e}^{-4t}, \\ B_5(\mu) &= -\tfrac{28}{27}\mathrm{e}^{-4t} + \tfrac{88}{27}\mathrm{e}^{2t} - \tfrac{20}{9}, \qquad B_6(\mu) = -2\mathrm{e}^{-4t} - \tfrac{4}{3}\mathrm{e}^{2t} + \tfrac{10}{3}. \end{aligned} \tag{3.97}$$

For $m_Q \gtrsim \mu$, these expressions can be expanded using

$$t \approx -(\alpha_s/4\pi)\, \ln(m_Q/\mu), \tag{3.98}$$

and one readily recovers the logarithmic terms in the one-loop expressions in (3.87). It is possible to combine the one-loop results with the leading-log solution by simply adding the matching corrections arising at $\mu = m_Q$, in a hybrid approach similar to (3.80). This leads to the final result:

$$\begin{aligned} B_4(\mu) &= -(4\alpha_s/3\pi) + \tfrac{34}{27}x^{2/\beta_0} - \tfrac{4}{27}x^{-1/\beta_0} - \tfrac{10}{9} + (16/3\beta_0)x^{2/\beta_0}\ln x, \\ B_5(\mu) &= (4\alpha_s/\pi) - \tfrac{28}{27}x^{2/\beta_0} + \tfrac{88}{27}x^{-1/\beta_0} - \tfrac{20}{9}, \\ B_6(\mu) &= (4\alpha_s/3\pi) - 2x^{2/\beta_0} - \tfrac{4}{3}x^{-1/\beta_0} + \tfrac{10}{3}, \qquad x = \alpha_s(\mu)/\alpha_s(m_Q). \end{aligned} \tag{3.99}$$

This concludes our discussion of the renormalization of heavy–light currents in HQET. A more complete treatment can be found in Ref. [83].

3.7. Flavor-conserving heavy-quark currents

The short-distance expansion of currents containing two heavy-quark fields proceeds along the same lines. However, a new feature is that the Wilson coefficients become functions of the quark velocity transfer $w = v \cdot v'$. Another major complication arising for flavor-changing currents is the presence of two different heavy mass scales. To disentangle these difficulties, we first discuss the case of flavor-conserving currents. Such currents are also of some phenomenological importance. For instance, they mediate the pair creation of heavy quarks in e^+e^- collisions [85].

Consider then the vector current $V^\mu = \bar{Q}\gamma^\mu Q$ composed of two identical heavy-quark fields. In analogy to (3.53), its expansion in terms of operators of the effective theory reads

$$V^\mu \cong \sum_{i=1}^{3} C_i(w, \mu)\, J_i + O(1/m_Q)\,, \tag{3.100}$$

but now there are three independent renormalized operators,

$$J_i = Z_{ij}^{-1}\, J_j^{\rm bare} = Z_{ij}^{-1}\, Z_h\, \bar h_{v'} \Gamma_j\, h_v\,, \tag{3.101}$$

$$\Gamma_1 = \gamma^\mu\,, \qquad \Gamma_2 = v^\mu\,, \qquad \Gamma_3 = v'^\mu\,. \tag{3.102}$$

At order α_s, the short-distance coefficients are obtained from a comparison of the diagrams depicted in Fig. 3.4, supplemented by wave function renormalization. We write the amplitudes as $\langle V^\mu\rangle = \bar u_Q \Gamma^\mu u_Q$, so that $\Gamma^\mu = \gamma^\mu$ at tree level. After a somewhat tedious calculation, one finds that the one-loop vertex function in the full theory is free of ultraviolet divergences [86],

$$\Gamma_{\rm QCD}^\mu = \left(1 + \frac{2\alpha_s}{3\pi}\{-[w r(w) - 1]\ln(m_Q^2/\lambda^2) + F(w)\}\right)\gamma^\mu - \frac{\alpha_s}{3\pi} r(w)\,(v^\mu + v'^\mu)\,. \tag{3.103}$$

As previously we have introduced a fictitious gluon mass λ to regulated the infrared divergences, which arise since the calculation is performed with on-shell quark states. The function $r(w)$ will play a very significant role in the further course of this review. It is defined as

$$r(w) = \frac{1}{\sqrt{w^2-1}}\,\ln(w + \sqrt{w^2-1})\,, \tag{3.104}$$

and satisfies $r(1) = 1$, and $r'(1) = -\frac{1}{3}$. The function $F(w)$ is more complicated. It reads

$$F(w) = \frac{w}{\sqrt{w^2-1}}[2L_2(-w_-) + \tfrac{1}{6}\pi^2 + \tfrac{1}{2}(w^2-1)r^2(w)] \\ - w r(w)\,\ln[2(w+1)] + \tfrac{3}{2}(w+1)r(w) - 2\,, \tag{3.105}$$

where $w_- = w - \sqrt{w^2-1}$, and

$$L_2(x) = -\int_0^x dt\,\frac{\ln(1-t)}{t} \tag{3.106}$$

is the dilogarithm. The normalization is such that $F(1) = 1$ for $w = 1$. Using this and the fact that $\bar u_Q(v)\, v^\mu\, u_Q(v) = \bar u_Q(v)\,\gamma^\mu\, u_Q(v)$, one finds that the radiative corrections vanish at zero recoil,

$$\Gamma_{\rm QCD}^\mu(v = v') = \gamma^\mu\,. \tag{3.107}$$

This is a consequence of the fact that the flavor-conserving vector current is exactly conserved.

Since in the effective theory the coupling of a heavy quark to a gluon does not involve a γ-matrix, it is easy to see that to all orders in perturbation theory the dimension-three operators J_i renormalize multiplicatively and irrespective of their Dirac structure. The one-loop matrix element of any of the bare current operators is given by [86]

$$\{1 - (2\alpha_s/3\pi)[w r(w) - 1][(1/\hat\epsilon) + \ln(\mu^2/\lambda^2)]\}\Gamma_i\,. \tag{3.108}$$

The matrix $\hat Z$ which defines the renormalized operators is proportional to the unit matrix, hence we shall simply denote it by Z. In the $\overline{\rm MS}$ scheme, it reads

$$Z = 1 - (2\alpha_s/3\pi\hat\epsilon)[w r(w) - 1]\,. \tag{3.109}$$

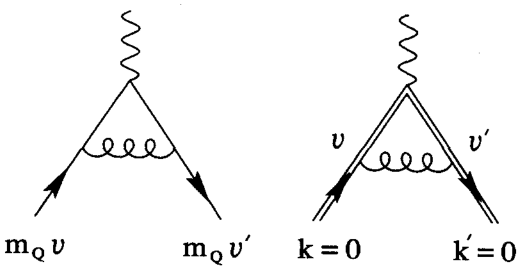


Fig. 3.4. Vertex correction diagrams arising in the matching calculation for flavor-conserving heavy-quark currents. The current changes the heavy quark velocity.

The renormalized vertex function in HQET becomes

$$\Gamma^\mu_{\rm HQET} = \left(1 - \frac{2\alpha_s}{3\pi}[wr(w) - 1]\ln(\mu^2/\lambda^2)\right) \sum_{i=1}^{3} C_i(w,\mu)\Gamma_i. \tag{3.110}$$

The dependence on λ matches with that in (3.103), so that the resulting one-loop expressions for the short-distance coefficients are independent of the infrared regulator. It follows that

$$C_1(w,\mu) = 1 - (2\alpha_s/3\pi)\{[wr(w) - 1]\ \ln(m_Q^2/\mu^2) - F(w)\},$$
$$C_2(w,\mu) = C_3(w,\mu) = -(\alpha_s/3\pi)r(w). \tag{3.111}$$

The matching calculation for the axial vector current proceeds similarly. In this case, the dimension three operators are obtained from (3.101) by setting

$$\Gamma_1 = \gamma^\mu\gamma_5, \qquad \Gamma_2 = v^\mu\gamma_5, \qquad \Gamma_3 = v'^\mu\gamma_5. \tag{3.112}$$

Using dimensional regularization with anticommuting γ_5, one finds [85,86]

$$C_1^5(w,\mu) = 1 - \frac{2\alpha_s}{3\pi}\Big([wr(w) - 1]\ \ln(m_Q^2/\mu^2) - F(w) + 2r(w)\Big),$$
$$C_2^5(w,\mu) = -C_3^5(w,\mu) = -\frac{\alpha_s}{3\pi}\left(r(w) + 2\frac{wr(w) - 1}{w - 1}\right). \tag{3.113}$$

We use a superscript 5 to distinguish these from the coefficients of the vector current.

For the renormalization group improvement of the one-loop results one needs the anomalous dimension matrix of the bilinear heavy-quark currents. It is proportional to the unit matrix, with a velocity dependent coefficient $\gamma^{\rm hh}(w)$. In the zero recoil limit the vector currents J_i are conserved in the effective theory, since $\mathrm{i}\partial \cdot J_i = 0$ by the equation of motion. This implies that

$$\gamma^{\rm hh}(1) = 0 \tag{3.114}$$

to all orders in perturbation theory. This constraint is satisfied by the one-loop coefficient

$$\gamma_0^{\rm hh}(w) = \tfrac{16}{3}[wr(w) - 1], \tag{3.115}$$

which is derived from (3.109) by means of (3.30). This famous velocity dependent anomalous dimension was obtained by Falk et al. [7].

Since in the effective theory the velocity of a heavy quark is conserved by the strong interactions, the heavy quark can be described by a Wilson line [8]. An external current can instantaneously change the velocity, resulting in a kink of that line. It is well known that such cusps lead to infrared singular behavior. The renormalization of cusp singularities of Wilson lines was investigated in detail by Korchemsky and Radyushkin in 1987 [87], prior to the development of HQET. In particular, they calculated the one- and two-loop coefficients of the so-called cusp anomalous dimension $\gamma^{\rm cusp}(\varphi)$, as a function of the cusp angle φ. But this anomalous dimension is precisely that of the bilinear heavy-quark currents, with the identification $\cosh\varphi = w$. This equivalence was pointed out in Ref. [88]. The two-loop coefficient is the same in the MS and $\overline{\rm MS}$ schemes, namely

$$\gamma_1^{\rm hh}(w) = [\tfrac{13}{3} - \tfrac{1}{2}\pi^2 - \tfrac{10}{9}n_{\rm f} - 6f(w)]\gamma_0^{\rm hh}(w) - 64I(w), \tag{3.116}$$

where ($w_- = w - \sqrt{w^2-1}$),

$$f(w) = wr(w) - 2 - \frac{w}{\sqrt{w^2-1}}[L_2(1-w_-^2) + (w^2-1)r^2(w)], \tag{3.117}$$

$$I(w) = \int\limits_0^{\varphi} {\rm d}\psi\,(\psi\coth\psi - 1)\left(\psi\coth^2\varphi + \frac{\sinh\varphi\cosh\varphi}{\sinh^2\varphi - \sinh^2\psi}\ln\frac{\sinh\varphi}{\sinh\psi}\right), \tag{3.118}$$

in terms of the cusp angle $\varphi = \cosh^{-1} w$. This function satisfies $I(1) = 0$, so that indeed the two-loop coefficient vanishes at zero recoil. Recently, the result for $\gamma_1^{\rm hh}(w)$ has been confirmed by Kilian, Manakos and Mannel in the framework of HQET [89].

Using these expressions, the coefficients a and S in (3.39) can be computed as a function of the velocity transfer w. For $n_{\rm f} = 3$ light quark flavors, one finds [86]

$$\begin{aligned} a_{\rm hh}(w) &= \tfrac{8}{27}[wr(w) - 1], \\ Z_{\rm hh}(w) \equiv \tfrac{1}{4}S_{\rm hh}(w) &= -\tfrac{1}{12}\gamma_0(w)[\tfrac{55}{54} + \tfrac{1}{12}\pi^2 + f(w)] - \tfrac{8}{9}I(w) \\ &= \tfrac{8}{81}(\tfrac{94}{9} - \pi^2)(w-1) - \tfrac{4}{135}(\tfrac{92}{9} - \pi^2)(w-1)^2 + \cdots. \end{aligned} \tag{3.119}$$

Both vanish at zero recoil. The next-to-leading order result for the Wilson coefficients is obtained from (3.45) in the factorized form $C_i(w,\mu) = \widehat{C}_i(m_{\rm Q},w)\,K_{\rm hh}(w,\mu)$. The μ-dependent function

$$K_{\rm hh}(w,\mu) = [\alpha_{\rm s}(\mu)]^{-a_{\rm hh}(w)}\{1 - [\alpha_{\rm s}(\mu)/\pi]Z_{\rm hh}(w)\} \tag{3.120}$$

is normalized at zero recoil: $K(1,\mu) = 1$. It is the same for any Dirac structure of the heavy-quark current. The renormalization-group invariant coefficients $\widehat{C}_i$, on the other hand, do depend on the structure of the current. For the vector current, for instance, they are given by

$$\begin{aligned} \widehat{C}_1(m_{\rm Q},w) &= [\alpha_{\rm s}(m_{\rm Q})]^{a_{\rm hh}(w)}\{1 + [\alpha_{\rm s}(m_{\rm Q})/\pi][Z_{\rm hh}(w) + \tfrac{2}{3}F(w)]\}, \\ \widehat{C}_2(m_{\rm Q},w) = \widehat{C}_3(m_{\rm Q},w) &= -[\alpha_{\rm s}(m_{\rm Q})]^{a_{\rm hh}(w)}[\alpha_{\rm s}(m_{\rm Q})/3\pi]r(w). \end{aligned} \tag{3.121}$$

Similar expressions follow for the axial vector current.

3.8. Flavor-changing heavy-quark currents

The operator product expansion of currents in the effective theory becomes considerably more complicated in the case of flavor-changing currents. The charged weak current $\bar{c}\gamma^\mu(1-\gamma_5)b$ is of this form, however, so it is necessary to consider this case in detail. The complications arise due to the fact that there are two different heavy-quark masses, m_b and m_c. Thus the calculation of the Wilson coefficients becomes a two-scale problem.

There are two ways of performing the transition from QCD to an effective low energy theory in which both the bottom and the charm quark are treated as heavy quarks in the sense of HQET: The transition can either be done in a single step, or by considering first an intermediate theory with a static bottom quark, but a dynamical charm quark. The latter becomes heavy in a second step. If one could solve perturbation theory to all orders, both treatments would lead to the same results for the Wilson coefficients. The calculation differs in both cases, however, and the results also differ if the perturbation series is truncated.

To see what the differences are, suppose first the two heavy quarks have similar masses, i.e. $m_c \sim m_b \sim m$, with m being some average mass. It is then natural to remove at the same time the dynamical degrees of freedom of both heavy quarks. This approach is similar to the case of flavor-conserving currents discussed in the previous section. Although our discussion will be completely general, let us consider the cases of the vector current $V^\mu = \bar{c}\gamma^\mu b$ and of the axial vector current $A^\mu = \bar{c}\gamma^\mu\gamma_5 b$ explicitly and denote them collectively by J^μ. Each of these currents obeys an expansion analogous to (3.100):

$$J^\mu(m_b, m_c) \cong \sum_{j=1}^{3} C_j(m_b, m_c, w, \mu, m)\, J_j(\mu) + \mathrm{O}(1/m), \tag{3.122}$$

where $J_j = \bar{h}^c_{v'}\Gamma_j h^b_v$. For the vector and axial vector currents, the matrices Γ_j have been given in (3.102) and (3.112), respectively. In the above expression we have indicated that matrix elements of the original current J^μ between states of the full theory depend on m_c and m_b, whereas matrix elements of the operators J_j between states of the effective theory are mass independent, but do depend on the renormalization scale μ. The short-distance coefficients are functions of the heavy-quark masses, the renormalization scale, and the matching scale m. The dependence on m would disappear if one could sum the perturbative series to all orders. The advantage of this first approach is its simplicity. At leading order in the $1/m$ expansion, only three current operators contribute. Matrix elements of higher dimension operators are suppressed by powers of Λ_{QCD}/m. The short-distance coefficients contain the dependence on m_b and m_c correctly to a given order in α_s, via matching at $\mu = m$. The only disadvantage is the residual dependence on the matching scale m, which arises when one calculates to finite order in perturbation theory. Although in a next-to-leading order calculation the scale in the leading anomalous scaling factor is determined, one has no control over the scale in the next-to-leading corrections proportional to $\alpha_s(m)$. This introduces an uncertainty of order $\alpha_s^2 \ln(m_b/m_c)$.

The alternative approach is to consider first, as an intermediate step for $m_b > \mu > m_c$, an effective theory with only a heavy bottom quark. Denoting the effective current operators of dimension $(3+k)$ in this theory by $\widetilde{J}_i^{(k)}(m_c, \mu)$, indicating that their matrix elements will depend both on the charm quark mass and the renormalization scale, we write the short-distance expansion as an expansion in

$1/m_b$:

$$J^\mu(m_b, m_c) \cong \sum_{k=0}^{\infty} \frac{1}{m_b^k} \sum_i D_i^{(k)}(m_b, \mu)\, \widetilde{J}_i^{(k)}(m_c, \mu) \,. \tag{3.123}$$

Since the velocity of the charm quark is still a dynamical degree of freedom, in this intermediate effective theory there are only two dimension-three operators, namely

$$\widetilde{J}_1^{(0)} = \bar{c}\,\Gamma_1\, h_v^b \,, \qquad \widetilde{J}_2^{(0)} = \bar{c}\,\Gamma_2\, h_v^b \,. \tag{3.124}$$

These are actually the same operators that appear in the expansion of a heavy–light vector or axial vector current, and it will turn out that the coefficients D_i are identical to the ones given in (3.69). A major complication arises from the fact that the charm quark is still a dynamical particle in the intermediate effective theory. Matrix elements of the higher dimension operators $\widetilde{J}_i^{(k)}$ will, in general, scale like m_c^k. This compensates the prefactor $1/m_b^k$ in (3.123). Consequently, these operators cannot be neglected even at leading order in the heavy-quark expansion. One would thus have to deal with an infinite number of operators in order to keep track of the full dependence on the heavy-quark masses. Ignoring this difficulty for the moment, we may use (3.123) to scale the currents from $\mu = m_b$ down to $\mu = m_c$, where we match onto the final effective theory with two heavy quarks. In performing this step, the operator basis collapses considerably. When terms of order Λ_{QCD}/m_Q (we use m_Q generically for m_c or m_b) are neglected on the level of matrix elements, only the three operators $J_j(\mu)$ in (3.122) remain. Each of the operators of the intermediate theory has an expansion in terms of these operators, with coefficients E_{ij}:

$$\widetilde{J}_i^{(k)}(m_c, m_c) \cong \sum_{j=1}^{3} m_c^k\, E_{ij}^{(k)}(m_c, w, \mu)\, J_j(\mu) + O(1/m_c) \,. \tag{3.125}$$

Combining this with (3.123), we obtain

$$J^\mu(m_b, m_c) \cong \sum_{j=1}^{3} C_j(m_b, m_c, w, \mu)\, J_j(\mu) + O(1/m_Q) \,, \tag{3.126}$$

with evolution coefficients

$$C_j(m_b, m_c, w, \mu) = \sum_{k=0}^{\infty} \left(\frac{m_c}{m_b}\right)^k \sum_i D_i^{(k)}(m_b, m_c)\, E_{ij}^{(k)}(m_c, w, \mu) \,. \tag{3.127}$$

In this expression the matching scale m of the first approach does not appear. The coefficients depend either on $\alpha_s(m_b)$ or $\alpha_s(m_c)$, i.e., the scaling in the intermediate region $m_b > \mu > m_c$ is properly taken into account. To achieve this, however, it would be necessary to consider an infinite number of operators in the intermediate effective theory. This is, of course, not manageable. It is important to realize, however, that the short-distance coefficients in (3.122) and (3.126) must agree, and that this equality must hold order by order in an expansion in the mass ratio m_c/m_b. Using this fact, one can combine the two approaches into a consistent next-to-leading order calculation.

Following Ref. [86], let us now discuss the calculation of the Wilson coefficients for the flavor-changing vector and axial vector currents following the first approach described above. As in the previous section, we will denote quantities referring to the axial vector current by a superscript 5. The only difference with respect to the calculation for the flavor-conserving currents is that the QCD vertex functions depend on two (instead of a single) heavy-quark masses. Accordingly, eq. (3.103) is replaced by

$$\Gamma^\mu_{\rm QCD} = \left(1 + \frac{\alpha_s}{\pi}\{\ln(m_b/m_c) - \tfrac{1}{4}\gamma_0^{hh}(w)\ln(m_c/\lambda) + \tfrac{2}{3}[f(w) \pm r(w) + g(z,w)]\}\right)\gamma^\mu$$
$$- (2\alpha_s/3\pi)[2r(w) \mp 1 + h_1^{(5)}(z,w)]v^\mu \mp (2\alpha_s/3\pi)h_2^{(5)}(z,w)v'^\mu, \qquad (3.128)$$

where we use a short-hand notation meaning that upper signs and no superscripts refer to the vector current, whereas lower signs and superscripts "5" refer to the axial vector current. $\gamma_0^{hh}(w)$ is the one-loop anomalous dimension of the flavor-conserving current given in (3.115), $z = m_c/m_b$ denotes the ratio of the heavy-quark masses, and the functions $r(w)$ and $f(w)$ have been defined in (3.104) and (3.117). The new functions $g(z,w)$ and $h_i(z,w)$ are such that they vanish in the limit $z \to 0$, the leading terms being of order $z\ln z$. Their analytic expressions are rather lengthy [85,86],

$$g(z,w) = w(w^2-1)^{-1/2}[L_2(1-zw_-) - L_2(1-zw_+)]$$
$$- z(1-2wz+z^2)^{-1}[(w^2-1)r(w) + (w-z)\ln z], \qquad (3.129)$$
$$h_2^{(5)}(z,w) = z(1-2wz+z^2)^{-2}\{2(w\mp 1)z(1\pm z)\ln z$$
$$- [(w\pm 1) - 2w(2w\pm 1)z + (5w \pm 2w^2 \mp 1)z^2 - 2z^3]r(w)\}$$
$$- z(1-2wz+z^2)^{-1}[\ln z - 1 \pm z],$$

where $w_\pm = w \pm (w^2-1)^{-1/2}$. The function $h_1^{(5)}$ is related to $h_2^{(5)}$ by

$$h_1^{(5)}(z,w) = h_2^{(5)}(z^{-1},w) - 2r(w) \pm 1. \qquad (3.130)$$

From now on we will simplify the notation further by omitting the argument w; i.e., we will write $r(w) = r$, $g(z,w) = g(z)$ etc.

Since matrix elements in the effective theory are independent of the heavy-quark masses, the one-loop vertex functions in HQET are the same as in the case of the flavor-conserving currents. The solution of the renormalization group equation proceeds exactly as described in section 3.7. We first display the next-to-leading order solution for the Wilson coefficient $C_1^{(5)}$:

$$C_1^{(5)}(\mu) = \left(\frac{\alpha_s(m)}{\alpha_s(\mu)}\right)^{a_{hh}}\left(1 - \frac{\alpha_s(\mu)}{\pi}Z_{hh}\right)$$
$$\times\left(1 + \frac{\alpha_s(m)}{\pi}\{\ln(m_b/m_c) + \tfrac{1}{4}\gamma_0^{hh}\ln(m/m_c) + Z_{hh} + \tfrac{2}{3}[f \pm r + g(z)]\}\right). \qquad (3.131)$$

The velocity dependent coefficients a_{hh} and Z_{hh} have been given in (3.119). The result seems to depend strongly on the arbitrary matching scale m. However, changes of m in the leading anomalous scaling factor are compensated by changes in the logarithmic term in parenthesis. This shows that, in fact, the choice of m is irrelevant in a next-to-leading order calculation. We are free to set $m = m_c$, which removes the $\ln(m/m_c)$ term. Adopting this choice, one obtains for the coefficient functions [86]

$$C_1^{(5)}(\mu) = \left(\frac{\alpha_s(m_c)}{\alpha_s(\mu)}\right)^{a_{hh}} \left(1 - \frac{\alpha_s(\mu)}{\pi} Z_{hh}\right)$$

$$\times \left(1 + \frac{\alpha_s(m)}{\pi}\{\ln(m_b/m_c) + Z_{hh} + \tfrac{2}{3}[f \pm r + g(z)]\}\right),$$

$$C_2^{(5)}(\mu) = -\left(\frac{\alpha_s(m_c)}{\alpha_s(\mu)}\right)^{a_{hh}} \frac{2\alpha_s(m)}{3\pi}[2r \mp 1 + h_1^{(5)}(z)],$$

$$C_3^{(5)}(\mu) = \mp\left(\frac{\alpha_s(m_c)}{\alpha_s(\mu)}\right)^{a_{hh}} \frac{2\alpha_s(m)}{3\pi} h_2^{(5)}(z). \tag{3.132}$$

The scale in the next-to-leading corrections is not determined at this order, however. Here m could be any combination of m_b and m_c (but not μ). For the actual values of the heavy-quark masses this scale ambiguity is somewhat of a problem, since $\alpha_s(m_b) \approx 0.20$ and $\alpha_s(m_c) \approx 0.32$ are quite different. A related problem is the appearance of a so-called "hybrid" logarithm $\alpha_s(m)\ln(m_b/m_c)$ in $C_1^{(5)}$. Asymptotically, at high orders in perturbation theory, such logarithms enter in the form

$$\left(\tfrac{1}{2}\beta_0 \frac{\alpha_s(m)}{\pi}\ln(m_b/m_c)\right)^n \approx \left(\frac{\ln(m_b/m_c)}{\ln(m/\Lambda)}\right)^n. \tag{3.133}$$

They can be important even for large n.

Both problems can be solved by summing the leading and subleading logarithms $(\alpha_s \ln z)^n$ and $\alpha_s(\alpha_s \ln z)^n$, where $z = m_c/m_b$, to all orders in perturbation theory. This requires to follow the second approach, in which one introduces an intermediate effective theory with a static bottom quark and a dynamical charm quark. By means of the renormalization group equation in this theory one can sum logarithms of m_b/μ in the intermediate region $m_b > \mu > m_c$, which is precisely what one needs. Unfortunately, as we have seen, it becomes increasingly difficult to keep track of higher-order terms in the mass ratio z. The following observation helps [86]: For any value of z, the product $|z \ln z| \le e^{-1} \approx 0.37$, such that $\frac{1}{2}\beta_0 z \ln z$ is a number of order unity, but can never become large. This implies that for a term of order z^k in (3.127) the first k powers of the large hybrid logarithms appearing at n-loop order in perturbation theory are effectively compensated by the prefactor z^k:

$$z^k\left(\frac{\beta_0}{2}\frac{\alpha_s}{\pi}\ln z\right)^n = \left(\frac{\beta_0}{2} z \ln z\right)^k \left(\frac{\alpha_s}{\pi}\right)^k \left(\frac{\beta_0}{2}\frac{\alpha_s}{\pi}\ln z\right)^{n-k}. \tag{3.134}$$

Consequently, a leading-log summation for the terms of order z leads to the same accuracy as a next-to-leading-log summation for the terms of order z^0. Similarly, a next-to-leading-log summation for the terms of order z, or a leading-log summation for the terms of order z^2, would lead to the same accuracy as a next-to-next-to-leading-log summation for the terms of order z^0. Based on this discussion, the strategy for a consistent next-to-leading order calculation is as follows [86]: At $\mu = m_c$, the Wilson coefficients C_j in (3.132) are expanded in powers of z: $C_j = \sum_{k=0}^{\infty} z^k C_j^{(k)}$. The expansion coefficients can be related to the Wilson coefficients of the intermediate effective theory by means of (3.127):

$$C_j^{(k)}(m_b, \mu = m_c) = \sum_i D_i^{(k)}(m_b, m_c)\, E_{ij}^{(k)}(\mu = m_c)\,; \qquad k \geq 0\,. \tag{3.135}$$

The coefficients $D_i^{(k)}(m_b, m_c)$ obey a renormalization group equation in the intermediate theory. For $k = 0$ this equation has to be solved to next-to-leading order, whereas for $k = 1$ a leading-log solution is sufficient. No renormalization group improvement is necessary for $k \geq 2$.

Let us outline the various steps in this program. As mentioned above, for $k = 0$ the coefficients D_i are equal to the coefficients appearing in the expansion of the heavy–light currents discussed in section 3.4. This is plausible since in the intermediate theory the charm quark is a light quark, although it is not a massless quark. But the matching is independent of the external states, so the quark mass does not play any role. Evaluating (3.69) for $m_Q = m_b$ and $\mu = m_c$, one obtains at next-to-leading order [for simplicity we omit the superscript (0)]

$$\begin{aligned} D_1^{(5)}(m_b, m_c) &= \left(\frac{\alpha_s(m_b)}{\alpha_s(m_c)}\right)^{-6/25} \left(1 + \frac{\alpha_s(m_b)}{\pi} Z_4 - \frac{\alpha_s(m_c)}{\pi}(Z_4 + \tfrac{4}{3})\right), \\ D_2^{(5)}(m_b, m_c) &= \pm \left(\frac{\alpha_s(m_b)}{\alpha_s(m_c)}\right)^{-6/25} \frac{2\alpha_s(m_b)}{3\pi}, \end{aligned} \tag{3.136}$$

where we have taken into account that the number of light quarks in the intermediate effective theory is $n_f = 4$. From (3.70) it follows that

$$Z_4 \equiv Z_{hl}(n_f = 4) = -\tfrac{9403}{7500} - \tfrac{7}{225}\pi^2 \approx -1.5608. \tag{3.137}$$

The coefficients $E_{ij}(\mu = m_c)$ follow from a one-loop comparison of the diagrams shown in Fig. 3.5. One obtains [86]

$$\begin{aligned} &E_{11}^{(5)}(\mu = m_c) = 1 + [2\alpha_s(m_c)/3\pi](2 + f \pm r), \\ &E_{12}^{(5)}(\mu = m_c) = -[4\alpha_s(m_c)/3\pi]\, r, \qquad E_{13}^{(5)}(\mu = m_c) = 0. \end{aligned} \tag{3.138}$$

Since D_2 in (3.136) is already of order α_s, we only need E_{2j} at tree level,

$$E_{2j}^{(5)}(\mu = m_c) = \delta_{j2} + O(\alpha_s)\,. \tag{3.139}$$

Combining these results, we obtain the renormalization-group improved version of the terms of order z^0 contained in the Wilson coefficients in (3.132), at $\mu = m_c$. For $C_1^{(5)}$ the replacement is

$$\begin{aligned} &1 + [\alpha_s(m)/\pi][\ln(m_b/m_c) + \tfrac{2}{3}(f \pm r)] \\ &\to \left(\frac{\alpha_s(m_b)}{\alpha_s(m_c)}\right)^{-6/25} \left(1 + \frac{\alpha_s(m_b) - \alpha_s(m_c)}{\pi} Z_4 + \frac{2\alpha_s(m_c)}{3\pi}(f \pm r)\right), \end{aligned} \tag{3.140}$$

and for $C_2^{(5)}$ one has

$$-\frac{2\alpha_s(m)}{3\pi}(2r \mp 1) \to -\left(\frac{\alpha_s(m_b)}{\alpha_s(m_c)}\right)^{-6/25} \left(\frac{4\alpha_s(m_c)}{3\pi} r \mp \frac{2\alpha_s(m_b)}{3\pi}\right). \tag{3.141}$$

There are no terms of order z^0 in $C_3^{(5)}$.

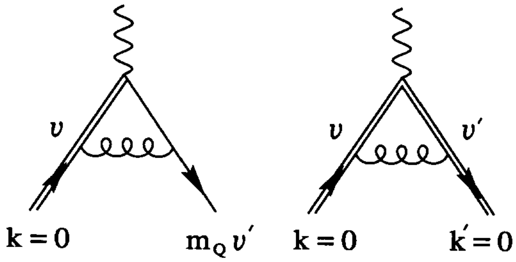


Fig. 3.5. Diagrams arising in the matching calculation of the Wilson coefficients E_{ij} and E_{ij}^5.

Consider now the terms of order z in (3.132). The aim is to sum the "subleading" logarithms $z \ln z$ contained in the functions $g(z)$ and $h_i^{(5)}(z)$ to all orders in perturbation theory. This is accomplished by calculating the coefficients $D_i^{(1)}$ in (3.123) in leading logarithmic approximation. Once again, there is a close analogy to the calculation of the coefficients of the dimension-four operators O_j for heavy–light currents, which was discussed in section 3.6. In the present case, however, we need to consider only those operators in (3.82) which lead to matrix elements proportional to the charm quark mass (substitute $\bar{c}$ for $\bar{q}$). The operators O_4 to O_6 are of this type, since a derivative acting on the charm quark field gives $m_c v'^\mu + \dots$. But in addition there are two operators not considered in section 3.6, where we have set $m_q = 0$ for simplicity. They are $O_7 = m_c\, J_1$ and $O_8 = m_c\, J_2$. Falk and Grinstein have calculated the coefficients of these operators in leading logarithmic approximation, as well as the coefficients $E_{ij}^{(1)}$ and $E_{ij}^{5\,(1)}$ from tree level matching [84]. Their results can be used to replace the terms of order $z \ln z$ in (3.132) by leading-log improved functions $z\, S_i(x)$, where $x = \alpha_s(m_c)/\alpha_s(m_b)$. Putting everything together, we obtain the short-distance coefficients in the factorized form

$$C_i^{(5)}(m_b, m_c, w, \mu) = \widehat{C}_i^{(5)}(m_b, m_c, w) K_{hh}(w, \mu), \tag{3.142}$$

where $K_{hh}(w, \mu)$ is the universal μ-dependent function defined in (3.120). The renormalization-group invariant coefficients $\widehat{C}_i^{(5)}$ are given by [86]

$$\begin{aligned}
\widehat{C}_1^{(5)} &= A\bigg(1 + \frac{\alpha_s(m_b) - \alpha_s(m_c)}{\pi} Z_4 + \frac{\alpha_s(m_c)}{\pi}[Z_{hh} + \tfrac{2}{3}(f \pm r)] \\
&\qquad + z S_1^{(5)}(x) + \frac{2\alpha_s(m)}{3\pi} G(z)\bigg), \\
\widehat{C}_2^{(5)} &= A\left(\pm\frac{2\alpha_s(m_b)}{3\pi} - \frac{4\alpha_s(m_c)}{3\pi}\, r + z\, S_2^{(5)}(x) - \frac{2\alpha_s(m)}{3\pi}\, H_1^{(5)}(z)\right), \\
\widehat{C}_3^{(5)} &= \mp A\{z S_3^{(5)}(x) + [2\alpha_s(m)/3\pi]\, H_2^{(5)}(z)\},
\end{aligned} \tag{3.143}$$

where

$$A = \left(\frac{\alpha_s(m_c)}{\alpha_s(m_b)}\right)^{6/25} [\alpha_s(m_c)]^{a_{hh}} \tag{3.144}$$

is the leading-logarithmic scaling factor. The subleading logarithms are taken into account by the functions [84]

$$
\begin{aligned}
S_1^{(5)}(x) &= w\left(\tfrac{17}{27} - \tfrac{5}{9}x^{-6/25} - \tfrac{2}{27}x^{-9/25} + \tfrac{8}{25}\ln x\right) \pm \left(\tfrac{1}{6} - \tfrac{5}{9}x^{-6/25} + \tfrac{4}{9}x^{-9/25} - \tfrac{1}{18}x^{-12/25}\right), \\
S_2^{(5)}(x) &= \mp w\left(\tfrac{14}{27} + \tfrac{10}{9}x^{-6/25} - \tfrac{44}{27}x^{-9/25}\right) + \left(\tfrac{2}{3} + \tfrac{5}{9}x^{-6/25} + \tfrac{2}{9}x^{-9/25} - \tfrac{13}{9}x^{-12/25}\right), \\
S_3^{(5)}(x) &= 1 - \tfrac{5}{3}x^{-6/25} + \tfrac{2}{3}x^{-9/25}.
\end{aligned}
\tag{3.145}
$$

Finally, in (3.143) we have introduced new functions

$$
\begin{aligned}
G(z,w) &= g(z,w) + 3wz\,\ln z, \\
H_1^{(5)}(z,w) &= h_1^{(5)}(z,w) - (3 \mp 2w)z\,\ln z, \\
H_2^{(5)}(z,w) &= h_2^{(5)}(z,w) + z\,\ln z,
\end{aligned}
\tag{3.146}
$$

the leading terms of which (for small z) are of order z or $z^2 \ln z$, but not of order $z \ln z$.

Eq. (3.143) is the final result for the Wilson coefficients at next-to-leading order in renormalization-group improved perturbation theory. The residual scale ambiguity arising from the next-to-leading terms proportional to $\alpha_s(m)$ is of order $\alpha_s^2(z\ln z)$ or $\alpha_s^2(z\ln z)^2$. This is of the same order as next-to-next-to-leading corrections of order α_s^2, which would require a three-loop calculation.

3.9. Power corrections to heavy-quark currents

Let us finally discuss the operator product expansion of bilinear heavy-quark currents at order $1/m_Q$. At this order one has to include a complete set of dimension-four operators O_i in (3.126):

$$
J^\mu \cong \sum_i C_i(w)\,J_i + \sum_j \left(\frac{B_j(w)}{2m_b} + \frac{B_j'(w)}{2m_c}\right) O_j + \text{nonlocal operators} + O(1/m_Q^2). \tag{3.147}
$$

For simplicity we only display the dependence of the Wilson coefficients on $w = v \cdot v'$ explicitly, since this will become important in what follows. We also do not show the nonlocal operators, which contain time-ordered products of one of the dimension-three currents J_i with a $1/m_b$ or $1/m_c$ insertion from the effective Lagrangian. As in the case of the heavy–light currents discussed in section 3.6, the coefficients of these nonlocal terms are simply the products of the coefficients of the component operators. Since in the present case the nonlocal operators do not mix into the local ones, it suffices to focus on the local operators O_i. For the vector and axial vector currents there are fourteen independent class I operators of dimension four. A convenient basis for the vector current operators is [90]

$$
\begin{aligned}
O_1 &= \bar h^{\rm c}_{v'}\, \gamma^\mu\, {\rm i}\not{D}\, h^{\rm b}_v\,, & O_8 &= -\bar h^{\rm c}_{v'}\, {\rm i}\overleftarrow{\not{D}}\, \gamma^\mu\, h^{\rm b}_v\,, \\
O_2 &= \bar h^{\rm c}_{v'}\, v^\mu\, {\rm i}\not{D}\, h^{\rm b}_v\,, & O_9 &= -\bar h^{\rm c}_{v'}\, {\rm i}\overleftarrow{\not{D}}\, v^\mu\, h^{\rm b}_v\,, \\
O_3 &= \bar h^{\rm c}_{v'}\, v'^\mu\, {\rm i}\not{D}\, h^{\rm b}_v\,, & O_{10} &= -\bar h^{\rm c}_{v'}\, {\rm i}\overleftarrow{\not{D}}\, v'^\mu\, h^{\rm b}_v\,, \\
O_4 &= \bar h^{\rm c}_{v'}\, {\rm i}D^\mu\, h^{\rm b}_v\,, & O_{11} &= -\bar h^{\rm c}_{v'}\, {\rm i}\overleftarrow{D^\mu}\, h^{\rm b}_v\,, \\
O_5 &= \bar h^{\rm c}_{v'}\, \gamma^\mu\, {\rm i}v'\cdot D\, h^{\rm b}_v\,, & O_{12} &= -\bar h^{\rm c}_{v'}\, {\rm i}v\cdot \overleftarrow{D}\, \gamma^\mu\, h^{\rm b}_v\,, \\
O_6 &= \bar h^{\rm c}_{v'}\, v^\mu\, {\rm i}v'\cdot D\, h^{\rm b}_v\,, & O_{13} &= -\bar h^{\rm c}_{v'}\, {\rm i}v\cdot \overleftarrow{D}\, v^\mu\, h^{\rm b}_v\,, \\
O_7 &= \bar h^{\rm c}_{v'}\, v'^\mu\, {\rm i}v'\cdot D\, h^{\rm b}_v\,, & O_{14} &= -\bar h^{\rm c}_{v'}\, {\rm i}v\cdot \overleftarrow{D}\, v'^\mu\, h^{\rm b}_v\,.
\end{aligned}
\tag{3.148}
$$

A similar set of operators can be constructed for the expansion of the axial vector current $A^\mu = \bar c\,\gamma^\mu\gamma_5\, b$.

As in the case of heavy–light currents, reparameterization invariance imposes restrictive relations between the coefficients of the dimension-three and four operators in (3.147). Since in the case of bilinear heavy-quark currents any local dimension-four operator in HQET must contain a covariant derivative acting on one of the heavy quark fields, none of the operators O_i is reparameterization invariant by itself. There exists a unique combination of the operators O_i with the dimension-three operators J_i which is reparameterization-invariant. It is thus possible to relate all coefficients B_j to the leading-order coefficients C_i.

Let us recall that reparameterization invariance requires that the heavy-quark velocity v always appear in a certain combination with the covariant derivative, which was denoted by $\mathcal{V}$ in (2.38). Furthermore, the heavy-quark spinor fields must appear in the form of $\widetilde{h}_v$ given in (2.42). A subtlety specific for heavy-quark currents is that not only the effective current operators, but also the velocity dependent coefficient functions have to be written in a reparameterization invariant way. This means that the variable $w = v\cdot v'$ which these functions depend on has to be replaced by the reparametrization invariant operator [90]

$$
\widehat{w} = \mathcal{V}'^\dagger\cdot\mathcal{V} = (v' - {\rm i}\overleftarrow{D}/m_{\rm c})\cdot(v + {\rm i}D/m_{\rm b})\,,
\tag{3.149}
$$

where it is understood that ${\rm i}D$ acts on $h^{\rm b}_v$, whereas ${\rm i}\overleftarrow{D}$ acts on $\bar h^{\rm c}_{v'}$. Hence the correct extension of the leading-order vector current is [15]

$$
V^\mu \stackrel{\rm RPI}{\to} \bar{\widetilde{h}}{}^{\rm c}_{v'}[C_1(\widehat{w})\,\gamma^\mu + C_2(\widehat{w})\,\mathcal{V}^\mu + C_3(\widehat{w})\,\mathcal{V}'^{\dagger\mu}]\widetilde{h}^{\rm b}_v.
\tag{3.150}
$$

Rewriting this in terms of the local operators J_i and O_j, and using the expansion

$$
C_i(\widehat{w}) = C_i(w) + \frac{\partial C_i(w)}{\partial w}\left(\frac{{\rm i}v'\cdot D}{m_{\rm b}} - \frac{{\rm i}v\cdot\overleftarrow{D}}{m_{\rm c}}\right) + {\rm O}(1/m_{\rm Q}^2)\,,
\tag{3.151}
$$

one can relate the coefficients B_j to the leading-order coefficients C_i. The result is [90]:

$$
B_1(w) = B_8'(w) = C_1(w)\,, \qquad B_2(w) = \tfrac{1}{2}B_4(w) = B_9'(w) = C_2(w)\,,
$$

$$
B_3(w) = B_{10}'(w) = \tfrac{1}{2}B_{11}'(w) = C_3(w)\,,
$$

[15] For the axial vector current one inserts γ_5 between the parentheses and $\widetilde{h}^{\rm b}_v$, and replaces the coefficients $C_i(\widehat{w})$ by $C_i^5(\widehat{w})$.

$$B_5(w) = B'_{12}(w) = 2\partial C_1(w)/\partial w, \qquad B_6(w) = B'_{13}(w) = 2\partial C_2(w)/\partial w,$$
$$B_7(w) = B'_{14}(w) = 2\partial C_3(w)/\partial w,$$
$$B_j(w) = 0, \quad j = 8, \dots, 14; \qquad B'_j(w) = 0, \quad j = 1, \dots, 7. \tag{3.152}$$

These relations are valid to all orders in perturbation theory. Of course, similar relations can be derived for any other current built of two heavy-quark fields.

At order $1/m_Q$, the effect of the operator $\widehat{w}$ in the short-distance coefficients can be explicitly evaluated at the level of matrix elements. The equation of motion allows one to replace the covariant derivatives in (3.151) by total derivatives acting on the currents. By means of (2.29), their matrix elements can be expressed in terms of the mass parameter $\bar{\Lambda}$. The total effect is that the reparameterization invariant operator $\widehat{w}$ can be replaced by a new variable [90]

$$\bar{w} = w + (\bar{\Lambda}/m_c + \bar{\Lambda}/m_b)(w - 1), \tag{3.153}$$

which is different from the velocity transfer w of the hadrons. This variable can be interpreted as the velocity transfer of free quarks. Consider the weak transition $H_b \to H_c + W^-$, where H_Q are hadrons containing the heavy quarks. Before the decay, the bottom quark in the initial state H_b moves on average with the hadron's velocity. When the W boson is emitted, the outgoing charm quark has in general some different velocity. Let us denote the product of these velocities by $\bar{w}$. Over short time scales the quark velocities remain unchanged. This is what is "seen" by hard gluons. After the W emission, however, the light degrees of freedom still have the initial hadron's velocity. But they have to combine with the outgoing heavy quark to form the final state H_c. Thus, a rearrangement is necessary, which happens over much larger, hadronic time scales by the exchange of soft gluons. In this process the velocity of the charm quark is changed by an amount of order $1/m_Q$ (its momentum is changed by an amount of order Λ_{QCD}). Hence the final hadron velocity transfer w will differ from the "short-distance" quark velocity transfer $\bar{w}$ by an amount of order $1/m_Q$. The precise relation between w and $\bar{w}$ is determined by momentum conservation and is given in (3.153). In fact, this relation is nothing but the condition $(p_{H_b} - p_{H_c})^2 = (p_b - p_c)^2$, i.e., that the momentum transfer to the hadrons equals the momentum transfer to free heavy quarks. At zero recoil, no rearrangement is needed, and indeed $w = \bar{w} = 1$ in this limit.

When the variable $\bar{w}$ is used as the argument in the coefficient functions, the operators O_5 to O_7 and O_{12} to O_{14} no longer appear in the expansion (3.147) since, for instance,

$$C_1(w)\, J_1 + 2\,\frac{\partial C_1(w)}{\partial w}\left(\frac{O_5}{2m_b} + \frac{O_{12}}{2m_c}\right) \cong C_1(\bar{w})\, J_1 + O(1/m_Q^2)\,. \tag{3.154}$$

This leads us to the final result for the vector current [90],

$$V^\mu \cong C_1(\bar{w})\left(J_1 + \frac{O_1}{2m_b} + \frac{O_8}{2m_c}\right) + C_2(\bar{w})\left(J_2 + \frac{O_2 + 2O_4}{2m_b} + \frac{O_9}{2m_c}\right)$$
$$+ C_3(\bar{w})\left(J_3 + \frac{O_3}{2m_b} + \frac{O_{10} + 2O_{11}}{2m_c}\right) + \cdots, \tag{3.155}$$

where the ellipsis represents nonlocal operators and terms of order $1/m_Q^2$. The generalization to other currents is straightforward. By means of this remarkable relation, the operator product expansion at order $1/m_Q$ is known with the same accuracy as the expansion to leading order.

3.10. Summary and numerical results

At the end of this lengthy chapter, let us briefly summarize the main results. The inclusion of short-distance corrections completes the construction of the effective Lagrangian of HQET and the operator product expansion of currents in the effective theory. We have given a comprehensive description of the results known so far at leading and subleading order in the $1/m_Q$ expansion, and at next-to-leading order in perturbation theory. These results build the basis for the applications of heavy-quark symmetry to be discussed in the following chapters.

The flavor-changing weak current $\bar{c}\gamma^\mu(1-\gamma_5)b$ will play a most important role, since it mediates transitions between B and $D^{(*)}$ mesons, which can be used to measure the element V_{cb} of the Cabibbo–Kobayashi–Maskawa matrix. Since the analytical expressions for the corresponding Wilson coefficients in (3.143) are rather complicated, we present in Tab. 3.1 the numerical values of these coefficients as a function of $\bar{w}$. The maximum quark velocity transfer is $\bar{w}_{max} \approx 1.8$. It can be related to the hadron velocity transfer w by means of (3.153). Throughout this work we shall use $m_b = 4.80$ GeV and $m_c = 1.45$ GeV for the heavy-quark masses. These are obtained by taking the spin-averaged meson masses $\overline{m}_M = \frac{1}{4}(3m_V + m_P)$ and subtracting 0.5 GeV, which we assume as a reasonable estimate for $\bar{\Lambda}$. For the scale m in the subleading corrections we take the geometric average $m \approx 2.23$ GeV. More uncertain is the value of the QCD scale parameter. Whereas high energy data, in particular the measurements of $\alpha_s(m_Z)$ at LEP, are best fit by using a large value $\Lambda_{\overline{MS}} \approx 0.22$ GeV for $n_f = 5$ quark flavors (corresponding to $\Lambda_{\overline{MS}} \approx 0.32$ GeV for $n_f = 4$), low energy data favor a smaller value of the scale parameter [91]. They are not incompatible with a recent study of the mass splitting between the 1s and 1p states in charmonium using lattice gauge theory, which predicts that $\Lambda_{\overline{MS}} \approx 160$ MeV for $n_f = 4$ [92]. One has to keep in mind, however, that the running coupling constant depends only logarithmically on the scale parameter, so that a large uncertainty in $\Lambda_{\overline{MS}}$ does not imply a large uncertainty in the physical coupling constants. In the numerical analysis we use $\Lambda_{\overline{MS}} = 0.25$ GeV for the scale parameter in the two-loop expression (3.18) for $\alpha_s(\mu)$ in the region between the bottom and the charm quark masses, where $n_f = 4$. The corresponding coupling constants are

$$\alpha_s(m_b) \approx 0.20, \qquad \alpha_s(m_c) \approx 0.32, \qquad \alpha_s(m) \approx 0.26. \tag{3.156}$$

They agree well with directs measurements of $\alpha_s(\mu)$ at low energies, which have an experimental uncertainty of about 10% [91]. More extensive tables of the coefficient functions obtained for various sets of input parameters can be found in Ref. [86].

Of particular interest are the values of the short-distance coefficients at zero recoil, since they determine the radiative corrections to the normalization of the Isgur–Wise function. The analytic expressions resulting in the limit $w = \bar{w} = 1$ can be found in Ref. [86]. Here we shall focus on two quantities which will play an important role in the further course of this review. We define

$$\eta_V = \sum_{i=1}^{3} C_i(\bar{w}=1), \qquad \eta_A = C_1^5(\bar{w}=1), \tag{3.157}$$

Table 3.1
Short-distance coefficients for b $\to$ c transitions.

$\bar{w}$	$\widehat{C}_1$	$\widehat{C}_2$	$\widehat{C}_3$	$\widehat{C}_1^5$	$\widehat{C}_2^5$	$\widehat{C}_3^5$
1.0	1.136	−0.085	−0.021	0.985	−0.122	0.042
1.1	1.107	−0.080	−0.021	0.965	−0.115	0.040
1.2	1.081	−0.077	−0.020	0.946	−0.109	0.038
1.3	1.056	−0.073	−0.019	0.927	−0.103	0.036
1.4	1.033	−0.070	−0.018	0.910	−0.098	0.035
1.5	1.011	−0.067	−0.018	0.894	−0.094	0.033
1.6	0.991	−0.064	−0.017	0.878	−0.089	0.032
1.7	0.972	−0.062	−0.017	0.864	−0.086	0.031
1.8	0.953	−0.059	−0.016	0.850	−0.082	0.030

and use the next-to-leading order expressions for the Wilson coefficients to obtain

$$\eta_V = \eta_A[1 + 2\alpha_s(m_b)/3\pi - \tfrac{14}{27}z(1 - 4x^{-9/25} + 3x^{-12/25})],$$

$$\eta_A = x^{6/25}\bigg(1 + \frac{\alpha_s(m_b) - \alpha_s(m_c)}{\pi} Z_4 - \frac{8\alpha_s(m_c)}{3\pi} + z(\tfrac{25}{54} - \tfrac{14}{27}x^{-9/25} + \tfrac{1}{18}x^{-12/25} + \tfrac{8}{25}\ln x) - \frac{2\alpha_s(m)}{\pi}\frac{z^2}{1-z}\ln z\bigg), \tag{3.158}$$

where $x = \alpha_s(m_c)/\alpha_s(m_b)$, and $Z_4 \approx -1.561$ from (3.137). These quantities are very stable under changes of the input parameters. For $\Lambda_{\overline{\rm MS}} = 0.25 \pm 0.05$ GeV and $z = 0.30 \pm 0.05$ one finds

$$\eta_V = 1.025 \pm 0.006, \qquad \eta_A = 0.986 \pm 0.006. \tag{3.159}$$

It is instructive to compare this, in retrospective, to different approximations for the short-distance coefficients often used in the literature. A very fashionable one is the leading logarithmic approximation [5,7]. It predicts that $\eta_V = \eta_A = x^{6/25} \approx 1.12$, far off the next-to-leading results shown above. This failure is not unexpected, of course, since $\ln(m_b/m_c) \approx 1.2$ is not a particularly large number. When one tries to improve this by including the subleading logarithms of order $z \ln z$ [84], one obtains $\eta_V \approx \eta_A \approx 1.19$, which is even worse. This clearly demonstrates the necessity to include all next-to-leading order corrections. On the other hand, a simple one-loop calculation which takes into account all corrections of order $\alpha_s(m)$, but without a renormalization group improvement, gives $\eta_V \approx 1.02$ and $\eta_A \approx 0.96$ [29,79], in fairly good agreement with (3.159).

For later purpose, it will be necessary to know the coefficients η_V and η_A in the limit of equal heavy-quark masses. They are

$$\eta_V = 1, \qquad \eta_A = 1 - 2\alpha_s(m_Q)/3\pi \qquad (\text{for} \quad z = 1). \tag{3.160}$$

In this limit the vector current is not renormalized.

4. Hadronic matrix elements

4.1. Covariant representation of states

The purpose of the operator product expansion of currents discussed in the previous chapters was to disentangle the short-distance physics related to length scales set by the Compton wavelengths of the heavy quarks from confinement effects relevant at large distances. This procedure makes explicit the m_Q-dependence of any Green's function of the full theory which contains one or more heavy-quark fields. Hadronic matrix elements of heavy-quark currents have a $1/m_Q$ expansion as shown in (3.1). The HQET matrix elements in this expansion contain the long-distance physics associated with the interactions of the cloud of light degrees of freedom among themselves and with the background color field provided by the heavy quarks. These hadronic quantities depend in a most complicated way on the "brown muck" quantum numbers of the external states, the quantum numbers of the current, and on the heavy-quark velocities. They are related to matrix elements of effective current operators in HQET. Recall from section 2.3 that these matrix elements are independent of the heavy-quark masses, if the states of the effective theory are taken to be the eigenstates of the leading-order Lagrangian $\mathcal{L}_{\rm HQET}$ in (2.22). Matrix elements evaluated using these states have a well defined behavior under spin–flavor symmetry transformations. When combined with the requirements of Lorentz covariance, restrictive constraints on their structure can be derived. We have already encountered an example of this in chapter 1, where heavy-quark symmetry and Lorentz covariance could be used to reduce a set of hadronic form factors to a single universal Isgur–Wise function. The way these relations were derived was somewhat cumbersome, however. We will now introduce a more elegant formalism, which allows one to derive the general form of matrix elements in a straightforward manner. The clue is to work with a covariant tensor representation of states with definite transformation properties under the Lorentz group and the heavy-quark spin–flavor symmetry [7,93,94].

The eigenstates of HQET can be thought of as the "would-be hadrons" built from an infinitely heavy quark dressed with light quarks, antiquarks and gluons. In such a state both the heavy quark and the cloud of light degrees of freedom have well defined transformation properties under the Lorentz group. The heavy quark can be represented by a spinor $u_{\rm h}(v,s)$ satisfying

$$\not{v}\,u_{\rm h}(v,s) = u_{\rm h}(v,s)\,, \tag{4.1}$$

and we identify the velocity v with that of the hadron. Because of heavy-quark symmetry the wave function of the state (when properly normalized) is independent of the flavor and spin of the heavy quark, and the states can be characterized by the quantum numbers of the "brown muck". In particular, for each configuration of light degrees of freedom with total angular momentum $j \ge 0$ and parity P there is a degenerate doublet of states with spin-parity $J^P = (j \pm \frac{1}{2})^P$. Following Falk [94], we discuss the cases of integral and half integral j separately. Hadronic states with integral j have odd fermion number and correspond to the baryons; states with half-integral j have even fermion number and correspond to mesons.

First consider the baryons. In this case the "brown muck" is an object with spin-parity j^P that can be represented by a totally symmetric, traceless tensor $A^{\mu_1\cdots\mu_j}$ subject to the transversality condition

$$v_{\mu_1} A^{\mu_1\cdots\mu_j} = 0. \tag{4.2}$$

States are said to have "natural" parity if $P = (-1)^j$, and "unnatural" parity otherwise. The composite heavy baryon can be represented by the tensor wave function

$$\psi^{\mu_1\cdots\mu_j} = u_{\rm h} A^{\mu_1\cdots\mu_j}. \tag{4.3}$$

Under a connected Lorentz transformation Λ, this object transforms as a spinor-tensor field

$$\psi^{\mu_1\cdots\mu_j} \to \Lambda^{\mu_1}_{\nu_1}\cdots\Lambda^{\mu_j}_{\nu_j} D(\Lambda)\psi^{\nu_1\cdots\nu_j}, \tag{4.4}$$

where $D(\Lambda) = \exp(-{\rm i}\frac{1}{4}\omega_{\mu\nu}\sigma^{\mu\nu})$ is the usual spinor representation of Λ. A heavy-quark spin rotation $\widetilde{\Lambda}$, on the other hand, acts only on $u_{\rm h}$; hence

$$\psi^{\mu_1\cdots\mu_j} \to D(\widetilde{\Lambda})\psi^{\mu_1\cdots\mu_j}. \tag{4.5}$$

Here $\widetilde{\Lambda}$ is restricted to spatial rotations (in the rest frame). The infinitesimal form of $D(\widetilde{\Lambda})$ was considered in (1.15).

The simplest but important case $j^P = 0^+$ corresponds to the ground-state $\Lambda_{\rm Q}$ baryon with total spin-parity $J^P = \frac{1}{2}^+$. It can be represented by a spinor u_Λ, on which we impose the mass-independent normalization condition

$$\bar u_\Lambda(v,r)\,u_\Lambda(v,s) = \delta_{rs}. \tag{4.6}$$

Since the light degrees of freedom are in a configuration of total spin zero, the spin of the baryon is carried by the heavy quark, and the spinor u_Λ coincides with the heavy-quark spinor. Hence:

$$\psi_\Lambda = u_\Lambda(v,s) = u_{\rm h}(v,s). \tag{4.7}$$

For $j \geq 0$, the object $\psi^{\mu_1\cdots\mu_j}$ does not transform irreducibly under the Lorentz group, but is a linear combination of two components with total spin $j \pm \frac{1}{2}$. These correspond to a degenerate doublet of physical states, which only differ in the orientation of the heavy-quark spin relative to the angular momentum of the light degrees of freedom. The nonrelativistic quark model suggests that one should identify the states with $j^P = 1^+$ with the $\Sigma_{\rm Q}$ ($J^P = \frac{1}{2}^+$) and $\Sigma_{\rm Q}^*$ ($J^P = \frac{3}{2}^+$) baryons. In the quark model these states contain a heavy quark and a light vector diquark with no orbital angular momentum. Unlike the $\Lambda_{\rm Q}$ baryons, they have unnatural parity. This implies that decays between $\Lambda_{\rm Q}$ and $\Sigma_{\rm Q}^{(*)}$ must be described by parity-odd form factors [95,96].

Next consider the heavy mesons. We shall only discuss the case $j^P = \frac{1}{2}^-$ in detail, and refer to Ref. [94] for the treatment of higher spins. Since quarks and antiquarks have opposite intrinsic parity, the corresponding physical states with "natural" parity are the ground-state pseudoscalar ($J^P = 0^-$) and vector ($J^P = 1^-$) mesons. As before, the heavy quark is represented by a spinor $u_{\rm h}(v,s)$ subject to the condition (4.1). The light degrees of freedom as a whole transform under the Lorentz group as an antiquark moving at velocity v. They are described by an antifermion spinor $\bar v_\ell(v,s')$ satisfying

$$\bar v_\ell(v,s')\,\rlap/v = -\bar v_\ell(v,s'). \tag{4.8}$$

The ground-state mesons can be represented by the composite object $\psi = u_{\rm h}\bar v_\ell$, which is a 4×4 Dirac matrix with two spinor indices, one for the heavy quark and one for the "brown muck". Under a connected Lorentz transformation Λ, the meson wave function ψ transforms as

$$\psi \to D(\Lambda)\psi D^{-1}(\Lambda), \tag{4.9}$$

whereas under a heavy-quark spin rotation $\tilde{\Lambda}$

$$\psi \to D(\tilde{\Lambda})\psi. \tag{4.10}$$

The composite ψ represents a linear combination of the physical pseudoscalar and vector meson states. It is easiest to identify these states in the rest frame, where u_h has only upper components, whereas $\bar{v}_\ell$ has only lower components. The nonvanishing components of ψ are thus contained in a 2×2 matrix, which can be written as a linear combination of the identity I and the Pauli matrices σ^i. Let us choose the quantization axis in 3-direction and work with the rest frame spinor basis

$$u_\mathrm{h}(\Uparrow) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad u_\mathrm{h}(\Downarrow) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad v_\ell(\uparrow) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad v_\ell(\downarrow) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}. \tag{4.11}$$

Then a basis of states is

$$\frac{1}{\sqrt{2}}(\Uparrow\downarrow + \Downarrow\uparrow) = -\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix}, \qquad \frac{1}{\sqrt{2}}(\Uparrow\downarrow - \Downarrow\uparrow) = -\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & \sigma^3 \\ 0 & 0 \end{pmatrix},$$

$$(\Uparrow\uparrow) = -\frac{1}{2}\begin{pmatrix} 0 & \sigma^1 + \mathrm{i}\sigma^2 \\ 0 & 0 \end{pmatrix}, \qquad (\Downarrow\downarrow) = -\frac{1}{2}\begin{pmatrix} 0 & \sigma^1 - \mathrm{i}\sigma^2 \\ 0 & 0 \end{pmatrix}. \tag{4.12}$$

Let us furthermore define two transverse polarization vectors $\epsilon_\pm$ and a longitudinal polarization vector ϵ_3 by

$$\epsilon_\pm^\mu = (1/\sqrt{2})(0, 1, \pm\mathrm{i}, 0), \qquad \epsilon_3^\mu = (0, 0, 0, 1). \tag{4.13}$$

We are then lead to identify the pseudoscalar (P) and vector (V) meson states as [7,94]

$$P(v=0) = -(1/\sqrt{2})\tfrac{1}{2}(1+\gamma^0)\gamma_5, \qquad V(v=0,\epsilon) = (1/\sqrt{2})\tfrac{1}{2}(1+\gamma^0)\not{\epsilon}. \tag{4.14}$$

The second state in (4.12) has longitudinal polarization, whereas the last two states have transverse polarization.

To get familiar with this representation, consider the action of the spin operator $\boldsymbol{\Sigma}$ on ψ. A matrix representation of the components $\boldsymbol{\Sigma}^i$ in the rest frame is $\Sigma^i = \frac{1}{2}\gamma_5\gamma^0\gamma^i$, and the action of the operator $\boldsymbol{\Sigma}^i$ on the meson wave function is $\boldsymbol{\Sigma}^i\psi = [\Sigma^i, \psi]$. Using this, we compute

$$\boldsymbol{\Sigma}^2 P = \boldsymbol{\Sigma}^3 P = 0, \qquad \boldsymbol{\Sigma}^2 V(\epsilon) = 2V(\epsilon),$$

$$\boldsymbol{\Sigma}^3 V(\epsilon_\pm) = \pm V(\epsilon_\pm), \qquad \boldsymbol{\Sigma}^3 V(\epsilon_3) = 0, \tag{4.15}$$

which shows that P has spin zero, and V has spin one. Next consider the action of the heavy-quark spin operator $\mathbf{S}$. It has the same matrix representation as $\boldsymbol{\Sigma}$, but only acts on the heavy-quark spinor in ψ: $\mathbf{S}^i\psi = S^i\psi$, with $S^i = \Sigma^i$. It follows that

$$S^3 P = \tfrac{1}{2}V(\epsilon_3), \qquad S^3 V(\epsilon_3) = \tfrac{1}{2}P, \qquad S^3 V(\epsilon_\pm) = \pm\tfrac{1}{2}V(\epsilon_\pm), \tag{4.16}$$

in accordance with the spin assignments in (4.12).

In a general frame, the tensor wave functions in (4.14) can be readily generalized in a Lorentz covariant way by replacing γ^0 with $\not{v}$. The polarization vector of a vector meson with velocity v satisfies $\epsilon \cdot v = 0$ and $\epsilon \cdot \epsilon^* = -1$, as well as

$$\sum_{\text{pol.}} \epsilon^\mu \epsilon^{*\nu} = v^\mu v^\nu - g^{\mu\nu}. \tag{4.17}$$

It can be written as a combination of three orthonormal vectors e^i that are orthogonal to v: $\epsilon^\mu = \boldsymbol{a} \cdot \boldsymbol{e}^\mu$. The generators of the spin symmetry are $S^i = \frac{1}{2}\gamma_5 \not{v} \not{e}^i$ [cf. (1.13)]. This can be used to derive the general relation between pseudoscalar and vector meson states in HQET:

$$\boldsymbol{a} \cdot \boldsymbol{S} P(v) = \tfrac{1}{2}\gamma_5 \not{v} \not{\epsilon} P(v) = \tfrac{1}{2} V(v,\epsilon). \tag{4.18}$$

This is the correct generalization of (1.41).

The covariant representation of states can be used to determine in a very efficient way the structure of hadronic matrix elements in the effective theory [7,93]. The goal is to find a minimal form factor decomposition consistent with Lorentz covariance, parity invariance of the strong interactions, and heavy-quark symmetry. The flavor symmetry is manifest when one uses mass-independent wave functions to represent hadron states which obey a mass-independent normalization as in (1.25) and (4.6). The correct transformation properties under the spin symmetry are guaranteed when one collects a spin doublet of states into a single object. Let us discuss this in detail for the ground-state pseudoscalar and vector mesons, although the method is completely general. One introduces a combined meson wave function $\mathcal{M}(v)$ that represents both $P(v)$ and $V(v,\epsilon)$ by

$$\mathcal{M}(v) = \frac{1+\not{v}}{2} \begin{cases} -\gamma_5; & \text{pseudoscalar meson,} \\ \not{\epsilon}; & \text{vector meson,} \end{cases} \tag{4.19}$$

where we have omitted the factor $1/\sqrt{2}$ from (4.14) for later convenience. We note that when one prefers to work with meson states obeying the relativistic normalization (1.26), one has to multiply $\mathcal{M}(v)$ by $m_{\rm M}^{1/2}$, where $m_{\rm M}$ is the physical meson mass. The covariant tensor wave function has the important property

$$\mathcal{M}(v) = P_+ \mathcal{M}(v) P_-, \tag{4.20}$$

where $P_\pm = \frac{1}{2}(1 \pm \not{v})$. This will often be used below to simplify expressions. Consider now a transition between two heavy mesons, $M(v) \to M'(v')$, mediated by a renormalized effective current operator $\bar{h}'\Gamma h$, which changes a heavy quark Q into another heavy quark Q′. Throughout this chapter we will use the short-hand notation $h \equiv h_v^{\rm Q}$ and $h' \equiv h_{v'}^{\rm Q'}$. According to the Feynman rules of HQET, the "heavy-quark part" of the decay amplitude is simply proportional to $\bar{u}'_{\rm h}\Gamma u_{\rm h}$; interactions of the heavy quarks with gluons do not modify the Dirac structure of Γ.[16] Since the heavy-quark spinors are part of the tensor wave functions associated with the hadron states, it follows that the amplitude must be proportional to $\overline{\mathcal{M}}'(v')\Gamma\mathcal{M}(v)$. This is a Dirac matrix with two indices representing the light degrees of freedom. Since the total matrix element is a Lorentz scalar, these indices must be contracted with those of a matrix Ξ. Hence we may write

$$\langle M'(v')|\bar{h}'\Gamma h|M(v)\rangle = \mathrm{Tr}\{\Xi(v,v',\mu)\overline{\mathcal{M}}'(v')\Gamma\mathcal{M}(v)\}. \tag{4.21}$$

[16] Recall that these gluons are soft. Hard gluon effects are factorized into the Wilson coefficients.

The matrix Ξ contains all long-distance dynamics. It is a most complicated object, only constrained by the symmetries of the effective theory. Heavy-quark symmetry requires that it be independent of the spins and masses of the heavy quarks, as well as of the Dirac structure of the current. Hence, Ξ can only be a function of the meson velocities [17] and of the renormalization scale μ. Lorentz covariance and parity invariance imply that Ξ must transform as a scalar with even parity. This allows the decomposition

$$\Xi(v,v',\mu) = \Xi_1 + \Xi_2 \not{v} + \Xi_3 \not{v}' + \Xi_4 \not{v}\not{v}', \tag{4.22}$$

with coefficients $\Xi_i = \Xi_i(w,\mu)$. But using the projection property (4.20) of the tensor wave functions, one finds that under the trace

$$\Xi(v,v',\mu) \to \Xi_1 - \Xi_2 - \Xi_3 + \Xi_4 \equiv -\xi(w,\mu). \tag{4.23}$$

Therefore [7],

$$\langle M'(v')|\bar{h}'\Gamma h|M(v)\rangle = -\xi(w,\mu)\,\mathrm{Tr}\{\overline{\mathcal{M}}'(v')\Gamma\mathcal{M}(v)\}. \tag{4.24}$$

The sign is chosen such that the universal form factor $\xi(w,\mu)$ coincides with the Isgur–Wise function, which emerged in chapter 1 as the single form factor that describes semileptonic weak decay processes of heavy mesons in the infinite quark-mass limit. Eq. (4.24) summarizes in a compact way the results derived in section 1.4. Using the explicit form of the meson wave functions one can readily recover the relations (1.28), (1.29), and (1.45). The only new feature is that the Isgur–Wise function depends on the renormalization scale μ. This is necessary to compensate the scale dependence of the Wilson coefficients, which multiply the renormalized current operators in the short-distance expansion.

Using the tensor methods described above, it is completely straightforward and systematic to find a minimal decomposition in terms of universal form factors for any matrix element in HQET. In particular, it is possible to evaluate matrix elements of higher dimension operators. By combining the results of this section with the explicit expressions for the renormalized currents derived in chapter 3, we now have the tools at hand to construct the $1/m_Q$ expansion of any hadronic matrix element of the full theory at leading and next-to-leading order. In the following sections, we will apply these methods to the weak decay form factors of heavy mesons. We shall come back to the discussion of baryons in section 4.5. We will then extend the formalism to heavy-to-light transitions and discuss the $1/m_Q$ expansion for meson decay constants, and the extraction of V_{ub} from a comparison of the pion spectrum in $B \to \pi\,\ell\,\bar{\nu}$ and $D \to \pi\,\ell\,\bar{\nu}$ decays.

4.2. Meson decay form factors

One of the most important applications of heavy-quark symmetry is to derive relations between the form factors parameterizing the exclusive weak decays $B \to D\,\ell\,\bar{\nu}$ and $B \to D^*\ell\,\bar{\nu}$. A detailed theoretical understanding of these processes is a necessary prerequisite for a reliable determination of the elements V_{cb} of the Cabibbo–Kobayashi–Maskawa matrix. In the limit of an exact spin–flavor symmetry, the meson form factors have been discussed in section 1.4. We will now refine this

[17] For this to be true it is essential that one identifies the velocities of the heavy quarks with the hadron velocities. This is a legitimate choice because of the reparameterization invariance of the effective theory.

analysis considerably by including the first-order power corrections in $1/m_c$ and $1/m_b$, as well as renormalization effects at next-to-leading order in perturbation theory. The original analysis of power corrections is due to Luke [30]. Radiative corrections at leading and subleading order have been included in a systematic way in Refs. [86,90].

We start by introducing a convenient set of six hadronic form factors $h_i(w)$, which parameterize the relevant meson matrix elements of the flavor-changing vector and axial vector currents $V^\mu = \bar c\gamma^\mu b$ and $A^\mu = \bar c\gamma^\mu\gamma_5 b$,

$$\begin{aligned}
\langle D(v')|V^\mu|B(v)\rangle &= h_+(w)\,(v+v')^\mu + h_-(w)\,(v-v')^\mu,\\
\langle D^*(v',\epsilon)|V^\mu|B(v)\rangle &= \mathrm{i}h_\mathrm{V}(w)\epsilon^{\mu\nu\alpha\beta}\epsilon^*_\nu v'_\alpha v_\beta,\\
\langle D^*(v',\epsilon)|A^\mu|B(v)\rangle &= h_{\mathrm{A}_1}(w)(w+1)\epsilon^{*\mu} - [h_{\mathrm{A}_2}(w)v^\mu + h_{\mathrm{A}_3}(w)v'^\mu]\epsilon^*\cdot v.
\end{aligned} \tag{4.25}$$

Here $w = v\cdot v'$ is the velocity transfer of the mesons. The results (1.29) and (1.45) obtained in section 1.4 from the consideration of the naive symmetry limit would correspond to

$$\begin{aligned}
&h_+(w) = h_\mathrm{V}(w) = h_{\mathrm{A}_1}(w) = h_{\mathrm{A}_3}(w) = \xi(w)\,,\\
&h_-(w) = h_{\mathrm{A}_2}(w) = 0\,.
\end{aligned} \tag{4.26}$$

But even at leading order in the $1/m_\mathrm{Q}$ expansion there are corrections to these relations from renormalization group effects. They can be taken into account by combining the operator product expansion of the flavor-changing currents $J^\mu = V^\mu$ or A^μ in (3.126) with the general form (4.24) of matrix elements of the dimension-three operators in the effective theory. According to (3.142), the μ-dependence of the Wilson coefficients of any bilinear heavy-quark current can be factorized into a universal function $K_\mathrm{hh}(w,\mu)$, which is normalized at zero recoil. The μ-dependence of this function has to cancel against that of the Isgur–Wise function. We can use this fact to define a renormalization-group invariant Isgur–Wise form factor by

$$\xi_\mathrm{ren}(w) \equiv \xi(w,\mu)K_\mathrm{hh}(w,\mu), \qquad \xi_\mathrm{ren}(1) = 1. \tag{4.27}$$

Neglecting terms of order $1/m_\mathrm{Q}$, we then obtain [86]

$$\langle M'(v')|J^\mu|M(v)\rangle = -\xi_\mathrm{ren}(w)\sum_{i=1}^{3}\widehat C_i^{(5)}(w)\,\mathrm{Tr}\{\overline{\mathcal{M}}'(v')\Gamma_i\mathcal{M}(v)\}. \tag{4.28}$$

For $J^\mu = V^\mu$ and A^μ, the matrices Γ_i are given in (3.102) and (3.112), respectively. It is now straightforward to evaluate the traces to find

$$\begin{aligned}
h_+(w) &= \{\widehat C_1(w) + \tfrac12(w+1)[\widehat C_2(w) + \widehat C_3(w)]\}\xi_\mathrm{ren}(w),\\
h_-(w) &= \tfrac12(w+1)[\widehat C_2(w) - \widehat C_3(w)]\xi_\mathrm{ren}(w),\\
h_\mathrm{V}(w) &= \widehat C_1(w)\xi_\mathrm{ren}(w),\\
h_{\mathrm{A}_1}(w) &= \widehat C_1^5(w)\xi_\mathrm{ren}(w),
\end{aligned}$$

$$h_{A_2}(w) = \widehat{C}_2^5(w)\xi_{\rm ren}(w),$$

$$h_{A_3}(w) = [\widehat{C}_1^5(w) + \widehat{C}_3^5(w)]\xi_{\rm ren}(w). \tag{4.29}$$

This is the correct generalization of (4.26) at leading order in the $1/m_Q$ expansion. It is still true that the meson form factors are all proportional to a single universal function $\xi_{\rm ren}(w)$. The relations between them are not as simple as shown in (4.26), however, since radiative corrections violate the spin–flavor symmetry. Their effects are contained in the various combinations of short-distance coefficients, which can be evaluated using the numerical results given in Tab. 3.1.

At this point it is instructive to reconsider the implications of current conservation. As discussed in section 1.4, the fact that the vector current operators of dimension three are conserved in the effective theory can be used to derive the normalization of the Isgur–Wise function at zero recoil. Following eqs. (1.30)–(1.33) one finds that $\xi(1,\mu) = 1$, and by definition the renormalized form factor $\xi_{\rm ren}(w)$ obeys the same normalization. An observation which will turn out to be important later is that this normalization also follows from the conservation of the flavor-conserving vector current in the full theory, which implies that $h_+(1) = 1$ in the limit of equal meson masses. From the above expressions we find that $h_+(1) = \eta_V\xi_{\rm ren}(1)$, where the QCD coefficient η_V is defined in (3.157). The fact that $\eta_V = 1$ in the limit of equal masses [cf. (3.160)] leads to the normalization of the Isgur–Wise function. The important point is that current conservation in the full theory is a more powerful constraint than current conservation in the effective theory. Whereas the latter leads to a relation for the Isgur–Wise function only, the condition $h_+(1) = 1$ can be used to derive separate constraints at every order in the $1/m_Q$ expansion.

Following Ref. [30], let us now discuss the structure of the leading power corrections to the results given above. We will present the discussion for a general transition between two heavy mesons, and later specialize it to $B \to D^{(*)}$ decays. At order $1/m_Q$ there appears in the operator product expansion of the currents in (3.147) a set of local and nonlocal effective current operators of dimension four. The local operators O_j contain a covariant derivative acting on one of the heavy-quark fields and thus have the generic form $\bar h'\Gamma iD_\alpha h$ or $\bar h'(-i\overleftarrow{D}_\alpha)\Gamma h$. The Dirac matrices Γ carry two Lorentz indices; for instance $\Gamma = \gamma^\mu\gamma^\alpha$ for O_1. Because of heavy-quark symmetry the structure of Γ is irrelevant, however. In generalization of (4.21), hadronic matrix elements of these operators can be written as traces over the meson wave functions in the following form:

$$\langle M'(v')|\bar h'\Gamma iD_\alpha h|M(v)\rangle = -\mathrm{Tr}\{\xi_\alpha(v,v',\mu)\overline{\mathcal{M}}'(v')\Gamma\mathcal{M}(v)\},$$

$$\langle M'(v')|\bar h'(-i\overleftarrow{D}_\alpha)\Gamma h|M(v)\rangle = -\mathrm{Tr}\{\bar\xi_\alpha(v',v,\mu)\overline{\mathcal{M}}'(v')\Gamma\mathcal{M}(v)\}. \tag{4.30}$$

These matrix elements are related to each other by Dirac conjugation combined with an interchange of the velocity variables. The tensor form factor $\xi_\alpha(v,v',\mu)$ carries the Lorentz index of the covariant derivative. Taking into account the projection properties of the tensor wave functions, one finds that the most general decomposition of this object involves three scalar functions:

$$\xi_\alpha(v,v',\mu) = \xi_+(w,\mu)(v+v')_\alpha + \xi_-(w,\mu)(v-v')_\alpha - \xi_3(w,\mu)\gamma_\alpha. \tag{4.31}$$

T-invariance of the strong interactions requires that these functions be real. This, in turn, implies that the conjugate $\bar\xi_\alpha(v',v,\mu)$ is given by the same expression but for a change of the sign of the second term.

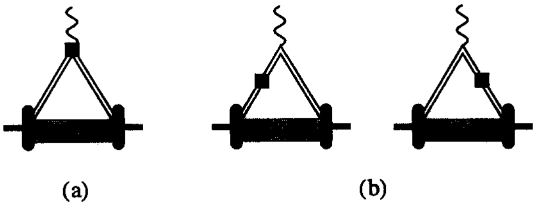


Fig. 4.1. Diagrams representing the first-order power corrections (black squares) to hadronic form factors: (a) corrections from local dimension four current operators; (b) corrections from insertions of higher dimension operators from the effective Lagrangian. The gray area represents the light degrees of freedom.

The three new functions $\xi_i(w,\mu)$ are not all independent. The equation of motion $\mathrm{i}v\cdot Dh = 0$ implies that $v^\alpha \xi_\alpha(v,v',\mu) \hat{=} 0$, where the symbol $\hat{=}$ is used to indicate that this relation holds under the traces in (4.30), i.e., between the projection operators provided by the meson wave functions. In terms of the scalar functions, this constraint is equivalent to

$$(w+1)\xi_+(w,\mu) - (w-1)\xi_-(w,\mu) + \xi_3(w,\mu) = 0. \tag{4.32}$$

A second relation can be derived by noting that $\mathrm{i}\partial_\alpha(\bar{h}'\Gamma h) = \bar{h}'\mathrm{i}\overleftarrow{D}_\alpha\Gamma h + \bar{h}'\Gamma\mathrm{i}D_\alpha h$. Using (2.29) to evaluate the matrix element of the total derivative of the current, one obtains

$$\xi_\alpha(v,v',\mu) - \bar{\xi}_\alpha(v',v,\mu) = \bar{\Lambda}(v-v')_\alpha\,\xi(w,\mu), \tag{4.33}$$

which can be used to relate $\xi_-(w,\mu)$ to the Isgur–Wise function. Recall that $\bar{\Lambda} = m_{\mathrm{M}} - m_{\mathrm{Q}} = m_{M'} - m_{Q'}$ can be identified with the effective mass of the light degrees of freedom as defined in (2.30). The two constraints can be combined to give

$$\xi_-(w,\mu) = \frac{\bar{\Lambda}}{2}\xi(w,\mu), \qquad \xi_+(w,\mu) = \frac{\bar{\Lambda}}{2}\frac{w-1}{w+1}\xi(w,\mu) - \frac{1}{w+1}\xi_3(w,\mu). \tag{4.34}$$

These relations suggest a close relation between these functions and the leading-order Isgur–Wise function. The origin of this relation is the reparameterization invariance of the effective theory. In fact, we have seen in section 3.9 that the local dimension-four operators O_j are renormalized by the same Wilson coefficients as the dimension three operators J_i. For their matrix elements this implies that the μ-dependence of the functions $\xi_i(w,\mu)$ must be the same as the μ-dependence of the Isgur–Wise function $\xi(w,\mu)$. This leads us to introduce a new, renormalization-group invariant function $\eta(w)$ by

$$\xi_3(w,\mu) \equiv \bar{\Lambda}\xi(w,\mu)\eta(w). \tag{4.35}$$

We expect that $\eta(w)$ will be a slowly varying function of order unity, and this is in fact confirmed by QCD sum rule calculations (see chapter 5). To summarize, meson matrix elements of any of the local operators O_j can be parameterized in terms of the product $\bar{\Lambda}\xi(w,\mu)$ and a single new function $\eta(w)$. A graphical representation of this first type of power corrections is shown in Fig. 4.1(a).

An important consequence of the relations (4.34) is that, at zero recoil, the tensor form factor reduces to

$$\xi_\alpha(v,v,\mu) = -\bar{\Lambda}\eta(1)(v_\alpha + \gamma_\alpha) \hat{=} 0, \tag{4.36}$$

which vanishes between the (equal velocity) projection operators provided by the meson wave functions. Consequently, in the limit $v = v'$ there are no contributions of the local dimension-four operators O_j to meson matrix elements. This statement, which is true to all orders in perturbation theory, is the first part of the so-called Luke's theorem [30,46,97].

A second class of $1/m_Q$ corrections comes from the presence of higher dimension operators in the effective Lagrangian. Insertions of $\mathcal{L}_1$ in (3.72) into matrix elements of the leading-order currents $J_i = \bar{h}'\Gamma_i h$ represent corrections to the meson wave functions, which appear since the states of the effective theory are different from the physical states of the full theory. For the heavy–light currents the corresponding operators T_k have been discussed in detail in section 3.6. For bilinear heavy quark currents one has to distinguish insertions of $\mathcal{L}_1$ on the incoming heavy-quark line from insertions of $\mathcal{L}_1'$ (the prime indicates that this is part of the effective Lagrangian for the heavy quark Q') on the outgoing heavy-quark line, as depicted in Fig. 4.1(b). The corresponding matrix elements are related to each other by Dirac conjugation and an exchange of variables. We only display the first one,

$$\begin{aligned}&\langle M'(v')|\mathrm{i}\int \mathrm{d}x\, T\{J_i(0), \mathcal{L}_1(x)\}|M(v)\rangle\\&= -2\bar{\Lambda}\chi_1(w,\mu)\mathrm{Tr}\{\overline{\mathcal{M}}'(v')\Gamma_i\mathcal{M}(v)\}\\&\quad - 2\bar{\Lambda}C_{\mathrm{mag}}(\mu)\mathrm{Tr}\{\chi_{\alpha\beta}(v,v',\mu)\overline{\mathcal{M}}'(v')\Gamma_i P_+\sigma^{\alpha\beta}\mathcal{M}(v)\}.\end{aligned} \tag{4.37}$$

Note that we have factored out $\bar{\Lambda}$ in order for the form factors to be dimensionless. The kinetic operator contained in $\mathcal{L}_1$ transforms as a Lorentz scalar. An insertion of it does not affect the Dirac structure of the matrix element. Hence, the corresponding function $\chi_1(w,\mu)$ effectively corrects the Isgur–Wise function. The chromo-magnetic operator, on the other hand, carries a nontrivial Dirac structure. An insertion of it on the incoming heavy-quark line brings a matrix $\sigma^{\alpha\beta}$ next to the meson wave function $\mathcal{M}(v)$, see Fig. 4.1b. In addition, a propagator separates this insertion from the heavy-quark current, resulting in a projection operator $P_+ = \frac{1}{2}(1+\not{v})$ on the right-hand side of Γ_i. This explains the structure of the second trace in (4.37). Noting that $v_\alpha P_+\sigma^{\alpha\beta}\mathcal{M}(v) = 0$, we write the general decomposition of the tensor form factor $\chi_{\alpha\beta}(v,v',\mu)$ as

$$\chi_{\alpha\beta}(v,v',\mu) = \mathrm{i}\chi_2(w,\mu)v'_\alpha\gamma_\beta + \chi_3(w,\mu)\sigma_{\alpha\beta}. \tag{4.38}$$

The terms proportional to $\chi_3(w,\mu)$ in (4.37) can be simplified, irrespective of the structure of the current, by means of the identity [54]

$$P_+\sigma^{\alpha\beta}\mathcal{M}(v)\sigma_{\alpha\beta} = 2d_{\mathrm{M}}\mathcal{M}(v), \tag{4.39}$$

where $d_{\mathrm{P}} = 3$ for pseudoscalar meson, and $d_{\mathrm{V}} = -1$ for a vector meson. We have already encountered these coefficients in (2.34). It follows that the function $\chi_3(w,\mu)$ always appears in combination with the Isgur–Wise function, but in a way that is different for pseudoscalar and vector mesons. It thus represents a spin-symmetry violating correction to the meson wave functions.

Given the above definitions and the expansion of the currents in (3.155), it is straightforward, if tedious, to calculate the first-order power corrections to the hadronic form factors $h_i(w)$. Of course, these physical form factors should be written in terms of renormalized universal functions $\chi_i^{\mathrm{ren}}(w)$, rather than the μ-dependent functions that parameterize the matrix elements in the effective theory.

For the renormalization of $\chi_1(w,\mu)$, which always appears in the same combination with the Isgur–Wise form factor, it is important to recall from section 3.9 that the Wilson coefficients depend on the velocity transfer variable $\bar{w}$, which according to (3.153) differs from the hadron velocity transfer by terms of order $1/m_Q$. Their μ-dependence is determined by the function [cf. (3.120)]

$$K_{hh}(\bar{w},\mu) = K_{hh}(w,\mu) + \left(\frac{\bar{\Lambda}}{m_Q} + \frac{\bar{\Lambda}}{m_{Q'}}\right)(w-1)\frac{\partial}{\partial w}K_{hh}(w,\mu) + \cdots . \tag{4.40}$$

The first term renormalizes the Isgur–Wise function, whereas the second one contributes to the renormalization of $\chi_1(w,\mu)$. We define [cf. (4.27)]

$$\begin{aligned} &K_{hh}(\bar{w},\mu)\,[\xi(w,\mu) + (\bar{\Lambda}/m_Q + \bar{\Lambda}/m_{Q'})\chi_1(w,\mu)] \\ &\quad = \xi_{ren}(w) + (\bar{\Lambda}/m_Q + \bar{\Lambda}/m_{Q'})\chi_1^{ren}(w), \end{aligned} \tag{4.41}$$

where

$$\chi_1^{ren}(w) = \chi_1(w,\mu)K_{hh}(w,\mu) + \xi_{ren}(w)(w-1)(\partial/\partial w)\ln K_{hh}(w,\mu). \tag{4.42}$$

The renormalization of the other two functions is straightforward. Factorizing the coefficient of the chromo-magnetic operator as in (3.45), i.e.

$$C_{mag}(\mu) = \widehat{C}_{mag}(m_Q)K_{mag}(\mu), \tag{4.43}$$

we define

$$\chi_i^{ren}(w) = \chi_i(w,\mu)K_{mag}(\mu)K_{hh}(w,\mu), \quad i=2,3. \tag{4.44}$$

So far $K_{mag}(\mu)$ is known in leading logarithmic approximation only, but this will turn out to be sufficient for our purposes.

For the presentation of the exact results for the meson form factors at order $1/m_Q$ we define functions $N_i(w)$, which contain the corrections to the limit of an exact spin–flavor symmetry, by

$$h_i(w) = N_i(w)\xi_{ren}(w). \tag{4.45}$$

It is furthermore convenient to introduce the dimensionless ratios

$$\varepsilon_c = \bar{\Lambda}/2m_c, \qquad \varepsilon_b = \bar{\Lambda}/2m_b, \tag{4.46}$$

as well as three new functions $L_{P,V}(w)$ and $L_3^Q(w)$,

$$\begin{aligned} \xi_{ren}(w)L_P(w) &= 2\chi_1^{ren}(w) - 4\widehat{C}_{mag}(m_Q)[(w-1)\chi_2^{ren}(w) - 3\chi_3^{ren}(w)], \\ \xi_{ren}(w)L_V(w) &= 2\chi_1^{ren}(w) - 4\widehat{C}_{mag}(m_Q)\chi_3^{ren}(w), \\ \xi_{ren}(w)L_3^Q(w) &= 4\widehat{C}_{mag}(m_Q)\chi_2^{ren}(w). \end{aligned} \tag{4.47}$$

One can show that L_P and L_V are corrections to the Isgur–Wise function which always appear for pseudoscalar and vector mesons, respectively, irrespective of the structure of the current [46]. We first present the results for N_+ and N_{A_1}, which will play a special role in the analysis below. They are [98]:

$$
\begin{aligned}
N_+(w) = &\{\widehat{C}_1(\bar{w}) + \tfrac{1}{2}(w+1)[\widehat{C}_2(\bar{w}) + \widehat{C}_3(\bar{w})]\}[1 + \varepsilon_c L_D(w) + \varepsilon_b L_B(w)] \\
&+ \varepsilon_c \tfrac{1}{2}(w-1)\{[1-2\eta(w)]\widehat{C}_2(\bar{w}) + [3-2\eta(w)]\widehat{C}_3(\bar{w})\} \\
&+ \varepsilon_b \tfrac{1}{2}(w-1)\{[3-2\eta(w)]\widehat{C}_2(\bar{w}) + [1-2\eta(w)]\widehat{C}_3(\bar{w})\},
\end{aligned}
\tag{4.48}
$$

$$
\begin{aligned}
N_{A_1}(w) = &\widehat{C}_1^5(\bar{w})[1 + \varepsilon_c L_{D^*}(w) + \varepsilon_b L_B(w)] \\
&+ \varepsilon_c \frac{w-1}{w+1}[\widehat{C}_1^5(\bar{w}) + 2\eta(w)\widehat{C}_3^5(\bar{w})] \\
&+ \varepsilon_b \frac{w-1}{w+1}\{[1-2\eta(w)]\widehat{C}_1^5(\bar{w}) + 2\eta(w)\widehat{C}_2^5(\bar{w})\}.
\end{aligned}
$$

Notice that the Wilson coefficients are functions of the variable $\bar{w}$, whereas the universal form factors depend on the meson velocity transfer w. The expressions for the remaining four form factors are more lengthy. To display them we omit the dependence on $\bar{w}$ and w. We find [98]

$$
\begin{aligned}
N_- = &\tfrac{1}{2}(w+1)(\widehat{C}_2 - \widehat{C}_3)(1 + \varepsilon_c L_D + \varepsilon_b L_B) \\
&- \varepsilon_c\{(1-2\eta)[\widehat{C}_1 - \tfrac{1}{2}(w-1)(\widehat{C}_2 - \widehat{C}_3)] + (w+1)\widehat{C}_3\} \\
&+ \varepsilon_b\{(1-2\eta)[\widehat{C}_1 + \tfrac{1}{2}(w-1)(\widehat{C}_2 - \widehat{C}_3)] + (w+1)\widehat{C}_2\}, \\
N_V = &\widehat{C}_1(1 + \varepsilon_c L_{D^*} + \varepsilon_b L_B) + \varepsilon_c(\widehat{C}_1 - 2\eta\widehat{C}_3) + \varepsilon_b[(1-2\eta)\widehat{C}_1 - 2\eta\widehat{C}_2], \\
N_{A_2} = &\widehat{C}_2^5(1 + \varepsilon_c L_{D^*} + \varepsilon_b L_B) + [\widehat{C}_1^5 + (w-1)\widehat{C}_2^5]\varepsilon_c L_3^c \\
&- \frac{2\varepsilon_c}{w+1}\Big((1+\eta)(\widehat{C}_1^5 + \widehat{C}_3^5) - \tfrac{1}{2}(w+1)(1+2\eta)\widehat{C}_2^5\Big) \\
&+ \frac{2\varepsilon_b}{w+1}\widehat{C}_2^5\Big(\tfrac{1}{2}(3w+1) - (w+2)\eta\Big), \\
N_{A_3} = &(\widehat{C}_1^5 + \widehat{C}_3^5)(1 + \varepsilon_c L_{D^*} + \varepsilon_b L_B) - [\widehat{C}_1^5 - (w-1)\widehat{C}_3^5]\varepsilon_c L_3^c \\
&+ \frac{\varepsilon_c}{w+1}[(w-1-2\eta)(\widehat{C}_1^5 - \widehat{C}_3^5) + 4w(1+\eta)\widehat{C}_3^5] \\
&+ \varepsilon_b\Big((1-2\eta)(\widehat{C}_1^5 + \widehat{C}_3^5) + \frac{2}{w+1}(w\eta - 1)\widehat{C}_2^5\Big).
\end{aligned}
\tag{4.49}
$$

These expressions are the main result of this chapter.

As discussed above, vector current conservation in the full theory implies that $h_+(1) = 1$ in the limit of equal heavy-quark masses. Given the above expressions, this is equivalent to $L_P(1) = 0$, which must hold for any value of m_Q in the Wilson coefficient $\widehat{C}_{\rm mag}(m_Q)$. It follows that [18]

$$\chi_1^{\rm ren}(1) = \chi_3^{\rm ren}(1) = 0. \tag{4.50}$$

This also implies the vanishing of the μ-dependent functions $\chi_1(w,\mu)$ and $\chi_3(w,\mu)$ at $w = 1$. It then follows that the matrix elements in (4.37) vanish at zero recoil, which is the second part of Luke's theorem. We shall discuss the phenomenological implications of this theorem in the next section.

It is difficult to extract much information from the complicated expressions for N_i without a prediction for the subleading universal functions η and $\chi_i^{\rm ren}$. For this reason a detailed numerical discussion is postponed until chapter 5, where these form factors are calculated using the QCD sum rule approach. Some important, general observations can be made without such an analysis, however. Using (4.50), one finds that the $1/m_Q$ corrections in N_+ and N_{A_1} vanish at zero recoil. The normalization of the corresponding meson form factors is predicted by HQET up to second-order power corrections:

$$h_+(1) = \eta_V + O(1/m_Q^2), \qquad h_{A_1}(1) = \eta_A + O(1/m_Q^2). \tag{4.51}$$

The perturbative coefficients η_V and η_A have been given in (3.159). Notice that there is no such result for the remaining form factors.

Further predictions can be made for ratios of meson form factors, in which some of the universal functions drop out [56]. An important example is the ratio

$$R_1(w) = h_V(w)/h_{A_1}(w), \tag{4.52}$$

which is readily seen to be independent of $\chi_i^{\rm ren}(w)$. In fact, at order $1/m_Q$ the following simple expression can be derived [98]:

$$R_1(w) = F_1(\bar w)\left(1 + \frac{2\bar\varepsilon_c}{w+1} + \frac{2\bar\varepsilon_b}{w+1}[1 - 2F_2(\bar w)\eta(w)]\right), \tag{4.53}$$

$$F_1(\bar w) = 1 + [4\alpha_s(m_c)/3\pi]r(\bar w), \tag{4.54}$$

and the second short-distance coefficient $F_2(\bar w)$ is almost independent of $\bar w$ over the kinematic range accessible in semileptonic decays: $F_2(\bar w) \approx 0.9$. Note that the unknown function $\eta(w)$ enters in the $1/m_b$ corrections only. Unless the values of this function were anomalously large, [19] we conclude that the ratio $R_1(w)$ can be predicted in an essentially model-independent way. Note that both the QCD and the $1/m_c$ corrections are positive. As a consequence, the deviations from the symmetry limit $R_1 = 1$ are rather substantial. For instance, assuming $\bar\Lambda = 0.5 \pm 0.2$ GeV and neglecting the $1/m_b$ corrections we obtain $R_1 = 1.33 \pm 0.08$ at zero recoil. In chapter 6 we will discuss an experimental test of this prediction.

4.3. Implications of Luke's theorem

The analysis of the previous section shows that the matrix elements which describe the leading $1/m_Q$ corrections to meson decay amplitudes vanish at zero recoil. As a consequence, in the limit

[18] Originally, this was derived by considering in addition the vector current matrix element between two vector mesons, which gives the second condition $L_V(1) = 1$ [30].

[19] We will see in chapter 5 that $\eta(w) \approx 0.6$, so that the corrections proportional to $\bar\varepsilon_b$ in (4.53) are indeed very small.

$v = v'$ there are no terms of order $1/m_Q$ in the hadronic matrix elements in (4.25). This is Luke's theorem [30]. It is independent of the structure of the Wilson coefficients and thus valid to all orders in perturbation theory [46,97].

There is considerable confusion in the literature about the implications of this result. It is often claimed that the theorem would protect any meson decay rate, or even all form factors that are normalized in the spin–flavor symmetry limit, from first-order power corrections at zero recoil. As pointed out in Ref. [18] this is not correct. The reason is simple but somewhat subtle. We have seen above that the form factors h_+ and h_{A_1} in (4.25) are the only ones protected by Luke's theorem; the others are multiplied by kinematic factors which vanish for $v = v'$. In fact, from (4.49) it is apparent that the $1/m_Q$ corrections to h_-, h_V, h_{A_2}, and h_{A_3} do not vanish at zero recoil. The fact that these functions are kinematically suppressed does not imply that they could not contribute to physical decay rates. This is often overlooked. Consider, as an example, the process $B \to D\ell\bar\nu$ in the limit of vanishing lepton mass. By angular momentum conservation, the two pseudoscalar mesons must be in a relative p-wave in order to match the helicities of the lepton pair. The amplitude is proportional to the velocity $|v_D|$ of the D meson in the B rest frame, which leads to a factor (w^2-1) in the decay rate. In such a situation, form factors which are kinematically suppressed can contribute. Indeed, an explicit calculation shows that the $B \to D\ell\bar\nu$ decay rate is proportional to

$$(w^2-1)^{3/2}\left|h_+(w) - \frac{m_B - m_D}{m_B + m_D}\,h_-(w)\right|^2 . \tag{4.55}$$

The two form factors contribute equally to the rate, although h_- is kinematically suppressed in (4.25). Consequently, the decay rate at zero recoil does receive corrections of order $1/m_Q$.[20] As emphasized by Neubert [99], the situation is different for $B \to D^*\ell\bar\nu$ transitions. Because the vector meson has spin one, the decay can proceed in an s-wave, and there is no helicity suppression near zero recoil. One finds that close to $w = 1$ the decay rate is proportional to $(w^2-1)^{1/2}|h_{A_1}(w)|^2$. Since this form factor is protected by Luke's theorem, these transitions are ideally suited for a precision measurement of V_{cb}. This will be discussed in detail in section 6.2.

To some extent the subtleties with Luke's theorem arise as an artifact of the form factor basis in (4.25), which is however the basis most convenient for calculations in HQET. When discussing physics, it is better to go back to the form factors introduced in (1.35) and (1.46). Recall that, in the limit of vanishing lepton mass, terms proportional to the momentum transfer q^μ do not contribute to semileptonic decay amplitudes, since the lepton current is conserved. Then only the form factors F_1, V, A_1 and A_2 are relevant. They correspond to the following combinations of heavy-quark form factors:

$$\begin{aligned} &RF_1(q^2) = h_+(w) - \frac{m_B - m_D}{m_B + m_D}\,h_-(w), \qquad R^*V(q^2) = h_V(w), \\ &R^{*-1}A_1(q^2) = \tfrac{1}{2}(w+1)\,h_{A_1}(w), \qquad R^*A_2(q^2) = h_{A_3}(w) + \frac{m_{D^*}}{m_B}\,h_{A_2}(w). \end{aligned} \tag{4.56}$$

The definition of $R^{(*)}$ and the relation between w and q^2 are given in (1.39). Only $A_1(q^2)$ is protected against $1/m_Q$ corrections at zero recoil, corresponding to maximum momentum transfer

[20] As is generally the case because of phase space, the rate vanishes at zero recoil. Strictly speaking, it is only possible to measure close to zero recoil and to extrapolate the spectrum to $w = 1$.

$q^2_{\rm max} = (m_{\rm B} - m_{\rm D^*})^2$ to the lepton pair. This means that in practice Luke's theorem reduces to the single but important result that [21]

$$\frac{m_{\rm B} + m_{\rm D^*}}{2\sqrt{m_{\rm B} m_{\rm D^*}}} A_1(q^2_{\rm max}) = h_{\rm A_1}(1) = \eta_{\rm A} + \delta_{1/m^2}, \tag{4.57}$$

where $\eta_{\rm A} \approx 0.99 \pm 0.01$ from (3.159), and δ_{1/m^2} represents higher-order power corrections.

4.4. *Second-order power corrections and the anatomy of* δ_{1/m^2}

The above result offers the possibility for a model-independent determination of $V_{\rm cb}$ from a measurement of the semileptonic decay rate $\rm B \to D^* \ell \bar\nu$ near zero recoil [16,99]. In this region the rate is proportional to $|h_{\rm A_1}(1)|^2$, which is known theoretically up to corrections of order $1/m^2_{\rm Q}$. One expects these higher-order power corrections to be of order $\varepsilon^2_{\rm c} \approx 3\%$, but of course such a naive estimate could be too optimistic. For a precision measurement of $V_{\rm cb}$, it is important to know the structure of $1/m^2_{\rm Q}$ corrections in more detail.

Although in principle straightforward, the analysis of higher-order power corrections in HQET is a tremendous enterprise. As shown in Fig. 4.2, three classes of corrections have to be distinguished: matrix elements of local dimension-five current operators, "mixed" corrections resulting from the combination of corrections to the current and to the Lagrangian, and corrections from one or two insertions of operators from the effective Lagrangian into matrix elements of the leading-order currents. Within these classes, one can distinguish corrections proportional to $1/m^2_{\rm b}$, $1/m^2_{\rm c}$, or $1/m_{\rm c} m_{\rm b}$. Falk and Neubert took the challenge and analyzed these corrections for both meson and baryon decay form factors [46,52]. We shall only briefly discuss the main results and implications of their analysis. More than thirty new universal functions are necessary to parameterize the second-order power corrections to meson form factors. [22] When radiative corrections are neglected, eleven combinations of these functions contribute to the hadronic form factors $h_i(w)$. The general results greatly simplify at zero recoil, however. There the equation of motion and the Ward identities of the effective theory can be used to prove that matrix elements of local dimension-five current operators, as well as matrix elements of time-ordered products containing a dimension-four current and an insertion from the effective Lagrangian, can be expressed in terms of the parameters λ_1 and λ_2, which describe the $1/m_{\rm Q}$ corrections to the physical meson masses in (2.34). This result is a generalization of the first part of Luke's theorem. In addition, the conservation of the flavor-conserving vector current in the full theory forces certain combinations of the universal functions to vanish at zero recoil, in analogy to the second part of Luke's theorem. The consequence is that whenever a form factor is protected by Luke's theorem, the structure of second-order power corrections at zero recoil becomes rather simple.

In particular, the coefficient δ_{1/m^2} in (4.57) is found to have the following structure [46]:

$$\delta_{1/m^2} = -\left(\frac{1}{2m_{\rm c}} - \frac{1}{2m_{\rm b}}\right)\left(\frac{\ell_{\rm V}}{2m_{\rm c}} - \frac{\ell_{\rm P}}{2m_{\rm b}}\right) + \frac{\Delta}{4m_{\rm c} m_{\rm b}}. \tag{4.58}$$

The coefficients $\ell_{\rm P,V}$ are related to the zero recoil matrix elements of the vector current between two pseudoscalar or vector mesons

[21] There are other important implications of this theorem for inclusive decays of hadrons containing a heavy quark. They are, however, not the subject of this review.

[22] $\Lambda_{\rm Q}$ baryon decays are simpler. In this case one has to introduce ten new functions.

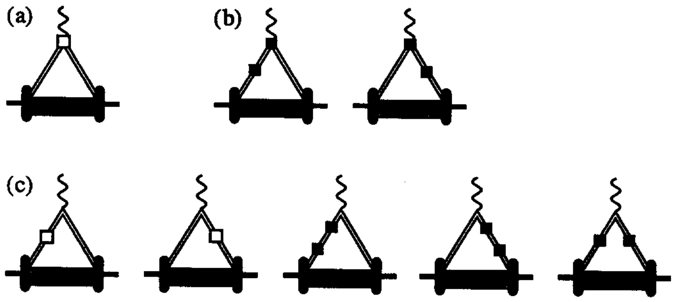


Fig. 4.2. Diagrams representing the second-order power corrections to meson form factors in HQET: (a) corrections to the current; (b) mixed corrections to the current and to the effective Lagrangian; (c) corrections to the effective Lagrangian. The black squares represent operators of order $1/m_b$ or $1/m_c$, the open ones denote operators of order $1/m_b^2$, $1/m_c^2$, or $1/m_c m_b$.

$$\langle D(v)|\bar{c}\gamma^\mu b|B(v)\rangle = 2\eta_V v^\mu\left[1 - \ell_P\left(\frac{1}{2m_c} - \frac{1}{2m_b}\right)^2 + \cdots\right],$$

$$\langle D^*(v,\epsilon)|\bar{c}\gamma^\mu b|B^*(v,\epsilon)\rangle = 2\eta_V v^\mu\left[1 - \ell_V\left(\frac{1}{2m_c} - \frac{1}{2m_b}\right)^2 + \cdots\right]. \tag{4.59}$$

The terms in parentheses describe the deviations from the flavor symmetry limit, in which the "brown muck" configurations in the initial and final meson are exactly the same. Due to the fact that the heavy-quark masses are different, the light degrees of freedom undergo some rearrangement, leading to a small form factor suppression. In the nonrelativistic constituent quark model, the deficit in the wave function overlap results from an m_Q-dependence of the reduced mass of the light constituent quark. An estimate of this effect in the ISGW model [19] gives [23]

$$\ell_P = \ell_V = 3m_q^2 \approx 0.37\ \mathrm{GeV}^2, \tag{4.60}$$

where we have used the constituent quark mass $m_q \approx 0.35$ GeV. This gives $\approx -2\%$ for the first term in (4.58). The important observation is that this contribution is parameterically suppressed and vanishes in the limit of an exact flavor symmetry.

The second term in (4.58) is suppressed by the bottom quark mass and is therefore expected to be small. The coefficient Δ can be written [46]

$$\Delta = \tfrac{1}{2}(m_V^2 - m_P^2) + \tfrac{4}{3}\lambda_1 + \cdots, \tag{4.61}$$

where the ellipsis represents corrections resulting from a double insertion of the chromo-magnetic operator. There are reasons to expect that such "second-order" hyperfine effects are strongly suppressed (see chapter 5). We shall neglect them. The contribution to Δ proportional to the vector-pseudoscalar mass splitting is small and positive. It can be extracted from the mass difference of the B and B* mesons. The main uncertainty, then, resides in the parameter λ_1. Several authors have calculated this quantity using QCD sum rules [54,107–109]. We shall discuss the results in detail in section 5.4.

[23] A similar result is obtained from lattice gauge theory [106].

For an estimate of Δ, we use the range $0 < -\lambda_1 < 0.5\ \mathrm{GeV}^2$ and obtain values between -1.4% and 0.8% for the second term in (4.58). This leads to our final estimate

$$-3\% < \delta_{1/m^2} < -1\%. \tag{4.62}$$

We can combine this with (4.57) to obtain one of the most important, and certainly most precise predictions of HQET,

$$h_{A_1}(1) = 0.97 \pm 0.04, \qquad A_1(q^2_{\max}) = 0.86 \pm 0.04. \tag{4.63}$$

In estimating the theoretical uncertainty, we have taken into account that corrections which have been neglected are of order $[\alpha_s(m_c)/\pi]^2 \approx 1\%$ and $\varepsilon_c^3 \approx 0.5\%$, and are thus expected to be very small.

4.5. Baryon-decay form factors

After the discovery of the Λ_b baryon [100,101], the prospects are good that weak decays of heavy baryons can soon be studied in detail. Eventually, such a program could lead to an independent measurement of the elements V_{cb} and V_{ub} of the Cabibbo–Kobayashi–Maskawa matrix. Heavy Λ_Q baryons are particularly simple in that they are composed of a heavy quark and light degrees of freedom with total angular momentum zero. For this reason the theoretical description is rather simpler, and the predictive power of HQET is large. It is, therefore, worthwhile to study the semileptonic process $\Lambda_b \to \Lambda_c \ell \bar\nu$ in detail.

According to (4.7), in the effective theory a Λ_Q baryon is represented by a spinor $u_\Lambda(v,s)$ that can be identified with the spinor of the heavy quark. For simplicity, we shall from now on omit the spin labels and write $u(v) \equiv u_\Lambda(v,s)$. The baryon matrix elements of the weak currents $V^\mu = \bar c\gamma^\mu b$ and $A^\mu = \bar c\gamma^\mu\gamma_5 b$ can be parameterized by six hadronic form factors, which we write as functions of the baryon velocity transfer $w = v\cdot v'$,

$$\begin{aligned}\langle \Lambda_c(v')|V^\mu|\Lambda_b(v)\rangle &= \bar u(v')[F_1(w)\gamma^\mu + F_2(w)v^\mu + F_3(w)v'^\mu]u(v),\\ \langle \Lambda_c(v')|A^\mu|\Lambda_b(v)\rangle &= \bar u(v')[G_1(w)\gamma^\mu + G_2(w)v^\mu + G_3(w)v'^\mu]\gamma_5 u(v).\end{aligned} \tag{4.64}$$

The aim is to construct an expansion of the hadronic form factors $F_i(w)$ and $G_i(w)$ in powers of $1/m_Q$, and to relate the coefficients in this expansion to universal, mass-independent functions of the velocity transfer. This is achieved by constructing the operator product expansion of the weak currents and evaluating the matrix elements of the effective current operators using the tensor methods of section 4.1. The analysis at leading order was done by several authors [102–105]. It was extended to order $1/m_Q$ by Georgi, Grinstein, and Wise [45]. The next-to-leading order QCD corrections have been included in Refs. [86,90]. Second-order power corrections, which we shall not discuss in detail, have been analyzed by Falk and Neubert [46]. One proceeds in complete analogy to the meson case, but things are much simpler due to the fact that the light degrees of freedom have spin zero, whereas they have spin $\frac{1}{2}$ in the meson case. Accordingly, there will be no Lorentz index for the "brown muck".

Let us then repeat, step by step, the analysis of section 4.2. The baryon matrix elements of the dimension-three current operators are described by a universal form factor $\zeta(w,\mu)$, which is the analog of the Isgur–Wise function in the baryon case. It is defined by

$$\langle \Lambda'(v')|\bar h'\Gamma h|\Lambda(v)\rangle = \zeta(w,\mu)\bar u(v')\Gamma u(v). \tag{4.65}$$

For simplicity we will refer to this function as the Isgur–Wise function, too, but we emphasize that it is not at all related to the function $\xi(w,\mu)$ which arises in the analysis of meson decays. Combining (4.65) with the short-distance expansion of the currents in (3.126), one can immediately derive expressions for the baryon form factors to leading order in $1/m_Q$. They are

$$F_i(w) = C_i(w,\mu)\zeta(w,\mu) = \widehat{C}_i(w)\zeta_{\rm ren}(w),$$
$$G_i(w) = C_i^5(w,\mu)\zeta(w,\mu) = \widehat{C}_i^5(w)\zeta_{\rm ren}(w), \tag{4.66}$$

where we have defined the renormalized Isgur–Wise function in analogy to (4.27) by

$$\zeta_{\rm ren}(w) \equiv \zeta(w,\mu)K_{\rm hh}(w,\mu). \tag{4.67}$$

As in the meson case, this function is normalized at zero recoil:

$$\zeta_{\rm ren}(1) = \zeta(1,\mu) = 1. \tag{4.68}$$

The power corrections of order $1/m_Q$ involve again contributions of two types. The first come from the local dimension-four operators O_j in the expansion of the current. Their matrix elements can be parameterized in a way similar to (4.30), e.g.

$$\langle \Lambda'(v')|\bar h'\Gamma \mathrm{i}D_\alpha h|\Lambda(v)\rangle = \zeta_\alpha(v,v',\mu)\bar u(v')\Gamma u(v). \tag{4.69}$$

But now the general structure of the tensor form factor is simpler:

$$\zeta_\alpha(v,v',\mu) = \zeta_+(w,\mu)(v+v')_\alpha + \zeta_-(w,\mu)(v-v')_\alpha. \tag{4.70}$$

In the baryon case there is no analog of the function $\xi_3(w,\mu)$. The two constraints following from the equation of motion are then sufficient to relate $\zeta_\pm(w,\mu)$ to the Isgur–Wise function. One finds [45]

$$\zeta_+(w,\mu) = \frac{\bar\Lambda}{2}\frac{w-1}{w+1}\zeta(w,\mu), \qquad \zeta_-(w,\mu) = \tfrac{1}{2}\bar\Lambda\zeta(w,\mu). \tag{4.71}$$

In the baryon case, matrix elements of the local dimension-four operators are completely determined in terms of $\bar\Lambda$ and the leading-order Isgur–Wise function. Of course, $\bar\Lambda = m_\Lambda - m_Q$ will have a different value than the corresponding parameter for heavy mesons. In fact, the difference

$$\bar\Lambda_{\rm baryon} - \bar\Lambda_{\rm meson} = m_\Lambda - m_{\rm M} \approx 0.33\ \mathrm{GeV}^2 \tag{4.72}$$

is independent of m_Q and can be extracted from the known hadron masses in the b-system.

The form factors also receive corrections from insertions of higher dimension operators in the effective Lagrangian into matrix elements of the leading-order currents $J_i = \bar h'\Gamma_i h$. However, baryon matrix elements with an insertion of the chromo-magnetic operator vanish, since $\chi_{\alpha\beta}(v,v',\mu)$ in (4.38) necessarily contains Dirac matrices. This is not allowed in the baryon case. Insertions of the kinetic operator preserve the Dirac structure of the current, hence effectively correcting the Isgur–Wise function $\zeta(w,\mu)$. The total effect is

$$\langle \Lambda'(v')|\mathrm{i}\int \mathrm{d}x\, T\{J_i(0),\mathcal{L}_1(x)\}|\Lambda(v)\rangle = 2\bar\Lambda\chi(w,\mu)\bar u(v')\Gamma_i u(v). \tag{4.73}$$

We define the renormalized function $\chi_{\rm ren}(w)$ in analogy to (4.42).

It is now straightforward to work out the power corrections to the hadronic form factors $F_i(w)$ and $G_i(w)$ at subleading order in the $1/m_Q$ expansion. We find it convenient to collect certain $1/m_Q$ corrections that always appear in combination with the Isgur–Wise function into a new, renormalized function

$$\widehat{\zeta}_{bc}(w) \equiv \zeta_{ren}(w) + (\varepsilon_c + \varepsilon_b)\left(2\chi_{ren}(w) + \frac{w-1}{w+1}\zeta_{ren}(w)\right), \tag{4.74}$$

with ε_Q as defined in (4.46). Because of the dependence on the heavy-quark masses this is no longer a universal form factor. The important thing, however, is that $\widehat{\zeta}_{bc}(w)$ is still normalized at zero recoil (see below). Its flavor dependence is irrelevant as long as one considers $\Lambda_b \to \Lambda_c$ transitions only. Let us then factorize the hadronic form factors in the form

$$F_i(w) = N_i(w)\widehat{\zeta}_{bc}(w), \qquad G_i(w) = N_i^5(w)\widehat{\zeta}_{bc}(w). \tag{4.75}$$

The exact next-to-leading order expressions for the correction factors are

$$\begin{aligned}
N_1(w) &= \widehat{C}_1(\bar{w})\left(1 + \frac{2}{w+1}(\varepsilon_c + \varepsilon_b)\right),\\
N_2(w) &= \widehat{C}_2(\bar{w})\left(1 + \frac{2w\varepsilon_b}{w+1}\right) - [\widehat{C}_1(\bar{w}) + \widehat{C}_3(\bar{w})]\frac{2\varepsilon_c}{w+1},\\
N_3(w) &= \widehat{C}_3(\bar{w})\left(1 + \frac{2w\varepsilon_c}{w+1}\right) - [\widehat{C}_1(\bar{w}) + \widehat{C}_2(\bar{w})]\frac{2\varepsilon_b}{w+1},\\
N_1^5(w) &= \widehat{C}_1^5(\bar{w}),\\
N_2^5(w) &= \widehat{C}_2^5(\bar{w})\left(1 + \frac{2\varepsilon_c}{w+1} + 2\varepsilon_b\right) - [\widehat{C}_1^5(\bar{w}) + \widehat{C}_3^5(\bar{w})]\frac{2\varepsilon_c}{w+1},\\
N_3(w) &= \widehat{C}_3^5(\bar{w})\left(1 + 2\varepsilon_c + \frac{2\varepsilon_b}{w+1}\right) + [\widehat{C}_1^5(\bar{w}) - \widehat{C}_2^5(\bar{w})]\frac{2\varepsilon_b}{w+1}.
\end{aligned} \tag{4.76}$$

These results are completely determined in terms of ε_i and the short-distance coefficient functions.

It is easy to show that Luke's theorem applies to baryon matrix elements, too. From (4.71) it follows that $\zeta_\alpha(v, v, \mu) = 0$. Furthermore, vector current conservation in the full theory implies that $F(1) \equiv \sum_i F_i(1) = 1$ in the limit of equal-baryon masses. From (4.76) one finds that $F(1) = \zeta_{ren}(1) + 4\varepsilon_Q\chi_{ren}(1)$ in this limit. In addition to the normalization of the Isgur–Wise function, this requires that

$$\chi_{ren}(1) = \chi(1, \mu) = 0. \tag{4.77}$$

This in turn implies that $\widehat{\zeta}_{bc}(1) = 1$, which justifies the definition of this function in the first place.

The numerical values of the correction factors $N_i^{(5)}(w)$ depend on the value of $\bar{\Lambda}$, which is not precisely known. For the purpose of illustration, we use $m_b = 4.8$ GeV and $m_c = 1.45$ GeV as in chapter 3, and take $\bar{\Lambda} = m_{\Lambda_b} - m_b \approx 0.84$ GeV. With these parameters, $\varepsilon_c \approx 0.29$ and $\varepsilon_b \approx 0.09$. This leads to the numbers shown in Tab. 4.1, which are given in dependence of the baryon velocity transfer w over the region accessible in $\Lambda_b \to \Lambda_c \ell\bar{\nu}$ decays. For completeness, we also show the quark

Table 4.1
Correction factors for the $\Lambda_b \to \Lambda_c$ decay form factors.

w	$\bar{w}$	N_1	N_2	N_3	N_1^5	N_2^5	N_3^5
1.00	1.0	1.57	−0.42	−0.12	0.99	−0.48	0.17
1.11	1.2	1.46	−0.37	−0.11	0.94	−0.43	0.15
1.22	1.4	1.38	−0.34	−0.10	0.91	−0.39	0.14
1.33	1.6	1.30	−0.31	−0.09	0.88	−0.35	0.13
1.44	1.8	1.25	−0.29	−0.09	0.85	−0.32	0.12

velocity transfer $\bar{w}$, which according to (3.153) differs from the velocity transfer of the baryons. It is used to evaluate the Wilson coefficients.

The numerical analysis shows that symmetry breaking corrections can be quite sizable in heavy baryon decays. This is not too surprising, since $\varepsilon_c \approx 30\%$ sets the natural scale of power corrections, and the QCD corrections are typically of order $\alpha_s(m_c)/\pi \approx 10\%$. We note, however, that upon contraction with the lepton current the form factors $F_2(w)$ and $G_2(w)$ become suppressed, relative to $F_3(w)$ and $G_3(w)$, by a factor $m_{\Lambda_c}/m_{\Lambda_b} \approx 0.4$. The apparently large corrections to these form factors thus become less important when one computes physical decay amplitudes. In addition, Luke's theorem protects the quantities $\sum_i F_i(w)$ and $G_1(w)$ from $1/m_Q$ corrections at zero recoil:

$$\sum_{i=1}^{3} F_i(1) = \eta_V + O(1/m_Q^2), \qquad G_1(1) = \eta_A + O(1/m_Q^2). \tag{4.78}$$

Because of the second relation it should eventually be possible to extract an accurate value of V_{cb} from the measurement of semileptonic Λ_b decays near zero recoil, where the decay rate is governed by the form factor G_1,

$$\lim_{w\to 1} \frac{1}{\sqrt{w^2-1}} \frac{d\Gamma(\Lambda_b \to \Lambda_c \ell\bar{\nu})}{dw} = \frac{G_F^2|V_{cb}|^2}{4\pi^3} m_{\Lambda_c}^3 (m_{\Lambda_b} - m_{\Lambda_c})^2 |G_1(1)|^2. \tag{4.79}$$

The deviations from the symmetry limit $G_1(1) = 1$ are expected to be small. The reason is that, as in the meson case, the second-order power corrections to Luke's theorem are parameterically suppressed. One finds an expression similar to (4.58), namely [46]

$$G_1(1) = \eta_A - \ell_\Lambda \left(\frac{1}{2m_c} - \frac{1}{2m_b}\right)^2 + \frac{\Delta'}{4m_c m_b}. \tag{4.80}$$

The first term accounts for the deficit in the overlap of the Λ_c and Λ_b wave functions at rest. The second term is of order $1/m_c m_b$ and expected to be small. A rough estimate gives $G_1(1) \approx 0.93$ with an accuracy of approximately $\pm 7\%$.

Besides a measurement of V_{cb} near zero recoil, there are several applications of the form factor relations derived in this section. When combined with a prediction for the baryon Isgur–Wise function, the results compiled in Tab. 4.1 determine the semileptonic decay rate completely. Of particular importance are predictions that are independent of the form of the Isgur–Wise function, such as various differential (in q^2 or $v \cdot v'$) asymmetries observable in the cascade $\Lambda_b \to \Lambda_c \ell\bar{\nu} \to \Lambda X \ell\bar{\nu}$. They depend on ratios of form factors, which are predicted in HQET up to corrections of order $1/m_Q^2$. As discussed in detail by Körner and Krämer, very simple results are obtained at $q^2 = 0$ [110]. There several asymmetries are related to the parameter

$$\epsilon = (g_V - g_A)/(g_V + g_A), \tag{4.81}$$

where g_V and g_A denote the vector and axial vector couplings at zero momentum transfer (not zero recoil). The asymmetry parameter ϵ can be shown to vanish both at leading and subleading order in the $1/m_Q$ expansion, when next-to-leading logarithmic radiative corrections are neglected [46,110]. This explains the small value

$$\epsilon \approx 2.4\% + O(1/m_Q^2) \tag{4.82}$$

obtained from the numbers in Tab. 4.1.[24] A contribution of order a few percent is expected from $1/m_Q^2$ corrections. Since the prediction (4.82) is basically free of theoretical uncertainties, a precise measurement of ϵ would yield important information about the size of second-order power corrections.

The analysis presented in this section can be readily extended to the case of excited baryons. Generally, the derivations become more cumbersome as the spin of the light degrees of freedom increases, and the number of universal form factors increases accordingly. But there are no fundamental problems. We note that from the phenomenological point of view only decays of the ground state Λ_b decays into excited baryons are likely to be of interest; excited bottom baryons will decay strongly or electromagnetically into the ground-state. For the discussion of such processes the reader is referred to Refs. [96,111].

4.6. *Meson decay constants*

In this and the following section, we will discuss examples of the heavy-quark expansion for matrix elements of heavy–light currents. Although the methods are the same as in the case of transitions between two heavy quarks, the description of heavy-to-light matrix elements suffers from a proliferation of the number of universal form factors, in particular when one goes beyond the leading order in $1/m_Q$. The reason is that the light state has to be represented by a tensor wave function which is more complicated than for heavy hadrons. An exception are matrix elements which describe the amplitude for a weak current to produce a meson out of the vacuum. They are conventionally parameterized in terms of decay constants f_M defined by

$$\langle 0|\bar q\gamma^\mu\gamma_5 Q|P(v)\rangle = \mathrm{i}f_P\sqrt{m_P}\,v^\mu, \qquad \langle 0|\bar q\gamma^\mu Q|V(v,\epsilon)\rangle = \mathrm{i}f_V\sqrt{m_V}\,\epsilon^\mu, \tag{4.83}$$

where we use the mass-independent normalization of meson states as in (1.25). Decay constants are hadronic parameters of primary interest, which play a fundamental role in the phenomenology of weak decays. In principle, they can be measured in leptonic decays such as $\pi^+ \to \mu^+\nu_\mu$. The values of f_π and f_K are in fact quite precisely known from such processes. A direct measurement of heavy-meson decay constants is difficult, however. The vector mesons D^* and B^* decay by the strong or electromagnetic interactions into pseudoscalar mesons, so that it will not be possible to observe their weak decays. The leptonic decay rates of pseudoscalar mesons, on the other hand, are helicity suppressed and very small. By angular momentum conservation a spin zero particle cannot decay

[24] The relation between ϵ and the baryon form factors can be found in Ref. [46]. We note that there is a sign error in eq. (5.13) of this reference.

into a pair of massless, left-handed leptons. Hence the decay amplitude must be proportional to the lepton mass m_ℓ. Indeed, one finds that

$$\Gamma(\mathrm{P} \to \ell\bar{\nu}) = \frac{G_F^2 |V_{ij}|^2}{8\pi} f_P^2 m_P m_\ell^2 \left(1 - \frac{m_\ell^2}{m_P^2}\right)^2, \tag{4.84}$$

where V_{ij} is the element of the Cabibbo–Kobayashi–Maskawa matrix associated with the flavor quantum numbers of the meson P. For light particles, the muonic decay mode gives the dominant branching fraction

$$\mathrm{Br}(\pi^- \to \mu^- \bar{\nu}_\mu) \approx 99.99\%, \qquad \mathrm{Br}(\mathrm{K}^- \to \mu^- \bar{\nu}_\mu) \approx 63.51\%. \tag{4.85}$$

For heavy mesons, on the other hand, the total decay width scales like m_P^5, whereas $f_P^2 m_P$ approaches a constant in the $m_P \to \infty$ limit (see below). Thus the leptonic branching fraction becomes increasingly smaller as m_P becomes larger. For the D meson, decays into muons and τ-leptons have similar probability (up to a factor 2), but muons are much easier to detect experimentally. For B mesons, decays into τ-leptons are the only ones for which there is a chance of discovery in the foreseeable future. The branching fractions are tiny, however. One expects

$$\begin{aligned} &\mathrm{Br}(\mathrm{D}^- \to \mu^- \bar{\nu}_\mu) \approx 3.6 \times 10^{-4} \left(\frac{f_D}{200\ \mathrm{MeV}}\right)^2, \\ &\mathrm{Br}(\mathrm{B}^- \to \tau^- \bar{\nu}_\tau) \approx 1.3 \times 10^{-4} \left(\frac{f_B}{190\ \mathrm{MeV}}\right)^2 \left(\frac{V_{ub}}{0.005}\right)^2. \end{aligned} \tag{4.86}$$

Presently there is no evidence for any such decays.[25] For the decay constant of the D meson an upper limit has been derived [113], which is not too far off the expected value of about 200 MeV,

$$f_D < 290\ \mathrm{MeV}, \quad 90\%\ \mathrm{CL}. \tag{4.87}$$

Meson decay constants crucially appear in the theoretical description of many important phenomena such as B–$\bar{\mathrm{B}}$ mixing or nonleptonic weak decays (see, for instance, Refs. [114,115]). The present ignorance about them puts important limitations on the precision to which one knows the parameters of the Cabibbo–Kobayashi–Maskawa matrix, or on various tests of the standard model [116]. In the absence of detailed experimental information, a reliable theoretical estimate of heavy-meson decay constants is therefore most desirable. Any such estimate has to rely on nonperturbative methods, which will be discussed in chapter 5. Currently these methods still have considerable uncertainties. An alternative strategy is to relate f_{D^*}, f_B, and f_{B^*} to the decay constant f_D of the D meson, which will presumably be measured in the near future. Heavy-quark symmetry predicts such relations in the limit $m_b, m_c \gg \Lambda_{QCD}$ [16,31,54].

The starting point is again the short-distance expansion of the vector and axial vector currents in the effective theory. To order $1/m_Q$, the result for the vector current is given in (3.81). The expansion of the axial vector current is obtained by replacing $\bar{q}$ by $-\bar{q}\gamma_5$. The tensor methods of section 4.1 can be used to parameterize the matrix elements of the effective current operators in terms of universal,

[25] However, leptonic decays of D_S mesons have been observed recently by the CLEO collaboration [112].

m_Q-independent hadronic parameters. Matrix elements of the dimension-three operators J_i have the following form:

$$\langle 0|\bar{q}\Gamma h|M(v)\rangle = \tfrac{1}{2}\mathrm{i}F(\mu)\mathrm{Tr}\{\Gamma\mathcal{M}(v)\}. \tag{4.88}$$

The factor $\frac{1}{2}\mathrm{i}$ is inserted for later convenience. The parameter $F(\mu)$ depends on the scale at which the effective current operators are renormalized, but it does not depend on the heavy-quark mass m_Q. Since the physical decay constants are observable parameters, they must be independent of the renormalization procedure. This means that the μ-dependence of $F(\mu)$ must cancel against that of the Wilson coefficients $C_i(\mu)$ in (3.69). This leads us to define a renormalization-group invariant, renormalized parameter F_{ren} by

$$F_{\mathrm{ren}} = F(\mu)K_{\mathrm{hl}}(\mu). \tag{4.89}$$

Using the explicit form of the expansion of the heavy–light currents one finds that

$$f_M\sqrt{m_M} = \widehat{C}_M(m_Q)F_{\mathrm{ren}} + O(1/m_Q), \tag{4.90}$$

$$\widehat{C}_P(m_Q) = [\alpha_s(m_Q)]^{-2/\beta_0}\left[1 + \frac{\alpha_s(m_Q)}{\pi}\left(Z_{\mathrm{hl}} + \frac{2}{3}\right)\right],$$

$$\widehat{C}_V(m_Q) = [\alpha_s(m_Q)]^{-2/\beta_0}\left(1 + \frac{\alpha_s(m_Q)}{\pi}Z_{\mathrm{hl}}\right). \tag{4.91}$$

Eq. (4.90) expresses the well known asymptotic scaling law that $f_M\sqrt{m_M}$ approaches a constant (modulo logarithms of m_Q) as $m_M \to \infty$. HQET provides a rigorous derivation of this result. It also allows for a systematic study of the symmetry-breaking corrections. Neglecting terms of order $1/m_Q$ one finds that (recall that $m_P = m_V$ to leading order in the $1/m_Q$ expansion) [16,31,54,76]

$$\frac{f_B}{f_D} = \sqrt{\frac{m_D}{m_B}}\left(\frac{\alpha_s(m_c)}{\alpha_s(m_b)}\right)^{6/25}\left(1 \cancel{+ 0.894}\,\frac{\alpha_s(m_c) - \alpha_s(m_b)}{\pi}\right) \approx 0.69,$$

$$f_V/f_P = 1 - 2\alpha_s(m_Q)/3\pi. \tag{4.92}$$

This is the asymptotic form of the relations we are looking for.

Following the analysis of Neubert [54], let us now discuss the $1/m_Q$ corrections to these relations. They come from matrix elements of the local dimension-four operators O_j in (3.82), as well as the nonlocal operators T_k in (3.83). Consider first the local operators. According to (2.29), meson-to-vacuum matrix elements in the effective theory carry the momentum $\bar{\Lambda}v$. Therefore,

$$\langle 0|\mathrm{i}\partial_\alpha(\bar{q}\Gamma h)|M(v)\rangle = \tfrac{1}{2}\mathrm{i}\bar{\Lambda}F(\mu)\mathrm{Tr}\{v_\alpha\Gamma\mathcal{M}(v)\}. \tag{4.93}$$

This relation can be used to evaluate matrix elements of the operators O_4, O_5, and $O_6 - O_3$. Matrix elements of the remaining local operators contain a covariant derivative acting on the heavy-quark field. They have the general structure

$$\langle 0|\bar{q}\Gamma\mathrm{i}D_\alpha h|M(v)\rangle = \tfrac{1}{2}\mathrm{i}\,\mathrm{Tr}\{[F_1(\mu)v_\alpha + F_2(\mu)\gamma_\alpha]\Gamma\mathcal{M}(v)\}. \tag{4.94}$$

The parameters $F_i(\mu)$ are constrained by the equations of motion $\mathrm{i}v\cdot Dh = 0$ and $\mathrm{i}\not{D}q = m_q q$, where we consider the general case $m_q \neq 0$ for a moment. Contracting the matrix element in (4.94) with

v^α one finds that $F_1(\mu) = F_2(\mu)$. Furthermore, we are free to consider the special case $\Gamma = \gamma^\alpha\Gamma'$, for which the equation of motion for the light quark implies

$$\bar q\gamma^\alpha\Gamma' \mathrm{i}D_\alpha h = \mathrm{i}\partial^\alpha[\bar q\gamma_\alpha\Gamma' h] + m_q\bar q\Gamma' h. \tag{4.95}$$

Evaluating the matrix elements of these operators, one readily obtains

$$F_1(\mu) = F_2(\mu) = -\tfrac{1}{3}(\bar\Lambda - m_q)F(\mu). \tag{4.96}$$

From now on we shall set, as previously, $m_q = 0$. The equations of motion for the heavy and light quark fields imply that matrix elements of the local dimension-four operators between a heavy meson and the vacuum are all determined in terms of the product $\bar\Lambda F(\mu)$.

Next one has to evaluate the matrix elements of the nonlocal operators T_k. They represent the $1/m_Q$ corrections to the hadronic wave function arising from the presence of higher dimension operators in the effective Lagrangian. Two additional parameters $G_1(\mu)$ and $G_2(\mu)$ are needed to parameterize these corrections. They are defined by

$$\begin{aligned}&\langle 0|\mathrm{i}\int \mathrm{d}x\, T\{J_i(0), \mathcal{L}_1(x)\}|M(v)\rangle \\ &\quad = \mathrm{i}F(\mu)[G_1(\mu) + 2d_M C_{\mathrm{mag}}(\mu)G_2(\mu)]\mathrm{Tr}\{\Gamma_i\mathcal{M}(v)\}.\end{aligned} \tag{4.97}$$

Once again we encounter the coefficients $d_P = 3$ and $d_V = -1$.

It is now straightforward to derive the expressions for f_P and f_V at order $1/m_Q$ in the heavy-quark expansion. One obtains

$$\begin{aligned}f_M\sqrt{m_M} &= \widehat{C}_M(m_Q)F_{\mathrm{ren}} \\ &\quad\times\left(1 + \frac{1}{m_Q}[G_1(\mu) + 2d_M C_{\mathrm{mag}}(\mu)G_2(\mu)] - \frac{\bar\Lambda}{6m_Q}[b(\mu) + d_M B(\mu)]\right).\end{aligned} \tag{4.98}$$

The short-distance coefficients $b(\mu)$ and $B(\mu)$ are combinations of the Wilson coefficients $B_j(\mu)$ of the local dimension-four operators [54]. They are given by

$$\begin{aligned}B(\mu) &= \tfrac{16}{9}C_{\mathrm{mag}}(\mu) - \tfrac{7}{9} - \tfrac{41}{27}\alpha_s/\pi, \\ b(\mu) &= (16/\beta_0)\ln[\alpha_s(\mu)/\alpha_s(m_Q)] + \alpha_s/\pi.\end{aligned} \tag{4.99}$$

As previously, the coefficients are given here in a hybrid approach, in which the scale in the next-to-leading corrections is ambiguous. At tree level, one has $B = 1$ and $b = 0$.

Since (4.98) must be μ-independent both for pseudoscalar and vector mesons, it follows that the combinations

$$\begin{aligned}\widehat{G}_1(m_Q) &\equiv G_1(\mu) - \tfrac{1}{6}\bar\Lambda b(\mu), \\ \widehat{G}_2(m_Q) &\equiv C_{\mathrm{mag}}(\mu)G_2(\mu) - \tfrac{1}{12}\bar\Lambda[B(\mu) - 1]\end{aligned} \tag{4.100}$$

must separately be renormalization-group invariant. Notice that, unlike the individual coefficients $B_i(\mu)$ in (3.99), the expressions for $B(\mu)$ and $b(\mu)$ are just of the right form for this to be possible. At leading logarithmic order, one can define renormalized universal parameters by

$$\begin{aligned} G_1^{\rm ren} &= G_1(\mu) - (8\bar\Lambda/3\beta_0)\ln\alpha_s(\mu)\,, \\ G_2^{\rm ren} &= [\alpha_s(\mu)]^{-3/\beta_0}[G_2(\mu) - 4\bar\Lambda/27]\,, \end{aligned} \tag{4.101}$$

so that

$$\begin{aligned} \widehat G_1(m_Q) &= G_1^{\rm ren} + \tfrac{1}{6}\bar\Lambda[(16/\beta_0)\ln\alpha_s(m_Q) - \alpha_s/\pi]\,, \\ \widehat G_2(m_Q) &= [\alpha_s(m_Q)]^{3/\beta_0}\left(1 + \frac{13}{6}\frac{\alpha_s}{\pi}\right)G_2^{\rm ren} + \frac{4\bar\Lambda}{27}\left(1 + \frac{41}{48}\frac{\alpha_s}{\pi}\right). \end{aligned} \tag{4.102}$$

In terms of these quantities, the final result has the same form as at tree level

$$f_M\sqrt{m_M} = \widehat C_M(m_Q)F_{\rm ren}\left[1 + \frac{\widehat G_1(m_Q)}{m_Q} + \frac{2d_M}{m_Q}\left(\widehat G_2(m_Q) - \frac{\bar\Lambda}{12}\right)\right]. \tag{4.103}$$

The virtue of this expression is that it relates the $1/m_Q$ corrections to meson decay constants to the matrix elements of particular operators in the effective theory. The spin-symmetry breaking corrections are those proportional to d_M. They determine the ratio of vector to pseudoscalar decay constants. Using that $m_V/m_P = 1 + O(1/m_Q^2)$, one obtains

$$\frac{f_V}{f_P} = \left(1 - \frac{2\alpha_s(m_Q)}{3\pi}\right)\left[1 - \frac{8}{m_Q}\left(\widehat G_2(m_Q) - \frac{\bar\Lambda}{12}\right)\right]. \tag{4.104}$$

For a numerical estimate of the power corrections it is necessary to calculate the hadronic parameters $\widehat G_i$ using some nonperturbative approach. We will discuss such a calculation in section 5.4.

4.7. Heavy-to-light transitions and the determination of V_{ub}

Of the three independent mixing angles in the Cabibbo–Kobayashi–Maskawa matrix, V_{ub} is the most poorly determined. Chiral symmetry provides the normalization of the $K \to \pi\ell\bar\nu$ decay form factor at zero momentum transfer, allowing a precise determination of V_{us} [117]. Heavy-quark symmetry provides the normalization of the $B \to D^*\ell\bar\nu$ decay form factors at zero velocity transfer, allowing a model independent determination of V_{cb}. Neither of these symmetries by itself is as powerful in heavy-to-light transitions such as $B \to \pi\ell\bar\nu$ or $B \to \rho\ell\bar\nu$. However, when combined they may help to get a model-independent determination of V_{ub} from exclusive semileptonic decays, which is reliable than the present extraction from the endpoint region of the lepton spectrum in inclusive $B \to X_u\ell\bar\nu$ transitions [118,119].

Isgur and Wise suggested that the exclusive semileptonic decay mode $B \to \pi\ell\bar\nu$ could be used for a reliable determination of V_{ub} [120]. The basis for this hope is the fact that, to leading order in the heavy-quark expansion and over a limited kinematic range, the corresponding form factors are related to those of the process $D \to \pi\ell\bar\nu$ by heavy-quark flavor symmetry. In addition, an approximate normalization of the decay form factors at zero recoil can be derived by considering the soft pion limit [121–125]. Below, we shall discuss this decay mode in some detail. For an analysis of the similar process $B \to \rho \ell \bar{\nu}$, the reader is referred to Ref. [126].

We start by introducing hadronic form factors for the relevant matrix element of the vector current,

$$\langle \pi(p) | \bar{q} \gamma^\mu b | B(v) \rangle = 2 f_1(v \cdot p) v^\mu + 2 f_2(v \cdot p) p^\mu / v \cdot p . \tag{4.105}$$

When the lepton mass is neglected, the differential decay rate is given by

$$\begin{aligned} &\mathrm{d}\Gamma(\mathrm{B} \to \pi \ell \bar{\nu}) / \mathrm{d}(v \cdot p) \\ &\quad = \frac{G_F^2 m_B^2}{12\pi^3} |V_{ub}|^2 v \cdot p \left[1 - \left(\frac{m_\pi}{v \cdot p} \right)^2 \right]^{3/2} \left| f_2(v \cdot p) + \frac{v \cdot p}{m_B} f_1(v \cdot p) \right|^2 . \end{aligned} \tag{4.106}$$

In order to derive the behavior of the decay rate in the heavy-quark limit, we use the tensor formalism to construct the relevant matrix elements of the dimension-three heavy–light current operators in HQET. They have the structure

$$\langle \pi(p) | \bar{q} \Gamma h | M(v) \rangle = -\mathrm{Tr}\{ \Pi(v, p, \mu) \Gamma \mathcal{M}(v) \} , \tag{4.107}$$

where Γ is an arbitrary Dirac matrix. Using the fact that $\mathcal{M}(v) \not{v} = -\mathcal{M}(v)$, we can write down the most general decomposition

$$\Pi(v, p, \mu) = \gamma_5 [A(v \cdot p, \mu) + (\not{p} / v \cdot p) B(v \cdot p, \mu)] . \tag{4.108}$$

Using the explicit form of the expansion of the heavy–light vector current in (3.53), one finds that

$$\begin{aligned} f_1(v \cdot p) &= \widehat{C}_P(m_Q) A_{\mathrm{ren}}(v \cdot p) + [\widehat{C}_P(m_Q) - \widehat{C}_V(m_Q)] B_{\mathrm{ren}}(v \cdot p) + \mathrm{O}(1/m_Q) , \\ f_2(v \cdot p) &= \widehat{C}_V(m_Q) B_{\mathrm{ren}}(v \cdot p) + \mathrm{O}(1/m_Q) , \end{aligned} \tag{4.109}$$

where $\widehat{C}_P(m_Q)$ and $\widehat{C}_V(m_Q)$ have been defined in (4.91), and we have introduced the renormalized universal functions

$$A_{\mathrm{ren}}(v \cdot p) = A(v \cdot p, \mu) K_{\mathrm{hl}}(\mu) , \qquad B_{\mathrm{ren}}(v \cdot p) = B(v \cdot p, \mu) K_{\mathrm{hl}}(\mu) . \tag{4.110}$$

From (4.106), it follows that in the heavy-quark limit the decay rate is determined by $B_{\mathrm{ren}}(v \cdot p)$ alone. Since this form factor is independent of the heavy-quark mass, it drops out when one considers the ratio of the $\mathrm{B} \to \pi \ell \bar{\nu}$ and $\mathrm{D} \to \pi \ell \bar{\nu}$ decay rates at the same value of the kinematic variable $v \cdot p$. One obtains

$$\left. \frac{\mathrm{d}\Gamma(B \to \pi \ell \bar{\nu}) / \mathrm{d}(v \cdot p)}{\mathrm{d}\Gamma(D \to \pi \ell \bar{\nu}) / \mathrm{d}(v \cdot p)} \right|_{\text{same } v \cdot p} = \left| \frac{V_{\mathrm{ub}}}{V_{\mathrm{cd}}} \right|^2 \left(\frac{m_B}{m_D} \right)^2 \left| \frac{\widehat{C}_V(m_b)}{\widehat{C}_V(m_c)} \right|^2 [1 + \kappa(v \cdot p)] , \tag{4.111}$$

where $\kappa(v \cdot p)$ contains terms of order $1/m_c$ and $1/m_b$. It is only at the level of these power corrections that hadronic uncertainties enter this ratio of decay rates. Thus, from such a measurement, V_{ub} can in principle be extracted in a rather reliable way.

In order to estimate the magnitude of the nonperturbative corrections contained in κ, it is necessary to study the contributions of order $1/m_Q$ to the decay form factors in HQET. Such a detailed analysis was performed in Ref. [125]. Here we will restrict ourselves to a simple estimate. The general structure of the leading power corrections is

$$\kappa(v\cdot p) = \frac{v\cdot p + m_{\mathrm{D}^*} - m_{\mathrm{D}}}{v\cdot p + m_{\mathrm{B}^*} - m_{\mathrm{B}}} R_{\mathrm{BD}}(v\cdot p), \tag{4.112}$$

$$R_{\mathrm{BD}}(v\cdot p) = 1 + \frac{\bar\Lambda}{2m_{\mathrm{c}}} r_{\mathrm{c}}(v\cdot p) - \frac{\bar\Lambda}{2m_{\mathrm{b}}} r_{\mathrm{b}}(v\cdot p) + \mathrm{O}(1/m_{\mathrm{Q}}^2), \tag{4.113}$$

where $r_{\mathrm{Q}}(v\cdot p)$ is a dimensionless function that depends only logarithmically on the heavy-quark mass. In (4.112), we have made explicit the dominant momentum dependence of κ close to zero recoil, where $v\cdot p \to m_\pi$. Assuming that $r_{\mathrm{Q}}(v\cdot p)$ is of order unity, we expect that the scale of power corrections is set by

$$\bar\Lambda/2m_{\mathrm{c}} - \bar\Lambda/2m_{\mathrm{b}} \approx 10\%. \tag{4.114}$$

For a more reliable estimate of the power corrections, it is necessary to perform a detailed calculation using a nonperturbative approach such as lattice gauge theory or QCD sum rules. However, the decay $\mathrm{B}\to \pi\ell\bar\nu$ is special in that it is related to $\mathrm{B}^* \to \ell\bar\nu$, and hence to the decay constant of the B^* meson, in the soft pion limit. This connection can be used to show that [125]

$$\lim_{p\to 0} \frac{\widehat{C}_{\mathrm{V}}(m_{\mathrm{b}})}{\widehat{C}_{\mathrm{V}}(m_{\mathrm{c}})} R_{\mathrm{BD}}(v\cdot p) = \frac{g_{\mathrm{BB}^*\pi}\, f_{\mathrm{B}^*}\sqrt{m_{\mathrm{B}^*}}}{g_{\mathrm{DD}^*\pi}\, f_{\mathrm{D}^*}\sqrt{m_{\mathrm{D}^*}}}, \tag{4.115}$$

where $g_{\mathrm{PV}\pi}$ is the strong interaction coupling constant of a pion to a pseudoscalar and a vector meson. A reliable calculation of the right-hand side of (4.115) may be easier to obtain than a calculation of the quantity $R_{\mathrm{BD}}(v\cdot p)$. It would help to get a more reliable estimate of the power corrections to (4.111).

We conclude this section with as discussion of the range of validity of the various expansions considered above. The heavy-quark expansion is valid as long as, in the rest frame of the initial heavy meson, the energy of the light degrees of freedom before and after the weak decay is small compared to twice the heavy-quark mass. Hence, one must require that $v\cdot p/2m_{\mathrm{Q}} \ll 1$. For $m_\pi \le v\cdot p \le \frac{1}{2}(m_{\mathrm{M}}^2 + m_\pi^2)/m_{\mathrm{M}}$, where m_{M} denotes the mass of the heavy meson, the ratio $v\cdot p/2m_{\mathrm{Q}}$ varies roughly between 0 and 0.3. Hence, we expect the heavy-quark expansion to hold over essentially the entire kinematic range accessible in semileptonic decays. This assertion is in fact supported by quark model calculations [127,128]. Another important question is over what range in $v\cdot p$ can one trust the leading term in the chiral expansion, which gives rise to the soft pion relation (4.115). Since the pion is a pseudo-Goldstone boson associated with the spontaneous breaking of chiral symmetry, we expect that the scale for the momentum dependence of the universal form factors of HQET is set by $\Lambda_\chi = 4\pi f_\pi$, which is the characteristic scale of chiral symmetry breaking. Although one should not take this naive dimensional argument too seriously, we may argue that the universal form factors are slowly varying functions of $v\cdot p/\Lambda_\chi$, and the leading chiral behavior should be a good approximation until $v\cdot p \sim 1$ GeV. Hence, we expect that (4.115) should not only hold near zero recoil, but rather over a wide kinematic range. QCD sum rule calculations of the q^2-dependence of $\mathrm{B}\to \pi\ell\bar\nu$ decay form factors in the full theory support this expectation [129].

5. QCD sum rules

5.1. Nonperturbative techniques

In the previous chapters we have presented the state of the art in the theoretical understanding of the weak decay form factors of hadrons containing a heavy quark. The establishment of heavy-quark symmetry as an exact limit of the strong interactions enables one to derive approximate relations between decay amplitudes, and normalization conditions for certain form factors, which are similar to the relations and normalization conditions that can be derived for goldstone boson scattering amplitudes from the low energy theorems of current algebra. HQET provides the framework for a systematic investigation of the corrections to the limit of an exact spin–flavor symmetry. The output of such a model-independent analysis is a short-distance expansion of decay amplitudes, in which the dependence on the heavy-quark masses is explicit. At each order in the $1/m_Q$ expansion, the long-distance physics is parameterized by a minimal set of universal form factors, which are independent of the heavy-quark masses. To next-to-leading order in $1/m_Q$, we have presented these expansions for the weak decay form factors of the ground-state heavy mesons and baryons, and for the meson decay constants. The results fully incorporate the relations between form factors imposed by heavy-quark symmetry, which are often nontrivial and unexpected. An important example is that the first-order power corrections to the axial vector form factor h_{A_1} in (4.25) can be related to those to the vector form factor h_+ and can be shown to vanish at zero recoil. Similar relations exist at order $1/m_Q^2$ and eventually lead to the very precise estimate in (4.63). Another virtue of the analysis in HQET is that it relates the universal form factors to matrix elements of particular quark–gluon operators, making the connection between measurable decay amplitudes and the underlying theory of strong interactions more transparent. An example is provided by the functions χ_2 and χ_3 in (4.38), which relate hyperfine corrections arising from the interaction of the heavy-quark spin with the light degrees of freedom to matrix elements of the chromo-magnetic operator.

As presented so far the analysis is completely model independent. Since hadronic decay processes are of genuine nonperturbative nature, however, it is clear that predictions that can be made based on symmetries only are limited. In particular, very little can be said on general grounds about the properties of the form factors of the effective theory. But there is a lot of information contained in these functions. Much like the hadron structure functions probed in deep inelastic scattering, the Isgur–Wise functions [26] are fundamental, nonperturbative quantities in QCD. They describe the properties of the light degrees of freedom in the background of the static color field provided by the heavy quarks. Since, in a way, a static color source is the most direct way to probe the strong interactions of quarks and gluons at large distances, a theoretical understanding of the Isgur–Wise functions would not only enlarge the predictive power of HQET significantly, but would also teach us in a very direct way about the nonperturbative nature of the strong interactions. This is, of course, a very challenging task, and it must be emphasized that dealing with the universal form factors one leaves the safe grounds of symmetries. [27] Such investigations can nevertheless be rewarding, however. Ultimately, they are necessary for a comprehensive understanding of the physics of heavy mesons and baryons.

[26] In this chapter we shall generally refer to the universal form factors of HQET as the Isgur–Wise functions.

[27] To put it in a concise way: This is where the brown muck hits the fan.

Several nonperturbative approaches have been proposed to address the complicated problem of calculating hadronic matrix elements. Were it not for its technical limitations, the method of choice would probably be lattice gauge theory [130], which allows to perform numerical simulations of QCD on a discretized space-time lattice. This is an approach from first principles, which has the potential to reproduce all features of the theory to arbitrary accuracy. In practice, however, the technical limitations are tremendous. It is beyond the scope of this review to give a detailed description of the current status of lattice gauge theory and its applications to weak interaction phenomenology. The reader is referred to the recent reviews Refs. [131-133] and references therein. We will instead point out some difficulties specific for the study of hadrons containing a single heavy quark. For a reliable simulation of hadronic interactions one needs lattices much larger than the confinement scale, i.e. $Na \gg R_{\rm had} \sim 1/\Lambda_{\rm QCD}$, where N is the number of lattice sites in each dimension, and a is the lattice spacing. In order for the lattice not to be too coarse to simulate the interactions of a heavy quark, on the other hand, one has to require that $a \ll 1/m_Q$, so that the Compton wave length of the heavy quark is large compared to the lattice spacing. Clearly, these two requirements are in conflict as $m_Q/\Lambda_{\rm QCD}$ becomes large. Presently it is possible to achieve lattice spacings $a \approx (2\ \text{GeV})^{-1}$, which is not small enough to simulate the strong interactions of quarks heavier than the charm quark. This problem initiated many of the developments which eventually lead to the modern formulation of HQET. Eichten [27], and Lepage and Thacker [28], pointed out the utility of the static approximation: In the effective theory the length scale for fluctuations of the effective heavy-quark field h_v is set by the residual momentum, and is thus of order of the confinement scale. It is then sufficient to require that $Na \gg R_{\rm had} \gg a$, which is the same condition as for light hadronic systems. Still, the calculation of heavy meson or heavy baryon form factors is a very complicated task. The Green functions of interest receive a large contamination from excited states, and sophisticated "smeared" operators must be constructed to enhance the overlap with the ground-state. At present, calculations are limited to the so-called "quenched" or "valence" approximation, in which the effects of sea quarks are neglected [134,135]. So far, results in the effective theory have only been obtained for the decay constants of heavy mesons, but not for their weak decay form factors.

In this chapter we discuss the alternative approach provided by QCD sum rules. Several forms of sum rules are discussed in the literature. They all rely, in one way or another, on quark-hadron duality to relate hadronic matrix elements to transition amplitudes of quarks and gluons in the underlying theory. In section 5.2 we discuss two inclusive sum rules, which lead to a lower and an upper bound for the slope of the Isgur-Wise function at zero recoil. They were derived by Bjorken [93] and Voloshin [136] by imposing that the inclusive sum of the probabilities of heavy-meson decays into hadronic states should be the same as the probability for the free quark transition. In the following four sections we introduce the QCD sum rule approach of Shifman, Vainshtein, and Zakharov (SVZ) [137], and discuss its applications to the calculation of the decay constants and form factors of heavy mesons. These sum rules are on less certain grounds in that they attempt to describe exclusive processes, thus relying on local duality of the quark and hadron pictures. Yet the SVZ approach is closely related to the underlying theory. A large number of hadronic parameters and form factors have been studied using this method, often with success (for an overview see Ref. [138]). Since the first QCD sum rule analysis of heavy-quark systems by Shuryak [26], meson decay constants and form factors have been extensively studied. Recently, important improvements have been obtained by combining the SVZ technique with the effective theory for heavy quarks. The QCD sum rules for heavy mesons are now among the most elaborate sum rules known in the literature. In section

5.7, we combine the results obtained so far into a prediction for the meson decay form factors that incorporates correctly the symmetry relations and QCD corrections discussed in chapters 3 and 4. These results build the basis for a comprehensive analysis of semileptonic B decays, which is the subject of chapter 6.

We shall not discuss in detail several other approaches that have been used to calculate the universal form factors of the effective theory, although they are interesting in their own rights. We briefly mention two of them. Neubert and Rieckert have proposed a method to extract the Isgur–Wise functions for heavy mesons and baryons from overlap integrals of light-cone wave functions, or more generally from any model that predicts the hadronic form factors at $q^2 = 0$ [18]. The idea is that at fixed momentum transfer the recoil $v \cdot v'$ is a function of the hadron masses m_1 and m_2. It can be varied over the entire kinematic range $v \cdot v' \geq 1$ by adjusting the mass ratio m_1/m_2. This allows one to extract the velocity dependent form factors of HQET, at leading and next-to-leading order in the $1/m_Q$ expansion, from a knowledge of the hadronic form factors at the single point $q^2 = 0$. In Ref. [18] this approach has been applied to the BSW model [17], and explicit expressions for the Isgur–Wise functions for heavy mesons have been obtained. A similar analysis for baryon decays was performed in Ref. [139]. Another interesting approach is to study the properties of hadrons containing a heavy quark in the context of an exactly solvable toy model of the strong interactions, such as the 't Hooft model, i.e., two-dimensional Yang–Mills theory in the large N_c limit [140]. Burkardt and Swanson [141], as well as Grinstein and Mende [142], have investigated heavy-meson decay constants and weak decay form factors in this model. Some of their results are similar to those obtained from detailed studies in the full theory; however, it must be emphasized that there is no simple way to extrapolate from two to four dimensions (for instance, there is no analog of the spin symmetry in the 't Hooft model), or from an infinite number of colors to $N_c = 3$.

5.2. *Properties of the Isgur–Wise function*

The leading-order Isgur–Wise function $\xi(w)$ plays a central role in the description of the weak decays of heavy mesons. It contains the long-distance physics associated with the strong interactions of the light degrees of freedom and cannot be calculated from first principles. Nevertheless, some important properties of this function can be derived on general grounds, such as its normalization at zero recoil, which is a consequence of current conservation. According to (1.36), the Isgur–Wise function is the elastic form factor of a ground-state heavy meson in the limit where short-distance and power corrections are negligible. As such, $\xi(w)$ must be a monotonically decreasing function of the velocity transfer $w = v \cdot v'$, which is analytic in the cut w-plane with a branch point at $w = -1$, corresponding to the threshold $q^2 = 4m_Q^2$ for heavy-quark pair production [cf. (1.37)]. However, being obtained from a limiting procedure, the Isgur–Wise function can have stronger singularities than the physical elastic form factor. In fact, the short-distance corrections contained in the function $K_{hh}(w, \mu)$ lead to an essential singularity at $w = -1$ in the renormalized Isgur–Wise function defined in (4.27).

When using a phenomenological parameterization of the universal form factor one should incorporate the above properties. Some legitimate forms suggested in the literature are

$$\xi_{BSW}(w) = \frac{2}{w+1} \exp\left(- (2\varrho^2 - 1)\, \frac{w-1}{w+1}\right), \qquad \xi_{ISGW}(w) = \exp[-\varrho^2 (w-1)],$$

$$\xi_{\rm pole}(w) = [2/(w+1)]^{2\varrho^2}. \tag{5.1}$$

The first function is the form factor derived in Ref. [18] from an analysis of the BSW model [17], the second one corresponds to the ISGW model [19], and the third one is a pole-type ansatz. Of particular interest is the behavior of the Isgur–Wise function close to zero recoil, which is determined by the slope parameter $\varrho^2 > 0$ defined by $\xi'(1) = -\varrho^2$, so that

$$\xi(w) = 1 - \varrho^2\,(w-1) + \mathrm{O}[(w-1)^2]. \tag{5.2}$$

It is important to realize that the kinematic region accessible in semileptonic decays is small ($1 < w < 1.6$). As long as ϱ^2 is the same, different functional forms of $\xi(w)$ will give similar results inside this region, although they can differ substantially outside. For instance, for $\varrho^2 = 0.8$ one finds $\xi_{\rm BSW}(1.6) \approx 0.67$, $\xi_{\rm ISGW}(1.6) \approx 0.62$, and $\xi_{\rm pole}(1.6) \approx 0.66$. The conclusion is that a precise knowledge of the slope parameter would basically determine the Isgur–Wise function in the physical region.

Bjorken has shown that ϱ^2 is related to the form factors of transitions of a ground-state heavy meson into excited states in which the light degrees of freedom carry quantum numbers $j^P = \frac{1}{2}^+$ or $\frac{3}{2}^+$, by a sum rule which is an expression of quark–hadron duality: In the infinite mass limit, the inclusive sum of the probabilities for decays into hadronic states is equal to the probability for the free quark transition. If one normalizes the latter probability to unity, the sum rule has the form [93,143]

$$\begin{aligned} 1 &= \tfrac{1}{2}(w+1)\Big(|\xi(w)|^2 + \sum_l |\xi^{(l)}(w)|^2\Big) \\ &\quad + (w-1)\Big(2\sum_m |\tau_{1/2}^{(m)}(w)|^2 + (w+1)^2 \sum_n |\tau_{3/2}^{(n)}(w)|^2\Big) + \mathrm{O}[(w-1)^2], \end{aligned} \tag{5.3}$$

where l, m, n label the radial excitations of states with the same spin-parity quantum numbers. The sums are understood in a generalized sense as sums over discrete states and integrals over continuum states. The terms in the first line on the right-hand side of the sum rule correspond to transitions into states with "brown muck" quantum numbers $j^P = \frac{1}{2}^-$. The ground state gives a contribution proportional to the Isgur–Wise function, and excited states contribute proportional to analogous functions $\xi^{(l)}(w)$. Because at zero recoil these states must be orthogonal to the ground-state, it follows that $\xi^{(l)}(1) = 0$, and one can conclude that the corresponding contributions to (5.3) are of order $(w-1)^2$. The contributions in the second line correspond to transitions into states with $j^P = \frac{1}{2}^+$ or $\frac{3}{2}^+$. Because of the change in parity these are p-wave transitions. The amplitudes are proportional to the velocity $|\boldsymbol{v}_{\rm f}| = (w^2-1)^{1/2}$ of the final state in the rest frame of the initial state, which explains the suppression factor $(w-1)$ in the decay probabilities. The functions $\tau_j(w)$ are the analogs of the Isgur–Wise function for these transitions. Their precise definition is given in Ref. [143]; it is not important for the following discussion, however. Transitions into excited states with quantum numbers other than the above proceed via higher partial waves and are suppressed by at least a factor $(w-1)^2$.

For $w = 1$, eq. (5.3) reduces to the normalization condition for the Isgur–Wise function. The Bjorken sum rule is obtained by expanding in powers of $(w-1)$ and keeping terms of first order. Taking into account the definition of the slope parameter ϱ^2 in (5.2), one finds that [93,143]

$$\varrho^2 = \tfrac{1}{4} + \sum_m |\tau_{1/2}^{(m)}(w)|^2 + 2\sum_n |\tau_{3/2}^{(n)}(w)|^2 > \tfrac{1}{4}. \tag{5.4}$$

Notice that the lower bound is due to the prefactor $\frac{1}{2}(w+1)$ of the first term in (5.3) and is of purely kinematic origin. In the analogous sum rule for Λ_Q baryons this factor is absent, and consequently the slope parameter of the baryon Isgur–Wise function $\zeta(w)$ is only subject to the trivial constraint $\varrho^2_{\text{baryon}} > 0$ [144].

Based on various model calculations there is a general belief that the contributions of the excited states in the Bjorken sum rule are sizable, and that ϱ^2 is substantially larger than $\frac{1}{4}$. For instance, Blok and Shifman have estimated the contributions of the lowest-lying excited states to (5.4) using QCD sum rules and find that $0.35 < \varrho^2 < 1.15$ [145]. The experimental observation that semileptonic B decays into excited D** meson have a large branching ratio of $\approx 2.5\%$ gives further support to the importance of such contributions [146].

Voloshin has derived another, less well known sum rule involving the form factors for transitions into excited states, which is the analog of the "optical sum rule" for the dipole scattering of light in atomic physics. It reads [136]

$$\tfrac{1}{2}(m_M - m_Q) = \sum_m E_{1/2}^{(m)} |\tau_{1/2}^{(m)}(w)|^2 + 2\sum_n E_{3/2}^{(n)} |\tau_{3/2}^{(n)}(w)|^2, \tag{5.5}$$

where E_j are the excitation energies relative to the mass m_M of the ground-state heavy meson. The important point is that one can combine this relation with the Bjorken sum rule to obtain an upper bound for the slope parameter ϱ^2,

$$\varrho^2 < \tfrac{1}{4} + (m_M - m_Q)/2E_{\text{min}}, \tag{5.6}$$

where E_{min} denotes the minimum excitation energy. In the quark model one expects [28] that $E_{\text{min}} \approx m_M - m_Q$, and one may use this as an estimate to obtain $\varrho^2 < 0.75$.

In the above discussion of the sum rules we have ignored renormalization effects. However, both the slope parameter ϱ^2 in (5.4) and the heavy-quark mass m_Q in (5.5) are renormalization-scheme dependent quantities. The question arises how the sum rules can be written in a renormalization-group invariant way. Although there exist some qualitative ideas how to account for the μ-dependence of the various parameters [136,143], it is currently not known how to include renormalization effects in a quantitative way. One should therefore consider the bounds on ϱ^2 as somewhat uncertain. We account for this by relaxing the upper bound derived from the Voloshin sum rule. Tentatively, then, we conclude that

$$0.25 < \varrho^2 < 1.0, \tag{5.7}$$

where it is expected that the actual value is close to the upper bound.

Recently, de Rafael and Taron claimed to have derived an upper bound $\varrho^2 < 0.48$ from general analyticity arguments [147]. If true, this had severely constrained the form of the Isgur–Wise function near zero recoil. It took quite some time until it became clear what went wrong with the derivation in Ref. [147]: The effects of resonances below the threshold for heavy-meson pair production invalidate

[28] Strictly speaking the lowest excited "state" contributing to the sum rule is $D+\pi$, which has an excitation energy spectrum with a threshold at m_π. However, one expects that this spectrum is broad, so that this contribution will not invalidate the upper bound for ϱ^2 derived here.

the argument [148–151]. It is possible to derive a new bound, $\varrho^2 < 6$, which takes into account the known properties of the Υ-states [152]. However, it is too loose to be of any phenomenological relevance, and one is thus left with the sum rule result (5.7).

5.3. The sum rules of SVZ

In 1979, Shifman, Vainshtein, and Zakharov (SVZ) proposed a new technique for dynamical, QCD-based calculations of hadron properties [137]. Their idea was to study the correlators of certain currents at small (but not too small) euclidean distances, where asymptotic freedom allows for perturbative calculations, while nonperturbative effects can be included as power corrections in an operator product expansion. In QCD these nonperturbative effects are related to the nontrivial vacuum structure. Their inclusion in the form of vacuum expectation values of local quark–gluon operators is the most important ingredient of the SVZ approach. The sum rules are obtained by using dispersion relations to relate the current correlators to spectral densities, which have an interpretation in terms of physical intermediate states because of quark–hadron duality.

Since their proposal, the SVZ sum rules have been applied very successfully to many hadronic systems. The decay constants of heavy mesons have been first investigated in Refs. [26,153]. Meson decay form factors were considered in Refs. [154–157]. Recently, the SVZ sum rules have been reformulated in the context of HQET, both at leading [158–160] and next-to-leading order [54,56] in the $1/m_{\rm Q}$ expansion. This approach offers many advantages over the conventional one. First, by construction it implements the Ward identities of the effective theory, which lead to normalization conditions at zero recoil. Second, and not less important, it enables one to perform a systematic renormalization group improvement of the current correlators. Finally, the simple form of the Feynman rules of HQET facilitates analytical calculations. As a consequence, it has become standard to include two-loop radiative corrections in the analysis of both the two- and three-current correlation functions, a task which would require a tremendous effort in the full theory. In the following sections we review the existing calculations for the decay constants and weak decay form factors of heavy mesons, both at leading and next-to-leading order in the $1/m_{\rm Q}$ expansion. Similar studies of baryon form factors can be found in Ref. [161].

In the remainder of this section we shall illustrate the procedure for the parameter $F_{\rm ren}$ defined in (4.89), which gives the leading term in the $1/m_{\rm Q}$ expansion of heavy-meson decay constants. We shall present the formalism as developed in Refs. [54,56], which makes use of the tensor methods of section 4.1. One starts by defining interpolating currents $J_{\rm M} = \bar h_v \Gamma_{\rm M} q$ for the ground-state pseudoscalar and vector mesons, with

$$\Gamma_{\rm M} = \begin{cases} -\gamma_5; & \text{pseudoscalar meson,} \\ \gamma_\mu - v_\mu; & \text{vector meson.} \end{cases} \tag{5.8}$$

These currents can create the ground-state meson $\mathrm{M}(v)$, as well as any excited state with the right quantum numbers as long as it contains a heavy quark Q with velocity v. Consider then the two-current correlator

$$\Pi(\omega) = \mathrm{i} \int \mathrm{d}^4x\, \mathrm{e}^{ik\cdot x}\, \langle 0|T\{J_{\rm M}^\dagger(x), J_{\rm M}(0)\}|0\rangle, \tag{5.9}$$

where $\omega \equiv 2v\cdot k$, in the effective theory. In the rest frame, where $v = (1,\mathbf{0})$, it describes the generation and later annihilation of a system containing a static heavy quark and some light degrees of freedom. There is no spatial propagation of the heavy quark. The energy levels of the system can be excited by injecting some energy k^0 into the two-point function. The imaginary part of $\Pi(\omega)$ gives the spectrum of these excitations. In a general frame the two-current correlator is an analytic function in $\omega = 2v \cdot k$, with discontinuities on the positive real axis.[29] To see this, recall that the corresponding correlator in the full theory with h_v replaced by Q is analytic in the external momentum p^2 with discontinuities for $p^2 > m_Q^2$, where we set the mass of the light quark to zero. Because of the redefinition of the phase of the heavy-quark field in the effective theory, the relation between k and p is $p = m_Q v + k$. Then, in the infinite mass limit,

$$(p^2 - m_Q^2)/m_Q \to 2v \cdot k = \omega. \tag{5.10}$$

In the deep euclidean region $\omega \ll 0$, asymptotic freedom allows for a perturbative calculation of the correlator in an expansion in powers of $\alpha_s(-\omega)$. However, since we are interested in the properties of the ground-state mesons we would like to get closer to the resonance region, where nonperturbative effects become increasingly important. The idea of the SVZ approach is to work in the transition region, where nonperturbative effects are already important, but they are still small and local enough to be described as power corrections in an operator product expansion. The coefficients in this expansion are related to vacuum expectation values of local quark–gluon operators, the so-called vacuum condensates κ_n [137]. In perturbation theory these condensates would vanish by definition, but in the SVZ approach they are included as a simple way to parameterize the nontrivial vacuum structure of QCD. Hence, in this transition region the correlator is approximated as

$$\Pi(\omega) \approx \Pi_{\rm pert}(\omega) + \Pi_{\rm cond}(\omega), \tag{5.11}$$

where

$$\Pi_{\rm pert}(\omega) = \int {\rm d}\nu\, \frac{\rho_{\rm pert}(\nu)}{\nu - \omega - {\rm i}\epsilon} + \text{subtractions}, \qquad \Pi_{\rm cond}(\omega) = \sum_n C_n \frac{\kappa_n}{(-\omega)^n}. \tag{5.12}$$

The perturbative contribution is written as a dispersion integral, which may require a local subtraction polynomial in ω to be well defined. Both the spectral density $\rho_{\rm pert}$ and the Wilson coefficients C_n have a perturbative expansion in $\alpha_s(-\omega)$.

The approximation (5.11) is useful as long as only the first few terms in the series of power corrections are important. Then the number of condensates is small, since only operators with the quantum numbers of the vacuum (i.e., no spin, color, flavor etc.) can have a nonzero expectation value. For our purposes the relevant condensates have dimension lower than six. They are the quark condensate, the gluon condensate, and the mixed quark–gluon condensate [137,138]:

$$\begin{aligned} &\langle \bar q q\rangle \approx -(0.23\ \text{GeV})^3, \qquad \langle \alpha_s GG\rangle \approx 0.04\ \text{GeV}^4, \\ &\langle g\bar q\sigma_{\mu\nu}G^{\mu\nu}q\rangle \equiv m_0^2\langle\bar q q\rangle, \quad m_0^2 \approx 0.8\ \text{GeV}^2. \end{aligned} \tag{5.13}$$

The numerical values refer to a renormalization scale $\mu = 1$ GeV. These numbers have emerged from the phenomenological analysis of many hadronic systems using the sum rule approach. They are believed to have an accuracy of about 30%.

[29] The factor 2 is inserted for convenience.

The idea of QCD sum rules is to relate the theoretical approximation of $\Pi(\omega)$ to a hadronic representation of the correlator, which is obtained by saturating it with a complete set of physical intermediate states. In HQET these are states $X(v)$ containing a single heavy quark with velocity v:

$$\Pi_{\rm hadr}(\omega) = \sum_X \frac{|\langle X(v)|\,J_{\rm M}\,|0\rangle|^2}{\omega_X - \omega - \mathrm{i}\epsilon} + \text{subtractions}. \tag{5.14}$$

The sum is understood as a sum over discrete states and an integral over continuum states, and ω_X is twice the effective "mass" of the state in HQET, i.e., the mass of the physical state minus the heavy-quark mass. Let us evaluate the contribution of the ground-state meson $\mathrm{M}(v)$ to the sum. Inserting $p^2 = m_{\rm M}^2$ into (5.10) we find that $\omega_{\rm M} = 2(m_{\rm M} - m_{\rm Q}) = 2\bar\Lambda$. The hadronic matrix elements are readily evaluated using (4.88) and the trace identity

$$\Big(\sum_{\rm pol}\Big)\mathrm{Tr}\{\bar\Gamma_{\rm M}\mathcal{M}(v)\}\,\mathrm{Tr}\{\overline{\mathcal{M}}(v)\Gamma_{\rm M}\} = -2\mathrm{Tr}\{\bar\Gamma_{\rm M}P_+\Gamma_{\rm M}\}, \tag{5.15}$$

where $P_+ = \frac{1}{2}(1+\not v)$, and a sum over polarizations is understood if M is a vector meson. One obtains

$$\Pi_{\rm hadr}(\omega) = -\tfrac{1}{2}\,\mathrm{Tr}\{\bar\Gamma_{\rm M}P_+\Gamma_{\rm M}\}\frac{F^2(\mu)}{2\bar\Lambda - \omega - \mathrm{i}\epsilon} + \cdots. \tag{5.16}$$

It can be readily seen from the Feynman rules of the effective theory that any contribution to the correlator must be proportional to the same trace over Dirac matrices. This is how the spin symmetry is incorporated into the formalism. It is then convenient to define a function $\pi(\omega)$ by

$$\Pi(\omega) = -\tfrac{1}{2}\,\mathrm{Tr}\{\bar\Gamma_{\rm M}P_+\Gamma_{\rm M}\}\pi(\omega). \tag{5.17}$$

Equating the theoretical and the hadronic representation of the correlator $\pi(\omega)$, one obtains the sum rule

$$\frac{F^2(\mu)}{2\bar\Lambda - \omega - \mathrm{i}\epsilon} = \int \mathrm{d}\nu\,\frac{\rho_{\rm pert}(\nu)}{\nu - \omega - \mathrm{i}\epsilon} - \sum_{X\neq \rm M}\frac{F_X^2(\mu)}{\omega_X - \omega - \mathrm{i}\epsilon} + \text{subtractions} + \pi_{\rm cond}(\omega), \tag{5.18}$$

where we have written the contributions of excited states in terms of reduced matrix elements $F_X(\mu)$, which are defined in analogy to $F(\mu)$. For simplicity we use the same symbol for the perturbative spectral density after the trace has been factored out.

Now comes an important assumption of the SVZ approach: In order to be able to evaluate the right-hand side of the sum rule without a detailed knowledge of the spectrum and the matrix elements of excited states, one employs local (in ω) quark–hadron duality. The picture is that the nonperturbative QCD interactions, which are responsible for the differences in the perturbative and the hadronic spectral densities, distort the spectrum in such a way that when one "smears" the physical spectrum over a few levels by integration with some smooth weight function, one obtains locally the same spectral density as in perturbation theory. This allows one to replace the sum over excited states by the integral over the perturbative spectral density above the so-called continuum threshold ω_0:

$$\sum_{X\neq \rm M}\frac{F_X^2(\mu)}{\omega_X - \omega - \mathrm{i}\epsilon} \approx \int_{\omega_0}^{\infty}\mathrm{d}\nu\,\frac{\rho_{\rm pert}(\nu)}{\nu - \omega - \mathrm{i}\epsilon}. \tag{5.19}$$

Effectively, this absorbs the ignorance about the structure of higher resonance contributions into a single parameter. A more refined description would be to retain a sum over the lowest lying states, and to approximate only very high and broad resonances by the perturbative continuum. However, this would introduce a set of unknown parameters F_X and ω_X, and the approach would lose much of its simplicity and predictive power.

Under the assumption (5.19) the sum rule simplifies to

$$\frac{F^2(\mu)}{2\bar{\Lambda}-\omega-\mathrm{i}\epsilon}=\int_0^{\omega_0}\mathrm{d}\nu\,\frac{\rho_{\mathrm{pert}}(\nu)}{\nu-\omega-\mathrm{i}\epsilon}+\text{subtractions}+\pi_{\mathrm{cond}}(\omega)\,. \tag{5.20}$$

This is still not very useful. In order to extract information about the ground-state one has to go to small values of $(-\omega)$, so that the weight function in the dispersion integral enhances the low energy contributions relative to the high energy ones. But on the other hand the theoretical calculation is reliable only when $(-\omega)$ is large. The observation of SVZ was that an "optimal balance" between these contradictory requirements can be achieved by means of a Borel improvement of the sum rule. The idea is the following: An alternative way to extrapolate the function $\pi(\omega)$ from large values of $(-\omega)$ to smaller ones is by taking derivatives. When $(-\omega)$ tends to infinity, an arbitrary number n of derivatives can be calculated in a reliable way. By considering the simultaneous limits $-\omega\to\infty$ and $n\to\infty$, with $T=-\omega/n$ fixed, one can explore the behavior of the function at scales T. This procedure defines the Borel transformation,

$$\widehat{B}_T^{(\omega)}f(\omega)\equiv\lim_{\substack{n\to\infty\\ -\omega\to\infty}}\frac{(-\omega)^{n+1}}{\Gamma(n+1)}\left(\frac{\mathrm{d}}{\mathrm{d}\omega}\right)^n f(\omega)\,,\qquad T=\frac{-\omega}{n}\ \text{fixed.} \tag{5.21}$$

$T>0$ is called the Borel parameter. The application of this transformation improves the sum rule (5.20) in a threefold way. To see this, note that

$$\widehat{B}_T^{(\omega)}\frac{1}{\nu-\omega-\mathrm{i}\epsilon}=\exp(-\nu/T)\,,\qquad \widehat{B}_T^{(\omega)}\frac{1}{(-\omega)^n}=\frac{1}{\Gamma(n)T^{n-1}}\,. \tag{5.22}$$

The first relation shows that the weight factor in the dispersion integral becomes an exponential, increasing the sensitivity to the ground-state. According to the second relation, the nonperturbative power corrections get multiplied by factors $1/n!$, which improves the convergence of the series by suppressing the contributions from higher dimension condensates. Finally, taking an infinite number of derivatives eliminates the unwanted subtraction polynomials in the dispersion relations. This leads to the final form of the sum rule,

$$F^2(\mu)\,\exp(-2\bar{\Lambda}/T)=\int_0^{\omega_0}\mathrm{d}\omega\,\rho_{\mathrm{pert}}(\omega)\,\mathrm{e}^{-\omega/T}+\widehat{B}_T^{(\omega)}\pi_{\mathrm{cond}}(\omega)\equiv K(\omega_0,T,\mu)\,. \tag{5.23}$$

We have replaced the integration variable ν by ω. Note that, as a result of the theoretical calculation, the right-hand side of the sum rule will depend on the renormalization scale μ. This dependence must match that of the hadronic parameter $F^2(\mu)$ on the left-hand side. The renormalization-group invariant form of the sum rule is obtained by introducing the renormalized parameter F_{ren} defined in (4.89). One obtains

$$F_{\mathrm{ren}}^2\,\exp(-2\bar{\Lambda}/T)=K_{\mathrm{hl}}^2(\mu)K(\omega_0,T,\mu)\,. \tag{5.24}$$

The μ-dependence on the right-hand side must cancel to all orders in perturbation theory. A second sum rule, which is independent of $F_{\rm ren}$, can be obtained by taking the logarithmic derivative with respect to the inverse Borel parameter:

$$\bar{\Lambda} = -\tfrac{1}{2}(\partial/\partial T^{-1})\ln K(\omega_0, T, \mu) . \tag{5.25}$$

Note that this is μ-independent. By virtue of the Borel improvement it is possible to explore these relations in the region of relatively small values of T, which was not accessible before.

The rest of the program is as follows: One derives the theoretical expression for the function $K(\omega_0, T, \mu)$ by calculating the perturbative spectral density and the first few power corrections using the Feynman rules of the effective theory. One then tries to find a sensible choice of the continuum threshold ω_0 such that the right-hand side of the sum rule (5.25) becomes approximately independent of the arbitrary Borel parameter T. In a last step one uses the so-determined values of ω_0 and $\bar{\Lambda}$ as input parameters for the sum rule (5.24) to obtain $F_{\rm ren}$. The existence of a set of parameters for which both sum rules become simultaneously stable (i.e., independent of T) is not always guaranteed. The SVZ approach is a self-consistent one: The existence of a stability region justifies a posteriori the assumption of local duality. Very important in this context is the notion of the so-called "sum rule window", which determines the region in T for which the assumptions and approximations inherent in the method are justified. When the Borel parameter becomes too small, the nonperturbative corrections blow up, leading to a break-down of the operator product expansion. On the other hand, since T plays the role of a temperature [30] in the integral over the spectral density, one has to keep it small enough so that only low lying excitations close to the ground-state give a sizable contribution. Hence the aim is to obtain optimal stability inside a window roughly given by $\Lambda_{\rm QCD} < T < 2\bar{\Lambda}$, where $2\bar{\Lambda}$ is twice the "mass" of the ground state.

Results obtained from a QCD sum rule analysis are meaningless without a careful discussion of the theoretical uncertainties. This is a somewhat subtle issue, which often receives too little attention. One should distinguish two classes of uncertainties: First, there are well defined approximations made in the theoretical calculation of the correlator, such as the truncation of the perturbative expansion and of the series of power corrections. The corresponding errors can often be estimated on dimensional grounds and by considering the convergence of the terms that are included in the calculation. Similarly, there will be uncertainties arising from the fact that the QCD parameters such as the vacuum condensates are not very accurately known. They can be estimated by varying these parameters within reasonable limits. Much harder to obtain is an estimate of the systematic uncertainties inherent in the method, which are related to the assumption of local duality. One should keep in mind that these systematic errors, which are not controlled by any small parameter, can sometimes be larger than the theoretical uncertainties of the first type. It is important that one carefully determines the sum rule window and checks whether or not a satisfactory stability can be obtained.

5.4. Meson masses and decay constants

Let us now analyze the sum rules (5.24) and (5.25) in detail. The theoretical calculation of the function $K(\omega_0, T, \mu)$ involves the evaluation of the diagrams shown in Fig. 5.1. The perturbative

[30] As suggested by Radyushkin, an alternative way to interpret the Borel parameter is to identify $t_{\rm E} = 1/T$ with a euclidean time [160]. This exhibits the relation between QCD sum rules and lattice gauge theory.

contributions consist of the bare quark loop and the complete set of two-loop diagrams, which give the corrections of order α_s. The leading nonperturbative contributions are proportional to the quark condensate (dimension three) and to the mixed quark–gluon condensate (dimension five). The contribution from the gluon condensate is found to vanish. For a consistent calculation at order α_s, one should include the one-loop radiative corrections to the quark condensate, whereas it is sufficient to compute the Wilson coefficients of higher dimension condensates at tree level. The ultraviolet divergences of the loop diagrams are removed upon the renormalization of the heavy–light currents J_M by the Z-factor given in (3.60). One finds that the renormalized correlator has indeed the same μ-dependence as $F^2(\mu)$, so that the right-hand side of (5.24) is μ-independent. One is free to evaluate it for any choice $\mu = \Lambda$, where Λ is a characteristic scale of the low energy effective theory. Let us then define a renormalization-group invariant function $\widehat{K}(\omega_0, T)$ by

$$F_{\rm ren}^2 \exp(-2\bar{\Lambda}/T) = [\alpha_s(\Lambda)]^{4/\beta_0}\left[1 - \frac{2\alpha_s(\Lambda)}{\pi}\left(Z_{\rm hl} + \frac{2}{3}\right)\right]\widehat{K}(\omega_0, T). \tag{5.26}$$

The perturbative factor is chosen to match the Wilson coefficient $\widehat{C}_{\rm P}(m_{\rm Q})$ in (4.91). From the calculation of the Feynman diagrams one obtains the result [158,159,162]

$$\widehat{K}(\omega_0, T) = \frac{3}{8\pi^2}\int\limits_0^{\omega_0} d\omega\, \omega^2\, e^{-\omega/T}\left(1 + \frac{\alpha_s(\Lambda)}{\pi}[\tfrac{13}{3} + \tfrac{4}{9}\pi^2 - 2\ln(\omega/\Lambda)]\right)$$
$$- \langle\bar{q}q\rangle(\Lambda)[1 + 2\alpha_s(\Lambda)/3\pi] + m_0^2\langle\bar{q}q\rangle/4T^2. \tag{5.27}$$

Note that changes in Λ in the prefactor in (5.26) are compensated by changes in the logarithmic term in the dispersion integral, and by the running of the quark condensate. By virtue of the next-to-leading order renormalization group improvement the result is independent of the choice of Λ. Following Ref. [158] we set $\Lambda = 1.15$ GeV, which will turn out to be approximately equal to $2\bar{\Lambda}$, i.e., twice the mass of the ground-state in the effective theory. This has several advantages over the other popular choice $\Lambda = T$: First, $\bar{\Lambda}$ is a fundamental mass parameter of HQET, whereas T is a mathematical parameter introduced by the Borel transformation. Second, at the lower end of the sum rule window the Borel parameter is too small for a perturbative expansion in $\alpha_s(T)$. Finally, keeping the coupling constant $\alpha_s(\Lambda)$ independent of T simplifies the analysis of the sum rules, for instance when taking derivatives with respect to T as in (5.25). However, it must be emphasized that from the theoretical point of view $\Lambda = T$ is as valid a choice as any other.

As emphasized in the previous section, the first step in the analysis of the sum rules is to find the "sum rule window". The lower value of T for which the calculation is reliable is determined by the requirement that the nonperturbative power corrections not be too large. To be specific, we would like these terms to be at most 30% of the perturbative contribution. This yields the lower bound $T \geq 0.6$ GeV. We should also not consider too large values of T, for which the integral over the perturbative spectral density becomes dominated by excited states. Since the spectral density $\rho_{\rm pert}(\omega)$ grows like ω^2, it is unavoidable that higher resonance contributions are substantial even after the Borel improvement. In order to reduce the sensitivity to how well these contributions are approximated by the assumption of duality, we require that the pole contribution of the heavy meson M give at least 30% of the quark loop. For typical threshold values $\omega_0 \approx 2$ GeV this implies $T \leq 1$ GeV. In Fig. 5.2(a) we show the numerical evaluation of the sum rule (5.25) for three different

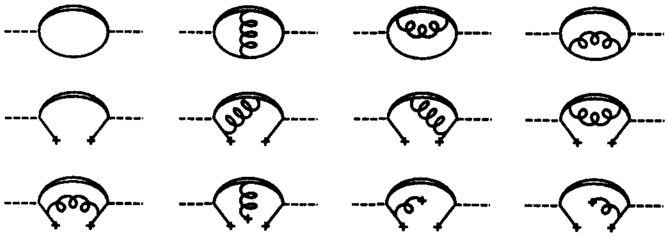


Fig. 5.1. Feynman diagrams contributing to the sum rule for meson decay constants in the effective theory. The interpolating currents are represented by dashed lines.

values of the continuum threshold. It supports the self-consistency of the procedure that the resulting value of $\bar{\Lambda}$ is indeed almost independent of the Borel parameter over the region where one can hope for stability. One obtains [31]

$$\omega_0 = 2.0 \pm 0.3 \text{ GeV}, \qquad \bar{\Lambda} = 0.57 \pm 0.07 \text{ GeV}. \tag{5.28}$$

The results are very stable under variations of the vacuum condensates within reasonable limits. For instance, using values in the ranges $(-\langle\bar{q}q\rangle)^{1/3} = 0.23 \pm 0.02$ GeV and $m_0^2 = 0.8 \pm 0.2$ GeV^2 changes the result for $\bar{\Lambda}$ by less than ± 0.03 GeV. Note that $\omega_0/2 \approx 2\bar{\Lambda}$, which is a very reasonable result meaning that the (effective) continuum starts at twice the mass of the ground-state.

In the next step, the numerical results in (5.28) are used as an input for the sum rule (5.26), from which one extracts the hadronic parameter F_{ren}. The result is shown in Fig. 5.2(b). One observes excellent stability over the entire "sum rule window". Again the main uncertainty is due to the choice of the continuum threshold. We find

$$F_{\text{ren}} = 0.40 \pm 0.06 \text{ GeV}^{3/2}. \tag{5.29}$$

Very similar results for $\bar{\Lambda}$ and F_{ren} have been obtained by Bagan et al. [159].

At this point several remarks are in order. The first concerns the truncation of the series of power corrections. Taking $T \approx 0.8$ GeV as a typical value, one finds that the contribution of the mixed condensate is three times smaller than that of the quark condensate. Also the contributions of the dimension six condensates to the sum rules are known [158], and they are suppressed by three orders of magnitude relative to the quark condensate. We conclude that the series of nonperturbative corrections converges rapidly and can safely be truncated after the mixed condensate. Unfortunately, the perturbative expansion is not on such safe grounds. The coefficient of the order-α_s correction to the perturbative spectral density is very large: $\frac{13}{3} + \frac{4\pi^2}{9} \approx 8.7$. In Feynman gauge, most of the effect ($\approx 80\%$) comes from the gluon exchange between the heavy and the light quark, i.e., from their Coulomb attraction [159]. Prior to the development of the sum rules in the effective theory it was assumed that $\alpha_s(m_Q)$ would be the coupling associated with these large radiative corrections [153]. The important new result is that this is not correct. By virtue of the next-to-leading order renormalization group improvement of the currents in HQET, we now know that the relevant coupling $\alpha_s(\Lambda)$ is much larger. To illustrate the effect of the next-to-leading corrections we show in Fig. 5.2

[31] We correct the value for $\bar{\Lambda}$ quoted in Ref. [54], which is 15% too small due to an unfortunate error in the numerical evaluation program.

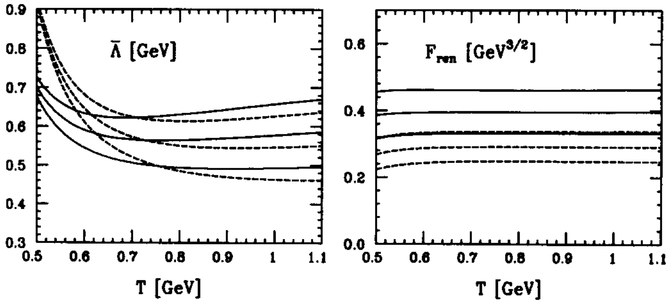


Fig. 5.2. Numerical evaluation of the sum rules (5.25) and (5.26) for different values of the continuum threshold. We show $\bar\Lambda$ and $F_{\rm ren}$ as functions of the Borel parameter. From top to bottom, the curves correspond to ω_0 = 2.3, 2.0, 1.7 GeV. The dashed lines are obtained by neglecting radiative corrections in the sum rule.

as dashed lines the numerical results obtained by setting $\alpha_s = 0$ in the function $\widehat K(\omega_0, T)$ in (5.27). While this has very little effect on $\bar\Lambda$, basically because of the logarithmic derivative in (5.25), the prediction for $F_{\rm ren}$ changes by 30%. One might argue that such a radiative correction is too large to be trustworthy, and that it might even indicate a break-down of the perturbative expansion. Indeed it would be interesting to know if a similar effect occurs at two-loop order. In any case, the effect is purely perturbative and not specific to QCD sum rules. It is a general property of the correlator of two heavy-light currents in the $m_Q \to \infty$ limit.

The sum rule prediction for the renormalized parameter $F_{\rm ren}$ can be related to the so-called static limit of the decay constant of the B meson, which is a somewhat artificial quantity often used in the literature on lattice gauge theory. It is defined as

$$f_{\rm B}^{\rm stat} \equiv (\widehat C_{\rm P}(m_{\rm b})/\sqrt{m_{\rm B}})\,F_{\rm ren}, \tag{5.30}$$

and is the decay constant of the B meson in the fictitious limit where $1/m_{\rm b}$ power corrections are absent. The short-distance coefficient $\widehat C_{\rm P}(m_{\rm b})$ relevant for a pseudoscalar meson has been given in (4.91). Successively decreasing the number of light flavors when scaling down from $m_{\rm b}$ to scales below $m_{\rm c}$, one finds that

$$\begin{aligned}\widehat C_{\rm P}(m_{\rm b}) &= \left(\frac{\alpha_s(m_{\rm b})}{\alpha_s(m_{\rm c})}\right)^{-6/25} [\alpha_s(m_{\rm c})]^{-2/9} \\ &\quad\times\left(1 - 0.894\frac{\alpha_s(m_{\rm b}) - \alpha_s(m_{\rm c})}{\pi} - 0.855\frac{\alpha_s(m_{\rm c})}{\pi}\right) \approx 1.37.\end{aligned} \tag{5.31}$$

When combined with the sum rule prediction (5.29) this leads to

$$f_{\rm B}^{\rm stat} = 240 \pm 40 \text{ MeV}. \tag{5.32}$$

Let us compare this prediction to some recent computations of $f_{\rm B}^{\rm stat}$ in lattice gauge theory, which use the static approximation for the heavy-quark propagator [27]. In these analyses one "measures" the parameter $F(\mu = a^{-1})$ in units of $a^{-3/2}$, with a being the lattice spacing. Two steps are necessary to convert this into a meaningful physical prediction. First, the lattice spacing has to be eliminated by normalizing the result to some other physical quantity. For this purpose the various groups use the decay constant of the pion, the string tension, the mass of the ρ-meson, or the mass splitting between the $1s$ and $1p$ states in charmonium. Second, a renormalization factor is required to relate the lattice results to the renormalized parameter $F_{\rm ren}$. As in the case of QCD sum rules, this perturbative part of the analysis suffers from the fact that the coefficient of the next-to-leading correction of order α_s is rather large [168,169]. After going through these steps, the different groups present the following results:[32]

$$f_{\rm B}^{\rm stat} = \begin{cases} 235 \pm 29 \text{ MeV} ; & \text{Ref. [163]}, \\ 260 \pm 47 \text{ MeV} ; & \text{Ref. [164]}, \\ 230 \pm 34 \text{ MeV} ; & \text{Ref. [165]}, \\ 319 \pm 11 \text{ MeV} ; & \text{Ref. [166]}, \\ 253^{+105}_{-21} \text{ MeV} ; & \text{Ref. [167]}. \end{cases} \tag{5.33}$$

What we said before about sum rules is equally true in this case: The quoted errors reflect only part of the systematic uncertainties, which arise from finite volume effects, nonzero lattice spacings, and using the quenched approximation. Clearly, there are substantial differences between the results of the different groups, which to some extent may be due to the different scale setting methods employed. On the other hand, there seems to be an overall agreement between lattice gauge theory and QCD sum rules that the static limit of the B meson decay constant is large, probably well above 200 MeV.

This result is surprising. Recall that the experimental upper limit for $f_{\rm D}$ is 290 MeV, so that taking (4.92) as a guideline one would expect that $f_{\rm B}$ should be smaller than 200 MeV. Furthermore, both QCD sum rule and lattice gauge calculations in the full theory, i.e. with dynamical heavy quarks, predict that $f_{\rm D}$ is smaller than the upper limit: $f_{\rm D} \approx 200$ MeV [153,158,159,163,167]. This indicates that $1/m_{\rm Q}$ corrections to the asymptotic relations (4.92) must be substantial. In HQET these corrections are parameterized by the coefficients G_1 and G_2 in (4.97). By including insertions of the higher dimension operators from the effective Lagrangian into the current correlator (5.9), it is possible to derive sum rules for these parameters [54]. In the same way that the analysis of F was closely related to that of $\bar\Lambda$, it turns out that the analysis of G_i involves the mass parameters λ_i, which describe the $1/m_{\rm Q}$ corrections to the physical meson masses [cf. (2.34)]. The spin-symmetry violating parameters G_2 and λ_2 can be extracted with rather good accuracy, since the corresponding sum rules show excellent stability. For the quantities G_1 and λ_1, however, the stability in the original sum rule analysis in Ref. [54], in which only the leading contributions were taken into account, was not satisfactory. The situation has been improved recently with the inclusion of two-loop radiative corrections [108,171].

We first discuss the corrections to the meson masses. The relevant equations have been given in (2.32)–(2.36). The QCD sum rule analysis of the parameter λ_2 yields [54,108]

$$\lambda_2 = 0.12 \pm 0.02 \text{ GeV}, \tag{5.34}$$

which refers to a renormalization scale $\mu = 2\bar\Lambda$. According to (2.36), this is one quarter of the vector-pseudoscalar mass splitting in the effective theory. In order to compare this to the mass splittings of the physical mesons, one has to include the Wilson coefficient $C_{\rm mag}(\mu)$ of the chromo-magnetic operator. We evaluate it at $\mu = 2\bar\Lambda$, and use $\alpha_s/\pi = 0.1$ in the next-to-leading correction in (3.80). This yields

[32] The result of Ref. [164] increases to 310 ± 56 MeV when the lattice spacing is determined without using f_π.

$$m_{B^*}^2 - m_B^2 = 0.45 \pm 0.08\ \mathrm{GeV}^2, \qquad m_{D^*}^2 - m_D^2 = 0.54 \pm 0.09\ \mathrm{GeV}^2, \tag{5.35}$$

in excellent agreement with the experimental numbers given in (1.22). The most recent sum rule result for λ_1 is $\lambda_1 = -0.5 \pm 0.1\ \mathrm{GeV}^2$ [108]. However, there are arguments that this value is too large [172]. Important in this context is the field-theory analog of the virial theorem, which relates the kinetic energy of a heavy quark inside a hadron H_Q to a matrix element of the gluon field strength tensor [109],

$$\frac{\langle H_Q(v')|\bar{h}_{v'} ig G^{\mu\nu} h_v|H_Q(v)\rangle}{\langle H_Q(v)|\bar{h}_v h_v|H_Q(v)\rangle} = -\tfrac{1}{3}\lambda_1 (v^\mu v'^\nu - v^\nu v'^\mu)[1 + O(w-1)]. \tag{5.36}$$

In the context of QCD sum rules, this theorem implies that λ_1 should not receive contributions from diagrams without gluons, making explicit an "intrinsic smallness" of this parameter. This requirement is not fulfilled in the analysis of Ref. [108]. In fact, a QCD sum rule based on the virial theorem give a smaller value [173], which agrees well with the result $\lambda_1 = -0.18 \pm 0.06\ \mathrm{GeV}^2$ that can be extracted from the analysis of Eletsky and Shuryak [107]. Taking all these estimates into account, we quote

$$\lambda_1 = -0.25 \pm 0.20\ \mathrm{GeV}^2. \tag{5.37}$$

Using this together with the sum rule prediction for $\bar{\Lambda}$ in (5.28), we can calculate the heavy-quark masses m_b and m_c. Recall from section 2.6 that these masses are well defined parameters of HQET and should be considered as generalized pole masses. From (2.32), we obtain

$$m_Q = \overline{m}_M - \bar{\Lambda} + \lambda_1/m_Q + O(1/m_Q^2), \tag{5.38}$$

where

$$\overline{m}_M = \tfrac{1}{4}(m_P + 3m_V) \tag{5.39}$$

denotes the spin averaged meson mass. Using the experimental values $\overline{m}_B = 5.31$ GeV and $\overline{m}_D = 1.97$ GeV, we obtain

$$\begin{aligned} m_b &= 4.71 \pm 0.07\ (\bar{\Lambda}) \pm 0.02\ (\lambda_1) \pm 0.004\ (\text{h.o.})\ \mathrm{GeV}, \\ m_c &= 1.30 \pm 0.07\ (\bar{\Lambda}) \pm 0.08\ (\lambda_1) \pm 0.05\ (\text{h.o.})\ \mathrm{GeV}, \\ m_b - m_c &= 3.34 \pm 0.06\ (\lambda_1) \pm 0.04\ (\text{h.o.})\ \mathrm{GeV}, \end{aligned} \tag{5.40}$$

where we have given separate estimates of the theoretical uncertainties resulting from variation of $\bar{\Lambda}$ and λ_1, as well as from higher-order power corrections, which we estimate as $(\bar{\Lambda}/2m_Q)^2$. Adding the errors in quadrature, we find $m_b = 4.71 \pm 0.07$ GeV and $m_c = 1.30 \pm 0.12$ GeV. These values compare well with other estimates for the pole masses obtained from QCD sum rules for quarkonium spectra [137,174,175], and from a calculation using lattice gauge theory [176].

Let us then turn to the discussion of power corrections to the meson decay constants, starting again with the spin-symmetry breaking corrections. According to (4.104) they are determined by $\bar{\Lambda}$ and the renormalized parameter $\widehat{G}_2(m_Q)$. For the bottom and charm systems, one finds [54]

$$\widehat{G}_2(m_b) = -26 \pm 4\ \mathrm{MeV}, \qquad \widehat{G}_2(m_c) = -44 \pm 7\ \mathrm{MeV}. \tag{5.41}$$

This can be combined with the result for $\bar{\Lambda}$ in (5.28) to predict the ratios of vector and pseudoscalar meson decay constants,

$$f_{B^*}/f_B = 1.07 \pm 0.02, \qquad f_{D^*}/f_D = 1.35 \pm 0.05. \tag{5.42}$$

These results compare well with a recent calculation of f_V/f_P in lattice gauge theory, which gives 1.12 ± 0.05 and 1.30 ± 0.06 for the two ratios [170]. We take this as an indication that hyperfine corrections are well accounted for by including the leading power corrections in the heavy-quark expansion. The most accurate determination of the spin-symmetry conserving parameter $\widehat{G}_1(m_Q)$ has been presented in Ref. [171]. The result is

$$\widehat{G}_1(m_b) = -0.81 \pm 0.15\ \text{GeV}, \qquad \widehat{G}_1(m_c) = -0.72 \pm 0.15\ \text{GeV}. \tag{5.43}$$

For the slope parameter $A(m_Q)$ defined by

$$f_P\sqrt{m_P} = \widehat{C}_P(m_Q)F_{\text{ren}}[1 + A(m_Q)/m_Q + \cdots], \tag{5.44}$$

where $A = G_1 + 6G_2 - \frac{1}{2}\bar{\Lambda}$ according to (4.103), this implies the rather large value $A(m_b) \approx A(m_c) \approx -1.2$ GeV. This direct calculation of $1/m_Q$ corrections in the effective theory may be confronted with indirect determinations of the parameter A, obtained by fitting the mass dependence of the decay constant as calculated from QCD sum rules in the full theory to (5.44). This yields $A \approx -(0.7-1.2)$ GeV [107,158,159]. An estimate of A can also be obtained from lattice gauge theory, by matching results obtained in the full theory (for $m_P < 2.6$ GeV) with the predictions of the static limit ($m_P = \infty$). Depending on whether a linear or a quadratic fit (in $1/m_P$) is used for the extrapolation, the authors of Ref. [170] find $A \approx -0.8$ GeV or $A \approx -1.6$ GeV from such an analysis.

It is clear from this discussion that there are substantial $1/m_Q$ corrections to the first relation in (4.92). Even for the decay constant of the B meson, a value $A \approx -1$ GeV corresponds to a 20% correction and would reduce f_B^{stat} in (5.32) to $f_B \approx 190$ MeV. Fortunately, we will see below that such a situation is not generic, but rather an anomaly specific for heavy-meson decay constants. The $1/m_Q$ corrections to the weak decay form factors will turn out to be much smaller.

5.5. The Isgur–Wise function

So far the discussion has focused on the simplest case of correlators of two heavy-light currents containing a heavy quark with the same velocity. For the investigation of the weak decay form factors one has to extend the analysis to the case of different velocities v and v'. One has to consider three-current correlators, which contain two interpolating currents J_M and $J_{M'}$ for the heavy mesons M(v) and M$'(v')$, as well as a heavy-quark current $J_\Gamma = \bar{h}'_{v'}\Gamma h_v$ that changes the velocities. For the analysis of the Isgur–Wise function, the correlator of interest is

$$\widetilde{\Xi} = \int \mathrm{d}x\,\mathrm{d}y\, \exp[\mathrm{i}(k'\cdot x - k\cdot y)]\langle 0|T\{J_{M'}^\dagger(x), J_\Gamma(0), J_M(y)\}|0\rangle. \tag{5.45}$$

Because of heavy-quark symmetry the flavor of the heavy quarks is completely irrelevant. What matters are the velocities v and v'. In fact, the Feynman rules of the effective theory allow one to write

$$\widetilde{\Xi} = \Xi(\omega, \omega'; w)\,\mathrm{Tr}\{\overline{\Gamma}_{M'}P'_+\Gamma P_+\Gamma_M\}, \tag{5.46}$$

where P_+ and P'_+ are the usual projection operators. All information about the Dirac quantum numbers of the states and of the heavy-quark current are contained in the trace. The coefficient function Ξ is analytic in the variables $\omega = 2v\cdot k$ and $\omega' = 2v'\cdot k'$. It furthermore depends on the velocity transfer $w = v\cdot v'$.

The evaluation of the correlator follows the standard lines discussed in section 5.3. In the "not so deep" euclidean region, Ξ is approximated by a perturbative calculation supplemented by nonperturbative power corrections proportional to the vacuum condensates. The perturbative contribution is written in form of a double dispersion integral in ω and ω' plus subtraction terms, which in this case can contain single dispersion integrals:

$$\Xi_{\rm theo} \approx \int \mathrm{d}\nu\,\mathrm{d}\nu'\,\frac{\rho_{\rm pert}(\nu,\nu';w)}{(\nu-\omega-\mathrm{i}\epsilon)(\nu'-\omega'-\mathrm{i}\epsilon)} + \text{subtractions} + \Xi_{\rm cond}. \tag{5.47}$$

On the other hand, the correlator can be written as a dispersion integral over physical intermediate states. We explicitly separate the double-pole contribution associated with the ground-state mesons from higher resonance contributions, which we describe by a physical spectral density $\rho_{\rm res}$:

$$\Xi_{\rm phen} = \Xi_{\rm pole} + \int \mathrm{d}\nu\,\mathrm{d}\nu'\,\frac{\rho_{\rm res}(\nu,\nu';w)}{(\nu-\omega-\mathrm{i}\epsilon)(\nu'-\omega'-\mathrm{i}\epsilon)} + \text{subtractions}. \tag{5.48}$$

Using the trace relation (5.15) twice, one finds that

$$\Xi_{\rm pole} = \frac{\xi(w,\mu)\,F^2(\mu)}{(2\bar\Lambda-\omega-\mathrm{i}\epsilon)(2\bar\Lambda-\omega'-\mathrm{i}\epsilon)}. \tag{5.49}$$

Notice that the decay constant $F(\mu)$ appears in this equation. There is, in fact, a close relationship between the sum rule for the Isgur–Wise function and that for the meson decay constant. This connection will eventually lead to the correct normalization of $\xi(w,\mu)$ at zero recoil [115,158,160].

The QCD sum rule is obtained by equating the phenomenological and theoretical representation of the correlator. This step is a little more subtle than in the case of the two-current sum rules. As pointed out by Blok and Shifman [145], there is no reason to expect that the perturbative and the hadronic spectral densities are locally dual to each other. A detailed analysis of the exactly solvable toy model of the harmonic oscillator shows that the physical spectral density is in fact very different from the perturbative one (at least to lowest order in perturbation theory). Whereas the latter is restricted to a small area in the first quadrant of the $\nu-\nu'$ plane, the physical states populate a much larger region. In addition, as one departs from the diagonal $\nu=\nu'$ these states contribute with alternating signs, whereas the perturbative spectral density is always positive. One can argue that such a situation is in fact quite general. In order to restore duality is it necessary to integrate the spectral densities over the "off-diagonal" variable $\omega_- = \nu-\nu'$, keeping the "diagonal" variable $\omega_+ = \frac{1}{2}(\nu+\nu')$ fixed. This procedure was in fact suggested in the original calculation of the sum rule for the Isgur–Wise function in Ref. [158]. Let us then rewrite the spectral functions in terms of $\omega_\pm$ and define

$$\int \mathrm{d}\omega_-\,\rho(\omega_+,\omega_-;w) \equiv \bar\rho(\omega_+,w). \tag{5.50}$$

Only for the integrated spectral densities $\bar\rho_{\rm pert}$ and $\bar\rho_{\rm res}$ is it justified to assume local (in ω_+) duality.

The rest is straightforward. One performs a double Borel transformation of the correlator in ω and ω', which introduces two Borel parameters τ and τ'. Because of heavy-quark symmetry they appear in a completely symmetric way, and it is natural to set

$$\tau = \tau' \equiv 2T. \tag{5.51}$$

The reason for the factor 2 will become clear below. After Borel transformation the weight function in the dispersion integrals is $\exp(-\omega_+/T)$, i.e. independent of ω_-, so that the "effective" spectral densities introduced above arise in a natural way by performing the integrals over ω_-. Assuming local duality in ω_+, one equates the theoretical and the phenomenological expressions to obtain the sum rule

$$\xi(w,\mu)F^2(\mu)\,\exp(-2\bar{\Lambda}/T) = \int\limits_0^{\omega_0} \mathrm{d}\omega_+\,\exp(-\omega_+/T)\,\bar{\rho}_{\rm pert}(\omega_+,w) + \widehat{\Xi}_{\rm cond}(T,w) \equiv K(\omega_0,T,\mu;w)\,, \tag{5.52}$$

where $\widehat{\Xi}_{\rm cond}$ denotes the Borel transformed contributions from the vacuum condensates.

Compare this sum rule to that for the parameter $F(\mu)$ in (5.23). For the Isgur–Wise function to be normalized at zero recoil, it is necessary that for $w = 1$ the function $K(\omega_0,T,\mu;w)$ reduce to $K(\omega_0,T,\mu)$. In particular, this implies that the continuum threshold and the Borel parameter must be the same in both sum rules. [33] This explains the factor 2 in the definition of T in (5.51). On an empirical basis it has been observed for a long time that the Borel parameter in a three-current sum rule should be chosen approximately twice as large as in the corresponding two-current sum rule [177]. HQET shows that in the infinite quark-mass limit this relation becomes exact.

To lowest order in perturbation theory, the correlator $K(\omega_0,T,\mu;w)$ was first calculated in Refs. [115,160]. Later, radiative corrections, in particular the two-loop corrections to the perturbative spectral density, have been included by Neubert [158,178]. In view of the large perturbative corrections to the two-current correlator encountered above, it is imperative to include such effects in the analysis of the Isgur–Wise function. However, prior to the development of HQET nobody succeeded in calculating the two-loop perturbative corrections to a correlator of three currents that contain heavy-quark fields. In the effective theory things are simpler because the Feynman rules contain no Dirac matrices, but still the calculation is quite involved and requires elaborate techniques such as Kotikov's method of differential equations [179]. Altogether, the Feynman diagrams shown in Fig. 5.3 need to be calculated. After the renormalization group improvement, the sum rule for the renormalized Isgur–Wise function defined in (4.27) can be written in a form similar to (5.26), namely

$$\xi_{\rm ren}(w) = [\alpha_s(\Lambda)]^{-a_{\rm hh}(w)}\left(1 - \frac{\alpha_s(\Lambda)}{\pi}Z_{\rm hh}(w)\right)\frac{\widehat{K}(\omega_0,T;w)}{\widehat{K}(\omega_0,T)}\,, \tag{5.53}$$

with $\widehat{K}(\omega_0,T)$ as given in (5.27). The perturbative coefficients $a_{\rm hh}(w)$ and $Z_{\rm hh}(w)$ have been introduced in (3.119). They both vanish at zero recoil. At next-to-leading order in perturbation theory, the exact expression for the renormalization-group invariant correlator in the numerator of the sum rule is [178]

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[33] We know of no compelling reason why after integration over ω_- the threshold in the three-current sum rule should not depend on the recoil, in which case the condition would only be that at zero recoil it must agree with that of the two-current sum rule. We shall not pursue this possibility any further, but refer to Ref. [158] for a critical discussion.

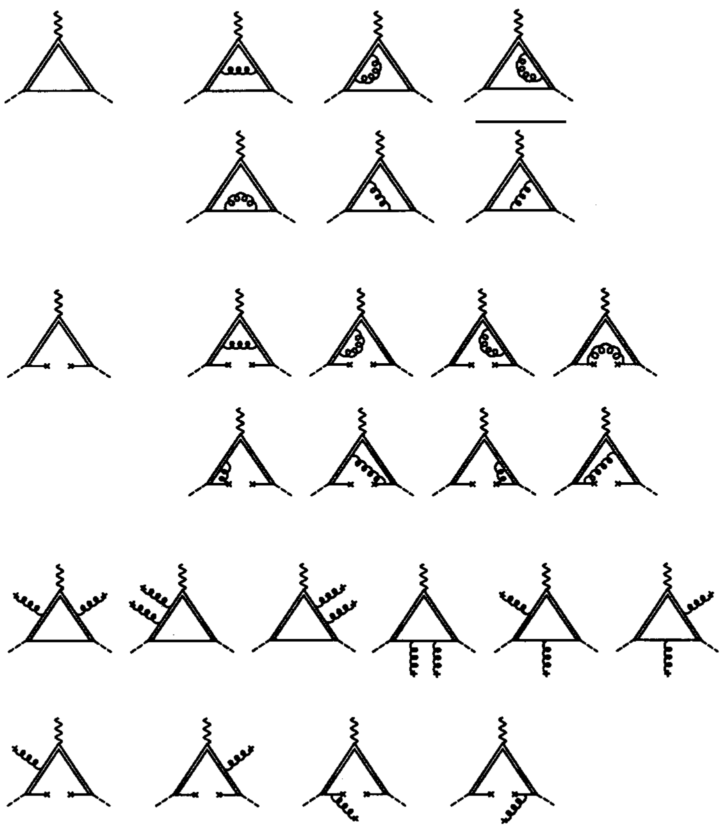


Fig. 5.3. Feynman diagrams contributing to the sum rule for the Isgur–Wise function. The velocity-changing heavy-quark current is represented by a wavy line.

$$\begin{aligned}
\widehat{K}(T,\omega_0;w) = {} & \frac{3}{8\pi^2}\left(\frac{2}{w+1}\right)^2 \int\limits_0^{\omega_0} \mathrm{d}\omega_+\,\omega_+^2\,\exp(-\omega_+/T) \\
& \times\left(1+\frac{\alpha_s(\Lambda)}{\pi}\{\tfrac{13}{3}+\tfrac{4}{9}\pi^2-[2-\gamma(w)]\ln(\omega_+/\Lambda)+c_{\mathrm{pert}}(w)\}\right) \\
& -\langle\bar{q}q\rangle(\Lambda)\left(1+\frac{\alpha_s(\Lambda)}{\pi}\{\tfrac{2}{3}-\gamma(w)[\ln(\Lambda/T)+\gamma_{\mathrm{E}}]+c_{\langle\bar{q}q\rangle}(w)\}\right) \\
& +\left(\frac{w-1}{w+1}\right)\frac{\langle\alpha_s GG\rangle}{48\pi T}+\frac{(2w+1)}{3}\,\frac{m_0^2\,\langle\bar{q}q\rangle}{4T^2},
\end{aligned} \tag{5.54}$$

where

$$\gamma(w) = \tfrac{4}{3}[wr(w)-1] \tag{5.55}$$

is proportional to the one-loop anomalous dimension of the heavy-quark current given in (3.115). The hard work in obtaining this result is hidden in the functions $c_i(w)$, both of which vanish at

$w = 1$. Their analytical expressions are

$$\begin{aligned}
c_{\rm pert}(w) &= \tfrac{1}{2}\gamma(w)\{4\ln 2 - 3 + \ln[\tfrac{1}{2}(w+1)]\} - \tfrac{4}{3}[wh(w) - 1] \\
&\quad + \ln[\tfrac{1}{2}(w+1)] + \tfrac{2}{3}(w^2-1)r^2(w) \\
&= (\tfrac{16}{9}\ln 2 - \tfrac{49}{54})(w-1) - (\tfrac{8}{15}\ln 2 - \tfrac{197}{600})(w-1)^2 + \cdots, \\
c_{\langle\bar qq\rangle}(w) &= \tfrac{1}{2}\gamma(w)\{4\ln 2 + \ln[\tfrac{1}{2}(w+1)]\} - \tfrac{4}{3}[wh(w) - 1] - \tfrac{2}{3}(w-1)r(w) \\
&= (\tfrac{16}{9}\ln 2 - \tfrac{56}{27})(w-1) - (\tfrac{8}{15}\ln 2 - \tfrac{112}{225})(w-1)^2 + \cdots,
\end{aligned} \tag{5.56}$$

with

$$h(w) = \frac{1}{\sqrt{w^2-1}}[L_2(1-w_-^2) - L_2(1-w_-)] + \frac{3}{4}\sqrt{w^2-1}\,r^2(w), \tag{5.57}$$

where $w_- = w - \sqrt{w^2-1}$. The first two terms in the expansion of $c_{\rm pert}(w)$ around zero recoil have been obtained independently by Bagan et al. [180]. Noting that $\gamma(1) = c_i(1) = 0$, one readily sees that the functions in (5.27) and (5.54) become identical at zero recoil. Hence the sum rule for the Isgur–Wise function has incorporated the correct normalization at $w = 1$. This is one of the advantages of working in the effective theory.

Before proceeding we note that the analysis needs to be modified when one considers very large values of w. In this case higher dimension condensates become enhanced by powers of w and cannot be neglected. The effect can be simulated by using so-called "soft" or "nonlocal" condensates, such as $\langle\bar q(x)q(0)\rangle = \langle\bar qq\rangle\, f(x^2)$, where $f(x^2)$ is some function that vanishes at large euclidean distances. Such an ansatz amounts to a partial summation of the operator product expansion. The gaussian model $f(x^2) = \exp(-x^2/\sigma^2)$ can be analyzed analytically. One finds that for very large values of w the condensate contributions become exponentially suppressed [158,160]. However, for a typical hadronic scale $\sigma \sim 1$ fm the effect becomes significant only for values $w \gg 1$ far outside the physical region. It is irrelevant for all practical purposes.

The sum rule (5.53) exhibits very good stability for $T > 0.7$ GeV, which is almost identical to the onset of stability in the two-current sum rule. In Fig. 5.4 we show the range of predictions for the Isgur–Wise function obtained by varying the continuum threshold between 1.7 GeV $< \omega_0 <$ 2.3 GeV, and the Borel parameter over the range 0.7 GeV $< T <$ 1.1 GeV. The dependence on the values of the vacuum condensates is very weak.

There are two reason why this analysis should be more reliable than that of the decay constants. The first is that the sum rule for the Isgur–Wise function is independent of the mass parameter $\bar\Lambda$, whereas the prediction for $F_{\rm ren}$ is proportional to $\exp(\bar\Lambda/T)$, and therefore depends exponentially on the heavy-quark mass since $\bar\Lambda = m_{\rm M} - m_{\rm Q}$. The second reason is related to QCD corrections. Whereas the numerical prediction for $F_{\rm ren}$ was increased by 30% due to the effect of radiative corrections, one finds that the net effect of order-α_s corrections to the Isgur–Wise function is very small, even at large recoil [178]. The reason is obvious: The large coefficient $\frac{13}{3} + \frac{4}{9}\pi^2$ is the same in (5.27) and (5.54) and cancels in the ratio in (5.53).

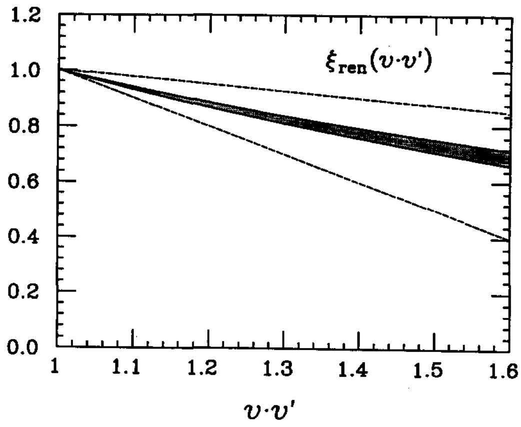


Fig. 5.4. Prediction for the Isgur–Wise function in the kinematic region accessible in semileptonic decays. The dashed lines indicate the bounds on the slope at $v \cdot v' = 1$ derived from the Bjorken and Voloshin sum rules, see (5.7).

From the sum rule one can also determine the slope parameter ϱ^2, i.e., the negative derivative of the renormalized Isgur–Wise function with respect to w at zero recoil. We find [34]

$$\varrho^2_{\rm ren} \approx 0.7, \tag{5.58}$$

where the subscript "ren" indicates that this is the slope of the renormalized, μ-independent function $\xi_{\rm ren}(w)$. In order to obtain a more physical slope parameter one must include the short-distance coefficients in the expansion of the currents. Consider, as an example, the axial vector form factor $h_{A_1}(w)$ in (4.25). At leading order in the $1/m_Q$ expansion it is given by $h_{A_1}(w) = \widehat{C}_1^5(w)\,\xi_{\rm ren}(w)$. The velocity dependence of the Wilson coefficient can be deduced from Tab. 3.1. Denoting the slope of the physical form factor by $\varrho^2_{A_1}$, we obtain

$$\varrho^2_{A_1} \approx 0.2 + \varrho^2_{\rm ren} \approx 0.9\,. \tag{5.59}$$

Similar results have been found in Refs. [145,180]. Note that the theoretical prediction for the slope parameter is in agreement with the bounds in (5.7) derived from the Bjorken and Voloshin sum rules.

5.6. Power corrections

It is also possible to derive sum rules for the subleading Isgur–Wise functions discussed in section 4.2, which parameterize the leading power corrections to meson form factors in the effective theory. The original analysis of these functions is due to Neubert [56]. It was later refined by including the complete set of radiative corrections [181,182]. The procedure is conceptually different for the $1/m_Q$ corrections associated with local or nonlocal higher dimension operators in the expansion of the currents. In the case of local effective current operators, one simply replaces the heavy-quark current J_Γ in (5.45) by the dimension-four current $\bar h'_{v'}\Gamma\,iD_\alpha h_v$. This leads to a set of three sum rules for the form factors $\xi_\pm(w,\mu)$ and $\xi_3(w,\mu)$ defined in (4.31). The virtue of working in the effective theory is that the equations of motion of HQET are automatically incorporated. For instance, one

[34] This result depends on the assumption of a velocity independent continuum threshold, see Ref. [158].

finds that the relations in (4.34) are exactly satisfied, at every order in the perturbative expansion and in the series of power corrections. It is thus sufficient to discuss the analysis of the form factor $\eta(w)$ defined in (4.35). When radiative corrections are neglected, a remarkably simple result for this function is obtained at zero recoil, namely $\eta(1) = \frac{1}{3}$ independent of all sum rule parameters [56]. The calculation of $\eta(w)$ including radiative corrections is rather complicated. One has to reevaluate the diagrams shown in Fig. 5.3 with a derivative coupling at the vertex of the heavy-quark current. In addition, there are graphs where a gluon originates from the covariant derivative contained in the current. It is useful to write the result as

$$\eta(w) = \tfrac{1}{3} + \Delta(w). \tag{5.60}$$

Then $\Delta(w)$ satisfies the QCD sum rule [182]

$$\begin{aligned}
&\Delta(w)\,[\xi(w)F^2\bar\Lambda\exp(-2\bar\Lambda/T)] \\
&\quad = \frac{\alpha_s T^4}{4\pi^3}\left(\frac{2}{w+1}\right)^2\{1+(w+1)[2+r(w)] - \tfrac{2}{3}[wr(w)-1]\}\delta_3(\omega_0/T) \\
&\qquad - (2\alpha_s\langle\bar qq\rangle T/9\pi)[7+(3-w)r(w)]\delta_0(\omega_0/T) \\
&\qquad + \frac{\langle\alpha_s GG\rangle}{72\pi}\left(\frac{w-1}{w+1}\right) + \frac{m_0^2\langle\bar qq\rangle}{18T}(w-1),
\end{aligned} \tag{5.61}$$

$$\delta_n(x) = \frac{1}{\Gamma(n+1)}\int_0^x dz\, z^n e^{-z}. \tag{5.62}$$

Eq. (5.61) shows that indeed $\Delta(1) = O(\alpha_s)$ in accordance with the above discussion.

At order α_s, one is not sensitive to the running of the quantities $\xi(w)$ and F appearing on the left-hand side of the sum rule. Their μ-dependence would show up at order α_s^2. For the numerical evaluation it is of advantage to eliminate the combination $\xi(w)F^2\bar\Lambda$ by means of the sum rule

$$\xi(w)F^2\bar\Lambda\exp(-2\bar\Lambda/T) = \frac{9T^4}{8\pi^2}\left(\frac{2}{w+1}\right)^2\delta_3(\omega_0/T) - \frac{2w+1}{3}\frac{m_0^2\langle\bar qq\rangle}{4T}, \tag{5.63}$$

which is obtained by taking the derivative with respect to T^{-1} in (5.54) and neglecting terms of order α_s. By taking the ratio of (5.61) and (5.63) one reduces to a minimum the systematic uncertainties in the calculation of $\Delta(w)$. In Fig. 5.5 we show the prediction for $\eta(w)$ obtained from the numerical analysis. Here and in the following we use the same input parameters as in the analysis of the Isgur–Wise function. For the coupling constant we take $\alpha_s/\pi \approx 0.12$, corresponding to the scale $\Lambda = 2\bar\Lambda$. The sum rule analysis confirms our guess that $\eta(w)$ should be a slowly varying function of order unity, which was the motivation for its introduction in section 4.2. Over the kinematic range accessible in semileptonic decays, we find that $\eta(w) = 0.62\pm0.05$. Note that the inclusion of radiative corrections has enhanced the lowest-order prediction by almost a factor 2.

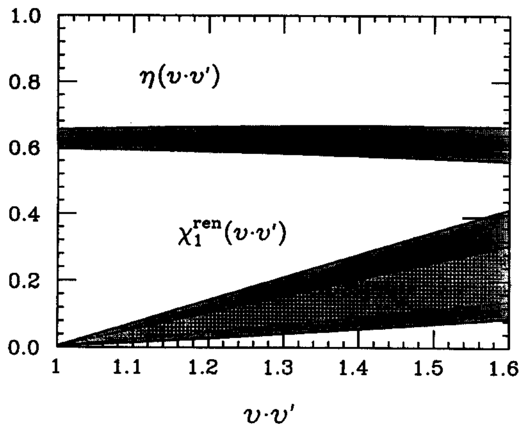
The remaining subleading Isgur–Wise functions $\chi_i(w)$ introduced in (4.37) parameterize the effects of nonlocal operators arising from insertions of higher dimension operators from the effective Lagrangian into matrix elements of the leading-order currents. In order to derive QCD sum rules for these functions, one has to consider a zero momentum insertion of the kinetic or the chromo-magnetic operator into the three-current correlator (5.45). A subtlety is that these insertions will not only correct the heavy-quark current, but also the heavy–light currents that interpolate the heavy mesons. Thus, instead of obtaining directly sum rules for $\chi_i(w)$, one also encounters the parameters G_i, which parameterize the corresponding corrections to the meson decay constants. These effects have to be disentangled by means of the sum rules for G_i discussed at the end of section 5.4. Since the calculations are quite tedious, we shall only present the main results and refer for details to Refs. [56,181]. Consider first the spin-symmetry violating form factors $\chi_2(w)$ and $\chi_3(w)$, which parameterize the effects of an insertion of the chromo-magnetic operator. They obey the sum rules

$$\begin{aligned}
&\chi_2(w)F^2\bar{\Lambda}\exp(-2\bar{\Lambda}/T)\\
&\quad= -\frac{\alpha_s T^4}{8\pi^3}\left(\frac{2}{w+1}\right)^2\left(\frac{1-r(w)}{w-1}+2\right)\delta_3(\omega_0/T)\\
&\qquad+\frac{\alpha_s\langle\bar{q}q\rangle T}{6\pi}\left(\frac{1-r(w)}{w-1}+\frac{1}{w+1}\right)\delta_0(\omega_0/T)-\frac{\langle\alpha_s GG\rangle}{96\pi}\left(\frac{2}{w+1}\right),\\
&\chi_3(w)F^2\bar{\Lambda}\exp(-2\bar{\Lambda}/T)\\
&\quad=\frac{\alpha_s T^4}{8\pi^3}\left(\frac{2}{w+1}\right)^2\{wr(w)-1+\ln[\tfrac{1}{2}(w+1)]\}\delta_3(\omega_0/T)\\
&\qquad+\frac{3\delta\omega_2}{32\pi^2}\omega_0^3\exp(-\omega_0/T)\left[\left(\frac{2}{w+1}\right)^2-\xi(w)\right]\\
&\qquad+\frac{\alpha_s\langle\bar{q}q\rangle T}{6\pi}[2-r(w)-\xi(w)]\delta_0(\omega_0/T)\\
&\qquad+\frac{\langle\alpha_s GG\rangle}{96\pi}\left(\frac{2}{w+1}-\xi(w)\right)-\frac{m_0^2\langle\bar{q}q\rangle}{48T}[1-\xi(w)].
\end{aligned}\tag{5.64}$$

Notice that each term on the right-hand side of the sum rule for $\chi_3(w)$ vanishes at zero recoil, so that the constraint $\chi_3(1)=0$ is explicitly satisfied. The parameter $\delta\omega_2$ accounts for a spin-symmetry violating correction to the continuum threshold of order $1/m_Q$. Since the chromo-magnetic interaction changes the masses of the ground-state mesons, it also changes the masses of excited states. Indeed, it turns out that these changes are similar [54]. In the numerical analysis we use $\delta\omega_2=-0.1$ GeV.

For the evaluation of the sum rules it is again of advantage to eliminate the explicit dependence of the left-hand sides on $F^2\bar{\Lambda}$, by using the zero recoil limit of (5.63). Since the leading terms in the sum rules are of order α_s, even in the two-loop calculation presented here one is not sensitive to the running of $\chi_2(w)$ and $\chi_3(w)$. In this case, a renormalization group improvement in leading logarithmic approximation is sufficient. According to (4.44), the renormalized functions are then simply given by

$$\chi_i^{\rm ren}(w)=[\alpha_s(\Lambda)]^{-1/3-a_{hh}(w)}\chi_i(w),\quad i=2,3,\tag{5.65}$$



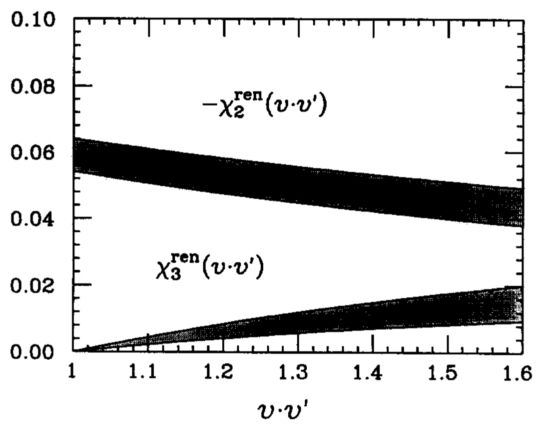


Fig. 5.5. Sum rule prediction for the form factor $\eta(v \cdot v')$, and for the function $\chi_1^{\rm ren}(v \cdot v')$ to be discussed below. The inner band for $\chi_1^{\rm ren}(v \cdot v')$ is obtained by reducing the intervals for ω_0 and T by a factor 2.

Fig. 5.6. Sum rule prediction for the spin-symmetry violating form factors $-\chi_2^{\rm ren}(v \cdot v')$ and $\chi_3^{\rm ren}(v \cdot v')$.

where we shall again use $\Lambda = 2\bar{\Lambda}$ as a characteristic scale of the low energy theory. The numerical results are shown in Fig. 5.6. The main conclusion to be drawn from this figure is that the spin-symmetry violating corrections to the meson form factors induced by the chromo-magnetic hyperfine interaction are small. Over the entire kinematic region, both form factors stay well below 10%.

Let us finally turn to the function $\chi_1(w)$, which does not violate the spin symmetry and thus effectively corrects the leading-order Isgur–Wise function, as shown in (4.41). Since $\chi_1(w)$ vanishes at zero recoil and drops out of the ratio of any two of the meson form factors, it does not affect the model-independent predictions of HQET. For this reason one may argue that a precise knowledge of this function is not very important. Accordingly, the QCD sum rule analysis of $\chi_1(w)$ is by far not as elaborate as for the other universal form factors. In particular, radiative corrections have not yet been calculated. Without such corrections one obtains:

$$\begin{aligned} &\chi_1(w) F^2 \bar{\Lambda} \exp(-2\bar{\Lambda}/T) \\ &\quad = -\frac{3T^4}{4\pi^2}\left(\frac{2}{w+1}\right)^2\left(\frac{w-1}{w+1}\right)\delta_3(\omega_0/T) - \frac{3m_0^2\langle\bar{q}q\rangle}{8T}\left(\frac{8w+1}{9} - \xi(w)\right). \end{aligned} \tag{5.66}$$

The condition $\chi_1(1) = 0$ imposed by Luke's theorem is explicitly satisfied. In leading logarithmic approximation, the renormalized form factor is obtained from (4.42) as

$$\chi_1^{\rm ren}(w) = [\alpha_s(\Lambda)]^{-a_{hh}(w)}\chi_1(w) + \frac{8}{27}\,\frac{r(w)-w}{w+1}\ln\left[\alpha_s(\Lambda)\right]\xi_{\rm ren}(w). \tag{5.67}$$

As previously we shall set $\Lambda = 2\bar{\Lambda}$. The stability of the sum rule (5.66) is not as good as in the other cases, which is not unexpected since the theoretical calculation is less sophisticated. The range of predictions obtained by varying the parameters ω_0 and T is wide. We show the numerical results in Fig. 5.5, together with the function $\eta(w)$. These two form factors have values of order unity, which is what one naively expects for the universal functions of HQET. It seems natural to separate them

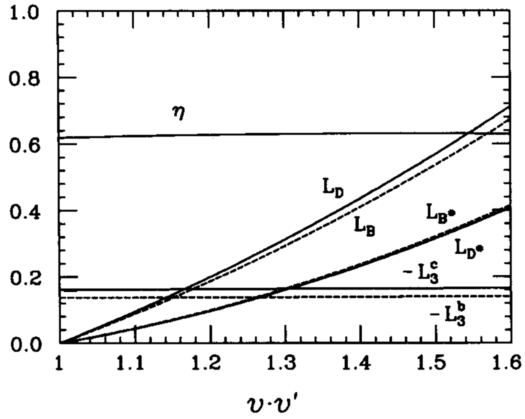


Fig. 5.7. Comparison of the form factors L_P, L_V, $-L_3^Q$, and η as functions of $v \cdot v'$. The solid lines refer to $m_Q = m_c$, the dashed ones to $m_Q = m_b$.

from the much smaller functions $\chi_2^{\rm ren}(w)$ and $\chi_3^{\rm ren}(w)$, which describe the hyperfine effects arising from the chromo-magnetic interaction of the heavy-quark spin with the light degrees of freedom.

5.7. *Predictions for the meson form factors*

The sum rule results of the previous two sections can be used to predict the meson form factors $h_i(w)$ in (4.25) to next-to-leading order in the $1/m_Q$ expansion. As pointed out in section 4.2, it is convenient to introduce the three combinations $L_{P,V}(w)$ and $L_3^Q(w)$ in favor of $\chi_i^{\rm ren}(w)$. According to their definition in (4.47), these functions depend logarithmically on the heavy-quark mass. Recall that L_P and L_V are corrections to the wave function of pseudoscalar and vector mesons, respectively, which are independent of the Dirac structure of the current. In Fig. 5.7 we compare these functions to the form factor $\eta(w)$. From now on we shall always work with the values $\omega_0 = 2$ GeV and $T = 1$ GeV, postponing for a moment the discussion of theoretical uncertainties. We conclude from this figure that close to zero recoil only η can lead to sizable $1/m_Q$ corrections. L_P and L_V are small because of Luke's theorem, and QCD sum rules predict that L_3^Q is small because it is related to chromo-magnetic hyperfine effects.

Recently, Baier and Grozin have extracted three of the four subleading form factors from QCD sum rules in the full theory [183], ignoring, however, effects of radiative corrections. Their results for L_P and L_V are very similar to ours. Their prediction for η is smaller than ours by a factor of 2, due to the neglect of the contributions of order α_s.

Let us now combine our results with the short-distance coefficients in Tab. 3.1 to compute the ratios $N_i(w) = h_i(w)/\xi_{\rm ren}(w)$ from (4.48) and (4.49). Together with the Isgur–Wise form factor these functions determine the meson form factors completely. As previously we use $m_b = 4.8$ GeV and $m_c = 1.45$ GeV for the heavy-quark masses, as well as $\bar\Lambda = 0.5$ GeV, which is still compatible with the sum rule result in (5.28). Our final numbers are given in Tab. 5.1. They are the main results of this chapter. For completeness, we show again the quark velocity transfer $\bar w$, which is related to the velocity transfer of the mesons by means of (3.153). It is used to evaluate the Wilson coefficients. Recall that in the limit of an exact heavy-quark symmetry one would have $N_- = N_{A_2} = 0$, and

Table 5.1
Predictions for the Isgur–Wise function and the ratios $N_i(w)$ over the kinematic range accessible in $B \to D$ and $B \to D^*$ transitions.

w	$\bar{w}$	ξ_{ren}	N_+	N_-	N_V	N_{A_1}	N_{A_2}	N_{A_3}
1.0	1.00	1.00	1.03	−0.03	1.33	0.99	−0.48	0.97
1.1	1.13	0.94	1.01	−0.03	1.29	0.97	−0.45	0.96
1.2	1.26	0.88	1.00	−0.03	1.26	0.96	−0.42	0.95
1.3	1.40	0.83	0.99	−0.03	1.24	0.96	−0.39	0.95
1.4	1.53	0.78	0.98	−0.03	1.22	0.95	−0.37	0.95
1.5	1 67	0.74	0.98	−0.04	1.20	0.95	−0.34	0.95
1.6	1.80	0.70	0.98	−0.04				

$N_i = 1$ otherwise. As a result of a conspiracy of QCD and power corrections, we find large symmetry breaking effects in the form factors h_V and h_{A_2}, whereas the corrections to the other four form factors are small. From a comparison with (4.29) and Tab. 3.1, the interested reader can separate the short- and long-distance contributions to the form factors. For instance, the short-distance corrections to h_V are given by $\widehat{C}_1$, the ones to h_{A_2} are given by $\widehat{C}_2^5$.

An alternative way to present the results is to go back to the conventional definition of meson form factors as functions of q^2, which are related to $h_i(w)$ by means of (4.56). This allows us to compare our predictions with those of the naive bound state models discussed at the end of section 1.4. We display the meson form factors F_1, V, A_1, and A_2 in Fig. 5.8, using the same conventions as in Fig. 1.3. In the limit of an exact heavy-quark symmetry these curves would be identical. According to this figure, the main effect of symmetry breaking corrections is to affect the normalization of the form factors but not their relative q^2-dependence. This is very different from the predictions of the quark models. Notice that the symmetry breaking effects are larger than in the models. This is not surprising. Since we have worked very hard to understand the origin of various sources of symmetry breaking corrections, we can hope to account for such effects in a much more detailed way than the naive models can. To give an example, in the Körner-Schuler model [20] one assumes that all form factors are the same at $q^2 = 0$, but their is no physical argument why this should be the case. In addition, none of the models accounts for the short-distance corrections discussed in chapter 3, but they are responsible for 50% of the enhancement of V relative to F_1 and A_1.

Let us come back, at this point, to a discussion of the theoretical uncertainties in the results presented above. With the exception of $\chi_1^{ren}(w)$, the QCD sum rule analysis of the universal functions and of the mass parameter $\bar{\Lambda}$ is very sophisticated. The results should have an accuracy of better than 20%. This means that close to zero recoil, where the Isgur–Wise function is normalized and $\chi_1^{ren}(w)$ does not contribute, the symmetry breaking corrections of order $1/m_Q$ can be estimated with this accuracy. The resulting uncertainty in the meson form factors is less than 10% for the form factors $h_V(1)$ and $h_{A_2}(1)$, which receive the largest corrections, and less than 3% for $h_-(1)$ and $h_{A_3}(1)$. There are no $1/m_Q$ corrections to $h_+(1)$ and $h_{A_1}(1)$ because of Luke's theorem. At large recoil the situation is different. There the main uncertainty comes from the Isgur–Wise function and from $\chi_1^{ren}(w)$, which enter all form factors in the same combination [cf. (4.41)]. We estimate that our results should have an accuracy of 15% at $w = w_{max}$, corresponding to $q^2 = 0$. However, the predictions for ratios of form factors are not affected by this and, therefore, are more reliable.

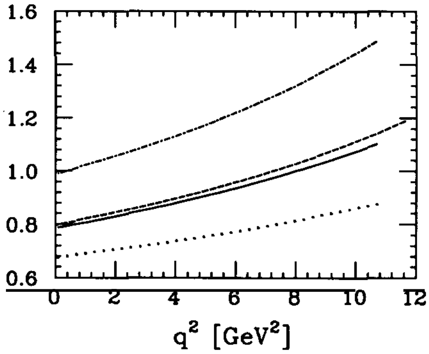


Fig. 5.8. The form factors F_1 (dashed), V (dashed-dotted), A_2 (dotted), and $[1 - q^2/(m_B + m_{D^*})^2]^{-1} A_1$ (solid) as functions of q^2.

In view of these remarks, and motivated by the results shown in Fig. 5.8, we propose to search for symmetry breaking effects by studying the form factors ratios

$$R_1 = \left(1 - \frac{q^2}{(m_B + m_{D^*})^2}\right) \frac{V(q^2)}{A_1(q^2)} = \frac{h_V(w)}{h_{A_1}(w)},$$

$$R_2 = \left(1 - \frac{q^2}{(m_B + m_{D^*})^2}\right) \frac{A_2(q^2)}{A_1(q^2)} = \frac{h_{A_3}(w) + r h_{A_2}(w)}{h_{A_1}(w)}, \tag{5.68}$$

where $r = m_{D^*}/m_B$, as a function of q^2 or w. We will see in the next chapter that R_1 and R_2 appear quite naturally in the description of $B \to D^* \ell \bar{\nu}$ transitions. Our prediction is that these ratios are almost constant, with values $R_1 \approx 1.3$ and $R_2 \approx 0.8$. According to (4.53), the result for R_1 is a model-independent prediction of HQET, which does not rely on a calculation of the subleading Isgur–Wise functions. The ratio R_2, on the other hand, can be shown to depend on $\eta(w)$ and $L_3^c(w)$ [56].

6. Phenomenology

6.1. Theoretical framework

We are now in a position to perform a comprehensive analysis of semileptonic B meson decays in the context of the new theoretical framework provided by HQET. The main difference to previous studies of these decays, which were based on phenomenological models such as those discussed at the end of chapter 1, is that we can now clearly separate the model-independent aspects of the analysis from the model-dependent ones. Under "model-independent" we shall understand quantities that can be absolutely predicted in the limit of an exact spin-flavor symmetry. This means that hadronic uncertainties enter the theoretical description only at the level of power corrections of order $1/m_Q$ or, in favorite cases, even of order $1/m_Q^2$. Typically, the normalization of decay rates at zero recoil,

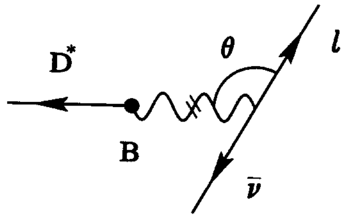


Fig. 6.1. Definition of the decay angle θ in $\mathrm{B} \to \mathrm{D}^* \ell\bar{\nu}$ decays.

as well as ratios of different decay amplitudes at the same value of q^2 (or $v \cdot v'$), are predicted by heavy-quark symmetry. Model dependence comes in when one wants to describe the q^2-dependence of decay rates. However, we stress that the form factor predictions derived in section 5.7 should be considered superior to any previous model calculation [35] in that they include perturbative QCD corrections and satisfy the relations imposed by heavy-quark symmetry to leading and next-to-leading order in $1/m_Q$.

The kinematics of the semileptonic decays $\mathrm{B} \to \mathrm{D}^{(*)}\ell\bar{\nu}$ has been analyzed in great detail in Refs. [20,185]. Here we shall only collect the basic formulas. For simplicity we shall consider the limit of vanishing lepton mass, $m_\ell = 0$, which is an excellent approximation when ℓ is e or μ. Then the lepton current is conserved, and terms proportional to the momentum transfer q^μ in the hadronic matrix elements do not contribute to the decay amplitudes. Consider first the transition $\mathrm{B} \to \mathrm{D}\ell\bar{\nu}$. The relevant hadronic matrix element is conventionally parameterized as shown in the second equation in (1.35). The differential decay rate is given by

$$\frac{\mathrm{d}\Gamma(\mathrm{B} \to \mathrm{D}\ell\bar{\nu})}{\mathrm{d}q^2} = \frac{G_F^2|V_{cb}|^2}{24\pi^3}|\boldsymbol{p}_D|^3|F_1(q^2)|^2, \tag{6.1}$$

where G_F is Fermi's constant, and $\boldsymbol{p}_D$ denotes the momentum of the D meson in the B rest frame (which for experiments working on the $\Upsilon(4s)$ resonance is very close to the laboratory frame). In the context of HQET it is more natural to introduce the velocity transfer $w = v \cdot v'$ in favor of the momentum transfer q^2, and the meson form factors $h_i(w)$ instead of the conventional form factors. Using the first relation in (4.56), as well as $|\boldsymbol{p}_D| = m_D(w^2-1)^{1/2}$, one finds

$$\mathrm{d}\Gamma(\mathrm{B} \to \mathrm{D}\ell\bar{\nu})/\mathrm{d}w$$
$$= \frac{G_F^2|V_{cb}|^2}{48\pi^3}(m_B+m_D)^2 m_D^3 (w^2-1)^{3/2}\left|h_+(w) - \frac{m_B-m_D}{m_B+m_D}h_-(w)\right|^2. \tag{6.2}$$

The relation between q^2 and w is shown in (1.39).

Next consider the decay $\mathrm{B} \to \mathrm{D}^*\ell\bar{\nu}$. In this case the relevant hadronic matrix element can be parameterized by four form factors [cf. (1.46)], three of which appear in the decay rate when the lepton mass is set to zero. Their contributions can be disentangled by measuring angular distributions or analyzing the polarization of the D^* mesons. It is convenient to introduce invariant helicity amplitudes $H_\pm(q^2)$ and $H_0(q^2)$ corresponding to transverse and longitudinal polarization. In the

[35] This includes, in particular, previous QCD sum rule calculations in the full theory [184].

parent rest frame, the D* meson and the virtual W boson go back to back and are forced to have the same helicity. It is useful to define θ as the angle between the lepton and the D* meson in the rest frame of the virtual W boson, i.e., in the center-of-momentum frame of the $\ell\bar\nu$ pair (see Fig. 6.1). The different helicity amplitudes lead to characteristic distributions in this angle. The double differential decay rate in q^2 and $\cos\theta$ is given by[36]

$$\begin{aligned} &\mathrm{d}^2\Gamma(\mathrm{B}\to\mathrm{D}^*\ell\bar\nu)/\mathrm{d}q^2\,\mathrm{d}\cos\theta \\ &\quad=\frac{G_F^2|V_{cb}|^2}{768\pi^3}|\boldsymbol{p}_\mathrm{D}|\frac{q^2}{m_\mathrm{B}^2}\{(1+\cos\theta)^2|H_+|^2+(1-\cos\theta)^2|H_-|^2+2\sin^2\theta|H_0|^2\}. \end{aligned} \tag{6.3}$$

Integrating over the angle, and summing over the final state polarizations, one obtains

$$\frac{\mathrm{d}\Gamma(\mathrm{B}\to\mathrm{D}^*\ell\bar\nu)}{\mathrm{d}q^2}=\frac{G_F^2|V_{cb}|^2}{96\pi^3}|\boldsymbol{p}_\mathrm{D}|\frac{q^2}{m_\mathrm{B}^2}\sum_{i=\pm,0}|H_i|^2. \tag{6.4}$$

The relation of the helicity amplitudes to the meson form factors in the conventional basis can be found in Refs. [20,185]. For our purposes, it is more convenient to consider H_i as functions of w, and to relate them to the meson form factors $h_i(w)$ introduced in (4.25). To this end, we define "reduced helicity amplitudes" $\widetilde{H}_i(w)$, for $i=\pm,0$, by

$$|H_i|^2\equiv(m_\mathrm{B}-m_{\mathrm{D}^*})^2(m_\mathrm{B}m_{\mathrm{D}^*}/q^2)(w+1)^2|h_{\mathrm{A}_1}(w)|^2|\widetilde{H}_i(w)|^2. \tag{6.5}$$

Then the differential distribution in w becomes

$$\begin{aligned} &\mathrm{d}\Gamma(\mathrm{B}\to\mathrm{D}^*\ell\bar\nu)/\mathrm{d}w \\ &\quad=\frac{G_F^2|V_{cb}|^2}{48\pi^3}(m_\mathrm{B}-m_{\mathrm{D}^*})^2m_{\mathrm{D}^*}^3\sqrt{w^2-1}(w+1)^2|h_{\mathrm{A}_1}(w)|^2\sum_{i=\pm,0}|\widetilde{H}_i(w)|^2. \end{aligned} \tag{6.6}$$

Since $h_{\mathrm{A}_1}(w)$ has been factored out, the reduced helicity amplitudes depend only on ratios of the meson form factors. Explicitly, they are given by

$$\begin{aligned} |\widetilde{H}_\pm(w)|^2&=\frac{1-2wr+r^2}{(1-r)^2}\left(1\mp\sqrt{\frac{w-1}{w+1}}R_1(w)\right)^2, \\ |\widetilde{H}_0(w)|^2&=\left(1+\frac{w-1}{1-r}[1-R_2(w)]\right)^2, \end{aligned} \tag{6.7}$$

where $r=m_{\mathrm{D}^*}/m_\mathrm{B}$, and $R_i(w)$ are the form factor ratios introduced in (5.68). By definition, the reduced helicity amplitudes are normalized at zero recoil:

$$|\widetilde{H}_i(1)|^2=1. \tag{6.8}$$

This is what makes them useful from the point of view of heavy-quark symmetry. At maximum recoil, $w_{\max}=\frac{1}{2}(r+r^{-1})$, corresponding to $q^2=0$, only virtual W bosons with longitudinal polarization contribute, and accordingly

[36] For decays into $\ell^+\nu$ one has to replace θ by $\pi-\theta$.

$$|\widetilde{H}_{\pm}(w_{\max})|^2 = 0. \tag{6.9}$$

Eq. (6.6) is ideally suited to distinguish the model-independent aspects of the analysis from the model-dependent ones. The w-dependence of the hadronic form factor $h_{A_1}(w)$ involves complicated nonperturbative physics and cannot yet be derived from first principles. Whenever it enters the analysis, one has to rely on some specific model calculation, such as the QCD sum rule analysis presented in chapter 5. Examples of model-dependent observables are the total semileptonic decay rates, the differential decay rates at large values of w (i.e., small momentum transfer), and integrated asymmetry parameters. They will be briefly discussed in section 6.4. On the other hand, in the limit of an unbroken heavy-quark symmetry the normalization of $h_{A_1}(w)$ at zero recoil is predicted in a model-independent way. When symmetry breaking effects are taken into account, one finds that $h_{A_1}(1)$ becomes slightly renormalized by perturbative corrections, but it does not receive corrections of order $1/m_Q$. Furthermore, as we have seen in section 4.4, the second-order power corrections are parameterically suppressed and can be estimated with an accuracy of $\pm 2\%$. The final result $h_{A_1}(1) = 0.97 \pm 0.04$ in (4.63) is one of the most important predictions of HQET. It can be used to obtain a reliable, model-independent measurement of V_{cb} from an extrapolation of the semileptonic decay rates to zero recoil. This will be discussed in the next section.

Heavy quark symmetry also determines the ratios of form factors, and hence the reduced helicity amplitudes, at any value of w. In the limit of an exact spin–flavor symmetry one has $R_1 = R_2 = 1$, implying that the helicity amplitudes reduce to trivial kinematic functions, for instance $|\widetilde{H}_0(w)|^2 \to 1$. But even beyond the leading order, HQET makes specific predictions for these quantities. The short-distance corrections to the form factor ratios can be calculated in a reliable way. They enhance the value of R_1 by 11%-13% (depending on w), whereas their effect on R_2 is below 1% and thus completely negligible [56]. To get an idea of the anatomy of the $1/m_Q$ corrections, it is instructive to make some mild approximations. In the case of R_1, one can safely neglect the contribution proportional to $1/m_b$ in (4.53). If the sum rule estimate for $\eta(w)$ is only approximately correct, this term is an order of magnitude smaller than the leading term proportional to $1/m_c$. In the case of R_2, it turns out to be a good approximation to use the tree level expressions for the short-distance coefficient functions. To further simplify the result, we set $m_c/m_b = 1/3$. This yields the approximate expressions [37]

$$\begin{aligned} R_1(w) &\approx \left(1 + \frac{4\alpha_s(m_c)}{3\pi} r(\bar{w})\right)\left(1 + \frac{1}{w+1}\frac{\bar{\Lambda}}{m_c}\right), \\ R_2(w) &\approx 1 - \frac{\bar{\Lambda}}{m_c}\left(\frac{2}{w+1}\eta(w) + \frac{1}{3}L_3^c(w)\right), \end{aligned} \tag{6.10}$$

where the function $r(\bar{w})$ is given in (3.104) and $\bar{w}$ is defined in (3.153). Essentially the only uncertainty in R_1 resides in the value of the parameter $\bar{\Lambda}$. However, it is an unambiguous prediction of HQET that this ratio must be considerably larger than unity, since both the QCD and $1/m_c$ corrections are positive and sizable. The power corrections to R_2, on the other hand, depend on the subleading form factors η and L_3^c. The QCD sum rule results in Fig. 5.7 suggest that the contribution

[37] The exact expressions can be found in Ref. [56]. In this discussion we assume that $1/m_Q^2$ corrections are small. Ultimately, one would like to test this assumption by confronting the predictions of this chapter with experimental data.

Table 6.1
Predictions for the form factor ratios R_1 and R_2. The zero recoil limit $w = 1$ corresponds to $q^2 = (m_B - m_{D^*})^2$, whereas w_{max} corresponds to $q^2 = 0$. The label HQET* is used to indicate that these results depend, to some extent, on the values of nonperturbative parameters such as $\bar{\Lambda}$.

	HQET*	ISGW	BSW	KS	Ref. [184]
$R_1(1)$	1.35	1.01	0.91	1.09	1.31
$R_1(w_{max})$	1.27	1.27	1.09	1.00	1.23
$R_2(1)$	0.79	0.91	0.85	1.09	0.95
$R_2(w_{max})$	0.85	1.14	1.06	1.00	1.05

proportional to η is the dominant one. A measurement of R_2 could provide valuable information about this function. From now on we shall use the theoretical predictions for $R_i(w)$ that can be derived from Tab. 5.1. In the kinematic region accessible in $B \to D^*\ell\bar{\nu}$ transitions, they can be approximated by

$$\begin{aligned} R_1(w) &\approx 1.35 - 0.22(w-1) + 0.09(w-1)^2, \\ R_2(w) &\approx 0.79 + 0.15(w-1) - 0.04(w-1)^2. \end{aligned} \tag{6.11}$$

Essentially the same results would be obtained from (6.10).

Although there is no reason to believe that it makes any sense to apply the heavy-quark expansion to the $D \to K^*\ell\bar{\nu}$ decay amplitude, we may still believe in a "continuity of signs" and guess that the tendency $R_1 > 1$ and $R_2 < 1$ should persist, and most likely even become more pronounced, when we imagine changing the heavy quark masses from m_b and m_c to m_c and m_s. This tendency is in fact consistent with the experimental values of the form factor ratios obtained from an analysis of the joint angular distribution in $D \to K^*\ell\bar{\nu}$ decays. Taking the weighted average of the results reported by the experiments E691 [186], E653 [187], and E687 [188], we get $R_1^{DK^*}(q^2 = 0) = 1.89 \pm 0.25$ and $R_2^{DK^*}(q^2 = 0) = 0.73 \pm 0.15$. Although we have no right to extrapolate (6.10) down to the strange quark mass, we take this observation as a confirmation of our prediction that symmetry breaking corrections enhance R_1 and suppress R_2.

The quark models discussed in section 1.4 make very different predictions for the form factor ratios. In Tab. 6.1, we compare the results obtained from the models of ISGW [19], BSW [17], and KS [20], as well as from QCD sum rules in the full theory [184], to our predictions based on HQET. None of the quark models reproduces the large value of R_1 at zero recoil, which is, however, a model-independent result of HQET. Part of the discrepancy comes from the fact that R_1 receives a substantial short-distance correction proportional to $\alpha_s(m_c)$, which is not included in any of the models. Notice also that at $q^2 = 0$ all models give values $R_2 \geq 1$, in contrast to our result based on heavy quark symmetry and the QCD sum rule analysis of the subleading form factor $\eta(w)$. We are thus not surprised that none of these models can account for the suppression of R_2, i.e., of the form factor $A_2(0)$ relative to $A_1(0)$, observed in $D \to K^*\ell\bar{\nu}$ transitions.

The form factor ratios determine the reduced helicity amplitudes in (6.7). In Fig. 6.2 we show how these quantities are affected by the symmetry breaking corrections. The fact that $R_1 > 1$ is reflected in a larger difference between $|\widetilde{H}_-|^2$ and $|\widetilde{H}_+|^2$ than would be obtained in the heavy-quark symmetry limit. In section 6.3 we will see that the so-called forward–backward asymmetry is enhanced by this

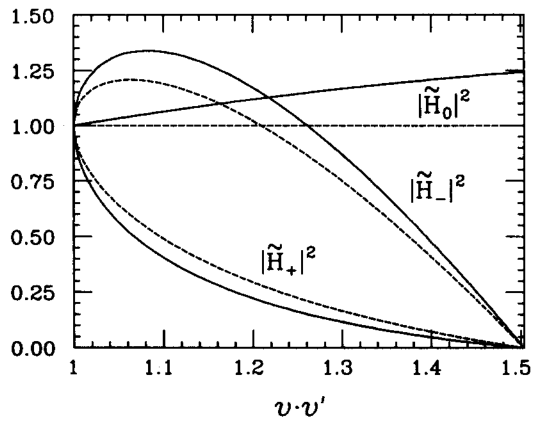


Fig. 6.2. Predictions for the reduced helicity amplitudes $|\widetilde{H}_i(w)|^2$. The dashed lines show the model-independent results obtained in the limit of an exact heavy-quark symmetry. The solid lines are obtained by including the symmetry breaking corrections as given in (6.11).

effect. Note that our prediction $R_2 < 1$ leads to an increase in $|\widetilde{H}_0|^2$ for $w > 1$, whereas the models in Tab. 6.1 predict that $|\widetilde{H}_0|^2 < 1$ for large values of w, corresponding to small momentum transfer. This difference is of relevance to the data analysis, because of an experimental cut which modifies the kinematic considerations presented above in a significant way: a cut in the lepton energy in the parent rest frame, which is enforced by reasons of particle identification. At fixed value of w (or q^2), the lepton energy E_ℓ is a function of the decay angle θ. The requirement that $E_\ell \geq E_{\rm cut}$ introduces a cutoff in the integration over $\cos\theta$, namely $-1 \leq \cos\theta \leq x_0(w)$, where

$$x_0(w) = \min\left\{1; \frac{1}{\sqrt{w^2-1}}\left(\frac{m_{\rm B} - 2E_{\rm cut}}{m_{\rm D^*}} - w\right)\right\}. \tag{6.12}$$

One finds that $x_0(w) < 1$ if w is above a threshold w_0 given by

$$w_0 = \frac{m_{\rm B} - 2E_{\rm cut}}{2m_{\rm D^*}} + \frac{m_{\rm D^*}}{2(m_{\rm B} - 2E_{\rm cut})}. \tag{6.13}$$

Then the differential decay rate $\mathrm{d}\Gamma/\mathrm{d}w$ is affected by w-dependent factors arising from the integration over $\cos\theta$. Fortunately, there is no such effect in the region near zero recoil, so that the measurement of $V_{\rm cb}$ to be discussed below is not directly affected. But nevertheless the impact of this cut is significant. For $E_{\rm cut} = 1$ GeV, which is the value used by the ARGUS and CLEO collaborations, one finds that $w_0 \approx 1.12$, meaning that over three quarters of the kinematic region the spectrum is modified. The problem is that the three helicity amplitudes are affected in a different way, since they are associated with different θ-dependence. To reconstruct the true decay rates requires an assumption about the ratios $|\widetilde{H}_\pm|/|\widetilde{H}_0|$. This is where model dependence enters the experimental analysis. Unfortunately, as we have seen, the quark models used so far for this purpose cannot be trusted to give reliable predictions for these ratios.

6.2. Model-independent determination of V_{cb}

One of the most important results of HQET is the prediction of the normalization of hadronic form factors at zero recoil. It can be used to obtain a model-independent measurement of the element V_{cb} of the Cabibbo–Kobayashi–Maskawa matrix. As pointed out in Ref. [99], the semileptonic decay $B \to D^* \ell \bar\nu$ is ideally suited for this purpose. Experimentally, this is a particularly clean mode since the reconstruction of the D^* mass provides a powerful rejection against background. From the theoretical point of view, it is ideal since the decay rate at zero recoil is protected by Luke's theorem against first-order power corrections in $1/m_Q$. Using the normalization of the reduced helicity amplitudes at $w = 1$, one finds from (6.6)

$$\lim_{w\to 1} \frac{1}{\sqrt{w^2-1}} \frac{d\Gamma(B \to D^* \ell \bar\nu)}{dw} = \frac{G_F^2 |V_{cb}|^2}{4\pi^3} (m_B - m_{D^*})^2 m_{D^*}^3 |h_{A_1}(1)|^2 . \tag{6.14}$$

The leading nonperturbative corrections to $h_{A_1}(1)$ are of order $1/m_Q^2$. The detailed analysis of section 4.4 gives the rather precise, model-independent result $h_{A_1}(1) = 0.97 \pm 0.04$. Ideally, then, one can extract V_{cb} with a theoretical uncertainty of about 4% from an extrapolation of the spectrum to $w = 1$.

It is instructive to compare this method to the extraction of V_{us} from an analysis of $K \to \pi \ell \bar\nu$ decays. On first sight, the current value $|V_{us}| = 0.2196 \pm 0.0023$ has a surprisingly small uncertainty given that the process is described by an hadronic form factor $f_+(q^2)$, which parameterizes the vector current matrix element between K and π mesons. The reason is, of course, again a symmetry of QCD, namely chiral symmetry. In the limit of equal light quark masses, the flavor-changing current $\bar u \gamma^\mu s$ is conserved, in which case it follows that $f_+(0) = 1$ at zero momentum transfer. The Ademollo–Gatto theorem states that the deviations, which arise since the mass difference $m_s - m_u$ is nonzero, are of second order in the symmetry breaking parameter [189]. Hence, $f_+(0) = 1 + O[(m_s - m_u)^2]$. Chiral perturbation theory provides the theoretical framework for a systematic analysis of these corrections in an expansion in m_q/Λ_χ, where $\Lambda_\chi = 4\pi f_\pi$ is the scale of chiral symmetry breaking. Because of the Ademollo–Gatto theorem, the terms linear in m_q can only come from so-called chiral logarithms and can be calculated in a reliable way. Further corrections from higher dimension operators in the chiral expansion can be estimated on dimensional grounds. From such a detailed analysis, Leutwyler and Roos conclude that [117]

$$f_+^{K^-\pi^0}(0) = 0.982 \pm 0.008, \qquad f_+^{K^0\pi^+}(0) = 0.961 \pm 0.008. \tag{6.15}$$

When combined with a measurement of the $K \to \pi \ell \bar\nu$ decay rate at $q^2 = 0$, this leads to the precise value of V_{us} quoted above.

Compare this to the extraction of V_{cb} using heavy-quark symmetry. In this case the relevant form factor is $h_{A_1}(w)$, which in the limit of infinite heavy-quark masses is normalized at zero velocity transfer, again as a consequence of current conservation. HQET provides the theoretical framework to analyze the corrections to this limit in a systematic way. The perturbative corrections due to hard gluons can be calculated reliably by using the powerful machinery of the renormalization group. Luke's theorem states that the nonperturbative corrections are of second order in the symmetry breaking parameter, i.e., of order $1/m_Q^2$. They can be estimated and are found to be parameterically suppressed. A detailed analysis leads to the prediction (4.63), which may be compared to (6.15). The purpose of pointing out this analogy is to convince the reader that heavy-quark symmetry is an important theoretical concept, with similar implications and predictive power as the more familiar chiral symmetry.

Presently, our proposal to measure V_{cb} from an extrapolation to zero recoil poses quite a challenge to the experimentalists. First, there is the fact that the decay rate vanishes at zero recoil because of phase space. Therefore the statistics gets worse as one tries to measure close to $w = 1$. However, we do not believe that this will be an important limitation of the method. The phase space suppression is proportional to $\sqrt{w^2 - 1}$ and is in fact a rather mild one. When going from the endpoint $w_{max} \approx 1.5$ down to $w = 1.05$, the change in the statistical error in $|V_{cb}|^2$ due to the variation of the phase space factor is not even a factor of two. This can also be seen from the differential distribution in w shown in Fig. 6.8 below. A more serious problem is related to the fact that, for experiments working on the $\Upsilon(4s)$ resonance, the zero recoil limit corresponds to a situation where both the B and the D* mesons are approximately at rest in the laboratory. Then the pion in the subsequent decay $D^* \to D\pi$ is very soft and can hardly be detected. Thus, the present experiments have to make cuts which disfavor the zero recoil region, leading to large systematic uncertainties for values of w smaller than about 1.15. This second problem would be absent at an asymmetric B-factory, where the rest frame of the parent B meson is boosted relative to the laboratory frame.

In view of these difficulties, one presently has to rely on an extrapolation over a wide range in w to obtain a measurement of V_{cb}. Following Ref. [99], let us rewrite the differential decay rate (6.6) as

$$\begin{aligned}
&\mathrm{d}\Gamma(\mathrm{B} \to \mathrm{D}^*\ell\bar{\nu})/\mathrm{d}w \\
&\quad = (G_F^2/48\pi^3)(m_B - m_{D^*})^2 m_{D^*}^3 \eta_A^2 \sqrt{w^2-1}\,(w+1)^2 \\
&\quad\quad \times \left(1 + \frac{4w}{w+1}\,\frac{1-2wr+r^2}{(1-r)^2}\right)|V_{cb}|^2 \widehat{\xi}^2(w),
\end{aligned} \tag{6.16}$$

where $\eta_A = 0.99$ is the short-distance correction to the form factor $h_{A_1}(w)$ at zero recoil [cf. (3.159)]. The new function $\widehat{\xi}(w)$ is given by

$$\widehat{\xi}^2(w) = \sum_i |\widetilde{H}_i(w)|^2 \left(1 + \frac{4w}{w+1}\,\frac{1-2wr+r^2}{(1-r)^2}\right)^{-1} \eta_A^{-2}|h_{A_1}(w)|^2. \tag{6.17}$$

Eq. (6.16) is written in such a way that the deviations from the heavy-quark symmetry limit are absorbed into the form factor $\widehat{\xi}(w)$, which in the absence of symmetry breaking corrections would be the Isgur–Wise function. Since everything except V_{cb} and $\widehat{\xi}(w)$ in (6.16) is known, a measurement of the differential decay rate is equivalent to a measurement of the product $|V_{cb}|\widehat{\xi}(w)$. However, the theory predicts the normalization of $\widehat{\xi}(w)$ at zero recoil:

$$\widehat{\xi}(1) = \eta_A^{-1} h_{A_1}(1) = 1 + \delta_{1/m^2} = 0.98 \pm 0.04, \tag{6.18}$$

where the uncertainty comes from power corrections of order $1/m_Q^2$. Using this information, $|V_{cb}|$ and $\widehat{\xi}(w)$ can be obtained separately from such a measurement.

In Ref. [99], this strategy has been applied for the first time to the combined samples of the 1989 ARGUS and CLEO data, as compiled in Ref. [190]. For the extrapolation to zero recoil the

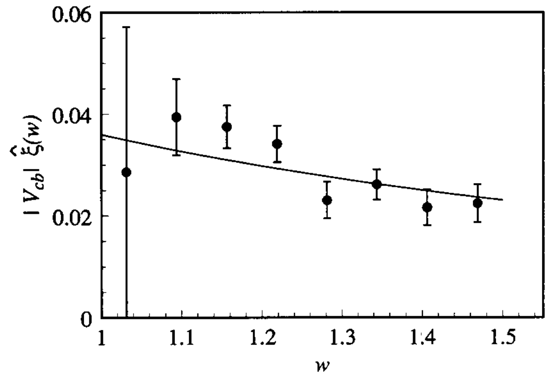


Fig. 6.3. ARGUS data for the product $|V_{cb}|\widehat{\xi}(v\cdot v')$ as a function of the recoil $v\cdot v'$ [146]. V_{cb} is obtained from an extrapolation to $v\cdot v'=1$. The fit curve is explained in the text.

parameterizations given in (5.1) have been used for the function $\widehat{\xi}(w)$, treating its slope at zero recoil as a free parameter. The result obtained for V_{cb} was [38]

$$|V_{cb}|(\tau_B/1.5\ \text{ps})^{1/2}=0.039\pm0.006. \tag{6.19}$$

Note that the most recent value for the lifetime of the B^0 meson is $\tau_B = 1.52\pm0.10$ ps [191]. Depending on which parameterization was used, the slope parameter, which we denote by $\widehat{\varrho}^2$, was found in the range between 1.1 and 1.4. A linear fit to the data yielded $|V_{cb}| = 0.036\pm0.005$ and $\widehat{\varrho}^2\approx0.8$.

Since this original analysis, the data have changed. In particular, the branching ratio for $D^{*+}\to D^0\pi^+$ has increased from 55% [192] to 68% [193]. This lowers the decay rate for $B\to D^*\ell\bar{\nu}$, and correspondingly decreases V_{cb} by 10%. However, the new data recently reported by ARGUS [146] and CLEO [194] give a larger branching ratio than the old data, indicating that further changes in the analysis must have taken place. It is thus not possible to simply rescale the result for V_{cb} obtained from the 1989 data.

In Fig. 6.3 we show the new ARGUS data [146] for the product $|V_{cb}|\widehat{\xi}(w)$. From an unconstrained fit using again the parameterizations in (5.1), the following value is obtained:

$$|V_{cb}|(\tau_B/1.5\ \text{ps})^{1/2}=0.049\pm0.008. \tag{6.20}$$

However, the fit gives very large values for the slope parameter $\widehat{\varrho}^2$, between 1.9 and 2.3. A linear fit to the data, on the other hand, gives $|V_{cb}| = 0.043\pm0.006$ and $\widehat{\varrho}^2\approx1.2$.

This poses the question whether it is possible to constrain the extrapolation to zero recoil. If $\widehat{\xi}(w)$ was truly the Isgur–Wise function, the Voloshin sum rule discussed in section 5.2 would clearly exclude values $\widehat{\varrho}^2\approx2$, making the result (6.20) meaningless. However, in general the slopes of $\widehat{\xi}(w)$ and $\xi(w)$ can differ because of symmetry breaking corrections. We cannot predict the w-dependence of these functions in a model-independent way, but we can hope to calculate the ratio

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[38] In the original paper we used $\tau_B = 1.18$ ps and $\eta_A = 0.95$.

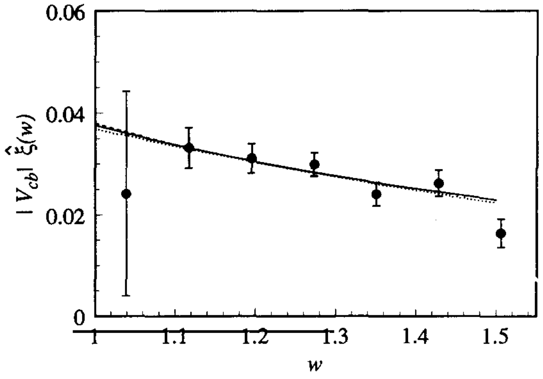


Fig. 6.4. CLEO data for the product $|V_{cb}|\widehat{\xi}(v\cdot v')$ as a function of the recoil $v\cdot v'$.

$$K(w) = \widehat{\xi}(w)/\xi_{ren}(w) \tag{6.21}$$

with less model dependence. Many of the uncertainties in the nonperturbative calculation will affect both functions in a similar way. We find that $K(w)$ is a slowly increasing function, with a slope $K'(1) \approx 0.09$ at $w = 1$. The reason is that the corrections to the heavy-quark symmetry limit enhance the sum of the helicity amplitudes appearing in the numerator in (6.17). As a consequence, the form factor $\widehat{\xi}(w)$ is not as steep as the Isgur–Wise function

$$\widehat{\varrho}^2 \approx \varrho^2 - 0.09. \tag{6.22}$$

Although this is a somewhat model-dependent statement, there can be no doubt that the two functions will be very similar. We conclude that the large values of $\widehat{\varrho}^2$ obtained from the ARGUS fit are inconsistent with the Voloshin sum rule, and cannot be tolerated on general grounds. We may speculate that the problem is caused by an underestimation of the systematic uncertainties in the zero recoil region. On the other hand, the data suggest that the slope parameter should be close to the upper bound $\varrho^2_{max} \approx 1$ in (5.7). For the purpose of illustration, we show in Fig. 6.3 a fit using the pole ansatz in (5.1) with $\widehat{\varrho}^2 = 1$. We find values significantly smaller than the result of the ARGUS fit,

$$|V_{cb}|(\tau_B/1.5\ \mathrm{ps})^{1/2} = 0.037 \pm 0.006. \tag{6.23}$$

Very recently, CLEO has published new results for the recoil spectrum with much higher statistics and better systematics [194]. Their data are shown in Fig. 6.4. The curves correspond to the various fit functions in (5.1). Without any constraints on the slope at zero recoil, the values obtained from the fits are

$$|V_{cb}|(\tau_B/1.5\ \mathrm{ps})^{1/2} = 0.038 \pm 0.007, \qquad 1.0 \pm 0.4 < \widehat{\varrho}^2 < 1.2 \pm 0.7. \tag{6.24}$$

It is reassuring that the result for the slope parameter is in agreement with the bound derived from the Voloshin sum rule. Because of the very careful analysis of systematic uncertainties in Ref. [194], we consider these new CLEO numbers to be the currently best available results for V_{cb} and $\widehat{\varrho}^2$.

Before we proceed, let us briefly consider the possibility of extracting V_{cb} from $B \to D\ell\bar{\nu}$ decays. As discussed in section 4.3, this mode is not protected by Luke's theorem, due to its helicity suppression at zero recoil. However, one can show that the symmetry-breaking corrections are parametrically suppressed by the factor [16]

$$S = \left(\frac{m_B - m_D}{m_B + m_D}\right)^2 \approx 0.23. \tag{6.25}$$

For this reason one may hope that the theoretical uncertainty in extracting V_{cb} from these transitions is not much worse than in the case of $B \to D^*\ell\bar{\nu}$ decays. From (6.2), we find for the extrapolation of the spectrum to zero recoil

$$\lim_{w\to 1} \frac{1}{(w^2-1)^{3/2}} \frac{d\Gamma(B \to D\ell\bar{\nu})}{dw} = \frac{G_F^2|V_{cb}|^2}{48\pi^3}(m_B + m_D)^2 m_D^3 \eta_V^2 (1 + S\cdot K)^2, \tag{6.26}$$

where

$$K = \delta_1 + (\bar{\Lambda}/2m_c + \bar{\Lambda}/2m_b)[(1+\delta_1) - 2(1+\delta_2)\eta(1)]. \tag{6.27}$$

Here η_V and δ_i are perturbative QCD corrections arising from finite renormalizations of the currents in the intermediate region $m_b > \mu > m_c$. Using the numbers given in Tab. 3.1, one obtains $\eta_V \approx 1.03$, $\delta_1 \approx 0.11$, and $\delta_2 \approx 0.09$. Hadronic uncertainties first appear at order $1/m_Q$. However, the important observation made in Refs. [56,182] is that they are parameterically suppressed. The appearance of the kinematic factor S involving the meson masses was anticipated by Voloshin and Shifman, who noted that when this factor is set to zero there are no $1/m_Q$ corrections to the decay rate at zero recoil [16]. An additional suppression occurs if the sum rule estimate $\eta(1) \approx 0.6$ is only approximately correct. Then the combination $[1.11 - 2.18\eta(1)]$ is very small. Assuming $\eta(1) = 0.6 \pm 0.2$ with a generous error, we find $S\cdot K = 1.5 \pm 2.3\%$, i.e. at most a few percent. This is of the same magnitude as the expected $1/m_Q^2$ corrections to the decay rate. We thus conclude that, at zero recoil, the normalization of the form factor combination relevant to $B \to D\ell\bar{\nu}$ transitions is known with an accuracy similar to that in (4.63):

$$h_+(1) - \frac{m_B - m_D}{m_B + m_D} h_-(1) = 1.05 \pm 0.06, \tag{6.28}$$

where we have doubled the error to account for the unknown contributions of order $1/m_Q^2$.

Unfortunately, from the experimental point of view a measurement of the spectrum in $B \to D\ell\bar{\nu}$ decays is a very difficult task. One has to deal with several sources of background, and in addition there is the handicap of the helicity suppression of the decay rate near $w = 1$. Until now, only the total branching ratio has been measured.

6.3. Asymmetries

We have seen in section 6.1 that heavy-quark symmetry makes definit predictions for the reduced helicity amplitudes $\widetilde{H}_i$, which are related to ratios of the weak decay form factors. The transverse helicity amplitudes only depend on the ratio R_1 and are predicted in an essentially model-independent way, up to corrections of order $1/m_Q^2$. The longitudinal helicity amplitude is constant in the heavy-quark symmetry limit, and it only receives small corrections as shown in Fig. 6.2. A measurement of these quantities would provide a direct test of the predictions of HQET.

According to (6.3), the helicity amplitudes are experimentally accessible by a measurement of angular distributions. This is a complicated task, however. It is easier to integrate over the decay angles, in which case information about the helicity amplitudes can still be obtained from certain asymmetry parameters. We first discuss the forward–backward asymmetry in the angular distribution in the lepton decay angle θ. At fixed value of w (or q^2), one defines

$$A_{\rm FB}(w) = \frac{\mathrm{d}\Gamma/\mathrm{d}w(\theta > \pi/2) - \mathrm{d}\Gamma/\mathrm{d}w(\theta < \pi/2)}{\mathrm{d}\Gamma/\mathrm{d}w(\theta > \pi/2) + \mathrm{d}\Gamma/\mathrm{d}w(\theta < \pi/2)}. \tag{6.29}$$

In this ratio the hadronic form factor $h_{A_1}(w)$, as well as other kinematic factors, drop out. One obtains

$$A_{\rm FB}(w) = \frac{3}{4}\,\frac{|\widetilde{H}_-|^2 - |\widetilde{H}_+|^2}{|\widetilde{H}_-|^2 + |\widetilde{H}_+|^2 + |\widetilde{H}_0|^2}, \tag{6.30}$$

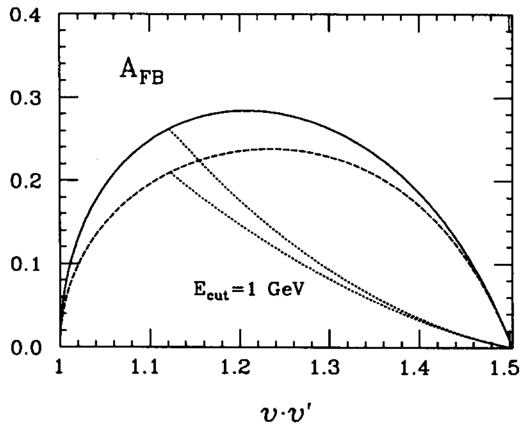
where we omit the dependence of $\widetilde{H}_i$ on w to simplify the notation. For kinematic reasons the asymmetry vanishes at the endpoints $w = 1$ and $w = w_{\rm max}$. Otherwise it is a positive quantity, since $|\widetilde{H}_-|^2 > |\widetilde{H}_+|^2$. This is a consequence of the left-handedness of the weak interactions at the quark level [20] (see, however, Ref. [195]). In Fig. 6.5 we show the theoretical prediction for $A_{\rm FB}(w)$ obtained by using the helicity amplitudes as given in Fig. 6.2. The corrections to the heavy-quark symmetry limit enhance the asymmetry, the reason being that the difference $|\widetilde{H}_-|^2 - |\widetilde{H}_+|^2$ is proportional to the form factor ratio R_1, which receives positive short-distance and $1/m_c$ corrections. We also show the effect of a cut in the lepton energy. Following the analysis of the CLEO collaboration [196], we use a symmetric cut in the angular integration, i.e. $-x_0 \le \cos\theta \le x_0$ with x_0 as given in (6.12), in order not to affect the contributions proportional to $|\widetilde{H}_-|^2$ and $|\widetilde{H}_+|^2$ in a different way. Clearly, the effect of such a cut is quite dramatic. In order to reconstruct the spectrum one has to make an assumption about the ratio $(|\widetilde{H}_-|^2 + |\widetilde{H}_+|^2)/|\widetilde{H}_0|^2$, which involves model dependence.

From the theoretical point of view, it would be ideal to measure the forward–backward asymmetry for transversely polarized D* mesons only. In this case

$$A_{\rm FB}^{\rm T} = \frac{3}{4}\,\frac{|\widetilde{H}_-|^2 - |\widetilde{H}_+|^2}{|\widetilde{H}_-|^2 + |\widetilde{H}_+|^2} = \frac{3}{2}\,\frac{X(w)}{1 + X^2(w)}, \tag{6.31}$$

$$X(w) = \sqrt{\frac{w-1}{w+1}}\,R_1(w). \tag{6.32}$$

Being a function of the ratio R_1 only, the "polarized" asymmetry $A_{\rm FB}^{\rm T}$ is predicted by HQET in an essentially model-independent way. According to (6.10), its measurement would provide valuable information about the mass parameter $\bar\Lambda$. Experimentally, such a measurement is, of course, more difficult than in the unpolarized case. But it has the advantage that the effect of a symmetric cut in $\cos\theta$ simply results in an overall kinematic factor, which can be corrected for in a model-independent way. In addition, the polarized forward–backward asymmetry is larger than the unpolarized one. The theoretical prediction is shown in Fig. 6.6.



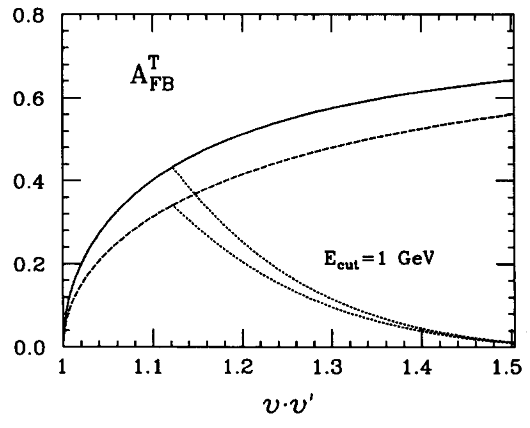
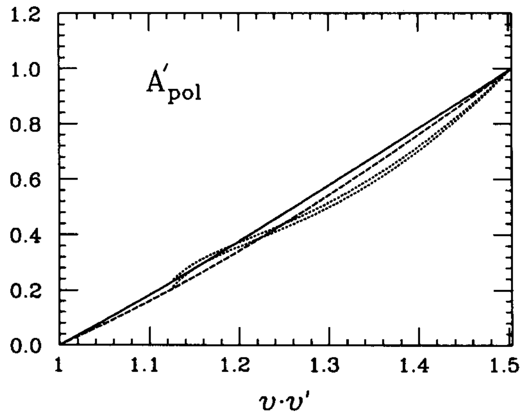


Fig. 6.5. The forward–backward asymmetry with (solid) and without (dashed) inclusion of symmetry breaking corrections. The dotted lines correspond to a cut $E_\ell > 1$ GeV in the lepton energy.

Fig. 6.6. The forward–backward asymmetry for transversely polarized D* mesons.



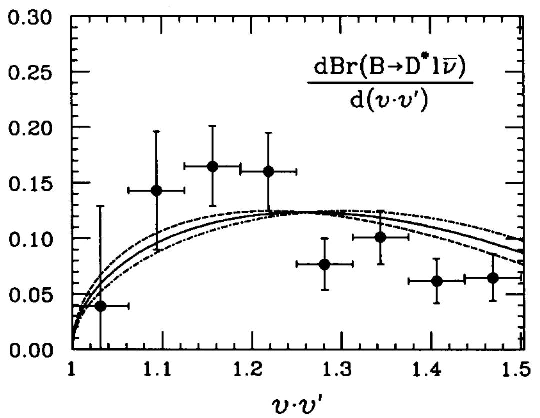


Fig. 6.7. Prediction for the polarization asymmetry defined in (6.35). The notation is the same as in Fig. 6.5; however, the dotted lines now refer to an asymmetric cut in the decay angle.

Fig. 6.8. The recoil spectrum in $B \to D^* \ell \bar{\nu}$ transitions. The data are taken from Ref. [146]. The theoretical curves correspond to $\varrho^2_{A_1} = 1.1$ (dashed), 0.8 (solid), and 0.5 (dashed-dotted). The integrated branching ratio is normalized to the ARGUS result of 5.2%.

Another interesting observable is the polarization asymmetry, which measures the difference in the relative production of D* mesons with longitudinal and transverse polarization. One usually defines a differential asymmetry by

$$A_{\rm pol}(w) = 2\frac{{\rm d}\Gamma_{\rm L}/{\rm d}w}{{\rm d}\Gamma_{\rm T}/{\rm d}w} - 1 = \frac{2|\widetilde{H}_0|^2}{|\widetilde{H}_-|^2 + |\widetilde{H}_+|^2} - 1. \tag{6.33}$$

It is related to the forward-backward asymmetries by

$$A_{\mathrm{pol}}(w) = 2A_{\mathrm{FB}}^{\mathrm{T}}(w)/A_{\mathrm{FB}}(w) - 3. \tag{6.34}$$

At the endpoints of the spectrum one has $A_{\mathrm{pol}}(1) = 0$ and $A_{\mathrm{pol}}(w_{\mathrm{max}}) = \infty$, showing that $A_{\mathrm{pol}}(w)$ is a strongly varying function of w. For this reason we prefer to introduce a related quantity $A'_{\mathrm{pol}}(w)$ by

$$A'_{\mathrm{pol}}(w) = \frac{A_{\mathrm{pol}}(w)}{1 + A_{\mathrm{pol}}(w)} = 1 - \frac{|\widetilde{H}_-|^2 + |\widetilde{H}_+|^2}{2|\widetilde{H}_0|^2}. \tag{6.35}$$

It satisfies $A'_{\mathrm{pol}}(1) = 0$ and $A'_{\mathrm{pol}}(w_{\mathrm{max}}) = 1$. Our theoretical prediction for this function is shown in Fig. 6.7. We find only small symmetry breaking effects in this case, since according to Fig. 6.2 the corrections to $(|\widetilde{H}_-|^2 + |\widetilde{H}_+|^2)$ and $|\widetilde{H}_0|^2$ are similar and compensate each other to a large extent. For a similar reason, an (asymmetric) cut in the lepton decay angle $(-1 \le \cos\theta \le x_0)$ has very little effect. For curiosity we note that the theoretical prediction for the asymmetry is perfectly reproduced by the linear relation $A'_{\mathrm{pol}}(w) = (w-1)/(w_{\mathrm{max}} - 1)$, but there is no deep reason for this.

Both the forward-backward and polarization asymmetries have been measured by the ARGUS and CLEO collaborations, however, not as a function of w or q^2, but integrated over the spectrum. The asymmetries then depend on the shape of the form factor $h_{A_1}(w)$ and cannot be predicted in a model-independent way. In the next section, we compare the experimental results to a theoretical calculation based on the QCD sum rule analysis of chapter 5. According to Fig. 6.5, we expect a mild model dependence in the case of the forward-backward asymmetry, which is a slowly varying function over a wide range in w. The integrated polarization asymmetry, on the other hand, will be more sensitive to the shape of $h_{A_1}(w)$.

6.4. Some model-dependent results

Let us then, finally, make some model-dependent predictions, which can be directly compared to the experimental results reported by the ARGUS and CLEO collaborations. We stress, however, that ultimately the aim must be to obtain high precision data that can be confronted with the model-independent predictions of the previous two sections.

As pointed out in section 6.1, the model dependence enters the analysis through the shape of the hadronic form factor $h_{A_1}(w)$. If this function was known, the decay rate (6.6) would essentially be determined. Here we shall use the QCD sum rule results collected in Tab. 5.1 to obtain an estimate of this form factor. We find that the result can be parameterized by the pole form

$$h_{A_1}(w) \approx 0.97[2/(w+1)]^{2\varrho^2_{A_1}}, \quad \varrho^2_{A_1} \approx 0.8. \tag{6.36}$$

The slope parameter is somewhat smaller than in (5.59) due to the effect of $1/m_Q$ corrections, which do not change the normalization of the form factor at zero recoil, but do affect its shape. The above result relies heavily on the QCD sum rule prediction for the Isgur–Wise function. Although the analysis presented in section 5.5 was quite sophisticated, one must not forget the systematic uncertainties inherent in the sum rule approach. To account for this, we shall consider the three cases $\varrho^2_{A_1} = 0.5$, 0.8, and 1.1. This last value is just about what can be tolerated in view of the Voloshin sum rule. The variation of our results with $\varrho^2_{A_1}$ will give an idea of the amount of model dependence.

Table 6.2
Model-dependent predictions for the semileptonic B decay branching ratios in units of $|V_{cb}/0.04|^2 \times (\tau_B/1.5\ \mathrm{ps})$.

$\varrho^2_{A_1}$	Br(D)	Br(D^*_+)	Br(D^*_-)	Br(D^*_0)	Br(D^*)
1.1	2.01%	0.49%	2.29%	3.24%	6.02%
0.8	2.50%	0.53%	2.60%	3.83%	6.96%
0.5	3.13%	0.59%	2.96%	4.54%	8.08%

Table 6.3
Model-dependent values for the product $|V_{cb}| \times (\tau_B/1.5\ \mathrm{ps})^{1/2}$. The errors are the experimental ones.

$\varrho^2_{A_1}$	from $B \to D\ell\bar{\nu}$	from $B \to D^*\ell\bar{\nu}$
1.1	0.037 ± 0.006	0.037 ± 0.002
0.8	0.034 ± 0.006	0.035 ± 0.002
0.5	0.030 ± 0.005	0.032 ± 0.002

We start by discussing the total branching ratios for $B^0 \to D^+\ell^-\bar{\nu}$ and $B^0 \to D^{*+}\ell^-\bar{\nu}$ decays. They provide an alternative, but model-dependent way to extract a value of V_{cb} from the data. Our results are given in Tab. 6.2. The branching ratios for decays into the different polarization states of the D^* meson are shown separately. Comparing these results to the experimental ones,

$$
\begin{aligned}
&\mathrm{Br}(B \to D\ell\bar{\nu}) = 1.75 \pm 0.42 \pm 0.35\%; \quad \text{Ref. [197]},\\
&\mathrm{Br}(B \to D^*\ell\bar{\nu}) = \begin{cases} 5.2 \pm 0.5 \pm 0.6\%; & \text{ARGUS [146]},\\ 4.50 \pm 0.44 \pm 0.44\%; & \text{CLEO [194]}, \end{cases}
\end{aligned}
\tag{6.37}
$$

where the first number corresponds to the weighted average of ARGUS and CLEO data, we find the values of V_{cb} given in Tab. 6.3. It is reassuring that the results obtained from the two decay modes are consistent with each other. This means that the relations between the form factors imposed by heavy-quark symmetry seem to work well. However, we observe that the model dependence in this extraction of V_{cb} is large. A similar spread of predictions, namely values between 0.034 and 0.042 (for $\tau_B = 1.5$ ps), has been observed by the ARGUS collaboration by comparing their results to various models [146]. Note that the experimental errors in Tab. 6.3 are small. It is already the theoretical uncertainty which limits this measurement, and one cannot be sure that the spread in the predictions provides a reliable estimate of the actual theoretical uncertainty. We conclude that this model-dependent extraction has very little potential of being improved in the future. Ultimately, a precise value for V_{cb}, with theoretical uncertainties that can be estimated in a controlled way, can only come from an analysis in the zero recoil region.

For completeness, we show in Fig. 6.8 the predicted shape of the spectrum for $B \to D^*\ell\bar{\nu}$ decays, in comparison to the ARGUS data [146]. One sees that the distribution is broad, which means that as far as statistics is concerned there is no problem in obtaining a large data sample close to zero recoil. It is the systematic uncertainties that give rise to the large error bars. The data seem to prefer larger values of the slope parameter $\varrho^2_{A_1}$ than are allowed by the Voloshin sum rule, but in view of the discussion of the previous section one should not take this observation too serious.

From the total branching ratios in Tab. 6.2, one can compute the integrated forward–backward and polarization asymmetries for $B \to D^*\ell\bar{\nu}$ decays, which are defined as

Table 6.4
Comparison of the integrated asymmetry parameters with experimental data.

$\varrho^2_{A_1}$	$\overline{A}_{FB}$ no cut	$\overline{A}_{FB}$ $E_{cut} = 1$ GeV	$\overline{A}_{pol}$ no cut	$\overline{A}_{pol}$ $E_{cut} = 1$ GeV
1.1	0.22	0.16	1.33	0.98
0.8	0.22	0.16	1.44	1.05
0.5	0.22	0.15	1.55	1.12
exp.	0.2 ± 0.1	0.14 ± 0.07	1.10 ± 0.45	0.7 ± 0.9
Ref.	[146]	[196]	[146]	[198]

$$\overline{A}_{FB} = \frac{3}{4}\frac{\Gamma_- - \Gamma_+}{\Gamma_- + \Gamma_+ + \Gamma_0}, \qquad \overline{A}_{pol} = \frac{2\Gamma_0}{\Gamma_- + \Gamma_+} - 1, \tag{6.38}$$

where $\Gamma_\pm$ and Γ_0 denote the total rates for producing transversely and longitudinally polarized D* mesons. However, the asymmetries are very sensitive to the cut applied in the lepton energy, and the experimental results are usually presented for a given value of the cutoff E_{cut}. It is thus necessary to take into account the effects of the cut in the theoretical calculation, as described in the previous sections. In Tab. 6.4, we compare our results to the available data. There is general agreement, but the experimental uncertainties are still very large. Notice that, as expected, the integrated forward–backward asymmetry is essentially independent of the shape of the form factor $h_{A_1}(w)$. On the other hand, we have seen in the previous section that the differential asymmetry is rather sensitive to symmetry breaking corrections which affect the ratios of form factors. This sensitivity persists when one integrates over the spectrum. Neglecting symmetry breaking corrections, we would obtain 0.19 and 0.13 (instead of 0.22 and 0.16) for the integrated asymmetries without and with applying a cut in the lepton energy, again independently of the slope parameter $\varrho^2_{A_1}$. The experiments are not yet sensitive to these kind of differences.

7. Concluding remarks

At the end of our review of heavy-quark symmetry and its role in the phenomenology of hadrons containing a heavy quark, let us quote from one of the founding papers. In 1980, Shuryak wrote [25]:

"The masses of light (u,d,s) quarks are rather different, but still there is an approximate SU(3) and a more accurate isotopic SU(2) symmetry on their substitution. The reason for this is that these masses are too small to be important. Something similar occurs for the heavy quarks (c,b,...), because their masses are too large on the usual hadronic scale. So, some symmetry for their substitution should exist between 0^- and 1^- mesons, Σ- and Λ-type baryons etc."

Later he added [26]:

"In this limit hadrons with one heavy quark resemble the hydrogen atom with its fixed center, and many problems of current models of hadronic structure ... are made trivial. Mesons made of one very heavy and one light quark are, in some sense, the simplest hadrons in which non-trivial QCD dynamics is essential, so study of them is of great importance. Of course, mesons made of two very heavy quarks are simpler, but they are next to trivial."

It took almost ten years after these remarkable observations until Isgur and Wise found that [12]:

"The new symmetries allow us to obtain absolutely normalized model-independent predictions in the heavy-quark limit of all the form factors for the $Q_1 \to Q_2$ induced weak pseudoscalar to pseudoscalar and pseudoscalar to vector transitions in terms of a single universal function $\xi(t)$ with $\xi(0) = 1$." [39]

This discovery opened the door for a new, model-independent description of heavy-quark systems in a well defined limit of QCD. It initiated the development of the heavy-quark effective theory, which by now has become a well established part of theoretical particle physics.

We have presented a comprehensive overview of the current status of these developments, starting from the intuitive physical picture of heavy-quark symmetry, which has emerged from the work of many authors over the last decade. Whereas Shuryak observed the similarities of the symmetries arising when hadronic systems are composed of light quarks with masses m_q "too light to be important", or when they contain a heavy quark with mass m_Q "too heavy to be important", we now have two conceptually clear descriptions of such systems in terms of effective field theories: chiral perturbation theory and the heavy-quark effective theory. They provide an appropriate description of QCD at low energies in an expansion in the symmetry breaking parameters m_q and $1/m_Q$, respectively, in such a way that the leading terms are determined from symmetry, up to a minimal set of reduced matrix elements such as the pion decay constant f_π or the Isgur–Wise function $\xi(v \cdot v')$. Having such a theoretical framework, one can work out the structure of the symmetry breaking corrections in a systematic way. Since, in both cases, one is dealing with nonperturbative hadronic physics, these corrections will not all be calculable; some of them are, however, and others can be constrained by the symmetries of the effective theories.

In chiral perturbation theory, the calculable corrections come from the so-called chiral logarithms, which lead to nonanalytic behavior of the form $m_q^n \ln^m(m_q/\Lambda)$, where m, n are integers. In addition, there are "local" contributions proportional to m_q^n from higher dimension operators in the chiral expansion, the coefficients of which contain complicated long-distance physics. At every order in m_q, it is possible to determine a minimal set of operators allowed by chiral symmetry, and to estimate their contributions by dimensional analysis. In heavy-quark effective theory, the corrections which can be reliably calculated arise from short-distance physics related to hard gluons probing the quantum numbers (spin, flavor, and velocity) of the heavy quarks. Through the running of the effective coupling constant $\alpha_s(m_Q)$, these effects lead to nonanalytic behavior of the form $(1/m_Q)^n[1/\ln(m_Q/\Lambda)]^m$. Again, there are further long-distance corrections proportional to $(1/m_Q)^n$, whose structure is constrained by heavy-quark symmetry, and whose magnitude can be estimated on dimensional grounds.

The theoretical tools to separate the short- and long-distance contributions are provided by the operator product expansion in combination with the renormalization group. We have discussed in detail the application of this machinery to hadronic matrix elements of heavy-quark currents, with all its intricacies. The main results are given in chapters 3 and 4, where we present the heavy-quark expansion for the weak decay form factors of heavy mesons and baryons, as well as for the meson decay constants, to next-to-leading order in $1/m_Q$ and in QCD perturbation theory. The resulting expressions are not always beautiful (at least, they are rather complicated), but recall from Shuryak that we are dealing with systems in which nontrivial QCD dynamics is essential. In the most important

[39] Here t is proportional to $(v_1 - v_2)^2$, so that $t = 0$ corresponds to zero recoil.

case of the weak decay form factors of heavy mesons, the reduced matrix elements appearing in the $1/m_Q$ expansion are the universal Isgur–Wise function $\xi(v\cdot v')$, a parameter $\bar\Lambda$ which can be identified with the "mass" of the light degrees of freedom in the hadronic bound state, and a set of four next-to-leading universal functions $\eta(v\cdot v')$ and $\chi_i(v\cdot v')$; $i=1,2,3$. Some of these functions are subject to nontrivial conditions at zero recoil, namely $\xi(1)=1$ and $\chi_1(1)=\chi_3(1)=0$. Otherwise, however, they contain the long-distance hadronic dynamics and cannot be calculated yet from first principles. We have presented theoretical predictions for these quantities derived from QCD sum rules. Although this is not a rigorous approach from first principles, some general results have been obtained which are likely to be model independent. For instance, one finds that hyperfine corrections arising from the coupling of the heavy-quark spin to the gluon field are small.

In the last part of the review, we have presented a new analysis of semileptonic B meson decays in the light of these developments. Emphasis is put on a clear separation of model-independent aspects from model-dependent ones. We have made several predictions which become exact in the heavy-quark symmetry limit, meaning that hadronic uncertainties enter the description only at the level of power corrections in $1/m_Q$. In many cases, the leading corrections can even be calculated. In particular, we have proposed a measurement of the element V_{cb} of the Cabibbo–Kobayashi–Maskawa matrix from the endpoint region of the recoil spectrum in $B\to D^*\ell\,\bar\nu$ decays, with a theoretical uncertainty of only 4%. Currently, this method is limited by the experimental systematic errors. It gives a value $|V_{cb}|\approx 0.04$, however, with an uncertainty of at least 15%. With the availability of data samples with higher statistics, and with a better control of the systematic errors in the zero recoil region, V_{cb} obtained by using this method will become the third-best known entry (after V_{ud} and V_{us}) in the quark mixing matrix. More generally, we may hope that the ongoing theoretical developments related to heavy-quark symmetry help to promote the phenomenology of heavy-quark systems from an era of naive bound state models to a solid theoretical description in terms of a systematic expansion in QCD. This would be worth the effort.

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