WEAK DECAYS OF HEAVY MESONS IN THE STATIC QUARK APPROXIMATION ☆

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When one or more quarks are heavy compared to hadronic scales, some new symmetries appear in the low energy effective lagrangian for QCD. We exploit these static quark symmetries to derive model-independent normalizations of some weak hadronic matrix elements involving heavy quarks, as well as many relationships between such matrix elements. We briefly discuss how some of these conditions can be used to improve determinations of Kobayashi–Maskawa angles.

The weak decays of heavy mesons can be a rich source of information about particle physics. For example, the semileptonic decays of B mesons play a crucial role in determining certain weak mixing angles, and the rare decays of B mesons are a sensitive probe of new physics beyond the minimal standard theory.

In a hadron containing a heavy quark Q ($m_Q \gg \Lambda_{QCD}$) along with light degrees of freedom, the heavy quark acts as an essentially static color source in the hadron's rest frame. It has recently been realized that this fact can be exploited to derive relationships between hadronic parameters [1]. A simple example is the matrix element of the axial current $A_\nu = \bar{q}\gamma_\nu\gamma_5 Q$ (where q is a light quark field) between the ground state pseudoscalar meson P with $\bar{q}$Q flavor quantum numbers and the vacuum[#1]

$$\langle 0 | A_\nu | P(\boldsymbol{p}) \rangle = \frac{f_P p_\nu}{\sqrt{2E_P}} . \tag{1}$$

Since the axial current is partially conserved, it receives no renormalization. However, there is a large logarithm, namely $\ln(m_Q^2/\Lambda_{QCD}^2)$, in the perturbative expansion of the matrix element (1) coming from loop momenta p in the region $\Lambda_{QCD} < p < m_Q$. This logarithm can be displayed explicitly by going over to an effective theory where the heavy quark is treated as a static color source. The axial current in this effective theory, $A'_\nu(\mu)$, does require renormalization and hence develops a dependence on the subtraction point μ. The relationship between the axial current in the effective theory and in the complete theory is

$$A_\nu = C(\mu) A'_\nu(\mu) + ... , \tag{2}$$

where the ellipses denote higher dimension operators suppressed by powers of m_Q. At $\mu = m_Q$

$$C(m_Q) = 1 + O(\alpha_s(m_Q)/\pi) . \tag{3}$$

At this subtraction point there is a large logarithm in the matrix element of $A'_\nu(\mu)$. We can transfer it from the matrix element to the coefficient function C by scaling the subtraction point μ down from m_Q to some hadronic scale around Λ_{QCD} using the renormalization group equations. The result is that in the leading logarithmic approximation

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#1 In this paper we will find it convenient to normalize states containing a heavy quark according to $\langle M(\boldsymbol{p}', s') | M(\boldsymbol{p}, s) \rangle = \delta_{s's}\delta^3(\boldsymbol{p}' - \boldsymbol{p})$. This is the origin of the factor $(2E_P)^{-1/2}$ on the right-hand side of eq. (1).

$$C(\mu)=\left(\frac{\alpha_s(m_Q)}{\alpha_s(\mu)}\right)^{-6/(33-2N_f)} . \tag{4}$$

For μ of the order of the QCD scale, $C(\mu)$ is the factor that takes into account the large logarithm in the matrix element (1). The strong interaction matrix element that multiplies $C(\mu)$ in eq. (4) to yield f_P of eq. (1) can be eliminated by comparing analogous systems containing different heavy quarks. In other words, at the QCD scale all heavy quarks become equivalent static sources in evaluating the hadronic matrix element. If Q' is another heavy quark of mass $m_{Q'}$ and P' is the $\bar{Q}'q$ meson analogous to P then [1]

$$\frac{f_{P'}}{f_P}=\left(\frac{m_P}{m_{P'}}\right)^{1/2}\left(\frac{\alpha_s(m_{P'})}{\alpha_s(m_P)}\right)^{-6/(33-2N_f)} , \tag{5}$$

where N_f is the number of flavors appropriate to the mass scale in the interval between $m_Q \simeq m_P$ and $m_{Q'} \simeq m_{P'}$. In this paper we derive other results which follow from the static nature of the heavy quark Q in hadrons containing it. Some of these will be relevant to the determination of the weak mixing angles.

The low energy effective field theory where Q is treated as a static source has symmetries that are not present in the complete theory with a fully dynamical quark. The first we discuss arises because the spin degree of freedom of a very heavy quark decouples [2]. As a consequence, in the effective theory the spin of the heavy quark generates a symmetry. In fact, since in the static limit the number of Q's and $\bar{Q}$'s are separately conserved, the quark spin $\boldsymbol{S}_Q$ and the antiquark spin $\boldsymbol{S}_{\bar{Q}}$ are the generators of independent SU(2) symmetries. These spin symmetries have implications for the hadronic spectrum. Suppose P is the lowest lying pseudoscalar meson with $\bar{q}Q$ flavor quantum numbers and V is the corresponding vector meson. Then

$$S_Q^3|P\rangle=-\tfrac{1}{2}|V\rangle, \quad S_Q^3|V\rangle=-\tfrac{1}{2}|P\rangle, \tag{6a,b}$$

where in eqs. (6) the states have zero three-momentum and $J^3|V\rangle=0$. Because P is transformed into V these states are degenerate in the symmetry limit; thus, any mass splitting must be of order $1/m_Q$. This result [2], which is commonly stated as a consequence of the non-relativistic quark model, is actually a general property of QCD. In particular, it holds even though the mesons P and V are not necessarily weakly bound states.

The matrix element analogous to (1) for vector mesons is

$$\langle 0|V_\nu|V(\boldsymbol{p},\boldsymbol{\epsilon})\rangle=\frac{f_V\epsilon_\nu}{\sqrt{2E_V}} , \tag{7}$$

where the vector current $V_\nu=\bar{q}\gamma_\nu Q$. In the effective theory with the heavy quark treated as a static color source, the vector current $V'_\nu(\mu)$ has the same anomalous dimension as the axial current. Consequently the relationship between V_ν and $V'_\nu(\mu)$ is the same as that between A_ν and $A'_\nu(\mu)$:

$$V_\nu=C(\mu)V'_\nu(\mu)+... , \tag{8}$$

with C given by eq. (4). In the low energy effective theory

$$\begin{aligned}\langle 0|A'_0(\mu)|P\rangle &=\langle 0|\bar{q}(0)\gamma_5 Q(0)|P\rangle \\ &=\langle 0|\bar{q}(0)\gamma_5 Q^{(+)}(0)|P\rangle \\ &=4\langle 0|\bar{q}(0)\gamma_5 Q^{(+)}(0)(S_Q^3)^2|P\rangle \\ &=2\langle 0|[S_Q^3,\bar{q}(0)\gamma_5 Q^{(+)}(0)]|V\rangle ,\end{aligned} \tag{9}$$

where $Q^{(+)}(x)$ is the positive (negative) frequency part of $Q(x)$ corresponding to its Q($\bar{Q}$) part. The replacement of Q by $Q^{(+)}$ here is allowed since in the static limit $Q^{(-)}$ cannot connect $|P\rangle$ to the vacuum. Using the explicit representation

$$S_Q^3=\frac{i}{4}\int d^3x : Q^{(+)}(x)^\dagger[\gamma^1,\gamma^2]Q^{(+)}(x):, \tag{10}$$

one can calculate the commutator [#2] in eq. (9). Using the gamma matrix identities

$$\gamma^\nu\gamma^\lambda\gamma^\sigma=g^{\nu\lambda}\gamma^\sigma+g^{\lambda\sigma}\gamma^\nu-g^{\sigma\nu}\gamma^\lambda-i\gamma_5\gamma_\mu\epsilon^{\mu\nu\lambda\sigma} , \tag{11}$$

and the operator equation $\gamma^0 Q^{(+)}(x)=Q^{(+)}(x)$, one arrives at the result

$$\langle 0|A'_0(\mu)|P\rangle=-\langle 0|V'_3(\mu)|V\rangle . \tag{12}$$

Combining eqs. (1), (2), (7), (8) and (12) yields the new relation

$$f_V=m_P f_P . \tag{13}$$

[#2] Since in the effective theory only momenta $\ll m_Q$ are relevant, each of $Q^{(+)}$ and $Q^{(-)}$ are essentially local field operators.

So far we have discussed relations between matrix elements. We now turn to the derivation of the absolute normalization of a matrix element with potentially important applications. We consider the transition matrix element

$$\langle P'(\boldsymbol{p}')|V_\nu|P(\boldsymbol{p})\rangle = \frac{f_+(p+p')_\nu + f_-(p-p')_\nu}{\sqrt{4E_{P'}E_P}}, \tag{14}$$

where in this case the vector current is

$$V_\nu = \bar{Q}'\gamma_\nu Q. \tag{15}$$

In eqs. (14) and (15), Q and Q′ are both heavy quarks with $m_Q > m_{Q'}$. The mesons P and P′ are the lowest lying pseudoscalar mesons with q̄Q and q̄Q′ flavor quantum numbers, respectively. Going over to an effective theory with Q treated as a static color source, but with Q′ still fully dynamical, we would have $V_\nu = C(\mu)V'_\nu(\mu)$ as before, with $C(\mu)$ given by eq. (4). However, since Q′ is also heavy we can go over to a new effective field theory where both the heavy quarks Q and Q′ are treated as static color sources. The vector current V''_ν in this effective theory is not renormalized by the physics at scales $p < m_{Q'}$ since at such scales both quarks are effectively static. Therefore

$$V'_\nu(m_{Q'}) = V''_\nu + \ldots, \tag{16}$$

where the ellipses denote terms which are suppressed by factors of $\alpha_s(m_{Q'})/\pi$ and powers of $\Lambda_{QCD}/m_{Q'}$. Combining these results gives that the relation between the vector current V_ν in the complete theory where both Q and Q′ are fully dynamical and the vector current V''_ν in the effective field theory where both Q and Q′ are treated as static color sources is (in the leading logarithmic approximation)

$$V_\nu = \left(\frac{\alpha_s(m_P)}{\alpha_s(m_{P'})}\right)^{-6/(33-2N_f)} V''_\nu. \tag{17}$$

In addition to the heavy quark spin symmetries already discussed, the low energy effective theory where both the heavy quarks Q and Q′ are treated as static has interactions which have an SU(2) flavor symmetry under

$$\begin{pmatrix} Q \\ Q' \end{pmatrix} \to U \begin{pmatrix} Q \\ Q' \end{pmatrix}, \tag{18}$$

where $U \in SU(2)$. This symmetry arises because once the quarks Q and Q′ are sufficiently heavy that they can be treated as static color sources (i.e., m_Q, $m_{Q'} \gg \Lambda_{QCD}$), the values of their masses become irrelevant (note, in particular, that $m_Q - m_{Q'}$ is also irrelevant to matrix elements we are considering and may itself be large compared to Λ_{QCD}). The existence of such a symmetry is important: it is what allows us to normalize the matrix element of V''_0 between P and P′ states. Since the operator $\int d^3x\, V''_0(x)$ is a generator of the SU(2) flavor symmetry,

$$\langle P'|V''_0|P\rangle = 1. \tag{19}$$

Combining eqs. (14), (17) and (19) gives

$$f_+(m_P + m_{P'}) + f_-(m_P - m_{P'}) = \sqrt{4m_{P'}m_P}\left(\frac{\alpha_s(m_P)}{\alpha_s(m_{P'})}\right)^{-6/(33-2N_f)}. \tag{20}$$

In eq. (20) the form factors $f_\pm$, which are functions of $t = (p-p')^2$, are to be evaluated at $t_m = (m_P - m_{P'})^2$ where both mesons are at rest, as required by the static limit. Note that this is the kinematic point emphasized by the quark model calculations of refs. [3,4].

Eq. (20) has important consequences for the determination of weak mixing angles using semileptonic B meson decay. In the standard six-quark model the coupling of quarks to W-bosons is given by the interaction lagrangian density

$$L_{int} = \frac{-g_2}{2\sqrt{2}}(\bar{u}, \bar{c}, \bar{t})\gamma^\mu(1-\gamma_5)V\begin{pmatrix} d \\ s \\ b \end{pmatrix}W_\mu + \text{h.c.} \tag{21}$$

In eq. (21), V is a 3×3 unitary matrix (the Kobayashi–Maskawa matrix [5]) that arises from diagonalization of the quark mass matrices. The rates for the semileptonic decays $\bar{B} \to D\ell\bar{\nu}_\ell$, where $\ell = e, \mu, \tau$, are determined by the product of the magnitude of V_{cb} and the hadronic form factors f_+ and f_- appearing in eq. (14). We now have from eq. (20) that

$$f_+(t_m)(m_B + m_D) + f_-(t_m)(m_B - m_D) = \sqrt{4m_B m_D}\left(\frac{\alpha_s(m_B)}{\alpha_s(m_D)}\right)^{-6/25}, \tag{22}$$

where $t_m = (m_B - m_D)^2$. Eq. (22) is *not* a model-dependent result. It is a consequence of QCD, corrections to it being suppressed by factors of $\alpha_s(m_D)/\pi$ and Λ_{QCD}/m_D. Thus using experimentally determined values for $|V_{cb}f_+|$ and f_-/f_+ with this relation allows a model independent determination of V_{cb} up to corrections to the static quark and leading logarithm approximations.

Since contracting $(p_{\bar{B}}-p_D)_\mu$ against the lepton current gives a factor of the lepton mass, electronic and muonic $\bar{B}\to D$ transitions are practically sensitive to only f_+. However, the f_- form factor can be measured in $\bar{B}\to D\tau\bar{\nu}_\tau$. Although its extraction will undoubtedly be difficult, the above discussion indicates that the required effort is warranted.

We can also obtain by similar methods many other rigorous absolutely normalized conditions governing weak transitions between heavy quarks. In the first place, of course, eq. (20) can be applied to the $T\to\bar{B}$ semileptonic decays. Eq. (20) also has a close analogue in baryon decays $Qq_1q_2\to Q'q_1q_2$. In addition, we can use the heavy quark spin symmetries to generate normalization predictions for other than ground state to ground state transitions: equations analogous to eq. (20) exist for $P\to V'$, $V\to P'$ and $V\to V'$ transitions[#3]. The situation for excited baryons is somewhat more complex. Since $\boldsymbol{S}_Q$ and $\boldsymbol{J}_\ell\equiv\boldsymbol{J}-\boldsymbol{S}_Q$ (the light quark and gluon angular momentum operator) are separately conserved, heavy quark baryons can be simultaneously assigned the corresponding quantum numbers s_Q, m_Q, j_ℓ and m_ℓ. Assuming that it has $J^P=\frac{1}{2}^+$, the ground state could therefore have either $j_\ell=0$ or 1, leading to a doublet or sextet of degenerate ground states. In the valence quark model the former option corresponds to having the two light quarks in an antisymmetric spin-zero state (so that, if they are nonstrange light quarks, the Pauli principle would force them to have $I=0$ and the state would be the Λ_Q); the latter option corresponds (once again in strangeness zero) to degenerate Σ_Q and Σ^*_Q spin multiplets. The first option is clearly indicated, in which case the spin of the ground states of Qq_1q_2 and $Q'q_1q_2$ would be the spin of the heavy quarks; the generators $\boldsymbol{S}_Q$ and $\boldsymbol{S}_{Q'}$ then produce no new conditions[#4].

[#3] The latter cases may be relevant to T* semileptonic decay.

[#4] They *can* produce relations between $\Lambda_Q\to\Sigma_{Q'}$ and $\Lambda_Q\to\Sigma^*_{Q'}$ as well as normalizations for $(\Sigma_Q,\Sigma^*_Q)\to(\Sigma_{Q'},\Sigma^*_{Q'})$. The quark model suggests [6] that Σ_Q and Σ^*_Q are strongly decaying, in which case the latter matrix elements would be uninteresting.

We hope to provide a more complete classification and calculation of possible normalizations like eq. (20) and relations like eqs. (5) and (13) elsewhere, but will give two more results of potentially immediate relevance before concluding this letter. It is already known that $b\to ce\bar{\nu}_e$ transitions are nearly saturated by the two channels $\bar{B}\to De\bar{\nu}_e$ and $\bar{B}\to D^*e\bar{\nu}_e$, with the latter dominant by about a factor of two. It is thus of interest to use $\boldsymbol{S}_c$ symmetry to generate a $P\to V$ relation from eq. (19):

$$\begin{aligned}1&=\langle P'|V_0''|P\rangle\\&=-2\langle V'|[S^3_{Q'},V_0'']|P\rangle\\&=-\langle V'|A_3''|P\rangle\,.\end{aligned}\tag{23}$$

Defining

$$\langle V'(\boldsymbol{p}',\boldsymbol{\epsilon})|A_\nu(0)|P(\boldsymbol{p})\rangle=\frac{1}{\sqrt{4E_{V'}E_P}}\times[fp^*_\nu+a_+(\epsilon^*\cdot p)(p+p')_\nu+a_-(\epsilon^*\cdot p)(p-p')_\nu]\,,\tag{24}$$

we find

$$f(t_m)=\sqrt{4m_{V'}m_P}\left(\frac{\alpha_s(m_P)}{\alpha_s(m_{V'})}\right)^{-6/(33-2N_f)}\,,\tag{25}$$

or in the case of $\bar{B}\to D^*$

$$f(t_m)=\sqrt{4m_Bm_{D^*}}\left(\frac{\alpha_s(m_B)}{\alpha_s(m_{D^*})}\right)^{-6/25}\tag{26}$$

$$=\sqrt{4m_Bm_{D^*}}\left(\frac{\ln(m^2_{D^*}/\Lambda^2_{QCD})}{\ln(m^2_B/\Lambda^2_{QCD})}\right)^{-6/25}\tag{27}$$

$$=7.1\ \mathrm{GeV}\,.\tag{28}$$

This route to an absolute normalization for V_{cb} is in many ways simpler than the $\bar{B}\to D$ transition:

(1) The $\bar{B}\to D^*e\bar{\nu}_e$ exclusive decay is easier to detect experimentally since reconstruction of the D^*–D mass difference provides a powerful constraint.

(2) Although the two axial form factors f and a_+ and a vector form factor g (see refs. [3,4]) all contribute to $\bar{B}\to D^*$ decays, they can all be measured in the electronic or muonic transitions so that the experimentally more difficult tau decays are not needed.

(3) The f form factor is the most important one in $\bar{B}\to D^*$. Moreover, it *alone* contributes to the rate as $t\to t_m$, so that it completely dominates the rate in the

zero-recoil ($t=t_m$) region of the Dalitz plot where we need to determine it.

Finally, we would like to emphasize a point made earlier by one of us [7]: measurements of Cabibbo suppressed $c \to d$ matrix elements can cupply us with information on the $b \to u$ matrix elements crucial to the determination of V_{ub}. This is because the static SU(2) flavor symmetry (plus ordinary isospin symmetry) tells us that the matrix elements for $P \to X$ and $P' \to X'$ (where X and X' are either identical or isospin partners) are related up to trivial Clebsch–Gordan factors by

$$\langle X(\boldsymbol{p}_X, \boldsymbol{\epsilon}) | (V^\mu - A^\mu)_{Q \to q} | P(\mathbf{0}) \rangle$$

$$= \left(\frac{\alpha_s(m_P)}{\alpha_s(m_{P'})}\right)^{-6/(33-2N_f)}$$

$$\times \langle X'(\boldsymbol{p}_X, \boldsymbol{\epsilon}) | (V^\mu - A^\mu)_{Q' \to q'} | P'(\mathbf{0}) \rangle , \qquad (29)$$

where q and q' are the light quarks in X and X'. Note that this relation holds for each $\boldsymbol{p}_X$ (with $|\boldsymbol{p}_X| < m_{Q'}$) and $\boldsymbol{\epsilon}$, so that it relates a $Q \to q$ matrix element at $t = (p_P - p_X)^2$ to one at $t' = (p_{P'} - p_X)^2$. From eq. (29) the relationships between all the invariant form factors appearing in $P \to X$ and those in $P' \to X'$ follow, and will always be of the form (n is a non-negative integer)

$$f_{Q \to q}(t)$$

$$= \left(\frac{m_P}{m_{P'}}\right)^{1/2-n} \left(\frac{\alpha_s(m_P)}{\alpha_s(m_{P'})}\right)^{-6/(33-2N_f)} f_{Q' \to q'}(t'). \qquad (30)$$

There is a special case [8] where the $\Lambda_{QCD}/m_{P'}$ corrections to this relation are more subtle than those considered previously. Since $m_\pi \simeq 0$, the V'–P' mass difference must be very small before eq. (29) applies to the entire $P' \to \pi$ transition. This is because the contribution of the V' pole to the $P' \to \pi$ transition contains a factor $(m_{V'} - m_{P'} + E_\pi)^{-1}$ which becomes independent of the heavy quark mass only for $m_{V'} - m_{P'} \ll E_\pi$. Since $m_{D^*} - m_D \simeq m_\pi$, this means that eq. (29) cannot be applied to $D \to \pi$ for small E_π; in practice this restriction should have very little effect. The existence of relation (29) makes measurements of the form factors of the Cabibbo suppressed decays $D \to \pi \ell^+ \nu_\ell$ and $D \to \rho \ell^+ \nu_\ell$ of the utmost importance to the eventual precise determination of V_{ub} from the corresponding exclusive $\bar{B} \to \pi \ell \bar{\nu}_\ell$ and $\bar{B} \to \rho \ell \bar{\nu}_\ell$ decays.

M_D is not sufficiently large for Λ_{QCD}/m_D corrections to predictions like (28) and (29) to be completely neglected (the model of refs. [3,4] suggests that they are of the order of 10%). Perturbative corrections of order $\alpha_s(m_D)/\pi \simeq 0.1$ should also be studied. Nevertheless, it is clear that having rigorous relations like those derived here for static quarks represents a large step forward: it reduces our dependence on models to estimating the deviations of real systems from the static limit.

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