# WEAK TRANSITION FORM FACTORS BETWEEN HEAVY MESONS ☆

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We present an extension of our earlier observations on symmetries in operation in the weak decays of heavy mesons containing a single heavy quark. The new symmetries allow us to obtain absolutely normalized model-independent predictions in the heavy quark limit of all of the form factors for the $Q_1 \to Q_2$ induced weak pseudoscalar to pseudoscalar and pseudoscalar to vector transitions in terms of a single universal function $\xi(t)$ with $\xi(0)=1$.

The properties of hadrons containing a single heavy quark Q ($m_Q \gg \Lambda_{QCD}$) along with light degrees of freedom are constrained by symmetries which are not manifest in QCD [1]. The first of these is a flavor symmetry which arises from the fact that the long-wavelength properties of the light degrees of freedom in such a hadron become independent of $m_Q$ for $m_Q \gg \Lambda_{QCD}$. Thus, for example, the *light* degrees of freedom of a $\bar{B}$ and D meson can be related by an (approximate) b↔c SU(2) symmetry even though $m_b$ and $m_c$ are very different. The second symmetry pointed out in ref. [1] is a related spacetime symmetry which arises in QCD because the spin of a heavy quark decouples from the gluon field [2]. This makes $\boldsymbol{S}_Q$, the heavy quark spin operator, the generator of another SU(2) group of symmetries of the light degrees of freedom in a meson containing a single heavy quark. Thus, for example, the *light* degrees of freedom in the $\bar{B}$ and $\bar{B}^*$ mesons are in (approximately) the same state since the spin orientation of the b quark does not affect their dynamics. These symmetries are manifest in an effective theory where the heavy quark acts in its hadron's rest frame like a spatially static triplet source of (short-distance) color electric field [2,3]. In the effective theory the heavy quark's couplings to the gluon degrees of freedom are independent of its mass and described by a Wilson line [3].

To apply these symmetries to the matrix elements of weak current, we must know these currents in the effective theory. For the currents of interest here (color indices are suppressed)

$$V^{ji}_\nu \equiv \bar{Q}_j \gamma_\nu Q_i \,, \tag{1}$$

$$A^{ji}_\nu \equiv \bar{Q}_j \gamma_\nu \gamma_5 Q_i \,, \tag{2}$$

one has [1,4] for $J_\nu = V_\nu$ or $A_\nu$

$$J^{ji}_\nu = C_{ji} \boldsymbol{J}^{ji}_\nu + \dots \,, \tag{3}$$

where the ellipses denote higher dimension operators suppressed by powers of $m_Q$, and where on the right-hand side of eq. (3) $\boldsymbol{J}^{ji}_\nu$ is the naive weak current in the effective theory. In the leading logarithmic approximation with $m_j < m_i$

$$C_{ji} = \left( \frac{\alpha_s(m_i)}{\alpha_s(m_j)} \right)^{6/(33-2N_f)} \tag{4}$$

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Note that it is the charges associated with the currents $\bar{Q}\gamma_\nu \frac{1}{2}\boldsymbol{\tau} Q$ with

$$Q \equiv \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix}, \tag{5}$$

the heavy quark field operators in the effective theory, which lead to the flavor symmetry.

In our earlier paper [1] we applied this symmetry for static quarks, where a $Q_1 \to Q_2$ transition simply substituted one static heavy quark for another. Here we exploit a powerful extension of this method which makes use of the fact that when $Q_1$ at *velocity* $\boldsymbol{v}$ makes a weak transition into $Q_2$ at *velocity* $\boldsymbol{v}'$, the amplitude for the light degrees of freedom to make any associated transition is independent of $m_1$ and $m_2$ if they are sufficiently large. I.e., the light degrees of freedom interact only with the (moving) color fields of $Q_1$ and $Q_2$ which depend only on the Lorentz boosts required of the (mass independent) rest frame color fields. In the effective theory the mass of a heavy quark Q is taken to infinity in such a way that $p^\mu_Q/m_Q$ is held fixed, but the four momenta of the light degrees of freedom are neglected compared to $m_Q$. In this limit the interactions of the gluons with the heavy quark do not alter its straight world line and are independent of its mass and spin. In the matrix elements of the currents in eqs. (1) and (2), changes in the heavy quark velocity and spin occur only due to the actions of the currents. In a typical transition $H_1(\boldsymbol{v}=0) \to H_2(\boldsymbol{v}')$ (where $H_i$ is a hadron containing the single heavy quark $Q_i$), the form factors will therefore be determined by the product of the amplitude for the heavy quarks to make the transition $Q_1(\boldsymbol{v}=0) \to Q_2(\boldsymbol{v}')$ and for the light quarks to be "excited" by the transition from the hadron $H_1$ at rest into the hadron $H_2$ moving with velocity $\boldsymbol{v}'$. The resulting symmetry is somewhat unusual in that it relates states of equal *velocity* but different mass, and therefore of different momentum.

Given this peculiarity, it is convenient to remove insofar as possible the dependence of the heavy quark and hadron formalism on the heavy quark mass $m_Q$ [#1]. We therefore work with heavy boson fields $\tilde{\phi} \equiv m_Q^{-1}\phi$ and heavy fermion fields $\tilde{\psi} \equiv m_Q^{-3/2}\psi$ with creation operators normalized according to

$$[\tilde{a}(\boldsymbol{v}', s'), \tilde{a}^\dagger(\boldsymbol{v}, s)] = 2\gamma\delta_{s's}\delta^3(\gamma'\boldsymbol{v}' - \gamma\boldsymbol{v}), \tag{6}$$

and

$$[\tilde{b}(\boldsymbol{v}', s'), \tilde{b}^\dagger(\boldsymbol{v}, s)] = \gamma\delta_{s's}\delta^3(\gamma'\boldsymbol{v}' - \gamma\boldsymbol{v}), \tag{7}$$

respectively, where $\gamma = E/m_Q = (1-\boldsymbol{v}^2)^{-1/2}$. In contrast to the usual flavor symmetry we then define symmetry operators $U$ by

$$U^\dagger \tilde{a}_i^\dagger(\boldsymbol{v}, s) U = R_{ij}\tilde{a}_j^\dagger(\boldsymbol{v}, s), \tag{8}$$

etc., where $R = \exp(\mathrm{i}\,\frac{1}{2}\boldsymbol{\sigma}\cdot\boldsymbol{\theta})$.

Let us now work out the consequences of this symmetry for pseudoscalar to pseudoscalar transitions via the heavy quark vector current (1). Since in the effective theory

$$\langle \tilde{P}_j(\boldsymbol{v}') | \bar{\tilde{Q}}_j(0)\gamma_\nu \tilde{Q}_i(0) | \tilde{P}_i(\boldsymbol{v}) \rangle,$$

is independent of $i$ and $j$, in terms of the usual meson states with normalizations

$$\langle M(\boldsymbol{p}', s') | M(\boldsymbol{p}, s) \rangle = 2E\delta_{s's}\delta^3(\boldsymbol{p}' - \boldsymbol{p}),$$

and the full vector current we have that

$$\rho_{ji} \equiv \frac{\langle P_j(p_j') | \bar{Q}_j(0)\gamma_\nu Q_i(0) | P_i(p_i) \rangle}{C_{ji}\sqrt{m_j m_i}}, \tag{9}$$

is independent of $i, j$. By definition

$$\langle P_1(p_1') | \bar{Q}_1(0)\gamma_\nu Q_1(0) | P_1(p_1) \rangle = f_{11}(t_{11})(p_1 + p_1')_\nu, \tag{10}$$

$$\langle P_2(p_2') | \bar{Q}_2(0)\gamma_\nu Q_2(0) | P_2(p_2) \rangle = f_{22}(t_{22})(p_2 + p_2')_\nu, \tag{11}$$

$$\langle P_2(p_2') | \bar{Q}_2(0)\gamma_\nu Q_1(0) | P_1(p_1) \rangle = f_+(t_{21})(p_1 + p_2')_\nu + f_-(t_{21})(p_1 - p_2')_\nu, \tag{12}$$

where $t_{rs} = (p_s - p_r')^2$. Thus using $\rho_{11} = \rho_{22} = \rho_{21}$ and equating coefficients of the velocities $\boldsymbol{v}'$ and $\boldsymbol{v}$ separately gives

$$f_{11}(t_{11}) \equiv \xi(t_{11}), \tag{13}$$

$$f_{22}(t_{22}) = \xi\left(\frac{m_1^2}{m_2^2}\, t_{22}\right), \tag{14}$$

[#1] Henceforth we will ignore differences between the heavy quark mass and the mass of the hadron of which it is a part.

$$f_\pm(t_{21}) = \pm\left(\frac{\alpha_s(m_1)}{\alpha_s(m_2)}\right)^{-6/(33-2N_f)}$$

$$\times\left(\frac{m_1 \pm m_2}{\sqrt{4m_1 m_2}}\right)\xi\left(\frac{m_1}{m_2}[t_{21}-t_m]\right), \qquad (15)$$

where $t_m \equiv (m_1-m_2)^2$ is the maximum momentum transfer in the $P_1 \to P_2 e\bar{\nu}_e$ transition. All four form factors are thus determined by a single universal function $\xi(t)$. Moreover, since $\bar{Q}_1\gamma_\nu Q_1$ is a conserved current, $\xi(0)=1$. In our earlier work [1], we found only the value of $f_+(m_1+m_2)+f_-(m_1-m_2)$ at $t=t_m$. Eq. (15) is, to the order we are working, consistent with the Ademollo–Gatto theorem which states that as $m_2 \to m_1$, $f_+(t_m)$ deviates from unity by terms quadratic in $\Delta m \equiv m_1 - m_2$. When $m_2$ is close to $m_1$, the linear term in $\Delta m$ which arises from expanding the ratio of strong interaction fine structure constants in $C_{21}$ is, with $\bar{m} \equiv \frac{1}{2}(m_1+m_2)$,

$$\frac{\alpha_s(\bar{m})}{\pi}\frac{\Delta m}{\bar{m}},$$

and so is a higher order effect. It can be compensated by one-loop contributions to the matching condition between the full field theory and the effective theory. Although there are other corrections of order $\alpha_s \bar{m})/\pi$, in our numerical results to be quoted below we will use $C'_{21} = C_{21}\{1-[\alpha_s(\bar{m})/\pi]\Delta m/\bar{m}\}$ to ensure compliance with the Ademollo–Gatto theorem in the approach to the equal mass limit. For $b \to c$ transitions, the resulting compensation of $C_{21}$ is almost complete.

By using the symmetry of the light degrees of freedom under a rotation of the heavy quark spin, analogous relations can be derived for $P_1 \to V_2$ transitions, where $V_2$ is the vector meson degenerate with $P_2$ as $m_2 \to \infty$. To use this symmetry we note that

$$h_Q|P_Q(\boldsymbol{p})\rangle = \tfrac{1}{2}|V_Q(\boldsymbol{p},0)\rangle, \qquad (16)$$

where $h_Q$ is the helicity operator of the heavy quark Q, and $|V_Q(\boldsymbol{p},0)\rangle$ is the helicity zero state of $V_Q$. Thus for $\Gamma$ any product of $\gamma$-matrices,

$$\langle V_2(\boldsymbol{p}',0)|\bar{Q}_2(0)\Gamma Q_1(0)|P_1(\boldsymbol{p})\rangle$$
$$= 2\langle P_2(\boldsymbol{p}')|[h_{Q_2}, \bar{Q}_2(0)\Gamma Q_1(0)]|P_1(\boldsymbol{p})\rangle. \qquad (17)$$

Since

$$[\hat{\boldsymbol{n}}\cdot\boldsymbol{J}_Q, Q^\dagger(0)] = [\hat{\boldsymbol{n}}\cdot\boldsymbol{S}_Q, Q^\dagger(0)]$$
$$= Q^\dagger(0)\hat{\boldsymbol{n}}\cdot\boldsymbol{\Sigma}, \qquad (18)$$

where $\Sigma^i = \frac{1}{8}i\epsilon^{ijk}[\gamma^j,\gamma^k]$, one has many useful commutation relations, e.g.,

$$[S^z_{Q_2}, A^{21}_0] = -\tfrac{1}{2}V^{21}_3, \qquad (19)$$
$$[S^z_{Q_2}, A^{21}_3] = -\tfrac{1}{2}V^{21}_0, \qquad (20)$$
$$[S^z_{Q_2}, A^{21}_\pm] = \mp\tfrac{1}{2}A^{21}_\mp, \qquad (21)$$
$$[S^z_{Q_2}, V^{21}_0] = -\tfrac{1}{2}A^{21}_3, \qquad (22)$$
$$[S^z_{Q_2}, V^{21}_3] = -\tfrac{1}{2}A^{21}_0, \qquad (23)$$

and

$$[S^z_{Q_2}, V^{21}_\pm] = \mp\tfrac{1}{2}V^{21}_\mp, \qquad (24)$$

which are more than sufficient to determine all of the form factors in $P_1 \to V_2$ transitions. With the definitions

$$\langle V_2(\boldsymbol{p}',\boldsymbol{\epsilon})|A^{21}_\nu(0)|P_1(\boldsymbol{p})\rangle$$
$$= f\epsilon^*_\nu + a_+(\epsilon^*\cdot p)(p+p')_\nu + a_-(\epsilon^*\cdot p)(p-p')_\nu, \qquad (25)$$

$$\langle V_2(\boldsymbol{p}',\boldsymbol{\epsilon})|V^{21}_\nu(0)|P_1(\boldsymbol{p})\rangle$$
$$= ig\epsilon_{\nu\rho\tau\sigma}\epsilon^{*\rho}(p+p')^\sigma(p-p')^\tau, \qquad (26)$$

The $P_1 \to V_2$ form factors can be related to the $P_1 \to P_2$ form factors and thereby to the universal function $\xi(t)$. The results, which can be most easily derived in the frame where $\boldsymbol{p}'=\boldsymbol{0}$, are

$$a_+(t_{21}) = -a_-(t_{21}) = -g(t_{21})$$
$$= -\left(\frac{\alpha_s(m_1)}{\alpha_s(m_2)}\right)^{-6/(33-2N_f)}$$
$$\times\frac{\xi((m_1/m_2)[t_{21}-t_m])}{\sqrt{4m_1 m_2}}, \qquad (27)$$

$$f(t_{21}) = \left(\frac{\alpha_s(m_1)}{\alpha_s(m_2)}\right)^{-6/(33-2N_f)}$$
$$\times[4m_1 m_2 + (t_m - t_{21})]$$
$$\times\frac{\xi((m_1/m_2)[t_{21}-t_m])}{\sqrt{4m_1 m_2}} \qquad (28)$$

We previously found [1] only the result

$$f(t_m) = \sqrt{4m_1 m_2}\left(\frac{\alpha_s(m_1)}{\alpha_s(m_2)}\right)^{-6/(33-2N_f)}$$

Table 1
Zero recoil values of $\bar{B}\to D$, $D^*$ form factors of ref. [5] compared to the results of this paper.

| | $f_+$ | $f/\sqrt{4m_B m_{D^*}}$ | $\sqrt{4m_B m_{D^*}}\, g$ | $\sqrt{4m_B m_{D^*}}\, a_+$ |
|---|---|---|---|---|
| ref. [5] (corrected [a]) | 1.15 | 1.06 | 1.08 | −0.97 |
| heavy quark limit | 1.16 | 1.01 | 1.01 | −1.01 |

[a] We have multiplied the results of ref. [5] by $C'_{cb}\simeq 1.02$ to make this comparison meaningful.

That $a_+ + a_- = 0$ in the heavy quark limit is transparent in the $\boldsymbol{p}' = \mathbf{0}$ frame: this combination of form factors is proportional to the amplitude for $A^{21}_\nu$ acting on $\langle V_2(\mathbf{0}, \boldsymbol{\epsilon})|$ to produce $P_1$ in a D-wave, but the axial current acting at the origin can only produce $Q_1$ in an S- or P-wave.

These results are all consistent with those of refs. [5,6] in the appropriate limit, apart from the short-distance factor $C_{21}$ which these authors neglected, and given that they worked to leading order in $\boldsymbol{v}$ and $\boldsymbol{v}'$. However, unlike this previous work based on the nonrelativistic quark model, the results of this paper are systematic consequences fo QCD, corrections to them being suppressed by $\alpha_s(m_Q)/\pi$ and $\Lambda_{QCD}/m_Q$. The results of ref. [5] may now be used to provide estimates for the departures of real systems from the $m_Q\to\infty$ limit of this paper. In table 1 we compare our model-independent, absolutely normalized predictions for $b\to c$ form factors at $t=t_m$ with the quark model results of ref. [5]. The deviations are all less than about 5%, which strongly suggests that the large-$m_Q$ limit is applicable to these decays. It should also be noted that present experimental evidence on the rates of $\bar{B}\to De\bar{\nu}_e$ and $\bar{B}\to D^*e\bar{\nu}_e$, and on the $D^*$ polarization in the latter decay are consistent with the predictions of this limit.

We have so far not directly addressed the question of the range of validity in $(t_m - t)$ of our results. This range is determined by two approximations. Since the Fock states of the heavy mesons are only independent of the heavy quark mass on length scales greater than $m_Q^{-1}$, the universality of the function $\xi$ will break down when these states are probed at distances smaller than this. This restricts the range of validity of our results to

$$\frac{t_m - t}{2m_1 m_2} \ll \left(\frac{m_2}{\Lambda_{QCD}}\right)^2 , \tag{29}$$

and gives rise to $\Lambda_{QCD}/m_Q$ corrections to our results at $t=t_m$. Even if valid up to such $t_m - t$ as an evaluation of the effects of the lowest dimension operator, our results may be *inapplicable* if higher order terms in eq. (3) become important at lower values of $t_m - t$. This could happen since hard transverse gluon exchange effects suppressed by $(\Lambda_{QCD}/m_Q)\ \alpha_s(m_Q)/\pi$ produce an asymptotically [#2] dominant power-law tail to the form factors [8] not present in $\xi(t)$. However, since $t_m - t = (m_1 m_2/m_\ell^2) Q_\ell^2$, where $m_\ell$ and $Q_\ell$ are the effective mass and momentum transfer of the light degrees of freedom, and since asymptopia is surely above $Q_\ell^2 = 1$ GeV $^2$, we can confidently apply our results over the full kinematic range explored by the $b\to c$ transition.

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[#2] See ref. [7] for a discussion of the momentum transfers required to reach asymptopia.

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