Pitman-closeness as a measure to evaluate the quality of forecasts

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Summary: We consider Pitman-closeness to evaluate the performance of forecasting methods. Optimal weights for the combination of forecasts are calculated with respect to this criterion. We show that these weights depend on the assumption on the distribution of the forcast errors. In the normal case they are identical with the optimal weights with respect to the MSE-criterion.

Key words: Pitman-closeness, evaluation of forcasting methods, combination of forecasts.

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1. Introduction

In forecasting theory there are many measures to evaluate the performance of different forecasting methods, e. g. *mean square error* (MSE), *mean absolute deviation* (MAD), *turning points* or *Theil's U*. Especially the MSE is one of the most popular measures in practical and theoretical investigations. Regarding the combination of forecasts, optimal weights are proposed relative to the MSE-criterion, e. g. Bates and Granger (1969). These weights depend on the variance and the structure of the covariances between the forecast errors.

In this article, another quality measure which is known as Pitman-closeness will be analysed. Pitman's measure of closeness (Pitman 1937) was introduced in 1937 and has often been discussed since the beginning of the 80s (in particular Keating et al. 1993). Pitman-closeness is defined to compare two estimators. We calculate the probability that the absolute value of the difference between the first estimator and the fixed, unknown parameter of interest, is smaller than that of the second estimator. Thus, the critical point of a subjective choice of the loss function like in the MSE-criterion is not that much important.

We also considered the application of Pitman-closeness in the context of combination of forecasts. A problem is that an assumption on the distribution of the forecast errors is needed, but this is not very popular. Compared to the MSE-criterion this criterion is more restrictive and the ranking of different forecasting methods depends on this assumption. When the errors are normally distributed, the Pitman-closeness is equivalent to the MSE-criterion. Thus, the combination with the smallest error variance is also the Pitman-closest-combination. For another model with rectangular distributed errors it will be shown that the optimal weights with respect to the MSE-criterion are no more optimal if we consider Pitman-closeness.

2. Pitman-closeness to evaluate forecasting methods

We will start with the definition of Pitman-closeness for the comparison of two estimators.

Definition 2.1. Let $\hat{\theta}_1$ and $\hat{\theta}_2$ be two estimators for an unknown parameter $\theta \in \Theta$ of an underlying distribution, where Θ denotes the parameter space. Then $\hat{\theta}_1$ is Pitman-closer (with respect to θ) than $\hat{\theta}_2$ if and only if

$$P\left(\left|\hat{\theta}_1 - \theta\right| < \left|\hat{\theta}_2 - \theta\right|\right) \geq 0.5 \quad \forall \theta \in \Theta$$

with strict inequality for at least one $\theta \in \Theta$.

To generalize Definition 2.1. we can use a loss function $L(\cdot,\theta)$ without any restriction. Definition 2.1. is a special case of symmetric loss functions $L(x,\theta)$ which are strictly decreasing for $x<0$ and strictly increasing for $x>0$.

The above definition characterizes a best estimator in a class S of estimators for a parameter $\theta \in \Theta$ as follows.

Definition 2.2. An estimator $\hat{\theta}^* \in S$ is called Pitman-closest-estimator in S with respect to the parameter $\theta \in \Theta$ if and only if $\forall \hat{\theta} \in S,\ \hat{\theta} \neq \hat{\theta}^*$

$$P\left(\left|\hat{\theta}^* - \theta\right| < \left|\hat{\theta} - \theta\right|\right) \geq 0.5 \quad \forall \theta \in \Theta$$

with strict inequality for at least one $\theta \in \Theta$.

In the following text the idea of Pitman-closeness will be applied to the combination of forecasts. Note that the parameter to be estimated in estimation theory is an unknown but fixed value, whereas in forecasting theory the value to be forecasted (estimated) is random. In the latter case we are interested in the problem if a certain forecast is Pitman-closer than another.

Let

F_{it}, i=1,2 be two forecasts for Y_t (t: time) with $Y_t = F_{it} + e_{it}$

e_{it}, i=1,2 be the corresponding forecast errors.

Definition 2.3. Let Y_t be the variable to be forecasted and F_{1t} und F_{2t} two forecasts. Then the forecast F_{1t} is called Pitman-closer (with respect to Y_t) than F_{2t} if and only if

$$P\left(\left|Y_t - F_{1t}\right| < \left|Y_t - F_{2t}\right|\right) > 0.5,$$

which is equivalent to

$$P\left(\left|e_{1t}\right| < \left|e_{2t}\right|\right) > 0.5 \ .$$

Thus we say F_{1t} is Pitman-closer than F_{2t} if it is more probable that the forecast F_{1t} has a smaller absolute forecast error than F_{2t}. This seems to be a plausible criterion for the comparison of forecasts. But how can this be optimized?

Definition 2.4. Let F^* be a forecast in a class $\Im$ of forecasts for a variable Y_t. Then F^* is called Pitman-closest-forecast in $\Im$ if and only if

$$P\left(\left|Y_t - F^*\right| < \left|Y_t - F\right|\right) > 0.5 \quad \forall F \in \Im, F \neq F^* \ ,$$

which is the same as

$$P\left(\left|e_{F^*}\right| < \left|e_F\right|\right) > 0.5 \quad \forall F \in \Im, F \neq F^* \ ,$$

where $e_{F^*} := Y_t - F^*$ and $e_F := Y_t - F$.

According to Definition 2.4., we have to make assumptions concerning the distributions of the errors in order to calculate the Pitman-closeness probability but this is very restrictive. This is a disadvantage to the MSE-criterion, which requires only the assumption of errors with zero mean and positive definite covariance matrix.

The next section considers the combination of forecasts where a new forecast is calculated as a weighted sum of given forecasts. The comparison of two individual forecasts can be achieved by assigning one for the weight of the special forecast and zero for the weights of all other forecasts.

3. Pitman-closeness and combined forecasts

Let $\Im^* = \{F_{1t},...,F_{nt}\}$ be a class of unbiased forecasts for Y_t, that is,

$E(F_{it})=E(Y_t)$, $i=1,...,n$,

$e_{it} = Y_t - F_{it}$, $e = (e_{1t},...,e_{nt})'$, $E(e)=\mathbf{0}$.

Further let $W = \{w: \ w = (w_1,...,w_n)', \sum_{i=1}^{n} w_i = 1\}$ be the set of possible vectors of weights, so that a combined forecast is also unbiased and $\Im = \{w'V, \ w \in W \text{ and } V = (F_{1t},...,F_{nt})'\}$ denotes the set of all these combinations.

3.1. Normally distributed errors

Assumption: $e \sim N_n(0,\Sigma)$

Theorem 3.1. Let $F_a = a'V$ and $F_b = b'V$, $F_a, F_b \in \Im$, a≠b. Then F_a is Pitman-closer (with respect to Y_t) than F_b if and only if

$$\sigma_a^2 < \sigma_b^2 \ ,$$

where $\sigma_a^2 := \mathrm{Var}(a'e) = a'\Sigma a$ and $\sigma_b^2 := \mathrm{Var}(b'e) = b'\Sigma b$.

Proof: Fountain (1991), Fountain and Keating (1994) and Fountain et al. (1993) considered Pitman-closeness and linear estimators. Hence, most of the reasonings in the proof can be used in the sense of linear combinations of forecasts.

F_a is Pitman-closer (with respect to Y_t) than F_b means

$$P(|Y_t - F_a| < |Y_t - F_b|) > 0.5$$

$$\Leftrightarrow \quad P(|(a_1 + ... + a_n)Y_t - a_1F_{1t} - ... - a_nF_{nt}| < |(b_1 + ... + b_n)Y_t - b_1F_{1t} - ... - b_nF_{nt}|) > 0.5$$

$$\Leftrightarrow \quad P(|a_1e_{1t} + ... + a_ne_{nt}| < |b_1e_{1t} + ... + b_ne_{nt}|) > 0.5$$

$$\Leftrightarrow \quad P(|a'e| < |b'e|) > 0.5 \quad \Leftrightarrow \quad P\left((a'e)^2 < (b'e)^2\right) > 0.5$$

$$\Leftrightarrow \quad P\left(e'aa'e < e'bb'e\right) > 0.5 \quad \Leftrightarrow \quad P\left(e'\left(aa'-bb'\right)e < 0\right) > 0.5$$

$$\Leftrightarrow \quad P\left(e'\Sigma^{-0.5}\Sigma^{0.5}\left(aa'-bb'\right)\Sigma^{0.5}\Sigma^{-0.5}e\right) > 0.5 \qquad (1)$$

Let

$C := \Sigma^{0.5}\left(aa'-bb'\right)\Sigma^{0.5}$. Then using spectral decomposition we get

$C = \lambda_1 v_1 v_1' + \lambda_2 v_2 v_2'$, where $\lambda_1, \lambda_2, v_1, v_2$ are eigenvalues and eigenvectors of C and

$\lambda_{1,2} = \dfrac{\|c\|^2 - \|d\|^2 \pm \|c+d\|\|c-d\|}{2}$, $(\lambda_1 > 0, \lambda_2 < 0)$, where $c := \Sigma^{0.5}a$ and $d := \Sigma^{0.5}b$. This result is described in a short form in Fountain and Keating (1994). For a detailed calculation of the eigenvalues and eigenvectors see Fountain (1991), where the matrix C=aa'-bb' (here C=cc'-dd') is used, but this leads to the same conclusions. Then,

equation (1) holds $\quad \Leftrightarrow \quad P\left(\lambda_1 e'\Sigma^{-0.5}v_1 v_1'\Sigma^{-0.5}e + \lambda_2 e'\Sigma^{-0.5}v_2 v_2'\Sigma^{-0.5}e < 0\right) > 0.5$.

With $X_1 := e'\Sigma^{-0.5}v_1 = v_1'\Sigma^{-0.5}e$ and $X_2 := e'\Sigma^{-0.5}v_2 = v_2'\Sigma^{-0.5}e$ this is equivalent to

$$P(\lambda_1 X_1^2 + \lambda_2 X_2^2 < 0) > 0.5 \;\Leftrightarrow\; P\left(\frac{X_1^2}{X_2^2} < -\frac{\lambda_2}{\lambda_1}\right) > 0.5 \;\Leftrightarrow\; P\left(-\sqrt{-\frac{\lambda_2}{\lambda_1}} < \frac{X_1}{X_2} < \sqrt{-\frac{\lambda_2}{\lambda_1}}\right) > 0.5. \quad (2)$$

From the definitions of X_1 and X_2 the following holds.

$E(X_i) = v_i'\Sigma^{-0.5}E(e) = 0$, $\quad Var(X_i) = v_i'\Sigma^{-0.5}\Sigma\Sigma^{-0.5}v_i = v_i'v_i = 1$ and

$Cov(X_1, X_2) = v_1'\Sigma^{-0.5}\Sigma\Sigma^{-0.5}v_2 = v_1'v_2 = 0$.

The random variables X_1 and X_2 are N(0,1) distributed with zero covariance.

This implies that $\dfrac{X_1}{X_2} \sim \text{Cauchy}(0,1)$.

Equation (2) can be written as

$$\left(\frac{1}{2} + \frac{1}{\pi}\arctan\sqrt{-\frac{\lambda_2}{\lambda_1}}\right) - \left(\frac{1}{2} + \arctan\left(-\sqrt{-\frac{\lambda_2}{\lambda_1}}\right)\right) > 0.5$$

$$\Leftrightarrow \quad \frac{2}{\pi}\arctan\sqrt{-\frac{\lambda_2}{\lambda_1}} > 0.5 \;\Leftrightarrow\; \arctan\sqrt{-\frac{\lambda_2}{\lambda_1}} > \frac{\pi}{4}$$

$$\Leftrightarrow \quad -\frac{\lambda_2}{\lambda_1} > 1 \Leftrightarrow -\|c\|^2 + \|d\|^2 + \|c+d\|\|c-d\| > \|c\|^2 - \|d\|^2 + \|c+d\|\|c-d\|$$

$$\Leftrightarrow \quad \|d\|^2 > \|c\|^2 \;\Leftrightarrow\; b'\Sigma^{0.5}\Sigma^{0.5}b > a'\Sigma^{0.5}\Sigma^{0.5}a \;\Leftrightarrow\; \sigma_b^2 > \sigma_a^2$$

Hence the combination with vector “a” is better than that with vector “b” if and only if the variance of its forecast error is smaller. Therefore, an optimal forecast in the class $\Im$ is given by a vector “a” of weights which minimizes σ_a^2.

Now we will present a general method for constructing an optimal combination.

Theorem 3.2. The optimal combination of forecasts in the class $\Im$, called Pitman-closest-forecast in $\Im$, is given by

$$F^* = a^{*\prime} V$$

where

$$a^* = \frac{\Sigma^{-1}\mathbf{1}}{\mathbf{1}'\Sigma^{-1}\mathbf{1}},$$

and $\mathbf{1}=(1,...,1)'$ is a n dimensional vector.

This theorem holds without a proof because the minimization of the combined error variance is similar to the minimization problem of the MSE-criterion, which is comprehensively discussed in the literature.

The following example presents some results for the comparison of forecasts with normally distributed errors. The calculations are made for a set $\Im^*$ of two individual forecasts.

Example: In addition to the covariance structure of the next chapter we are using covariance matrices of the following form:

$$\Sigma = \begin{pmatrix} \frac{z^2}{3} & \frac{1}{9}\alpha z \\ \frac{1}{9}\alpha z & \frac{1}{3} \end{pmatrix},$$

where $\alpha \in [-1,1]$ and z>0.

The Pitman-closest-combination (which is equal to the optimal MSE-combination) is compared for different α‘s and z‘s with both of the individual forecasts and their arithmetic mean. The Pitman-closeness probabilities, showing that the Pitman-closest-combination performs better than a given forecast are summarized in the table below:

Table 1:

	$P(\lvert Y_t - F_{a*}\rvert < \lvert Y_t - F_b\rvert)$										
	arithmetic mean (b=(0.5, 0.5)')			first individual forecast (b=(1, 0)')				second individual forecast (b=(0, 1)')			
α	z=2	z=3	z=4	z=1	z=2	z=3	z=4	z=1	z=2	z=3	z=4
−1	0.6205	0.6959	0.7491	0.6959	0.7837	0.8361	0.8694	0.6959	0.6325	0.6082	0.5955
−0.5	0.6157	0.6892	0.7419	0.6700	0.765	0.8227	0.8593	0.6700	0.6038	0.5790	0.5883
0	0.6142	0.6871	0.7397	0.6476	0.75	0.8128	0.8524	0.6476	0.578	0.5526	0.5396
0.5	0.6157	0.6892	0.7419	0.6273	0.7384	0.8065	0.8483	0.6273	0.5533	0.5268	0.5134
1	0.6205	0.6959	0.7491	0.6082	0.7304	0.8041	0.8488	0.6082	0.5281	****	0.5141

For the arithmetic mean, when z=1 and for the second individual forecast, when α=1 and z=3, the probabilities are missing. In these cases the optimal combinations result in the special individual and the arithmetic mean forecast and thus, since they are identical with the competing forecasts, no comparison is made. The table documents the performance of Pitman-closest-combinations. An interesting contrast is that larger probabilities are attained under the comparison of the optimal combination with forecasts that have larger variance.

3.2. Rectangular distributed errors

If the errors are not normally distributed, the calculation of the Pitman-closeness probability is very difficult. Thus a simulation study for rectangular distributed errors is performed. At first, the case of the combination of two forecasts will be considered.

Then $e = (e_{1t}, e_{2t})'$, where $e_{1t} \sim R_{[-gk,gk]}$, $e_{2t} \sim R_{[-k,k]}$, $g,k>0$.

Remark: For the above case we only need to consider the distributions in a special proportion. If $X_i \sim R_{[-z_i,z_i]}$, then $Y_i = dX_i \sim R_{[-dz_i,dz_i]}$, i=1,2 , d>0. Using this the Pitman-closeness probability is given by $P(\lvert a_1Y_1 + a_2Y_2\rvert < \lvert b_1Y_1 + b_2Y_2\rvert)$ where a_1, a_2, b_1, b_2 are constants. This is equivalent to

$$P(\lvert a_1dX_1 + a_2dX_2\rvert < \lvert b_1dX_1 + b_2dX_2\rvert) = P(\lvert a_1X_1 + a_2X_2\rvert < \lvert b_1X_1 + b_2X_2\rvert).$$

The Pitman-closeness probabilty for the comparison of two different combinations of Y_1 and Y_2 is the same as the probability for the comparison of the same combination of X_1 and X_2. This result obviously also holds for variables that do not have the rectangular distribution.

For the joint distribution of the errors a Farlie-Gumbel-Morgenstern-distribution

$h_{12}(x_1,x_2):=h_1(x_1)\, h_2(x_2)[1+\alpha(1-H_1(x_1))(1-H_2(x_2))]$

is proposed, where h_1 and h_2 are the density functions and H_1 and H_2 are the distribution functions of the rectangular distributed errors, and $\alpha\in[-1,1]$ measures the dependence between the errors.

Now consider the variances and covariances of the errors. Since the errors are rectangular distributed we have

$$\mathrm{Var}(e_{1t})=\frac{g^2}{3}k^2\ ,\ \mathrm{Var}(e_{2t})=\frac{1}{3}k^2\ .$$

The covariance of two random variables which are common FGM-distributed can easily be calculated as it is described in Schucany et al. (1978), using

$$\mathrm{Cov}(e_{1t},e_{2t})=\alpha\delta_1\delta_2\,,$$

where $\delta_j:=\int x[2H_j(x)-1]h_j(x)dx \quad ,j=1,2.$

Here it holds that

$$\delta_1=\frac{1}{3}gk \text{ and } \delta_2=\frac{1}{3}k\,,$$

which implies $\mathrm{Cov}(e_{1t},e_{2t})=\frac{1}{9}\alpha g k^2$, $\mathrm{Corr}(e_{1t},e_{2t})=\frac{1}{3}\alpha$.

To get a well defined density function we require $\alpha\in[-1,1]$. We can see that the absolute value of the correlation coefficient is smaller than 1/3. Consequently, the FGM-model is valid only for “small” correlations between the forecast errors; nevertheless we get some interesting conclusions.

3.2.1. Description of the simulation study

- The simulation was performed for different proportions between rectangular distributions (e.g. g=1, 2).
- For given proportions the covariance matrices of the forecast errors and through this the optimal weights with respect to the MSE-criterion were calculated (for α=–1, –0.7. –0.5, –0.3, 0, 0.3, 0.5, 0.7, 1).
- Then for weights a_1 and $a_2=1-a_1$ the Pitman-closeness probability $P\left(|a_1e_{1t}+a_2e_{2t}|<|b_1e_{1t}+b_2e_{2t}|\right)$ is calculated, where b_1 and b_2 are defined as fixed optimal MSE-weights and a_1 was varied.

- Because the optimal combination of forecasts with respect to the MSE can be outperformed under Pitman-closeness we searched the optimal weights with respect to Pitman-closeness for different α's.

3.2.2. Interpretation of the simulation

First, the simulation results for the case g=2 will be adressed. The results for other choices of g are similar. The following three figures show the Pitman-closeness probabilities plotted against the weights $a_1 \in [-0.1, 0.3]$ for the forecast with larger variance.

Figure A shows the probabilities for α=0.7 and the optimal MSE-weight b_1=0.13115. In Figures B and C the values for α=–0.7 and α=0 with the optimal MSE-weights b_1=0.24719 and b_1=0.2 are given, respectively.

Figure A: α=0.7

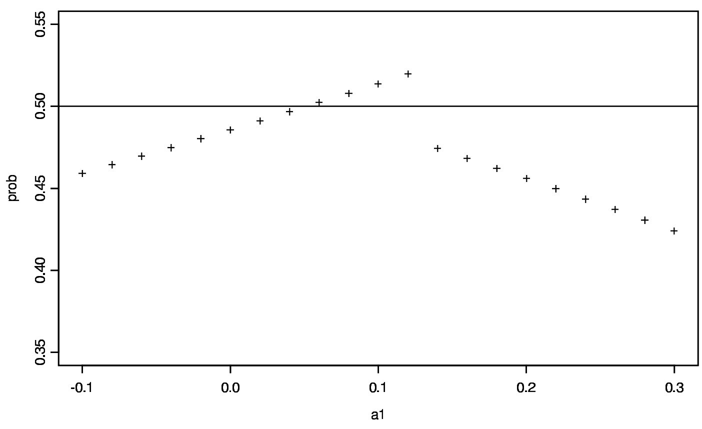
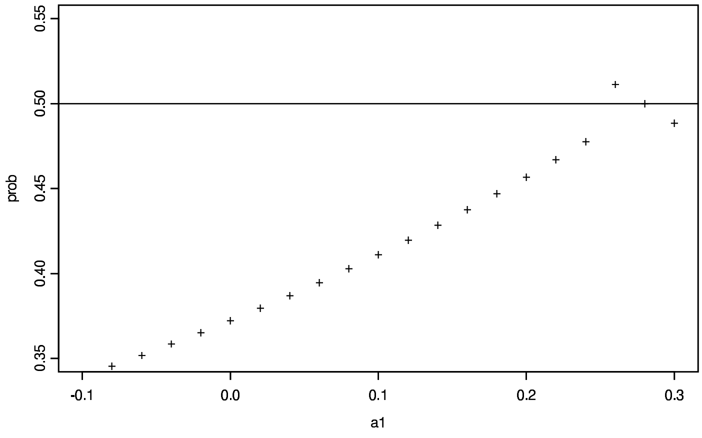
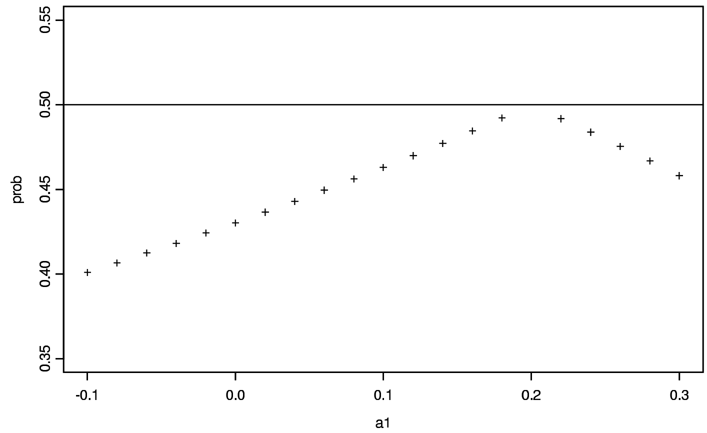


Figure B: α=–0.7



<u>Figure C:</u> α=0



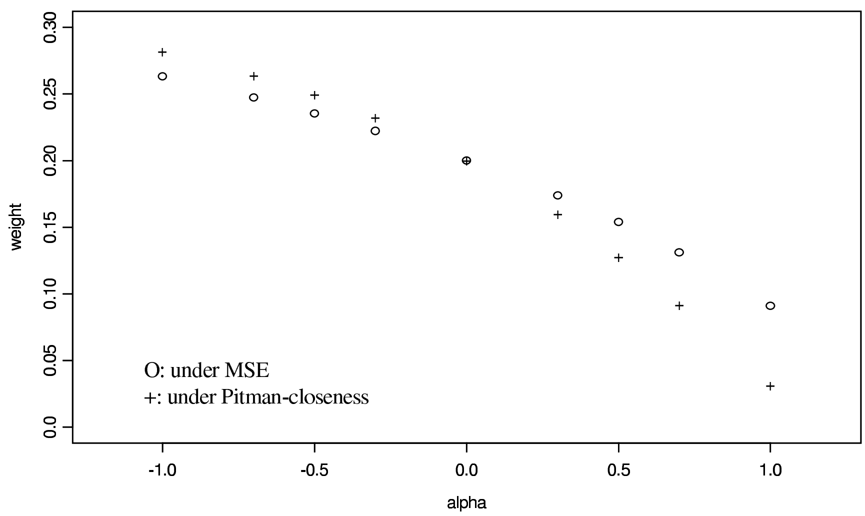
For α=0.7 the optimal MSE-weight b_1 can be outperformed in the interval $[x_\alpha, b_1)$. The same conclusions can be made for other positive values of α. Therefore, we have to look for the optimal Pitman-closeness weight in this interval.

For α=–0.7 the optimal MSE-weight is outperformed in an interval $(b_1, y_\alpha]$, which happens whenever α is negative.

For α=0 the optimal MSE-weight is also the optimal Pitman-closeness weight.

The following figure shows the optimal weights for the forecast with larger variance of the two evaluation criteria and verifies the results given above.

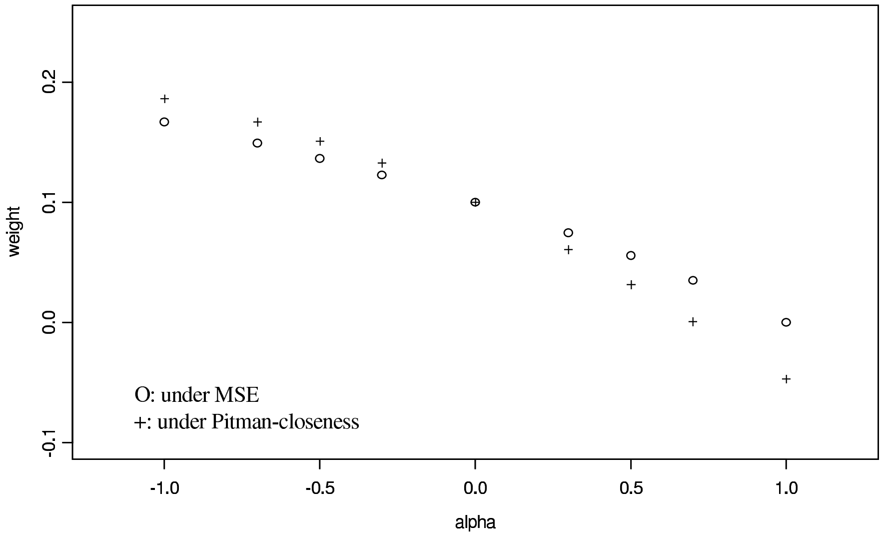
<u>Figure D:</u> Optimal weights (g=2)



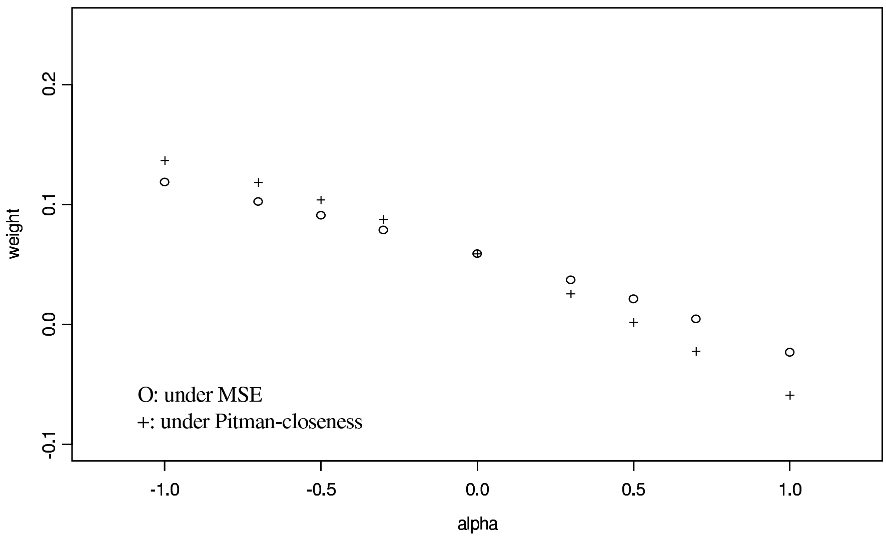
For negative values of α the optimal weights of Pitman-closeness exceed the optimal weights of the MSE-criterion, whereas for positive α's the optimal weights of the MSE-criterion exceed the optimal weights of Pitman-closeness. If the absolute value of α gets larger, then the discrepancy between the optimal weights of the criteria increases.

For other g>1 we can draw the same conclusions as before. The comparison of the optimal weights of the two criteria for other g values is shown in the following figures. The exact values are presented in Table 2 (Appendix).

<u>Figure E:</u> Optimal weights (g=3)



<u>Figure F:</u> Optimal weights (g=4)



It is obvious that the structure in the graphs remains the same for different values of g. For g=1, which implies that the errors are identically distributed, the two forecasts will be weighted with 0.5 independent of α. Finally, we can say that in the case of positively correlated errors the Pitman-closeness weight for the forecast with larger variance is smaller and in the case of negatively correlated errors larger than the MSE-weight for an optimal combination.

3.2.3. Combination of three forecasts with rectangular distributed errors

Similar to the study of the combination of two forecasts, some results for the combination of three forecasts are calculated. Using the same notation as above the Farlie-Gumbel-Morgenstern-density is given by

$$\begin{aligned} h_{123}(x_1,x_2,x_3) := h_1(x_1)\, h_2(x_2)\, h_3(x_3)[1 &+ \alpha_{12}(1-H_1(x_1))(1-H_2(x_2)) + \alpha_{13}(1-H_1(x_1))(1-H_3(x_3)) \\ &+ \alpha_{23}(1-H_2(x_2))(1-H_3(x_3)) + \alpha_{123}(1-H_1(x_1))(1-H_2(x_2))(1-H_3(x_3))] \ . \end{aligned}$$

H_3 and h_3 denote distribution and density function of the third error respectively. The α's must be chosen in such a way that h_{123} is a well defined density function. Integrating out one of the variables, results in a two dimensional FGM-distribution. As above the α_{ij} (i,j=1,2,3, i<j) explain the covariances between the individual forecast errors.

Since we have more parameters now we will only do some calculations for two parameter combinations. As before, the optimal MSE-weights are fixed and compared under Pitman-closeness with varying weights. The following two figures show the results of the simulation performed. The symbols a_1 and a_2 represent the weights such that $a_3=1-a_1-a_2$, and prob stands for the Pitman-closeness probability.

Figure G: $\alpha_{123}=0$, $\alpha_{12}=\alpha_{13}=\alpha_{23}=-0.3$

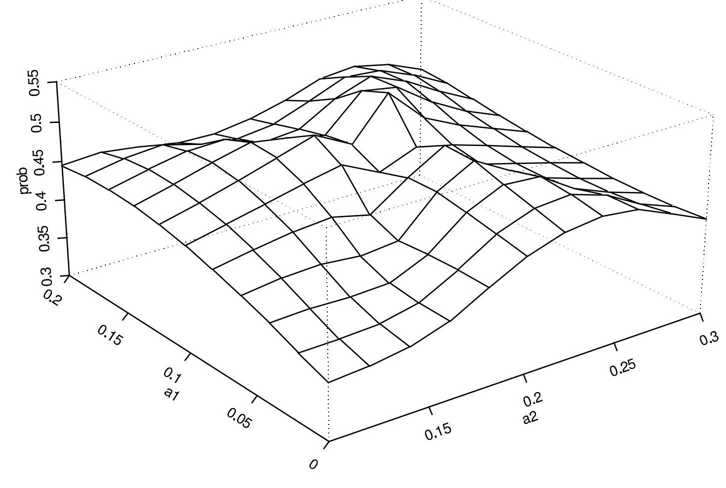
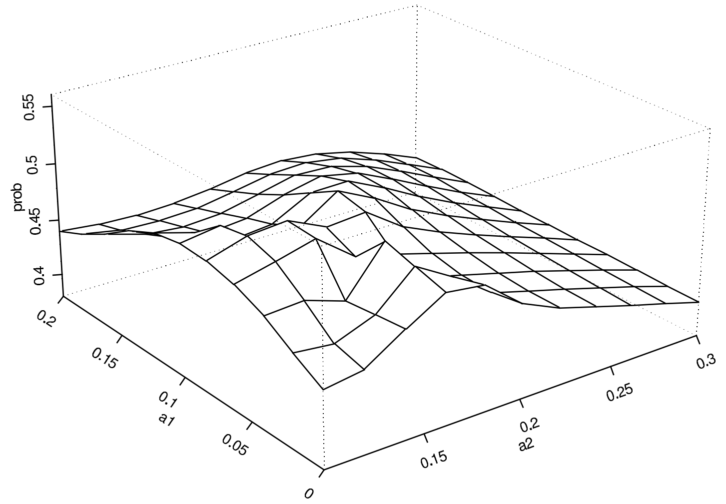


Figure H: α_{123}=0, α_{12}=α_{13}=α_{23}=0.3



In the first example the optimal combination under the MSE-criterion with weights (0.1053, 0.2947, 0.69) is outperformed by combinations with weights (0.08, 0.22, 0.7), (0.12, 0.22, 0.66), (0.12, 0.24, 0.64), (0.14, 0.24, 0.62) and (0.14, 0.26, 0.6), and in the second example the optimal weights under the MSE-criterion (0.0557, 0.1606, 0.7737) are outperformed by (0.02, 0.18, 0.8), (0.02, 0.2, 0.78), (0.04, 0.16, 0.8), (0.04, 0.18, 0.78), (0.06, 0.14, 0.8) and (0.08, 0.2, 0.72).

The graphs include only the scatter points calculated using a step width of 0.02 for the weights. The scatter lines help to get a better view. Some jump discontinuities like in the figures in Section 3.2.2. may appear.

4. Conluding remarks

First we can say that the idea of the Pitman-closeness criterion seems to be more plausible than that of the MSE-criterion but its use has some difficulties. The application of the Pitman-closeness criterion for the evaluation of forecasts, especially for the combination of forecasts, is associated with an assumption of the error distribution. This appears to be a restriction. If the errors are normally distributed, a theoretical derivation of the optimal weights for a combined forecast is relatively easy. In this case they are equivalent to the optimal MSE-weights. For other distributions we have to work with computer simulations. This could be difficult for distributions which are defined in an interval including infinity and for combinations of more than three methods. It is shown that the optimal weights, in contrast to the MSE-criterion, depend on the error distribution. For the combination of two or three forecasts with rectangular distributed errors there is a difference between the two criteria and therefore the MSE optimality does not hold. This could also happen using other distributions.

Furthermore, the rectangular distribution does not include as many restrictions as the normal distribution whereas the model of the Farlie-Gumbel-Morgenstern-distribution restricts the study to correlation coefficients in the interval [−1/3, 1/3].

5. References

[1] **Bates, J. M., Granger, C. W. J.** (1969): The Combination of Forecasts, *Operational Research Quarterly 20, 451-468.*

[2] **Fountain, R. L.** (1991): Pitman Closeness Comparison of Linear Estimators: A Canonical Form, *Communications in Staistics Theory and Methods A20, 3535-3550.*

[3] **Fountain, R. L., Keating, J.P.** (1994): The Pitman comparison of unbiased linear estimators, *Statistics & Probability Letters 19, 131-136.*

[4] **Keating, J. P., Mason, R.L., Sen, P.K.** (1993): Pitman's Measure of Closeness, *Philadelphia: SIAM.*

[5] **Pitman, E. J. G.** (1937): The closest estimates of statistical parameters, *Proceedings of the Cambridge Philosophical Society 33, 212-222.*

[6] **Schucany, W. R., Parr, W. C., Boyer, J. E.** (1978): Correlation structure in Farlie-Gumbel-Morgenstern distributions, *Biometrika 65, 650-652.*

6. Appendix: Table 2: Optimal weights of the combination of two forecasts with normally distributed errors

g	1		2		3		4	
α	"MSE"	"Pitman"	"MSE"	"Pitman"	"MSE"	"Pitman"	"MSE"	"Pitman"
−1	0.5	0.5	0.26316	0.28158	0.16667	0.18642	0.11864	0.13680
−0.7	0.5	0.5	0.24719	0.26374	0.14912	0.16683	0.10247	0.11862
−0.5	0.5	0.5	0.23529	0.24917	0.13636	0.15106	0.09091	0.10417
−0.3	0.5	0.5	0.22222	0.23192	0.12264	0.13274	0.07865	0.08764
0	0.5	0.5	0.2	0.2	0.1	0.1	0.05882	0.05882
0.3	0.5	0.5	0.17391	0.15951	0.07447	0.06089	0.03704	0.02570
0.5	0.5	0.5	0.15385	0.12736	0.05556	0.03178	0.02128	0.00195
0.7	0.5	0.5	0.13115	0.09124	0.03488	0.00083	0.00441	-0.02242
1	0.5	0.5	0.09091	0.03079	0	-0.04689	-0.02326	-0.05878