Random walks on Galton-Watson trees with random conductances

Dissertation
zur Erlangung des akademischen Grades eines
Doktors der Naturwissenschaften
(Dr. rer. nat.)

Der Fakultät für Mathematik
der Technischen Universität Dortmund
vorgelegt von

TABEA GLATZEL

im Februar 2025

Dissertation

Random walks on Galton-Watson trees with random conductances

Fakultät für Mathematik

Technische Universität Dortmund

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Tag der mündlichen Prüfung: 22.05.2025

Acknowledgements

I could not have finished this thesis without the help of several people in my life. At this point, I would like to express my heartfelt thanks to them for their support.

First and foremost, I want to thank my supervisor Prof. Dr. Jan Nagel for giving me the opportunity to work on this interesting research topic and guiding me through my PhD journey. His expertise and knowledge as well as his kindness and helpfulness were invaluable. I am immensely grateful for his encouragement and outstanding feedback and his belief in my academic potential at all times. In particular, I thank him for always being there when I needed his support and for always taking the time for discussions about anything I was unsure about.

My sincere thanks also go to Prof. Dr. Nina Gantert who warmly welcomed me in Munich. It was a pleasure to work with her and to learn from her. I am very grateful for her helpful suggestions and comments and that she agreed to be the second reviewer of this thesis.

I would also like to thank Prof. Dr. Jeannette Woerner and Prof. Dr. Michael Voit, who taught and supported me when I was a student and aroused my interest in probability theory.

My warmest thanks go to my colleagues from Lehrstuhl IV and the entire sixth floor for creating a great working atmosphere. I would especially like to thank Sarah and Helene, who are not only colleagues but now also dear friends. Thank you for your moral support and encouragement and for always being there to talk. Many thanks for helping me in any way possible.

Above all, I am deeply thankful to my family and friends. Mum and dad, Jonas and Waheed, thank you for your unwavering love and support and for always believing in me. Kathi and Katrin, thank you for your encouragement and for always being there for me. I could not have done all this without you. I love you.

To all those who contributed in any way to this thesis, thank you so much.

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Chapter 1

Introduction

Random walks in random environments are suitable to model the movement of particles in inhomogeneous environments. Over the past fifty years, their asymptotic and potentially anomalous behaviour has been intensively studied, creating an interesting and still active research area. Random walks in a random environment first appeared in biophysical literature as a model for DNA replication, introduced by [Che67]. Their mathematical investigation began in the 1970s with the work of [Sol75], who studied the asymptotic behaviour of an one-dimensional random walk in a random environment on the integers. The behaviour of this random process has been extensively studied and is now well understood. Since then, several extensions and generalizations of this model have been considered, such as higher dimensions, other graphs or time-continuous random walks.

A prominent example of random walks in random environments are random walks on Galton-Watson trees, which are often seen as a mean-field model for random walks on high-dimensional percolation clusters, since close to criticality the environment looks locally tree-like. A Galton-Watson tree is the family tree generated by a Galton-Watson branching process. Starting from a single vertex, each vertex in the tree has a random number of descendants according to the offspring distribution and independently for different vertices. If the offspring mean is strictly larger than one, there is a positive probability that the branching process survives. The corresponding Galton-Watson tree is then called supercritical. By conditioning on the survival of the branching process this yields an infinite tree. An example of a realization of the first generations of a Galton-Watson tree is shown in Figure 1.1.

The most obvious approach to define a random walk on the tree is to consider a simple random walk. Given a Galton-Watson tree T, the simple random walk $(X_n)_{n\geq 0}$ on T is a time-discrete Markov chain which starts at the root and at each time step it moves to one neighbour uniformly chosen among all neighbours of its current state. The asymptotic behaviour of the simple random walk on infinite supercritical Galton-Watson trees has been well-studied since the 1990s.

The first basic question in studying the long-term behaviour is whether the random walk is transient or recurrent. The decisive factor here is the probability of returning to

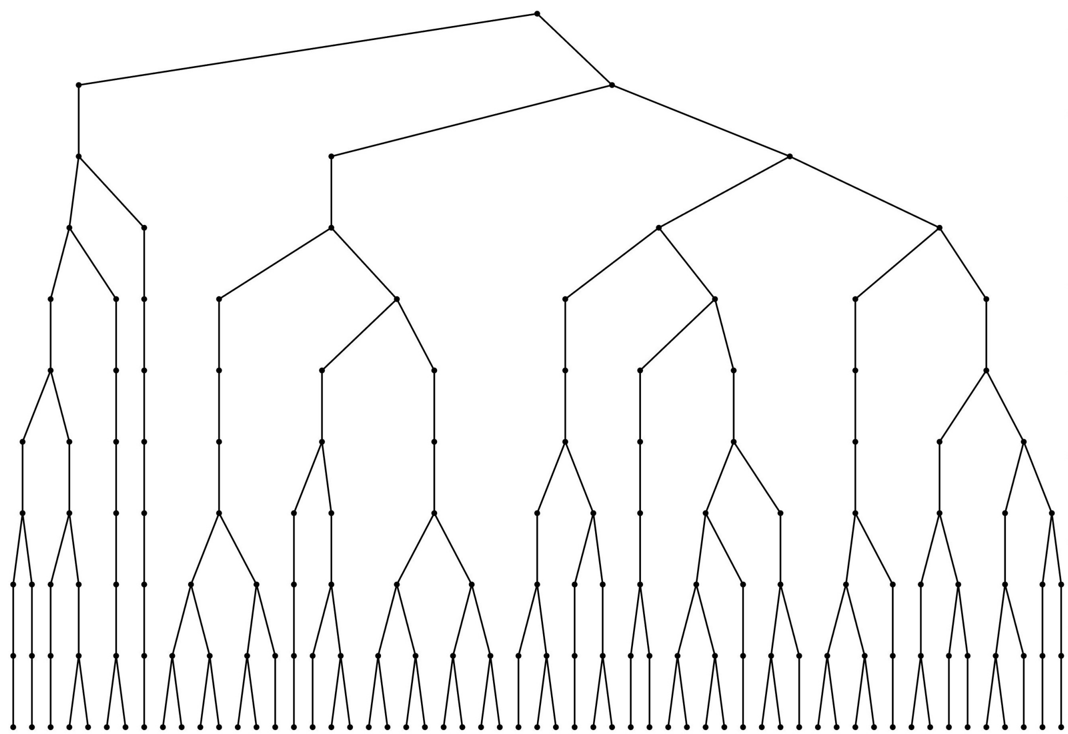


Figure 1.1: A realization of the first generations of a Galton-Watson tree with offspring law $\nu = \frac{1}{2}\delta_1 + \frac{1}{2}\delta_2$. Here, δ_a denotes the Dirac measure in a.

the starting point. If the probability that the random walk on T returns to the root is strictly less than one, it visits the root only a finite number of times. The random walk is then said to be transient. Otherwise, the random walk returns to the root infinitely often and is called recurrent. It was first shown in the unpublished work of [GK84] that the simple random walk on infinite supercritical Galton-Watson trees is almost surely transient. That is, the random walk is transient on almost all trees when we condition on the survival of the tree. This fact is also a consequence of Theorem 4.3 together with Proposition 6.4 in [Lyo90].

Transience implies that the random walk returns to each vertex in the the tree only a finite number of times and therefore the distance of the walker to the root tends to infinity. This raises the question how fast the random walk moves away from the root. Therefore, the next natural step is to study the rate of escape. It was proven by [LPP95] that the simple random walk moves away from the root with a linear rate. In other words, there exists a deterministic constant v such that

$$\lim_{n\to\infty} \frac{|X_n|}{n} = v \quad \text{almost surely,} \tag{1.0.1}$$

where $|X_n|$ denotes the distance of X_n to the root. The limit holds almost surely with respect to the annealed law, that is, averaged over the random walk and the tree, and conditioned on T being infinite. We say that the simple random walk on Galton-Watson trees satisfies a law of large numbers. The limit v is called the rate of escape or the speed of the random walk. Theorem 3.2 in [LPP95] implies that the speed can be calculated explicitly in terms of the offspring distribution. Moreover, it is strictly positive. The proof relies on the construction of an invariant measure for the environment observed by the walker which allows the application of the ergodic theorem to the resulting ergodic Markov chain.

After that, the question arises whether the random walk satisfies a functional central limit theorem. It was shown by [Pia98] that there exists a positive constant σ^2 such that the process

$$\frac{1}{\sqrt{n\sigma^2}}\big(|X_{\lfloor tn\rfloor}| - \lfloor tn\rfloor v\big), \quad t \in [0,1]$$

converges in law to a standard Brownian motion. An explicit expression for the value σ^2 in terms of the offspring distribution is not known. Only if the number of offspring is constant, it is possible to calculate the volatility explicitly. The proof is based on the existence of a regeneration structure which yields independent increments with good moments.

In this work we focus on the random conductance model, a generalization where the transition probabilities of the random walk are no longer uniform but depend on the environment. The underlying graph is still an infinite supercritical Galton-Watson tree. Additionally, the edges in the tree are randomly assigned i.i.d. edge weights, which we call conductances. The random walk in the environment formed by the tree and the conductances is then a Markov chain which crosses an edge with a probability proportional to the conductance of that edge. It was shown by [GMPV12] that the random walk is a.s. transient and that it has a deterministic and positive speed. In other words, a convergence as in (1.0.1) still holds when we now also average over the conductances, whereby the limit depends on both the offspring law of the tree and the conductance law in a highly non-trivial way. As for the simple random walk, the main tool for proving a law of large numbers is to consider the environment from the current location of the walker. The key is the identification of the stationary measure for the environment process, which then allows an application of the ergodic theorem. As a consequence, the speed of the walk can be expressed as an expectation under the invariant measure. The authors also provide a formula for the speed that includes the law of effective conductances. Unfortunately, both expressions for the speed do not allow an explicit calculation. Only if the marginal distribution of the conductances is degenerated, which corresponds to the simple random walk, an explicit formula for the speed is known since the work of [LPP95]. Compared to the simple random walk, random conductances can only slow down the walker. More precisely, it was shown by [GMPV12] that in the case of non-degenerated conductances, the speed of the random walk with random conductances is strictly smaller than the speed of the simple random walk. A central limit theorem for the random walk on Galton-Watson trees with random conductances is only known under very strong conditions on the offspring law. It was proven by [Bar13] that the distance of the walker to the root satisfies a central limit theorem if the number of descendants of each vertex is larger than the ratio of the maximum to minimum edge weight. If this ratio is large, it means that each vertex in the tree requires a great number of offspring. One of the main results of this thesis is a central limit theorem for more general offspring distributions. Moreover, our aim is to study the influence of the environment law on limit statements as above. In other words, we will investigate how the limiting speed and the fluctuations of the random walk depend on the distribution of the environment. We will in particular study the effect of very small edge weights on these quantities.

Closely related to our model is the biased random walk on Galton-Watson trees. In this model each edge between the n-th and the $(n+1)$-th generation is assigned the weight λ^{-n}, where $\lambda > 0$ is called the bias parameter. The random walk then takes an edge proportional to its weight. This means that it moves back towards the root with a probability proportional to λ, whereas the probability of moving to a descendant is proportional to 1. Hence, for any $\lambda > 1$ the random walk has a tendency to the root, with higher values λ pushing the random walk more to the root. A great number of offspring has the converse effect. Therefore, it is not surprising that the question of transience or recurrence depends on the offspring mean m and on the bias parameter λ. It was proven by [Lyo90] that the λ-biased random walk is a.s. transient if $\lambda < m$, a.s. null recurrent if $\lambda = m$ and positive recurrent if $\lambda > m$. In the transient case the authors of [LPP96] showed the random walk with a bias towards the root has a deterministic and positive speed $v_\lambda > 0$. On the other hand, a biased random walk that is pushed away from the root, i.e. $\lambda < 1$, may have zero speed when it spends too much time in leaves. An expression for the speed which depends on the bias parameter in a highly non-trivial way was shown by [Aïd14]. When $\lambda = 1$, the considered random walk coincides with the simple random walk studied in [LPP95]. A quenched central limit theorem was proven by [PZ08] for $\lambda \leq m$.

The model can be further generalized by randomizing the bias. More precisely, each vertex v in the Galton-Watson tree is marked with a random weight $A(v)$, independently for different vertices and with the same marginal distribution. The random walk then moves towards the root with a probability proportional to 1 while it moves to a descendant with a probability proportional to the weight of this vertex. When each vertex gets the deterministic mark $A(v) = \lambda^{-1}$, the random walk corresponds to the λ-biased random walk. The main difference to our model is that i.i.d. weights are randomly assigned to the vertices and not to the edges. In other words, in our model the conductances are realizations of a collection of independent random variables, while here the ratios of conductances are realizations of i.i.d. random variables. A criterion for transience, depending on the distribution of the weights and the offspring mean, is proven in [LP92]. [Aïd08] gives conditions for the random walk to have a positive speed. A central limit theorem is shown in [Far11].

This thesis is organized as follows. We start in Chapter 2 with the description of the random conductance model and introduce random walks on infinite supercritical Galton-Watson trees with random conductances. It is known since the work of [GMPV12] that the distance of the random walk to the root satisfies a law of large numbers with limit the speed of the walk, which is deterministic and positive, but cannot be computed explicitly. We will investigate the influence of the environment law on the speed in the following two chapters.

In Chapter 3 we study the regularity of the speed as a function of the marginal distribution of the conductances. We in particular investigate the influence of very small edge weights on the speed. In order to do this, we assign a small conductance $\varepsilon > 0$ to a positive fraction of edges and study the behaviour of the speed as $\varepsilon \to 0$. When $\varepsilon > 0$ is small, finite subtrees formed by edges with larger conductances act like traps in the environment. The random walk can only leave these subtrees by crossing an edge with conductance ε, which happens rarely. Whereas, if the small conductances are reduced to zero, the random walk cannot enter these finite subtrees at all and therefore such a slowdown does not occur for $\varepsilon = 0$. This suggests that the speed is not continuous at $\varepsilon = 0$. Verifying this discontinuity is the main goal of this chapter. Note that for $\varepsilon = 0$ the random walk can only move on a subtree of the original tree. This subtree is again a Galton-Watson tree, but it might be finite. As usual for trees with positive extinction probability, the speed $v(0)$ is given as the almost sure limit of $|X_n|/n$ under the annealed law when the traversable tree is conditioned to be infinite. Provided that the tree formed by larger conductances is supercritical, we show in Theorem 3.1.2 that

$$\lim_{\varepsilon \to 0} v(\varepsilon) = \beta v(0)$$

for a constant $\beta \in (0, 1)$. This implies that the limit of the speed for conductances approaching zero is strictly smaller than the speed of the random walk as usually defined on trees with positive extinction probability. For the proof we recall the construction of the invariant measure for the environment process and a formula for the speed in Section 3.2.

In Chapter 4 we focus on the behaviour of the speed as a function of the offspring distribution. The main result of this chapter is that the speed is a continuous function of the offspring law. We consider a sequence of offspring distributions ν_n such that for all n the corresponding Galton-Watson trees are supercritical and have no leaves. Furthermore, we assume that ν_n converges weakly to a measure ν. In Theorem 4.1.1 we show that

$$\lim_{n \to \infty} v(\nu_n) = v(\nu).$$

That is, the limit of the speed is given by the speed of the random walk on a Galton-Watson tree with offspring law ν.

In Chapter 5 the first main result is a functional central limit theorem for the random walk on a supercritical Galton-Watson tree without leaves when the edges of the tree are assigned i.i.d. uniformly elliptic conductances. Provided that the conductance law has at least one atom, we show in Theorem 5.1.1 that there exists some constant $\sigma^2 > 0$ such that

$$\Big(\frac{1}{\sqrt{n\sigma^2}}\big(|X_{\lfloor tn\rfloor}| - \lfloor tn\rfloor v\big)\Big)_{t\in[0,1]} \xrightarrow[n\to\infty]{d} \big(\sigma B_t\big)_{t\in[0,1]}$$

under the annealed law, where $(B_t)_{t\in[0,1]}$ is a standard Brownian motion. The volatility σ^2 of the limit depends in a highly non-trivial way on the environment law. We study the effect of small edge weights on the fluctuations of the random walk. As in Chapter 3, we assign a small conductance $\varepsilon > 0$ to a positive fraction of edges and investigate the behaviour of the volatility as $\varepsilon \to 0$. Provided that the tree formed by larger conductances is supercritical, we show in Theorem 5.1.3 that the volatility for $\varepsilon \to 0$ is bounded away from zero

$$\liminf_{\varepsilon\to 0} \sigma^2(\varepsilon) > 0.$$

This implies that the slowdown induced by the small edge weights is not too strong. The proof of the central limit theorem relies on the existence of a renewal structure with good moments to decouple the increments of the random walk. This approach is a standard technique for proving a central limit theorem for random walks in random environments. In Section 5.2, we define the specific regeneration times. The required bounds on escape probabilities and moments of regeneration times are also given in that section. In order to control the volatility for small ε, we need the bounds to hold uniformly in ε. This is the key challenge in the proofs.

Chapter 2

Description of the model

In this chapter we introduce random walks on Galton-Watson trees with random conductances. This includes the description of the environment, which is randomly chosen but kept fixed over time, and the definition of the random walk, which, given the environment, is a Markov chain whose transition probabilities depend on the environment. We start with introducing the usual terminology for trees that we will use throughout this thesis. Next, we define the environment law, that is, a probability measure on the set of weighted rooted trees (endowed with a suitable σ-algebra) such that under this law the tree is a Galton-Watson tree with i.i.d. conductances. In the third section we are then ready to introduce the random walk on Galton-Watson trees. Given a realization of the environment, the considered random walk is a Markov chain on the set of vertices of the tree whose transition probabilities are given by the conductances. We introduce the quenched and the annealed law. The former is the distribution of the random walk when we fix a realization of the environment, the latter is obtained when we average over all realizations of the environment. The term conductances comes from a useful connection between random walks on weighted graphs and electrical networks, which we will discuss in the last section.

2.1 Weighted trees

Let us start with some preliminaries on trees. A tree T is a non-oriented, connected graph without loops. We use the same notation T for the set of vertices as for the tree itself. The set of undirected edges is denoted by $\mathcal{E}(T)$. We call two vertices u, $v \in T$ neighbours if they are connected by an edge, i.e. $(u, v) \in \mathcal{E}(T)$. In this case we write $u \sim v$. The degree of v, denoted by $\deg(v)$ (or $\deg_T(v)$), indicates the number of neighbours of v in T. If the degree is finite for each vertex, the tree is called locally finite.

In this work we only consider rooted trees. This means that one specific vertex is set to be the root of the tree. We write ρ for the root and (T, ρ) for the rooted tree. If the root is known from the context we still write T instead of (T, ρ). We denote the graph distance of a vertex v to the root by $|v|$. The n-th generation of T is given by all vertices

$v \in T$ that have graph distance n to the root. We write

$$G_n = G_n(T) = \{v \in T : |v| = n\} \tag{2.1.1}$$

for the n-th generation of T. The neighbour of v that lies on the path to the root is called the ancestors of v, denoted by v^*, the other neighbours are called descendants. We may define trees as a subtree of the Ulam-Harris tree $\mathbb{T} = \bigcup_{n=0}^{\infty} \mathbb{N}^n$, where by convention $\mathbb{N}^0 = \{\emptyset\}$. This means that a vertex of a tree is identified by a finite sequence of integers. A rooted tree T is then a subset of $\mathbb{T}$ satisfying the following properties:

(1) $\emptyset \in T$,

(2) $v_1 \dots v_m \in T$ implies $v_1 \dots v_k \in T$ for all $m \geq 1$, $1 \leq k < m$,

(3) $v_1 \dots v_m \in T$ implies $v_1 \dots v_{m-1} w \in T$ for all $w \in \{1, \dots v_m\}$, $m \geq 1$,

see e.g. [LG05]. We let $\mathcal{T}$ be the set of all locally finite, rooted trees. We equip $\mathcal{T}$ with the topology induced by the metric

$$d_{\mathcal{T}} \colon \mathcal{T} \times \mathcal{T} \to \mathbb{R}, \quad d_{\mathcal{T}}((T, \rho), (T', \rho')) = \exp\big(-\sup\{n \geq 0 : T_{|n}(\rho) = T'_{|n}(\rho')\}\big),$$

where $T_{|n}(\rho)$ denotes the subtree of (T, ρ) consisting of the first n generations. It can be easily verified that $(\mathcal{T}, d_{\mathcal{T}})$ is a separable metric space. To see the separability, note that the set of finite trees $\{T \in \mathcal{T} : |T| < \infty\}$ is a countable and dense subset of $\mathcal{T}$.

In order to obtain weighted trees, we label each edge $e \in \mathcal{E}(T)$ in a tree T with a non-negative weight, which we call the conductance of e. We denote it by $\xi(e)$. The set of weighted, rooted trees is defined by

$$\Omega = \big\{(T, \rho, \xi) : (T, \rho) \in \mathcal{T},\, \xi \in [0, \infty)^{\mathcal{E}(T)}\big\}.$$

An element $\omega = (T, \rho, \xi) \in \Omega$ is called an environment.

Next, we define a metric on Ω. To quantify the distance between two weighted trees, we have to compare both the trees and the conductance configurations. Note that some edges may belong to only one of the trees. For this reason, we let ξ_{UH} be the conductance configuration of the Ulam-Harris tree such that each edge e that is also in the tree T gets the weight $\xi(e)$, while edges that do not belong to the tree get the weight zero. Formally, given an environment $\omega = (T, \rho, \xi) \in \Omega$, we set

$$\xi_{UH}(e) = \begin{cases} \xi(e), & e \in \mathcal{E}(T) \\ 0, & e \notin \mathcal{E}(T). \end{cases}$$

for all edges $e \in \mathcal{E}(\mathbb{T})$. Consequently, we have $\xi_{UH} \in \Xi = [0, \infty)^{\mathcal{E}(\mathbb{T})}$. We equip the set Ξ with the metric

$$d_{\Xi} \colon \Xi \times \Xi \to \mathbb{R}, \quad d_{\Xi}(\xi, \xi') = \sum_{e \in \mathcal{E}(\mathbb{T})} \frac{1}{2^{\varphi(e)}} \frac{|\xi(e) - \xi'(e)|}{1 + |\xi(e) - \xi'(e)|}.$$

Here, $\varphi\colon \mathcal{E}(\mathbb{T}) \to \mathbb{N}$ denotes some fixed bijection, which exists since the set of all edges in the Ulam-Harris tree is countable. Finally, this allows us to define a metric on set of environments Ω as follows

$$d\colon \Omega \times \Omega \to [0, \infty), \quad d(\omega, \omega') = d_{\mathcal{T}}((T, \rho), (T', \rho')) + d_{\Xi}(\xi_{UH}, \xi'_{UH}).$$

We note that the metric space (Ω, d) is separable. To see this, observe that the set of all finite trees with rational conductances $\{(T, \rho, \xi) \in \Omega : |T| < \infty, \, \xi \in \mathbb{Q}^{\mathcal{E}(T)}\}$ defines a countable and dense subset of Ω.

2.2 Random environments

In this section we introduce Galton-Watson trees with random conductances. To start with, we consider a Galton-Watson branching process. We can interpret the process as a model for the evolution of a population over time. At the beginning, this population consists of a single individual. Each individual lives for exactly one time step and has a random number of descendants, independent of all the other individuals. The genealogical tree associated with this branching process yields a random tree, which we call a Galton-Watson tree. We denote the offspring distribution by ν, that is, the probability that an individual has k descendants is given by $\nu(\{k\})$. Throughout this thesis we assume $\nu(\{0\}) = 0$, which means that every vertex has at least one descendant. We say that the tree has no leaves. This assumption ensures that the considered trees are infinite. Moreover, we assume that the tree is supercritical and that the offspring mean is finite. The former means that the average number of descendents of a vertex is strictly larger than one, i.e. $m_1 = \int x \, \mathrm{d}\nu(x) \in (1, \infty)$.

In order to obtain weighted trees, we label each edge e in a tree T with a random conductance $\xi(e)$, independently for different edges and with the same marginal law μ. More precisely, $\xi = (\xi(e))_{e \in \mathcal{E}(T)}$ is a family of independent and identically distributed random variables. We assume the marginal distribution μ to be uniformly elliptic, so that the conductances are almost surely bounded and bounded away from zero. This means that there exists a constant $\kappa \geq 1$ such that

$$\mu([\kappa^{-1}, \kappa]) = 1. \tag{2.2.1}$$

To make this precise, we denote the Borel σ-algebra on the set of environments Ω by $\mathcal{G}$. We then define P as the law on $(\Omega, \mathcal{G})$ such that under P, T is a Galton-Watson tree with offspring law ν and given T, all edges in T are labelled with independent and identically distributed conductances with marginal law μ. We write E for the corresponding expectation. Formally, we may construct P as a pushforward law of a product measure as follows. We endow the set of rooted trees $\mathcal{T}$ with its Borel σ-algebra and we let GW be the law on $(\mathcal{T}, \mathcal{B}(\mathcal{T}))$ so that under GW, T is a Galton-Watson tree with offspring distribution ν. Moreover, we equip the product space $\mathcal{T} \times \Xi$ with the product topology and its Borel σ-algebra. We set $\tilde{\mathrm{P}} = \mathrm{GW} \otimes \mu^{\otimes \mathcal{E}(\mathbb{T})}$. That is, under $\tilde{\mathrm{P}}$, T is a Galton-Watson tree and all edges in the Ulam-Harris tree are labelled with independent and identically distributed conductances with marginal law μ. Since we only want to assign a weight to the edges in the underlying tree, we introduce the mapping

$$\pi\colon \mathcal{T} \times \Xi \to \Omega, \quad (T, \rho, (\xi(e)_{e\in\mathcal{E}(\mathbb{T})})) \mapsto (T, \rho, (\xi(e)_{e\in\mathcal{E}(T)})). \tag{2.2.2}$$

Note that π is continuous and therefore measurable. Then the environment law P is the pushforward measure of $\tilde{\mathrm{P}}$ induced by π, i.e.

$$\mathrm{P}(A) = \tilde{\mathrm{P}}(\pi^{-1}(A)) \tag{2.2.3}$$

for measurable subsets $A \subseteq \Omega$. This way of defining the environment law simplifies some of our proofs.

2.3 Random walks in random environments

We consider a time discrete random walk $(X_n)_{n\geq 0}$ on weighted trees. When we fix a realization of the environment, the random walk moves along the edges of the underlying tree, taking an edge with a probability proportional to its conductance. More precisely, given an environment $\omega = (T, \rho, \xi)$, we let P_ω^v be the law of the Markov chain $(X_n)_{n\geq 0}$ on the tree T, starting at a vertex $v \in T$ and with transition probabilities

$$P_\omega^v(X_{n+1} = y \mid X_n = x) = \begin{cases} \frac{\xi(x,y)}{C(x)}, & x \sim y \\ 0, & \text{otherwise} \end{cases} \tag{2.3.1}$$

where $C(x) = \sum_{w\sim x} \xi(x, w)$ is the sum of all edge weights incident to a vertex x. We call P_ω^v the quenched law and we write P_ω for the law of $(X_n)_{n\geq 0}$ starting at the root. The quenched law is a probability measure on the space of trajectories $(\mathbb{T}^{\mathbb{N}_0}, \mathcal{F})$, where $\mathcal{F}$ is the σ-algebra generated by the finite dimensional projections, that is,

$$\mathcal{F} = \sigma\big(\{\pi_m^{-1}(A) : m \geq 0,\, A \subseteq \mathbb{T}^{m+1}\}\big) \tag{2.3.2}$$

with

$$\pi_m\colon \mathbb{T}^{\mathbb{N}_0} \to \mathbb{T}^{m+1}, \quad \pi_m(x) = (x_0, \ldots, x_m).$$

We equip the space of trajectories with the metric

$$d_{\mathrm{tra}}\colon \mathbb{T}^{\mathbb{N}_0} \times \mathbb{T}^{\mathbb{N}_0} \to [0,\infty), \quad d_{\mathrm{tra}}(x,y) = \exp\big(-\sup\{n \geq 0 : \pi_n(x) = \pi_n(y)\}\big),$$

which induces the product topology. Note that $\mathcal{F}$ coincides with the Borel σ-algebra on $\mathbb{T}^{\mathbb{N}_0}$, since $\mathbb{T}$ is countable (see e.g. Theorem 5.10 in [Els18]).

Moreover, we introduce the annealed law $\mathbb{P}^v$ that averages over all realizations of the environment. Formally, $\mathbb{P}^v$ is a measure on the product space $(\Omega \times \mathbb{T}^{\mathbb{N}_0}, \mathcal{G} \otimes \mathcal{F})$ defined by

$$\mathbb{P}^v(A \times B) = \int_A P_\omega^v(B)\, \mathrm{dP}(\omega) \tag{2.3.3}$$

for measurable subsets $A \times B$. As before, we write $\mathbb{P}$ if the random walk starts at the root and the corresponding expectation is denoted by $\mathbb{E}^v$ (and $\mathbb{E}$ if $v = \rho$). The following lemma shows that the integral in (2.3.3) is well-defined.

Lemma 2.3.1. *For every set $B \in \mathcal{F}$, the mapping*

$$\Omega \to [0,1], \quad \omega \mapsto P_\omega^v(B)$$

is measurable with respect to $\mathcal{G}$.

We remark that Lemma 2.3.1 implies that the mapping $(\Omega, \mathcal{F}) \to [0,1]$, $(\omega, B) \mapsto P_\omega(B)$ defines a transition kernel.

Proof. We let

$$\tilde{\mathcal{F}} := \left\{\pi_m^{-1}(A) : m \geq 0,\ A \subseteq \mathbb{T}^{m+1}\right\}$$

be a generator of $\mathcal{F}$, recall (2.3.2). For every $B = \pi_m^{-1}(A) \in \tilde{\mathcal{F}}$ we have

$$P_\omega^v(B) = \sum_{(v_0,\dots,v_m)\in A} P_\omega^v(X_0 = v_0, \dots, X_m = v_m) = \sum_{\substack{(v_0,\dots,v_m)\in A,\\ v_i \sim v_{i+1}\, \forall i}} \prod_{i=0}^{m-1} \frac{\xi(v_i, v_{i+1})}{C(v_i)},$$

which implies that the mapping $\omega \mapsto P_\omega^v(B)$ is measurable for all $B \in \tilde{\mathcal{F}}$. Moreover, the family of sets

$$\mathcal{M} = \left\{B \in \mathcal{F} : \omega \mapsto P_\omega^v(B) \text{ is measurable}\right\}$$

defines a monotone class. To see this, let $B = \bigcup_{n\geq 1} B_n$ be a countable monotone union with $B_n \in \mathcal{M}$. Then the mapping

$$\omega \mapsto P_\omega^v(B) = \lim_{n\to\infty} P_\omega^v(B_n)$$

is measurable as the limit of measurable functions. Analogously, we obtain that $\mathcal{M}$ is closed under countable monotone intersections. Finally, by the monotone class theorem (Theorem 3.4 in [Bil95]) we obtain $\mathcal{M} = \mathcal{F}$, which concludes the proof. □

The next lemma shows that we can integrate a non-negative measurable function with respect to the annealed law by first integrating with respect to the quenched law and then integrating with respect to the environment measure.

Lemma 2.3.2. *Let $f\colon \Omega \times \mathbb{T}^{\mathbb{N}_0} \to [0,\infty)$ be measurable with respect to $\mathcal{G} \otimes \mathcal{F}$. Then the following hold:*

(a) For $\omega \in \Omega$ fixed, the mapping $x \mapsto f(\omega, x)$ is measurable with respect to $\mathcal{F}$.

(b) The mapping $\omega \mapsto \int f(\omega, x)\, \mathrm{d}P_\omega^v(x)$ is measurable with respect to $\mathcal{G}$.

(c) We have

$$\int f \,\mathrm{d}\mathbb{P}^v = \int \int f(\omega, x)\, \mathrm{d}P_\omega^v(x)\, \mathrm{dP}(\omega).$$

Proof. We first show the statement for indicator functions $f = \mathbb{1}_D$ with $D = A \times B \in \mathcal{G} \times \mathcal{F}$. Since $\mathbb{1}_D(\omega, x) = \mathbb{1}_A(\omega)\mathbb{1}_B(x)$, the mapping $x \mapsto \mathbb{1}_D(\omega, x)$ is trivially measurable with respect to $\mathcal{F}$. Integrating $\mathbb{1}_D$ with respect to the quenched law yields

$$\int \mathbb{1}_D(\omega, x)\, \mathrm{d}P_\omega^v(x) = \mathbb{1}_A(\omega) P_\omega^v(B).$$

Thus, Lemma 2.3.1 implies that the mapping $\omega \mapsto \int \mathbb{1}_D(\omega, x)\, \mathrm{d}P_\omega^v(x)$ is measurable with respect to $\mathcal{G}$. Finally, recalling the definition of the annealed law in (2.3.3), we have

$$\int \mathbb{1}_D \,\mathrm{d}\mathbb{P}^v = \mathbb{P}^v(A \times B) = \int \mathbb{1}_A(\omega) P_\omega^v(B)\, \mathrm{dP}(\omega) = \int \int \mathbb{1}_D(\omega, x)\, \mathrm{d}P_\omega^v(x)\, \mathrm{dP}(\omega).$$

Let us now introduce the family of sets

$$\mathcal{D} = \big\{ D \in \mathcal{G} \otimes \mathcal{F} : \mathbb{1}_D \text{ satisfies (a)–(c) from the lemma} \big\}.$$

We have already shown $\mathcal{G} \times \mathcal{F} \subseteq \mathcal{D}$. Moreover, it is easy to check that $\mathcal{D}$ defines a Dynkin system and therefore we obtain $\mathcal{D} = \mathcal{G} \otimes \mathcal{F}$. In other words, for all $D \in \mathcal{G} \otimes \mathcal{F}$ the indicator function $\mathbb{1}_D$ satisfies (a)–(c) from the lemma.

Since we can approximate every measurable function $f \geq 0$ by a sequence of simple functions f_n with $f_n \uparrow f$, the statement follows by the monotone convergence theorem. □

A similar result holds for integrable functions.

Lemma 2.3.3. *Let $f\colon \Omega \times \mathbb{T}^{\mathbb{N}_0} \to \mathbb{R}$ be integrable with respect to the annealed law $\mathbb{P}^v$. Then the following hold:*

(a) For P*-almost all ω, the mapping $x \mapsto f(\omega, x)$ is integrable with respect to P_ω^v.*

(b) The mapping $\omega \mapsto \int f(\omega, x)\, \mathrm{d}P_\omega^v(x)$ is integrable with respect to P.

(c) We have

$$\int f \,\mathrm{d}\mathbb{P}^v = \int \int f(\omega, x)\, \mathrm{d}P_\omega^v(x)\, \mathrm{dP}(\omega).$$

Proof. From Lemma 2.3.2 we know that the functions

$$x \mapsto |f(\omega, x)| \quad \text{and} \quad \omega \mapsto \int |f(\omega, x)| \, \mathrm{d}P_\omega^v(x)$$

are measurable. Moreover, we have

$$\int \int |f(\omega, x)| \, \mathrm{d}P_\omega^v(x) \, \mathrm{dP}(\omega) = \int |f| \, \mathrm{d}\mathbb{P}^v < \infty,$$

which shows the integrability of the function $\omega \mapsto \int f(\omega, x) \, \mathrm{d}P_\omega^v(x)$. In addition, the finiteness of the above integral implies

$$\int |f(\omega, x)| \, \mathrm{d}P_\omega^v(x) < \infty \quad \mathrm{P} - \text{almost surely},$$

that is, the function $x \mapsto f(\omega, x)$ is integrable with respect to the quenched law for P-almost all ω. When we write $f = f^+ - f^-$, Lemma 2.3.2 yields the desired identity in (c). □

Let us turn to the long-term behaviour of the random walk. Given an environment $\omega = (T, \rho, \xi)$, the random walk on T starting at the root is called transient if it has positive probability of never returning to the root. Otherwise, we call it recurrent. It is shown (in a more general setting) in Proposition 2.1 in [GMPV12] that the random walk defined above is a.s. transient. For uniformly elliptic conductances the transience is a direct consequence of Rayleigh's Monotonicity Principle (see Lemma 2.4.5 below) and the transience of the simple random walk on Galton-Watson trees.

Proposition 2.3.4. *The random walk $(X_n)_{n\geq 0}$ is transient for* P*-almost all environments ω.*

The transience implies that the distance of the random walk to the root tends to infinity, which raises the question how fast it moves away from the root. It is proven (in a more general form) in Theorem 4.1 in [GMPV12] that the limit of $|X_n|/n$ exists a.s.

Theorem 2.3.5. *The limit*

$$\lim_{n \to \infty} \frac{|X_n|}{n} = v = v(\nu, \mu)$$

exists P_ω-almost surely for P*-almost all ω. Moreover, v is deterministic and strictly positive.*

The limit $v(\nu, \mu)$ is called the speed of the random walk. Unfortunately, we cannot calculate the speed explicitly except when the marginal law of the conductances is degenerated. In this case the process coincides with the simple random walk and an explicit formula for the speed is known since the work of [LPP95]. The speed depends on both the offspring distribution and the marginal law of the conductances. We study the regularity of $v(\nu, \mu)$ as a function of μ and ν in Chapters 3 and 4, respectively.

A standard method for proving such a law of large numbers for random walks in random environments is to consider the environment observed by the walker. The key challenge here is the identification of the invariant measure for this process which is done by [GMPV12]. Applying the ergodic theorem to the resulting ergodic Markov chain then yields the law of large numbers. Consequently, the speed is given as an expectation under the invariant measure and it is therefore not surprising that it cannot be calculated explicitly. The authors also provide a formula for the speed which includes effective conductances. The effective conductivity is a quantity from the theory of electrical networks. In the next section we will introduce it and discuss its significance in the context of random walks.

2.4 Random walks and electrical networks

We can think of a weighted tree as an electrical network whose edges correspond to electrical conductors. Then several useful connections can be observed between the physical concept of voltage and current and properties of the random walk. We present in this section some of these results, which are required for the proofs in the later chapters. The references are [DS84] and [LP16]. Unless otherwise stated, the proofs of the results in this section can be found therein and are not given here.

Let $\omega = (T, \rho, \xi) \in \Omega$ be an environment. We begin with introducing the effective conductance between a vertex $v \in T$ and a set of vertices $A \subseteq T$. As mentioned above, the weighted tree can be seen as an electrical network. The edges in the tree correspond to electrical conductors, whereby the electrical resistance of a conductor is given by the reciprocal of the appropriate edge weight. For this reason, the edge weights are called conductances.

When we impose a voltage between v and A, the amount of current flowing into the circuit depends on all the conductances in the network in between. We may think of the circuit between v and A as a single conductor in which all voltages and currents remain unchanged. The conductivity of this equivalent conductor is called the effective conductance between v and A, denoted by $\mathcal{C}_\omega(v, A)$. There is also a probabilistic definition of effective conductances via an escape probability. To make this precise, we set

$$\eta_A = \inf\{n \geq 0 : X_n \in A\} \quad \text{and} \quad \eta_A^+ = \inf\{n \geq 1 : X_n \in A\}. \tag{2.4.1}$$

That is, η_A is the hitting time of the set A and η_A^+ denotes the first time after zero at which the random walk visits the set A. If the starting point of the random walk is located in A, η_A^+ indicates the return time to A. Note that this is the only situation where η_A^+ differs from η_A. If $A = \{z\}$ contains only a single vertex $z \in T$, we write η_z for the first hitting time of z and η_z^+ for the first hitting time of z after zero. Following [LP16] with modified notation, the effective conductance between v and A is given by

$$\mathcal{C}_\omega(v, A) = C(v) P_\omega^v(\eta_A < \eta_v^+). \tag{2.4.2}$$

Here, $P_\omega^v(\eta_A < \eta_v^+)$ is the probability that a random walk starting at v hits A before it returns to v. The effective resistance between v and A is defined by $\mathcal{R}_\omega(v, A) = \mathcal{C}_\omega(v, A)^{-1}$. Using this probabilistic interpretation of the effective conductance, one can show the following.

Lemma 2.4.1. *Let A and B be disjoint sets of vertices and $v \notin A \cup B$. Then we have*

$$P_\omega^v(\eta_B < \eta_A) \leq \frac{\mathcal{C}_\omega(v, B)}{\mathcal{C}_\omega(v, A \cup B)} \leq \frac{\mathcal{C}_\omega(v, B)}{\mathcal{C}_\omega(v, A)}.$$

In particular, if it is not possible for the random walk to visit A and B during a single excursion starting from v, we have

$$P_\omega^v(\eta_B < \eta_A) = \frac{\mathcal{C}_\omega(v, B)}{\mathcal{C}_\omega(v, A \cup B)} = \frac{\mathcal{C}_\omega(v, B)}{\mathcal{C}_\omega(v, A) + \mathcal{C}_\omega(v, B)}.$$

The proof of the first estimate can be found in [BGP03], see Fact 2 therein. It is not hard to see that their arguments also imply the second identity.

On infinite trees we can also define the effective conductance from a vertex v to infinity. We set

$$\mathcal{C}_\omega(v, \infty) = \lim_{n \to \infty} \mathcal{C}_\omega(v, G_n), \tag{2.4.3}$$

recall that G_n denotes the n-th generation of the tree T. The limit is well-defined, since for $n > |v|$ the effective conductance $\mathcal{C}_\omega(v, G_n)$ is monotonically decreasing in n. In view of (2.4.2), it is positive if and only if the random walk has positive probability of never returning to its starting point, which implies the following result.

Theorem 2.4.2. *Given an environment $\omega = (T, \rho, \xi)$, the random walk on T is transient if and only if $\mathcal{C}_\omega(\rho, \infty) > 0$.*

Using this fact, the authors of [GMPV12] showed that the random walk on supercritical Galton-Watson trees with random conductances is almost surely transient, as stated in Proposition 2.3.4.

We have seen that the effective conductance is an important quantity for escape probabilities and the question of recurrence and transience of a random walk. Next, let us focus on how we can calculate effective conductances. To start with, we provide two useful facts about electrical networks.

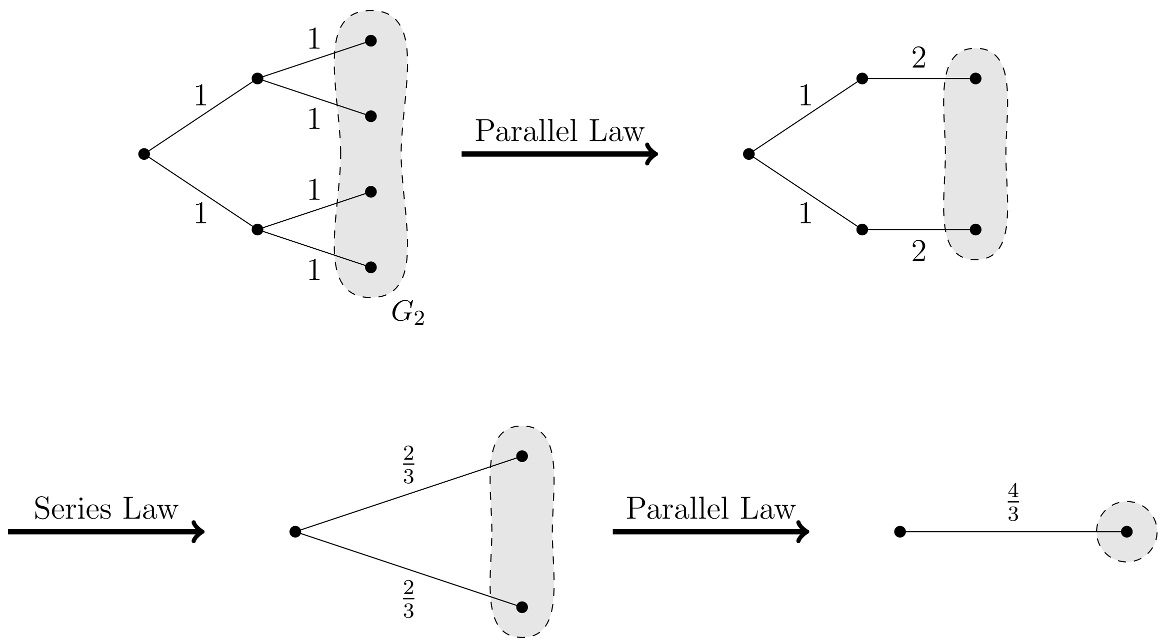


Figure 2.1: Transformation of the first two generations of a binary tree with unit conductances to an equivalent single conductor.

Lemma 2.4.3 (Parallel Law)**.** *Two conductors ξ_1 and ξ_2 in parallel are equivalent to one single conductor $\xi_1 + \xi_2$. In other words, suppose that a vertex $z \in T \setminus A$ has two neighbours z_1 and z_2 that are located in the set A. If we replace the vertices z_1, z_2 and the edges (z, z_1), (z, z_2) by a single vertex z' and a single edge (z, z') with conductance $\xi(z, z_1)+\xi(z, z_2)$, then the effective conductance between the vertex v and the set of vertices A in the modified environment is equal to the effective conductance $\mathcal{C}_\omega(v, A)$ in the original environment.*

Lemma 2.4.4 (Series Law)**.** *Two conductors ξ_1 and ξ_2 in series are equivalent to one single conductor $(\xi_1^{-1} + \xi_2^{-1})^{-1}$. More precisely, suppose that $z \in T \setminus (\{v\} \cup A)$ is a vertex with $\deg(z) = 2$ and neighbours z_1 and z_2. If we replace the two edges (z, z_1) and (z, z_2) by a single edge (z_1, z_2) with conductance $(\xi(z, z_1)^{-1} + \xi(z, z_2)^{-1})^{-1}$, then the effective conductance between the vertex v and the set of vertices A in the modified environment is equal to the effective conductance $\mathcal{C}_\omega(v, A)$ in the original environment.*

We remark that the formulation of the last to lemmas is adapted to our setting where the underlying network is a tree. For general networks we refer to [LP16]. Let us consider an example to illustrate how these rules can be used to compute escape probabilities of a random walk. We let T be the binary tree with unit conductances. Using the Parallel Law and the Series Law, we can gradually reduce the first two generations of the tree to a single conductor, as shown in Figure 2.1. The resulting conductance of the equivalent conductor is the effective conductance from ρ to G_2, i.e. $\mathcal{C}_\omega(\rho, G_2) = \frac{4}{3}$. Due to (2.4.2), the probability that a random walk starting at the root will hit the second generation of the tree before returning to the root is

$$P_\omega^\rho(\eta_{G_2} < \eta_\rho^+) = \frac{\mathcal{C}_\omega(\rho, G_2)}{C(\rho)} = \frac{2}{3}.$$

The next result from electrical network theory deals with the question what happens to the effective conductance when we change edge weights in the tree.

Lemma 2.4.5 (Rayleigh's Monotonicity Principle)**.** *Let $\omega = (T, \rho, \xi)$ and $\omega' = (T, \rho, \xi')$ be two environments where ξ and ξ' are conductance configurations with $\xi(e) \leq \xi'(e)$ for all edges $e \in \mathcal{E}(T)$. Furthermore, we let $A \subseteq T$ be a set of vertices and $v \notin A$. Then we have*

$$\mathcal{C}_\omega(v, A) \leq \mathcal{C}_{\omega'}(v, A).$$

This shows that increasing edge weights can only increase the effective conductance. In particular, removing an edge decreases the effective conductance. If this edge is not incident to v, this also decreases the probability that a random walk starting at v hits A before it returns to v.

Chapter 3

The speed of random walk on Galton-Watson trees with vanishing conductances

As stated in Theorem 2.3.5, the random walk on infinite supercritical Galton-Watson trees with i.i.d. conductances moves away from the root with a linear rate. The speed of the walk is given as an expectation of ratios of effective conductances and cannot be computed explicitly. We want to investigate how the speed depends on the distribution of the environment. In this chapter we study the regularity of the speed as a function of the conductance law. In particular, we investigate the effect on the speed when a positive fraction of edge weights approaches zero. In the next chapter we will then focus on the behaviour of the speed as a function of the offspring law.

The regularity of the speed as a function of the local transition probabilities is a prominent question for random walks in random environments. It has been well studied for biased random walks on Galton-Watson trees. The biased random walk has a positive limiting speed when the bias parameter is smaller than the offspring mean. The speed depends on the bias in a highly non-trivial way. [BAFS14] studied the monotonicity of the speed as a function of the bias. The behaviour of the speed when the bias parameter is close to the recurrent regime has been investigated by [BAHOZ13]. Results on the differentiability of the speed as function of the bias parameter can be found in [BT20]. The speed also depends on the offspring law of the underlying Galton-Watson tree. [MSZ15] studied the monotonicity of the speed with respect to the offspring distribution when the bias is kept fixed. For an overview we refer to [BAF16].

This chapter is organized as follows. In the next section we present the main results of this chapter. After that, to prepare the proofs, we construct the invariant measure for the environment seen from the random walk that [GMPV12] used to prove the law of large numbers. Moreover, we recall a formula for the speed which is crucial to compute the limit of the speed when a positive fraction of edges approaches zero. The proofs are given in the last section.

The results of this chapter are presented in the paper [GN21].

3.1 Main results

Let T be an infinite supercritical Galton-Watson tree with i.i.d. conductances and let $(X_n)_{n\geq 0}$ be the random walk on T starting at the root. The speed of the random walk is then given as the almost sure limit

$$\lim_{n\to\infty} \frac{|X_n|}{n} = v(\nu, \mu) \quad \mathbb{P} - \text{almost surely,}$$

see Theorem 2.3.5. As mentioned in the introduction, the speed depends, among others, on the marginal law of the conductances. We are interested in the effect on the speed when we change the distribution of the conductances. So, let $(\mu_n)_{n\in\mathbb{N}}$ be a sequence of uniformly elliptic measures that converges weakly to a measure μ. Our goal is to calculate the limit of the speed $v(\nu, \mu_n)$.

Let us first consider the case where all measures μ_n have the same ellipticity constant $\kappa \geq 1$, that is, $\mu_n([\kappa^{-1}, \kappa]) = 1$ for all $n \in \mathbb{N}$. This guarantees that the weak limit μ is also uniformly elliptic. Then we obtain the convergence of the speed

$$\lim_{n\to\infty} v(\nu, \mu_n) = v(\nu, \mu).$$

This shows that the speed is a continuous function of the marginal law of the conductances as long as we stay in the framework of uniformly elliptic measures. This first result is stated in the following proposition.

Proposition 3.1.1. *For any $\kappa \geq 1$, the mapping $\mu \mapsto v(\nu, \mu)$ is continuous on the set of uniformly elliptic measures satisfying* (2.2.1), *equipped with the weak topology.*

Next, we investigate what happens when we leave the set of uniformly elliptic measures. More precisely, we study the speed when some of the conductances approach zero. In order to do this, we let μ be a uniformly elliptic measure with ellipticity constant $\kappa \geq 1$. For $\varepsilon \geq 0$ and $\alpha \in (0, 1)$ we introduce

$$\mu_\varepsilon = \alpha\delta_\varepsilon + (1-\alpha)\mu, \tag{3.1.1}$$

where δ_ε denotes the Dirac measure in ε. Without loss of generality we assume $\varepsilon < \kappa^{-1}$. Note that this defines a weakly convergent sequence of measures

$$\mu_\varepsilon \xrightarrow[\varepsilon\to 0]{w} \alpha\delta_0 + (1-\alpha)\mu = \mu_0,$$

and for any $\varepsilon > 0$ the measure μ_ε is uniformly elliptic whereas the weak limit μ_0 is not. We write P_ε for the environment measure P if the marginal law of the conductances is given by μ_ε for $\varepsilon \geq 0$. If $\varepsilon = 0$, a positive fraction of edges is assigned weight zero.

Since the random walk cannot cross these edges, it can only move on a subtree of the original tree, which might be finite. We denote the traversable tree formed by the edges with positive conductances by T_1. Note that T_1 is a Galton-Watson tree with offspring distribution $\bar{\nu}$ and with i.i.d. conductances with marginal law μ, where $\bar{\nu}$ is given by

$$\begin{aligned}\bar{\nu}(\{k\}) = \mathrm{P}_0(\deg_{T_1}(\rho) = k) &= \sum_{n=k}^{\infty} \mathrm{P}_0(\deg_{T_1}(\rho) = k \mid \deg_T(\rho) = n)\mathrm{P}_0(\deg_T(\rho) = n) \\ &= \sum_{n=k}^{\infty} \nu(\{k\}) \binom{n}{k} (1-\alpha)^k \alpha^{n-k}. \end{aligned} \tag{3.1.2}$$

If $(1-\alpha)m_1 \leq 1$, the subtree T_1 dies out with probability one. Then the distance of the random walk to the root cannot tend to infinity and therefore we set $v(\nu, \mu_0) = 0$. Otherwise, if $(1-\alpha)m_1 > 1$, the tree T_1 is supercritical, which means that T_1 has a positive probability to survive. Since the traversable tree T_1 is a Galton-Watson tree with offspring law $\bar{\nu}$ and conductance law μ, we set $v(\nu, \mu_0) = v(\bar{\nu}, \mu)$. Here, consistently with Remark 4.1 in [GMPV12] and the definition in [LPP95], the speed $v(\bar{\nu}, \mu)$ is given as the almost sure limit of $|X_n|/n$ under $\bar{\mathbb{P}}$, where $\bar{\mathbb{P}}$ is the annealed law when we condition on the survival of the tree. More precisely, we let $\bar{\mathrm{P}}$ be the law on Ω such that under $\bar{\mathrm{P}}$, T is a Galton-Watson tree with offspring distribution $\bar{\nu}$ and i.i.d. conductances with marginal law μ. The conditioned annealed law $\bar{\mathbb{P}}$ is then defined as in (2.3.3) with P replaced by $\bar{\mathrm{P}}(\,\cdot \mid |T| = \infty)$. The following theorem gives the limit of the speed as ε tends to zero.

Theorem 3.1.2. *We consider a random walk on a supercritical Galton-Watson tree with random conductances. We assume that the second moment $m_2 < \infty$ of the offspring distribution ν is finite. Then for μ_ε as in* (3.1.1) *we have*

$$\lim_{\varepsilon \searrow 0} v(\nu, \mu_\varepsilon) = \hat{\mathrm{P}}_0(|T_1| = \infty) \cdot v(\nu, \mu_0), \tag{3.1.3}$$

where $\hat{\mathrm{P}}_0$ is the invariant measure for the environment seen from the random walk (see Section 3.2).

We notice that $\hat{\mathrm{P}}_0(|T_1| = \infty) < 1$, since each edge in the tree has a positive probability of having the weight ε. Theorem 3.1.2 therefore shows that the limit of the speed on Galton-Watson trees with vanishing conductances is smaller than speed of the random walk as usually defined on trees with a positive extinction probability. This slowdown effect occurs, since for small ε finite subtrees formed by the edges with larger conductances act like traps in the environment. To leave such a subtree, the walker has to move along an edge with conductance ε, which happens rarely when ε is small. For $\varepsilon = 0$ and conditioned on the survival of T_1, the random walk cannot be slowed down by these finite subtrees, since it cannot enter them at all. However, for $\varepsilon = 0$ a different slowdown effect is created by the leaves of the tree. The limit in Theorem 3.1.2 shows that the latter effect is in some sense weaker.

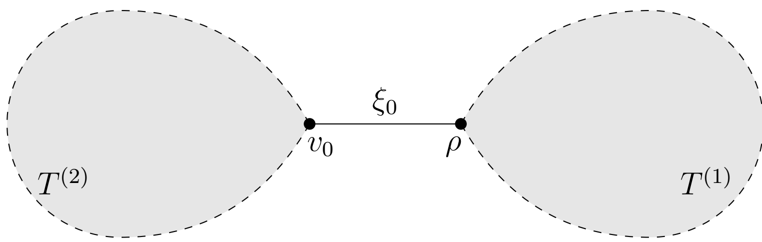


Figure 3.1: Under $\mathrm{P}_\varepsilon^{\mathrm{aug}}$, the subtrees $T^{(1)}$ and $T^{(2)}$ are independent and identically distributed weighted Galton-Watson trees with law P_ε. The conductance of the additional edge (ρ, v_0) is denoted by ξ_0.

3.2 The invariant measure

A standard argument for proving a law of large numbers for random walks in random environments is to consider the environment from the point of view of the particle. This is the process where we shift the root of the tree to the position of the walker.

When we consider the environment seen from the particle, the root indicates its current position. In order to determine the distance to the starting point, we need to know the past of the random walk. For this reason, we consider bi-infinite random walks on the tree. We equip the space of bi-infinite paths $\mathbb{T}^{\mathbb{Z}}$ with the product topology and its Borel σ-algebra $\hat{\mathcal{F}}$. Given an environment $\omega \in \Omega$, we let $\hat{P}_\omega$ be the law of the bi-infinite random walk $(X_n)_{n\in\mathbb{Z}}$ on T such that $X_0 = \rho$ and $(X_n)_{n\geq 0}$ and $(X_{-n})_{n\geq 0}$ are independent with marginal law as defined by (2.3.1). Thereby, the random walk in positive time is interpreted as the future, whereas the random walk in negative time represents the past of the random walk.

Let us now construct the invariant measure, following [GMPV12] with modified notation. The invariant measure should represent what the random walk typically sees after a long time. First, we note that the root of a Galton-Watson tree has no ancestor, which means that in distribution the root has one neighbour less than all the other vertices in the tree. For this reason, we introduce augmented Galton-Watson trees. An augmented Galton-Watson tree is a random tree where we attach to the root of a Galton-Watson tree an additional edge (ρ, v_0) and we let the additional vertex v_0 be the root of an independent Galton-Watson tree, see Figure 3.1. Formally, we let $\mathrm{P}_\varepsilon^{\mathrm{aug}}$ be the law on Ω such that under $\mathrm{P}_\varepsilon^{\mathrm{aug}}$, T is an augmented Galton-Watson tree, that is,

$$\mathrm{P}_\varepsilon^{\mathrm{aug}}((T,\rho,\xi) \in A) = \sum_{k\in\mathbb{N}} \nu(\{k-1\})\mathrm{P}_\varepsilon((T,\rho,\xi) \in A \mid \deg(\rho) = k) \tag{3.2.1}$$

for measurable sets $A \in \mathcal{G}$. We denote by $\mathrm{E}_\varepsilon^{\mathrm{aug}}$ the corresponding expectation.

Moreover, under the invariant measure, the conductance configuration of the edges adjacent to the root should correspond to what the random walk typically sees after a long time. The root is therefore weighted by the average conductance of adjoining edges.

We define the measure $\hat{\mathrm{P}}_\varepsilon$ via the corresponding expectation

$$\hat{\mathrm{E}}_\varepsilon[f(T,\rho,\xi)] = \mathrm{E}_\varepsilon^{\mathrm{aug}}\left[f(T,\rho,\xi)\frac{C(\rho)}{\gamma_\varepsilon \deg(\rho)}\right], \tag{3.2.2}$$

where $\gamma_\varepsilon = \int x \, \mathrm{d}\mu_\varepsilon(x)$ denotes the mean conductance of an edge. We let $\hat{\mathbb{P}}_\varepsilon$ be the corresponding annealed law, defined analogously to (2.3.3) on $\Omega \times \mathbb{T}^{\mathbb{Z}}$, and we write $\hat{\mathbb{E}}_\varepsilon$ for the associated expectation.

The environment seen from the random walk is the Markov process $(T, X_n, \xi)_{n\in\mathbb{Z}}$ with state space Ω and transition operator

$$Gf(T,\rho,\xi) = \begin{cases} \frac{1}{C(\rho)}\sum_{v\sim\rho}\xi(\rho,v)f(T,v,\xi), & C(\rho)>0 \\ f(T,\rho,\xi), & C(\rho)=0 \end{cases} \tag{3.2.3}$$

for a function $f\colon \Omega \to \mathbb{R}$. A straightforward calculation shows that G is reversible with respect to $\hat{\mathrm{P}}_\varepsilon$, i.e.

$$\hat{\mathrm{E}}_\varepsilon\big[f(T,\rho,\xi)Gg(T,\rho,\xi)\big] = \hat{\mathrm{E}}_\varepsilon\big[Gf(T,\rho,\xi)g(T,\rho,\xi)\big] \tag{3.2.4}$$

for $\hat{\mathrm{P}}_\varepsilon$-square-integrable functions f, g, see Lemma 3.1 in [GMPV12]. In particular, substituting $g \equiv 1$ in (3.2.4) shows that $\hat{\mathrm{P}}_\varepsilon$ is an invariant measure for the environment observed by the particle. Then the sequence $(T, X_n, \xi)_n$ of the tree seen from the random walk is stationary under $\hat{\mathbb{P}}_\varepsilon$.

Note that the measures introduced in this section are also well-defined for $\varepsilon = 0$ and (3.2.4) holds as well. That is, the measure $\hat{\mathrm{P}}_0$ is still invariant for the environment seen from the random walk. However, it is not the unique invariant measure. For example, a measure under which the event $\{C(\rho) = 0\}$ has probability 1 is trivially invariant. The following lemma shows that the measure $\hat{\mathrm{P}}_0$ is the weak limit of $\hat{\mathrm{P}}_\varepsilon$. It is therefore the correct measure to consider for the limit of the speed.

Lemma 3.2.1. *As $\varepsilon \to 0$, we have $\hat{\mathrm{P}}_\varepsilon \xrightarrow{w} \hat{\mathrm{P}}_0$ and $\hat{\mathbb{P}}_\varepsilon \xrightarrow{w} \hat{\mathbb{P}}_0$ weakly.*

For vertices $u, v, z \in T$, we define the signed distance from u to v with respect to z as

$$[u - v]_z = d_T(u,z) - d_T(v,z), \tag{3.2.5}$$

where $d_T(u,z)$ denotes the graph distance between the vertices u and z in the tree T. The signed distance indicates how many steps more are required from z to u than from z to v. In particular, $[u-v]_z$ does not change when the reference vertex z is replaced by a vertex $\tilde{z}$ provided that the path from z to $\tilde{z}$ is disjoint from the path from u to v. If $(x_{-n})_{n\geq 0}$ is a transient path on T, each vertex of the path from u to v is visited only a finite number of times. Consequently, $[u-v]_{x_{-n}}$ is constant for n sufficiently large and the following limit is well-defined:

$$[u-v]_{x_{-\infty}} = \lim_{n\to\infty}[u-v]_{x_{-n}}. \tag{3.2.6}$$

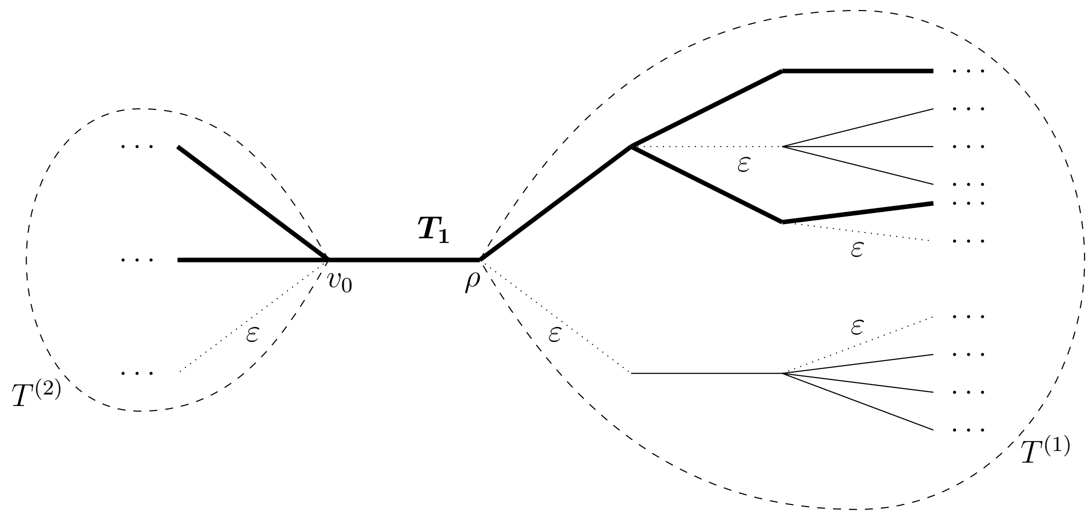


Figure 3.2: T_1 is the subtree formed by the edges with conductance larger than ε containing the root. $T^{(1)}$ and $T^{(2)}$ are independent and identically distributed weighted Galton-Watson trees (see Figure 3.1) and edges with conductance ε are indicated by dotted lines.

Here, $x_{-\infty} = (x_{-\infty,0}, x_{-\infty,1}, \dots)$ denotes the boundary point towards which $(x_{-n})_{n\geq 0}$ converges, that is, $x_{-\infty}$ is the infinite self-avoiding path that intersects $(x_{-n})_{n\geq 0}$ infinitely often. Note that $x_{-\infty}$ defines a measurable function of the trajectory, but it is not continuous. The limit $[u-v]_{x_{-\infty}}$ is called the horodistance from u to v relative to the boundary point $x_{-\infty}$. As observed in [LPP95], and applied to our setting by [GMPV12] (see (5.1) therein), the ergodicity of the environment seen from the particle under $\hat{\mathbb{P}}_\varepsilon$ implies that the speed is given by

$$\lim_{n\to\infty} \frac{|X_n|}{n} = \hat{\mathbb{E}}_\varepsilon\big[[X_1 - X_0]_{X_{-\infty}}\big]. \tag{3.2.7}$$

The limit holds $\mathbb{P}_\varepsilon$-almost surely, or equivalently $\hat{\mathbb{P}}_\varepsilon$-almost surely, for any $\varepsilon > 0$. Note that the boundary point $X_{-\infty}$ and thus also the distance $[X_1 - X_0]_{X_{-\infty}}$ is only well-defined if the random walk in negative time $(X_{-n})_{n\geq 0}$ is transient. For any $\varepsilon > 0$ the underlying tree is infinite and the random walk is almost surely transient, see Proposition 2.3.4. Whereas, if $\varepsilon = 0$, the random walk can only move on a subtree of the original tree, which might be finite. If this is the case, the random walk in negative time cannot be transient. However, if this subtree survives, $[X_1 - X_0]_{X_{-\infty}}$ is still well-defined for $\varepsilon = 0$.

We let T_1 be the subtree that contains the root after removing all edges with conductance ε, see Figure 3.2. That is, if $\varepsilon = 0$, T_1 denotes the traversable tree. In order to determine the limit of the speed for $\varepsilon \to 0$, we distinguish whether the subtree T_1 is finite or infinite. In view of (3.2.7), for $\varepsilon > 0$ the speed is then given by

$$v(\nu, \mu_\varepsilon) = \hat{\mathbb{E}}_\varepsilon\big[[X_1 - X_0]_{X_{-\infty}} \mathbb{1}_{\{|T_1|=\infty\}}\big] + \hat{\mathbb{E}}_\varepsilon\big[[X_1 - X_0]_{X_{-\infty}} \mathbb{1}_{\{|T_1|<\infty\}}\big]. \tag{3.2.8}$$

In order to determine the limit of the speed, we study both expectations separately. Their limits are given in the following two propositions.

Proposition 3.2.2. *On the event $\{|T_1| = \infty\}$, the mean of $[X_1 - X_0]_{X_{-\infty}}$ converges to its mean under $\hat{\mathbb{P}}_0$,*

$$\lim_{\varepsilon \searrow 0} \hat{\mathbb{E}}_\varepsilon\big[[X_1 - X_0]_{X_{-\infty}} \mathbb{1}_{\{|T_1|=\infty\}}\big] = \hat{\mathbb{E}}_0\big[[X_1 - X_0]_{X_{-\infty}} \mathbb{1}_{\{|T_1|=\infty\}}\big].$$

Proposition 3.2.3. *On the event $\{|T_1| < \infty\}$, the mean of $[X_1 - X_0]_{X_{-\infty}}$ vanishes,*

$$\lim_{\varepsilon \searrow 0} \hat{\mathbb{E}}_\varepsilon\big[[X_1 - X_0]_{X_{-\infty}} \mathbb{1}_{\{|T_1|<\infty\}}\big] = 0.$$

The next lemma provides a formula for the speed for $\varepsilon = 0$.

Lemma 3.2.4. *If T_1 is supercritical, the speed for $\varepsilon = 0$ is given by*

$$v(\nu, \mu_0) = \hat{\mathbb{E}}_0\big[[X_1 - X_0]_{X_{-\infty}} \mid |T_1| = \infty\big],$$

which is strictly positive.

Combining the last two propositions with the representation for the speed in (3.2.8) and Lemma 3.2.4 implies Theorem 3.1.2.

3.3 Proofs

3.3.1 Continuity of the speed on the set of uniformly elliptic measures: proof of Proposition 3.1.1

We let $(\mu_n)_{n\in\mathbb{N}}$ and μ be uniformly elliptic measures with common ellipticity constant $\kappa \geq 1$ such that $\mu_n \xrightarrow{w} \mu$ weakly. We denote the law of the augmented Galton-Watson tree by $\mathrm{P}_n^{\mathrm{aug}}$ and $\mathrm{P}^{\mathrm{aug}}$ if the marginal law of the conductances is given by μ_n and μ, respectively.

It was proven by [GMPV12] that the speed can be expressed as an expectation of a ratio of effective conductances

$$v(\nu, \mu_n) = 1 - \frac{2}{\gamma_n} \mathrm{E}_n^{\mathrm{aug}}\left[\xi_0 \frac{\mathcal{C}_{\omega^*}(\rho, \infty)}{\mathcal{C}_\omega(\rho, \infty)}\right], \tag{3.3.1}$$

see Theorem 4.1 in [GMPV12]. Here, $\gamma_n = \int x \,\mathrm{d}\mu_n(x)$ denotes the mean conductance of an edge and ξ_0 is the conductance of the additional edge (ρ, v_0) in the augmented tree. Moreover, T^* denotes the subtree composed of $T^{(2)}$ and the additional edge and we write $\omega^* = (T^*, \rho, (\xi(e))_{e\in\mathcal{E}(T^*)})$ for the corresponding environment, see Figure 3.3. In order to determine the limit of the speed as n tends to infinity, we first note that from the uniform ellipticity of μ_n and the weak convergence $\mu_n \xrightarrow{w} \mu$ we get

$$\lim_{n\to\infty} \gamma_n = \lim_{n\to\infty} \int x \,\mathrm{d}\mu_n(x) = \int x \,\mathrm{d}\mu(x) = \gamma. \tag{3.3.2}$$

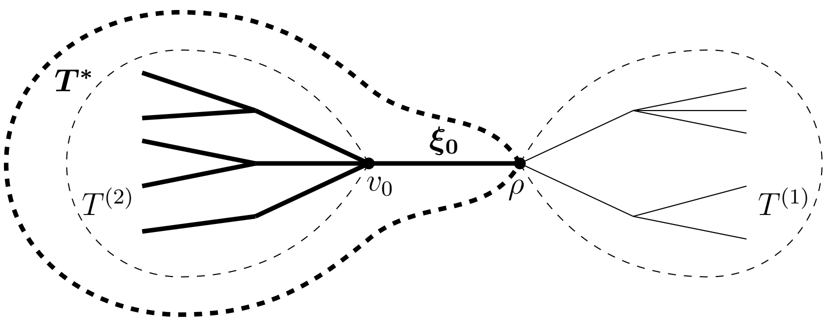


Figure 3.3: The subtree T^* consists of the tree $T^{(2)}$ and the additional edge (ρ, v_0) (indicated by the thick lines). $T^{(1)}$ and $T^{(2)}$ are independent and identically distributed weighted Galton-Watson trees (see Figure 3.1).

Next, we consider the expectation in (3.3.1). Analogously to the construction of the environment law in Section 2.2, the augmented law $\mathrm{P}_n^{\mathrm{aug}}$ is the pushforward measure of $\tilde{\mathrm{P}}_n^{\mathrm{aug}} = \mathrm{GW}^{\mathrm{aug}} \otimes \mu_n^{\otimes \mathcal{E}(\mathbb{T})}$ under π, recall (2.2.3). Here, $\mathrm{GW}^{\mathrm{aug}}$ denotes the law on $\mathcal{T}$ such that under $\mathrm{GW}^{\mathrm{aug}}$, T is an augmented Galton-Watson tree. Since the effective conductance only depends on the conductance configuration of the tree T, we can take the expectation with respect to $\tilde{\mathrm{P}}_n^{\mathrm{aug}}$ without changing its value. This implies

$$\begin{aligned}\mathrm{E}_n^{\mathrm{aug}}\left[\xi_0 \frac{\mathcal{C}_{\omega^*}(\rho,\infty)}{\mathcal{C}_\omega(\rho,\infty)}\right] = \tilde{\mathrm{E}}_n^{\mathrm{aug}}\left[\xi_0 \frac{\mathcal{C}_{\omega^*}(\rho,\infty)}{\mathcal{C}_\omega(\rho,\infty)}\right] &= \tilde{\mathrm{E}}_n^{\mathrm{aug}}\left[\tilde{\mathrm{E}}_n^{\mathrm{aug}}\left[\xi_0 \frac{\mathcal{C}_{\omega^*}(\rho,\infty)}{\mathcal{C}_\omega(\rho,\infty)} \;\middle|\; T\right]\right] \\ &= \tilde{\mathrm{E}}^{\mathrm{aug}}\left[\tilde{\mathrm{E}}_n^{\mathrm{aug}}\left[\xi_0 \frac{\mathcal{C}_{\omega^*}(\rho,\infty)}{\mathcal{C}_\omega(\rho,\infty)} \;\middle|\; T\right]\right],\end{aligned}$$

where we used for the last equality that the distribution of T is independent of n. Together with (3.3.2), we arrive at

$$\lim_{n\to\infty} v(\nu,\mu_n) = 1 - \frac{2}{\gamma} \lim_{n\to\infty} \tilde{\mathrm{E}}^{\mathrm{aug}}\left[\tilde{\mathrm{E}}_n^{\mathrm{aug}}\left[\xi_0 \frac{\mathcal{C}_{\omega^*}(\rho,\infty)}{\mathcal{C}_\omega(\rho,\infty)} \;\middle|\; T\right]\right]. \tag{3.3.3}$$

By Rayleigh's Monotonicity Principle (Lemma 2.4.5) we have $\mathcal{C}_{\omega^*}(\rho,\infty) \le \mathcal{C}_\omega(\rho,\infty)$, since T^* is a subtree of T. Together with the uniform ellipticity of μ_n, this implies that conditional expectation inside is bounded by κ. Consequently, the proof is complete once we have shown that the sequence of the conditional expectations converges almost surely, i.e.

$$\tilde{\mathrm{E}}_n^{\mathrm{aug}}\left[\xi_0 \frac{\mathcal{C}_{\omega^*}(\rho,\infty)}{\mathcal{C}_\omega(\rho,\infty)} \;\middle|\; T\right] \xrightarrow[n\to\infty]{a.s.} \tilde{\mathrm{E}}^{\mathrm{aug}}\left[\xi_0 \frac{\mathcal{C}_{\omega^*}(\rho,\infty)}{\mathcal{C}_\omega(\rho,\infty)} \;\middle|\; T\right]. \tag{3.3.4}$$

Dominated convergence then implies

$$\lim_{n\to\infty} v(\nu,\mu_n) = 1 - \frac{2}{\gamma} \tilde{\mathrm{E}}^{\mathrm{aug}}\left[\tilde{\mathrm{E}}^{\mathrm{aug}}\left[\xi_0 \frac{\mathcal{C}_{\omega^*}(\rho,\infty)}{\mathcal{C}_\omega(\rho,\infty)} \;\middle|\; T\right]\right] = 1 - \frac{2}{\gamma} \tilde{\mathrm{E}}^{\mathrm{aug}}\left[\xi_0 \frac{\mathcal{C}_{\omega^*}(\rho,\infty)}{\mathcal{C}_\omega(\rho,\infty)}\right] = v(\nu,\mu).$$

Let us prove (3.3.4). The Doob-Dynkin lemma implies that there exist measurable functions h_n, $h\colon \mathcal{T} \to \mathbb{R}$ such that

$$\tilde{\mathrm{E}}_n^{\mathrm{aug}}\left[\xi_0 \frac{\mathcal{C}_{\omega^*}(\rho,\infty)}{\mathcal{C}_{\omega}(\rho,\infty)} \,\middle|\, T\right] = h_n(T), \quad \tilde{\mathrm{E}}^{\mathrm{aug}}\left[\xi_0 \frac{\mathcal{C}_{\omega^*}(\rho,\infty)}{\mathcal{C}_{\omega}(\rho,\infty)} \,\middle|\, T\right] = h(T).$$

Under $\tilde{\mathrm{P}}_n^{\mathrm{aug}}$ (and $\tilde{\mathrm{P}}^{\mathrm{aug}}$), the conductances are independent of the tree and therefore we obtain for almost all trees $t \in \mathcal{T}$

$$h_n(t) = \tilde{\mathrm{E}}_n^{\mathrm{aug}}\left[\xi_0 \frac{\mathcal{C}_{(t^*,\rho,\xi)}(\rho,\infty)}{\mathcal{C}_{(t,\rho,\xi)}(\rho,\infty)}\right], \quad h(t) = \tilde{\mathrm{E}}^{\mathrm{aug}}\left[\xi_0 \frac{\mathcal{C}_{(t^*,\rho,\xi)}(\rho,\infty)}{\mathcal{C}_{(t,\rho,\xi)}(\rho,\infty)}\right], \tag{3.3.5}$$

see e.g. Corollary 4.38 in [Bre92]. The weak convergence of the marginal law of the conductances implies

$$\tilde{\mathrm{P}}_n^{\mathrm{aug}} = \mathrm{GW}^{\mathrm{aug}} \otimes \mu_n^{\otimes \mathcal{E}(\mathbb{T})} \xrightarrow[n\to\infty]{w} \mathrm{GW}^{\mathrm{aug}} \otimes \mu^{\otimes \mathcal{E}(\mathbb{T})} = \tilde{\mathrm{P}}^{\mathrm{aug}}.$$

Hence, it remains to show that the mapping $\xi \mapsto \xi_0 \frac{\mathcal{C}_{(t^*,\rho,\xi)}(\rho,\infty)}{\mathcal{C}_{(t,\rho,\xi)}(\rho,\infty)}$ is continuous and bounded. Then $h_n(t) \to h(t)$ follows for almost all trees $t \in \mathcal{T}$, that is, $h_n(T) \to h(T)$ almost surely.

As above, the boundedness is a direct consequence of $\mathcal{C}_{(t^*,\rho,\xi)}(\rho,\infty) \le \mathcal{C}_{(t,\rho,\xi)}(\rho,\infty)$ and the uniform ellipticity of μ_n.

To show the continuity we let $G_n(t)$ be the n-th generation of a tree $t \in \mathcal{T}$. Dirichlet's Principle (see e.g. [LP16]) implies that the effective conductance $\mathcal{C}_{(t,\rho,\xi)}(\rho, G_n(t))$ can be expressed as the minimum of linear functions in ξ, thus it is concave on $\{(\xi(e))_{e\in\mathcal{E}(t)} : \xi(e) > 0 \,\forall e \in \mathcal{E}(t)\}$. Hence, its limit, the effective conductance from the root to infinity

$$\mathcal{C}_{(t,\rho,\xi)}(\rho,\infty) = \lim_{n\to\infty} \mathcal{C}_{(t,\rho,\xi)}(\rho, G_n(t)),$$

is also concave. Consequently, the mapping

$$\left\{(\xi(e))_{e\in\mathcal{E}(t)} : \xi(e) > 0 \,\forall e \in \mathcal{E}(t)\right\} \to (-\infty, 0], \quad \xi \mapsto -\mathcal{C}_{(t,\rho,\xi)}(\rho,\infty)$$

is convex and bounded from above, which implies that it is continuous (see e.g. [RV73, Section 41]). This shows that both effective conductances in (3.3.5) are continuous, and so the mapping $\xi \mapsto \xi_0 \frac{\mathcal{C}_{(t^*,\rho,\xi)}(\rho,\infty)}{\mathcal{C}_{(t,\rho,\xi)}(\rho,\infty)}$ is continuous.

To see that this completes the proof of Proposition 3.1.1, let us briefly summarize the results. From the continuity and boundedness of the mapping $\xi \mapsto \xi_0 \frac{\mathcal{C}_{(t^*,\rho,\xi)}(\rho,\infty)}{\mathcal{C}_{(t,\rho,\xi)}(\rho,\infty)}$ and the weak convergence $\tilde{\mathrm{P}}_n^{\mathrm{aug}} \xrightarrow{w} \tilde{\mathrm{P}}^{\mathrm{aug}}$ we get the almost sure convergence of the sequence of conditional expectations in (3.3.3). Dominated convergence then implies

$$v(\nu, \mu_n) \xrightarrow[n\to\infty]{} v(\nu, \mu).$$

$\square$

3.3.2 The limit of the speed for vanishing conductances: proof of Theorem 3.1.2

As mentioned in (3.2.7), for any $\varepsilon > 0$ the speed can be expressed as the expectation of $[X_1 - X_0]_{X_{-\infty}}$ under the invariant measure $\hat{\mathbb{P}}_\varepsilon$, i.e.

$$v(\nu, \mu_\varepsilon) = \hat{\mathbb{E}}_\varepsilon\big[[X_1 - X_0]_{X_{-\infty}}\big]. \tag{3.3.6}$$

First, we note that the convergence of the speed is not directly implied by the weak convergence of the invariant measures $\hat{\mathbb{P}}_\varepsilon \xrightarrow{w} \hat{\mathbb{P}}_0$ in Lemma 3.2.1, since $[X_1 - X_0]_{X_{-\infty}}$ is not a continuous function of the trajectory of the bi-infinite random walk. Moreover, as explained in Section 3.2 we have to be carefully since the limit $X_{-\infty}$ is possibly not well-defined if $\varepsilon = 0$. We therefore distinguish whether the subtree T_1 is finite or infinite,

$$v(\nu, \mu_\varepsilon) = \hat{\mathbb{E}}_\varepsilon\big[[X_1 - X_0]_{X_{-\infty}} \mathbb{1}_{\{|T_1| = \infty\}}\big] + \hat{\mathbb{E}}_\varepsilon\big[[X_1 - X_0]_{X_{-\infty}} \mathbb{1}_{\{|T_1| < \infty\}}\big].$$

As stated in Proposition 3.2.2 the first summand converges to the corresponding mean under $\hat{\mathbb{P}}_0$. Proposition 3.2.3 implies that the second expectation vanishes as $\varepsilon \to 0$. Consequently, we have

$$\begin{aligned}\lim_{\varepsilon \to 0} v(\nu, \mu_\varepsilon) &= \hat{\mathbb{E}}_0\big[[X_1 - X_0]_{X_{-\infty}} \mathbb{1}_{\{|T_1| = \infty\}}\big] \\ &= \hat{\mathbb{E}}_0\big[[X_1 - X_0]_{X_{-\infty}} \,\big|\, |T_1| = \infty\big] \hat{\mathrm{P}}_0(|T_1| = \infty).\end{aligned}$$

By Lemma 3.2.4 we get

$$\lim_{\varepsilon \to 0} v(\nu, \mu_\varepsilon) = v(\nu, \mu_0) \hat{\mathrm{P}}_0(|T_1| = \infty),$$

which concludes the proof. □

It remains to show Proposition 3.2.2, Proposition 3.2.3 and Lemma 3.2.4.

Proof of Proposition 3.2.2

We assume the tree T_1 to be supercritical, otherwise the statement is trivial. The idea of the proof is to make use of the weak convergence of the invariant measures $\hat{\mathbb{P}}_\varepsilon \xrightarrow{w} \hat{\mathbb{P}}_0$. Unfortunately, $[X_1 - X_0]_{X_{-\infty}} \mathbb{1}_{\{|T_1| = \infty\}}$ is not a continuous function of the trajectory of the random walk. We therefore introduce

$$D_M = [X_1 - X_0]_{X_{-M}} = d_T(X_1, X_{-M}) - d_T(X_0, X_{-M}) \tag{3.3.7}$$

as an approximation for $D_\infty = [X_1 - X_0]_{X_{-\infty}}$ which is continuous, since it only depends on finitely many coordinates. Moreover, we approximate $\mathbb{1}_{\{|T_1| = \infty\}}$ by the indicator functions $\mathbb{1}_{\{|T_1| > N\}}$. We then write

$$\begin{aligned}\hat{\mathbb{E}}_\varepsilon\big[D_\infty \mathbb{1}_{\{|T_1| = \infty\}}\big] = \hat{\mathbb{E}}_\varepsilon\big[D_M \mathbb{1}_{\{|T_1| > N\}}\big] &+ \hat{\mathbb{E}}_\varepsilon\big[D_M\big(\mathbb{1}_{\{|T_1| = \infty\}} - \mathbb{1}_{\{|T_1| > N\}}\big)\big] \\ &+ \hat{\mathbb{E}}_\varepsilon\big[(D_\infty - D_M)\mathbb{1}_{\{|T_1| = \infty\}}\big].\end{aligned} \tag{3.3.8}$$

We will now treat the three summands separately.

First summand in (3.3.8)**:** The random variable $D_M \mathbb{1}_{\{|T_1|>N\}}$ is uniformly bounded and it depends only on a finite number of generations and on finitely many coordinates, thus it is a continuous function. The weak convergence of the invariant measures in Lemma 3.2.1 implies

$$\lim_{\varepsilon\to 0} \hat{\mathbb{E}}_\varepsilon\big[D_M \mathbb{1}_{\{|T_1|>N\}}\big] = \hat{\mathbb{E}}_0\big[D_M \mathbb{1}_{\{|T_1|>N\}}\big].$$

By definition (3.2.6), D_M converges almost surely to D_∞ on any infinite tree T_1. Hence, we obtain

$$\lim_{M\to\infty} \lim_{N\to\infty} \lim_{\varepsilon\to 0} \hat{\mathbb{E}}_\varepsilon\big[D_M \mathbb{1}_{\{|T_1|>N\}}\big] = \lim_{M\to\infty} \hat{\mathbb{E}}_0\big[D_M \mathbb{1}_{\{|T_1|=\infty\}}\big] = \hat{\mathbb{E}}_0\big[D_\infty \mathbb{1}_{\{|T_1|=\infty\}}\big] \tag{3.3.9}$$

by dominated convergence.

Second summand in (3.3.8)**:** Since $|D_M| \le 1$, we have the following bound for the second expectation in (3.3.8)

$$\big|\hat{\mathbb{E}}_\varepsilon\big[D_M\big(\mathbb{1}_{\{|T_1|=\infty\}} - \mathbb{1}_{\{|T_1|>N\}}\big)\big]\big| \le \hat{\mathrm{E}}_\varepsilon\big[\mathbb{1}_{\{|T_1|>N\}} - \mathbb{1}_{\{|T_1|=\infty\}}\big].$$

Recalling the definition of $\hat{\mathrm{P}}_\varepsilon$ in (3.2.2), we obtain

$$\begin{aligned}\big|\hat{\mathbb{E}}_\varepsilon\big[D_M\big(\mathbb{1}_{\{|T_1|=\infty\}} - \mathbb{1}_{\{|T_1|>N\}}\big)\big]\big| &\le \mathrm{E}^{\mathrm{aug}}_\varepsilon\left[\big(\mathbb{1}_{\{|T_1|>N\}} - \mathbb{1}_{\{|T_1|=\infty\}}\big)\frac{C(\rho)}{\gamma_\varepsilon \deg(\rho)}\right] \\ &\le \frac{\kappa}{\gamma_\varepsilon}\mathrm{E}^{\mathrm{aug}}_\varepsilon\big[\mathbb{1}_{\{|T_1|>N\}} - \mathbb{1}_{\{|T_1|=\infty\}}\big],\end{aligned}$$

where we used for the last inequality that $C(\rho)$ is bounded by $\kappa \deg(\rho)$. We observe that the distribution of the indicator functions does not depend on the value ε, since T_1 is the subtree formed by the edges with conductance at least κ^{-1}. This implies

$$\mathrm{E}^{\mathrm{aug}}_\varepsilon\big[\mathbb{1}_{\{|T_1|>N\}} - \mathbb{1}_{\{|T_1|=\infty\}}\big] = \mathrm{E}^{\mathrm{aug}}_0\big[\mathbb{1}_{\{|T_1|>N\}} - \mathbb{1}_{\{|T_1|=\infty\}}\big]$$

and therefore, since $\gamma_\varepsilon \to \gamma_0 > 0$ as $\varepsilon \to 0$,

$$\lim_{N\to\infty} \limsup_{\varepsilon\to 0} \big|\hat{\mathbb{E}}_\varepsilon\big[D_M\big(\mathbb{1}_{\{|T_1|=\infty\}} - \mathbb{1}_{\{|T_1|>N\}}\big)\big]\big| \le \frac{\kappa}{\gamma_0} \lim_{N\to\infty} \mathrm{E}^{\mathrm{aug}}_0\big[\mathbb{1}_{\{|T_1|>N\}} - \mathbb{1}_{\{|T_1|=\infty\}}\big] = 0 \tag{3.3.10}$$

by dominated convergence.

Third summand in (3.3.8)**:** In view of (3.3.8) and the limits (3.3.9) and (3.3.10), the proof is complete once we have shown that the third summand in (3.3.8) vanishes as $M \to \infty$, uniformly in ε, that is

$$\lim_{M\to\infty} \limsup_{\varepsilon\to 0} \hat{\mathbb{E}}_\varepsilon\big[|D_\infty - D_M| \mathbb{1}_{\{|T_1|=\infty\}}\big] = 0. \tag{3.3.11}$$

Recalling the definition of D_M in (3.3.7), we distinguish where the random walk in negative time is located after M steps. Since $|D_\infty - D_M| \leq 2$, we have

$$\begin{aligned}\hat{\mathbb{E}}_\varepsilon\big[|D_\infty - D_M|\mathbb{1}_{\{|T_1|=\infty\}}\big] &\leq \hat{\mathbb{E}}_\varepsilon\big[|D_\infty - D_M|\mathbb{1}_{\{|T_1|=\infty, X_{-M}\in T_1, X_{-M}\neq\rho\}}\big] \\ &\quad + 2\hat{\mathbb{P}}_\varepsilon(X_{-M}\notin T_1) + 2\hat{\mathbb{P}}_\varepsilon(X_{-M}=\rho, |T_1|=\infty). \qquad (3.3.12)\end{aligned}$$

We start with studying the second summand. The indicator function $\mathbb{1}_{\{X_{-M}\notin T_1\}}$ only depends on the first M coordinates, thus it is a continuous and bounded function. The weak convergence of the invariant measures in Lemma 3.2.1 implies

$$\hat{\mathbb{P}}_\varepsilon(X_{-M}\notin T_1) \xrightarrow[\varepsilon\to 0]{} \hat{\mathbb{P}}_0(X_{-M}\notin T_1) = 0. \qquad (3.3.13)$$

Let us turn to the third term in (3.3.12). As before, we approximate $\mathbb{1}_{\{|T_1|=\infty\}}$ by the continuous functions $\mathbb{1}_{\{|T_1|>N\}}$. Using the weak convergence of the invariant measures again, we obtain

$$\begin{aligned}\limsup_{\varepsilon\to 0}\hat{\mathbb{P}}_\varepsilon(X_{-M}=\rho, |T_1|=\infty) &\leq \limsup_{\varepsilon\to 0}\hat{\mathbb{P}}_\varepsilon(X_{-M}=\rho, |T_1|>N) \\ &= \hat{\mathbb{P}}_0(X_{-M}=\rho, |T_1|>N)\end{aligned}$$

and therefore, letting $N\to\infty$,

$$\limsup_{\varepsilon\to 0}\hat{\mathbb{P}}_\varepsilon(X_{-M}=\rho, |T_1|=\infty) \leq \hat{\mathbb{P}}_0(X_{-M}=\rho, |T_1|=\infty) \xrightarrow[M\to\infty]{} 0. \qquad (3.3.14)$$

Note that the probability on the right-hand side vanishes, since the random walk in negative time is transient under $\hat{\mathbb{P}}_0$, conditioned on $\{|T_1|=\infty\}$.

Combining (3.3.12) with (3.3.13) and (3.3.14), it remains to show

$$\lim_{M\to\infty}\limsup_{\varepsilon\to 0}\hat{\mathbb{E}}_\varepsilon\big[|D_\infty - D_M|\mathbb{1}_{\{|T_1|=\infty, X_{-M}\in T_1, X_{-M}\neq\rho\}}\big] = 0. \qquad (3.3.15)$$

On the event that $X_{-k}\neq\rho$ for all $k\geq M$, the $X_{-M}, X_{-M-1}, \ldots$ are all vertices in the same subtree of a descendant of ρ, which implies $D_M = D_\infty$. Hence, the expectation above can be bounded as follows

$$\begin{aligned}&\hat{\mathbb{E}}_\varepsilon\big[|D_\infty - D_M|\mathbb{1}_{\{|T_1|=\infty, X_{-M}\in T_1, X_{-M}\neq\rho\}}\big] \\ &\qquad \leq 2\hat{\mathbb{P}}_\varepsilon\big(X_{-k}=\rho \text{ for some } k\geq M, |T_1|=\infty, X_{-M}\in T_1, X_{-M}\neq\rho\big). \qquad (3.3.16)\end{aligned}$$

This means that we need to bound the probability of returning to the root after a large time M, uniformly in ε. We have

$$\begin{aligned}&\hat{\mathbb{P}}_\varepsilon\big(X_{-k}=\rho \text{ for some } k\geq M, |T_1|=\infty, X_{-M}\in T_1, X_{-M}\neq\rho\big) \\ &\qquad = \hat{\mathrm{E}}_\varepsilon\left[\sum_{v\in T_1,\, v\neq\rho}\hat{P}_\omega(X_{-k}=\rho \text{ for some } k\geq M, X_{-M}=v)\mathbb{1}_{\{|T_1|=\infty\}}\right]. \qquad (3.3.17)\end{aligned}$$

Lemma 2.4.1 yields for $v \neq \rho$

$$\begin{aligned}\hat{P}_\omega(X_{-k} = \rho \text{ for some } k \geq M \mid X_{-M} = v) &= \hat{P}^v_\omega(X_{-k} = \rho \text{ for some } k \geq 0) \\ &= \hat{P}^v_\omega(\eta_\rho < \infty) \\ &\leq \frac{\mathcal{C}_\omega(v,\rho)}{\mathcal{C}_\omega(v,\infty)}.\end{aligned}$$

Plugging this bound in (3.3.17), we obtain

$$\begin{aligned}&\hat{\mathbb{P}}_\varepsilon\big(X_{-k} = \rho \text{ for some } k \geq M, |T_1| = \infty, X_{-M} \in T_1, X_{-M} \neq \rho\big) \\ &\quad\leq \hat{\mathrm{E}}_\varepsilon\left[\sum_{v \in T_1,\, v \neq \rho} \frac{\mathcal{C}_\omega(v,\rho)}{\mathcal{C}_\omega(v,\infty)} \hat{P}_\omega(X_{-M} = v) \mathbb{1}_{\{|T_1|=\infty\}}\right] \\ &\quad= \hat{\mathbb{E}}_\varepsilon\left[\frac{\mathcal{C}_\omega(X_{-M},\rho)}{\mathcal{C}_\omega(X_{-M},\infty)} \mathbb{1}_{\{|T_1|=\infty, X_{-M}\in T_1, X_{-M}\neq\rho\}}\right],\end{aligned}$$

where the equality holds due to Lemma 2.3.2. Using the Cauchy-Schwarz inequality, we arrive at

$$\begin{aligned}&\hat{\mathbb{P}}_\varepsilon\big(X_{-k} = \rho \text{ for some } k \geq M, |T_1| = \infty, X_{-M} \in T_1, X_{-M} \neq \rho\big)^2 \\ &\quad\leq \hat{\mathbb{E}}_\varepsilon\big[\mathcal{C}_\omega(X_{-M},\rho)^2 \mathbb{1}_{\{|T_1|=\infty, X_{-M}\neq\rho\}}\big] \hat{\mathbb{E}}_\varepsilon\big[\mathcal{C}_\omega(X_{-M},\infty)^{-2} \mathbb{1}_{\{X_{-M}\in T_1, |T_1|=\infty\}}\big]. \quad (3.3.18)\end{aligned}$$

Concerning the first expectation, we note that the effective conductance between the root and X_{-M} only depends on the path between these two vertices. Hence, the approximation $\mathcal{C}_\omega(X_{-M},\rho)\mathbb{1}_{\{|T_1|>N, X_{-M}\neq\rho\}}$ is continuous. Moreover, by Rayleigh's Monotonicity Principle (Lemma 2.4.5) in combination with the Series Law (Lemma 2.4.4) and the boundedness of the conductances we get

$$\mathcal{C}_\omega(X_{-M},\rho) \leq \kappa |X_{-M}|^{-1} \leq \kappa. \quad (3.3.19)$$

The weak convergence of the invariant measures in Lemma 3.2.1 then implies

$$\begin{aligned}\limsup_{\varepsilon\to 0} \hat{\mathbb{E}}_\varepsilon\big[\mathcal{C}_\omega(X_{-M},\rho)^2 \mathbb{1}_{\{|T_1|=\infty, X_{-M}\neq\rho\}}\big] &\leq \limsup_{\varepsilon\to 0} \hat{\mathbb{E}}_\varepsilon\big[\mathcal{C}_\omega(X_{-M},\rho)^2 \mathbb{1}_{\{|T_1|>N, X_{-M}\neq\rho\}}\big] \\ &= \hat{\mathbb{E}}_0\big[\mathcal{C}_\omega(X_{-M},\rho)^2 \mathbb{1}_{\{|T_1|>N, X_{-M}\neq\rho\}}\big].\end{aligned}$$

Letting $N \to \infty$, we obtain

$$\limsup_{\varepsilon\to 0} \hat{\mathbb{E}}_\varepsilon\big[\mathcal{C}_\omega(X_{-M},\rho)^2 \mathbb{1}_{\{|T_1|=\infty, X_{-M}\neq\rho\}}\big] \leq \hat{\mathbb{E}}_0\big[\mathcal{C}_\omega(X_{-M},\rho)^2 \mathbb{1}_{\{|T_1|=\infty, X_{-M}\neq\rho\}}\big]$$

by monotone convergence. Due to (3.3.19) and the transience of the random walk under $\hat{\mathbb{P}}_0$ conditioned on $\{|T_1| = \infty\}$, the effective conductance $\mathcal{C}_\omega(X_{-M},\rho)$ converges almost surely to zero as $M \to \infty$. By dominated convergence we get

$$\lim_{M\to\infty} \limsup_{\varepsilon\to 0} \hat{\mathbb{E}}_\varepsilon\big[\mathcal{C}_\omega(X_{-M},\rho)^2 \mathbb{1}_{\{|T_1|=\infty, X_{-M}\neq\rho\}}\big] = 0. \quad (3.3.20)$$

In view of (3.3.15) and (3.3.16), the proof is complete once we have shown that the second expectation in (3.3.18) remains bounded. Recall that $\mathcal{R}_\omega(v,\infty)=\mathcal{C}_\omega(v,\infty)^{-1}$ is the effective resistance from v to infinity. Since the process $(T,X_n,\xi)_{n\in\mathbb{Z}}$ is stationary under $\hat{\mathbb{P}}_\varepsilon$, we have

$$\begin{aligned}\hat{\mathbb{E}}_\varepsilon\big[\mathcal{R}_\omega(X_{-M},\infty)^2\mathbb{1}_{\{X_{-M}\in T_1,|T_1|=\infty\}}\big]&=\hat{\mathbb{E}}_\varepsilon\Big[\mathcal{R}_\omega(\rho,\infty)^2\mathbb{1}_{\{\rho\in T_1^{X_M},|T_1^{X_M}|=\infty\}}\Big]\\&=\hat{\mathbb{E}}_\varepsilon\big[\mathcal{R}_\omega(\rho,\infty)^2\mathbb{1}_{\{X_M\in T_1,|T_1|=\infty\}}\big]\\&\le\hat{\mathrm{E}}_\varepsilon\big[\mathcal{R}_\omega(\rho,\infty)^2\mathbb{1}_{\{|T_1|=\infty\}}\big],\end{aligned}$$

where T_1^v denotes the subtree of T that contains the vertex $v\in T$ after removing all edges with conductance ε. Note that $T_1^\rho=T_1$. Recalling the definition of the invariant measure $\hat{\mathrm{P}}_\varepsilon$ in (3.2.2), we can bound the above expectation by

$$\begin{aligned}\hat{\mathrm{E}}_\varepsilon\big[\mathcal{R}_\omega(\rho,\infty)^2\mathbb{1}_{\{|T_1|=\infty\}}\big]&=\mathrm{E}_\varepsilon^{\mathrm{aug}}\Big[\mathcal{R}_\omega(\rho,\infty)^2\mathbb{1}_{\{|T_1|=\infty\}}\frac{C(\rho)}{\gamma_\varepsilon\deg(\rho)}\Big]\\&\le\frac{\kappa}{\gamma_\varepsilon}\mathrm{E}_\varepsilon^{\mathrm{aug}}\big[\mathcal{R}_\omega(\rho,\infty)^2\mathbb{1}_{\{|T_1|=\infty\}}\big]\\&\le\frac{\kappa}{\gamma_\varepsilon}\mathrm{E}_\varepsilon^{\mathrm{aug}}\Big[\mathcal{R}_\omega(\rho,\infty)^2\Big(\mathbb{1}_{\{|T_1^{(1)}|=\infty\}}+\mathbb{1}_{\{|T_1^{(2)}|=\infty,\,\kappa^{-1}\le\xi(\rho,v_0)\le\kappa\}}\Big)\Big].\end{aligned}$$

Here, $T_1^{(i)}$ is the subtree of $T^{(i)}$ formed by the edges with conductance at least κ^{-1} that contains the root of $T^{(i)}$ $(i=1,\,2)$, see Figure 3.1 for the definition of $T^{(1)}$ and $T^{(2)}$. Furthermore, we denote the backbone tree of $T_1^{(i)}$ by $T_1^{(i),\mathrm{Bb}}$. The backbone of a tree is the subtree where all vertices that do not have an infinite line of descent are removed. The associated environment is called $\omega_1^{(i),\mathrm{Bb}}$. By Rayleigh's Monotonicity Principle (Lemma 2.4.5) removing edges can only increase the effective resistance, which implies

$$\mathcal{R}_\omega(\rho,\infty)\le\mathcal{R}_{\omega_1^{(1),\mathrm{Bb}}}(\rho,\infty)$$

and

$$\mathcal{R}_\omega(\rho,\infty)\le\mathcal{R}_{\omega_1^{(2),\mathrm{Bb}}}(v_0,\infty)+\xi(\rho,v_0)^{-1}\le\mathcal{R}_{\omega_1^{(2),\mathrm{Bb}}}(v_0,\infty)+\kappa.$$

Since $\mathcal{R}_{\omega_1^{(1),\mathrm{Bb}}}(\rho,\infty)\mathbb{1}_{\{|T_1^{(1)}|=\infty\}}$ and $\mathcal{R}_{\omega_1^{(2),\mathrm{Bb}}}(v_0,\infty)\mathbb{1}_{\{|T_1^{(2)}|=\infty\}}$ have the same distribution under $\mathrm{P}_\varepsilon^{\mathrm{aug}}$, which does not depend on ε, we obtain

$$\begin{aligned}&\hat{\mathrm{E}}_\varepsilon\big[\mathcal{R}_\omega(\rho,\infty)^2\mathbb{1}_{\{|T_1|=\infty\}}\big]\\&\quad\le\frac{\kappa}{\gamma_\varepsilon}\mathrm{E}_\varepsilon^{\mathrm{aug}}\big[\mathcal{R}_{\omega_1^{(1),\mathrm{Bb}}}(\rho,\infty)^2\mathbb{1}_{\{|T_1^{(1)}|=\infty\}}\big]+\frac{\kappa}{\gamma_\varepsilon}\mathrm{E}_\varepsilon^{\mathrm{aug}}\big[\big(\mathcal{R}_{\omega_1^{(2),\mathrm{Bb}}}(v_0,\infty)+\kappa\big)^2\mathbb{1}_{\{|T_1^{(2)}|=\infty\}}\big]\\&\quad\le\frac{2\kappa}{\gamma_\varepsilon}\mathrm{E}_0^{\mathrm{aug}}\big[\mathcal{R}_{\omega_1^{(1),\mathrm{Bb}}}(\rho,\infty)^2\mathbb{1}_{\{|T_1^{(1)}|=\infty\}}\big]+\frac{2\kappa^2}{\gamma_\varepsilon}\mathrm{E}_0^{\mathrm{aug}}\big[\mathcal{R}_{\omega_1^{(1),\mathrm{Bb}}}(\rho,\infty)\mathbb{1}_{\{|T_1^{(1)}|=\infty\}}\big]+\frac{\kappa^3}{\gamma_\varepsilon}.\end{aligned}$$

Under $\mathrm{P}_0^{\mathrm{aug}}$, $T_1^{(1)}$ is a supercritical Galton-Watson tree. Thus, conditioned on the survival of $T_1^{(1)}$, the backbone tree $T_1^{(1),\mathrm{Bb}}$ is a supercritical Galton-Watson tree without leaves (see e.g. Proposition 4.10 in [Lyo92]) and with uniformly elliptic conductances. Lemma 3.3.1 below then implies that the moments of the effective resistance are finite. Since $\gamma_\varepsilon \to \gamma_0 > 0$ as $\varepsilon \to 0$, we finally get

$$\limsup_{\varepsilon\to 0} \hat{\mathrm{E}}_\varepsilon\big[\mathcal{R}_\omega(\rho,\infty)^2 \mathbb{1}_{\{|T_1|=\infty\}}\big] \le \frac{2\kappa}{\gamma_0}\mathrm{E}_0^{\mathrm{aug}}\big[\mathcal{R}_{\omega_1^{(1),\mathrm{Bb}}}(\rho,\infty)^2 \bigm| |T_1^{(1)}| = \infty\big] + \frac{2\kappa^2}{\gamma_0}\big[\mathcal{R}_{\omega_1^{(1),\mathrm{Bb}}}(\rho,\infty) \bigm| |T_1^{(1)}| = \infty\big] + \frac{\kappa^3}{\gamma_0} \le C$$

for some constant $C = C(\nu, \alpha, \kappa) < \infty$. To conclude, this yields (3.3.15) and therefore (3.3.11).

In total, combining (3.3.8) with (3.3.9), (3.3.10) and (3.3.11) implies

$$\lim_{\varepsilon\to 0} \hat{\mathbb{E}}_\varepsilon\big[D_\infty \mathbb{1}_{\{|T_1|=\infty\}}\big] = \hat{\mathbb{E}}_0\big[D_\infty \mathbb{1}_{\{|T_1|=\infty\}}\big],$$

which is what we wanted to show. □

For the proof of Proposition 3.2.2 we need moment bounds on the effective resistance. The following lemma shows that the effective resistance of Galton-Watson trees has finite moments of any order.

Lemma 3.3.1. *Let T be a supercritical Galton-Watson tree without leaves and with uniformly elliptic conductances. Then for all $p > 0$ we have*

$$\mathrm{E}\big[\mathcal{R}_\omega(\rho,\infty)^p\big] < \infty.$$

Proof. Since the marginal distribution of the conductances is uniformly elliptic, there exists some constant $\kappa \ge 1$ such that the conductance of each edge is at least κ^{-1}. By Rayleigh's Monotonicity Principle (Lemma 2.4.5) the effective resistance increases when the conductances of all edges are reduced to κ^{-1}, which implies

$$\mathcal{R}_\omega(\rho,\infty) \le \mathcal{R}_{(T,\rho,(\kappa^{-1})_{e\in\mathcal{E}(T)})}(\rho,\infty) = \kappa\mathcal{R}_{(T,\rho,(1)_{e\in\mathcal{E}(T)})}(\rho,\infty). \tag{3.3.21}$$

Concerning the last equality, note that multiplying the conductance of each edge with the same positive value changes the effective conductance by the same factor. Hence, it suffices to show that the effective resistance of a Galton-Watson tree with unit conductances has finite moments of any order.

Let $\omega = (T, \rho, (1)_{e\in\mathcal{E}(T)})$ be an environment with unit conductances. We denote by N the number of offspring of the root and by $v_1, \ldots, v_N$ the vertices of the first generation. Furthermore, we let $T(v)$ be the subtree rooted at v composed of v and all its descendants and we write $\omega(v) = (T(v), v, (\xi(e))_{e\in\mathcal{E}(T(v))})$ for the corresponding environment. See Figure 3.4 for an example. Using the Parallel Law and the Series Law (Lemma 2.4.3 and

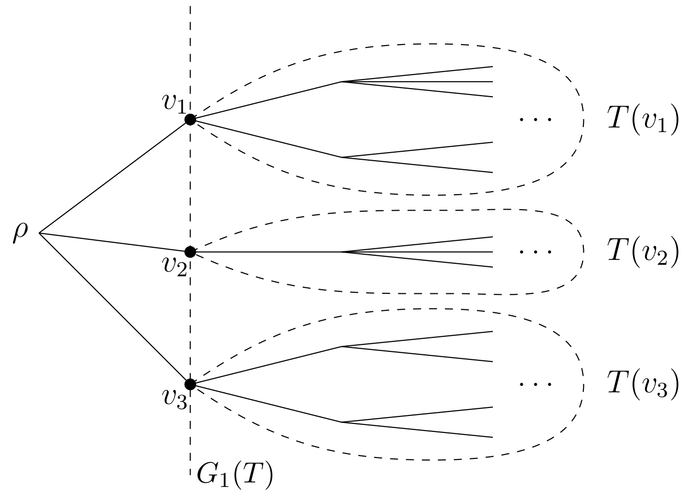


Figure 3.4: The vertices v_i in the first generation $G_1(T)$ are marked by dots. $T(v_i)$ is the subtree formed by v_i and all its descendants (marked by the dashed balloons).

2.4.4), we get for $n \geq 2$ the following recursion

$$\mathcal{R}_\omega(\rho, G_n(T)) = \left(\sum_{i=1}^{N} \frac{1}{1 + \mathcal{R}_\omega\big(v_i, G_{n-1}(T(v_i))\big)} \right)^{-1}.$$

For convenience, we write $\mathcal{R}_n = \mathcal{R}_\omega(\rho, G_n(T))$ for the effective resistance between the root and the n-th generation of T. Moreover, we write $X \preccurlyeq Y$ if a random variable Y stochastically dominates a random variable X. Since the subtrees $T(v_1), \dots, T(v_N)$ are i.i.d. Galton-Watson trees, we obtain

$$\mathcal{R}_n \preccurlyeq \left(\sum_{i=1}^{N} \frac{1}{1 + \mathcal{R}_{n-1}^{(i)}} \right)^{-1},$$

where $\mathcal{R}_{n-1}^{(1)}, \dots, \mathcal{R}_{n-1}^{(N)}$ denote independent copies of $\mathcal{R}_{n-1}$. Bounding the harmonic mean by the arithmetic mean gives rise to

$$\mathcal{R}_n \preccurlyeq \frac{1}{N^2} \sum_{i=1}^{N} \big(1 + \mathcal{R}_{n-1}^{(i)}\big) = \frac{1}{N} + \frac{1}{N^2} \sum_{i=1}^{N} \mathcal{R}_{n-1}^{(i)}. \tag{3.3.22}$$

The effective resistance between ρ and infinity is given by the limit of $\mathcal{R}_n$,

$$\mathcal{R}_\infty = \mathcal{R}_\omega(\rho, \infty) = \lim_{n \to \infty} \mathcal{R}_n.$$

We preform an induction to show that $\mathcal{R}_\infty$ has finite moments of any order. Lemma 9.1 in [LPP95] implies that the first moment $\mathrm{E}[\mathcal{R}_\infty]$ is bounded. Now, assume $\mathrm{E}[\mathcal{R}_\infty^{m-1}] < \infty$

for some $m \geq 2$. Using the stochastic domination in (3.3.22), we have

$$\mathrm{E}[\mathcal{R}_n^m] \leq \mathrm{E}\left[\frac{1}{N^m}\left(1 + \frac{1}{N}\sum_{i=1}^{N} \mathcal{R}_{n-1}^{(i)}\right)^m\right] = \mathrm{E}\left[\frac{1}{N^m}\sum_{k=0}^{m}\binom{m}{k}\left(\frac{1}{N}\sum_{i=1}^{N}\mathcal{R}_{n-1}^{(i)}\right)^k\right].$$

Applying Jensen's inequality, we obtain

$$\begin{aligned}
\mathrm{E}[\mathcal{R}_n^m] &\leq \sum_{k=0}^{m}\binom{m}{k}\mathrm{E}\left[N^{-m-1}\sum_{i=1}^{N}\left(\mathcal{R}_{n-1}^{(i)}\right)^k\right] \\
&= \sum_{k=0}^{m}\binom{m}{k}\mathrm{E}\left[N^{-m-1}\sum_{i=1}^{N}\mathrm{E}\Big[\left(\mathcal{R}_{n-1}^{(i)}\right)^k \,\Big|\, N\Big]\right] \\
&= \mathrm{E}[N^{-m}]\sum_{k=0}^{m}\binom{m}{k}\mathrm{E}[\mathcal{R}_{n-1}^k],
\end{aligned}$$

where we used in the last equality that the $\mathcal{R}_{n-1}^{(i)}$ are identically distributed and independent of N. Since the effective resistance $\mathcal{R}_n$ is monotone in n, we arrive at

$$\mathrm{E}[\mathcal{R}_n^m] \leq \mathrm{E}[N^{-m}]\sum_{k=0}^{m-1}\binom{m}{k}\mathrm{E}[\mathcal{R}_\infty^k] + \mathrm{E}[N^{-m}]\mathrm{E}[\mathcal{R}_{n-1}^m], \tag{3.3.23}$$

which holds for all $n \geq 2$. Note that we have $\mathrm{E}[\mathcal{R}_\infty^k] < \infty$ for $1 \leq k \leq n-1$ by the induction hypothesis. Furthermore, for $n = 1$, the Parallel Law (Lemma 2.4.3) implies $\mathcal{R}_1 = N^{-1}$ and therefore

$$\mathrm{E}[\mathcal{R}_1^m] = \mathrm{E}[N^{-m}] \leq \mathrm{E}[N^{-m}]\sum_{k=0}^{m-1}\binom{m}{k}\mathrm{E}[\mathcal{R}_\infty^k].$$

This implies a recursion of the from $x_1 \leq a$ and $x_n \leq a + bx_{n-1}$ for $n \geq 2$, for some $a, b > 0$. Iterating this gives rise to $x_n \leq a\sum_{k=0}^{n-1} b^k$. In our setting, iterating (3.3.23) leads to

$$\mathrm{E}[\mathcal{R}_n^m] \leq \left(\mathrm{E}[N^{-m}]\sum_{k=0}^{m-1}\binom{m}{k}\mathrm{E}[\mathcal{R}_\infty^k]\right)\sum_{k=0}^{n-1}\mathrm{E}[N^{-m}]^k.$$

Since $0 < \mathrm{E}[N^{-m}] < 1$, the right-hand side of above expression converges as $n \to \infty$. We conclude that

$$\mathrm{E}[\mathcal{R}_\infty^m] = \lim_{n\to\infty}\mathrm{E}[\mathcal{R}_n^m] \leq \left(\mathrm{E}[N^{-m}]\sum_{k=0}^{m-1}\binom{m}{k}\mathrm{E}[\mathcal{R}_\infty^k]\right)\frac{1}{1-\mathrm{E}[N^{-m}]} < \infty. \tag{3.3.24}$$

In view of (3.3.21), this completes the proof. □

Proof of Proposition 3.2.3

The proof is based on the stationarity and the ergodicity of the process $(T, X_n, \xi)_{n\in\mathbb{Z}}$ under $\hat{\mathbb{P}}_\varepsilon$. Recall that T_1^v is the subtree of T that contains v after removing all edges with conductance ε, and that we write T_1 for T_1^ρ. By the ergodic theorem we have

$$\hat{\mathbb{E}}_\varepsilon\big[[X_1 - X_0]_{X_{-\infty}}\mathbb{1}_{\{|T_1|<\infty\}}\big] = \lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}[X_{k+1}-X_k]_{X_{-\infty}}\mathbb{1}_{\{|T_1^{X_k}|<\infty\}} \tag{3.3.25}$$

almost surely. The sum in (3.3.25) can only increase when the random walk moves on finite subtrees T_1^v. We let a_k denote the times at which the random walk enters a finite tree and b_k are the times of leaving a finite tree. Formally, we set

$$b_0 = \inf\left\{n \geq 1 : X_n \notin T_1^{X_0}\right\}$$

and recursively for $k \geq 1$

$$\begin{aligned} a_k &= \inf\left\{n \geq b_{k-1} : |T_1^{X_n}| < \infty\right\},\\ b_k &= \inf\left\{n > a_k : X_n \notin T_1^{X_{a_k}}\right\}. \end{aligned}$$

These times are well-defined, since a_k and b_k are almost surely finite for all k. To see this, observe that whenever the random walk reaches a new maximal generation, it has a positive probability of encountering and traversing an edge with conductance ε to a vertex v such that T_1^v is finite. Moreover, every finite subtree T_1^v is left after a finite amount of time. In particular, this implies $b_k \to \infty$ as $k \to \infty$ and the limit in (3.3.25), since it exists, is equal to the limit along the subsequence $(b_k)_{k\geq 0}$

$$\begin{aligned} \hat{\mathbb{E}}_\varepsilon\big[[X_1 - X_0]_{X_{-\infty}}\mathbb{1}_{\{|T_1|<\infty\}}\big] &= \lim_{n\to\infty}\frac{1}{b_n}\sum_{k=0}^{b_n-1}[X_{k+1}-X_k]_{X_{-\infty}}\mathbb{1}_{\{|T_1^{X_k}|<\infty\}}\\ = \lim_{n\to\infty}\frac{1}{b_n}\sum_{k=0}^{b_0-1}[X_{k+1}-X_k]_{X_{-\infty}}\mathbb{1}_{\{|T_1^{X_k}|<\infty\}} &+ \lim_{n\to\infty}\frac{1}{b_n}\sum_{k=1}^{n}\sum_{i=a_k}^{b_k-1}[X_{i+1}-X_i]_{X_{-\infty}}. \end{aligned} \tag{3.3.26}$$

The first sum on the right-hand side of (3.3.26) is bounded by b_0 and therefore we have

$$\lim_{n\to\infty}\frac{1}{b_n}\sum_{k=0}^{b_0-1}[X_{k+1}-X_k]_{X_{-\infty}}\mathbb{1}_{\{|T_1^{X_k}|<\infty\}} = 0 \quad \hat{\mathbb{P}}_\varepsilon - \text{a.s.} \tag{3.3.27}$$

Let us proceed with the second term in (3.3.26). We can bound the distance that the random walk can gain on a finite tree by its number of vertices. This yields

$$\sum_{i=a_k}^{b_k-1}[X_{i+1}-X_i]_{X_{-\infty}} = [X_{b_k} - X_{a_k}]_{X_{-\infty}} \leq |X_{b_k} - X_{a_k}| \leq \left|T_1^{X_{a_k}}\right|,$$

where we used in the first equality that the signed distance is additive, that is, for vertices u, v, w and z we have $[u-v]_z + [v-w]_z = [u-w]_z$. We define

$$L_k = \sup\left\{n \geq 0 : X_k, \dots, X_{k+n} \in T_1^{X_k}\right\} + \sup\left\{n \geq 0 : X_k, \dots, X_{k-n} \in T_1^{X_k}\right\} + 1$$

so that L_k counts the number of steps of the random walk in $T_1^{X_k}$ until leaving it. Note that for $a_k \leq i \leq b_k - 1$ the random walk is located in the same finite subtree $T_1^{X_i} = T_1^{X_{a_k}}$ and the time spent in this tree is given by $L_i = b_k - a_k$. This implies

$$\begin{aligned}
\lim_{n\to\infty} \frac{1}{b_n} \sum_{k=1}^{n} \sum_{i=a_k}^{b_k-1} [X_{i+1} - X_i]_{X_{-\infty}} &\leq \lim_{n\to\infty} \frac{1}{b_n} \sum_{k=1}^{n} \left|T_1^{X_{a_k}}\right| \\
&= \lim_{n\to\infty} \frac{1}{b_n} \sum_{k=1}^{n} \sum_{i=a_k}^{b_k-1} \frac{1}{L_i} \left|T_1^{X_i}\right| \\
&= \lim_{n\to\infty} \frac{1}{b_n} \sum_{k=a_1}^{b_n-1} \frac{1}{L_k} \left|T_1^{X_k}\right| \mathbb{1}_{\{|T_1^{X_k}|<\infty\}} \\
&\leq \lim_{n\to\infty} \frac{1}{b_n} \sum_{k=0}^{b_n-1} \frac{1}{L_k} \left|T_1^{X_k}\right| \mathbb{1}_{\{|T_1^{X_k}|<\infty\}}.
\end{aligned}$$

By the ergodic theorem these averages converge almost surely to their mean and therefore we arrive at

$$\begin{aligned}
\lim_{n\to\infty} \frac{1}{b_n} \sum_{k=1}^{n} \sum_{i=a_k}^{b_k-1} [X_{i+1} - X_i]_{X_{-\infty}} &\leq \hat{\mathbb{E}}_\varepsilon \left[\frac{1}{L_0} |T_1| \mathbb{1}_{\{|T_1|<\infty\}} \right] \\
&\leq \hat{\mathbb{E}}_\varepsilon \left[\frac{1}{L_0^2} \right]^{\frac{1}{2}} \hat{\mathrm{E}}_\varepsilon \left[|T_1|^2 \mathbb{1}_{\{|T_1|<\infty\}} \right]^{\frac{1}{2}}. \qquad (3.3.28)
\end{aligned}$$

Here, the second step is obtained by applying the Cauchy-Schwarz inequality. Since we have $L_0 \geq b_0 > 0$, the first expectation on the right-hand side can be bounded by

$$\hat{\mathbb{E}}_\varepsilon \left[\frac{1}{L_0^2} \right] \leq \hat{\mathbb{E}}_\varepsilon \left[\frac{1}{b_0^2} \right] = \sum_{k=1}^{\infty} \frac{1}{k^2} \hat{\mathbb{P}}_\varepsilon (b_0 = k). \qquad (3.3.29)$$

Recall that b_0 indicates when the random walk leaves the tree T_1, we have

$$\begin{aligned}
\hat{\mathbb{P}}_\varepsilon (b_0 = k) &= \hat{\mathbb{P}}_\varepsilon \big(X_0, \dots, X_{k-1} \in T_1, X_k \notin T_1 \big) \\
&\leq \hat{\mathbb{P}}_\varepsilon \big(X_0, \dots, X_{k-1} \in T_1, X_k \notin T_1, |T_1| > 1 \big) + \hat{\mathrm{P}}_\varepsilon \big(|T_1| = 1 \big).
\end{aligned}$$

If the subtree T_1 consists only of the root, the random walk can only leave T_1 in its first step. The probability of this event is given by

$$\hat{\mathrm{P}}_\varepsilon \big(|T_1| = 1 \big) = \frac{1}{\gamma_\varepsilon} \int \frac{C(\rho)}{\deg(\rho)} \mathbb{1}_{\{|T_1|=1\}} \, \mathrm{dP}_\varepsilon^{\mathrm{aug}} = \frac{1}{\gamma_\varepsilon} \int \frac{\varepsilon \deg(\rho)}{\deg(\rho)} \, \mathrm{dP}_\varepsilon^{\mathrm{aug}} = \frac{\varepsilon}{\gamma_\varepsilon}.$$

Otherwise, if $|T_1| > 1$, the random walk always sees at least one edge with conductance larger than κ^{-1} as long as it is located in the subtree T_1. This yields the following upper bound:

$$\begin{aligned}
&\hat{\mathbb{P}}_\varepsilon\big(X_0, \dots, X_{k-1} \in T_1, X_k \notin T_1, |T_1| > 1\big) \\
&\quad = \sum_{i=1}^\infty \hat{\mathbb{P}}_\varepsilon\big(X_0, \dots, X_{k-1} \in T_1, X_k \notin T_1, |T_1| > 1 | \deg(X_{k-1}) = i\big)\hat{\mathbb{P}}_\varepsilon(\deg(X_{k-1}) = i) \\
&\quad \le \sum_{i=1}^\infty \nu(\{i-1\})\hat{\mathbb{P}}_\varepsilon\big(X_k \notin T_1 \;\big|\; X_0, \dots, X_{k-1} \in T_1, |T_1| > 1, \deg(X_{k-1}) = i\big) \\
&\quad \le \sum_{i=1}^\infty \nu(\{i-1\}) \frac{(i-1)\varepsilon}{(i-1)\varepsilon + \kappa^{-1}}.
\end{aligned}$$

Dominated convergence implies

$$\hat{\mathbb{P}}_\varepsilon(b_0 = k) \le \sum_{i=1}^\infty \nu(\{i-1\}) \frac{(i-1)\varepsilon}{(i-1)\varepsilon + \kappa^{-1}} + \frac{\varepsilon}{\gamma_\varepsilon} \xrightarrow[\varepsilon \to 0]{} 0.$$

Concerning the last summand, note that $\gamma_\varepsilon \to \gamma_0 > 0$ as $\varepsilon \to 0$. Together with (3.3.29), we arrive at

$$\lim_{\varepsilon \to 0} \hat{\mathbb{E}}_\varepsilon\left[\frac{1}{L_0^2}\right] \le \sum_{k=1}^\infty \frac{1}{k^2} \lim_{\varepsilon \to 0} \hat{\mathbb{P}}_\varepsilon(b_0 = k) = 0 \tag{3.3.30}$$

again by dominated convergence.

Consequently, the proof is complete once we have shown that the second expectation in (3.3.28) remains bounded. Recalling the definition of the invariant measure in (3.2.2), we have

$$\begin{aligned}
\hat{\mathrm{E}}_\varepsilon\big[|T_1|^2 \mathbb{1}_{\{|T_1| < \infty\}}\big] &= \mathrm{E}_\varepsilon^{\mathrm{aug}}\left[|T_1|^2 \mathbb{1}_{\{|T_1| < \infty\}} \frac{C(\rho)}{\gamma_\varepsilon \deg(\rho)}\right] \\
&\le \frac{\kappa}{\gamma_\varepsilon} \mathrm{E}_\varepsilon^{\mathrm{aug}}\big[|T_1|^2 \mathbb{1}_{\{|T_1| < \infty\}}\big] \\
&= \frac{\kappa}{\gamma_\varepsilon} \mathrm{E}_0^{\mathrm{aug}}\big[|T_1|^2 \mathbb{1}_{\{|T_1| < \infty\}}\big].
\end{aligned} \tag{3.3.31}$$

To see the last equality, we remember that T_1 is formed by the edges with conductance at least κ^{-1} and therefore it does not depend on ε. Under $\mathrm{P}_0^{\mathrm{aug}}$, T is an augmented Galton-Watson tree. That is, T consists of two independent Galton-Watson trees $T^{(1)}$ and $T^{(2)}$ whose roots a connected by an edge (ρ, v_0), see Figure 3.1. We recall that $T_1^{(i)}$ denotes the subtree of $T^{(i)}$ that is formed by the edges with conductance at least κ^{-1} that contains the root of $T^{(i)}$ ($i = 1, 2$). Then we can write

$$\begin{aligned} \mathrm{E}_0^{\mathrm{aug}}\big[|T_1|^2 \mathbb{1}_{\{|T_1|<\infty\}}\big] &= \mathrm{E}_0^{\mathrm{aug}}\Big[|T_1|^2\Big(\mathbb{1}_{\{|T_1^{(1)}|<\infty, |T_1^{(2)}|<\infty, \kappa^{-1}\le \xi(\rho,v_0)\le\kappa\}} + \mathbb{1}_{\{|T_1^{(1)}|<\infty, \xi(\rho,v_0)=0\}}\Big)\Big] \\ &\le \mathrm{E}_0^{\mathrm{aug}}\Big[\big(|T_1^{(1)}| + |T_1^{(2)}|\big)^2 \mathbb{1}_{\{|T_1^{(1)}|<\infty, |T_1^{(2)}|<\infty\}}\Big] + \mathrm{E}_0^{\mathrm{aug}}\Big[|T_1^{(1)}|^2 \mathbb{1}_{\{|T_1^{(1)}|<\infty\}}\Big]. \end{aligned} \tag{3.3.32}$$

Under $\mathrm{P}_0^{\mathrm{aug}}$, the tree $T_1^{(1)}$ is a Galton-Watson tree. Actually, when we condition on extinction, $T_1^{(1)}$ is a subcritical Galton-Watson tree. This is a consequence of the duality principle for branching processes (see Section 12 in [AN72]). Lemma 3.3.2 below implies that the second summand in (3.3.32) is finite

$$\mathrm{E}_0^{\mathrm{aug}}\Big[|T_1^{(1)}|^2 \mathbb{1}_{\{|T_1^{(1)}|<\infty\}}\Big] \le \mathrm{E}_0\big[|T_1|^2 \;\big|\; |T_1| < \infty\big] < \infty.$$

Using that under $\mathrm{P}_0^{\mathrm{aug}}$, $T^{(1)}$ and $T^{(2)}$ are independent and identically distributed Galton-Watson trees, we obtain for the first expectation in (3.3.32)

$$\begin{aligned} &\mathrm{E}_0^{\mathrm{aug}}\Big[\big(|T_1^{(1)}| + |T_1^{(2)}|\big)^2 \mathbb{1}_{\{|T_1^{(1)}|<\infty, |T_1^{(2)}|<\infty\}}\Big] \\ &\quad \le 2\mathrm{E}_0\big[|T_1|^2 \;\big|\; |T_1| < \infty\big] + 2\mathrm{E}_0\big[|T_1| \;\big|\; |T_1| < \infty\big]^2 < \infty. \end{aligned}$$

The finiteness of the occurring moments is again implied by Lemma 3.3.2 below. Due to (3.3.31) we get

$$\lim_{\varepsilon\to 0} \hat{\mathrm{E}}_\varepsilon\big[|T_1|^2 \mathbb{1}_{\{|T_1|<\infty\}}\big] \le \frac{\kappa}{\gamma_0}\mathrm{E}_0^{\mathrm{aug}}\big[|T_1|^2 \mathbb{1}_{\{|T_1|<\infty\}}\big] < \infty.$$

Combining this with (3.3.30) shows that the right-hand of (3.3.28) vanishes as $\varepsilon \to 0$. Finally, together with (3.3.26) and (3.3.27), this implies

$$\lim_{\varepsilon\to 0} \hat{\mathbb{E}}_\varepsilon\big[[X_1 - X_0]_{X_{-\infty}} \mathbb{1}_{\{|T_1|<\infty\}}\big] = 0,$$

which is what we wanted to show. □

The following lemma gives an expression for the second moment of the size of a subcritical Galton-Watson tree. The formula can also be derived from the results of [Pak71]. Nevertheless, we include a proof for completeness.

Lemma 3.3.2. *Let T be a subcritical Galton-Watson tree. We assume the second moment of the offspring distribution to be finite. Then we have*

$$\mathrm{E}\big[|T|^2\big] = \frac{m_2 - m_1^2 - m_1 + 1}{(1 - m_1)^3} < \infty,$$

where m_k denotes the k-th moment of the offspring distribution ($k = 1, 2$).

Proof. We let N denote the number of offspring of the root. The size of the n-th generation is given by

$$Z_0 = 1, \quad Z_n = \sum_{i=1}^{Z_{n-1}} X_{n,i},$$

where $(X_{n,i})_{i,n\in\mathbb{N}}$ is a doubly infinite array of independent and identically distributed random variables with $X_{n,i} \overset{d}{=} N$ for all $n, i \in \mathbb{N}$. The number of vertices in the tree is then given by $|T| = \sum_{n=0}^{\infty} Z_n$ and we have

$$\mathrm{E}\big[|T|^2\big] = \sum_{n=0}^{\infty} \mathrm{E}\big[Z_n^2\big] + 2\sum_{m=0}^{\infty}\sum_{n=m+1}^{\infty} \mathrm{E}[Z_m Z_n] \tag{3.3.33}$$

by monotone convergence. For $m < n$, using the independence of $(X_{n,i})_{i\in\mathbb{N}}$ and $(Z_k)_{1\le k\le n-1}$, we obtain the following recursion for the mixed moments

$$\begin{aligned}\mathrm{E}[Z_m Z_n] &= \mathrm{E}\left[Z_m \sum_{i=1}^{Z_{n-1}} X_{n,i}\right] = \mathrm{E}\left[Z_m \sum_{i=1}^{Z_{n-1}} \mathrm{E}[X_{n,i} \mid Z_1, \dots, Z_{n-1}]\right] \\ &= \mathrm{E}\left[Z_m \sum_{i=1}^{Z_{n-1}} \mathrm{E}[X_{n,i}]\right] = m_1 \mathrm{E}[Z_m Z_{n-1}],\end{aligned}$$

where $m_1 = \mathrm{E}[N]$ is the offspring mean. Iterating this leads to

$$\mathrm{E}[Z_m Z_n] = m_1^{n-m}\mathrm{E}[Z_m^2]$$

for $m < n$. Together with (3.3.33), we obtain

$$\begin{aligned}E[|T|^2] &= \sum_{n=0}^{\infty} \mathrm{E}[Z_n^2] + 2\sum_{m=0}^{\infty} m_1^{-m}\mathrm{E}[Z_m^2] \sum_{n=m+1}^{\infty} m_1^n \\ &= \sum_{n=0}^{\infty} \mathrm{E}[Z_n^2] + \frac{2m_1}{1-m_1}\sum_{n=0}^{\infty} \mathrm{E}[Z_n^2] \\ &= \frac{1+m_1}{1-m_1}\sum_{n=0}^{\infty} \mathrm{E}[Z_n^2].\end{aligned}$$

Here, in the second equality we used that $m_1 < 1$ holds, since the Galton-Watson tree is subcritical by assumption. From Section 2 in [AN72] we derive the following expression for the second moment of the generation sizes

$$\mathrm{E}[Z_n^2] = \frac{m_2 m_1^{n-1}(1-m_1^n) - m_1^{n+1} + m_1^{2n}}{1-m_1}, \tag{3.3.34}$$

where $m_2 = E[N^2]$ is the second moment of the offspring distribution. Using $m_1 < 1$ again, we conclude that

$$\begin{aligned}
E[|T|^2] &= \frac{1+m_1}{(1-m_1)^2} \sum_{n=0}^{\infty} (m_2 m_1^{n-1}(1-m_1^n) - m_1^{n+1} + m_1^{2n}) \\
&= \frac{1+m_1}{(1-m_1)^2} \left(\frac{m_2}{m_1(1-m_1)} - \frac{m_2}{m_1(1-m_1^2)} - \frac{m_1}{1-m_1} + \frac{1}{1-m_1^2} \right) \\
&= \frac{m_2 - m_1^2 - m_1 + 1}{(1-m_1)^3}.
\end{aligned}$$

□

Proof of Lemma 3.2.4

We start with some notations. Recall that $\bar{\mathrm{P}}$ is the law on Ω such that under $\bar{\mathrm{P}}$, T is a Galton-Watson tree with offspring law $\bar{\nu}$ (see (3.1.2)) and conductance law μ. We let $\bar{\mathrm{P}}^{\mathrm{aug}}$ be the corresponding law of an augmented Galton-Watson tree, defined as in (3.2.1) with P_ε replaced by $\bar{\mathrm{P}}$. Analogously to (3.2.2), the invariant measure $\bar{\mathrm{P}}^{\mathrm{inv}}$ for the environment seen from the particle is then given via

$$\bar{\mathrm{E}}^{\mathrm{inv}}[f(T,\rho,\xi)] = \bar{\mathrm{E}}^{\mathrm{aug}}\left[f(T,\rho,\xi) \frac{C(\rho)}{\gamma \deg(\rho)} \right]$$

with $\gamma = \int x \,\mathrm{d}\mu(x)$. We write $\bar{\mathbb{P}}^{\mathrm{inv}}$ for the corresponding annealed law on $\Omega \times \mathbb{T}^{\mathbb{Z}}$, defined analogously to (2.3.3). As usual, $\bar{\mathrm{E}}^{\mathrm{aug}}$, $\bar{\mathrm{E}}^{\mathrm{inv}}$ and $\bar{\mathbb{E}}^{\mathrm{inv}}$ denote the associated expectations. The speed for $\varepsilon = 0$ is then given by

$$v(\nu,\mu_0) \coloneqq v(\bar{\nu},\mu) = \bar{\mathbb{E}}^{\mathrm{inv}}\big[[X_1 - X_0]_{X_{-\infty}} \bigm| |T| = \infty\big],$$

see Section 3.1 and Remark 4.1 in [GMPV12].

We obtain the desired representation for the speed once we have shown that the distribution of $(T_1, \rho, (\xi_e)_{e\in\mathcal{E}(T_1)})$ under $\hat{\mathrm{P}}_0(\,\cdot \mid |T_1| = \infty)$ is equal to the distribution of $(T, \rho, (\xi_e)_{e\in\mathcal{E}(T)})$ under $\bar{\mathrm{P}}^{\mathrm{inv}}(\,\cdot \mid |T| = \infty)$. We let $t \in \mathcal{T}$ be a tree with m generations, $k = \deg_t(\rho)$ the degree of the root and $A \subseteq (0,\infty)^{\otimes \mathcal{E}(t)}$ a measurable subset of conductance configurations. By Bayes' theorem we have

$$\begin{aligned}
&\hat{\mathrm{P}}_0(T_{1|m} = t,\, (\xi_e)_{e\in\mathcal{E}(t)} \in A \mid |T_1| = \infty) \\
&= \hat{\mathrm{P}}_0(T_{1|m} = t,\, (\xi_e)_{e\in\mathcal{E}(t)} \in A) \hat{\mathrm{P}}_0(|T_1| = \infty \mid T_{1|m} = t,\, (\xi_e)_{e\in\mathcal{E}(t)} \in A) \hat{\mathrm{P}}_0\big(|T_1| = \infty\big)^{-1} \\
&= \hat{\mathrm{P}}_0(\deg_{T_1}(\rho) = k) \hat{\mathrm{P}}_0(T_{1|m} = t \mid \deg_{T_1}(\rho) = k) \hat{\mathrm{P}}_0((\xi_e)_{e\in\mathcal{E}(t)} \in A \mid T_{1|m} = t) \\
&\qquad \times \hat{\mathrm{P}}_0(|T_1| = \infty \mid T_{1|m} = t,\, (\xi_e)_{e\in\mathcal{E}(t)} \in A) \hat{\mathrm{P}}_0(|T_1| = \infty)^{-1},
\end{aligned} \tag{3.3.35}$$

where $T_{1|m}$ denotes the first m generations of the tree T_1. We consider the probabilities successively, starting with the first one. Recalling the definitions of $\hat{\mathrm{P}}_0$ in (3.2.2) and $\bar{\nu}$ in (3.1.2), we calculate for $k \geq 1$

$$\begin{aligned}
\hat{\mathrm{P}}_0(\deg_{T_1}(\rho) = k) &= \mathrm{E}_0^{\mathrm{aug}}\left[\frac{C(\rho)}{\gamma_0 \deg_T(\rho)} \mathbb{1}_{\{\deg_{T_1}(\rho)=k\}}\right] \\
&= \sum_{n=k}^{\infty} \mathrm{E}_0^{\mathrm{aug}}\left[\frac{C(\rho)}{(1-\alpha)\gamma n} \;\middle|\; \deg_{T_1}(\rho) = k, \deg_T(\rho) = n\right] \\
&\qquad\qquad \times \hat{\mathrm{P}}_0(\deg_{T_1}(\rho) = k \mid \deg_T(\rho) = n)\hat{\mathrm{P}}_0(\deg_T(\rho) = n) \\
&= \sum_{n=k}^{\infty} \frac{k}{(1-\alpha)n}\binom{n}{k}(1-\alpha)^k \alpha^{n-k} \nu(\{n-1\}) \\
&= \sum_{n=k}^{\infty} \binom{n-1}{k-1}(1-\alpha)^{k-1}\alpha^{n-1-(k-1)}\nu(\{n-1\}) \\
&= \bar{\nu}(\{k-1\}) = \bar{\mathrm{P}}^{\mathrm{aug}}(\deg(\rho) = k) = \bar{\mathrm{P}}^{\mathrm{inv}}(\deg(\rho) = k), \qquad (3.3.36)
\end{aligned}$$

where we used in the second equality that $\gamma_0 = \int x \, \mathrm{d}\mu_0(x) = (1-a)\gamma$. If $k = 0$, we have

$$\hat{\mathrm{P}}_0(\deg_{T_1}(\rho) = 0) = 0 = \bar{\mathrm{P}}^{\mathrm{inv}}(\deg(\rho) = 0).$$

The second probability in (3.3.35) is equal to $\bar{\mathrm{P}}^{\mathrm{inv}}(T_{|m} = t \mid \deg(\rho) = k)$. To see this, we observe that conditioned on $\{\deg_{T_1}(\rho) = k\}$, T_1 consists of k i.i.d. Galton-Watson trees, each rooted at one neighbour of ρ and with offspring law $\bar{\nu}$. Proceeding similarly as in (3.3.36), we can write the third probability in (3.3.35) as

$$\begin{aligned}
&\hat{\mathrm{P}}_0((\xi_e)_{e\in\mathcal{E}(t)} \in A \mid T_{1|m} = t) \\
&= \mathrm{E}_0^{\mathrm{aug}}\left[\frac{C(\rho)}{\gamma_0 \deg_T(\rho)} \mathbb{1}_{\{(\xi_e)_{e\in\mathcal{E}(t)}\in A, T_{1|m}=t\}}\right]\hat{\mathrm{P}}_0(T_{1|m} = t)^{-1} \\
&= \sum_{n=k}^{\infty} \frac{k}{(1-\alpha)n} \mathrm{E}_0^{\mathrm{aug}}\left[\frac{C(\rho)}{\gamma \deg_{T_1}(\rho)} \mathbb{1}_{\{(\xi_e)_{e\in\mathcal{E}(t)}\in A, T_{1|m}=t, \deg_T(\rho)=n\}}\right]\hat{\mathrm{P}}_0(T_{1|m} = t)^{-1} \\
&= \sum_{n=k}^{\infty} \frac{k}{(1-\alpha)n} \mathrm{E}_0^{\mathrm{aug}}\left[\frac{C(\rho)}{\gamma \deg_{T_1}(\rho)} \mathbb{1}_{\{(\xi_e)_{e\in\mathcal{E}(t)}\in A\}} \;\middle|\; T_{1|m} = t\right]\mathrm{P}_0^{\mathrm{aug}}(T_{1|m} = t, \deg_T(\rho) = n) \\
&\qquad \times \hat{\mathrm{P}}_0(T_{1|m} = t)^{-1},
\end{aligned}$$

where we used in the second step that we have $\deg_{T_1}(\rho) = k$ on the event $\{T_{1|m} = t\}$. Moreover, to see the last equality, observe that the random variables $\frac{C(\rho)}{\gamma \deg_{T_1}(\rho)} \mathbb{1}_{\{(\xi_e)_{e\in\mathcal{E}(t)}\in A\}}$ and $\mathbb{1}_{\{\deg_T(\rho)=n\}}$ are independent under $\mathrm{P}_0^{\mathrm{aug}}(\,\cdot \mid T_{1|m} = t)$. The same calculation as in

(3.3.36) yields

$$\begin{aligned}
&\sum_{n=k}^{\infty} \frac{k}{(1-\alpha)n} \mathrm{P}_0^{\mathrm{aug}}(T_{1|m} = t, \deg_T(\rho) = n) \\
&\quad = \sum_{n=k}^{\infty} \frac{k}{(1-\alpha)n} \mathrm{P}_0^{\mathrm{aug}}(\deg_{T_1}(\rho) = k, \deg_T(\rho) = n) \mathrm{P}_0^{\mathrm{aug}}(T_{1|m} = t \mid \deg_{T_1}(\rho) = k) \\
&\quad = \sum_{n=k}^{\infty} \frac{k}{(1-\alpha)n} \nu(\{n-1\}) \binom{n}{k} (1-\alpha)^k \alpha^{n-k} \mathrm{P}_0^{\mathrm{aug}}(T_{1|m} = t \mid \deg_{T_1}(\rho) = k) \\
&\quad = \bar{\nu}(\{k-1\}) \mathrm{P}_0^{\mathrm{aug}}(T_{1|m} = t \mid \deg_{T_1}(\rho) = k)
\end{aligned}$$

and consequently

$$\begin{aligned}
\hat{\mathrm{P}}_0((\xi_e)_{e \in \mathcal{E}(t)} \in A \mid T_{1|m} = t) = \mathrm{E}_0^{\mathrm{aug}}&\left[\frac{C(\rho)}{\gamma \deg_{T_1}(\rho)} \mathbb{1}_{\{(\xi_e)_{e \in \mathcal{E}(t)} \in A\}} \,\middle|\, T_{1|m} = t \right] \bar{\nu}(\{k-1\}) \\
&\times \mathrm{P}_0^{\mathrm{aug}}(T_{1|m} = t \mid \deg_{T_1}(\rho) = k) \hat{\mathrm{P}}_0(T_{1|m} = t)^{-1}.
\end{aligned}$$

Under $\mathrm{P}_0^{\mathrm{aug}}(\,\cdot \mid T_{1|m} = t)$, the conductances $(\xi_e)_{e \in \mathcal{E}(t)}$ are i.i.d. with marginal law μ, which implies

$$\begin{aligned}
\hat{\mathrm{P}}_0((\xi_e)_{e \in \mathcal{E}(t)} \in A \mid T_{1|m} = t) &= \bar{\mathrm{E}}^{\mathrm{aug}} \left[\frac{C(\rho)}{\gamma \deg(\rho)} \mathbb{1}_{\{(\xi_e)_{e \in \mathcal{E}(t)} \in A\}} \,\middle|\, T_{|m} = t \right] \bar{\mathrm{P}}^{\mathrm{aug}}(\deg(\rho) = k) \\
&\qquad \times \bar{\mathrm{P}}^{\mathrm{aug}}\big(T_{|m} = t \,\big|\, \deg(\rho) = k\big) \bar{\mathrm{P}}^{\mathrm{inv}}(T_{|m} = t)^{-1} \\
&= \bar{\mathrm{E}}^{\mathrm{aug}} \left[\frac{C(\rho)}{\gamma \deg(\rho)} \mathbb{1}_{\{(\xi_e)_{e \in \mathcal{E}(t)} \in A, T_{|m} = t\}} \right] \bar{\mathrm{P}}^{\mathrm{inv}}(T_{|m} = t)^{-1} \\
&= \bar{\mathrm{P}}^{\mathrm{inv}}((\xi_e)_{e \in \mathcal{E}(t)} \in A \,\big|\, T_{|m} = t).
\end{aligned}$$

In order to compute the last two probabilities in (3.3.35), we let q be the extinction probability of a Galton-Watson tree with offspring law $\bar{\nu}$. Then we have

$$\begin{aligned}
\hat{\mathrm{P}}_0(|T_1| = \infty \mid T_{1|m} = t,\, (\xi_e)_{e \in \mathcal{E}(t)} \in A) &= 1 - q^{|G_m(t)|} \\
&= \bar{\mathrm{P}}^{\mathrm{inv}}(|T| = \infty \mid T_{|m} = t,\, (\xi_e)_{e \in \mathcal{E}(t)} \in A)
\end{aligned}$$

and

$$\begin{aligned}
\hat{\mathrm{P}}_0(|T_1| = \infty) &= \sum_{k=1}^{\infty} \hat{\mathrm{P}}_0(|T_1| = \infty \mid \deg_{T_1}(\rho) = k) \hat{\mathrm{P}}_0(\deg_{T_1}(\rho) = k) \\
&= \sum_{k=1}^{\infty} (1 - q^k) \bar{\nu}(\{k-1\}) \\
&= \sum_{k=1}^{\infty} \bar{\mathrm{P}}^{\mathrm{inv}}(|T| = \infty \mid \deg(\rho) = k) \bar{\mathrm{P}}^{\mathrm{inv}}(\deg(\rho) = k) \\
&= \bar{\mathrm{P}}^{\mathrm{inv}}(|T| = \infty).
\end{aligned}$$

We finally obtain

$$\begin{aligned}
&\hat{\mathrm{P}}_0(T_{1|m} = t,\, (\xi_e)_{e\in\mathcal{E}(t)} \in A \mid |T_1| = \infty)\\
&\quad= \bar{\mathrm{P}}^{\mathrm{inv}}(\deg(\rho) = k)\bar{\mathrm{P}}^{\mathrm{inv}}(T_{|m} = t \mid \deg(\rho) = k)\bar{\mathrm{P}}^{\mathrm{inv}}((\xi_e)_{e\in\mathcal{E}(t)} \in A \mid T_{|m} = t)\\
&\qquad\quad \times \bar{\mathrm{P}}^{\mathrm{inv}}(|T| = \infty \mid T_{|m} = t,\, (\xi_e)_{e\in\mathcal{E}(t)} \in A)\bar{\mathrm{P}}^{\mathrm{inv}}(|T| = \infty)^{-1}\\
&\quad= \bar{\mathrm{P}}^{\mathrm{inv}}(T_{|m} = t,\, (\xi_e)_{e\in\mathcal{E}(t)} \in A \mid |T| = \infty),
\end{aligned}$$

which completes the first part of the proof.

It remains to show that $v(\nu, \mu_0)$ is strictly positive. As stated in [GMPV12], the speed of the random walk on Galton-Watson trees with leaves can be expressed as an expectation of a ratio of effective conductances

$$v(\nu, \mu_0) = v(\bar{\nu}, \mu) = 1 - \frac{2}{\gamma}\bar{\mathrm{E}}^{\mathrm{aug}}\left[\xi_0 \frac{\mathcal{C}_{\omega^*}(\rho, \infty)}{\mathcal{C}_{\omega}(\rho, \infty)} \,\middle|\, |T| = \infty\right], \tag{3.3.37}$$

see Remark 4.1 in [GMPV12]. Here, ξ_0 denotes the conductance of the additional edge (ρ, v_0) in the augmented tree (see Figure 3.1) and ω^* is the environment formed by T^* and the corresponding conductances (see Figure 3.3 for the definition of T^*). Moreover, in Theorem 4.2 in [GMPV12] the authors proved that the speed on Galton-Watson trees without leaves and with random conductances is strictly positive. We will use their arguments to show that this result also holds when we allow leaves. Using the Parallel Law and the Series Law (Lemma 2.4.3 and 2.4.4), we have

$$\begin{aligned}
\bar{\mathrm{E}}^{\mathrm{aug}}\left[\xi_0 \frac{\mathcal{C}_{\omega^*}(\rho, \infty)}{\mathcal{C}_{\omega}(\rho, \infty)} \,\middle|\, |T| = \infty\right] &= \bar{\mathrm{E}}^{\mathrm{aug}}\left[\xi_0 \frac{(\xi_0^{-1} + \mathcal{C}_{\omega^{(2)}}(v_0, \infty)^{-1})^{-1}}{(\xi_0^{-1} + \mathcal{C}_{\omega^{(2)}}(v_0, \infty)^{-1})^{-1} + \mathcal{C}_{\omega^{(1)}}(\rho, \infty)} \,\middle|\, |T| = \infty\right]\\
&< \bar{\mathrm{E}}^{\mathrm{aug}}\left[\xi_0 \frac{\mathcal{C}_{\omega^{(2)}}(v_0, \infty)}{\mathcal{C}_{\omega^{(2)}}(v_0, \infty) + \mathcal{C}_{\omega^{(1)}}(\rho, \infty)} \,\middle|\, |T| = \infty\right],
\end{aligned}$$

where $\omega^{(i)}$ denotes the environment formed by $T^{(i)}$ and the corresponding conductances (see Figure 3.1 for the definition of $T^{(i)}$, $i \in \{1, 2\}$). We note that the inequality above is strict, since it is strict on the event $\{\mathcal{C}_{\omega^{(1)}}(\rho, \infty) > 0, \mathcal{C}_{\omega^{(2)}}(\rho, \infty) > 0\}$, which has positive probability under $\bar{\mathrm{P}}^{\mathrm{aug}}(\,\cdot \mid |T| = \infty)$. The tree T survives if one of the subtrees $T^{(1)}$ or $T^{(2)}$ survives. In particular, the survival of the tree is independent of ξ_0, which implies

$$\bar{\mathrm{E}}^{\mathrm{aug}}\left[\xi_0 \frac{\mathcal{C}_{\omega^*}(\rho, \infty)}{\mathcal{C}_{\omega}(\rho, \infty)} \,\middle|\, |T| = \infty\right] < \bar{\mathrm{E}}^{\mathrm{aug}}[\xi_0]\bar{\mathrm{E}}^{\mathrm{aug}}\left[\frac{\mathcal{C}_{\omega^{(2)}}(v_0, \infty)}{\mathcal{C}_{\omega^{(2)}}(v_0, \infty) + \mathcal{C}_{\omega^{(1)}}(\rho, \infty)} \,\middle|\, |T| = \infty\right] = \frac{\gamma}{2}.$$

Here, the last equality holds due to symmetry. In combination with the representation for the speed in (3.3.37) we finally get $v(\bar{\nu}, \mu) > 0$, which completes the proof. □

3.3.3 Weak convergence of the invariant measure: proof of Lemma 3.2.1

Recalling the definitions of the invariant measure $\hat{\mathrm{P}}_\varepsilon$ in (3.2.2) and the corresponding annealed law $\hat{\mathbb{P}}_\varepsilon$ (see (2.3.3)), $\hat{\mathrm{P}}_\varepsilon$ is the marginal law of ω under $\hat{\mathbb{P}}_\varepsilon$. Thus, the weak convergence of $\hat{\mathrm{P}}_\varepsilon$ is implied by the weak convergence of $\hat{\mathbb{P}}_\varepsilon$ and it suffices to show $\hat{\mathbb{P}}_\varepsilon \xrightarrow{w} \hat{\mathbb{P}}_0$ weakly.

To show the weak convergence, we introduce for any $m \geq 1$ a restricted annealed law $\hat{\mathbb{P}}^*_\varepsilon$ on $(\Omega \times \mathbb{T}^{2m+1}, \mathcal{G} \otimes \hat{\mathcal{F}}_m)$ where $\hat{\mathcal{F}}_m$ denotes the Borel σ-algebra on $\mathbb{T}^{2m+1}$. Given an environment $\omega \in \Omega$ and $m \geq 1$, we let $\hat{P}^*_\omega$ be the pushforward measure of $\hat{P}_\omega$ induced by

$$\hat{\pi}_m \colon (\mathbb{T}^{\mathbb{Z}}, \hat{\mathcal{F}}) \to (\mathbb{T}^{2m+1}, \hat{\mathcal{F}}_m), \quad \hat{\pi}_m((x_n)_{n\in\mathbb{Z}}) = (x_{-m}, \ldots, x_m). \tag{3.3.38}$$

The restricted annealed law $\hat{\mathbb{P}}^*_\varepsilon$ on $(\Omega \times \mathbb{T}^{2m+1}, \mathcal{G} \otimes \hat{\mathcal{F}}_m)$ is then defined by

$$\hat{\mathbb{P}}^*_\varepsilon(A \times B) = \int_A \hat{P}^*_\omega(B)\, \mathrm{d}\hat{\mathrm{P}}_\varepsilon(\omega) \tag{3.3.39}$$

for measurable sets $A \in \mathcal{G}$, $B \in \hat{\mathcal{F}}_m$. We observe that $\hat{\mathbb{P}}^*_\varepsilon$ is the pushforward measure of $\hat{\mathbb{P}}_\varepsilon$ under

$$\hat{\Pi}_m \colon (\Omega \times \mathbb{T}^{\mathbb{Z}}, \mathcal{G} \otimes \hat{\mathcal{F}}) \to (\Omega \times \mathbb{T}^{2m+1}, \mathcal{G} \otimes \hat{\mathcal{F}}_m), \quad \hat{\Pi}_m(\omega, x) = \big(\omega, \hat{\pi}_m(x)\big), \tag{3.3.40}$$

which is a consequence of the following short calculation:

$$\hat{\mathbb{P}}^*_\varepsilon(A \times B) = \int_A \hat{P}_\omega\big(\hat{\pi}_m^{-1}(B)\big)\, \mathrm{d}\hat{\mathrm{P}}(\omega) = \hat{\mathbb{P}}_\varepsilon\big(A \times \hat{\pi}_m^{-1}(B)\big) = \hat{\mathbb{P}}_\varepsilon\big(\hat{\Pi}_m^{-1}(A \times B)\big).$$

The proof of Lemma 3.2.1 is now done in two steps. We first show that the restricted law $\hat{\mathbb{P}}^*_\varepsilon$ converges weakly to $\hat{\mathbb{P}}^*_0$. Afterwards, we may apply the Portmanteau Theorem to transfer the convergence to the annealed law $\hat{\mathbb{P}}_\varepsilon$.

Weak convergence of $\hat{\mathbb{P}}^*_\varepsilon$

We let $f \colon \Omega \times \mathbb{T}^{2m+1} \to \mathbb{R}$ be a continuous and bounded function. We have

$$\begin{aligned}\left| \int f \, \mathrm{d}\hat{\mathbb{P}}^*_\varepsilon - \int f \, \mathrm{d}\hat{\mathbb{P}}^*_0 \right| \leq & \left| \int f \mathbb{1}_{\{|T_{|m}| \leq M\}} \, \mathrm{d}\hat{\mathbb{P}}^*_\varepsilon - \int f \mathbb{1}_{\{|T_{|m}| \leq M\}} \, \mathrm{d}\hat{\mathbb{P}}^*_0 \right| \\ & + \left| \int f \mathbb{1}_{\{|T_{|m}| > M\}} \, \mathrm{d}\hat{\mathbb{P}}^*_\varepsilon \right| + \left| \int f \mathbb{1}_{\{|T_{|m}| > M\}} \, \mathrm{d}\hat{\mathbb{P}}^*_0 \right|,\end{aligned}$$

where $T_{|m}$ denotes the subtree of T consisting of the first m generations of T. From the boundedness of the function f we get the following upper bound for the second summand

$$\left| \int f \mathbb{1}_{\{|T_{|m}| > M\}} \, \mathrm{d}\hat{\mathbb{P}}^*_\varepsilon \right| \leq \|f\|_\infty \mathrm{P}^{\mathrm{aug}}_\varepsilon(|T_{|m}| > M),$$

where $\|\cdot\|_\infty$ denotes the sup norm. We observe that the probability on the right-hand side does not depend on ε. Due to the local finiteness of the tree T, the probability tends to zero as $M \to \infty$. For the third summand we derive an upper bound analogously. Consequently, we can write

$$\left| \int f \, \mathrm{d}\hat{\mathbb{P}}_\varepsilon^* - \int f \, \mathrm{d}\hat{\mathbb{P}}_0^* \right| \leq \left| \int f \mathbb{1}_{\{|T_{|m}| \leq M\}} \, \mathrm{d}\hat{\mathbb{P}}_\varepsilon^* - \int f \mathbb{1}_{\{|T_{|m}| \leq M\}} \, \mathrm{d}\hat{\mathbb{P}}_0^* \right| + \delta(M) \tag{3.3.41}$$

for $\delta(M)$, independent of ε, with $\delta(M) \to 0$ as $M \to \infty$.

Let us now study the integral of f on the event $\{|T_{|m}| \leq M\}$. We let $T^{\mathrm{reg}(M)}$ be the M-regular tree, i.e. $\deg(v) = M$ for all $v \in T^{\mathrm{reg}(M)}$. Since on the event $\{|T_{|m}| \leq M\}$ the number of possible trajectories of length m is finite, we have by Lemma 2.3.3

$$\begin{aligned}
&\int f(\omega, x) \mathbb{1}_{\{|T_{|m}| \leq M\}} \, \mathrm{d}\hat{\mathbb{P}}_\varepsilon^*(\omega, x) \\
&= \sum_{z_{-m},\dots,z_m \in T_{|m}^{\mathrm{reg}(M)}} \int f(\omega, (z_{-m}, \dots, z_m)) \mathbb{1}_{\{z_{-m},\dots,z_m \in T, \, |T_{|m}| \leq M\}} \mathbb{1}_{\{x_{-m}=z_{-m},\dots,x_m=z_m\}} \, \mathrm{d}\hat{\mathbb{P}}_\varepsilon^*(\omega, x) \\
&= \sum_{z_{-m},\dots,z_m} \int f(\omega, (z_{-m}, \dots, z_m)) \mathbb{1}_{\{z_{-m},\dots,z_m \in T, \, |T_{|m}| \leq M\}} \int \mathbb{1}_{\{x_{-m}=z_{-m},\dots,x_m=z_m\}} \, \mathrm{d}\hat{P}_\omega^*(x) \, \mathrm{d}\hat{\mathrm{P}}_\varepsilon(\omega) \\
&= \sum_{z_{-m},\dots,z_m} \int f(\omega, (z_{-m}, \dots, z_m)) \mathbb{1}_{\{z_{-m},\dots,z_m \in T, \, |T_{|m}| \leq M\}} \hat{P}_\omega(X_{-m} = z_{-m}, \dots, X_m = z_m) \, \mathrm{d}\hat{\mathrm{P}}_\varepsilon(\omega) \\
&= \sum_{z_{-m},\dots,z_m} \frac{1}{\gamma_\varepsilon} \int h_{z_{-m},\dots,z_m,M}(\omega) \, \mathrm{d}\mathrm{P}_\varepsilon^{\mathrm{aug}}(\omega),
\end{aligned} \tag{3.3.42}$$

where $h_{z_{-m},\dots,z_m,M} \colon \Omega \to \mathbb{R}$ is given by

$$\begin{aligned}
h_{z_{-m},\dots,z_m,M}(\omega) =& f(\omega, (z_{-m}, \dots, z_m)) \mathbb{1}_{\{z_{-m},\dots,z_m \in T, \, |T_{|m}| \leq M\}} \frac{C(\rho)}{\deg(\rho)} \\
&\times \hat{P}_\omega(X_{-m} = z_{-m}, \dots, X_m = z_m).
\end{aligned} \tag{3.3.43}$$

In view of (3.3.41), we are done once we have shown that $\mathrm{P}_\varepsilon^{\mathrm{aug}}$ converges weakly to $\mathrm{P}_0^{\mathrm{aug}}$ and that the function $h_{z_{-m},\dots,z_m,M}$ is continuous and bounded, since $\gamma_\varepsilon \to \gamma_0$ as $\varepsilon \to 0$.

Weak convergence of $\mathrm{P}_\varepsilon^{\mathrm{aug}}$: As in the proof of Proposition 3.1.1, $\mathrm{P}_\varepsilon^{\mathrm{aug}}$ is the pushforward measure of $\tilde{\mathrm{P}}_\varepsilon^{\mathrm{aug}} = \mathrm{GW}^{\mathrm{aug}} \otimes \mu_\varepsilon^{\otimes \mathcal{E}(\mathbb{T})}$ under π defined in (2.2.2). We have

$$\mu_\varepsilon = \alpha \delta_\varepsilon + (1-\alpha)\mu \xrightarrow[\varepsilon \to 0]{w} \alpha \delta_0 + (1-\alpha)\mu = \mu_0,$$

which implies the weak convergence $\tilde{\mathrm{P}}_\varepsilon^{\mathrm{aug}} \xrightarrow{w} \tilde{\mathrm{P}}_0^{\mathrm{aug}}$. Due to the continuity of π, the weak convergence $\mathrm{P}_\varepsilon^{\mathrm{aug}} \xrightarrow{w} \mathrm{P}_0^{\mathrm{aug}}$ follows.

Continuity and boundedness of $h_{z_{-m},\ldots,z_m,M}$: The boundedness of $h_{z_{-m},\ldots,z_m,M}$ is a direct consequence from the boundedness of the conductances. To verify the continuity, we first observe that the indicator function $\mathbb{1}_{\{z_{-m},\ldots,z_m\in T,\,|T_{|m}|\le M\}}$ is continuous, since it depends only on a finite number of generations. Thus, it remains to show that the mapping $\omega \mapsto \frac{C(\rho)}{\deg(\rho)}\hat{P}_\omega(X_{-m}=z_{-m},\ldots,X_m=z_m)$ is continuous. We let $\omega^{(n)}=\big(T^{(n)},\rho^{(n)},\xi^{(n)}\big)$ be a sequence of environments converging to $\omega=(T,\rho,\xi)$ as $n\to\infty$. Recalling the definition of the metric on Ω in (2.1), this implies that for all $m\in\mathbb{N}$ we have $\big(T^{(n)}_{|m},\rho^{(n)}\big)=(T_{|m},\rho)$ for n sufficiently large and

$$\big(\xi^{(n)}(e)\big)_{e\in\mathcal{E}(T_{|m})}\xrightarrow[n\to\infty]{}\big(\xi(e)\big)_{e\in\mathcal{E}(T_{|m})}.$$

Let us fix $n_0\in\mathbb{N}$ such that $\big(T^{(n)}_{|m},\rho^{(n)}\big)=(T_{|m},\rho)$ for all $n\ge n_0$. Then for $n\ge n_0$ and for a valid trajectory $z_{-m},\ldots,z_m$ (in particular, $z_0=\rho$ and $z_{-m},\ldots,z_m\in T_{|m}$) we have

$$\begin{aligned}
&\hat{P}_{\omega^{(n)}}(X_{-m}=z_{-m},\ldots,X_m=z_m)\\
&\quad=P_{\omega^{(n)}}(X_0=z_0,\ldots,X_m=z_m)P_{\omega^{(n)}}(X_0=z_0,\ldots,X_{-m}=z_{-m})\\
&\quad=\prod_{k=0}^{m-1}\frac{\xi^{(n)}(z_k,z_{k+1})}{C^{(n)}(z_k)}\prod_{k=0}^{m-1}\frac{\xi^{(n)}\big(z_{-k},z_{-(k+1)}\big)}{C^{(n)}(z_{-k})},
\end{aligned}$$

where $C^{(n)}(v)=\sum_{w\sim v}\xi^{(n)}(v,w)$, analogously to $C(v)=\sum_{w\sim v}\xi(v,w)$. We define

$$\begin{aligned}
k_0^+&=\inf\{k\in\{0,\ldots,m-1\}:C(z_k)=0\},\\
k_0^-&=\inf\{k\in\{0,\ldots,m-1\}:C(z_{-k})=0\}.
\end{aligned}$$

If $k_0^+=k_0^-=\infty$, the path $z_{-m},\ldots,z_m$ has positive probability under $\hat{P}_\omega$. Due to the convergence of $\xi^{(n)}$, we easily obtain

$$\frac{C^{(n)}(\rho)}{\deg(\rho)}\hat{P}_{\omega^{(n)}}(X_{-m}=z_{-m},\ldots,X_m=z_m)\xrightarrow[n\to\infty]{}\frac{C(\rho)}{\deg(\rho)}\hat{P}_\omega(X_{-m}=z_{-m},\ldots,X_m=z_m).$$

Otherwise, $z_{-m},\ldots,z_m$ is not a possible trajectory under $\hat{P}_\omega$ and therefore we have to show

$$\frac{C^{(n)}(\rho)}{\deg(\rho)}\hat{P}_{\omega^{(n)}}(X_{-m}=z_{-m},\ldots,X_m=z_m)\xrightarrow[n\to\infty]{}0.$$

If $0<k_0^+<\infty$, there exists a z_k along the path in positive time with $C(z_k)=0$. Due to definition of k_0^+ we have $\xi^{(n)}\big(z_{k_0^+-1},z_{k_0^+}\big)\to\xi\big(z_{k_0^+-1},z_{k_0^+}\big)=0$ and $C^{(n)}\big(z_{k_0^+-1}\big)\to C\big(z_{k_0^+-1}\big)>0$. This implies

$$0\le\frac{C^{(n)}(\rho)}{\deg(\rho)}\hat{P}_{\omega^{(n)}}(X_{-m}=z_{-m},\ldots,X_m=z_m)\le\frac{C^{(n)}(\rho)}{\deg(\rho)}\frac{\xi^{(n)}\big(z_{k_0^+-1},z_{k_0^+}\big)}{C^{(n)}\big(z_{k_0^+-1}\big)}\xrightarrow[n\to\infty]{}0.$$

If $0 < k_0^- < \infty$, we obtain analogously

$$\frac{C^{(n)}(\rho)}{\deg(\rho)}\hat{P}_{\omega^{(n)}}(X_{-m} = z_{-m}, \dots, X_m = z_m) \xrightarrow[n\to\infty]{} 0.$$

Otherwise, if $k_0^+ = k_0^- = 0$, we have $C(\rho) = 0$ and in particular $\xi^{(n)}(\rho, z_1) \to \xi(\rho, z_1) = 0$. We then get

$$0 \leq \frac{C^{(n)}(\rho)}{\deg(\rho)}\hat{P}_{\omega^{(n)}}(X_{-m} = z_{-m}, \dots, X_m = z_m) \leq \frac{C^{(n)}(\rho)}{\deg(\rho)}\frac{\xi^{(n)}(\rho, z_1)}{C^{(n)}(\rho)} = \frac{\xi^{(n)}(\rho, z_1)}{\deg(\rho)} \xrightarrow[n\to\infty]{} 0,$$

which completes the proof of the continuity of $h_{z_{-m},\dots,z_m,M}$. In view of (3.3.41), this yields the weak convergence $\hat{\mathbb{P}}^*_\varepsilon \xrightarrow{w} \hat{\mathbb{P}}^*_0$.

Weak convergence of $\hat{\mathbb{P}}_\varepsilon$

The set of environments Ω is a metrizable and separable space (see Section 2.1). Hence, there exists a countable base $\mathcal{B}$ for the topology on Ω. For the topology on $\mathbb{T}^{\mathbb{Z}}$ the family

$$\mathcal{C} = \big\{\{x \in \mathbb{T}^{\mathbb{Z}} : x_{-m} = v_{-m}, \dots, x_m = v_m\} : m \geq 1,\, v_{-m}, \dots, v_m \in \mathbb{T}\big\}$$

defines a countable base. Thus, the sets $B \times C$ with $B \in \mathcal{B}$, $C \in \mathcal{C}$ form a countable base for the product topology on $\Omega \times \mathbb{T}^{\mathbb{Z}}$ (see e.g. Theorem 15.1 in [Mun00]).

Let $G \in \mathcal{G} \otimes \hat{\mathcal{F}}$ be an open set. Then there exist open basis sets $B_i \times C_i$ with $B_i \in \mathcal{B}$ and

$$C_i = \big\{x \in \mathbb{T}^{\mathbb{Z}} : x_{-m_i} = v^i_{-m_i}, \dots, x_{m_i} = v^i_{m_i}\big\} \in \mathcal{C}$$

such that $G = \bigcup_{i=1}^\infty (B_i \times C_i)$. The continuity of probability measures implies

$$\hat{\mathbb{P}}_0\left(\bigcup_{i=1}^r (B_i \times C_i)\right) \xrightarrow[r\to\infty]{} \hat{\mathbb{P}}_0(G).$$

Thus, given an arbitrary $\delta > 0$, we can choose $r \in \mathbb{N}$ such that

$$\hat{\mathbb{P}}_0\left(\bigcup_{i=1}^r (B_i \times C_i)\right) > \hat{\mathbb{P}}_0(G) - \delta.$$

We set $m^* = \max_{i=1,\dots,r} m_i$ and

$$C_i^* = \big\{x \in \mathbb{T}^{2m^*+1} : x_{-m_i} = v^i_{-m_i}, \dots, x_m = v^i_{m_i}\big\} \in \hat{\mathcal{F}}_{m^*},$$

Then we have $\hat{\pi}_{m^*}^{-1}(C_i^*) = C_i$ and therefore

$$\bigcup_{i=1}^r (B_i \times C_i) = \bigcup_{i=1}^r \big(B_i \times \hat{\pi}_{m^*}^{-1}(C_i^*)\big) = \bigcup_{i=1}^r \hat{\Pi}_{m^*}^{-1}(B_i \times C_i^*) = \hat{\Pi}_{m^*}^{-1}\left(\bigcup_{i=1}^r (B_i \times C_i^*)\right),$$

where $\hat{\pi}_{m^*}$ and $\hat{\Pi}_{m^*}$ are the projections defined in (3.3.38) and (3.3.40), respectively. Recall that $\hat{\mathbb{P}}^*_\varepsilon$ is the pushforward measure of $\hat{\mathbb{P}}_\varepsilon$ under $\hat{\Pi}_{m^*}$, we obtain

$$\begin{aligned}\liminf_{\varepsilon\to 0} \hat{\mathbb{P}}_\varepsilon(G) &\geq \liminf_{\varepsilon\to 0} \hat{\mathbb{P}}_\varepsilon\left(\bigcup_{i=1}^r (B_i \times C_i)\right) = \liminf_{\varepsilon\to 0} \hat{\mathbb{P}}^*_\varepsilon\left(\bigcup_{i=1}^r (B_i \times C_i^*)\right) \\ &\geq \hat{\mathbb{P}}^*_0\left(\bigcup_{i=1}^r (B_i \times C_i^*)\right) = \hat{\mathbb{P}}_0\left(\bigcup_{i=1}^r (B_i \times C_i)\right) \\ &> \hat{\mathbb{P}}_0(G) - \delta,\end{aligned}$$

where we used the weak convergence of $\hat{\mathbb{P}}^*_\varepsilon$ and the Portmanteau Theorem (see Theorem 2.1 in [Bil99]) for the second inequality. Since $\delta > 0$ was arbitrary we have

$$\liminf_{\varepsilon\to 0} \hat{\mathbb{P}}_\varepsilon(G) \geq \hat{\mathbb{P}}_0(G) \tag{3.3.44}$$

for all open sets $G \in \mathcal{G} \otimes \hat{\mathcal{F}}$. Due to the Portmanteau Theorem, this is equivalent to the weak convergence $\hat{\mathbb{P}}_\varepsilon \xrightarrow{w} \hat{\mathbb{P}}_0$. $\square$

Chapter 4

The speed of random walk on Galton-Watson trees with changing offspring law

The random walk on infinite supercritical weighted Galton-Watson trees satisfies a law of large numbers with the speed of the walk as limit. In the previous chapter we investigated the regularity of the speed as a function of the marginal law of the conductances. However, the speed does not only depend on the distribution of the conductances, but also on the offspring law. In this chapter we study its behaviour as a function of the offspring distribution.

We present the main results of this chapter in the first section. The proofs are given in Section 4.2.

4.1 Main results

We let T be an infinite supercritical Galton-Watson tree with i.i.d. conductances and $(X_n)_{n\geq 0}$ is the random walk on T starting at the root. Recall that the speed of the walk is given by

$$\lim_{n\to\infty}\frac{|X_n|}{n}=v\quad \mathbb{P}-\text{almost surely},$$

see Theorem 2.3.5. In order to study the regularity of the speed as a function of the offspring law, we consider a sequence of offspring distributions $(\nu_n)_{n\in\mathbb{N}}$ that converges weakly to a measure ν. In particular, this is equivalent to

$$\nu_n(\{k\})\xrightarrow[n\to\infty]{}\nu(\{k\})\quad\text{for all } k\geq 0.$$

We write P_n for the environment law P if the offspring distribution is given by ν_n. As before, we assume that under P_n, the tree T is supercritical without leaves. The convergence above implies that the tree with offspring law ν also has no leaves, but it might no longer be supercritical. If $\nu(\{1\}) = 1$, i.e. if the asymptotic offspring law ν is given by the Dirac measure δ_1, each vertex has exactly one descendant and the Galton-Watson tree is not supercritical. The random walk is then almost surely recurrent and therefore it has zero speed, i.e. $v(\delta_1, \mu) = 0$. For any other distribution ν we have $\int x \, \mathrm{d}\nu(x) > 1$, which means that the Galton-Watson tree with offspring law ν is still supercritical. Hence, by Proposition 2.3.4 the random walk is almost surely transient and Theorem 2.3.5 implies that it has positive speed $v(\nu, \mu) > 0$. In both settings the speed $v(\nu, \mu)$ is given by the limit of the speed $v(\nu_n, \mu)$ when n tends to infinity, as stated in the following theorem. Thus, the speed is a continuous function of the offspring law.

Theorem 4.1.1. *We consider a random walk on a supercritical Galton-Watson tree without leaves and with uniformly elliptic conductances. We let $(\nu_n)_{n\in\mathbb{N}}$ be a sequence of offspring distributions that converges weakly to a measure ν. We assume that $\nu_n(\{0\}) = 0$, $\int x \, \mathrm{d}\nu_n(x) \in (1, \infty)$ for all $n \geq 1$ and $\int x \, \mathrm{d}\nu(x) < \infty$. Then we have*

$$\lim_{n\to\infty} v(\nu_n, \mu) = v(\nu, \mu).$$

In order to prove Theorem 4.1.1 the invariant measure for the environment seen from the random walk introduced in Section 3.2 and the formula for the speed given in (3.2.7) are crucial. We let $\mathrm{P}_n^{\mathrm{aug}}$ be the augmented environment law as defined in (3.2.1) with P_ε replaced by P_n such that under $\mathrm{P}_n^{\mathrm{aug}}$, T is an augmented Galton-Watson tree with offspring law ν_n and i.i.d. conductances with marginal law μ. The invariant measure $\hat{\mathrm{P}}_n$ is then defined analogously to (3.2.2) and the corresponding annealed law is denoted by $\hat{\mathbb{P}}_n$, recall (2.3.3). Furthermore, we let $\mathrm{P}^{\mathrm{aug}}$, $\hat{\mathrm{P}}$ and $\hat{\mathbb{P}}$ be the corresponding measures when the offspring law is given by ν.

Lemma 4.1.2. *As $n \to \infty$, we have $\hat{\mathrm{P}}_n \xrightarrow{w} \hat{\mathrm{P}}$ and $\hat{\mathbb{P}}_n \xrightarrow{w} \hat{\mathbb{P}}$ weakly.*

As shown by [GMPV12], from the ergodicity of the environment observed by the particle under $\hat{\mathbb{P}}_n$ we get the following formula for the speed

$$\lim_{k\to\infty} \frac{|X_k|}{k} = \hat{\mathbb{E}}_n\big[[X_1 - X_0]_{X_{-\infty}}\big] \tag{4.1.1}$$

for any $n \geq 1$ (see also (3.2.7)). Recall that $[u - v]_{x_{-\infty}}$ denotes the horodistance from u to v relative to the boundary point $x_{-\infty}$, see definition (3.2.6). We note that the distance $[X_1 - X_0]_{X_{-\infty}}$ is only well-defined when the random walk in negative time is transient. As mentioned above, $\nu = \delta_1$ is the only weak limit so that the Galton-Watson tree with offspring law ν is not supercritical and thus the random walk is not transient. We therefore treat this case separately. The limits are given in the following two propositions.

Proposition 4.1.3. *If the Galton-Watson tree with offspring distribution ν is supercritical, the mean of $[X_1 - X_0]_{X_{-\infty}}$ converges to its mean under $\hat{\mathbb{P}}$,*

$$\lim_{n\to\infty} \hat{\mathbb{E}}_n\big[[X_1 - X_0]_{X_{-\infty}}\big] = \hat{\mathbb{E}}\big[[X_1 - X_0]_{X_{-\infty}}\big].$$

Proposition 4.1.4. *If $\nu = \delta_1$, the mean of $[X_1 - X_0]_{X_{-\infty}}$ vanishes,*

$$\lim_{n\to\infty} \hat{\mathbb{E}}_n\big[[X_1 - X_0]_{X_{-\infty}}\big] = 0.$$

Since the random walk on a Galton-Watson tree with offspring law δ_1 is recurrent, we have

$$v(\delta_1, \mu) = 0.$$

To see the recurrence, note that by Theorem 2.4.2, it is sufficient to show that the effective conductance from the root to infinity is zero for almost all environments. For any environment ω where each vertex has exactly one descendant the Series Law (Lemma 2.4.4) combined with Rayleigh's Monotonicity Principle (Lemma 2.4.5) yields $\mathcal{C}_\omega(\rho, \infty) = 0$, since the conductances are bounded.

Thus, in combination with the representation for the speed in (4.1.1), these two propositions provide the proof of Theorem 4.1.1.

We remark that the convergence of the speed in Theorem 4.1.1 is a direct consequence of the weak convergence of the offspring law when the marginal law of the conductances is degenerated, i.e. $\mu = \delta_a$ for some $a > 0$. Using the explicit formula for the speed of the simple random walk on Galton-Watson tree in Theorem 3.2 in [LPP95], we obtain

$$\lim_{n\to\infty} v(\nu_n, \delta_a) = \lim_{n\to\infty} \mathrm{E}_n\left[\frac{N-1}{N+1}\right] = \mathrm{E}\left[\frac{N-1}{N+1}\right] = v(\nu, \delta_a),$$

where N denotes the number of offspring of the root.

4.2 Proofs

4.2.1 Continuity of the speed: proof of Theorem 4.1.1

Recalling (4.1.1), for any $n \geq 1$ the speed of the random walk is given by

$$v(\nu_n, \mu) = \hat{\mathbb{E}}_n\big[[X_1 - X_0]_{X_{-\infty}}\big] \quad \hat{\mathbb{P}}_n - \text{a.s.} \tag{4.2.1}$$

We first note that the statement does not follow immediately from the weak convergence of the invariant measures, since $[X_1 - X_0]_{X_{-\infty}}$ is not a continuous function of the trajectory. Moreover, as discussed above, we have to distinguish whether the Galton-Watson tree with offspring law ν is supercritical or not. If $\nu \neq \delta_1$, it is still supercritical. Combining Proposition 4.1.3 with the representation for the speed in (4.2.1), we obtain

$$\lim_{n\to\infty} v(\nu_n, \mu) = \hat{\mathbb{E}}\big[[X_1 - X_0]_{X_{-\infty}}\big] = v(\nu, \mu).$$

Otherwise, if $\nu = \delta_1$, Proposition 4.1.4 implies that the expectation in (4.2.1) vanishes as $n \to \infty$. Since the random walk on a Galton-Watson tree with offspring law δ_1 is recurrent, we get

$$\lim_{n\to\infty} v(\nu_n, \mu) = 0 = v(\delta_1, \mu),$$

which completes the proof. □

It remains to prove Proposition 4.1.3 and Proposition 4.1.4.

Proof of Proposition 4.1.3

The main ingredient of the proof is the weak convergence of the invariant measures. As in the proof of Proposition 3.2.2 in Chapter 3, we approximate $D_\infty = [X_1 - X_0]_{X_{-\infty}}$ by the continuous and bounded functions $D_M = [X_1 - X_0]_{X_{-M}}$, recall (3.3.7). We have

$$\lim_{M\to\infty}\lim_{n\to\infty} \hat{\mathbb{E}}_n\big[D_M\big] = \lim_{M\to\infty} \hat{\mathbb{E}}\big[D_M\big] = \hat{\mathbb{E}}\big[D_\infty\big],$$

where we used the weak convergence of the invariant measure in Lemma 4.1.2 for the first equality. The second step holds by dominated convergence, since D_M converges to D_∞ almost surely. Thus, it remains to show

$$\lim_{M\to\infty}\limsup_{n\to\infty} \hat{\mathbb{E}}_n\big[|D_M - D_\infty|\big] = 0. \tag{4.2.2}$$

On the event that $X_{-k} \neq \rho$ for all $k \geq M$ we have $D_M = D_\infty$, which implies

$$\hat{\mathbb{E}}_n\big[|D_M - D_\infty|\big] \leq 2\hat{\mathbb{P}}_n(X_{-k} = \rho \text{ for some } k \geq M, X_{-M} \neq \rho) + 2\hat{\mathbb{P}}_n(X_{-M} = \rho). \tag{4.2.3}$$

From the weak convergence of the invariant measure and the transience of the random walk we get

$$\lim_{n\to\infty} \hat{\mathbb{P}}_n(X_{-M} = \rho) = \hat{\mathbb{P}}(X_{-M} = \rho) \xrightarrow[M\to\infty]{} 0.$$

Concerning the first summand in (4.2.3), we bound the probability of hitting the root in terms of effective conductances as in the proof of Proposition 3.2.2. Applying the Cauchy-Schwarz inequality afterwards we arrive at

$$\begin{aligned}
&\hat{\mathbb{P}}_n(X_{-k} = \rho \text{ for some } k \geq M, X_{-M} \neq \rho) \\
&\quad\leq \hat{\mathbb{E}}_n\left[\frac{\mathcal{C}_\omega(X_{-M}, \rho)}{\mathcal{C}_\omega(X_{-M}, \infty)}\mathbb{1}_{\{X_{-M}\neq\rho\}}\right] \\
&\quad\leq \hat{\mathbb{E}}_n\big[\mathcal{C}_\omega(X_{-M}, \rho)^2\mathbb{1}_{\{X_{-M}\neq\rho\}}\big]^{\frac{1}{2}}\hat{\mathbb{E}}_n\big[\mathcal{R}_\omega(X_{-M}, \infty)^2\big]^{\frac{1}{2}}.
\end{aligned} \tag{4.2.4}$$

Analogously to (3.3.20), the first term vanishes as $M \to \infty$, uniformly in n,

$$\lim_{M\to\infty}\lim_{n\to\infty} \hat{\mathbb{E}}_n\big[\mathcal{C}_\omega(X_{-M}, \rho)^2\mathbb{1}_{\{X_{-M}\neq\rho\}}\big] = \lim_{M\to\infty} \hat{\mathbb{E}}\big[\mathcal{C}_\omega(X_{-M}, \rho)^2\mathbb{1}_{\{X_{-M}\neq\rho\}}\big] = 0.$$

Consequently, we are done once we have shown that the second expectation in (4.2.4) remains bounded. Using that the process $(T, X_k, \xi)_{k\in\mathbb{Z}}$ is stationary under $\hat{\mathbb{P}}_n$, we obtain

$$\hat{\mathbb{E}}_n\big[\mathcal{R}_\omega(X_{-M},\infty)^2\big] = \hat{\mathbb{E}}_n\big[\mathcal{R}_\omega(\rho,\infty)^2\big] = \mathrm{E}_n^{\mathrm{aug}}\left[\mathcal{R}_\omega(\rho,\infty)^2 \frac{C(\rho)}{\gamma \deg(\rho)}\right] \le \frac{\kappa}{\gamma}\mathrm{E}_n^{\mathrm{aug}}\big[\mathcal{R}_\omega(\rho,\infty)^2\big].$$

By Rayleigh's Monotonicity Principle (Lemma 2.4.5), removing edges can only increase the effective resistance such that

$$\mathrm{E}_n^{\mathrm{aug}}\big[\mathcal{R}_\omega(\rho,\infty)^2\big] \le \mathrm{E}_n\big[\mathcal{R}_\omega(\rho,\infty)^2\big].$$

Under P_n, T is a supercritical Galton-Watson tree without leaves and with uniformly elliptic conductances. Hence, combining the bounds in (3.3.21) and (3.3.24) from the proof of Lemma 3.3.1 gives the following upper bound for the second moment of the effective resistance

$$\mathrm{E}_n\big[\mathcal{R}_\omega(\rho,\infty)^2\big] \le \frac{\kappa^2 \mathrm{E}_n[N^{-2}](1+\mathrm{E}_n[N^{-1}])}{(1-\mathrm{E}_n[N^{-2}])(1-\mathrm{E}_n[N^{-1}])},$$

where N denotes the number of offspring of the root. We notice that N^{-k} is a continuous function of the environment, bounded by 1, which implies

$$\lim_{n\to\infty} \mathrm{E}_n\big[N^{-k}\big] = \mathrm{E}\big[N^{-k}\big],$$

since $\mathrm{P}_n \xrightarrow{w} \mathrm{P}$. Moreover, for a supercritical Galton-Watson tree without leaves we have $\mathrm{E}[N^{-k}] < 1$. We finally obtain the boundedness of the second expectation in (4.2.4)

$$\lim_{n\to\infty} \hat{\mathbb{E}}_n\big[\mathcal{R}_\omega(X_{-M},\infty)^2\big] \le \frac{\kappa^3 \mathrm{E}[N^{-2}](1+\mathrm{E}[N^{-1}])}{\gamma(1-\mathrm{E}[N^{-2}])(1-\mathrm{E}[N^{-1}])} < \infty.$$

In total, this yields (4.2.2) and completes the proof. □

Proof of Proposition 4.1.4

We have

$$\lim_{n\to\infty} \hat{\mathbb{E}}_n\big[[X_1 - X_0]_{X_{-\infty}}\big] = \lim_{n\to\infty} \hat{\mathbb{E}}_n\big[[X_1 - X_0]_{X_{-\infty}} \,\big|\, \deg(\rho) = 2\big], \tag{4.2.5}$$

since $\hat{\mathrm{P}}_n(\deg(\rho) = 2) = \nu_n(\{1\}) \to 1$. Recall that $X_{-\infty}$ is the boundary point to which the random walk in negative time $(X_{-k})_{k\ge 0}$ converges. That is, $X_{-\infty}$ is the infinite path that does not backtrack and that intersects $(X_{-k})_{k\ge 0}$ infinitely often. We write $X_{-\infty,0}, X_{-\infty,1}, X_{-\infty,2}, \dots$ for the vertices along $X_{-\infty}$. Note that the random walk in negative time is transient for all $n \ge 1$ and therefore $X_{-\infty}$ is well-defined. We have

$$[X_1 - X_0]_{X_{-\infty}} = \begin{cases} -1, & X_1 = X_{-\infty,1} \\ +1, & X_1 \neq X_{-\infty,1}. \end{cases}$$

On the event $\{\deg(\rho) = 2\}$, we let v_+ and v_- be the two neighbours of the root. The expectation in (4.2.5) is then given by

$$\begin{aligned}\hat{\mathbb{E}}_n\big[[X_1 - X_0]_{X_{-\infty}} \,\big|\, \deg(\rho) = 2\big] = &\ \hat{\mathbb{P}}_n(X_1 = v_+, X_{-\infty,1} = v_- \mid \deg(\rho) = 2)\\ &+ \hat{\mathbb{P}}_n(X_1 = v_-, X_{-\infty,1} = v_+ \mid \deg(\rho) = 2)\\ &- \hat{\mathbb{P}}_n(X_1 = X_{-\infty,1} = v_+ \mid \deg(\rho) = 2)\\ &- \hat{\mathbb{P}}_n(X_1 = X_{-\infty,1} = v_- \mid \deg(\rho) = 2).\end{aligned}$$

Let us study the first probability. For convenience, we set $\hat{\mathrm{P}}_{A,n} = \hat{\mathrm{P}}_n(\,\cdot\,|\,A)$ and $\hat{\mathrm{P}}_A = \hat{\mathrm{P}}(\,\cdot\,|\,A)$ with $A = \{\deg(\rho) = 2\}$. We write $\hat{\mathrm{E}}_{A,n}$ and $\hat{\mathrm{E}}_A$ for the corresponding expectations. Moreover, we denote by $T_{|1}$ the subtree up to the first generation of T and $\omega_{|1} = (T_{|1}, \rho, \xi(e)_{e\in\mathcal{E}(T_{|1})})$ is the associated environment. Recalling the definition of the annealed law in (2.3.3), we have

$$\begin{aligned}\hat{\mathbb{P}}_n(X_1 = v_+, X_{-\infty,1} = v_- \mid \deg(\rho) = 2) &= \hat{\mathrm{E}}_n\big[\hat{P}_\omega(X_1 = v_+, X_{-\infty,1} = v_-) \,\big|\, \deg(\rho) = 2\big]\\ &= \hat{\mathrm{E}}_{A,n}\big[\hat{P}_\omega(X_1 = v_+, X_{-\infty,1} = v_-)\big]\\ &= \hat{\mathrm{E}}_{A,n}\big[\hat{\mathrm{E}}_{A,n}\big[P_\omega(X_1 = v_+)P_\omega(X_{-\infty,1} = v_-) \,\big|\, \omega_{|1}\big]\big].\end{aligned}$$

Here, in the last step we used that $(X_k)_{k\geq 0}$ and $(X_{-k})_{k\geq 0}$ are independent under $\hat{P}_\omega$. The probability that the random walk moves to v_+ in its first step is $\sigma(\omega_{|1})$-measurable. Hence, we obtain

$$\begin{aligned}\hat{\mathbb{P}}_n(X_1 = v_+, X_{-\infty,1} = v_- \mid \deg(\rho) = 2) &= \hat{\mathrm{E}}_{A,n}\left[\frac{\xi(\rho, v_+)}{C(\rho)}\hat{\mathrm{E}}_{A,n}\big[P_\omega(X_{-\infty,1} = v_-) \,\big|\, \omega_{|1}\big]\right]\\ &= \hat{\mathrm{E}}_{A}\left[\frac{\xi(\rho, v_+)}{C(\rho)}\hat{\mathrm{E}}_{A,n}\big[P_\omega(X_{-\infty,1} = v_-) \,\big|\, \omega_{|1}\big]\right]\end{aligned}$$

with $C(\rho) = \xi(\rho, v_+) + \xi(\rho, v_-)$. To see the last equality, observe that the distribution of $\omega_{|1}$ under $\hat{\mathrm{P}}_{A,n}$ does not depend on n. Conducting the same calculations for the other probabilities in (4.2.5) we arrive at

$$\begin{aligned}&\hat{\mathbb{E}}_n\big[[X_1 - X_0]_{X_{-\infty}} \,\big|\, \deg(\rho) = 2\big]\\ &= \hat{\mathrm{E}}_A\left[\frac{\xi(\rho, v_+) - \xi(\rho, v_-)}{C(\rho)}\Big(\hat{\mathrm{E}}_{A,n}\big[P_\omega(X_{-\infty,1} = v_-) \,\big|\, \omega_{|1}\big] - \hat{\mathrm{E}}_{A,n}\big[P_\omega(X_{-\infty,1} = v_+) \,\big|\, \omega_{|1}\big]\Big)\right].\end{aligned}$$

Using Lemma 4.2.1 below, we finally obtain

$$\lim_{n\to\infty} \hat{\mathbb{E}}_n\big[[X_1 - X_0]_{X_{-\infty}} \,\big|\, \deg(\rho) = 2\big] = 0$$

by dominated convergence, which completes the proof. □

Lemma 4.2.1. *If $\nu_n \xrightarrow{w} \delta_1$, with the notation from the proof of Proposition 4.1.4 we have*

$$\lim_{n\to\infty} \hat{\mathrm{E}}_{A,n}\big[P_\omega(X_{-\infty,1} = v_+) \,\big|\, \omega_{|1}\big] = \lim_{n\to\infty} \hat{\mathrm{E}}_{A,n}\big[P_\omega(X_{-\infty,1} = v_-) \,\big|\, \omega_{|1}\big] = \frac{1}{2}.$$

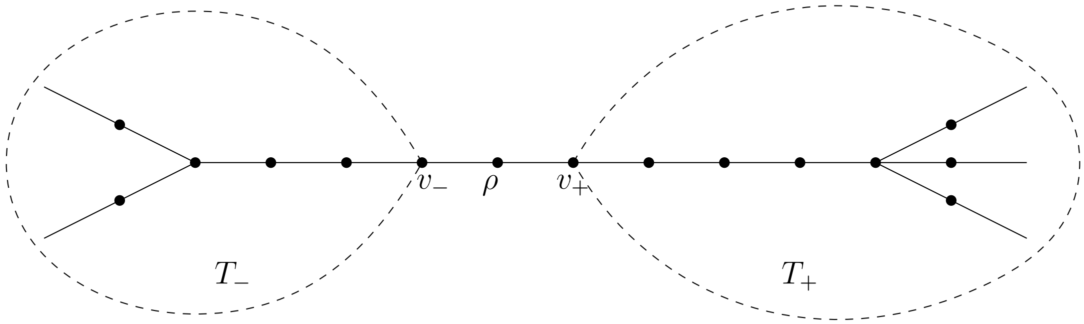


Figure 4.1: Under $\hat{\mathrm{P}}_{A,n}$, the root ρ has two neighbours denoted by v_+ and v_-. The subtrees T_+ and T_- are independent Galton-Watson trees with offspring law ν_n.

Proof. To start with, we introduce some notations. We let $\omega = (T, \rho, \xi)$ be an environment where the root has two neighbours v_+ and v_-. We denote the subtree rooted at v_+ composed of v_+ and all its descendants by $T_+ = T(v_+)$. Analogously, we set $T_- = T(v_-)$, see Figure 4.1. We write $\omega_+ = (T_+, v_+, (\xi(e))_{e\in\mathcal{E}(T_+)})$ and $\omega_- = (T_-, v_-, (\xi(e))_{e\in\mathcal{E}(T_-)})$ for the corresponding environments. We assume that the effective resistances $\mathcal{R}_{\omega_+}(v_+, \infty)$ and $\mathcal{R}_{\omega_-}(v_-, \infty)$ are both finite. Note that this is satisfied for almost all environments under $\hat{\mathrm{P}}_{A,n}$. Furthermore, we let G_m^+ and G_m^- be the m-th generation of the subtrees T_+ and T_-, respectively. Using the continuity of probability measures, we can write

$$\begin{aligned} P_\omega(X_{-\infty,1} = v_+) &= P_\omega\big(\exists k \in \mathbb{N} : \forall m \in \mathbb{N} : \eta_{G_m^+} < \eta_{G_k^-}\big) \\ &= \lim_{k\to\infty} P_\omega\big(\forall m \in \mathbb{N} : \eta_{G_m^+} < \eta_{G_k^-}\big) \\ &= \lim_{k\to\infty} \lim_{m\to\infty} P_\omega\big(\eta_{G_m^+} < \eta_{G_k^-}\big), \end{aligned}$$

where η_B is the hitting time of a set of vertices $B \subseteq T$, recall (2.4.1). Using the Series Law (Lemma 2.4.4) and the uniform ellipticity of the conductances, we obtain by Lemma 2.4.1

$$\begin{aligned} P_\omega\big(\eta_{G_m^+} < \eta_{G_k^-}\big) &= \frac{\mathcal{C}_\omega(\rho, G_m^+)}{\mathcal{C}_\omega(\rho, G_m^+) + \mathcal{C}_\omega(\rho, G_k^-)} \\ &= \frac{\mathcal{R}_\omega(\rho, G_m^-)}{\mathcal{R}_\omega(\rho, G_m^+) + \mathcal{R}_\omega(\rho, G_k^-)} \\ &= \frac{\xi(\rho, v_-)^{-1} + \mathcal{R}_{\omega_-}(v_-, G_m^-)}{\xi(\rho, v_+)^{-1} + \mathcal{R}_{\omega_+}(v_+, G_m^+) + \xi(\rho, v_-)^{-1} + \mathcal{R}_{\omega_-}(v_-, G_k^-)} \\ &\leq \frac{\kappa + \mathcal{R}_{\omega_-}(v_-, G_m^-)}{2\kappa^{-1} + \mathcal{R}_{\omega_+}(v_+, G_m^+) + \mathcal{R}_{\omega_-}(v_-, G_k^-)} \end{aligned} \tag{4.2.6}$$

and therefore

$$P_\omega(X_{-\infty,1} = v_+) \leq \frac{\kappa + \mathcal{R}_{\omega_-}(v_-, \infty)}{2\kappa^{-1} + \mathcal{R}_{\omega_+}(v_+, \infty) + \mathcal{R}_{\omega_-}(v_-, \infty)}.$$

Since the right-hand side is independent of the conductances in the first generation, we get

$$
\begin{aligned}
\hat{\mathrm{E}}_{A,n}\big[P_\omega(X_{-\infty,1}=v_+)\;\big|\;\omega_{|1}\big] &\le \hat{\mathrm{E}}_{A,n}\left[\frac{\kappa+\mathcal{R}_{\omega_-}(v_-,\infty)}{2\kappa^{-1}+\mathcal{R}_{\omega_+}(v_+,\infty)+\mathcal{R}_{\omega_-}(v_-,\infty)}\;\middle|\;\omega_{|1}\right] \\
&= \hat{\mathrm{E}}_{A,n}\left[\frac{\kappa+\mathcal{R}_{\omega_-}(v_-,\infty)}{2\kappa^{-1}+\mathcal{R}_{\omega_+}(v_+,\infty)+\mathcal{R}_{\omega_-}(v_-,\infty)}\right]. \qquad (4.2.7)
\end{aligned}
$$

We notice that $\mathcal{R}_{\omega_+}(v_+,\infty)$ and $\mathcal{R}_{\omega_-}(v_-,\infty)$ are identically distributed under $\hat{\mathrm{P}}_{A,n}$, which implies

$$
\hat{\mathrm{E}}_{A,n}\left[\frac{\kappa^{-1}+\mathcal{R}_{\omega_-}(v_-,\infty)}{2\kappa^{-1}+\mathcal{R}_{\omega_+}(v_+,\infty)+\mathcal{R}_{\omega_-}(v_-,\infty)}\right]=\frac{1}{2}
$$

by symmetry. Together with (4.2.7), we obtain

$$
\begin{aligned}
\hat{\mathrm{E}}_{A,n}\big[P_\omega(X_{-\infty,1}=v_+)\;\big|\;\omega_{|1}\big] &\le \frac{1}{2}+\hat{\mathrm{E}}_{A,n}\left[\frac{\kappa-\kappa^{-1}}{2\kappa^{-1}+\mathcal{R}_{\omega_+}(v_+,\infty)+\mathcal{R}_{\omega_-}(v_-,\infty)}\right] \\
&\le \frac{1}{2}+\hat{\mathrm{E}}_{A,n}\left[\frac{\kappa}{2\kappa^{-1}+\mathcal{R}_{\omega_+}(v_+,\infty)}\right] \\
&\le \frac{1}{2}+\hat{\mathrm{E}}_{A,n}\left[\frac{\kappa}{2\kappa^{-1}+\mathcal{R}_{\omega_+}(v_+,G_m^+)}\right]. \qquad (4.2.8)
\end{aligned}
$$

To see the last inequality, observe that the effective resistance $\mathcal{R}_{\omega_+}(v_+,G_m^+)$ is monotonically increasing in m, which implies

$$
\mathcal{R}_{\omega_+}(v_+,\infty)\ge\mathcal{R}_{\omega_+}(v_+,G_m^+)
$$

for any arbitrary $m\ge 1$. If the m-th generation of T_+ consists of a single vertex, the Series Law (Lemma 2.4.4) combined with the uniform ellipticity of the conductances yields $\mathcal{C}_{\omega_+}(v_+,G_m^+)\le\kappa m^{-1}$ and therefore

$$
\hat{\mathrm{E}}_{A,n}\left[\frac{\kappa}{2\kappa^{-1}+\mathcal{R}_{\omega_+}(v_+,G_m^+)}\mathbb{1}_{\{|G_m^+|=1\}}\right]\le\kappa\hat{\mathrm{E}}_{A,n}\big[\mathcal{C}_{\omega_+}(v_+,G_m^+)\mathbb{1}_{\{|G_m^+|=1\}}\big]\le\kappa^2m^{-1}.
$$

In case that the m-th generation of the subtree T_+ consists of more than one vertex, we establish the following bound

$$
\begin{aligned}
\hat{\mathrm{E}}_{A,n}\left[\frac{\kappa}{2\kappa^{-1}+\mathcal{R}_{\omega_+}(v_+,G_m^+)}\mathbb{1}_{\{|G_m^+|>1\}}\right] &\le \kappa^2\hat{\mathrm{P}}_{A,n}(|G_m^+|>1)=\kappa^2\mathrm{P}_n(|G_m(T)|>1) \\
&= \kappa^2(1-\nu_n(\{1\})^m),
\end{aligned}
$$

where we used in the second step that under $\hat{\mathrm{P}}_{A,n}$, T_+ is Galton-Watson trees with law P_n. Note that the right-hand side of the above estimate vanishes as $n\to\infty$, since $\nu_n\xrightarrow{w}\delta_1$

by assumption. In view of (4.2.8), combining the last two estimates implies

$$\begin{aligned}\lim_{n\to\infty} \hat{\mathrm{E}}_{A,n}\big[P_\omega(X_{-\infty,1}=v_+) \bigm| \omega_{|1}\big] &\le \frac{1}{2}+\kappa^2 m^{-1}+\lim_{n\to\infty}\kappa^2(1-\nu_n(\{1\})^m)\\ &= \frac{1}{2}+\kappa^2 m^{-1}\end{aligned}$$

for any arbitrary $m \ge 1$. Consequently, we obtain

$$\lim_{n\to\infty} \hat{\mathrm{E}}_{A,n}\big[P_\omega(X_{-\infty,1}=v_+) \bigm| \omega_{|1}\big] \le \frac{1}{2}. \tag{4.2.9}$$

When we bound the conductances in (4.2.6) in the other direction, we get the following lower bound

$$P_\omega(X_{-\infty,1}=v_+) \ge \frac{\kappa^{-1}+\mathcal{R}_{\omega_-}(v_-,\infty)}{2\kappa+\mathcal{R}_{\omega_+}(v_+,\infty)+\mathcal{R}_{\omega_-}(v_-,\infty)}$$

and an analogous calculation shows

$$\lim_{n\to\infty} \hat{\mathrm{E}}_{A,n}\big[P_\omega(X_{-\infty,1}=v_+) \bigm| \omega_{|1}\big] \ge \frac{1}{2} - \lim_{n\to\infty}\hat{\mathrm{E}}_{A,n}\left[\frac{\kappa-\kappa^{-1}}{2\kappa+\mathcal{R}_{\omega_+}(v_+,\infty)+\mathcal{R}_{\omega_-}(v_-,\infty)}\right] = \frac{1}{2}.$$

Together with (4.2.9), this completes the proof. □

4.2.2 Weak convergence of the invariant measure: proof of Lemma 4.1.2

For any $m \ge 1$ we let $\hat{\mathbb{P}}_n^*$ and $\hat{\mathbb{P}}^*$ be the restricted annealed law on $(\Omega \times \mathbb{T}^{2m+1}, \mathcal{G} \otimes \hat{\mathcal{F}}_m)$ defined as in (3.3.39) with $\hat{\mathrm{P}}_\varepsilon$ replaced by $\hat{\mathrm{P}}_n$ and $\hat{\mathrm{P}}$, respectively. It suffices to show that $\hat{\mathbb{P}}_n^*$ converges weakly to $\hat{\mathbb{P}}^*$. The weak convergence of the annealed law $\hat{\mathbb{P}}_n$ then follows by applying the Portmanteau Theorem analogously to the proof of Lemma 3.2.1. This in turn implies the convergence of $\hat{\mathrm{P}}_n$, since $\hat{\mathrm{P}}_n$ is the marginal law of ω under $\hat{\mathbb{P}}_n$.

Weak convergence of $\hat{\mathbb{P}}_n^*$

Let $f\colon \Omega\times\mathbb{T}^{2m+1}\to\mathbb{R}$ be a continuous and bounded function. We have

$$\begin{aligned}\left|\int f\,\mathrm{d}\hat{\mathbb{P}}_n^* - \int f\,\mathrm{d}\hat{\mathbb{P}}^*\right| &\le \left|\int f\mathbb{1}_{\{|T_{|m}|\le M\}}\,\mathrm{d}\hat{\mathbb{P}}_n^* - \int f\mathbb{1}_{\{|T_{|m}|\le M\}}\,\mathrm{d}\hat{\mathbb{P}}^*\right|\\ &\quad+\left|\int f\mathbb{1}_{\{|T_{|m}|> M\}}\,\mathrm{d}\hat{\mathbb{P}}_n^*\right| + \left|\int f\mathbb{1}_{\{|T_{|m}|> M\}}\,\mathrm{d}\hat{\mathbb{P}}^*\right|,\end{aligned} \tag{4.2.10}$$

where $T_{|m}$, as usual, denotes the first m generations of T. To derive an upper bound on the second and third summand, we need to determine the probability of the event

$\{|T_{|m}| > M\}$. Due to the weak convergence of the offspring distributions, we have for all $k \geq 1$

$$\mathrm{P}_n^{\mathrm{aug}}(\deg(\rho) = k) = \nu_n(\{k-1\}) \xrightarrow[n\to\infty]{} \nu(\{k-1\}) = \mathrm{P}^{\mathrm{aug}}(\deg(\rho) = k).$$

We let $\mathcal{T}_{m,M} = \{T \in \mathcal{T} : |T| \leq M,\, |v| \leq m \,\forall v \in T\}$ be the set of trees with at most M vertices and m generations. Then for any tree $t \in \mathcal{T}_{m,M}$

$$\begin{aligned}\lim_{n\to\infty} \mathrm{P}_n^{\mathrm{aug}}(T_{|m} = t) &= \lim_{n\to\infty} \mathrm{P}_n^{\mathrm{aug}}\big(\deg_T(v) = \deg_t(v)\,\forall v \in t_{|m-1}\big)\\ &= \lim_{n\to\infty} \prod_{v\in t_{|m-1}} \mathrm{P}_n^{\mathrm{aug}}(\deg_T(v) = \deg_t(v))\\ &= \prod_{v\in t_{|m-1}} \mathrm{P}^{\mathrm{aug}}(\deg_T(v) = \deg_t(v))\\ &= \mathrm{P}^{\mathrm{aug}}(T_{|m} = t)\end{aligned}$$

and therefore

$$\lim_{n\to\infty} \mathrm{P}_n^{\mathrm{aug}}(|T_{|m}| \leq M) = \lim_{n\to\infty} \sum_{t\in\mathcal{T}_{m,M}} \mathrm{P}_n^{\mathrm{aug}}(T_{|m} = t) = \mathrm{P}^{\mathrm{aug}}(|T_{|m}| \leq M). \tag{4.2.11}$$

Recalling the definition of the invariant measure in (3.2.2), we get the following upper bound on the second summand in (4.2.10)

$$\begin{aligned}\lim_{n\to\infty} \left| \int f \mathbb{1}_{\{|T_{|m}|>M\}} \,\mathrm{d}\hat{\mathbb{P}}_n^* \right| &\leq \lim_{n\to\infty} \|f\|_\infty \hat{\mathrm{P}}_n(|T_{|m}| > M)\\ &\leq \|f\|_\infty \frac{\kappa}{\gamma} \mathrm{P}^{\mathrm{aug}}(|T_{|m}| > M)\end{aligned}$$

where $\|\cdot\|_\infty$ denotes the sup norm. Obviously, this bound also holds for the third summand in (4.2.10). Since the tree is locally finite, the probability on the right-hand side vanishes as $M \to \infty$ and we can write

$$\lim_{n\to\infty} \left| \int f \,\mathrm{d}\hat{\mathbb{P}}_n^* - \int f \,\mathrm{d}\hat{\mathbb{P}}^* \right| \leq \lim_{n\to\infty} \left| \int f \mathbb{1}_{\{|T_{|m}|\leq M\}} \,\mathrm{d}\hat{\mathbb{P}}_n^* - \int f \mathbb{1}_{\{|T_{|m}|\leq M\}} \,\mathrm{d}\hat{\mathbb{P}}^* \right| + \delta(M) \tag{4.2.12}$$

for $\delta(M)$, independent of n, with $\delta(M) \to 0$ as $M \to \infty$.

As in the proof of Lemma 3.2.1 (see (3.3.42)), we have the following representation for the integral of f on the event $\{|T_{|m}| \leq M\}$

$$\int f(\omega, x) \mathbb{1}_{\{|T_{|m}|\leq M\}} \,\mathrm{d}\hat{\mathbb{P}}_n^*(\omega, x) = \sum_{z_{-m},\ldots,z_m} \frac{1}{\gamma} \int h_{z_{-m},\ldots,z_m,M}(\omega) \,\mathrm{d}\mathrm{P}_n^{\mathrm{aug}}(\omega), \tag{4.2.13}$$

where $h_{z_{-m},\ldots,z_m,M} \colon \Omega \to \mathbb{R}$ is the continuous and bounded function defined in (3.3.43). Thus, we are done once we have proven $\mathrm{P}_n^{\mathrm{aug}} \xrightarrow{w} \mathrm{P}^{\mathrm{aug}}$ weakly.

Weak convergence of $\mathrm{P}_n^{\mathrm{aug}}$

Recall that $\mathrm{P}_n^{\mathrm{aug}}$ is the pushforward measure of $\tilde{\mathrm{P}}_n^{\mathrm{aug}} = \mathrm{GW}_n^{\mathrm{aug}} \otimes \mu^{\otimes \mathcal{E}(\mathbb{T})}$ under π (see (2.2.2)). Here, $\mathrm{GW}_n^{\mathrm{aug}}$ is the probability measure on $\mathcal{T}$ such that under $\mathrm{GW}_n^{\mathrm{aug}}$, T is an augmented Galton-Watson tree with offspring law ν_n. We show that $\mathrm{GW}_n^{\mathrm{aug}}$ converges weakly, which implies the weak convergence of $\tilde{\mathrm{P}}_n^{\mathrm{aug}}$. Since the mapping π is continuous, this also implies the weak convergence of $\mathrm{P}_n^{\mathrm{aug}}$. The proof is done in two steps. First, we show

$$\int f \,\mathrm{d}\,\mathrm{GW}_n^{\mathrm{aug}} \xrightarrow[n\to\infty]{} \int f \,\mathrm{d}\,\mathrm{GW}^{\mathrm{aug}} \tag{4.2.14}$$

for all continuous and bounded functions that only depend on finitely many generations. Afterwards, we conclude that the convergence in (4.2.14) still holds when we integrate uniformly continuous and bounded functions, which is equivalent to the weak convergence of $\mathrm{GW}_n^{\mathrm{aug}}$.

We let $f\colon \mathcal{T} \to \mathbb{R}$ be a continuous, bounded and local function. That is, there exists some $m \geq 0$ such that

$$f(T) = f(T') \quad \text{for all } T,\, T' \in \mathcal{T} \text{ with } T_{|m} = T'_{|m}.$$

We have

$$\begin{aligned}\left| \int f \,\mathrm{d}\,\mathrm{GW}_n^{\mathrm{aug}} - \int f \,\mathrm{d}\,\mathrm{GW}^{\mathrm{aug}} \right| &\leq \left| \int f \mathbb{1}_{\{|T_{|m}| \leq M\}} \,\mathrm{d}\,\mathrm{GW}_n^{\mathrm{aug}} - \int f \mathbb{1}_{\{|T_{|m}| \leq M\}} \,\mathrm{d}\,\mathrm{GW}^{\mathrm{aug}} \right| \\ &\quad + \left| \int f \mathbb{1}_{\{|T_{|m}| > M\}} \,\mathrm{d}\,\mathrm{GW}_n^{\mathrm{aug}} \right| + \left| \int f \mathbb{1}_{\{|T_{|m}| > M\}} \,\mathrm{d}\,\mathrm{GW}^{\mathrm{aug}} \right|. \end{aligned} \tag{4.2.15}$$

The following calculation shows that the first summand vanishes as $n \to \infty$:

$$\begin{aligned} \lim_{n\to\infty} \int f(T) \mathbb{1}_{\{|T_{|m}| \leq M\}} \,\mathrm{d}\,\mathrm{GW}_n^{\mathrm{aug}}(T) &= \lim_{n\to\infty} \sum_{t \in \mathcal{T}_{m,M}} \int f(t) \mathbb{1}_{\{T_{|m} = t\}} \,\mathrm{d}\,\mathrm{GW}_n^{\mathrm{aug}}(T) \\ &= \lim_{n\to\infty} \sum_{t \in \mathcal{T}_{m,M}} f(t)\, \mathrm{GW}_n^{\mathrm{aug}}(T_{|m} = t) \\ &= \sum_{t \in \mathcal{T}_{m,M}} f(t)\, \mathrm{GW}^{\mathrm{aug}}(T_{|m} = t) \\ &= \int f(T) \mathbb{1}_{\{|T_{|m}| \leq M\}} \,\mathrm{d}\,\mathrm{GW}^{\mathrm{aug}}(T). \end{aligned}$$

Due to (4.2.11), we obtain for the second summand

$$\lim_{n\to\infty} \left| \int f \mathbb{1}_{\{|T_{|m}| > M\}} \,\mathrm{d}\,\mathrm{GW}_n^{\mathrm{aug}} \right| \leq \|f\|_\infty \,\mathrm{GW}^{\mathrm{aug}}(|T_{|m}| > M) \xrightarrow[M\to\infty]{} 0$$

for the second summand in (4.2.15). Since this bound is also valid for the third summand, (4.2.14) holds for all continuous, bounded and local functions.

Now, let $f\colon \mathcal{T} \to \mathbb{R}$ be a uniformly continuous and bounded function. We approximate f by the following sequence of local functions

$$f_m\colon \mathcal{T} \to \mathbb{R}, \quad f_m(T) \coloneqq f(T_{|m}).$$

The continuity and boundedness of f are transferred to f_m. Moreover, f_m converges uniformly to f, which is implied by the uniform continuity of f. To see this, observe that for any $\varepsilon > 0$ there exists $m_0 \geq 0$ such that

$$|f_m(T) - f(T)| = |f(T_{|m}) - f(T)| < \varepsilon \quad \text{for all } T \in \mathcal{T},\, m \geq m_0.$$

Finally, using that (4.2.14) holds for local, continuous and bounded functions, we obtain

$$\lim_{n\to\infty} \int f \,\mathrm{d}\,\mathrm{GW}_n^{\mathrm{aug}} = \lim_{n\to\infty} \lim_{m\to\infty} \int f_m \,\mathrm{d}\,\mathrm{GW}_n^{\mathrm{aug}} = \lim_{m\to\infty} \int f_m \,\mathrm{d}\,\mathrm{GW}^{\mathrm{aug}} = \int f \,\mathrm{d}\,\mathrm{GW}^{\mathrm{aug}}.$$

by dominated convergence. Note that the uniform convergence of f_m allows us to interchange the limits in the second equality. Finally, the Portmanteau Theorem implies $\mathrm{GW}_n^{\mathrm{aug}} \xrightarrow{w} \mathrm{GW}^{\mathrm{aug}}$. Together with (4.2.12) and (4.2.13), this yields the weak convergence of the restricted annealed law $\hat{\mathbb{P}}_n^*$. As mentioned above, an analogous application of the Portmanteau Theorem as in the proof of Lemma 3.2.1 completes the proof. □

Chapter 5

A central limit theorem for the random walk on Galton-Watson trees with random conductances

As discussed in the previous chapters, it is well known since the work of [GMPV12] that the random walk on Galton-Watson trees with random conductances moves away from the root with a linear rate. Given a law of large numbers, the natural question arises whether the distance of the random walk to the root satisfies a central limit theorem. In this chapter we will prove a functional central limit theorem when the edges of the tree are assigned randomly uniformly elliptic conductances. Moreover, we will investigate the effect of small conductances on the fluctuations of the random walk.

On Galton-Watson trees the validity of a central limit theorem has been first studied for the simple random walk, which corresponds to the case when all edge weights are identical to the same value. [Pia98] showed that the distance of the simple random walk to the root satisfies a central limit theorem under the annealed law. The proof is based on the existence of a regeneration structure with good moment bounds. One important generalization of this model is the biased random walk, which has a positive limiting speed when the bias parameter is smaller than the offspring mean. A quenched central limit theorem for the distance of this walk on Galton-Watson trees without leaves was proven by [PZ08]. [Bow18] extended the result to the setting of biased random walks on Galton-Watson trees with leaves. A further extension was proven by [Far11] who considered random walks on randomly biased Galton-Watson trees. The other main generalization is the random conductance model considered in this work. Here, a central limit theorem is only known under very strong conditions on the offspring distribution: The number of descendants of each vertex has to be larger than the ratio of the maximum to the minimum edge weight, see [Bar13]. In our setting this is not satisfied. Especially when we consider small edge weights, this would require a great number of offspring.

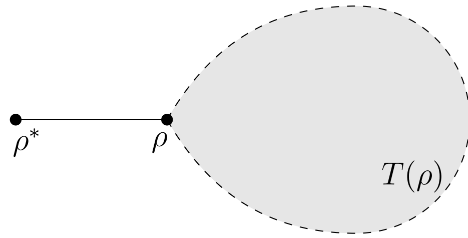


Figure 5.1: Under P, the subtree $T(\rho)$ is a weighted Galton-Watson tree.

This chapter is organized as follows. At the beginning we present the main results of this chapter, the functional central limit theorem in Theorem 5.1.1 and an estimate on the volatility in Theorem 5.1.3. The proof of the central limit theorem relies on the existence of a regeneration structure which yields independent increments with good moment bounds. We define the regeneration times in Section 5.2. Furthermore, the required moment bounds for the regeneration times and distances are stated in this section. All proofs are given in the last section.

The results of this chapter can be found in the preprint [GN24].

5.1 Main results

We let P be the law on Ω such that under P, T is a Galton-Watson tree with offspring law ν, with i.i.d. conductances and with an additional edge (ρ^*, ρ) added to the root ρ. That is, under P, the root has an artificial ancestor which we denote by ρ^*, see Figure 5.1. Note that this slightly different definition of the environment law is intended to simplify the notation in the proofs, but it is not relevant for the validity of the central limit theorem. We set $|\rho^*| = -1$ and $G_{-1} = \{\rho^*\}$. Furthermore, we let $\mathbb{T}^* = \mathbb{T} \cup \{-1\}$ be the Ulam-Harris tree with an artificial ancestor (denoted by -1) added to its root. A tree with an additional ancestor at its root is then a subset of the extended Ulam-Harris tree $\mathbb{T}^*$. Additionally to the assumptions on the offspring distribution in Section 2.2, we assume that ν has finite moments of any order, i.e. $m_k = \int x^k \, \mathrm{d}\nu(x) < \infty$ for all $k \geq 1$. Since we are interested in the behaviour of the random walk when a positive faction of edges has small weights, the marginal law of the conductances is given by

$$\mu_\varepsilon = \alpha\delta_\varepsilon + (1-\alpha)\mu \tag{5.1.1}$$

for some $\alpha \in [0, 1)$, $\varepsilon > 0$ and a uniformly elliptic measure μ satisfying (2.2.1) for some $\kappa \geq 1$. Without loss of generality we assume $\varepsilon < \kappa^{-1}$. In addition, we assume $\mu(\{1\}) > 0$, which means that each edge in the tree has a positive probability to have the weight one. We write P_ε instead of P if we want to emphasize the dependency of the environment measure on ε.

We let $(X_n)_{n\geq 0}$ be the random walk on T starting at the root. The main result in this chapter is a functional central limit theorem for the process

$$W_t^n = \frac{1}{\sqrt{n}}\big(|X_{\lfloor tn\rfloor}| - \lfloor tn\rfloor v(\nu,\mu_\varepsilon)\big), \quad \text{for } t\in[0,1]. \tag{5.1.2}$$

We denote by $\mathbb{D}[0,1]$ the set of real functions on $[0,1]$ that are right-continuous and have left-hand limits. We endow this set with the Skorokhod metric d_S and its Borel σ-Algebra. The Skorokhod metric is defined by

$$d_S(x,y) = \inf_{\lambda\in\Lambda} \max\{\|x\circ\lambda - y\|_\infty, \|\lambda - \mathrm{id}\|_\infty\}, \tag{5.1.3}$$

where $\|\cdot\|_\infty$ denotes the sup norm, id is the identity map on $[0,1]$ and Λ is the set of all monotonically increasing, continuous and bijective functions $\lambda\colon [0,1]\to[0,1]$. The process $(W_t^n)_{t\in[0,1]}$ then defines a random variable which takes values in $\mathbb{D}[0,1]$.

Theorem 5.1.1. *We consider a random walk on a supercritical Galton-Watson tree without leaves and with random conductances. We assume that all moments $m_k<\infty$ of the offspring distribution ν exist and that the marginal distribution of the conductances is given by $\mu_\varepsilon = \alpha\delta_\varepsilon + (1-\alpha)\mu$ for some fixed $\varepsilon>0$. Then there exists a constant $\sigma_\varepsilon^2 = \sigma^2(\nu,\mu_\varepsilon)>0$ such that*

$$\big(W_t^n\big)_{t\in[0,1]} \xrightarrow[n\to\infty]{d} \big(\sigma_\varepsilon B_t\big)_{t\in[0,1]}$$

under $\mathbb{P}$, with $(B_t)_{t\in[0,1]}$ a standard Brownian motion.

Remark 5.1.2. *(a) In Theorem 5.1.1 it is possible to set $\alpha=0$. This implies that the central limit theorem is valid for any marginal distribution of the conductances which is uniformly elliptic and has at least one atom. We assume that* 1 *is the atom but since we can rescale the distribution the choice of the atom is arbitrary.*

(b) For the construction of a regeneration structure with independent increments the assumption that there exists at least one edge weight that occurs with a positive probability is crucial. Unfortunately, this means that a continuous marginal law of the conductances is not possible.

The volatility depends on the offspring distribution and on the marginal law of the conductances in a highly non-trivial way. In particular, it depends on ε. We investigate the effect on the fluctuations of the random walk when a positive fraction of edge has small weights. We show that the volatility σ_ε^2 is bounded away from zero as $\varepsilon\to 0$ when the subtree formed by the edges with larger conductances is supercritical.

Theorem 5.1.3. *If the subtree $T_1(\rho)$ is supercritical, i.e. if $(1-\alpha)m_1>1$, we have for σ_ε^2 from Theorem 5.1.1*

$$\liminf_{\varepsilon\to 0} \sigma_\varepsilon^2 > 0.$$

Here, $T_1(\rho)$ denotes the subtree of $T(\rho)$ formed by the edges with conductance at least κ^{-1} that contains the root.

The question naturally arises whether the volatility is continuous in $\varepsilon = 0$. If $\varepsilon = 0$, a positive fraction of edges gets the weight zero and hence the random walk can only run on a subtree of the original tree. The traversable tree is again a Galton-Watson tree, but it might be finite. Provided that this tree is supercritical, the speed $v(\nu, \mu_0)$ of the random walk is given as the almost sure limit of $|X_n|/n$ under $\bar{\mathbb{P}}$, where $\bar{\mathbb{P}}$ is the annealed law when we condition on the survival of the traversable tree, see Section 3.1. Accordingly, we may then define σ_0^2 as the limiting volatility in such a conditioned environment. But a central limit theorem with volatility σ_0^2, i.e. for a random walk on Galton-Watson trees with leaves, is only known for the simple random walk, which corresponds to the case $\mu = \delta_1$. Similar to the speed, we then expect a discontinuity of the volatility in $\varepsilon = 0$. For small $\varepsilon > 0$ finite subtrees formed by edges with larger conductances act like traps in the environment that can only be left via an edge with weight ε. The time spent in these subtrees is of order ε^{-1}. Whereas, when the small edge weights are reduced to zero the random walk cannot enter these subtrees at all, which allows larger fluctuation of the walk. Therefore, we expect that

$$\liminf_{\varepsilon \to 0} \sigma_\varepsilon^2 < \sigma_0^2.$$

The comparable statement for the speed was specified in Theorem 3.1.2, where we showed that

$$\lim_{\varepsilon \to 0} v(\nu, \mu_\varepsilon) = \beta v(\nu, \mu_0)$$

for a constant $\beta \in [0, 1)$.

5.2 The regeneration structure

A standard argument for proving a central limit theorem for random walks in random environments is to construct a renewal structure to decouple the increments of the random walk. This method goes back to [Kes77]. Similar formulations for random walks on Galton-Watson trees can be found, for example, in [LPP95] and [Pia98]. The advantage of constructing regeneration times is that we obtain an i.i.d. sequence of increments between regenerations which allows us to apply classical limit theorems, provided that the regeneration times and distances are sufficiently integrable. Since we have random conductances, we need an additional condition on the local environment of the regeneration point to obtain independence similar to [vdHHN20].

The regeneration times are constructed as follows: We wait until the random walk reaches a generation by crossing an edge with conductance one for the first time. This time is called the first potential regeneration time and is denoted by σ_1. If the random walk never visits the ancestor of the vertex X_{σ_1} again, σ_1 is the first regeneration time, which we denote by τ_1. Otherwise, if the random walk returns to the previous generation

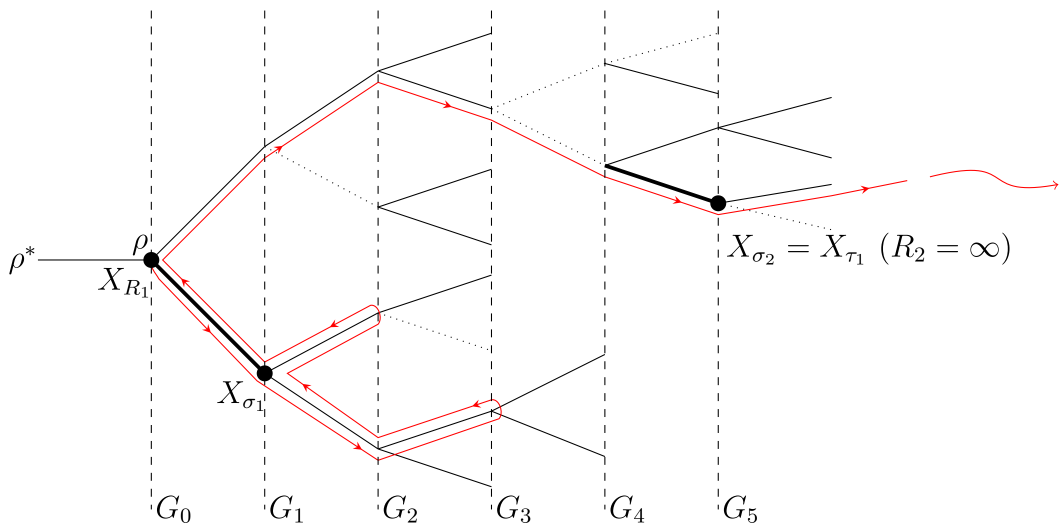


Figure 5.2: An illustration of the sample path of the random walk (in red) on a part of the tree. Edges with conductances larger than ε are indicated by solid lines (in black), whereby edges with conductance 1 are highlighted in thick; edges with conductance ε are indicated by dotted lines. The dashed lines represent the generations of the tree. Potential regeneration points of the walk and returns to the corresponding ancestors are marked by dots. Observe that the hitting time of G_1 is the first potential regeneration time σ_1, since G_1 is reached via an edge with conductance 1. But it is not the first regeneration because the random walk visits ρ again. The hitting time of G_4 is not the second potential regeneration time, although the random walk reaches a new generation, because it has passed an edge with conductance ε in its previous step. Finally, the second potential regeneration time σ_2 is the hitting time of G_5. This is also the first regeneration of the random walk, i.e. $\tau_1 = \sigma_2$, since the random walk never returns to the ancestor of the vertex X_{σ_2}.

at some time $R_1 > \sigma_1$, σ_1 is not a regeneration time. Instead, we wait until the random walk first reaches a new maximal generation by traversing an edge with conductance one. We refer to this time as the second potential regeneration time σ_2. If the random walk never visits the ancestor of the vertex X_{σ_2} again, then σ_2 is the first regeneration time τ_1. Otherwise, we repeat the steps described previously. The transience of the random walk on the tree and the positive density of edges with conductance one imply that the first regeneration time τ_1 is almost surely finite. By repeating the above procedure we obtain an infinite sequence $(\tau_k)_{k\geq 1}$ of regeneration times. Note that this construction guarantees that the random walk visits disjoint parts of the tree between regeneration times. In addition, the only edge that influences the increments before and after the regeneration time has a fixed conductance. Both are crucial to obtain independent increments. A sketch of this construction is presented in Figure 5.2. Let us now define the regeneration times formally.

Definition 5.2.1. *Given an environment $\omega = (T, \rho, \xi)$ and a random walk $(X_n)_{n\geq 0}$ on T, we define a sequence of stopping times*

$$\sigma_1 = \inf\{n \geq 1 : |X_n| > |X_m| \text{ for all } m < n,\, \xi(X_{n-1}, X_n) = 1\},$$
$$R_1 = \inf\{n \geq \sigma_1 : |X_n| = |X_{\sigma_1}| - 1\}$$

and recursively for $k > 1$

$$\sigma_k = \inf\{n \geq R_{k-1} : |X_n| > |X_m| \text{ for all } m < n,\, \xi(X_{n-1}, X_n) = 1\},$$
$$R_k = \inf\{n \geq \sigma_k : |X_n| = |X_{\sigma_k}| - 1\}.$$

We then have $\sigma_1 \leq R_1 \leq \sigma_2 \leq R_2 \leq \dots$

These stopping times are finite until $R_k = \infty$ for the first time, which indicates that the random walk never returns to the ancestor of the k-th potential regeneration point. As described above, this is the first regeneration of the random walk. Accordingly, we set

$$K = \inf\{k \geq 1 : R_k = \infty\}, \quad \tau_1 = \sigma_K \tag{5.2.1}$$

and τ_1 is called the first regeneration time. In Lemma 5.2.3 below we show that $K < \infty$ holds almost surely, which means that τ_1 is well-defined. In order to define a sequence of regeneration times, we let θ_m be the time shift of an infinite path $(x_n)_{n\geq 0}$ path such that

$$\theta_m(x_n)_{n\geq 0} = (x_{n+m})_{n\geq 0}. \tag{5.2.2}$$

For $k > 1$ the k-th regeneration time is defined by

$$\tau_k = \tau_{k-1} + \tau_1 \circ \theta_{\tau_{k-1}},$$

where $\tau_1 \circ \theta_{\tau_{k-1}}$ is the first regeneration time of the shifted path. We show in Lemma 5.2.4 below that these regeneration times are well-defined. In particular, this creates an infinite sequence $(\tau_k)_{k\geq 1}$ of regeneration times.

For the existence of the regeneration structure defined above, a uniform bound for the probability that the random walk starting at the root never hits the ancestor of the root is crucial. We denote this hitting time by

$$\eta_* = \eta_{\rho^*} = \inf\{n \geq 0 : X_n = \rho^*\}.$$

Moreover, given an environment $\omega = (T, \rho, \xi)$, we recall that $T(v)$ is the subtree consisting of the vertex $v \in T$ and all its descendants. We write $T_1(v)$ for the subtree of $T(v)$ formed by the edges with conductance larger than ε that contains the vertex v. This subtree is again a Galton-Watson tree, which is supercritical if $(1-\alpha)m_1 > 1$, where m_1 denotes the offspring mean. The next lemma gives the required uniform bound.

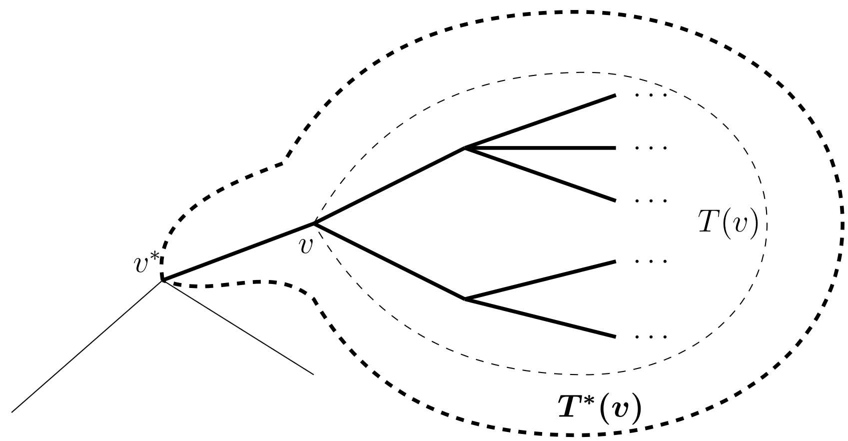


Figure 5.3: $T(v)$ is the subtree formed by v and all its descendants, $T^*(v)$ (indicated by the thick edges) is the subtree composed of $T(v)$ and v^*.

Lemma 5.2.2 (Annealed escape probability)**.** *There exists some constant* $c_\varepsilon = c_\varepsilon(\nu, \alpha, \kappa) > 0$ *such that*

$$\mathbb{P}(\eta_* = \infty \mid \xi(\rho, \rho^*) = 1) \geq c_\varepsilon > 0.$$

If the subtree $T_1(\rho)$ *is supercritical, i.e. if* $(1-\alpha)m_1 > 1$*, there exists some constant* $c = c(\nu, \alpha, \kappa) > 0$*, independent of* ε*, such that*

$$\mathbb{P}(\eta_* = \infty \mid \xi(\rho, \rho^*) = 1) \geq c > 0.$$

The next crucial step is to verify that the regeneration times defined above actually exist. The following lemma gives the existence of the first regeneration time τ_1.

Lemma 5.2.3. *The random variable*

$$K = \inf\{k \geq 1 : R_k = \infty\}$$

is almost surely finite with respect to the annealed law.

To show that all further regeneration times are almost surely finite as well, we have to introduce some more notations. Given an environment $\omega = (T, \rho, \xi)$ and a vertex $v \in T$, $v \neq \rho^*$, we denote the subtree composed of $T(v)$ and the ancestor v^* of v by $T^*(v)$ and the corresponding environment by $\omega^*(v) = \big(T^*(v), v, (\xi(e))_{e \in \mathcal{E}(T^*(v))}\big)$, see Figure 5.3. Having in mind that we think of a tree as a subset of the extended Ulam-Harris tree $\mathbb{T}^*$, which means that each vertex corresponds to a sequence of integers, we may identify the subtree $(T^*(v), v)$ with the set

$$[T^*(v)] = \{z - v : z \in T^*(v)\} \subset \mathbb{T}^*.$$

Here, the difference between two vertices $z = (z_1, \ldots, z_n)$ and $v = (z_1, \ldots, z_l)$ is defined by

$$z - v = \begin{cases} (z_{l+1}, \ldots, z_n), & z \in T(v) \\ -1, & z = v^*. \end{cases}$$

In other words, $[T^*(v)]$ shifts the subtree $T^*(v)$ such that the root v is identified with $\emptyset$ in the extended Ulam-Harris tree. This keeps the structure of the tree unchanged, but the tree becomes independent of the position of the vertex v in the original tree. We write $[\omega^*(v)]$ for corresponding environment. Next, we introduce the time shift Θ_{τ_k} of a set $B \in \mathcal{G} \otimes \mathcal{F}$:

$$B \circ \Theta_{\tau_k} = \big\{(\omega, (X_n)_{n\geq 0}) : ([\omega^*(X_{\tau_k})], (X_n - X_{\tau_k})_{n\geq\tau_k}) \in B\big\}.$$

Note that $X_n - X_{\tau_k}$, which denotes the position of X_n in $[T^*(X_{\tau_k})]$, is well-defined by the definition of τ_k. Moreover, for $k \geq 1$ we let $\mathcal{G}_k$ be the σ-algebra generated by the sets

$$\big\{\tau_k = m, X_0 = v_0, \ldots, X_m = v_m, \omega\backslash\omega(v_m) \in A\big\}. \tag{5.2.3}$$

Here, for two environments $\omega = \big(T, \rho, (\xi(e))_{e\in\mathcal{E}(T)}\big)$ and $\omega' = \big(T', \rho', (\xi(e))_{e\in\mathcal{E}(T')}\big)$ such that T' is a subtree of T we write $T\backslash T'$ for the tree that we get when we remove from T all edges of T' and all isolated vertices. The associated environment is denoted by $\omega\backslash\omega' = \big(T\backslash T', \rho, (\xi(e))_{e\in\mathcal{E}(T\backslash T')}\big)$. The following lemma gives the existence of an infinite sequence of regeneration times $(\tau_k)_{k\geq 1}$.

Lemma 5.2.4 (Existence of regeneration times)**.** *For any $k \geq 1$, the k-th regeneration time τ_k is almost surely finite with respect to the annealed law. Moreover, we have*

$$\mathbb{P}\big(B \circ \Theta_{\tau_k} \bigm| \mathcal{G}_k\big) = \mathbb{P}\big(B \bigm| \eta_* = \infty, \xi(\rho, \rho^*) = 1\big)$$

for $B \in \mathcal{G} \otimes \mathcal{F}$.

In order to use the regeneration structure to prove a central limit theorem it is crucial that the time differences between regenerations are independent, just as the increments in between. At time τ_k, the random walk hits the generation $|X_{\tau_k}|$ for the first time and after time τ_k, the random walk will never visit the generation $|X_{\tau_k}| - 1$ again. This means that $(X_n)_{n<\tau_k}$ and $(X_n)_{n\geq\tau_k}$ visit disjoint parts of the tree. Moreover, the only edge that influences the increments before and after τ_k has a fixed conductance. Hence, the time until the next regeneration $\tau_{k+1} - \tau_k$ is independent of what happened before τ_k. The inter-regeneration times therefore form an independent sequence, just as the inter-regeneration distances. This is formalized in the following proposition.

Proposition 5.2.5 (Stationarity and independence)**.** *Under $\mathbb{P}$, the sequence*

$$\Big([\omega^*(X_{\tau_n})\backslash\omega(X_{\tau_{n+1}})], (X_k - X_{\tau_n})_{\tau_n \le k \le \tau_{n+1}}, \tau_{n+1} - \tau_n\Big)_{n\ge 1}$$

is stationary and independent. Furthermore, the marginal distribution of this sequence is given by

$$\begin{aligned}&\mathbb{P}\Big([\omega^*(X_{\tau_n})\backslash\omega(X_{\tau_{n+1}})] \in A_1, (X_k - X_{\tau_n})_{\tau_n \le k \le \tau_{n+1}} \in A_2, \tau_{n+1} - \tau_n \in A_3\Big)\\ &\quad = \mathbb{P}\Big(\omega\backslash\omega(X_{\tau_1}) \in A_1, (X_k)_{k\le\tau_1} \in A_2, \tau_1 \in A_3 \,\Big|\, \eta_* = \infty, \xi(\rho,\rho^*) = 1\Big).\end{aligned}$$

Remark 5.2.6. *Proposition 5.2.5 implies in particular that the inter-regeneration times $(\tau_{n+1} - \tau_n)_{n\ge1}$ are i.i.d. just as the increments between regeneration points $(|X_{\tau_{n+1}}| - |X_{\tau_n}|)_{n\ge1}$.*

Using the regeneration structure, a central limit theorem can be shown, provided that the regeneration times and the regeneration distances are sufficiently integrable. The required moment bounds are given in the next two lemmas.

Lemma 5.2.7 (Moment bounds on regeneration distances)**.** *For $\varepsilon > 0$ fixed and for any $q \ge 1$ there exists a constant $C_\varepsilon = C_\varepsilon(\nu,\alpha,\kappa) > 0$ such that*

$$\mathbb{E}\big[|X_{\tau_1}|^q\big] \le C_\varepsilon \quad \textit{and} \quad \mathbb{E}\big[(|X_{\tau_2}| - |X_{\tau_1}|)^q\big] \le C_\varepsilon.$$

If the subtree $T_1(\rho)$ is supercritical, i.e. if $(1-\alpha)m_1 > 1$, the bounds above hold for a constant $C = C(\nu,\alpha,\kappa) > 0$ independent of ε.

Lemma 5.2.8 (Moment bounds on regeneration times)**.** *For $\varepsilon > 0$ fixed and for any $q \ge 1$ there exists a constant $C_\varepsilon = C_\varepsilon(\nu,\alpha,\kappa) > 0$ such that*

$$\mathbb{E}\big[\tau_1^q\big] \le C_\varepsilon \quad \textit{and} \quad \mathbb{E}\big[(\tau_2 - \tau_1)^q\big] \le C_\varepsilon.$$

Remark 5.2.9. *(a) We require that the moment bounds on the regeneration distances in Lemma 5.2.7 hold uniformly for $\varepsilon > 0$ in the supercritical case. This allows to study the behaviour of the volatility as $\varepsilon \to 0$, see Theorem 5.1.3.*

(b) We assume in this chapter that the offspring law has finite moments of any order. If instead we only assume that $m_r = \int x^r \,\mathrm{d}\nu(x) < \infty$ holds for some $r > 24$, the statement in Lemma 5.2.8 is still valid for $q = 2$, which is sufficient to prove Theorem 5.1.1.

(c) When $\varepsilon > 0$ is small, finite subtrees formed by edges with larger conductances act like traps in the environment. The time spent in such a subtree is of order ε^{-1}, but the random walk is unlikely to regenerate there. Therefore, we cannot expect to find moment bounds on the regeneration times that hold uniformly in ε.

We note that the constants in this chapter may depend on several model parameters. As mentioned above, uniform bounds for escape probabilities and moments of regeneration distances are crucial to control the variance for small ε. For this reason, we use a subscript to explicitly indicate when constants depend on ε. Moreover, the constants may vary from line to line.

5.3 Proofs

5.3.1 Central limit theorem: proof of Theorem 5.1.1

To prove the functional central limit theorem we follow the arguments of [Szn00]. It is done in three steps. First, we consider a jump process such that its jump sizes have the same distribution as the inter-regeneration distances of the random walk on the tree, but the jumps occur after uniform deterministic time intervals. After an analogous standardization as in the definition of the process W_t^n in (5.1.2), Donsker's theorem implies that this process converges to a Brownian motion. Next, we apply a random change of time to this process such that the process jumps after random time intervals, which have the same distribution as the inter-regeneration times. Lastly, we transfer the convergence to the (standardized) random walk on the tree.

Jump process with i.i.d. increments

We define

$$Z_k = |X_{\tau_k}| - |X_{\tau_{k-1}}| - (\tau_k - \tau_{k-1})v \quad \text{for } k \geq 1$$

with $\tau_0 = 0$ and

$$v = \frac{\mathbb{E}\big[|X_{\tau_2}| - |X_{\tau_1}|\big]}{\mathbb{E}[\tau_2 - \tau_1]}.$$

We will show at the end of this proof that v is the speed of the random walk.

By Proposition 5.2.5 and Lemma 5.2.7 the random variables $Z_2, Z_3, \dots$ are i.i.d. and square-integrable with $\mathbb{E}[Z_2] = 0$ and $\mathrm{Var}(Z_2) = \mathbb{E}[Z_2^2] =: \tilde{\sigma}_\varepsilon^2 > 0$. Applying Donsker's Theorem (see e.g. Theorem 14.1 in [Bil99]) implies

$$\left(\frac{1}{\sqrt{n}} \sum_{k=2}^{\lfloor nt \rfloor} Z_k\right)_{t\in[0,1]} \xrightarrow[n\to\infty]{d} (\tilde{\sigma}_\varepsilon B_t)_{t\in[0,1]}. \tag{5.3.1}$$

Moreover, the distance in the Skorokhod metric

$$d_S\Bigg(\left(\frac{1}{\sqrt{n}} \sum_{k=2}^{\lfloor nt \rfloor} Z_k\right)_{t\in[0,1]}, \left(\frac{1}{\sqrt{n}} \sum_{k=1}^{\lfloor nt \rfloor} Z_k\right)_{t\in[0,1]}\Bigg) \leq \frac{1}{\sqrt{n}}|Z_1| \leq \frac{1}{\sqrt{n}}(1+v)\tau_1$$

vanishes in probability, since τ_1 is $\mathbb{P}$-almost surely finite. By Theorem 3.1 in [Bil99] we obtain

$$\left(\frac{1}{\sqrt{n}} \sum_{k=1}^{\lfloor nt \rfloor} Z_k\right)_{t\in[0,1]} \xrightarrow[n\to\infty]{d} (\tilde{\sigma}_\varepsilon B_t)_{t\in[0,1]}. \tag{5.3.2}$$

Random change of time

The next step is to correct the jump times of the process considered above by applying a random time change. Afterwards, we can transfer the convergence result to this process using the arguments of Section 14 in [Bil99].

We let k_n be the number of regenerations until time n, i.e. k_n is the integer such that $\tau_{k_n} \leq n < \tau_{k_n+1}$. Obviously, we have

$$\frac{\tau_{k_n}}{k_n} \leq \frac{n}{k_n} < \frac{\tau_{k_n+1}}{k_n}.$$

Using the strong law of large numbers we obtain

$$\frac{\tau_{k_n}}{k_n} = \frac{1}{k_n} \sum_{k=1}^{k_n-1} (\tau_{k+1} - \tau_k) + \frac{\tau_1}{k_n} \xrightarrow[n\to\infty]{} \mathbb{E}[\tau_2 - \tau_1] \quad \mathbb{P}-\text{a.s.}$$

and

$$\frac{\tau_{k_n+1}}{k_n} = \frac{1}{k_n} \sum_{k=1}^{k_n} (\tau_{k+1} - \tau_k) + \frac{\tau_1}{k_n} \xrightarrow[n\to\infty]{} \mathbb{E}[\tau_2 - \tau_1] \quad \mathbb{P}-\text{a.s.}$$

Consequently, we get

$$\frac{k_n}{n} \xrightarrow[n\to\infty]{} \frac{1}{\mathbb{E}[\tau_2 - \tau_1]} \quad \mathbb{P}-\text{a.s.} \tag{5.3.3}$$

This implies

$$\frac{k_{\lfloor nt \rfloor}}{n} = t \cdot \frac{\lfloor nt \rfloor}{nt} \frac{k_{\lfloor nt \rfloor}}{\lfloor nt \rfloor} \xrightarrow[n\to\infty]{} \frac{t}{\mathbb{E}[\tau_2 - \tau_1]} \quad \mathbb{P}-\text{a.s.}$$

for $t \in [0,1]$, which shows that the functions $t \mapsto \frac{k_{\lfloor nt \rfloor}}{n}$, $t \in [0,1]$, form a sequence of monotone functions converging to a continuous limit ($\mathbb{P}$-almost everywhere). Using a counterpart of Dini's theorem (see Problem 127 in Part II of [PS72]), we get that the convergence holds uniformly, i.e.

$$\sup_{t\in[0,1]} \left| \frac{k_{\lfloor nt \rfloor}}{n} - \frac{t}{\mathbb{E}[\tau_2 - \tau_1]} \right| \xrightarrow[n\to\infty]{} 0 \quad \mathbb{P}-\text{a.s.}$$

As a result we obtain

$$d_S\left(\left(\frac{k_{\lfloor nt \rfloor}}{n} \right)_{t\in[0,1]}, \left(\frac{t}{\mathbb{E}[\tau_2 - \tau_1]} \right)_{t\in[0,1]} \right) \leq \sup_{t\in[0,1]} \left| \frac{k_{\lfloor nt \rfloor}}{n} - \frac{t}{\mathbb{E}[\tau_2 - \tau_1]} \right| \xrightarrow[n\to\infty]{} 0 \quad \mathbb{P}-\text{a.s.},$$

which implies

$$\left(\frac{k_{\lfloor nt \rfloor}}{n} \right)_{t\in[0,1]} \xrightarrow[n\to\infty]{d} \left(\frac{t}{\mathbb{E}[\tau_2 - \tau_1]} \right)_{t\in[0,1]}.$$

Together with the convergence result in (5.3.2), Theorem 3.9 in [Bil99] yields

$$\bigg(\Big(\frac{1}{\sqrt{n}}\sum_{k=1}^{\lfloor nt\rfloor} Z_k\Big)_{t\in[0,1]}, \Big(\frac{k_{\lfloor nt\rfloor}}{n}\Big)_{t\in[0,1]}\bigg) \xrightarrow[n\to\infty]{d} \bigg((\tilde{\sigma}_\varepsilon B_t)_{t\in[0,1]}, \Big(\frac{t}{\mathbb{E}[\tau_2-\tau_1]}\Big)_{t\in[0,1]}\bigg).$$

After applying the random time change to the jump process, we use the arguments of Section 14 in [Bil99] to conclude that

$$\Big(\frac{1}{\sqrt{n}}\sum_{k=1}^{k_{\lfloor nt\rfloor}} Z_k\Big)_{t\in[0,1]} \xrightarrow[n\to\infty]{d} \Big(\tilde{\sigma}_\varepsilon B_{\frac{t}{\mathbb{E}[\tau_2-\tau_1]}}\Big)_{t\in[0,1]}.$$

Due to the scale invariance of the Brownian motion, we obtain

$$\Big(\frac{1}{\sqrt{n}}\sum_{k=1}^{k_{\lfloor nt\rfloor}} Z_k\Big)_{t\in[0,1]} \xrightarrow[n\to\infty]{d} (\sigma_\varepsilon B_t)_{t\in[0,1]} \tag{5.3.4}$$

with $\sigma_\varepsilon^2 = \mathbb{E}[Z_2^2]\mathbb{E}[\tau_2-\tau_1]^{-1}$.

Comparison with the trajectories of the random walk on the tree

It remains to add the path between the regenerations. In order to transfer the convergence result to this process we show that the distance of these two processes in the Skorokhod metric vanishes in probability. We have

$$\begin{aligned} d_S&\bigg(\Big(\frac{1}{\sqrt{n}}\big(|X_{\lfloor nt\rfloor}|-\lfloor nt\rfloor v\big)\Big)_{t\in[0,1]}, \Big(\frac{1}{\sqrt{n}}\sum_{k=1}^{k_{\lfloor nt\rfloor}} Z_k\Big)_{t\in[0,1]}\bigg)\\ &\le \sup_{t\in[0,1]}\left|\frac{1}{\sqrt{n}}\big(|X_{\lfloor nt\rfloor}|-\lfloor nt\rfloor v\big) - \frac{1}{\sqrt{n}}\sum_{k=1}^{k_{\lfloor nt\rfloor}} Z_k\right|\\ &\le \frac{1}{\sqrt{n}}\sup_{t\in[0,1]}\Big(\big||X_{\lfloor nt\rfloor}|-|X_{\tau_{k_{\lfloor nt\rfloor}}}|\big| + \big|\lfloor nt\rfloor - \tau_{k_{\lfloor nt\rfloor}}\big|v\Big)\\ &\le \frac{1+v}{\sqrt{n}}\sup_{t\in[0,1]}\big(\lfloor nt\rfloor - \tau_{k_{\lfloor nt\rfloor}}\big). \end{aligned}$$

To see the last inequality, note that the distance the random walk can gain in a given amount of time is bounded by the number of steps it takes. By the definition of k_n we deduce

$$\sup_{t\in[0,1]}\big(\lfloor nt\rfloor - \tau_{k_{\lfloor nt\rfloor}}\big) \le \sup_{t\in[0,1]}\big(\tau_{k_{\lfloor nt\rfloor}+1} - \tau_{k_{\lfloor nt\rfloor}}\big) \le \max_{k\in\{0,\dots,n\}}|\tau_{k+1}-\tau_k|$$

and therefore

$$d_S\Big(\Big(\frac{1}{\sqrt{n}}\big(|X_{\lfloor nt\rfloor}|-\lfloor nt\rfloor v\big)\Big)_{t\in[0,1]},\Big(\frac{1}{\sqrt{n}}\sum_{k=1}^{k_{\lfloor nt\rfloor}}Z_k\Big)_{t\in[0,1]}\Big)$$
$$\leq (1+v)\Big(\frac{|\tau_1|}{\sqrt{n}}+\frac{\max_{k\in\{1,\dots,n\}}|\tau_{k+1}-\tau_k|}{\sqrt{n}}\Big). \tag{5.3.5}$$

We observe that the first summand vanishes in probability, since τ_1 is $\mathbb{P}$-almost surely finite. Furthermore, the random variables $(|\tau_{k+1}-\tau_k|)_{k\in\{1,\dots,n\}}$ are independent and identically distributed, non-negative and square-integrable by Proposition 5.2.5 and Lemma 5.2.8, which implies for all $\delta>0$

$$\mathbb{P}\Big(\frac{\max_{k\in\{1,\dots,n\}}|\tau_{k+1}-\tau_k|}{\sqrt{n}}\geq\delta\Big)\xrightarrow[n\to\infty]{}0.$$

This shows that (5.3.5) vanishes in probability. Due to the convergence result in (5.3.4) and Theorem 3.1 in [Bil99], we finally arrive at

$$\Big(\frac{1}{\sqrt{n}}\big(|X_{\lfloor nt\rfloor}|-\lfloor nt\rfloor v\big)\Big)_{t\in[0,1]}\xrightarrow[n\to\infty]{d}(\sigma_\varepsilon B_t)_{t\in[0,1]}$$

with

$$\sigma_\varepsilon^2=\frac{\mathbb{E}[(|X_{\tau_2}|-|X_{\tau_1}|-(\tau_2-\tau_1)v)^2]}{\mathbb{E}[\tau_2-\tau_1]}. \tag{5.3.6}$$

To complete the proof, we show that v is the almost sure limit of $|X_n|/n$ under the annealed law $\mathbb{P}$. By the law of large numbers and (5.3.3) we obtain

$$\frac{|X_{\tau_{k_n}}|}{n}=\frac{|X_{\tau_{k_n}}|}{k_n}\frac{k_n}{n}\xrightarrow[n\to\infty]{}\frac{\mathbb{E}[|X_{\tau_2}|-|X_{\tau_1}|]}{\mathbb{E}[\tau_2-\tau_1]}=v\quad\mathbb{P}-\text{a.s.}$$

and

$$0\leq\frac{|X_n|-|X_{\tau_{k_n}}|}{n}\leq\frac{n-\tau_{k_n}}{n}=1-\frac{\tau_{k_n}}{k_n}\frac{k_n}{n}\xrightarrow[n\to\infty]{}0\quad\mathbb{P}-\text{a.s.}$$

This implies

$$v(\nu,\mu_\varepsilon)=\lim_{n\to\infty}\frac{|X_n|}{n}=\lim_{n\to\infty}\left(\frac{|X_{\tau_{k_n}}|}{n}+\frac{|X_n|-|X_{\tau_{k_n}}|}{n}\right)=\frac{\mathbb{E}[|X_{\tau_2}|-|X_{\tau_1}|]}{\mathbb{E}[\tau_2-\tau_1]}=v\quad\mathbb{P}-\text{a.s.}$$

□

5.3.2 The volatility for conductances approaching zero: proof of Theorem 5.1.3

To emphasize the dependence on ε, we write P_ε and $\mathbb{P}_\varepsilon$ for the environment law and annealed law, respectively, when the marginal law of the conductances is given by μ_ε as in (5.1.1). The corresponding expectations are denoted by E_ε and $\mathbb{E}_\varepsilon$. As shown in the proof of Theorem 5.1.1, the volatility is given by

$$\sigma_\varepsilon^2(\nu,\mu_\varepsilon) = \frac{\mathbb{E}_\varepsilon\big[(|X_{\tau_2}| - |X_{\tau_1}| - (\tau_2-\tau_1)v(\nu,\mu_\varepsilon))^2\big]}{\mathbb{E}_\varepsilon[\tau_2-\tau_1]}$$

(see (5.3.6)), where

$$v(\nu,\mu_\varepsilon) = \frac{\mathbb{E}_\varepsilon[|X_{\tau_2}| - |X_{\tau_1}|]}{\mathbb{E}_\varepsilon[\tau_2-\tau_1]}$$

is the speed of the walk. The proof of Theorem 5.1.3 is based on the next two lemmas.

Lemma 5.3.1. *If $T_1(\rho)$ is supercritical, i.e. if $(1-\alpha)m_1 > 1$, there exists a constant $c > 0$, independent of ε, such that*

$$\mathbb{E}_\varepsilon\big[(|X_{\tau_2}| - |X_{\tau_1}| - (\tau_2-\tau_1)v(\nu,\mu_\varepsilon))^2\big] \geq c > 0.$$

Lemma 5.3.2. *If $T_1(\rho)$ is supercritical, i.e. if $(1-\alpha)m_1 > 1$, we have*

$$\limsup_{\varepsilon\to 0} \mathbb{E}_\varepsilon[\tau_2-\tau_1] < \infty.$$

Combining these two lemmas with the representation for the volatility above, we obtain

$$\liminf_{\varepsilon\to 0} \sigma_\varepsilon^2(\nu,\mu_\varepsilon) \geq \liminf_{\varepsilon\to 0} \frac{c}{\mathbb{E}_\varepsilon[\tau_2-\tau_1]} = \frac{c}{\limsup_{\varepsilon\to 0}\mathbb{E}_\varepsilon[\tau_2-\tau_1]} > 0.$$

□

Proof of Lemma 5.3.1

We start with some notations. We let d be the minimal integer such that the probability of a vertex having d descendants is positive, i.e. $d = \inf\{k \geq 1 : \nu(\{k\}) > 0\}$. Thinking of a tree as a subset of the Ulam-Harris tree, we denote the first descendant of the root ρ by v_1 and the first descendant of v_1 by v_2. Moreover, we introduce the set of environments $A = A_1 \cap A_2$ with

$$A_1 = \big\{\omega \in \Omega : \deg(v) = d+1 \text{ for all } v \in \{\rho\} \cup G_1(T)\big\},$$
$$A_2 = \big\{\omega \in \Omega : \xi(v^*, v) = 1 \text{ for all } v \in \{\rho\} \cup G_1(T) \cup G_2(T)\big\}$$

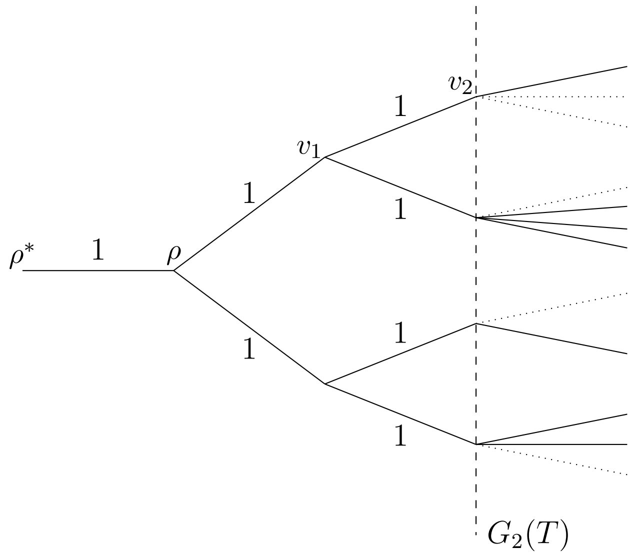


Figure 5.4: For each environment $\omega \in A$, the structure of the first two generations is fixed. The root ρ and each vertex in the first generation have d descendants (here $d = 2$); each edge adjacent to one of these vertices has conductance 1. The first descendant of ρ is denoted by v_1; the first descendant of v_1 is denoted by v_2.

(see Figure 5.4 for an example of an environment $\omega \in A$) and the set of trajectories

$$B = \big\{X_0 = \rho,\, X_1 = v_1,\, X_2 = v_2\big\} \cap \big\{|X_n| \geq 2\,\forall n \geq 2\big\}.$$

If $(\omega, (X_n)_{n\geq 0}) \in A \times B$, the random walk regenerates for the first time when it hits v_1 and for the second time when it hits v_2. More precisely, we have

$$\tau_1 = 1, \quad |X_{\tau_1}| = |v_1| = 1, \quad \tau_2 = 2, \quad |X_{\tau_2}| = |v_2| = 2.$$

This implies

$$\begin{aligned}\mathbb{E}_\varepsilon\big[(|X_{\tau_2}| - |X_{\tau_1}| - (\tau_2 - \tau_1)v(\nu, \mu_\varepsilon))^2\big] &\geq \mathbb{E}_\varepsilon\Big[\big(|X_{\tau_2}| - |X_{\tau_1}| - (\tau_2 - \tau_1)v(\nu, \mu_\varepsilon)\big)^2 \mathbb{1}_{A\times B}\Big] \\ &= (1 - v(\nu, \mu_\varepsilon))^2 \mathbb{P}_\varepsilon(A \times B) \\ &= (1 - v(\nu, \mu_\varepsilon))^2 \mathrm{E}_\varepsilon\big[P_\omega(B)\mathbb{1}_A(\omega)\big]. \qquad (5.3.7)\end{aligned}$$

Let $\omega \in A$ be an environment. Using the Markov property of $(X_n)_{n\geq 0}$, we get

$$P_\omega(B) = c_d P_\omega^{v_2}(|X_n| \geq 2\,\forall n \geq 0) = c_d P_\omega^{v_2}(\eta_{v_1} = \infty)$$

with $c_d = \left(\frac{1}{d+1}\right)^2$ and therefore

$$\begin{aligned}\mathrm{E}_\varepsilon\big[P_\omega(B)\mathbb{1}_A(\omega)\big] &= c_d \mathrm{E}_\varepsilon\big[P_\omega^{v_2}(\eta_{v_1} = \infty)\mathbb{1}_A(\omega)\big] \\ &= c_d \mathrm{E}_\varepsilon\big[P_\omega^{v_2}(\eta_{v_1} = \infty) \bigm| A\big]\mathrm{P}_\varepsilon(A) \\ &= c_d \mathbb{P}_\varepsilon(\eta_* = \infty \mid \xi(\rho, \rho^*) = 1)\mathrm{P}_\varepsilon(A).\end{aligned}$$

A short calculation shows that

$$\mathrm{P}_\varepsilon(A) = \mathrm{P}_\varepsilon(A_1)\mathrm{P}_\varepsilon(A_2 \mid A_1) = \nu(\{d\})^{1+d}\mu(\{1\})^{1+d+d^2} > 0$$

is strictly positive. Moreover, Lemma 5.2.2 implies

$$\mathbb{P}_\varepsilon(\eta_* = \infty \mid \xi(\rho, \rho^*) = 1) \geq \tilde{c} > 0$$

for some constant $\tilde{c} = \tilde{c}(\nu, \alpha, \kappa) > 0$. Consequently, there exists a constant $c = c(\nu, \alpha, \kappa) > 0$, independent of ε, such that

$$\mathrm{E}_\varepsilon\big[P_\omega(B_n)\mathbb{1}_A(\omega)\big] \geq c.$$

Plugging this in (5.3.7), we obtain

$$\mathbb{E}_\varepsilon\big[(|X_{\tau_2}| - |X_{\tau_1}| - (\tau_2 - \tau_1)v(\nu, \mu_\varepsilon))^2\big] \geq c(1 - v(\nu, \mu_\varepsilon))^2.$$

We establish a uniform bound for the speed of the random walk with random conductances by comparing it with the speed of the simple random walk. It was proven by [GMPV12] that random conductances can only slow down the random walk, see Theorem 4.4 therein. For the simple random walk an explicit formula for the speed is stated in Theorem 3.2 in [LPP95]. Combining both results implies

$$v(\nu, \mu_\varepsilon) \leq v(\nu, \delta_1) = \sum_{k=1}^{\infty} \nu(\{k\}) \frac{k-1}{k+1} < 1,$$

where $v(\nu, \delta_1)$ is the speed of the simple random walk. Finally, we conclude that

$$\mathbb{E}_\varepsilon\big[(|X_{\tau_2}| - |X_{\tau_1}| - (\tau_2 - \tau_1)v(\nu, \mu_\varepsilon))^2\big] \geq c(1 - v(\nu, \delta_1))^2 > 0,$$

which completes the proof. □

Proof of Lemma 5.3.2

In Chapter 3 we studied how the speed $v(\nu, \mu_\varepsilon)$ of the random walk depends on ε. If $T_1(\rho)$ is supercritical, Theorem 3.1.2 implies that the limit of the speed for conductances approaching zero is given by

$$\lim_{\varepsilon \to 0} v(\nu, \mu_\varepsilon) = \beta v(\nu, \mu_0)$$

for some constant $\beta \in (0, 1)$. By Lemma 3.2.4 the speed $v(\nu, \mu_0)$ is strictly positive. Consequently, we have

$$\lim_{\varepsilon \to 0} v(\nu, \mu_\varepsilon) = \lim_{\varepsilon \to 0} \frac{\mathbb{E}_\varepsilon[|X_{\tau_2}| - |X_{\tau_1}|]}{\mathbb{E}_\varepsilon[\tau_2 - \tau_1]} > 0.$$

We may apply Lemma 5.2.7 to uniformly bound the numerator such that we obtain

$$0 < \lim_{\varepsilon\to 0} v(\nu,\mu_\varepsilon) \le \liminf_{\varepsilon\to 0} \frac{C}{\mathbb{E}_\varepsilon[\tau_2-\tau_1]} = \frac{C}{\limsup_{\varepsilon\to 0}\mathbb{E}_\varepsilon[\tau_2-\tau_1]}$$

for some constant $C>0$. This can only hold if

$$\limsup_{\varepsilon\to 0}\mathbb{E}_\varepsilon[\tau_2-\tau_1] < \infty,$$

which completes the proof. □

5.3.3 Annealed escape probability: proof of Lemma 5.2.2

We start with the supercritical case. If $T_1(\rho)$ is supercritical, i.e. $(1-\alpha)m_1 > 1$, the probability that $T_1(\rho)$ survives is strictly positive and we can bound the annealed escape probability as follows

$$\begin{aligned}\mathbb{P}\big(\eta_* = \infty \bigm| \xi(\rho,\rho^*) = 1\big) &\ge \mathbb{P}\big(\eta_* = \infty, |T_1(\rho)| = \infty \bigm| \xi(\rho,\rho^*) = 1\big)\\ &= \mathbb{P}\big(\eta_* = \infty \bigm| \xi(\rho,\rho^*) = 1, |T_1(\rho)| = \infty\big)\mathrm{P}\big(|T_1(\rho)| = \infty\big).\end{aligned} \tag{5.3.8}$$

Here, we used in the second step that the conductance of the edge (ρ,ρ^*) is independent of the subtree $T_1(\rho)$. Recalling the definition of the annealed law $\mathbb{P}$ in (2.3.3), we can write

$$\begin{aligned}\mathbb{P}\big(\eta_* < \infty \bigm| \xi(\rho,\rho^*) = 1, |T_1(\rho)| = \infty\big) &= \frac{\mathbb{P}\big(\eta_* < \infty, \xi(\rho,\rho^*) = 1, |T_1(\rho)| = \infty\big)}{\mathbb{P}\big(\xi(\rho,\rho^*) = 1, |T_1(\rho)| = \infty\big)}\\ &= \frac{\mathrm{E}\big[P_\omega(\eta_* < \infty)\mathbb{1}_{\{\xi(\rho,\rho^*)=1, |T_1(\rho)|=\infty\}}\big]}{\mathrm{P}\big(\xi(\rho,\rho^*) = 1, |T_1(\rho)| = \infty\big)}\\ &= \mathrm{E}\big[P_\omega(\eta_* < \infty) \bigm| \xi(\rho,\rho^*) = 1, |T_1(\rho)| = \infty\big].\end{aligned} \tag{5.3.9}$$

We let $\omega = (T,\rho,\xi)$ be an environment where the root has an additional ancestor ρ^* with $\xi(\rho,\rho^*) = 1$ and $|T_1(\rho)| = \infty$. Due to the continuity of probability measures, the quenched probability in (5.3.9) is the following limit

$$P_\omega(\eta_* < \infty) = \lim_{k\to\infty} P_\omega(\eta_* < \eta_k),$$

where $\eta_k = \inf\{n \ge 0 : |X_n| = k\}$ denotes the hitting time of the k-th generation of the tree. Applying Lemma 2.4.1, we can express the probability that the random walk visits the ancestor of the root before it reaches the k-th generation as a ratio of effective conductances

$$P_\omega(\eta_* < \eta_k) = \frac{\mathcal{C}_\omega(\rho,\rho^*)}{\mathcal{C}_\omega(\rho,\rho^*) + \mathcal{C}_\omega(\rho,G_k)} = \frac{1}{1+\mathcal{C}_\omega(\rho,G_k)}.$$

We denote the backbone tree of $T_1(\rho)$, i.e. the subtree where all vertices that do not have an infinite line of descent are removed, by T_1^{Bb} and the corresponding environment by $\omega_1^{\mathrm{Bb}} = \big(T_1^{\mathrm{Bb}}, \rho, (\xi(e))_{e\in\mathcal{E}(T_1^{\mathrm{Bb}})}\big)$. Rayleigh's Monotonicity Principle (Lemma 2.4.5) implies that removing edges can only decrease the effective conductance, so that

$$\mathcal{C}_\omega(\rho, G_k) \geq \mathcal{C}_{\omega_1^{\mathrm{Bb}}}(\rho, G_k).$$

Note that the effective conductance on the right-hand side is now independent of ε, which is crucial for deriving escape estimates uniform in ε. We obtain

$$P_\omega(\eta_* < \eta_k) \leq \frac{1}{1+\mathcal{C}_{\omega_1^{\mathrm{Bb}}}(\rho, G_k)} = \frac{\mathcal{R}_{\omega_1^{\mathrm{Bb}}}(\rho, G_k)}{1+\mathcal{R}_{\omega_1^{\mathrm{Bb}}}(\rho, G_k)}$$

and

$$P_\omega(\eta_* < \infty) \leq \frac{\mathcal{R}_{\omega_1^{\mathrm{Bb}}}(\rho, \infty)}{1+\mathcal{R}_{\omega_1^{\mathrm{Bb}}}(\rho, \infty)}.$$

The effective resistance $\mathcal{R}_{\omega_1^{\mathrm{Bb}}}(\rho, \infty)$ depends only on the subtree rooted at ρ and is therefore independent of the conductance of the edge (ρ, ρ^*). Due to (5.3.9), we get

$$\begin{aligned}
\mathbb{P}\big(\eta_* < \infty \;\big|\; \xi(\rho,\rho^*) = 1, |T_1(\rho)| = \infty\big) &\leq \mathrm{E}\bigg[\frac{\mathcal{R}_{\omega_1^{\mathrm{Bb}}}(\rho, \infty)}{1+\mathcal{R}_{\omega_1^{\mathrm{Bb}}}(\rho, \infty)} \;\bigg|\; \xi(\rho,\rho^*) = 1, |T_1(\rho)| = \infty\bigg] \\
&= \mathrm{E}\bigg[\frac{\mathcal{R}_{\omega_1^{\mathrm{Bb}}}(\rho, \infty)}{1+\mathcal{R}_{\omega_1^{\mathrm{Bb}}}(\rho, \infty)} \;\bigg|\; |T_1(\rho)| = \infty\bigg] \\
&\leq \frac{\mathrm{E}\big[\mathcal{R}_{\omega_1^{\mathrm{Bb}}}(\rho, \infty) \;\big|\; |T_1(\rho)| = \infty\big]}{1+\mathrm{E}\big[\mathcal{R}_{\omega_1^{\mathrm{Bb}}}(\rho, \infty) \;\big|\; |T_1(\rho)| = \infty\big]}.
\end{aligned}$$

Here, the last step is obtained by applying Jensen's inequality, since the mapping $x \mapsto \frac{x}{1+x}$ is concave on $[0, \infty)$. Conditioned on the survival of $T_1(\rho)$, the backbone tree T_1^{Bb} is a supercritical Galton-Watson tree without leaves and with uniformly elliptic conductances (see Proposition 4.10 in [Lyo92]). Hence, Lemma 3.3.1 implies that the mean resistance is bounded:

$$\mathrm{E}\Big[\mathcal{R}_{\omega_1^{\mathrm{Bb}}}(\rho, \infty) \;\Big|\; |T_1(\rho)| = \infty\Big] \leq C$$

for some constant $C = C(\nu, \alpha, \kappa) < \infty$ independent of ε. This leads to

$$\mathbb{P}\big(\eta_* < \infty \;\big|\; \xi(\rho,\rho^*) = 1, |T_1(\rho)| = \infty\big) \leq \frac{C}{1+C} < 1$$

and, due to (5.3.8), we conclude that

$$\mathbb{P}\big(\eta_* = \infty \;\big|\; \xi(\rho,\rho^*) = 1\big) \geq \bigg(1 - \frac{C}{1+C}\bigg)\mathrm{P}\big(|T_1(\rho)| = \infty\big) > 0.$$

Let us proceed with the subcritical case. If $T_1(\rho)$ is (sub-)critical, i.e. $(1-\alpha)m_1 \leq 1$, the tree $T_1(\rho)$ dies out with probability one. For this reason, we cannot expect to find a uniform lower bound on the annealed escape probability. But we can still use the same arguments. We let $\omega = (T, \rho, \xi)$ be an environment with $\xi(\rho, \rho^*) = 1$. Applying Lemma 2.4.1 yields

$$P_\omega(\eta_* < \infty) = \frac{1}{1 + \mathcal{C}_\omega(\rho, \infty)} = \frac{\mathcal{R}_\omega(\rho, \infty)}{1 + \mathcal{R}_\omega(\rho, \infty)}.$$

The effective resistance $\mathcal{R}_\omega(\rho, \infty)$ only depends on the subtree rooted at ρ, which is a supercritical Galton-Watson tree with uniformly elliptic conductances. Note that here the ellipticity constant depends on ε. Lemma 3.3.1 then implies

$$\mathrm{E}[\mathcal{R}_\omega(\rho, \infty)] \leq C_\varepsilon$$

for some constant $C_\varepsilon = C_\varepsilon(\nu, \alpha, \kappa) > 0$ depending on ε. We finally obtain with Jensen's inequality

$$\mathbb{P}\big(\eta_* < \infty \bigm| \xi(\rho, \rho^*) = 1\big) = \mathrm{E}\big[P_\omega(\eta_* < \infty) \bigm| \xi(\rho, \rho^*) = 1\big] \leq \frac{C_\varepsilon}{1 + C_\varepsilon} < 1,$$

which completes the proof. □

5.3.4 Existence of regeneration times

Proof of Lemma 5.2.3

Recall the definitions of the annealed law $\mathbb{P}$ in (2.3.3) and the stopping times in Definition 5.2.1. We can express the return probability to the ancestor of the first potential regeneration point as

$$\mathbb{P}(R_1 < \infty) = \mathrm{E}\big[P_\omega(R_1 < \infty)\big] = \mathrm{E}\left[\sum_{v \in \mathbb{T}} P_\omega(R_1 < \infty, \sigma_1 < \infty, X_{\sigma_1} = v) \mathbb{1}_{\{v \in T, \xi(v, v^*) = 1\}}\right].$$

Since $(X_n)_{n \geq 0}$ is a Markov chain under the quenched law P_ω, we use the strong Markov property in σ_1 to compute the probability inside

$$\begin{aligned}\mathbb{P}(R_1 < \infty) &= \mathrm{E}\left[\sum_{v \in \mathbb{T}} P_\omega(R_1 < \infty | \sigma_1 < \infty, X_{\sigma_1} = v) P_\omega(\sigma_1 < \infty, X_{\sigma_1} = v) \mathbb{1}_{\{v \in T, \xi(v, v^*) = 1\}}\right] \\ &= \mathrm{E}\left[\sum_{v \in \mathbb{T}} P_\omega^v(\eta_{v^*} < \infty) P_\omega(\sigma_1 < \infty, X_{\sigma_1} = v) \mathbb{1}_{\{v \in T, \xi(v, v^*) = 1\}}\right].\end{aligned}$$

Here, $\eta_{v^*} = \inf\{n \geq 0 : X_n = v^*\}$ is the time when the random walk hits the ancestor of v for the first time. We notice that the number of vertices in the Ulam-Harris tree is countable. Thus, monotone convergence implies

$$\begin{aligned}
\mathbb{P}(R_1 < \infty) &= \sum_{v \in \mathbb{T}} \mathrm{E}\big[P_\omega^v(\eta_{v^*} < \infty) P_\omega(\sigma_1 < \infty, X_{\sigma_1} = v) \mathbb{1}_{\{v \in T, \xi(v,v^*)=1\}}\big] \\
&= \sum_{v \in \mathbb{T}} \mathrm{E}\big[P_\omega^v(\eta_{v^*} < \infty) P_\omega(\sigma_1 < \infty, X_{\sigma_1} = v) \bigm| v \in T, \xi(v,v^*) = 1\big] \mathrm{P}(v \in T, \xi(v,v^*) = 1).
\end{aligned}$$

The random variable $P_\omega^v(\eta_{v^*} < \infty)$ is $\sigma(\omega^*(v))$-measurable and $P_\omega(\sigma_1 < \infty, X_{\sigma_1} = v)$ is $\sigma(\omega \backslash \omega(v))$-measurable. Hence, they are independent under $\mathrm{P}(\,\cdot \mid v \in T, \xi(v,v^*) = 1)$ and we get

$$\begin{aligned}
&\mathbb{P}(R_1 < \infty) \\
&= \sum_{v \in \mathbb{T}} \mathrm{E}\big[P_\omega^v(\eta_{v^*} < \infty) \bigm| v \in T, \xi(v,v^*) = 1\big] \mathrm{E}\big[P_\omega(\sigma_1 < \infty, X_{\sigma_1} = v) \bigm| v \in T, \xi(v,v^*) = 1\big] \\
&\qquad \times \mathrm{P}(v \in T, \xi(v,v^*) = 1) \\
&= \mathrm{E}\big[P_\omega(\eta_* < \infty) \bigm| \xi(\rho,\rho^*) = 1\big] \sum_{v \in \mathbb{T}} \mathrm{E}\big[P_\omega(\sigma_1 < \infty, X_{\sigma_1} = v) \mathbb{1}_{\{v \in T, \xi(v,v^*)=1\}}\big] \\
&= \mathbb{P}\big(\eta_* < \infty \mid \xi(\rho,\rho^*) = 1\big) \mathbb{P}(\sigma_1 < \infty) \\
&= \mathbb{P}\big(\eta_* < \infty \mid \xi(\rho,\rho^*) = 1\big),
\end{aligned}$$

where we used the transience of the random walk and the fact that the random walk reaches a new generation via an edge with conductance one almost surely for the last equality. Analogously, we obtain for $k > 1$

$$\begin{aligned}
\mathbb{P}(R_k < \infty) &= \mathbb{P}(\eta_* < \infty \mid \xi(\rho,\rho^*) = 1) \mathbb{P}(\sigma_k < \infty) \\
&\leq \mathbb{P}(\eta_* < \infty \mid \xi(\rho,\rho^*) = 1) \mathbb{P}(R_{k-1} < \infty),
\end{aligned}$$

which implies

$$\mathbb{P}(R_k < \infty) \leq \mathbb{P}(\eta_* < \infty \mid \xi(\rho,\rho^*) = 1)^k. \tag{5.3.10}$$

By Lemma 5.2.2 the probability $\mathbb{P}(\eta_* < \infty \mid \xi(\rho,\rho^*) = 1)$ is strictly less than one and therefore

$$\begin{aligned}
\mathbb{P}(K = \infty) &= \mathbb{P}(R_n < \infty \text{ for all } n \in \mathbb{N}) \\
&= \lim_{n \to \infty} \mathbb{P}(R_n < \infty) \\
&\leq \lim_{n \to \infty} \mathbb{P}(\eta_* < \infty \mid \xi(\rho,\rho^*) = 1)^n = 0,
\end{aligned}$$

which concludes the proof. □

Proof of Lemma 5.2.4

We first prove the lemma for $k = 1$. From Lemma 5.2.3 we directly get that the first regeneration time τ_1 is almost surely finite. To show the second part of the statement we follow the arguments of [Guo16].

Given $m \in \mathbb{N}$, vertices $v_0, \ldots, v_m \in \mathbb{T}$ and a measurable subset of environments $A_1 \subseteq \Omega$, we let

$$A = A(m, v_0, \ldots, v_m, A_1) = \big\{\tau_1 = m, X_0 = v_0, \ldots, X_m = v_m, \omega\backslash\omega(v_m) \in A_1\big\}$$

be an element of the generator of $\mathcal{G}_1$ (see (5.2.3)). We set $v = v_m$ and

$$A_2^l = \big\{(X_n)_{n\geq 0} : \sigma_l = m, X_0 = v_0, \ldots, X_m = v_m\big\}$$

for $l \geq 1$. Furthermore, we let $B = B_1 \times B_2 \in \mathcal{G} \times \mathcal{F}$ be a set of the generator of $\mathcal{G} \otimes \mathcal{F}$. Recall that the first regeneration time occurs at one of the potential regeneration times σ_l, namely the last finite one (see definition (5.2.1)). This implies

$$\begin{aligned}
&\mathbb{P}(B \circ \Theta_{\tau_1} \cap A)\\
&= \sum_{l\in\mathbb{N}} \mathbb{P}\big(B \circ \Theta_{\tau_1} \cap A \cap \{\tau_1 = \sigma_l\}\big)\\
&= \sum_{l\in\mathbb{N}} \mathbb{P}\big(B \circ \Theta_{\tau_1} \cap A \cap \{\sigma_l < \infty, R_l = \infty\}\big)\\
&= \sum_{l\in\mathbb{N}} \mathrm{E}\big[P_\omega\big(\{(X_n - v)_{n\geq\sigma_l} \in B_2\} \cap A_2^l \cap \{R_l = \infty\}\big)\mathbb{1}_{A_1}(\omega\backslash\omega(v))\mathbb{1}_{B_1}([\omega^*(v)])\mathbb{1}_{\{v\in T,\xi(v,v^*)=1\}}\big].
\end{aligned}$$

Using the strong Markov property at time σ_l we get

$$\begin{aligned}
&\mathbb{P}(B \circ \Theta_{\tau_1} \cap A)\\
&= \sum_{l\in\mathbb{N}} \mathrm{E}\big[P_\omega^v((X_n - v)_{n\geq 0} \in B_2, \eta_{v^*} = \infty)P_\omega(A_2^l)\mathbb{1}_{A_1}(\omega\backslash\omega(v))\mathbb{1}_{B_1}([\omega^*(v)])\mathbb{1}_{\{v\in T,\xi(v,v^*)=1\}}\big]\\
&= \sum_{l\in\mathbb{N}} \mathrm{E}\big[P_\omega^v((X_n - v)_{n\geq 0} \in B_2, \eta_{v^*} = \infty)P_\omega(A_2^l)\mathbb{1}_{A_1}(\omega\backslash\omega(v))\mathbb{1}_{B_1}([\omega^*(v)]) \,\big|\, v \in T, \xi(v, v^*) = 1\big]\\
&\qquad \times \mathrm{P}(v \in T, \xi(v, v^*) = 1).
\end{aligned}$$

The random variables $P_\omega(A_2^l)\mathbb{1}_{A_1}(\omega\backslash\omega(v))$ and $P_\omega^v\big((X_n-v)_{n\geq 0} \in B_2, \eta_{v^*} = \infty\}\big)\mathbb{1}_{B_1}([\omega^*(v)])$ are independent under $\mathrm{P}(\,\cdot \mid v \in T, \xi(v, v^*) = 1)$. This implies

$$\begin{aligned}
&\mathbb{P}(B \circ \Theta_{\tau_1} \cap A)\\
&\quad= \sum_{l\in\mathbb{N}} \mathrm{E}\big[P_\omega^v\big((X_n - v)_{n\geq 0} \in B_2, \eta_{v^*} = \infty\big)\mathbb{1}_{B_1}([\omega^*(v)])\big| v \in T, \xi(v, v^*) = 1\big]\\
&\qquad\qquad \times \mathrm{E}\big[P_\omega(A_2^l)\mathbb{1}_{A_1}(\omega\backslash\omega(v)) \,\big|\, v \in T, \xi(v, v^*) = 1\big]\mathrm{P}(v \in T, \xi(v, v^*) = 1)\\
&\quad= \mathrm{E}\big[P_\omega\big((X_n)_{n\geq 0} \in B_2, \eta_* = \infty\big)\mathbb{1}_{B_1}(\omega) \,\big|\, \xi(\rho, \rho^*) = 1\big] \sum_{l\in\mathbb{N}} \mathrm{E}\big[P_\omega(A_2^l)\mathbb{1}_{A_1}(\omega\backslash\omega(v))\big]\\
&\quad= \mathbb{P}\big(B \cap \{\eta_* = \infty\} \,\big|\, \xi(\rho, \rho^*) = 1\big) \sum_{l\in\mathbb{N}} \mathrm{E}\big[P_\omega(A_2^l)\mathbb{1}_{A_1}(\omega\backslash\omega(v))\big].
\end{aligned}$$

The generator $\mathcal{G}\times\mathcal{F}$ of the σ-algebra $\mathcal{G}\otimes\mathcal{F}$ is a π-system, which implies that the identity above holds for arbitrary sets $B\in\mathcal{G}\otimes\mathcal{F}$. In particular, we can substitute B by the set of all events $\Omega\times\mathbb{T}^{\mathbb{N}_0}$ such that we obtain

$$\mathbb{P}(A)=\mathbb{P}(\eta_*=\infty\mid\xi(\rho,\rho^*)=1)\sum_{l\in\mathbb{N}}\mathrm{E}\big[P_\omega(A_2^l)\mathbb{1}_{A_1}(\omega\backslash\omega(v))\big]$$

for all A in the generator of $\mathcal{G}_1$ and therefore

$$\begin{aligned}\mathbb{P}(B\circ\Theta_{\tau_1}\cap A)&=\mathbb{P}(A)\frac{\mathbb{P}\big(B\cap\{\eta_*=\infty\}\bigm|\xi(\rho,\rho^*)=1\big)}{\mathbb{P}(\eta_*=\infty\mid\xi(\rho,\rho^*)=1)}\\&=\mathbb{P}(A)\mathbb{P}(B\mid\eta_*=\infty,\xi(\rho,\rho^*)=1).\end{aligned}$$

Using the uniqueness of extension of measures again, we obtain that the above holds for all $A\in\mathcal{G}_1$. This implies

$$\begin{aligned}\mathbb{E}\big[\mathbb{P}(B\mid\eta_*=\infty,\xi(\rho,\rho^*)=1)\mathbb{1}_A\big]&=\mathbb{P}(B\mid\eta_*=\infty,\xi(\rho,\rho^*)=1)\mathbb{P}(A)\\&=\mathbb{P}(B\circ\Theta_{\tau_1}\cap A)=\mathbb{E}\big[\mathbb{1}_{B\circ\Theta_{\tau_1}}\mathbb{1}_A\big]\end{aligned}$$

for any $A\in\mathcal{G}_1$ and we conclude that

$$\mathbb{P}(B\circ\Theta_{\tau_1}\mid\mathcal{G}_1)=\mathbb{E}\big[\mathbb{1}_{B\circ\Theta_{\tau_1}}\bigm|\mathcal{G}_1\big]=\mathbb{P}(B\mid\eta_*=\infty,\xi(\rho,\rho^*)=1)$$

for $B\in\mathcal{G}\otimes\mathcal{F}$.

Let us now show by induction that the lemma is true for general $k\in\mathbb{N}$. Suppose the statement holds for $k-1$. Then we have

$$\begin{aligned}\mathbb{P}(\tau_k<\infty)&=\mathbb{P}(\tau_{k-1}+\tau_1\circ\theta_{\tau_{k-1}}<\infty,\tau_{k-1}<\infty)\\&=\mathbb{P}(\tau_1\circ\theta_{\tau_{k-1}}<\infty\mid\tau_{k-1}<\infty)\mathbb{P}(\tau_{k-1}<\infty)\\&=\mathbb{P}(\tau_1<\infty\mid\eta_*=\infty,\xi(\rho,\rho^*)=1).\end{aligned}$$

Due to Lemma 5.2.2 and the fact that the first regeneration time τ_1 is almost surely finite, we obtain

$$\begin{aligned}0\le\mathbb{P}(\tau_k=\infty)&=\mathbb{P}(\tau_1=\infty\mid\eta_*=\infty,\xi(\rho,\rho^*)=1)\\&\le\mathbb{P}(\tau_1=\infty)\mathbb{P}(\eta_*=\infty\mid\xi(\rho,\rho^*)=1)^{-1}\mathbb{P}(\xi(\rho,\rho^*)=1)^{-1}=0.\end{aligned}$$

The proof of second part of the statement works with the same arguments as for $k=1$, merely the notation becomes even more cumbersome. □

Proof of Proposition 5.2.5

Lemma 5.2.4 implies

$$\begin{aligned}
&\mathbb{P}\big([\omega^*(X_{\tau_n})\backslash\omega(X_{\tau_{n+1}})] \in A_1, (X_k - X_{\tau_n})_{\tau_n \le k \le \tau_{n+1}} \in A_2, \tau_{n+1} - \tau_n \in A_3\big) \\
&= \mathbb{E}\big[\mathbb{P}\big([\omega^*(X_{\tau_n})\backslash\omega(X_{\tau_{n+1}})] \in A_1, (X_k - X_{\tau_n})_{\tau_n \le k \le \tau_{n+1}} \in A_2, \tau_{n+1} - \tau_n \in A_3 \,\big|\, \mathcal{G}_n\big)\big] \\
&= \mathbb{E}\big[\mathbb{P}\big(\omega\backslash\omega(X_{\tau_1}) \in A_1, (X_k)_{k \le \tau_1} \in A_2, \tau_1 \in A_3 \,\big|\, \eta_* = \infty, \xi(\rho, \rho^*) = 1\big)\big] \\
&= \mathbb{P}\big(\omega\backslash\omega(X_{\tau_1}) \in A_1, (X_k)_{k \le \tau_1} \in A_2, \tau_1 \in A_3 \,\big|\, \eta_* = \infty, \xi(\rho, \rho^*) = 1\big).
\end{aligned}$$

It remains to show the independence. For convenience, we set

$$S_n = \big([\omega^*(X_{\tau_n})\backslash\omega(X_{\tau_{n+1}})], (X_k - X_{\tau_n})_{\tau_n \le k \le \tau_{n+1}}, \tau_{n+1} - \tau_n\big).$$

Given $n \ge 1$ and measurable sets $B_1, \ldots, B_n$, we have

$$\begin{aligned}
\mathbb{P}(S_1 \in B_1, \ldots, S_n \in B_n) &= \mathbb{E}\big[\mathbb{1}_{\{S_1 \in B_1\}} \cdot \ldots \cdot \mathbb{1}_{\{S_n \in B_n\}}\big] \\
&= \mathbb{E}\big[\mathbb{1}_{\{S_1 \in B_1\}} \cdot \ldots \cdot \mathbb{1}_{\{S_{n-1} \in B_{n-1}\}} \mathbb{E}\big[\mathbb{1}_{\{S_n \in B_n\}} \,\big|\, \mathcal{G}_n\big]\big].
\end{aligned}$$

Lemma 5.2.4 implies

$$\mathbb{P}(S_n \in B_n | \mathcal{G}_n) = \mathbb{P}\big((\omega\backslash\omega(X_{\tau_1}), (X_k)_{k \le \tau_1}, \tau_1) \in B_n | \eta_* = \infty, \xi(\rho, \rho^*) = 1\big).$$

Since the right-hand side of the above expression is deterministic, we obtain

$$\begin{aligned}
\mathbb{P}(S_1 \in B_1, \ldots, S_n \in B_n) &= \mathbb{P}(S_n \in B_n \mid \mathcal{G}_n)\mathbb{P}(S_1 \in B_1, \ldots, S_{n-1} \in B_{n-1}) \\
&= \mathbb{E}\big[\mathbb{P}(S_n \in B_n \mid \mathcal{G}_n)\big]\mathbb{P}(S_1 \in B_1, \ldots, S_{n-1} \in B_{n-1}) \\
&= \mathbb{P}(S_n \in B_n)\mathbb{P}(S_1 \in B_1, \ldots, S_{n-1} \in B_{n-1}).
\end{aligned}$$

By induction we get

$$\mathbb{P}(S_1 \in B_1, \ldots, S_n \in B_n) = \mathbb{P}(S_1 \in B_1) \cdot \ldots \cdot \mathbb{P}(S_n \in B_n),$$

which concludes the proof. □

5.3.5 Moment bounds on regeneration distances: proof of Lemma 5.2.7

To show the moment bounds on the regeneration distances, we follow the arguments of [SZ99] with modifications as in [vdHHN20]. Let us assume that the subtree $T_1(\rho)$ is supercritical, i.e. $(1-\alpha)m_1 > 1$ and $T_1(\rho)$ has a strictly positive probability to survive. If the subtree $T_1(\rho)$ is (sub-)critical, the proof works analogously but the moment bounds then depend on ε, since the lower bound in Lemma 5.2.2 depends on ε in that case.

Lemma 5.2.2 and Proposition 5.2.5 imply

$$\begin{aligned}\mathbb{E}\big[(|X_{\tau_2}|-|X_{\tau_1}|)^q\big] &= \mathbb{E}\big[(|X_{\tau_1}|-|X_0|)^q \,\big|\, \eta_*=\infty, \xi(\rho,\rho^*)=1\big] \\ &\le \mathbb{E}\big[(|X_{\tau_1}|-|X_0|)^q\big]\mathbb{P}(\eta_*=\infty \mid \xi(\rho,\rho^*)=1)^{-1}\mathbb{P}(\xi(\rho,\rho^*)=1)^{-1} \\ &\le C\mathbb{E}\big[|X_{\tau_1}|^q\big].\end{aligned}$$

It is therefore sufficient to bound $\mathbb{E}\big[|X_{\tau_1}|^q\big]$. Since the first regeneration time occurs at one of the potential regeneration times, namely at the last finite one (see definition (5.2.1)), we have

$$\begin{aligned}\mathbb{E}\big[|X_{\tau_1}|^q\big] &= \sum_{k\in\mathbb{N}}\mathbb{E}\big[|X_{\tau_1}|^q \mathbb{1}_{\{\sigma_k<\infty, R_k=\infty\}}\big] \le \sum_{k\in\mathbb{N}}\mathbb{E}\big[|X_{\sigma_k}|^q\mathbb{1}_{\{\sigma_k<\infty\}}\big] \\ &\le \sum_{k\in\mathbb{N}}\mathbb{E}\big[|X_{\sigma_k}|^{2q}\mathbb{1}_{\{\sigma_k<\infty\}}\big]^{\frac{1}{2}}\mathbb{P}(\sigma_k<\infty)^{\frac{1}{2}},\end{aligned} \tag{5.3.11}$$

where the last step is obtained by applying the Cauchy-Schwarz inequality. For σ_k to be finite, the random walk has to visit a new generation via an edge with conductance one and return to its ancestor at least $k-1$ times. As we have seen in the proof of Lemma 5.2.3 (see (5.3.10)), we can bound the probability of this event by

$$\mathbb{P}(\sigma_k<\infty) \le \mathbb{P}(R_{k-1}<\infty) \le \mathbb{P}(\eta_*<\infty \mid \xi(\rho,\rho^*)=1)^{k-1}.$$

Lemma 5.2.2 implies that there exists a constant $c=c(\nu,\alpha,\kappa)\in(0,1)$ such that

$$\mathbb{P}(\sigma_k<\infty) \le (1-c)^{k-1}. \tag{5.3.12}$$

To bound the expectation in (5.3.11), we decompose the trajectory of the random walk. The path to the first regeneration consists, on the one hand, of excursions starting from potential regeneration points at which the random walk does not regenerate. On the other hand, the random walk may reach a new maximal generation outside of such excursions, but not via edge with conductance one, so that the hitting time is not a potential regeneration time. To specify this decomposition of the trajectory into excursions and the steps outside of them, we introduce

$$\begin{aligned} M_k &= |X_{\sigma_k}|, \\ N_k &= \max\{m\ge 1 : \eta_m < R_k\}\end{aligned}$$

so that M_k is the distance of the k-th potential regeneration point to the root and N_k is the maximal generation that the random walk visits before it returns to the ancestor of the vertex X_{σ_k}. Furthermore, we define

$$H_1 = M_1, \quad H_k = M_k - N_{k-1}.$$

That is, H_k indicates the number of generations that the random walk visits after it has passed the previous maximal generation N_{k-1} and until it reaches the next potential regeneration point. Lastly, we set

$$\tilde{N}_0 = 0, \quad \tilde{N}_k = N_k - M_k$$

so that $\tilde{N}_k$ indicates the number of generations that the random walk reaches for the first time after it has passed the k-th potential regeneration point and until it returns to its ancestor. In other words, $\tilde{N}_k$ counts the number of new generations that are visited during the excursion starting from the k-th potential regeneration point. The distance from the k-th regeneration point to the root can now be expressed as

$$|X_{\sigma_k}| = M_k = \sum_{i=0}^{k-1} (\tilde{N}_i + H_{i+1}).$$

Due to the bounds in (5.3.11) and (5.3.12), we obtain

$$\mathbb{E}\big[|X_{\tau_1}|^q\big] \leq \sum_{k\in\mathbb{N}} \mathbb{E}\left[\left(\sum_{i=0}^{k-1}(\tilde{N}_i + H_{i+1})\right)^{2q} \mathbb{1}_{\{\sigma_k<\infty\}}\right]^{\frac{1}{2}} (1-c)^{\frac{k-1}{2}}.$$

Applying Jensen's inequality twice we arrive at

$$\begin{aligned}
\mathbb{E}\big[|X_{\tau_1}|^q\big] &\leq \sum_{k\in\mathbb{N}} \mathbb{E}\left[k^{2q-1} \sum_{i=0}^{k-1} \big(\tilde{N}_i + H_{i+1}\big)^{2q} \mathbb{1}_{\{\sigma_k<\infty\}}\right]^{\frac{1}{2}} (1-c)^{\frac{k-1}{2}} \\
&\leq \sum_{k\in\mathbb{N}} \mathbb{E}\left[(2k)^{2q-1} \sum_{i=0}^{k-1} \big(\tilde{N}_i^{2q} + H_{i+1}^{2q}\big) \mathbb{1}_{\{\sigma_k<\infty\}}\right]^{\frac{1}{2}} (1-c)^{\frac{k-1}{2}} \\
&\leq \sum_{k\in\mathbb{N}} (2k)^{\frac{2q-1}{2}} \left(\sum_{i=0}^{k-1} \mathbb{E}\big[\tilde{N}_i^{2q} \mathbb{1}_{\{R_i<\infty\}}\big] + \sum_{i=0}^{k-1} \mathbb{E}\big[H_{i+1}^{2q} \mathbb{1}_{\{R_i<\infty\}}\big]\right)^{\frac{1}{2}} (1-c)^{\frac{k-1}{2}},
\end{aligned} \tag{5.3.13}$$

where we used for the last inequality that $\{\sigma_k < \infty\} \subseteq \{R_{k-1} < \infty\} \subseteq \{R_i < \infty\}$ holds for every $i \leq k-1$. The right-hand side remains finite if we can uniformly bound the occurring moments.

Uniform bound for $\mathbb{E}\big[H_{i+1}^{2q} \mathbb{1}_{\{R_i<\infty\}}\big]$

Recall that H_i counts the number of new generations the random walker visits after it has exceeded the previous maximal generation and until it reaches the next potential regeneration point. In other words, after reaching the previous maximal generation, we wait until the random walk enters a new generation for the first time via an edge with conductance one. For this reason, the idea is to stochastically dominate H_i by a geometric random variable. To do this, we have to derive a uniform lower bound for the probability that the random walker hits a new generation via an edge with conductance one.

Provided the return time R_i is finite, we denote the times at which the walker reaches a new maximum generation (regardless of the conductance of the crossed edge) by

$$s_k^i = \inf\{m \geq R_i : |X_m| = N_i + k\}.$$

Furthermore, we introduce random variables $Y_1^i, Y_2^i, \dots$ which indicate whether the walker reached the new generation via an edge with conductance one or not. That is,

$$Y_k^i = \begin{cases} 1, & \text{if } \xi(X_{s_k^i}, X_{s_k^i}^*) = 1 \\ 0, & \text{otherwise.} \end{cases}$$

We denote by

$$V_1 = V_1(T) = \big\{v \in T : \xi(v, z) = 1 \text{ for all } z \in G_1(T(v))\big\}$$

the set of all vertices $v \in T$ where every edge connecting v with a descendant has conductance one. Provided that the random walk is located at a vertex $v \in V_1$ at time s_k^i and its next step leads to a descendant, we have $Y_{k+1}^i = 1$. This yields the following lower bound

$$\begin{aligned} \mathbb{P}\big(\{Y_{k+1}^i = 1\} \cap B_{i,k}\big) &= \sum_{v \in \mathbb{T}} \mathbb{P}\big(\big\{Y_{k+1}^i = 1, X_{s_k^i} = v\big\} \cap B_{i,k}\big) \\ &\geq \sum_{v \in \mathbb{T}} \mathbb{P}\big(\big\{|X_{s_k^i+1}| = |v| + 1, X_{s_k^i} = v, v \in V_1\big\} \cap B_{i,k}\big) \\ &= \sum_{v \in \mathbb{T}} \mathrm{E}\big[P_\omega\big(\big\{|X_{s_k^i+1}| = |v| + 1, X_{s_k^i} = v\big\} \cap B_{i,k}\big)\mathbb{1}_{\{v \in V_1\}}\big], \end{aligned}$$

where $B_{i,k} = \{Y_1^i = \ldots = Y_k^i = 0, R_i < \infty\}$. Using the strong Markov property in s_k^i we obtain

$$\mathbb{P}\big(\{Y_{k+1}^i = 1\} \cap B_{i,k}\big) \geq \sum_{v \in \mathbb{T}} \mathrm{E}\big[P_\omega^v\big(|X_1| = |v| + 1\big)P_\omega\big(\{X_{s_k^i} = v\} \cap B_{i,k}\big)\mathbb{1}_{\{v \in V_1\}}\big].$$

When the random walk starts at a vertex $v \in V_1$, it moves to a descendant with a probability of at least $\frac{1}{\kappa+1}$. This gives rise to the following bound

$$\begin{aligned} \mathbb{P}\big(\{Y_{k+1}^i = 1\} \cap B_{i,k}\big) &\geq \frac{1}{\kappa+1} \sum_{v \in \mathbb{T}} \mathrm{E}\big[P_\omega\big(\{X_{s_k^i} = v\} \cap B_{i,k}\big)\mathbb{1}_{\{v \in V_1\}}\big] \\ &= \frac{1}{\kappa+1} \sum_{v \in \mathbb{T}} \mathrm{E}\big[P_\omega\big(\{X_{s_k^i} = v\} \cap B_{i,k}\big)\mathbb{1}_{\{v \in V_1\}} \bigm| v \in T\big]\mathrm{P}(v \in T). \end{aligned}$$

Since $P_\omega\big(\{X_{s_k^i} = v\} \cap B_{i,k}\big)$ and $\mathbb{1}_{\{v \in V_1\}}$ are independent under $\mathrm{P}(\,\cdot \mid v \in T)$, we obtain

$$\begin{aligned}
&\mathbb{P}\big(\{Y_{k+1}^i = 1\} \cap B_{i,k}\big) \\
&\quad \geq \frac{1}{\kappa+1} \sum_{v \in \mathbb{T}} \mathrm{E}\big[P_\omega\big(\{X_{s_k^i} = v\} \cap B_{i,k}\big) \;\big|\; v \in T\big] \mathrm{P}(v \in V_1 \mid v \in T)\mathrm{P}(v \in T) \\
&\quad = \frac{1}{\kappa+1} \mathrm{P}(\rho \in V_1)\mathbb{P}(B_{i,k})
\end{aligned}$$

and

$$\mathbb{P}\big(Y_{k+1}^i = 1 \;\big|\; B_{i,k}\big) \geq \frac{1}{\kappa+1}\mathrm{P}(\rho \in V_1) =: p \in (0,1).$$

This implies

$$\begin{aligned}
&\mathbb{P}\big(H_{i+1}\mathbb{1}_{\{R_i<\infty\}} > m\big) = \mathbb{P}\big(H_{i+1} > m, R_i < \infty\big) \\
&\quad \leq \mathbb{P}\big(Y_k^i = 0 \text{ for all } k \leq m \;\big|\; R_i < \infty\big) \\
&\quad = \mathbb{P}\big(Y_1^i = 0 \mid R_i < \infty\big) \cdot \ldots \cdot \mathbb{P}\big(Y_m^i = 0 \;\big|\; R_i < \infty, Y_1^i = 0, \ldots, Y_{m-1}^i = 0\big) \\
&\quad \leq (1-p)^m
\end{aligned}$$

and therefore

$$\mathbb{E}\big[(H_{i+1}\mathbb{1}_{\{R_i<\infty\}})^{2q}\big] \leq C_1 < \infty \tag{5.3.14}$$

for some constant $C_1 \geq 0$ independent of ε.

Uniform bound for $\mathbb{E}\big[\tilde{N}_i^{2q}\mathbb{1}_{\{R_i<\infty\}}\big]$

In view of (5.3.13), the proof is completed once we have shown that $\mathbb{E}[\tilde{N}_i^{2q}\mathbb{1}_{\{R_i<\infty\}}]$ is uniformly bounded. This is the major challenge in the proof. Recall that $\tilde{N}_i$ counts the number of generations the random walk visits during the excursion starting from the i-th potential regeneration point. Provided that $R_i < \infty$, a large value $\tilde{N}_i$ means that the random walk has to go a long distance back towards the root. We will show that the probability of backtracking a long way decays exponentially. To prove this, the key is a regularity estimate for trees similar as in [GK84].

Due to the construction of the potential regeneration points, the excursions take place in disjoint parts of the tree. Therefore, the random variables $\tilde{N}_i\mathbb{1}_{\{R_i<\infty\}}$ are i.i.d. and it suffices to show

$$\mathbb{E}\big[\tilde{N}_1^{2q}\mathbb{1}_{\{R_1<\infty\}}\big] \leq C_2 < \infty.$$

for a constant $C_2 \geq 0$ that is independent of ε. Using that $1 + x \leq \mathrm{e}^x$ holds for all $x \in \mathbb{R}$, we get for all $s > 0$

$$\mathbb{E}\big[\tilde{N}_1^{2q}\mathbb{1}_{\{R_1<\infty\}}\big] \leq C\mathbb{E}\big[\mathrm{e}^{s\tilde{N}_1}\mathbb{1}_{\{R_1<\infty\}}\big], \tag{5.3.15}$$

where $C = C(s,q)$ is some constant. Consequently, it suffices to bound

$$\mathbb{E}\big[\mathrm{e}^{s\tilde{N}_1}\mathbb{1}_{\{R_1<\infty\}}\big] = \sum_{n\in\mathbb{N}} \mathrm{e}^{sn}\mathbb{P}(\tilde{N}_1 = n, R_1 < \infty) \tag{5.3.16}$$

for some $s > 0$. This means we have to show that the event $\{\tilde{N}_1 = n, R_1 < \infty\}$ has exponentially small probability. We introduce

$$T = \max\{m \geq 1 : \eta_m < \eta_{X_0^*}\} - |X_0|$$

so that T counts the number of generations the random walk visits until it hits the ancestor of its starting point for the first time. We calculate

$$\begin{aligned}
\mathbb{P}(\tilde{N}_1 = n, R_1 < \infty) &= \sum_{v\in\mathbb{T}} \mathrm{E}\big[P_\omega(\tilde{N}_1 = n, R_1 < \infty, \sigma_1 < \infty, X_{\sigma_1} = v)\mathbb{1}_{\{v\in T, \xi(v,v^*)=1\}}\big] \\
&= \sum_{v\in\mathbb{T}} \mathrm{E}\big[P_\omega^v(T = n, \eta_{v^*} < \infty)P_\omega(\sigma_1 < \infty, X_{\sigma_1} = v)\mathbb{1}_{\{v\in T,\xi(v,v^*)=1\}}\big] \\
&= \sum_{v\in\mathbb{T}} \mathrm{E}\big[P_\omega^v(T = n, \eta_{v^*} < \infty)P_\omega(\sigma_1 < \infty, X_{\sigma_1} = v) \bigm| v \in T, \xi(v,v^*) = 1\big] \\
&\qquad \times \mathrm{P}(v \in T, \xi(v,v^*) = 1),
\end{aligned}$$

where the second step holds due to the Markov property. Under $\mathrm{P}(\,\cdot\,|v \in T, \xi(v,v^*) = 1)$, the random variables $P_\omega^v(T = n, \eta_{v^*} < \infty)$ and $P_\omega(\sigma_1 < \infty, X_{\sigma_1} = v)$ are independent. This implies

$$\begin{aligned}
&\mathbb{P}(\tilde{N}_1 = n, R_1 < \infty) \\
&= \sum_{v\in\mathbb{T}} \mathrm{E}\big[P_\omega^v(T = n, \eta_{v^*} < \infty) \bigm| v \in T, \xi(v,v^*) = 1\big]\mathrm{E}\big[P_\omega(\sigma_1 < \infty, X_{\sigma_1} = v)\mathbb{1}_{\{v\in T,\xi(v,v^*)=1\}}\big] \\
&= \mathrm{E}\big[P_\omega(T = n, \eta_* < \infty) \bigm| \xi(\rho,\rho^*) = 1\big]\mathbb{P}(\sigma_1 < \infty) \\
&= \mathrm{E}\left[\sum_{v\in G_n(T)} P_\omega(\eta_n < \eta_* < \eta_{n+1}, X_{\eta_n} = v) \,\middle|\, \xi(\rho,\rho^*) = 1\right] \\
&\leq \mathrm{E}\left[\sum_{v\in G_n(T)} P_\omega(\eta_* \circ \theta_{\eta_n} < \infty, X_{\eta_n} = v) \,\middle|\, \xi(\rho,\rho^*) = 1\right].
\end{aligned}$$

Here, $\eta_* \circ \theta_{\eta_n} = \inf\{m \geq \eta_n : X_m = \rho^*\}$ denotes the time at which the shifted path first hits ρ^*, recall definition (5.2.2). Again, due to the strong Markov property, we arrive at

$$\mathbb{P}(\tilde{N}_1 = n, R_1 < \infty) \leq \mathrm{E}\left[\sum_{v\in G_n(T)} P_\omega^v(\eta_* < \infty)P_\omega(X_{\eta_n} = v) \,\middle|\, \xi(\rho,\rho^*) = 1\right]. \tag{5.3.17}$$

Quenched return probability

We need a quenched bound for the return probability $P_\omega^v(\eta_* < \infty)$. We let $\omega = (T, \rho, \xi)$ be an environment and $v \in G_n(T)$ a vertex of the n-th generation. Given $L \le n$, we set $r = \lfloor \frac{n}{L} \rfloor$ and denote the ancestor of v in $G_{[i]} = G_{iL}(T)$ by v^i for $i = 0, \dots, r$. Recall that $\eta_{v^i} = \inf\{n \ge 0 : X_n = v^i\}$ denotes the hitting time of vertex v^i, we obtain due to the strong Markov property

$$P_\omega^v(\eta_* < \infty) \le P_\omega^v(\eta_{v^0} < \infty, \dots, \eta_{v^r} < \infty) \le \prod_{i=1}^{r} P_\omega^{v^i}(\eta_{v^{i-1}} < \infty). \tag{5.3.18}$$

We denote the unique path between v^{i-1} and v^i by $v^{i-1} = v_0^i, v_1^i, \dots, v_L^i = v^i$. Again, due to the Markov property, we have

$$\begin{aligned} P_\omega^{v^i}(\eta_{v^{i-1}} < \infty) = P_\omega^{v_L^i}\big(\eta_{v_{L-1}^i} < \infty, \dots, \eta_{v_0^i} < \infty\big) &= \prod_{j=1}^{L} P_\omega^{v_j^i}\big(\eta_{v_{j-1}^i} < \infty\big) \\ &= \prod_{j=1}^{L} \frac{\xi(v_{j-1}^i, v_j^i)}{\xi(v_{j-1}^i, v_j^i) + \mathcal{C}_{\omega(v_j^i)}(v_j^i, \infty)}, \end{aligned}$$

where we used Lemma 2.4.1 for the last equality. Since, by Rayleigh's Monotonicity Principle (Lemma 2.4.5), removing edges can only reduce the effective conductance, we obtain

$$\mathcal{C}_{\omega(v_j^i)}(v_j^i, \infty) \ge \mathcal{C}_{\omega(v_j^i)\setminus\omega(v^i)}(v_j^i, \infty)$$

and therefore

$$\begin{aligned} P_\omega^{v^i}(\eta_{v^{i-1}} < \infty) &\le \prod_{j=1}^{L} \frac{\xi(v_{j-1}^i, v_j^i)}{\xi(v_{j-1}^i, v_j^i) + \mathcal{C}_{\omega(v_j^i)\setminus\omega(v^i)}(v_j^i, \infty)} \\ &= \prod_{j=1}^{L} P_{\omega\setminus\omega(v^i)}^{v_j^i}\big(\eta_{v_{j-1}^i} < \infty\big) = P_{\omega\setminus\omega(v^i)}^{v^i}\big(\eta_{v^{i-1}} < \infty\big). \end{aligned}$$

Due to (5.3.18), we conclude that

$$P_\omega^v(\eta_* < \infty) \le \prod_{i=1}^{r} P_{\omega\setminus\omega(v^i)}^{v^i}(\eta_{v^{i-1}} < \infty).$$

By removing parts of the environment we obtain independence of $P_{\omega\setminus\omega(v^i)}^{v^i}(\eta_{v^{i-1}} = \infty)$ for different i under P, which is crucial for the rest of the proof.

We can derive an exponentially small bound for the probability above if the underlying environment is good enough in some sense. Roughly speaking, we call an environment good if for all vertices $v \in G_{[r]}$ the fraction of ancestors v^i with a sufficiently high indirect escape probability is large enough. To be more precise, we introduce the set of good environments

$$B_r(\delta,\beta,L) = \left\{ \omega \in \Omega : \sum_{i=1}^{r} \mathbb{1}_{\left\{P^{v^i}_{\omega\setminus\omega(v^i)}(\eta_{v^{i-1}}=\infty)\geq\delta\right\}} \geq \beta r \text{ for all } v \in G_{[r]} \right\}$$

for β, $\delta \in (0,1)$. That is, for $\omega \in B_r(\delta,\beta,L)$ every vertex $v \in G_{[r]}(T)$ has at least βr ancestors v^i for which the indirect escape probability $P^{v^i}_{\omega\setminus\omega(v^i)}(\eta_{v^{i-1}} = \infty)$ is at least δ. We have to show that $B_r(\delta,\beta,L)$ has a sufficiently high probability, provided that δ and β are small and L is large enough. Then we are done. To see this, observe that for $\omega \in B_r(\delta,\beta,L)$ and $v \in G_n(T)$ with $r = \lfloor \frac{n}{L} \rfloor$ we have

$$P^v_\omega(\eta_* < \infty) \leq \prod_{i=1}^{r} \left((1-\delta)\mathbb{1}_{\left\{P^{v^i}_{\omega\setminus\omega(v^i)}(\eta_{v^{i-1}}=\infty)\geq\delta\right\}} + \mathbb{1}_{\left\{P^{v^i}_{\omega\setminus\omega(v^i)}(\eta_{v^{i-1}}=\infty)<\delta\right\}} \right) \leq (1-\delta)^{\beta r}.$$

In view of (5.3.17), the above estimate implies

$$\begin{aligned}
&\mathbb{P}(\tilde{N}_1 = n, R_1 < \infty) \\
&\leq (1-\delta)^{\beta r} \mathrm{E}\left[\sum_{v\in G_n(T)} P_\omega(X_{\eta_n} = v) \,\middle|\, \{\xi(\rho,\rho^*)=1\} \cap B_r(\delta,\beta,L) \right] \\
&\quad + \mathrm{E}\left[\sum_{v\in G_n(T)} P_\omega(X_{\eta_n} = v) \,\middle|\, \{\xi(\rho,\rho^*)=1\} \cap B_r(\delta,\beta,L)^c \right] \mathrm{P}(B_r(\delta,\beta,L)^c \mid \xi(\rho,\rho^*)=1) \\
&= (1-\delta)^{\beta r} \mathrm{E}\big[P_\omega(\eta_n < \infty) \,\big|\, \{\xi(\rho,\rho^*)=1\} \cap B_r(\delta,\beta,L) \big] \\
&\quad + \mathrm{E}\big[P_\omega(\eta_n < \infty) \,\big|\, \{\xi(\rho,\rho^*)=1\} \cap B_r(\delta,\beta,L)^c \big] \mathrm{P}(B_r(\delta,\beta,L)^c \mid \xi(\rho,\rho^*)=1) \\
&\leq (1-\delta)^{\beta r} + \mathrm{P}(B_r(\delta,\beta,L)^c \mid \xi(\rho,\rho^*)=1).
\end{aligned} \tag{5.3.19}$$

Since our aim is to show that the probability $\mathbb{P}(\tilde{N}_1 = n, R_1 < \infty)$ decays exponentially (recall (5.3.16)), it remains to show that $B_r(\delta,\beta,L)^c$ has exponentially small probability.

$B_r(\delta,\beta,L)^c$ has exponentially small probability

We use similar arguments as [GK84] and [DGPZ02] to show that $B_r(\delta,\beta,L)^c$ has exponentially small probability. We introduce

$$A_r(\delta,v,L) = \sum_{i=1}^{r} \mathbb{1}_{\{P^{v^i}_{\omega\setminus\omega(v^i)}(\eta_{v^{i-1}}=\infty)\geq\delta)\}}$$

for $v \in G_{[r]}$ and

$$Z_r(\delta,\theta,L) = \sum_{v\in G_{[r]}} \mathrm{e}^{-\theta A_r(\delta,v,L)}.$$

The Markov inequality implies

$$\begin{aligned}\mathrm{P}(B_r(\delta,\beta,L)^c \mid \xi(\rho,\rho^*)=1) &= \mathrm{P}\Big(\min_{v\in G_{[r]}} A_r(\delta,v,L) < \beta r\Big)\\ &\le \mathrm{e}^{\theta\beta r}\mathrm{E}\big[\mathrm{e}^{-\theta\min_v A_r(\delta,v,L)}\big]\\ &\le \mathrm{e}^{\theta\beta r}\mathrm{E}[Z_r(\delta,\theta,L)]. \qquad (5.3.20)\end{aligned}$$

This means we have to show that the expectation of $Z_r(\delta,\theta,L)$ decays exponentially, provided that θ is large enough. Using the independence of $P^{v^i}_{\omega\backslash\omega(v^i)}(\eta_{v^{i-1}}=\infty)$, we obtain the following recursion

$$\begin{aligned}\mathrm{E}[Z_r(\delta,\theta,L)] &= \mathrm{E}\Bigg[\sum_{v\in G_{[r-1]}}\sum_{z\in G_L(T(v))}\mathrm{e}^{-\theta A_r(\delta,z,L)}\Bigg]\\ &= \mathrm{E}\Bigg[\sum_{v\in G_{[r-1]}}\mathrm{e}^{-\theta A_{r-1}(\delta,v,L)}\sum_{z\in G_L(T(v))}\mathrm{e}^{-\theta\mathbb{1}_{\{P^z_{\omega\backslash\omega(z)}(\eta_v=\infty)\ge\delta\}}}\Bigg]\\ &= \mathrm{E}\Bigg[\sum_{v\in G_{[r-1]}}\mathrm{E}\Bigg[\mathrm{e}^{-\theta A_{r-1}(\delta,v,L)}\mathrm{E}\Bigg[\sum_{z\in G_L(T(v))}\mathrm{e}^{-\theta\mathbb{1}_{\{P^z_{\omega\backslash\omega(z)}(\eta_v=\infty)\ge\delta\}}}\,\Bigg|\,\omega\backslash\omega(v)\Bigg]\,\Bigg|\,G_{[r-1]}\Bigg]\Bigg]\\ &= \mathrm{E}\Bigg[\sum_{v\in G_{[r-1]}}\mathrm{E}\Bigg[\mathrm{e}^{-\theta A_{r-1}(\delta,v,L)}\mathrm{E}\Bigg[\sum_{z\in G_L(T(v))}\mathrm{e}^{-\theta\mathbb{1}_{\{P^z_{\omega\backslash\omega(z)}(\eta_v=\infty)\ge\delta\}}}\Bigg]\,\Bigg|\,G_{[r-1]}\Bigg]\Bigg]\\ &= \mathrm{E}\Bigg[\sum_{z\in G_L}\mathrm{e}^{-\theta\mathbb{1}_{\{P^z_{\omega\backslash\omega(z)}(\eta_\rho=\infty)\ge\delta\}}}\Bigg]\mathrm{E}\Bigg[\sum_{v\in G_{[r-1]}}\mathrm{E}\Bigg[\mathrm{e}^{-\theta A_{r-1}(\delta,v,L)}\,\Bigg|\,G_{[r-1]}\Bigg]\Bigg]\\ &= \mathrm{E}[Z_1(\delta,\beta,L)]\mathrm{E}[Z_{r-1}(\delta,\theta,L)].\end{aligned}$$

Iterating this leads to

$$\mathrm{E}[Z_r(\delta,\theta,L)] = \mathrm{E}[Z_1(\delta,\theta,L)]^r. \qquad (5.3.21)$$

Recall that m_1 denotes the offspring mean, the expectation of $Z_1(\delta,\theta,L)$ can be bounded as follows

$$\begin{aligned}\mathrm{E}[Z_1(\delta,\theta,L)] &= \mathrm{E}\Bigg[\sum_{z\in G_L}\Big(\mathrm{e}^{-\theta}\mathbb{1}_{\{P^z_{\omega\backslash\omega(z)}(\eta_\rho=\infty)\ge\delta\}}+\mathbb{1}_{\{P^z_{\omega\backslash\omega(z)}(\eta_\rho=\infty)<\delta\}}\Big)\Bigg]\\ &\le \mathrm{e}^{-\theta}m_1^L+\mathrm{E}\Bigg[\sum_{z\in G_L}\mathbb{1}_{\{P^z_{\omega\backslash\omega(z)}(\eta_\rho=\infty)<\delta\}}\Bigg]\\ &= \mathrm{e}^{-\theta}m_1^L+\mathrm{E}\Bigg[\sum_{v\in G_1}\mathrm{E}\Bigg[\sum_{z\in G_{L-1}(T(v))}\mathbb{1}_{\{P^z_{\omega\backslash\omega(z)}(\eta_\rho=\infty)<\delta\}}\,\Bigg|\,G_1\Bigg]\Bigg]\\ &= \mathrm{e}^{-\theta}m_1^L+m_1\mathrm{E}\Bigg[\sum_{z\in G_{L-1}}\mathbb{1}_{\{P^z_{\omega\backslash\omega(z)}(\eta_*=\infty)<\delta\}}\Bigg].\end{aligned}$$

In Lemma 5.3.3 below we show that there exists a constant $C = C(\nu, \alpha, \kappa) > 0$ and a function γ with $\gamma(L, \nu, \alpha) \to 0$ for $L \to \infty$ such that

$$\mathrm{E}[Z_1(\delta, \theta, L)] \le \mathrm{e}^{-\theta} m_1^L + m_1 \gamma(L-1, \nu, \alpha) + C \frac{\delta \kappa}{1 - \delta(1 + \kappa^2 (L-1))} m_1^L =: \zeta(\theta, \delta, L, \kappa).$$

We can therefore fix L sufficiently large such that $m_1 \gamma(L-1, \nu, \alpha) < 1$ holds. We then choose θ large and δ small enough so that the right-hand side is strictly less than one. Finally, due to (5.3.20) and (5.3.21), we obtain

$$\mathrm{P}(B_r(\delta, \beta, L)^c \mid \xi(\rho, \rho^*) = 1) \le \big(\mathrm{e}^{\theta\beta} \zeta(\theta, \delta, L, \kappa)\big)^r.$$

Provided that β is small enough, this bound decays exponentially in r and then also in n, which completes the proof. To see this, let us briefly summarize the results. Recall that due to (5.3.19), the above estimate implies that the event $\{\tilde{N}_1 = n, R_1 < \infty\}$ has exponentially small probability and therefore

$$\mathbb{E}\big[\tilde{N}_1^{2q} \mathbb{1}_{\{R_1 < \infty\}}\big] \le C_2 < \infty$$

(see (5.3.15) and (5.3.16)). In view of (5.3.13), we conclude from this and from the uniform bound in (5.3.14) that $\mathbb{E}[|X_{\tau_1}|^q]$ remains finite. □

Lemma 5.3.3. *There exists a function γ with $\gamma(L, \nu, \alpha) \to 0$ as $L \to \infty$ and a constant $C = C(\nu, \alpha, \kappa) < \infty$, both independent of ε, such that for $0 < \delta < \frac{1}{1+\kappa^2 L}$*

$$\mathrm{E}\left[\sum_{v \in G_L} \mathbb{1}_{\{P^v_{\omega \setminus \omega(v)}(\eta_* = \infty) < \delta\}}\right] \le \gamma(L, \nu, \alpha) + C \frac{\delta \kappa}{1 - \delta(1 + \kappa^2 L)} m_1^L.$$

Proof. A quenched bound for the indirect escape probability $P^v_{\omega \setminus \omega(v)}(\eta_* = \infty)$ highly depends on the realization of the tree. The possibilities of the random walk to escape depend in particular on the structure of the backbone tree T_1^{Bb}. For this reason, we differentiate according to the size of the L-th generation of T_1^{Bb},

$$\begin{aligned}\mathrm{E}\left[\sum_{v \in G_L} \mathbb{1}_{\{P^v_{\omega \setminus \omega(v)}(\eta_* = \infty) < \delta\}}\right] &= \mathrm{E}\left[\sum_{v \in G_L} \mathbb{1}_{\{P^v_{\omega \setminus \omega(v)}(\eta_* = \infty) < \delta\}} \mathbb{1}_{\{|G_L \cap T_1^{\mathrm{Bb}}| = 0\}}\right] \\ &\quad + \mathrm{E}\left[\sum_{v \in G_L} \mathbb{1}_{\{P^v_{\omega \setminus \omega(v)}(\eta_* = \infty) < \delta\}} \mathbb{1}_{\{|G_L \cap T_1^{\mathrm{Bb}}| = 1\}}\right] \\ &\quad + \mathrm{E}\left[\sum_{v \in G_L} \mathbb{1}_{\{P^v_{\omega \setminus \omega(v)}(\eta_* = \infty) < \delta\}} \mathbb{1}_{\{|G_L \cap T_1^{\mathrm{Bb}}| > 1\}}\right]. \end{aligned} \tag{5.3.22}$$

As a reminder, T_1^{Bb} denotes the backbone of $T_1(\rho)$, i.e. the subtree where all vertices that do not have an infinite line of descent are removed. For convenience, we write T_1 for the subtree $T_1(\rho)$. We study the three summands in (5.3.22) separately. Let us start with the first one.

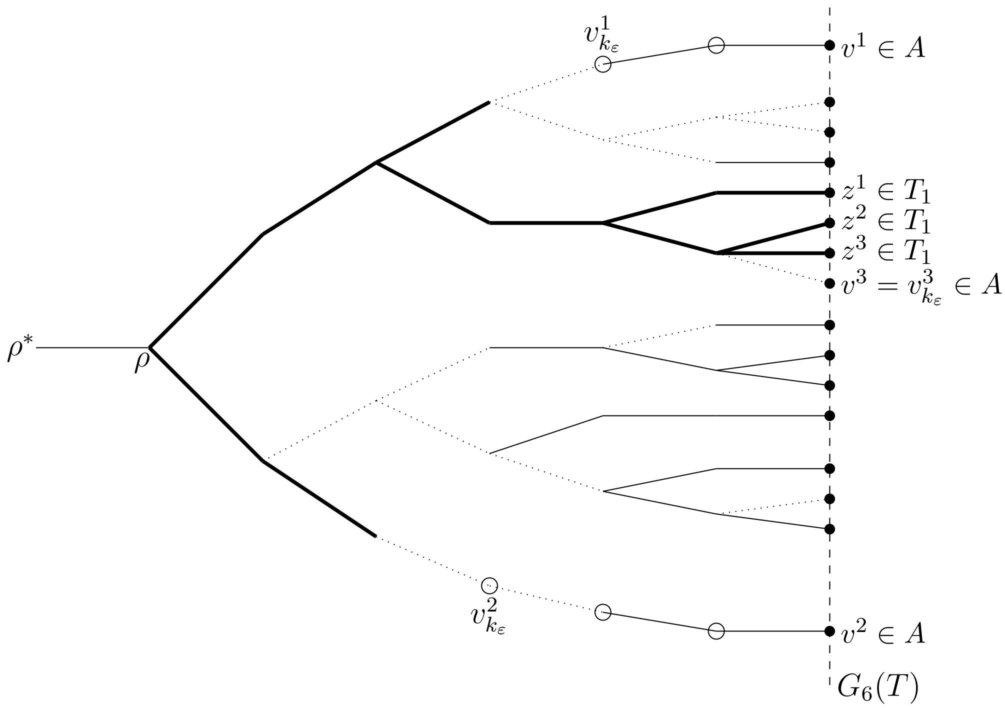


Figure 5.5: Edges with conductance ε are indicated by dotted lines; edges with conductance larger than ε are indicated by solid lines; vertices in the sixth generation $G_6(T)$ are marked by dots on the dashed line. $T_1 = T_1(\rho)$ is the subtree of $T(\rho)$ formed by the edges with conductance larger than ε containing the root (indicated by the thick lines). In the sixth generation $G_6(T)$ the vertices z^1, z^2, z^3 are in T_1. For a vertex $v \notin T_1$ the edge $(v_{k_\varepsilon - 1}, v_{k_\varepsilon})$ is the first edge of the path from ρ to v with conductance ε. A vertex v is located in the set A if $v = v_{k_\varepsilon}$ or if every vertex on the path from v_{k_ε} to v^* has degree 2. In the sixth generation the vertices v^1, v^2, v^3 are in A. The ancestors of v^i which must have degree 2 for v^i being in A are circled.

The case $|G_L \cap T_1^{\mathrm{Bb}}| = 0$: We start with considering the expected number of vertices that have an indirect escape probability less than δ on the event that the L-th generation of the backbone tree T_1^{Bb} is empty. This means that the subtree T_1 has to be finite. We denote the unique path between ρ and a vertex $v \in G_L$ by $\rho = v_0, v_1, \ldots, v_L = v$. Provided that $v \notin T_1$, we let k_ε be the index such that $(v_{k_\varepsilon - 1}, v_{k_\varepsilon})$ is the first edge of the path from ρ to v path with conductance ε, i.e.

$$k_\varepsilon = k_\varepsilon(v) = \min\{k \geq 1 : \xi(v_{k-1}, v_k) = \varepsilon\}.$$

Moreover, we introduce the set

$$A = \big\{v \in T \backslash T_1 : k_\varepsilon(v) = |v| \text{ or } \deg(v_k) = 2 \text{ for all } k \in \{k_\varepsilon(v), \ldots, |v| - 1\}\big\}$$

so that A contains a vertex $v \in T \backslash T_1$ if $v = v_{k_\varepsilon}$ or if every vertex on the path from v_{k_ε} to v^* has degree 2, see Figure 5.5 for an example. Whether the indirect escape probability

$P^v_{\omega\setminus\omega(v)}(\eta_* = \infty)$ is small depends on where the random walk starts: in T_1, in A or in neither. We therefore distinguish between these cases,

$$\begin{aligned}\mathrm{E}\Bigg[\sum_{v\in G_L}\mathbb{1}_{\{P^v_{\omega\setminus\omega(v)}(\eta_*=\infty)<\delta\}}\mathbb{1}_{\{|G_L\cap T_1^{\mathrm{Bb}}|=0\}}\Bigg] &= \mathrm{E}\Bigg[\sum_{v\in G_L}\mathbb{1}_{\{P^v_{\omega\setminus\omega(v)}(\eta_*=\infty)<\delta\}}\mathbb{1}_{\{|T_1|<\infty\}}\mathbb{1}_{\{v\in T_1\}}\Bigg]\\ &+\mathrm{E}\Bigg[\sum_{v\in G_L}\mathbb{1}_{\{P^v_{\omega\setminus\omega(v)}(\eta_*=\infty)<\delta\}}\mathbb{1}_{\{|T_1|<\infty\}}\mathbb{1}_{\{v\in A\}}\Bigg]\\ &+\mathrm{E}\Bigg[\sum_{v\in G_L}\mathbb{1}_{\{P^v_{\omega\setminus\omega(v)}(\eta_*=\infty)<\delta\}}\mathbb{1}_{\{|T_1|<\infty\}}\mathbb{1}_{\{v\notin T_1\cup A\}}\Bigg].\end{aligned} \tag{5.3.23}$$

We will treat the three summands separately in order of their appearance.

Bound for the first expectation in (5.3.23)**:** When the random walk starts at vertex $v \in T_1$, we cannot expect the probability of never hitting ρ^* to be small, since in this case the walker has to take an edge with conductance ε to escape on the pruned tree $T\setminus T(v)$. We therefore bound the first expectation in (5.3.23) by the expected number of vertices in the L-th generation of the subtree T_1 conditioned on this tree dying out, that is

$$\mathrm{E}\Bigg[\sum_{v\in G_L}\mathbb{1}_{\{P^v_{\omega\setminus\omega(v)}(\eta_*=\infty)<\delta\}}\mathbb{1}_{\{|T_1|<\infty\}}\mathbb{1}_{\{v\in T_1\}}\Bigg] \le \mathrm{E}\Bigg[\sum_{v\in G_L}\mathbb{1}_{\{v\in T_1\}}\ \Bigg|\ |T_1|<\infty\Bigg] = (m')^L \tag{5.3.24}$$

with $m' = \mathrm{E}\big[\sum_{z\in G_1}\mathbb{1}_{\{\kappa^{-1}\le\xi(\rho,z)\le\kappa\}}\big||T_1|<\infty\big]$. Conditioned on extinction, the subtree T_1 is a subcritical Galton-Watson tree, which implies that m' is strictly less than one (see e.g. Theorem 3.7 and Exercise 3.17 in [vdH17]). Consequently, the above bound and thus also the first expectation on the right-hand side of (5.3.23) vanishes as L tends to infinity.

Bound for the second expectation in (5.3.23)**:** When the random walk starts at a vertex $v \in A$, we cannot expect the probability of never hitting ρ^* to be small either. This is because in this case the random walk first has to enter the subtree T_1 in order to escape on the pruned tree $T\setminus T(v)$, which in turn implies that the walker has to cross an edge with conductance ε. We therefore bound the second expectation in (5.3.23) by the expected number of vertices $v \in G_L$ in A. A vertex v can be located in A in two different ways. Either its ancestor has degree 2 and is already in A, or its ancestor belongs to T_1

and the edge connecting them has conductance ε. This gives the following upper bound

$$\begin{aligned}
&\mathrm{E}\Bigg[\sum_{v\in G_L} \mathbb{1}_{\{P^v_{\omega\setminus\omega(v)}(\eta_*=\infty)<\delta\}}\mathbb{1}_{\{|T_1|<\infty\}}\mathbb{1}_{\{v\in A\}}\Bigg] \\
&\quad\le \mathrm{E}\Bigg[\sum_{v\in G_L} \mathbb{1}_{\{|T_1|<\infty\}}\big(\mathbb{1}_{\{v^*\in A,\,\deg(v^*)=2\}}+\mathbb{1}_{\{v^*\in T_1,\,\xi(v^*,v)=\varepsilon\}}\big)\Bigg] \\
&\quad= \mathrm{E}\Bigg[\sum_{v\in G_{L-1}} \mathbb{1}_{\{|T_1|<\infty\}}\sum_{z\in G_1(T(v))}\big(\mathbb{1}_{\{v\in A,\,\deg(v)=2\}}+\mathbb{1}_{\{v\in T_1,\,\xi(v,z)=\varepsilon\}}\big)\Bigg] \\
&\quad= \mathrm{E}\Bigg[\sum_{v\in G_{L-1}} \mathbb{1}_{\{|T_1|<\infty\}}\mathbb{1}_{\{v\in A,\,\deg(v)=2\}}\Bigg]+\mathrm{E}\Bigg[\sum_{v\in G_{L-1}} \mathbb{1}_{\{|T_1|<\infty\}}\sum_{z\in G_1(T(v))}\mathbb{1}_{\{v\in T_1,\,\xi(v,z)=\varepsilon\}}\Bigg].
\end{aligned} \tag{5.3.25}$$

The first expectation in (5.3.25) is equal to

$$\begin{aligned}
\mathrm{E}\Bigg[\sum_{v\in G_{L-1}} \mathbb{1}_{\{|T_1|<\infty\}}\mathbb{1}_{\{v\in A,\,\deg(v)=2\}}\Bigg] &= \mathrm{E}\Bigg[\sum_{v\in G_{L-1},\,v\in A}\mathrm{E}\big[\mathbb{1}_{\{|T_1|<\infty\}}\mathbb{1}_{\{\deg(v)=2\}}\,\big|\,\omega_{|L-1}\big]\Bigg] \\
&= \mathrm{E}\Bigg[\sum_{v\in G_{L-1},\,v\in A}\mathrm{P}(\deg(v)=2)\mathrm{E}\big[\mathbb{1}_{\{|T_1|<\infty\}}\,\big|\,\omega_{|L-1}\big]\Bigg] \\
&= \nu(\{1\})\mathrm{E}\Bigg[\sum_{v\in G_{L-1}}\mathbb{1}_{\{|T_1|<\infty\}}\mathbb{1}_{\{v\in A\}}\Bigg],
\end{aligned} \tag{5.3.26}$$

where $\omega_{|L-1}$ denotes the first $L-1$ generations of the environment. To see the second equality, note that the indicator functions $\mathbb{1}_{\{\deg(v)=2\}}$ and $\mathbb{1}_{\{|T_1|<\infty\}}$ are independent, because the first one is $\sigma(\omega(v))$-measurable, while the second one is $\sigma(\omega\setminus\omega(v))$-measurable, since $v\in A$ in particular implies $v\notin T_1$.

Concerning the second summand in (5.3.25), we note that the tree T_1 can only die out when the subtrees $T_1(y)$ become extinct for all y in the $(L-1)$-th generation of T_1. If this holds for all $y\in G_{L-1}\cap T_1$ except v, this provides the following upper bound

$$\begin{aligned}
&\mathrm{E}\Bigg[\sum_{v\in G_{L-1}} \mathbb{1}_{\{|T_1|<\infty\}}\sum_{z\in G_1(T(v))}\mathbb{1}_{\{v\in T_1,\,\xi(v,z)=\varepsilon\}}\Bigg] \\
&\quad\le \mathrm{E}\Bigg[\sum_{v\in G_{L-1},v\in T_1}\mathrm{E}\Bigg[\mathbb{1}_{\{|T_1(y)|<\infty\,\forall y\in G_{L-1}\cap T_1\setminus\{v\}\}}\sum_{z\in G_1(T(v))}\mathbb{1}_{\{\xi(v,z)=\varepsilon\}}\,\Bigg|\,\omega_{|L-1}\Bigg]\Bigg] \\
&\quad= \mathrm{E}\Bigg[\sum_{v\in G_{L-1},v\in T_1}\mathrm{E}\big[\mathbb{1}_{\{|T_1(y)|<\infty\,\forall y\in G_{L-1}\cap T_1\setminus\{v\}\}}\,\big|\,\omega_{|L-1}\big]\mathrm{E}\Bigg[\sum_{z\in G_1(T(v))}\mathbb{1}_{\{\xi(v,z)=\varepsilon\}}\Bigg]\Bigg].
\end{aligned}$$

Here, for the equality we used that the random variables $\mathbb{1}_{\{|T_1(y)|<\infty \text{ for all } y\in G_{L-1}\cap T_1\setminus\{v\}\}}$ and $\sum_{z\in G_1(T(v))}\mathbb{1}_{\{\xi(v,z)=\varepsilon\}}$ are independent. Let $m_{1|\varepsilon}=\mathrm{E}\big[\sum_{z\in G_1}\mathbb{1}_{\{\xi(\rho,z)=\varepsilon\}}\big]$ denote the expected number of descendants of the root which are connected via an edge with conductance ε. We calculate

$$\begin{aligned}
&\mathrm{E}\Bigg[\sum_{v\in G_{L-1}}\mathbb{1}_{\{|T_1|<\infty\}}\sum_{z\in G_1(T(v))}\mathbb{1}_{\{\xi(v,z)=\varepsilon\}}\Bigg]\\
&\quad\le m_{1|\varepsilon}\mathrm{E}\Bigg[\sum_{v\in G_{L-1},v\in T_1}\mathrm{E}\big[\mathbb{1}_{\{|T_1(y)|<\infty\,\forall y\in G_{L-1}\cap T_1\setminus\{v\}\}}\;\big|\;\omega_{|L-1}\big]\Bigg]\\
&\quad= m_{1|\varepsilon}\mathrm{E}\Bigg[\sum_{v\in G_{L-1},v\in T_1}\mathrm{E}\big[\mathbb{1}_{\{|T_1(y)|<\infty\,\forall y\in G_{L-1}\cap T_1\}}\;\big|\;\omega_{|L-1}\big]\mathrm{P}(|T_1(v)|<\infty)^{-1}\Bigg]\\
&\quad= m_{1|\varepsilon}\mathrm{E}\Bigg[\sum_{v\in G_{L-1}}\mathbb{1}_{\{v\in T_1\}}\mathbb{1}_{\{|T_1|<\infty\}}\Bigg]\mathrm{P}(|T_1|<\infty)^{-1}\\
&\quad= m_{1|\varepsilon}\mathrm{E}\Bigg[\sum_{v\in G_{L-1}}\mathbb{1}_{\{v\in T_1\}}\;\Bigg|\;|T_1|<\infty\Bigg]\\
&\quad= m_{1|\varepsilon}(m')^{L-1}.
\end{aligned}$$

To see the last equality, recall that conditioned on extinction, the subtree T_1 is a subcritical Galton-Watson tree and the mean size of the $(L-1)$-th generation is given by $(m')^{L-1}$, see (5.3.24). In view (5.3.25), combining this bound with (5.3.26) we arrive at

$$\mathrm{E}\Bigg[\sum_{v\in G_L}\mathbb{1}_{\{|T_1|<\infty\}}\mathbb{1}_{\{v\in A\}}\Bigg]\le\nu(\{1\})\mathrm{E}\Bigg[\sum_{v\in G_{L-1}}\mathbb{1}_{\{|T_1|<\infty\}}\mathbb{1}_{\{v\in A\}}\Bigg]+m_{1|\varepsilon}(m')^{L-1}.\qquad(5.3.27)$$

This means that we have a recursion of the form $x_{n+1}\le ax_n+cb^n$, $x_0=0$ for some a, b, $c\ge 0$. Iterating this leads to $x_{n+1}\le c\sum_{i=0}^n a^ib^{n-i}$. In our setting, iterating (5.3.27) yields

$$\mathrm{E}\Bigg[\sum_{v\in G_L}\mathbb{1}_{\{|T_1|<\infty\}}\mathbb{1}_{\{v\in A\}}\Bigg]\le m_{1|\varepsilon}\sum_{i=0}^{L-1}\nu(\{1\})^i(m')^{L-1-i}=m_{1|\varepsilon}(m')^{L-1}\sum_{i=0}^{L-1}\left(\frac{\nu(\{1\})}{m'}\right)^i.\qquad(5.3.28)$$

If $m'\neq\nu(\{1\})$, the geometric sum formula implies

$$m_{1|\varepsilon}(m')^{L-1}\sum_{i=0}^{L-1}\left(\frac{\nu(\{1\})}{m'}\right)^i=m_{1|\varepsilon}\frac{(m')^L-\nu(\{1\})^L}{m'-\nu(\{1\})}\xrightarrow[L\to\infty]{}0.$$

To see the convergence, observe that m' and $\nu(\{1\})$ are strictly less than one. Otherwise, if $m' = \nu(\{1\})$, we have

$$m_{1|\varepsilon}(m')^{L-1} \sum_{i=0}^{L-1} \left(\frac{\nu(\{1\})}{m'} \right)^i = L m_{1|\varepsilon}(m')^{L-1} \xrightarrow[L\to\infty]{} 0.$$

This shows that the second expectation on the right-hand side of (5.3.23) vanishes as $L \to \infty$.

Bound for the third expectation in (5.3.23)**:** Roughly speaking, when the random walk starts at a vertex $v \notin T_1 \cup A$, it has a good probability of never visiting ρ^*. This is because, on the one hand, starting at a vertex outside of T_1 implies that the walker has to cross an edge with conductance ε to reach ρ^* at some time. On the other hand, there exists a path in $T \backslash T(v)$ to escape, since the random walk starts at a vertex that is not located in the set A. For this reason, we derive a lower bound for the indirect escape probability to bound the third expectation in (5.3.23).

We let $\omega = (T, \rho, \xi)$ be an environment with $|G_L \cap T_1^{\mathrm{Bb}}| = 0$ and $v \in G_L$ a vertex in the L-th generation with $v \notin T_1$ and $v \notin A$. Recall that $(v_{k_\varepsilon - 1}, v_{k_\varepsilon})$ denotes the first edge with conductance ε on the path from ρ to v, Lemma 2.4.1 implies

$$P^v_{\omega\backslash\omega(v)}\big(\eta_* = \infty\big) \geq P^{v_{k_\varepsilon}}_{\omega\backslash\omega(v)}\big(\eta_{v_{k_\varepsilon-1}} = \infty\big) = \frac{\mathcal{C}_{\omega(v_{k_\varepsilon})\backslash\omega(v)}(v_{k_\varepsilon}, \infty)}{\varepsilon + \mathcal{C}_{\omega(v_{k_\varepsilon})\backslash\omega(v)}(v_{k_\varepsilon}, \infty)}.$$

Let us denote the environment where each conductance in ω is replaced by $a > 0$ by $\omega^a = (T, \rho, (a)_{e \in \mathcal{E}(T)})$. By Rayleigh's Monotonicity Principle (Lemma 2.4.5), reducing all edge weights to ε can only decrease the effective conductance. This implies

$$\mathcal{C}_{\omega(v_{k_\varepsilon})\backslash\omega(v)}(v_{k_\varepsilon}, \infty) \geq \mathcal{C}_{\omega^\varepsilon(v_{k_\varepsilon})\backslash\omega^\varepsilon(v)}(v_{k_\varepsilon}, \infty)$$

and, due to the monotonicity of the mapping $x \mapsto \frac{x}{x+\varepsilon}$ on $[0, \infty)$, we obtain

$$\begin{aligned} P^v_{\omega\backslash\omega(v)}\big(\eta_* = \infty\big) &\geq \frac{\mathcal{C}_{\omega^\varepsilon(v_{k_\varepsilon})\backslash\omega^\varepsilon(v)}(v_{k_\varepsilon}, \infty)}{\varepsilon + \mathcal{C}_{\omega^\varepsilon(v_{k_\varepsilon})\backslash\omega^\varepsilon(v)}(v_{k_\varepsilon}, \infty)} = P^{v_{k_\varepsilon}}_{\omega^\varepsilon\backslash\omega^\varepsilon(v)}\big(\eta_{v_{k_\varepsilon-1}} = \infty\big) \\ &= P^{v_{k_\varepsilon}}_{\omega^1\backslash\omega^1(v)}\big(\eta_{v_{k_\varepsilon-1}} = \infty\big) = \frac{\mathcal{C}_{\omega^1(v_{k_\varepsilon})\backslash\omega^1(v)}(v_{k_\varepsilon}, \infty)}{1 + \mathcal{C}_{\omega^1(v_{k_\varepsilon})\backslash\omega^1(v)}(v_{k_\varepsilon}, \infty)} \\ &= \frac{1}{1 + \mathcal{R}_{\omega^1(v_{k_\varepsilon})\backslash\omega^1(v)}(v_{k_\varepsilon}, \infty)}. \end{aligned}$$

To see the third step, observe that multiplying the conductance of each edge in the tree with the same positive value does not change the transition probabilities of the random walk. Since $v \notin A$, there is a vertex on the path from v_{k_ε} to v that has at least two descendants, i.e. there exists an index $k' \in \{k_\varepsilon, \ldots, L-1\}$ such that $\deg(v_{k'}) > 2$. In particular, this implies that there exists a vertex $u \neq v$ with $u \in G_L \cap T(v_{k_\varepsilon})$. Thinking of the tree as a subset of the Ulam-Harris tree, let u be the first vertex in $G_L \cap T(v_{k_\varepsilon})$ with $u \neq v$. We use Rayleigh's Monotonicity Principle (Lemma 2.4.5) and the Series Law (Lemma 2.4.4) to bound the effective resistance as follows

$$\mathcal{R}_{\omega^1(v_{k_\varepsilon})\setminus\omega^1(v)}(v_{k_\varepsilon},\infty) \leq \mathcal{R}_{\omega^1}(v_{k_\varepsilon},u) + \mathcal{R}_{\omega^1(u)}(u,\infty) \leq L + \mathcal{R}_{\omega^1(u)}(u,\infty).$$

This implies

$$P^v_{\omega\setminus\omega(v)}(\eta_* = \infty) \geq \frac{1}{1 + L + \mathcal{R}_{\omega^1(u)}(u,\infty)}.$$

We conclude that $P^v_{\omega\setminus\omega(v)}(\eta_* = \infty) < \delta$ can only hold if

$$\mathcal{R}_{\omega^1(u)}(u,\infty) > \frac{1-\delta(1+L)}{\delta}.$$

This leads to the following estimate for the third expectation in (5.3.23)

$$\begin{aligned} &\mathrm{E}\Bigg[\sum_{v\in G_L} \mathbb{1}_{\{P^v_{\omega\setminus\omega(v)}(\eta_*=\infty)<\delta\}}\mathbb{1}_{\{|T_1|<\infty\}}\mathbb{1}_{\{v\notin T_1\cup A\}}\Bigg] \\ &\quad\leq \mathrm{E}\Bigg[\sum_{v\in G_L} \mathbb{1}_{\{v\notin T_1\cup A\}}\mathrm{E}\big[\mathbb{1}_{\{P^v_{\omega\setminus\omega(v)}(\eta_*=\infty)<\delta\}} \,\big|\, \omega_{|L}\big]\Bigg] \\ &\quad\leq \mathrm{E}\Bigg[\sum_{v\in G_L} \mathbb{1}_{\{v\notin T_1\cup A\}}\mathrm{E}\Big[\mathbb{1}_{\left\{\mathcal{R}_{\omega^1(u)}(u,\infty)>\frac{1-\delta(1+L)}{\delta}\right\}}\Big]\Bigg] \\ &\quad= \mathrm{P}\bigg(\mathcal{R}_{\omega^1}(\rho,\infty) > \frac{1-\delta(1+L)}{\delta}\bigg)\mathrm{E}\Bigg[\sum_{v\in G_L}\mathbb{1}_{\{v\notin T_1\cup A\}}\Bigg]. \end{aligned} \tag{5.3.29}$$

For $\delta > 0$ small enough, the Markov inequality implies

$$\mathrm{P}\bigg(\mathcal{R}_{\omega^1}(\rho,\infty) > \frac{1-\delta(1+L)}{\delta}\bigg) \leq \frac{\delta}{1-\delta(1+L)}\mathrm{E}\big[\mathcal{R}_{\omega^1}(\rho,\infty)\big].$$

The first moment of the effective resistance is finite by Lemma 9.1 in [LPP95], since ω^1 is a supercritical Galton-Watson tree with unit conductance. The expectation in (5.3.29) can be bounded by the mean size of the L-th generation

$$\mathrm{E}\Bigg[\sum_{v\in G_L}\mathbb{1}_{\{v\notin T_1\cup A\}}\Bigg] \leq E[|G_L|] = m_1^L,$$

recall that m_1 is the offspring mean. In view of (5.3.29), we conclude from the last two estimates that

$$\mathrm{E}\Bigg[\sum_{v\in G_L} \mathbb{1}_{\{P^v_{\omega\setminus\omega(v)}(\eta_*=\infty)<\delta\}}\mathbb{1}_{\{|T_1|<\infty\}}\mathbb{1}_{\{v\notin T_1\cup A\}}\Bigg] \leq C_1\frac{\delta}{1-\delta(1+L)}m_1^L \tag{5.3.30}$$

for some constant $C_1 = C_1(\nu) < \infty$.

Finally, due to (5.3.24), (5.3.28) and (5.3.30), we obtain the following estimate for the first expectation in (5.3.22):

$$\mathrm{E}\Bigg[\sum_{v\in G_L} \mathbb{1}_{\{P^v_{\omega\backslash\omega(v)}(\eta_*=\infty)<\delta\}}\mathbb{1}_{\{|G_L\cap T_1^{\mathrm{Bb}}|=0\}}\Bigg] \leq h(L,\nu,\alpha) + C_1\frac{\delta}{1-\delta(1+L)}m_1^L \qquad (5.3.31)$$

for some function h that vanishes as L tends to infinity. This bound is independent of ε and becomes arbitrarily small when we first choose L large enough and then choose δ sufficiently small.

We remark that a (sub-)critical tree T_1 dies out with probability one, which implies $|G_L \cap T_1^{\mathrm{Bb}}| = \emptyset$ and therefore

$$\mathrm{E}\Bigg[\sum_{v\in G_L} \mathbb{1}_{\{P^v_{\omega\backslash\omega(v)}(\eta_*=\infty)<\delta\}}\mathbb{1}_{\{|G_L\cap T_1^{\mathrm{Bb}}|\geq 1\}}\Bigg] = 0.$$

Consequently, in the (sub-)critical case we are done at this point. For the rest of the proof we therefore assume the tree T_1 to be supercritical.

The case $|G_L \cap T_1^{\mathrm{Bb}}| = 1$: We proceed studying the second summand in (5.3.22). That is, we consider the expected number of vertices that have an indirect escape probability less than δ on the event that the L-th generation of the backbone tree T_1^{Bb} contains exactly one vertex. Whether the indirect escape probability $P^v_{\omega\backslash\omega(v)}(\eta_* = \infty)$ is small depends on where the random walk starts: at the vertex in the backbone tree or at some other vertex. We therefore distinguish between these cases,

$$\begin{aligned}\mathrm{E}\Bigg[\sum_{v\in G_L} \mathbb{1}_{\{P^v_{\omega\backslash\omega(v)}(\eta_*=\infty)<\delta\}}\mathbb{1}_{\{|G_L\cap T_1^{\mathrm{Bb}}|=1\}}\Bigg] &= \mathrm{E}\Bigg[\sum_{v\in G_L} \mathbb{1}_{\{P^v_{\omega\backslash\omega(v)}(\eta_*=\infty)<\delta\}}\mathbb{1}_{\{|G_L\cap T_1^{\mathrm{Bb}}|=1\}}\mathbb{1}_{\{v\in T_1^{\mathrm{Bb}}\}}\Bigg]\\ &\quad + \mathrm{E}\Bigg[\sum_{v\in G_L} \mathbb{1}_{\{P^v_{\omega\backslash\omega(v)}(\eta_*=\infty)<\delta\}}\mathbb{1}_{\{|G_L\cap T_1^{\mathrm{Bb}}|=1\}}\mathbb{1}_{\{v\notin T_1^{\mathrm{Bb}}\}}\Bigg].\end{aligned} \qquad (5.3.32)$$

As usual, we treat both summands separately.

Bound for the first expectation in (5.3.32)**:** When the random walk starts at the single vertex in the L-th generation of the backbone tree T_1^{Bb}, we cannot expect the probability of never hitting ρ^* to be small. This is because in this case the walker has to leave the backbone tree to escape on the pruned tree $T\backslash T(v)$, which implies in particular that the walker has to cross an edge with conductance ε in order to escape. We therefore bound the first expectation in (5.3.32) by the probability that the L-th generation of the backbone tree T_1^{Bb} contains only a single vertex

$$\begin{aligned}
&\mathrm{E}\Bigg[\sum_{v\in G_L}\mathbb{1}_{\{P^v_{\omega\setminus\omega(v)}(\eta_*=\infty)<\delta\}}\mathbb{1}_{\{|G_L\cap T_1^{\mathrm{Bb}}|=1\}}\mathbb{1}_{\{v\in T_1^{\mathrm{Bb}}\}}\Bigg]\\
&\quad\le \mathrm{E}\Bigg[\sum_{v\in G_L}\mathbb{1}_{\{v\in T_1^{\mathrm{Bb}}\}}\,\Bigg|\,|G_L\cap T_1^{\mathrm{Bb}}|=1\Bigg]\mathrm{P}\big(|G_L\cap T_1^{\mathrm{Bb}}|=1\big)\\
&\quad= \mathrm{P}\big(|G_L\cap T_1^{\mathrm{Bb}}|=1\big). \qquad (5.3.33)
\end{aligned}$$

When we condition on the survival of T_1, the backbone tree T_1^{Bb} is again a supercritical Galton-Watson tree. This implies

$$\lim_{L\to\infty}\mathrm{P}\big(|G_L\cap T_1^{\mathrm{Bb}}|=1\big)\le\lim_{L\to\infty}\mathrm{P}\big(|G_L\cap T_1^{\mathrm{Bb}}|=1\,\big|\,|T_1|=\infty\big)=0$$

and thus the first expectation in (5.3.32) vanishes as L tends to infinity.

Bound for the second expectation in (5.3.32)**:** When the random walk starts outside of the backbone tree T_1^{Bb}, it can escape via the backbone tree. This indicates that the random walk has a good probability of never hitting the vertex ρ^*. Just as for the third expectation in the previous case, we derive a lower bound for the indirect escape probability to bound the second expectation in (5.3.32).

We let $\omega=(T,\rho,\xi)$ be an environment with $|G_L\cap T_1^{\mathrm{Bb}}|=1$ and $v\in G_L$ a vertex in the L-th generation with $v\notin T_1^{\mathrm{Bb}}$. Then there exists exactly one vertex $u\in G_L\cap T_1$ such that the subtree $T_1(u)$ survives. Applying Lemma 2.4.1, we can express the indirect escape probability as a ratio of effective conductances

$$P^v_{\omega\setminus\omega(v)}\big(\eta_*=\infty\big)\ge\mathrm{P}^\rho_{\omega\setminus\omega(v)}\big(\eta_*=\infty\big)=\frac{\mathcal{C}_{\omega\setminus\omega(v)}(\rho,\infty)}{\xi(\rho,\rho^*)+\mathcal{C}_{\omega\setminus\omega(v)}(\rho,\infty)}\ge\frac{1}{1+\kappa\mathcal{R}_{\omega\setminus\omega(v)}(\rho,\infty)}.$$

Using Rayleigh's Monotonicity Principle (Lemma 2.4.5) and the Series Law (Lemma 2.4.4), we can bound the effective resistance as follows

$$\mathcal{R}_{\omega\setminus\omega(v)}(\rho,\infty)\le\mathcal{R}_\omega(\rho,u)+\mathcal{R}_{\omega(u)}(u,\infty)\le\kappa L+\mathcal{R}_{\omega_1^{\mathrm{Bb}}(u)}(u,\infty),$$

where $\omega_1^{\mathrm{Bb}}(u)$ is the environment formed by the backbone tree of $T_1(u)$ and the corresponding conductances. This implies

$$P^v_{\omega\setminus\omega(v)}\big(\eta_*=\infty\big)\ge\frac{1}{1+\kappa^2L+\kappa\mathcal{R}_{\omega_1^{\mathrm{Bb}}(u)}(u,\infty)} \qquad (5.3.34)$$

and therefore $\mathrm{P}^v_{\omega\setminus\omega(v)}\big(\eta_*=\infty\big)<\delta$ can only hold if

$$\mathcal{R}_{\omega_1^{\mathrm{Bb}}(u)}(u,\infty)>\frac{1-\delta(1+\kappa^2L)}{\delta\kappa}.$$

We calculate

$$
\begin{aligned}
&\mathrm{E}\Bigg[\sum_{v\in G_L} \mathbb{1}_{\{P^v_{\omega\setminus\omega(v)}(\eta_*=\infty)<\delta\}}\mathbb{1}_{\{|G_L\cap T_1^{\mathrm{Bb}}|=1\}}\mathbb{1}_{\{v\notin T_1^{\mathrm{Bb}}\}}\Bigg]\\
&= \mathrm{E}\Bigg[\sum_{v\in G_L}\sum_{u\in G_L\cap T_1} \mathrm{E}\big[\mathbb{1}_{\{P^v_{\omega\setminus\omega(v)}(\eta_*=\infty)<\delta\}}\mathbb{1}_{\{|G_L\cap T_1^{\mathrm{Bb}}|=1\}}\mathbb{1}_{\{v\notin T_1^{\mathrm{Bb}}\}}\mathbb{1}_{\{|T_1(u)|=\infty\}} \;\big|\; \omega_{|L}\big]\Bigg]\\
&\le \mathrm{E}\Bigg[\sum_{v\in G_L}\sum_{u\in G_L\cap T_1} \mathrm{E}\Bigg[\mathbb{1}_{\Big\{\mathcal{R}_{\omega_1^{\mathrm{Bb}}(u)}(u,\infty)>\frac{1-\delta(1+\kappa^2L)}{\delta\kappa},\,|T_1(u)|=\infty\Big\}}\mathbb{1}_{\{|T_1(z)|<\infty\,\forall z\in G_L\cap T_1\setminus\{u\},\,v\notin T_1^{\mathrm{Bb}}\}} \;\Bigg|\; \omega_{|L}\Bigg]\Bigg]\\
&= \mathrm{E}\Bigg[\sum_{v\in G_L}\sum_{u\in G_L\cap T_1} \mathrm{E}\big[\mathbb{1}_{\{|T_1(z)|<\infty\,\forall z\in G_L\cap T_1\setminus\{u\},\,v\notin T_1^{\mathrm{Bb}}\}} \;\big|\; \omega_{|L}\big]\mathrm{E}\Bigg[\mathbb{1}_{\Big\{\mathcal{R}_{\omega_1^{\mathrm{Bb}}(u)}(u,\infty)>\frac{1-\delta(1+\kappa^2L)}{\delta\kappa},\,|T_1(u)|=\infty\Big\}}\Bigg]\Bigg]\\
&= \mathrm{E}\Bigg[\sum_{v\in G_L}\sum_{u\in G_L\cap T_1} \mathrm{E}\big[\mathbb{1}_{\{|T_1(z)|<\infty\,\forall z\in G_L\cap T_1\setminus\{u\},\,v\notin T_1^{\mathrm{Bb}}\}} \;\big|\; \omega_{|L}\big]\mathrm{P}(|T_1(u)|=\infty)\Bigg]\\
&\qquad \times \mathrm{P}\Bigg(\mathcal{R}_{\omega_1^{\mathrm{Bb}}}(\rho,\infty) > \frac{1-\delta(1+\kappa^2L)}{\delta\kappa} \;\Bigg|\; |T_1|=\infty\Bigg),
\end{aligned}
$$

where we used the independence of the indicator functions for the second last equality. For $\delta>0$ sufficiently small, the Markov inequality implies

$$
\mathrm{P}\Bigg(\mathcal{R}_{\omega_1^{\mathrm{Bb}}}(\rho,\infty) > \frac{1-\delta(1+\kappa^2L)}{\delta\kappa} \;\Bigg|\; |T_1|=\infty\Bigg) \le \frac{\delta\kappa}{1-\delta(1+\kappa^2L)}\mathrm{E}\big[\mathcal{R}_{\omega_1^{\mathrm{Bb}}}(\rho,\infty) \;\big|\; |T_1|=\infty\big].
$$

Conditioned on the survival of the tree T_1, the backbone tree T_1^{Bb} is a supercritical Galton-Watson tree with uniformly elliptic conductances, thus by Lemma 3.3.1 the first moment of the effective resistance is finite. Using the independence of the indicator functions again yields

$$
\begin{aligned}
&\mathrm{E}\Bigg[\sum_{v\in G_L} \mathbb{1}_{\{P^v_{\omega\setminus\omega(v)}(\eta_*=\infty)<\delta\}}\mathbb{1}_{\{|G_L\cap T_1^{\mathrm{Bb}}|=1\}}\mathbb{1}_{\{v\notin T_1^{\mathrm{Bb}}\}}\Bigg]\\
&\quad\le C_2\frac{\delta\kappa}{1-\delta(1+\kappa^2L)}\mathrm{E}\Bigg[\sum_{v\in G_L}\sum_{u\in G_L\cap T_1}\mathrm{E}\big[\mathbb{1}_{\{|T_1(u)|=\infty,\,|T_1(z)|<\infty\,\forall z\in G_L\cap T_1\setminus\{u\},\,v\notin T_1^{\mathrm{Bb}}\}} \;\big|\; \omega_{|L}\big]\Bigg]\\
&\quad= C_2\frac{\delta\kappa}{1-\delta(1+\kappa^2L)}\mathrm{E}\Bigg[\sum_{v\in G_L}\mathbb{1}_{\{|G_L\cap T_1^{\mathrm{Bb}}|=1,\,v\notin T_1^{\mathrm{Bb}}\}}\Bigg]\\
&\quad\le C_2\frac{\delta\kappa}{1-\delta(1+\kappa^2L)}m_1^L \qquad (5.3.35)
\end{aligned}
$$

for some constant $C_2=C_2(\nu,\alpha,\kappa)<\infty$.

In view of (5.3.32), due to the bounds in (5.3.33) and (5.3.35), we conclude that

$$\mathrm{E}\bigg[\sum_{v\in G_L}\mathbb{1}_{\{P^v_{\omega\backslash\omega(v)}(\eta_*=\infty)<\delta\}}\mathbb{1}_{\{|G_L\cap T_1^{\mathrm{Bb}}|=1\}}\bigg]\leq \mathrm{P}\big(|G_L\cap T_1^{\mathrm{Bb}}|=1\big)+C_2\frac{\delta\kappa}{1-\delta(1+\kappa^2L)}m_1^L. \tag{5.3.36}$$

This bound is independent of ε and becomes arbitrarily small when we first choose L large and then δ small enough.

The case $|G_L\cap T_1^{\mathrm{Bb}}|>1$: Lastly, we consider the expected number of vertices that have an indirect escape probability less than δ on the event that the L-th generation of the backbone tree T_1^{Bb} consists of at least two vertices. This means that there is always a path in the backbone tree along which the random walk can escape no matter at which vertex in the L-th generation it starts. More precisely, for every vertex $v\in G_L$ there exists a vertex $u\neq v$ with $u\in G_L\cap T_1^{\mathrm{Bb}}$ and therefore we have the same lower bound for the indirect escape probability as in (5.3.34). This implies that $\mathrm{P}^v_{\omega\backslash\omega(v)}(\eta_*=\infty)<\delta$ can only hold if

$$\mathcal{R}_{\omega_1^{\mathrm{Bb}}(u)}(u,\infty)>\frac{1-\delta(1+\kappa^2L)}{\delta\kappa}.$$

Let us denote the vertices in the L-th generation of T by $z_1,z_2,\dots$ (ordered as in the Ulam-Harris tree). Similar arguments as in the previous case lead to

$$\begin{aligned}
&\mathrm{E}\bigg[\sum_{v\in G_L}\mathbb{1}_{\{P^v_{\omega\backslash\omega(v)}(\eta_*=\infty)<\delta\}}\mathbb{1}_{\{|G_L\cap T_1^{\mathrm{Bb}}|>1\}}\bigg]\\
&=\mathrm{E}\bigg[\sum_{v\in G_L}\mathbb{1}_{\{P^v_{\omega\backslash\omega(v)}(\eta_*=\infty)<\delta\}}\mathbb{1}_{\{|G_L\cap T_1^{\mathrm{Bb}}|>1\}}\sum_{i=1}^{|G_L|}\mathbb{1}_{\{z_i\in T_1,\,z_i\neq v,\,|T_1(z_i)|=\infty,\,|T_1(z_j)|<\infty\,\forall j<i\}}\bigg]\\
&\leq\mathrm{E}\bigg[\sum_{v\in G_L}\sum_{i=1}^{|G_L|}\mathbb{1}_{\{z_i\in T_1,\,z_i\neq v\}}\mathrm{E}\Big[\mathbb{1}_{\big\{\mathcal{R}_{\omega_1^{\mathrm{Bb}}(z_i)}(z_i,\infty)>\frac{1-\delta(1+\kappa^2L)}{\delta\kappa}\delta,\,|T_1(z_i)|=\infty\big\}}\mathbb{1}_{\{|T_1(z_j)|<\infty\,\forall j<i\}}\,\Big|\,\omega_{|L}\Big]\bigg]\\
&=\mathrm{E}\bigg[\sum_{v\in G_L}\sum_{i=1}^{|G_L|}\mathbb{1}_{\{z_i\in T_1,\,z_i\neq v\}}\mathrm{E}\big[\mathbb{1}_{\{|T_1(z_j)|<\infty\,\forall j<i\}}\,\big|\,\omega_{|L}\big]\mathrm{E}\Big[\mathbb{1}_{\big\{\mathcal{R}_{\omega_1^{\mathrm{Bb}}(z_i)}(z_i,\infty)>\frac{1-\delta(1+\kappa^2L)}{\delta\kappa},\,|T_1(z_i)|=\infty\big\}}\Big]\bigg]\\
&=\mathrm{E}\bigg[\sum_{v\in G_L}\sum_{i=1}^{|G_L|}\mathbb{1}_{\{z_i\in T_1\cap G_L,\,z_i\neq v\}}\mathrm{E}\big[\mathbb{1}_{\{|T_1(z_j)|<\infty\,\forall j<i\}}\,\big|\,\omega_{|L}\big]\mathrm{P}(|T_1(z_i)|=\infty)\bigg]\\
&\quad\times\mathrm{P}\bigg(\mathcal{R}_{\omega_1^{\mathrm{Bb}}}(\rho,\infty)>\frac{1-\delta(1+\kappa^2L)}{\delta\kappa}\,\bigg|\,|T_1|=\infty\bigg).
\end{aligned}$$

By an analogous application of the Markov inequality and Lemma 3.3.1 as before, we conclude that

$$\begin{aligned}
&\mathrm{E}\Bigg[\sum_{v\in G_L}\mathbb{1}_{\{P^v_{\omega\setminus\omega(v)}(\eta_*=\infty)<\delta\}}\mathbb{1}_{\{|G_L\cap T_1^{\mathrm{Bb}}|>1\}}\Bigg]\\
&\quad\le C_2\frac{\delta\kappa}{1-\delta(1+\kappa^2L)}\mathrm{E}\Bigg[\sum_{v\in G_L}\sum_{i=1}^{|G_L|}\mathbb{1}_{\{z_i\in T_1\cap G_L,\, z_i\neq v,\, |T_1(z_i)|=\infty,\, |T_1(z_j)|<\infty\,\forall j<i\}}\Bigg]\\
&\quad\le C_2\frac{\delta\kappa}{1-\delta(1+\kappa^2L)}m_1^L \qquad (5.3.37)
\end{aligned}$$

for some constant $C_2 = C_2(\nu,\alpha,\kappa) < \infty$ and $\delta > 0$ small enough. For L fixed this bound becomes arbitrarily small when we choose δ sufficiently small.

In total, adding up (5.3.31), (5.3.36) and (5.3.37) yields

$$\begin{aligned}
\mathrm{E}\Bigg[\sum_{v\in G_L}\mathbb{1}_{\{P^v_{\omega\setminus\omega(v)}(\eta_*=\infty)<\delta\}}\Bigg] &\le h(L,\nu,\alpha)+C_1\frac{\delta}{1-\delta(1+L)}m_1^L\\
&\quad+\mathrm{P}\big(|G_L\cap T_1^{\mathrm{Bb}}|=1\big)+C_2\frac{\delta\kappa}{1-\delta(1+\kappa^2L)}m_1^L\\
&\quad+C_2\frac{\delta\kappa}{1-\delta(1+\kappa^2L)}m_1^L\\
&\le \gamma(L,\nu,\alpha)+C\frac{\delta\kappa}{1-\delta(1+\kappa^2L)}m_1^L
\end{aligned}$$

for some constant $C = C(\nu,\alpha,\kappa) < \infty$ and some function γ that vanishes as $L\to\infty$. We note that this bound is independent of ε and becomes arbitrarily small when we first choose L large enough and afterwards δ sufficiently small. □

5.3.6 Moment bounds on regeneration times: proof of Lemma 5.2.8

By Lemma 5.2.2 and Proposition 5.2.5 we obtain

$$\mathbb{E}\big[(\tau_2-\tau_1)^q\big]=\mathbb{E}\big[\tau_1^q \mid \eta_*=\infty,\xi(\rho,\rho^*)=1\big]\le\mathbb{E}[\tau_1^q]\mathbb{P}\big(\eta_*=\infty,\xi(\rho,\rho^*)=1\big)^{-1}\le C_\varepsilon\mathbb{E}[\tau_1^q].$$

This shows that it suffices to bound $\mathbb{E}[\tau_1^q]$. It holds

$$\mathbb{E}[\tau_1^q]=\sum_{n=1}^{\infty}(n^q-(n-1)^q)\mathbb{P}(\tau_1\ge n).$$

This can be easily verified by the following short calculation:

$$\begin{aligned}\sum_{n=1}^{\infty}(n^q-(n-1)^q)\mathbb{P}(\tau_1\geq n)&=\sum_{n=1}^{\infty}(n^q-(n-1)^q)\sum_{m=n}^{\infty}\mathbb{P}(\tau_1=m)\\&=\sum_{m=1}^{\infty}\mathbb{P}(\tau_1=m)\sum_{n=1}^{m}(n^q-(n-1)^q)\\&=\sum_{m=1}^{\infty}m^q\mathbb{P}(\tau_1=m)=\mathbb{E}[\tau_1^q].\end{aligned}$$

Consequently, we obtain the boundedness of the q-th moment of the first regeneration time once we have shown that $P(\tau_1 \geq n) \leq C_\varepsilon n^{-r}$ holds for r sufficiently large. The first regeneration time is large either when the first regeneration occurs in a generation far away from the root, or when the random walk stays in the first generations for a long time,

$$\begin{aligned}\mathbb{P}(\tau_1\geq n^3)&=\mathbb{P}(\tau_1\geq n^3,\tau_1\geq\eta_n)+\mathbb{P}(\tau_1\geq n^3,\tau_1<\eta_n)\\&\leq\mathbb{P}(\tau_1\geq\eta_n)+\mathbb{P}(\eta_n\geq n^3).\end{aligned}\tag{5.3.38}$$

Applying Markov's inequality and Lemma 5.2.7 leads to the following estimate for the first summand:

$$\mathbb{P}(\tau_1\geq\eta_n)=\mathbb{P}(|X_{\tau_1}|\geq n)\leq\frac{\mathbb{E}[|X_{\tau_1}|^r]}{n^r}\leq C_\varepsilon n^{-r}\tag{5.3.39}$$

for any arbitrary $r > 0$ and some constant $C_\varepsilon = C_\varepsilon(r) < \infty$.

To derive a polynomially small bound for the second summand in (5.3.38) we follow the arguments of [Aïd10]. The hitting time of the n-th generation is large either when the random walk visits a great number of distinct vertices or when the walker returns many times to a single vertex. To formalize this, let ϑ_k be the k-th distinct vertex visited by the random walk. Furthermore, we introduce the local time of a vertex z

$$L(z)=\sum_{k=0}^{\infty}\mathbb{1}_{\{X_k=z\}}.$$

That is, $L(z)$ counts the number of visits in z. We now estimate the probability that the walker requires more than n^3 steps to reach the n-th generation as follows

$$\mathbb{P}(\eta_n>n^3)\leq\mathbb{P}\big(|\{X_0,\dots,X_{\eta_n}\}|>n^2\big)+\mathbb{P}\big(\exists k\leq n^2:L(\vartheta_k)>n\big).\tag{5.3.40}$$

We determine upper bounds for both probabilities separately. Let us start with the first one.

Bound for $\mathbb{P}\big(|\{X_0,\dots,X_{\eta_n}\}| > n^2\big)$

The random walk can only visit more than n^2 distinct vertices before reaching the n-th generation if the walker sees more than n distinct vertices in at least one generation. This in turn implies that the random walk has to return to the ancestor of each of these vertices. Using a uniform bound for the probability of hitting the ancestor of the root as in Lemma 5.2.2, we can show that the probability of this event decays exponentially in n. To specify this, we set $t_1^k = \eta_k$ and recursively for $i > 1$

$$t_i^k = \inf\{m > t_{i-1}^k : |X_m| = k,\, X_m \neq X_l \text{ for all } l < m\},$$

so that t_i^k is the hitting time of the i-th distinct vertex in generation k. We then have

$$\begin{aligned}\mathbb{P}(|\{X_0,\dots,X_{\eta_n}\}| > n^2) &\le \mathbb{P}(\exists k \le n : |\{X_0,\dots,X_{\eta_n}\} \cap G_k| \ge n)\\ &\le \sum_{k=1}^{n} \mathbb{P}(|\{X_0,\dots,X_{\eta_n}\} \cap G_k| \ge n)\\ &\le \sum_{k=1}^{n} \mathbb{P}(t_n^k < \infty). \end{aligned} \tag{5.3.41}$$

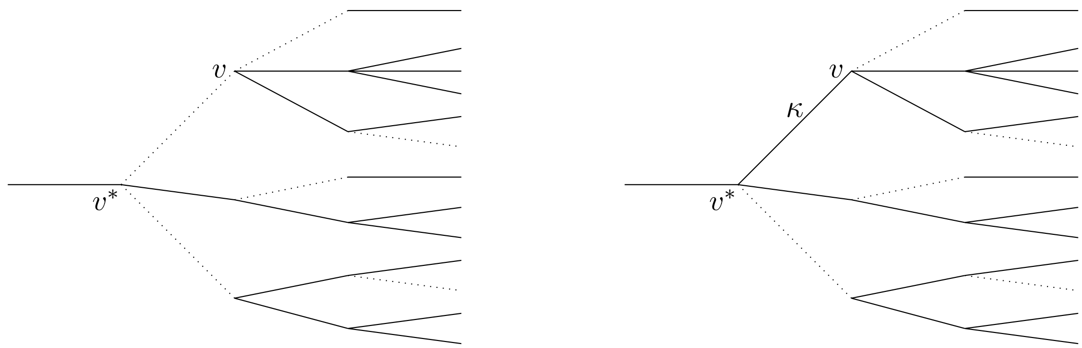
For t_n^k to be finite, t_{n-1}^k has to be finite as well and the random walk has to return to the ancestor of the vertex $X_{t_{n-1}^k}$. Due to the Markov property, this implies

$$\begin{aligned}\mathbb{P}(t_n^k < \infty) &= \sum_{v\in\mathbb{T}_k} \mathrm{E}\big[P_\omega(t_n^k < \infty, X_{t_{n-1}^k} = v)\mathbb{1}_{\{v\in T\}}\big]\\ &\le \sum_{v\in\mathbb{T}_k} \mathrm{E}\big[P_\omega(t_{n-1}^k < \infty, X_{t_{n-1}^k} = v, \eta_{v^*}\circ\theta_{t_{n-1}^k} < \infty)\mathbb{1}_{\{v\in T\}}\big]\\ &= \sum_{v\in\mathbb{T}_k} \mathrm{E}\big[P_\omega^v(\eta_{v^*} < \infty)P_\omega\big(t_{n-1}^k < \infty, X_{t_{n-1}^k} = v\big)\mathbb{1}_{\{v\in T\}}\big]. \end{aligned} \tag{5.3.42}$$

Here, $\mathbb{T}_k$ denotes the k-th generation of the Ulam-Harris tree and $\eta_{v^*}\circ\theta_{t_{n-1}^k}$ is the time at which the shifted path first hits v^*, recall definition (5.2.2). We note that the quenched probabilities in (5.3.42) are not independent, since the edge (v^*, v) influences both. Therefore, the idea is to fix its weight. If the conductance of the edge (v^*, v) is raised to κ, the probability of hitting the ancestor v^* when starting at v can only increase. To formalize this, we denote the modification of a given environment $\omega = (T, \rho, \xi)$ by $\tilde{\omega}_v = (T, \rho, \tilde{\xi})$ with

$$\tilde{\xi}(z^*, z) = \begin{cases} \xi(z^*, z), & z \neq v\\ \kappa, & z = v \end{cases}$$

see Figure 5.6 for an example. Lemma 2.4.1 and the monotonicity of the mapping $x \mapsto \frac{x}{x+b}$



(a) Environment ω (b) Modified environment $\tilde{\omega}_v$

Figure 5.6: In the modified environment $\tilde{\omega}_v$ the conductance of the edge (v^*, v) is replaced by κ.

on $[0, \infty)$ then imply

$$P_\omega^v(\eta_{v^*} < \infty) = \frac{\xi(v^*, v)}{\xi(v^*, v) + \mathcal{C}_{\omega(v)}(v, \infty)} \leq \frac{\kappa}{\kappa + \mathcal{C}_{\omega(v)}(v, \infty)} = P_{\tilde{\omega}_v}^v(\eta_{v^*} < \infty).$$

Observe that this upper bound is independent of $P_\omega(t_{n-1}^k < \infty, X_{t_{n-1}^k} = v)$. Together with (5.3.42), we obtain

$$\begin{aligned}
\mathbb{P}(t_n^k < \infty) &\leq \sum_{v \in \mathbb{T}_k} \mathrm{E}\big[P_{\tilde{\omega}_v}^v(\eta_{v^*} < \infty) P_\omega(t_{n-1}^k < \infty, X_{t_{n-1}^k} = v) \mid v \in T\big] \mathrm{P}(v \in T) \\
&= \sum_{v \in \mathbb{T}_k} \mathrm{E}\big[P_{\tilde{\omega}_v}^v(\eta_{v^*} < \infty) \mid v \in T\big] \mathrm{E}\big[P_\omega(t_{n-1}^k < \infty, X_{t_{n-1}^k} = v) \mid v \in T\big] \mathrm{P}(v \in T) \\
&= \mathrm{E}\big[P_{\tilde{\omega}_\rho}^\rho(\eta_* < \infty)\big] \sum_{v \in \mathbb{T}_k} \mathrm{E}\big[P_\omega\big(t_{n-1}^k < \infty, X_{t_{n-1}^k} = v\big) \mathbb{1}_{\{v \in T\}}\big] \\
&= \mathrm{E}\big[P_\omega^\rho(\eta_* < \infty \mid \xi(\rho^*, \rho) = \kappa)\big] \mathbb{P}(t_{n-1}^k < \infty) \\
&= \mathbb{P}(\eta_* < \infty \mid \xi(\rho^*, \rho) = \kappa) \mathbb{P}(t_{n-1}^k < \infty).
\end{aligned}$$

Iterating this yields

$$\mathbb{P}(t_n^k < \infty) \leq \mathbb{P}(\eta_* < \infty \mid \xi(\rho^*, \rho) = \kappa)^{n-1}.$$

Analogously to Lemma 5.2.2 we can show

$$\mathbb{P}(\eta_* = \infty \mid \xi(\rho^*, \rho) = \kappa) \geq c_\varepsilon$$

for some constant $c_\varepsilon = c_\varepsilon(\nu, \alpha, \kappa) > 0$. Finally, due to (5.3.41), we obtain the following bound for the first probability in (5.3.40)

$$\mathbb{P}\big(|\{X_0, \ldots, X_{\eta_n}\}| > n^2\big) \leq n(1 - c_\varepsilon)^{n-1}, \tag{5.3.43}$$

which decays exponentially in n.

Bound for $\mathbb{P}\big(\exists k \leq n^2 : L(\vartheta_k) > n\big)$

Next, we study the probability that the random walk returns to one of the vertices $\vartheta_1, \dots, \vartheta_{n^2}$ at least n times. Lemma 5.3.4 below gives an estimate for the probability of returning to the root many times. With this result we can derive a bound for the probability of visiting the vertex ϑ_k at least n times, which decays polynomially in n. We have

$$\mathbb{P}\big(\exists k \leq n^2 : L(\vartheta_k) > n\big) \leq \sum_{k=1}^{n^2} \mathbb{P}(L(\vartheta_k) > n). \tag{5.3.44}$$

We set $\eta_v^{(1)} = \eta_v$ and recursively for $k > 1$

$$\eta_v^{(k)} = \inf\{m > \eta_v^{(k-1)} : X_m = v\}$$

so that $\eta_v^{(k)}$ indicates when the random walk hits the vertex v the k-th time. Now we calculate

$$\begin{aligned}\mathbb{P}(L(\vartheta_k) > n) &= \sum_{v\in\mathbb{T}} \mathrm{E}\big[P_\omega(\eta_v^{(n+1)} < \infty, \vartheta_k = v)\mathbb{1}_{\{v\in T\}}\big] \\ &= \sum_{v\in\mathbb{T}} \mathrm{E}\big[P_\omega(\eta_v < \infty, \vartheta_k = v)P_\omega^v(\eta_v^+ < \infty)^n \mathbb{1}_{\{v\in T\}}\big] \\ &\leq \sum_{v\in\mathbb{T}} \mathrm{E}\big[P_\omega(\eta_v < \infty, \vartheta_k = v)\big(1 - P_\omega^v(\eta_v^+ = \infty, \eta_{v^*} = \infty)\big)^n \mathbb{1}_{\{v\in T\}}\big],\end{aligned} \tag{5.3.45}$$

where $\eta_v^+ = \inf\{m \geq 1 : X_m = v\}$ denotes the first hitting time of the vertex v after zero. Note again that both quenched probabilities in (5.3.45) are influenced by the edge (v^*, v), so they are not independent. We recall that $\tilde{\omega}_v$ is the environment where the conductance of the edge (v^*, v) is replaced by κ. The following calculation shows that this modification of the environment reduces the probability of never returning to v and never visiting the ancestor v^*:

$$\begin{aligned}P_\omega^v(\eta_v^+ = \infty, \eta_{v^*} = \infty) &= \sum_{z\sim v, z\neq v^*} P_\omega^v(\eta_v^+ = \infty, X_1 = z) \\ &= \sum_{z\sim v, z\neq v^*} P_\omega^v(X_1 = z)P_\omega^z(\eta_v = \infty) \\ &= \sum_{z\sim v, z\neq v^*} \frac{\xi(v,z)}{\xi(v^*,v) + \sum_{x\sim v, x\neq v^*}\xi(v,x)} P_{\tilde{\omega}_v}^z(\eta_v = \infty) \\ &\geq \sum_{z\sim v, z\neq v^*} \frac{\xi(v,z)}{\kappa + \sum_{x\sim v, x\neq v^*}\xi(v,x)} P_{\tilde{\omega}_v}^z(\eta_v = \infty) \\ &= \sum_{z\sim v, z\neq v^*} P_{\tilde{\omega}_v}^v(X_1 = z)P_{\tilde{\omega}_v}^z(\eta_v = \infty) \\ &= P_{\tilde{\omega}_v}^v(\eta_v^+ = \infty, \eta_{v^*} = \infty).\end{aligned}$$

In particular, this lower bound is independent of $P_\omega(\eta_v < \infty, \vartheta_k = v)$. Together with (5.3.45), this implies

$$\begin{aligned}
&\mathbb{P}(L(\vartheta_k) > n) \\
&\le \sum_{v\in\mathbb{T}} \mathrm{E}\big[P_\omega(\eta_v < \infty, \vartheta_k = v)\big(1 - P^v_{\tilde\omega_v}(\eta_v^+ = \infty, \eta_{v^*} = \infty)\big)^n \bigm| v \in T\big]\mathrm{P}(v \in T) \\
&= \sum_{v\in\mathbb{T}} \mathrm{E}\big[P_\omega(\eta_v < \infty, \vartheta_k = v) \bigm| v \in T\big] E\big[\big(1 - P^v_{\tilde\omega_v}(\eta_v^+ = \infty, \eta_{v^*} = \infty)\big)^n \bigm| v \in T\big]\mathrm{P}(v \in T) \\
&= \mathrm{E}\big[\big(1 - P^\rho_{\tilde\omega_\rho}(\eta_\rho^+ = \infty, \eta_* = \infty)\big)^n\big] \sum_{v\in\mathbb{T}} \mathrm{E}\big[P_\omega(\eta_v < \infty, \vartheta_k = v)\mathbb{1}_{\{v\in T\}}\big] \\
&= \mathrm{E}\big[P^\rho_{\tilde\omega_\rho}(\eta_\rho^+ < \infty)^n\big] \\
&= \mathbb{P}(L(\rho) > n \mid \xi(\rho^*, \rho) = \kappa).
\end{aligned}$$

By Lemma 5.3.4 below this bound decays polynomially in n. In view of (5.3.44), this implies that for any $\beta > 0$ there exists a constant $C_\varepsilon = C_\varepsilon(\beta)$ such that

$$\mathbb{P}\big(\exists k \le n^2 : L(\vartheta_k) > n\big) \le C_\varepsilon n^{2-\beta}.$$

Combining this estimate with the exponentially small bound in (5.3.43) gives the polynomial decay of the probability in (5.3.40),

$$\mathbb{P}(\eta_n > n^3) \le C'_\varepsilon n^{2-\beta}.$$

In total, due to (5.3.38) and (5.3.39), we conclude that

$$\mathbb{P}(\tau_1 \ge n^3) \le C_\varepsilon n^{-r} + C'_\varepsilon n^{2-\beta}, \tag{5.3.46}$$

where r, $\beta > 0$ can be chosen arbitrarily. This shows that the q-th moment of the first regeneration time $\mathbb{E}[\tau_1^q]$ remains bounded, which completes the proof. □

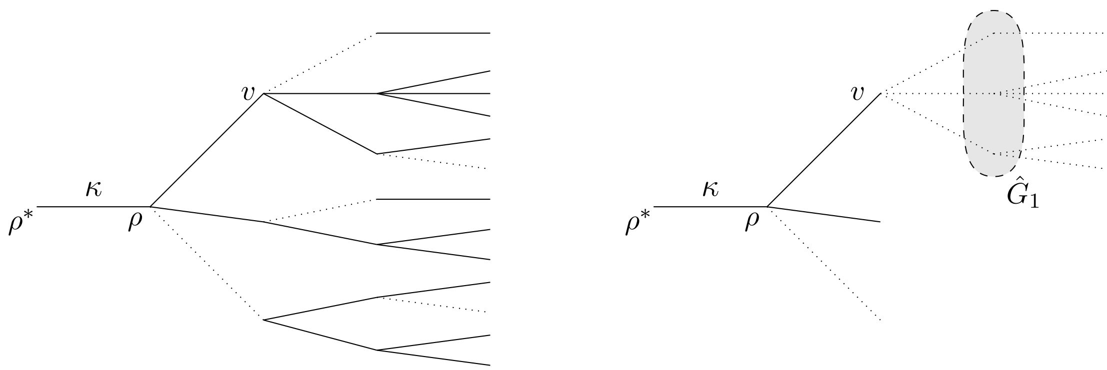
For the proof of Lemma 5.2.8, we need a bound for the probability that the random walk returns at least n times to the root. The required estimate is given in the following lemma.

Lemma 5.3.4 (Bound on local time)**.** *For any $\beta > 0$ there exists some constant $C_\varepsilon > 0$ such that for all $n \ge 1$*

$$\mathbb{P}(L(\rho) > n \mid \xi(\rho, \rho^*) = \kappa) \le C_\varepsilon n^{-\beta}.$$

Proof. Recall that $L(\rho)$ counts the number of visits in the root, $\eta_\rho^+ = \inf\{m \ge 1 : X_m = \rho\}$ is the first hitting time of the root after zero and $\tilde\omega_\rho$ denotes the environment where the conductance of the edge (ρ^*, ρ) is replaced by κ. We have

$$\mathbb{P}(L(\rho) > n \mid \xi(\rho^*, \rho) = \kappa) = \mathrm{E}\big[P_{\tilde\omega_\rho}(L(\rho) > n)\big] = \mathrm{E}\big[P_{\tilde\omega_\rho}(\eta_\rho^+ < \infty)^n\big].$$



(a) Environment $\tilde{\omega}_\rho$ (b) Modified environment $\hat{\omega}$

Figure 5.7: v is the first descendant of ρ. In the modified environment $\hat{\omega}$ the conductance of each edge in $T(v)$ is replaced by ε and all edges in $T(z)$ for $z \in G_1(T) \setminus \{v\}$ are removed. The first generation $\hat{G}_1$ of the subtree $T(v)$ is marked gray. Edges with conductance ε are indicated by dotted lines; edges with conductance larger than ε are indicated by solid lines.

We start with determining a bound for the quenched return probability $P_{\tilde{\omega}_\rho}(\eta_\rho^+ < \infty)$ by modifying the environment $\tilde{\omega}_\rho$ once again. We denote the first descendant of the root by v and we reduce the conductance of each edge in the subtree $T(v)$ to ε. For convenience, we write $\hat{G}_1$ for the first generation of this subtree. Furthermore, we remove all edges in the other subtrees $T(z)$ rooted at a descendant z of ρ with $z \neq v$. The new environment is called $\hat{\omega} = (T, \rho, \hat{\xi})$ with

$$\hat{\xi}(z^*, z) = \begin{cases} \kappa, & z = \rho \\ \xi(z^*, z), & |z| = 1 \\ \varepsilon, & z,\, z^* \in T(v) \\ 0, & \text{otherwise} \end{cases}$$

see Figure 5.7 for an example. The following calculation shows that a random walk in this new environment has a higher probability of returning to the root:

$$\begin{aligned} P_{\tilde{\omega}_\rho}(\eta_\rho^+ < \infty) &= P_{\tilde{\omega}_\rho}(X_1 = v) P^v_{\tilde{\omega}_\rho}(\eta_\rho < \infty) + \sum_{z \sim \rho,\, z \neq v} P_{\tilde{\omega}_\rho}(X_1 = z) P^z_{\tilde{\omega}_\rho}(\eta_\rho < \infty) \\ &\leq P_{\hat{\omega}}(X_1 = v) \frac{\xi(\rho, v)}{\xi(\rho, v) + \mathcal{C}_{\tilde{\omega}_\rho(v)}(v, \infty)} + \sum_{z \sim \rho,\, z \neq v} P_{\hat{\omega}}(X_1 = z) \\ &\leq P_{\hat{\omega}}(X_1 = v) \frac{\xi(\rho, v)}{\xi(\rho, v) + \mathcal{C}_{\hat{\omega}(v)}(v, \infty)} + \sum_{z \sim \rho,\, z \neq v} P_{\hat{\omega}}(X_1 = z) P^z_{\hat{\omega}}(\eta_\rho < \infty) \\ &= P_{\hat{\omega}}(\eta_\rho^+ < \infty). \end{aligned}$$

To see the third step, observe that we have $P^z_{\hat{\omega}}(\eta_\rho < \infty) = 1$ for all $z \sim \rho$, $z \neq v$ and $\mathcal{C}_{\hat{\omega}(v)}(v, \infty) \leq \mathcal{C}_{\tilde{\omega}_\rho(v)}(v, \infty)$ by Rayleigh's Monotonicity Principle (Lemma 2.4.5). This implies

$$\mathrm{E}\big[P^{\rho}_{\tilde{\omega}_\rho}(\eta^+_\rho < \infty)^n\big] \le \mathrm{E}\big[P^{\rho}_{\hat{\omega}}(\eta^+_\rho < \infty)^n\big] = \mathrm{E}\big[P_{\hat{\omega}}(L(\rho) > n)\big]. \tag{5.3.47}$$

If the random walk enters the subtree $T(v)$ sufficiently often, it is unlikely that walker will return to the root many times. We introduce

$$\hat{H}_n = \sum_{k=2}^{\eta_\rho^{(n+1)}} \mathbb{1}_{\{X_{k-2}=\rho, X_{k-1}=v, X_k \in \hat{G}_1\}}$$

so that $\hat{H}_n$ indicates how often the random walk moves from the root to a vertex in $\hat{G}_1$ before it returns to the root for the n-th time. Now we write

$$\mathrm{E}\big[P_{\hat{\omega}}(L(\rho) > n)\big] = \mathrm{E}\big[P_{\hat{\omega}}(L(\rho) > n, \hat{H}_n \ge n\delta_n)\big] + \mathrm{E}\big[P_{\hat{\omega}}(L(\rho) > n, \hat{H}_n < n\delta_n)\big], \tag{5.3.48}$$

where we choose $\delta_n = c_\varepsilon n^{-\frac{1}{3}}$ with $c_\varepsilon = \frac{\varepsilon^2}{\kappa^2+\varepsilon\kappa}$. We derive bounds for both summands separately.

Bound for the first expectation in (5.3.48)**:** The main idea is to compare the return probabilities of the random walk on $\hat{\omega}$ with those of a simple random walk on a Galton-Watson tree. In the environment $\hat{\omega}$, each edge in the subtree $T(v)$ has the same conductance. For this reason, if the random walk is only observed at the vertices of $T(v)$, it has the same distribution as a simple random walk on $T(v)$. On the event $\{L(\rho) > n, \hat{H}_n > n\delta_n\}$, the random walk on T returns at least $n\delta_n$ times from $\hat{G}_1$ to ρ. This in turn means that the walk restricted to $T(v)$ returns from $\hat{G}_1$ to v at least $n\delta_n$ times. Therefore, it suffices to investigate the probability that a simple random walk on a Galton-Watson tree hits the root at least $n\delta_n$ times. Using the estimates for the law of the first regeneration time of the simple random walk shown by [Pia98], we obtain a stretched exponential bound for this probability.

We specify this in the following. Given an environment $\omega = (T, \rho, \xi)$, we denote the first descendant of the root by v. We define a sequence of stopping times

$$\zeta_0 = \inf\{m \ge 0 : X_m \in T(v)\}$$

and recursively for $k \ge 1$

$$\zeta_k = \inf\{m \ge \zeta_{k-1} : X_m \in T(v),\, X_m \ne X_{\zeta_{k-1}}\}$$

so that ζ_k denote the points in time when the random walk is located in the subtree $T(v)$ (in a different vertex than at time ζ_{k-1}). The process $X^\zeta = (X_{\zeta_k})_{k\ge 0}$ is then a random walk on the subtree $T(v)$. Since $(X_k)_{k\ge 0}$ starts at ρ, the process X^ζ starts at v. To determine its transition probabilities we let x, $y \in T(v)$ be two neighbours. We note that it is not possible for the random walk X^ζ to visit more than one neighbour of x during a single excursion starting from x. Hence, Lemma 2.4.1 implies

$$P_{\hat\omega}(X_{\zeta_{n+1}} = y \mid X_{\zeta_n} = x) = \frac{\mathcal{C}_{\hat\omega}(x,y)}{\mathcal{C}_{\hat\omega}(x, Z_{T(v)}(x))},$$

where $Z_{T(v)}(x) = \{z \in T(v) : z \sim x\}$ denotes the set of neighbours of the vertex x in the subtree $T(v)$. Using the Parallel Law (Lemma 2.4.3), we have

$$\mathcal{C}_{\hat\omega}(x, Z_{T(v)}(x)) = \sum_{z\sim x,\, z\in T(v)} \hat\xi(x,z) = |Z_{T(v)}(x)|\varepsilon$$

such that

$$P_{\hat\omega}(X_{\zeta_{n+1}} = y \mid X_{\zeta_n} = x) = \frac{\varepsilon}{|Z_{T(v)}(x)|\varepsilon} = \frac{1}{|Z_{T(v)}(x)|}.$$

This shows that X^ζ is a simple random walk on $T(v)$ starting at v.

On the event $\{L(\rho) > n, \hat H_n \geq n\delta_n\}$, the random walk X^ζ returns to the vertex v at least $\lfloor n\delta_n \rfloor$ times. Consequently, the first regeneration time $\tau_1(X^\zeta)$ of the process X^ζ cannot occur before time $\lfloor n\delta_n\rfloor$, which implies

$$\mathrm{E}\big[P_{\hat\omega}(L(\rho) > n, \hat H_n \geq n\delta_n)\big] \leq \mathrm{E}\big[P^v_{\hat\omega}\big(\tau_1(X^\zeta) \geq \lfloor n\delta_n\rfloor\big)\big].$$

[Pia98] showed that there exists a positive constant $c > 0$ such that for all $n \geq 1$

$$\mathrm{E}\big[P^v_{\hat\omega}\big(\tau_1(X^\zeta) \geq \lfloor n\delta_n\rfloor\big)\big] \leq \mathrm{e}^{-c\lfloor n\delta_n\rfloor^{\frac{1}{3}}}.$$

Recall that $\delta_n = c_\varepsilon n^{-\frac{1}{3}}$, we get the following stretched exponential bound for the first probability in (5.3.48):

$$\mathrm{E}\big[P_{\hat\omega}(L(\rho) > n, \hat H_n \geq n\delta_n)\big] \leq \mathrm{e}^{-c\lfloor n\delta_n\rfloor^{\frac{1}{3}}} \leq \mathrm{e}^{-\tilde c_\varepsilon n^{\frac{2}{9}}} \tag{5.3.49}$$

for some constant $\tilde c_\varepsilon > 0$.

Bound for the second expectation in (5.3.48)**:** We introduce

$$R_n = n - \hat H_n = \sum_{k=2}^{\eta_\rho^{(n+1)}} \mathbb{1}_{\{X_{k-2}=X_k=\rho\}} = \sum_{k=1}^{n} \mathbb{1}_{\{\eta_\rho^{(k+1)} - \eta_\rho^{(k)} = 2\}},$$

which is well-defined on the event $\{L(\rho) > n\} = \{\eta_\rho^{(n+1)} < \infty\}$. That is, R_n counts the number of excursions of length 2 starting from the root until the random walk returns to the root for the n-th time. Given an environment $\omega = (T, \rho, \xi)$ and the modification

$\hat{\omega}$ introduced above, the random variables $\eta_\rho^{(2)} - \eta_\rho^{(1)}, \dots, \eta_\rho^{(n+1)} - \eta_\rho^{(n)}$ are independent and identically distributed under $P_{\hat{\omega}}(\,\cdot \mid L(\rho) > n)$. As a sum of i.i.d. Bernoulli random variables, R_n is binomially distributed with parameters n and success probability

$$P_{\hat{\omega}}\big(\eta_\rho^{(2)} - \eta_\rho^{(1)} = 2 \,\big|\, L(\rho) > n\big) = P_{\hat{\omega}}^\rho\big(\eta_\rho^+ = 2 \,\big|\, \eta_\rho^+ < \infty\big) = \frac{P_{\hat{\omega}}(\eta_\rho^+ = 2)}{P_{\hat{\omega}}(\eta_\rho^+ < \infty)}.$$

This implies

$$\begin{aligned}
&P_{\hat{\omega}}(L(\rho) > n, \hat{H}_n < n\delta_n) \le P_{\hat{\omega}}\big(L(\rho) > n,\, R_n \ge n(1-\delta_n)\big) \\
&\quad = P_{\hat{\omega}}\big(R_n \ge n(1-\delta_n) \,\big|\, L(\rho) > n\big) P_{\hat{\omega}}(L(\rho) > n) \\
&\quad = \sum_{k=\lceil n(1-\delta_n)\rceil}^{n} \binom{n}{k} \left(\frac{P_{\hat{\omega}}(\eta_\rho^+ = 2)}{P_{\hat{\omega}}(\eta_\rho^+ < \infty)}\right)^k \left(1 - \frac{P_{\hat{\omega}}(\eta_\rho^+ = 2)}{P_{\hat{\omega}}(\eta_\rho^+ < \infty)}\right)^{n-k} P_{\hat{\omega}}(\eta_\rho^+ < \infty)^n \\
&\quad = \sum_{k=\lceil n(1-\delta_n)\rceil}^{n} \binom{n}{k} P_{\hat{\omega}}(\eta_\rho^+ = 2)^k \big(P_{\hat{\omega}}(\eta_\rho^+ < \infty) - P_{\hat{\omega}}(\eta_\rho^+ = 2)\big)^{n-k} \\
&\quad \le P_{\hat{\omega}}\big(B_n \ge n(1-\delta_n)\big),
\end{aligned}$$

where B_n is a binomially distributed random variable with parameters n and $P_{\hat{\omega}}(\eta_\rho^+ = 2)$. The probability that the random walk returns to the root after two steps can be bounded as follows

$$\begin{aligned}
P_{\hat{\omega}}(\eta_\rho^+ = 2) &= 1 - P_{\hat{\omega}}(X_2 \ne \rho) = 1 - P_{\hat{\omega}}(X_1 = v) P_{\hat{\omega}}^v(X_1 \in \hat{G}_1) \\
&= 1 - \frac{\xi(\rho, v)}{\kappa + \sum_{z\sim\rho, z\ne\rho^*} \xi(\rho, z)} \cdot \frac{\varepsilon|\hat{G}_1|}{\xi(\rho, v) + \varepsilon|\hat{G}_1|} \\
&\le 1 - \frac{\varepsilon}{\kappa + |G_1(T)|\kappa} \cdot \frac{\varepsilon}{\kappa + \varepsilon} \le 1 - \hat{p}
\end{aligned}$$

with

$$\hat{p} = \frac{c_\varepsilon}{2|G_1(T)|}, \quad c_\varepsilon = \frac{\varepsilon^2}{\kappa^2 + \varepsilon\kappa}.$$

Thus, B_n can be stochastically dominated by a binomially distributed random variable $\hat{B}_n$ with parameters n and $1 - \hat{p}$, which implies

$$\begin{aligned}
P_{\hat{\omega}}(L(\rho) > n, \hat{H}_n < n\delta_n) &\le P\big(\hat{B}_n \ge n(1-\delta_n)\big) \\
&= P\big(\hat{B}_n - n(1-\hat{p}) \ge n(\hat{p} - \delta_n)\big). \qquad (5.3.50)
\end{aligned}$$

An application of Hoeffding's inequality yields a stretched exponential bound for the probability above if $\hat{p} - \delta_n$ is strictly positive. To guarantee this, we introduce the set

$$D_n = \Big\{\omega = (T, \rho, \xi) \in \Omega : |G_1(T)| \le \frac{1}{3} n^{\frac{1}{3}}\Big\}.$$

Then for $\omega \in D_n$ (or equivalently $\hat{\omega} \in D_n$) we have

$$\hat{p} - \delta_n = \frac{c_\varepsilon}{2|G_1(T)|} - c_\varepsilon n^{-\frac{1}{3}} \geq \frac{c_\varepsilon}{2} n^{-\frac{1}{3}} > 0$$

and by Hoeffding's inequality

$$P_{\hat{\omega}}(L(\rho) > n, \hat{H}_n < n\delta_n) \leq P\Big(\hat{B}_n - n(1-\hat{p}) \geq \frac{c_\varepsilon}{2} n^{\frac{2}{3}}\Big) \leq \mathrm{e}^{-\frac{c_\varepsilon^2}{2} n^{\frac{1}{3}}}.$$

Since this bound holds for all $\omega \in D_n$, we conclude that

$$\mathrm{E}\big[P_{\hat{\omega}}(L(\rho) > n, \hat{H}_n < n\delta_n) \mathbb{1}_{D_n}\big] \leq \mathrm{e}^{-\frac{c_\varepsilon^2}{2} n^{\frac{1}{3}}}.$$

Moreover, the Markov inequality yields

$$\mathrm{E}\big[P_{\hat{\omega}}(L(\rho) > n, \hat{H}_n < n\delta_n) \mathbb{1}_{D_n^c}\big] \leq P\Big(|G_1(T)| > \frac{1}{3} n^{\frac{1}{3}}\Big) \leq 3^{3\beta} E[|G_1(T)|^{3\beta}] n^{-\beta} \quad (5.3.51)$$

for $\beta > 0$ arbitrary, since the offspring law has finite moments of any order by assumption. Combining the last two estimates implies

$$\mathrm{E}[P_{\hat{\omega}}(L(\rho) > n, \hat{H}_n < n\delta_n)] \leq C_\varepsilon n^{-\beta}$$

for some constant $C_\varepsilon = C_\varepsilon(\beta) > 0$.

Together with (5.3.47) and the stretched exponential bound in (5.3.49), this implies that for any $\beta > 0$ there exists some constant $C'_\varepsilon = C'_\varepsilon(\beta) > 0$ such that

$$\mathrm{E}\big[P^{\rho}_{\tilde{\omega}_\rho}(\eta^+_\rho < \infty)^n\big] \leq \mathrm{E}\big[P_{\hat{\omega}}(L(\rho) > n)\big] \leq C'_\varepsilon n^{-\beta},$$

which is what we wanted to show. □

Finally, we note that, as mentioned in Remark 5.2.9, the assumption that the offspring law has finite moments of any order can be weakened without affecting the validity of the functional central limit theorem in Theorem 5.1.1. If instead we assume that the r-th moment of the offspring law is finite for some $r > 24$, the bound in Lemma 5.3.4 still holds for some $\beta > 8$ (see (5.3.51)). This in turn implies $\mathbb{P}(\tau_1 \geq n) \leq C_\varepsilon n^{-r}$ for some $r > 2$ (see (5.3.46)), so that the second moment $\mathbb{E}[\tau_1^2]$ is finite, which is sufficient to prove Theorem 5.1.1.

List of Symbols

Trees and environments

$\deg(v)$	degree of v	13
$d_T(u,v)$	graph distance between u and v	29
$\|v\|$	graph distance between v and the root	13
$[u-v]_{x_{-\infty}}$	horodistance from u to v, relative to the boundary point $x_{-\infty}$	29
$u \sim v$	u and v are adjacent	13
v^*	ancestor of v	14
$\xi(e)$	conductance of e	16
$C(v)$	sum of conductances at v	16
$\mathcal{C}_\omega(v,A)$	effective conductance in ω between v and A	21
$\mathcal{R}_\omega(v,A)$	effective resistance in ω between v and A	21
$\mathcal{E}(T)$	edge set of T	13
$G_n(T)$	n-th generation of T	14
$T_{\|n}$	the first n generations of T	14
$T^{(1)}$, $T^{(2)}$	subtrees that form the augmented tree by connecting their roots	28
$T(v)$	subtree formed by v and all its descendants	39
$[T(v)]$	shifted subtree $T(v)$ such that v is identified with $\emptyset$ in $\mathbb{T}$	75
$T^*(v)$	subtree formed by $T(v)$ and (v^*, v)	75
T^*	subtree formed by $T^{(2)}$ and (ρ, v_0)	31
T_1^v	subtree of T that contains v after removing all ε-edges	38
$T_1(v)$	subtree of $T(v)$ that contains v after removing all ε-edges	74
T_1	subtree $T_1(\rho)$	27
T^{Bb}	backbone tree of T	38
$T \backslash T'$	subtree where T' is removed from T	76
$\mathbb{T}$	Ulam-Harris tree	14
$\mathcal{T}$	set of rooted trees	14
Ω	set of environments	14

Probability measures and related objects

δ_a	Dirac-measure in a	8
ν	offspring law	15
$\bar{\nu}$	offspring law of T_1	27
m_k	k-th moment of ν	15
μ	conductance law, uniformly elliptic	15
μ_ε	conductance law $\mu_\varepsilon = \alpha\delta_\varepsilon + (1-\alpha)\mu$	26
κ	ellipticity constant of μ	15
γ	first moment of μ	29
GW	law of a Galton-Watson tree with offspring law ν	15
P	law of a Galton-Watson tree with offspring law ν and conductance law μ	16
$\mathrm{P}^{\mathrm{aug}}$	law of an augmented Galton-Watson tree	28
$\hat{\mathrm{P}}$	invariant measure for the environment observed by the particle	29
P_ω	quenched law of the random walk on ω	16
$\hat{P}_\omega$	quenched law of the bi-infinite random walk on ω	28
$\mathbb{P}$	annealed law	17
$\hat{\mathbb{P}}$	invariant annealed law	29
$\hat{\mathbb{P}}^*$	pushforward measure of $\hat{\mathbb{P}}$ under the projection $\hat{\Pi}_m$	51
$\mathcal{G}$	Borel σ-algebra on Ω	15
$\mathcal{F}$	Borel σ-algebra on $\mathbb{T}^{\mathbb{N}_0}$	16
$\hat{\mathcal{F}}$	Borel σ-algebra on $\mathbb{T}^{\mathbb{Z}}$	28

Hitting times and related objects

η_A	first hitting time of a set of vertices A	20
η_v	first hitting time of the vertex v	20
η_v^+	first hitting time of the vertex v after 0	20
η_*	first hitting time of ρ^*	74
η_k	first hitting time of the k-th generation	85
σ_k	k-th potential regeneration time	73
R_k	time of the first return to $X^*_{\sigma_k}$ after time σ_k	73
τ_k	k-th regeneration time	74
$L(v)$	local time of v	112

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