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Tobias Kley, Stanislav Volgushev, Holger Dette, Marc Hallin

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QUANTILE SPECTRAL PROCESSES: ASYMPTOTIC ANALYSIS AND INFERENCE

By Tobias Kley[(a)*,†], Stanislav Volgushev[(a)*,‡], Holger Dette[(a)*,§], and Marc Hallin[(b)*,¶]

[(a)] *Ruhr Universität Bochum*
[(b)] *Université libre de Bruxelles and Princeton University*

Abstract Quantile- and copula-related spectral concepts recently have been considered by various authors. Those spectra, in their most general form, provide a full characterization of the copulas associated with the pairs (X_t, X_{t-k}) in a process $(X_t)_{t\in\mathbb{Z}}$, and account for important dynamic features, such as changes in the conditional shape (skewness, kurtosis), time-irreversibility, or dependence in the extremes, that their traditional counterpart cannot capture. Despite various proposals for estimation strategies, no asymptotic distributional results are available so far for the proposed estimators, which constitutes an important obstacle for their practical application. In this paper, we provide a detailed asymptotic analysis of a class of smoothed rank-based cross-periodograms associated with the *copula spectral density kernels* introduced in Dette et al. (2011). We show that, for a very general class of (possibly non-linear) processes, properly scaled and centered smoothed versions of those cross-periodograms, indexed by couples of quantile levels, converge weakly, as stochastic processes, to Gaussian processes. A first application of those results is the construction of asymptotic confidence intervals for *copula spectral density kernels*. The same convergence results also provide asymptotic distributions (under serially dependent observations) for a new class of rank-based spectral methods involving the Fourier transforms of rank-based serial statistics such as the *Spearman, Blomqvist* or *Gini autocovariance coefficients*.

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1. Introduction. Spectral analysis and frequency domain methods play a central role in the nonparametric analysis of time series data. The classical frequency domain representation is based on the *spectral density*—call it the L^2-*spectral density* in order to distinguish it from other *spectral densities* to be defined in the sequel—which is traditionally defined as the Fourier transform of the autocovariance function of the process under study. Fundamental tools for the estimation of spectral densities are the *periodogram* and its *smoothed* versions. The classical periodogram—similarly call it the L^2-*periodogram*—can be defined either as the discrete Fourier transform of empirical autocovariances, or through L^2-projections of the observed series on a harmonic basis. The success of periodograms in time series analysis is rooted in their fast and simple computation (through the *fast Fourier transform* algorithm) and their interpretation in terms of cyclic behavior, both of a stochastic and of deterministic nature, which in specific applications are more illuminating than time-domain representations. L^2-periodograms are particularly attractive in the analysis of Gaussian time series, since the distribution of a Gaussian process is completely characterized by its spectral density. Classical references are Priestley (1981), Brillinger (1975) or Chapters 4 and 10 of Brockwell and Davis (1987).

Being intrinsically connected to means and covariances, the L^2-spectral density and L^2-periodogram inherit the nice features (such as optimality properties in the analysis of Gaussian series) of L^2 methods, but also their weaknesses: they are lacking robustness against outliers and heavy tails, and are unable to capture important dynamic features such as changes in the conditional shape (skewness, kurtosis), time-irreversibility, or dependence in the extremes. This was realized by many researchers, and various extensions and modifications of the L^2-periodogram have been proposed to remedy those drawbacks.

Robust non-parametric approaches to frequency domain estimation have been considered first: see Kleiner, Martin and Thomson (1979) for an early contribution, and Chapter 8 of Maronna, Martin and Yohai (2006) for an overview. More recently, Klüppelberg and Mikosch (1994) proposed a weighted ("self-normalized") version of the periodogram; see also Mikosch (1998). Hill and McCloskey (2013) used a robust version of autocovariances to obtain a robustified periodogram. In the context of signal detection, Katkovnik (1998) introduced a periodogram based on robust loss functions. The objective of all those attempts is a robustification of classical tools: they essentially aim at protecting existing L^2 spectral methods against the impact of possible outliers or violations of distributional assumptions.

Other attempts, more recent and somewhat less developed, consist in developing alternative spectral concepts and tools, mostly related with quantiles or copulas, and accounting for more general dynamic features. A first step in that direction was taken by Hong (1999), who proposes a generalized spectral density with covariances replaced by joint characteristic functions. In the specific problem of testing pairwise independence, Hong (2000) introduces a test statistic based on the Fourier transforms of (empirical) joint distribution functions and copulas at different lags. Recently, there has been a renewed surge of interest in that type of concept, with the introduction, under the names of *Laplace-, quantile-* and *copula* spectral density and spectral density *kernels*, of various quantile-related spectral concepts, along with the corresponding sample-based periodograms and smoothed periodograms. That strand of literature includes Li (2008, 2012, 2013), Hagemann (2011), Dette et al. (2011, 2013), and Lee and Rao (2012). A Fourier analysis of extreme events, which is related in spirit but quite different in many other respects, was considered by Davis, Mikosch and Zhao (2013). Finally, in the time domain, Linton and Whang (2007), Davis and Mikosch (2009), and Han et al. (2014) introduced the related concepts of *quantilograms* and *extremograms*. A more detailed account of some of those contributions is given in Section 2.

A deep understanding of the distributional properties of any statistical tool is crucial for its successful application. The construction of confidence intervals, testing procedures, and efficient estimation all rest on results concerning finite-sample or asymptotic properties of related estimators—here the appropriate (smoothed) periodograms associated with the quantile-related spectral density under study. Obtaining such asymptotic results, unfortunately, is not trivial, and, to the best of our knowledge, no results on the asymptotic distribution of the aforementioned (smoothed) quantile and copula periodograms are available so far.

In the case of i.i.d. observations, Hong (2000) derived the asymptotic distribution of an empirical version of the integrated version of his quantile spectral density, while Lee and Rao (2012) investigated the distributions of Cramér-von Mises type statistics based on empirical joint distributions. No results on the asymptotic distribution of the periodogram itself are given, though. Li (2008, 2012) does not consider asymptotics for smoothed versions of his quantile periodograms, while Hagemann (2011) and Dette et al. (2011, 2013) only obtain consistency results. This is perhaps not so surprising: the asymptotic distribution of classical L^2-spectral density estimators for general non-linear processes also has remained an active domain of research for several decades—see Brillinger (1975) for early results, Shao and Wu (2007), Liu and Wu (2010) or Giraitis and Koul (2013) for more recent references.

The present paper has two major objectives. First, it aims at providing a rigorous analysis of the asymptotic properties of a general class of smoothed rank-based copula cross-periodograms generalizing the quantile periodograms introduced by Hagemann (2011) and, in an integrated version, by Hong (2000). In Section 3, we show that, for general non-linear time series, properly centered and smoothed versions of those cross-periodograms, indexed by couples of quantile levels, converge in distribution to centered Gaussian processes. A first application of those results is the construction of asymptotic confidence intervals which we discuss in detail in Section 5.

The second objective of this paper is to introduce a new class of rank-based frequency domain methods that can be described as a nonstandard rank-based Fourier analysis of the serial features of time series. Examples of such methods are discussed in detail in Section 4, where we study a class of spectral densities, such as the *Spearman, Blomqvist and Gini spectra*, and the corresponding periodograms, associated with rank-based autocovariance concepts. Denoting by F the marginal distribution function of X_t, the *Spearman spectral density*, for instance, is defined as $\sum_{k\in\mathbb{Z}} e^{\mathrm{i}\omega k}\rho_k^{\mathrm{Sp}}$, where $\rho_k^{\mathrm{Sp}} := \mathrm{Corr}(F(X_t), F(X_{t-k}))$ denotes the lag-k *Spearman autocorrelation*. We show that estimators of those spectral densities can be obtained as functionals of the rank-based copula periodograms investigated in this paper. This connection, and our process-level convergence results on the rank-based copula periodograms, allow us to establish the asymptotic normality of the smoothed versions of the newly defined rank-based periodograms. Those results can be considered as frequency domain versions of Hájek's celebrated asymptotic representation and normality results for (non-serial) linear rank statistics under non-i.i.d. observations (Hájek, 1968).

The paper is organized as follows. Section 2 provides precise definitions of the spectral concepts to be considered throughout, and motivates the use of our quantile-related methods by a graphical comparison of the copula spectra of white noise, QAR(1) and ARCH(1) processes, respectively—all of which share the same helplessly flat L^2 spectral density. Section 3 is devoted to the asymptotics of rank-based copula (cross-)periodograms and their smoothed versions, presenting the main results of this paper: the convergence to a Gaussian process of the smoothed copula rank-based periodogram process (Theorem 3.5), based on an equally interesting asymptotic representation result (Theorem 3.6). Section 4 is dealing with the relation with Spearman, Gini, and Blomqvist autocorrelation coefficients and the related spectra. Based on a short Monte-Carlo study, Section 5 discusses the practical performances of the methods proposed, and Section 6 provides some conclusions and directions for future research. Proofs are concentrated in an appendix (Section 7) and an online supplement (Section 8).

2. Copula spectral density kernels and rank-based periodograms.

In this section, we provide more precise definitions of the various quantile- and copula-related spectra mentioned in the introduction, along with the corresponding periodograms.

Throughout, let $(X_t)_{t\in\mathbb{Z}}$ denote a strictly stationary process, of which we observe a finite stretch $X_0, ..., X_{n-1}$, say. Denote by F the marginal distribution function of X_t, and by $q_\tau := F^{-1}(\tau)$, $\tau \in (0,1)$ the corresponding quantile function. Our main object of interest is the *copula spectral density kernel*

$$\mathfrak{f}_{q_{\tau_1},q_{\tau_2}}(\omega) := \frac{1}{2\pi}\sum_{k\in\mathbb{Z}} \mathrm{e}^{-\mathrm{i}\omega k}\gamma_k^U(\tau_1,\tau_2), \qquad \omega\in\mathbb{R},\ (\tau_1,\tau_2)\in[0,1]^2, \tag{2.1}$$

based on the *copula cross-covariances*

$$\gamma_k^U(\tau_1,\tau_2) := \mathrm{Cov}\Big(I\{U_t\le\tau_1\}, I\{U_{t-k}\le\tau_2\}\Big), \qquad k\in\mathbb{Z},\ U_t := F(X_t).$$

where $U_t := F(X_t)$. Those copula spectral density kernels were introduced in Dette et al. (2011), and generalize the τ*-th quantile spectral densities* of Hagemann (2011), with which they coincide for $\tau_1 = \tau_2 = \tau$. An integrated version of this copula spectral density kernel was also considered by Hong (2000). The same copula spectral density kernel also takes the form

$$\mathfrak{f}_{q_{\tau_1},q_{\tau_2}}(\omega) := \frac{1}{2\pi}\sum_{k\in\mathbb{Z}} \mathrm{e}^{-\mathrm{i}\omega k}\Big(\mathbb{P}(X_k\le q_{\tau_1}, X_0\le q_{\tau_2}) - \tau_1\tau_2\Big), \qquad \omega\in\mathbb{R},\quad (\tau_1,\tau_2)\in(0,1)^2, \tag{2.2}$$

where $\mathbb{P}(X_k \le q_{\tau_1}, X_0 \le q_{\tau_2})$—the joint distribution function of (X_k, X_0) taken at (q_{τ_1}, q_{τ_2})—is, by definition, the copula of the pair (X_k, X_0) evaluated at (τ_1, τ_2), while $\tau_1\tau_2$ is the independence copula evaluated at the same (τ_1, τ_2). The copula spectral density kernel thus can be interpreted as the Fourier transform of the difference between pairwise copulas at lag k and the independence copula, which justifies the notation and the terminology.

As argued by Dette et al. (2011), the copula spectral densities provide a complete description of the pairwise copulas of a time series. Similar to the regression setting, where joint distributions and quantiles provide important generalizations of covariances and means, the copula spectral density kernel, by accounting for much more than the covariance between X_k an X_0, extends and supplements the classical L^2-spectral density.

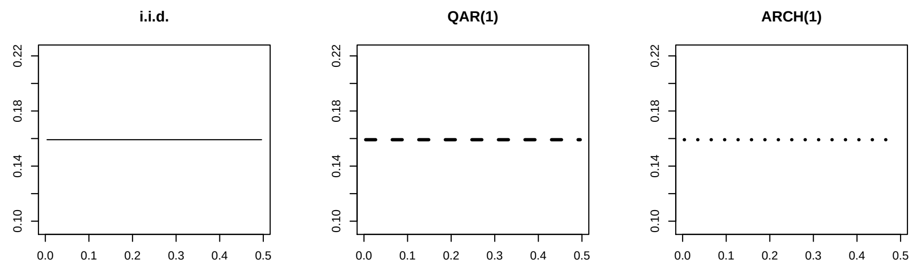

FIGURE 1. *Traditional L^2-spectra $(2\pi)^{-1}\sum_{k\in\mathbb{Z}}\mathrm{Cov}(Y_{t+k},Y_t)e^{-i\omega k}$. The process (Y_t) in the left-hand picture is independent standard normal white noise; in the middle picture, $Y_t = X_t/\operatorname{Var}(X_t)^{1/2}$ where (X_t) is QAR(1) as defined in (5.16); in the right-hand picture, $Y_t = X_t/\operatorname{Var}(X_t)^{1/2}$ where (X_t) is the ARCH(1) process defined in (5.18). All curves are plotted against $\omega/(2\pi)$.*

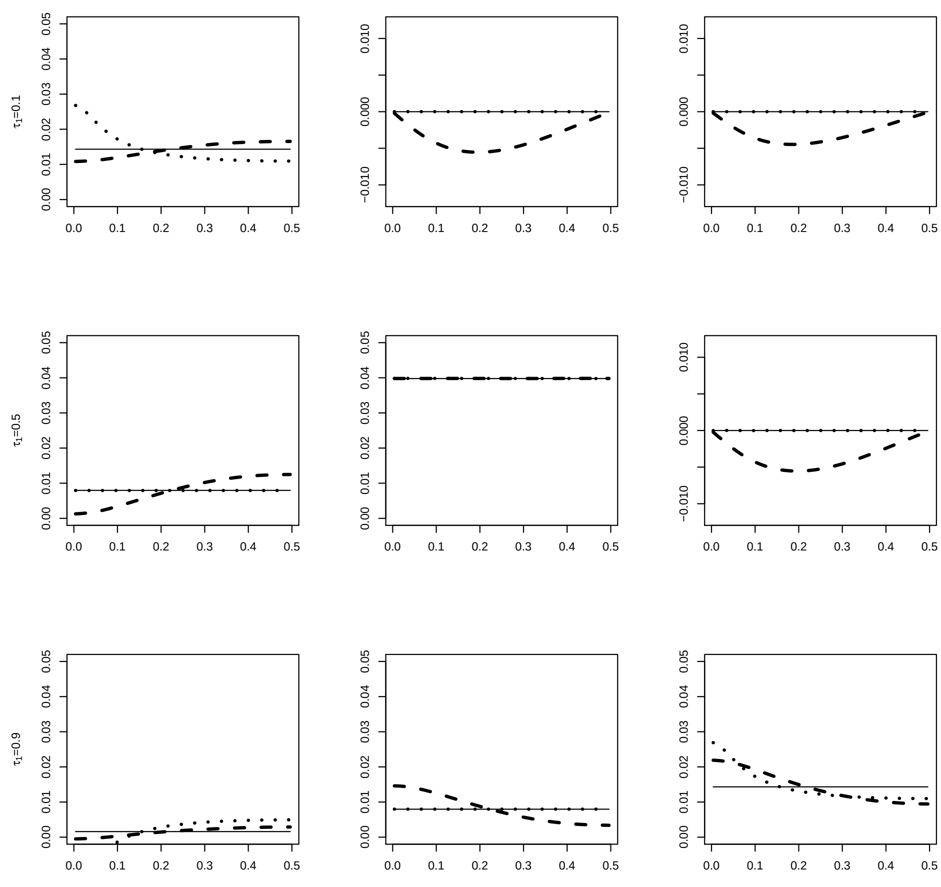

FIGURE 2. *Copula spectra $(2\pi)^{-1}\sum_{k\in\mathbb{Z}}\mathrm{Cov}(I\{F(Y_{t+k})\le\tau_1\},I\{F(Y_t)\le\tau_2\})e^{-i\omega k}$ for $\tau_1,\tau_2 = 0.1, 0.5,$ and 0.9. Real parts (imaginary parts) are shown in subfigures with $\tau_2\le\tau_1$ $(\tau_2>\tau_1)$. Solid, dashed, and dotted lines correspond to the white noise, QAR(1) and ARCH(1) processes in Figure 1. All curves are plotted against $\omega/(2\pi)$.*

As an illustration, the L^2-spectra and copula spectral densities are shown in Figures 1 and 2, respectively, for three different processes: (a) a Gaussian white noise process, (b) a QAR(1) process, and (c) an ARCH(1) process [the same processes are also considered in the simulations of Section 5]. All processes were standardized so that the marginal distributions have unit variance. Although their dynamics obviously are quite different, those three processes are uncorrelated, and thus they all exhibit the same L^2-spectrum. This very clearly appears in Figure 1. In Figure 2, the copula spectral densities associated with various values of τ_1 and τ_2 are shown for the same processes. Obviously, the three copula spectral densities differ considerably from each other, and therefore provide a much richer information about the dynamics of the three processes at hand. For a more detailed discussion of the advantages of the copula spectrum compared to the classical one, see Hong (2000), Dette et al. (2011, 2013), Hagemann (2011), and Lee and Rao (2012).

The consistent estimation of $\mathfrak{f}_{q_{\tau_1},q_{\tau_2}}(\omega)$ was independently considered in Hagemann (2011) for the special case $\tau_1 = \tau_2 \in (0,1)$ and by Dette et al. (2011, 2013) for general couples $(\tau_1, \tau_2) \in [0,1]^2$ of quantile levels, under different assumptions such as $m(n)$-decomposability and β-mixing.

Hagemann's estimator, called the *τ-th quantile periodogram*, is a traditional L^2 periodogram where observations are replaced with the indicators $I\{\hat{F}_n(X_t) \le \tau\} = I\{R_{n;t} \le n\tau\}$, where $\hat{F}_n(x) := n^{-1}\sum_{t=0}^{n-1} I\{X_t \le x\}$ denotes the empirical marginal distribution function and $R_{n;t}$ the rank of X_t among $X_0, \ldots, X_{n-1}$. Dette et al. (2011, 2013) introduce their Laplace rank-based periodograms by substituting an L^1 approach for the L^2 one, and considering the cross-periodograms associated with arbitrary couples (τ_1, τ_2) of quantile levels. See Remark 2.1 for details.

In this paper, we stick to the L^2 approach, but extend Hagemann's concept by considering, as in Dette et al. (2011), the cross-periodograms associated with couples (τ_1, τ_2). More precisely, we define the *rank-based copula periodogram* $I_{n,R}$—shortly, the *CR-periodogram*—as the collection

$$I_{n,R}^{\tau_1,\tau_2}(\omega) := \frac{1}{2\pi n} d_{n,R}^{\tau_1}(\omega) d_{n,R}^{\tau_2}(-\omega), \quad \omega \in \mathbb{R}, \quad (\tau_1, \tau_2) \in [0,1]^2, \tag{2.3}$$

with

$$d_{n,R}^{\tau}(\omega) := \sum_{t=0}^{n-1} I\{\hat{F}_n(X_t) \le \tau\} \mathrm{e}^{-\mathrm{i}\omega t} = \sum_{t=0}^{n-1} I\{R_{n;t} \le n\tau\} \mathrm{e}^{-\mathrm{i}\omega t},$$

Those cross-periodograms, as well as Hagemann's τ-th quantile periodograms, are measurable functions of the marginal ranks $R_{n;t}$, whence the terminology and the notation.

Classical periodograms and rank-based Laplace periodograms converge, as $n \to \infty$, to random variables whose expectations are the corresponding spectral densities; but they fail estimating those spectral densities in a consistent way. Similarly, the CR-periodogram $I_{n,R}^{\tau_1,\tau_2}(\omega)$ fails to estimate $\mathfrak{f}_{q_{\tau_1},q_{\tau_2}}(\omega)$ consistently. More precisely, we show (see Proposition 3.4 for details) that, under suitable assumptions, denoting by $\rightsquigarrow$ the *Hoffman-Jørgensen convergence* in $\ell^\infty([0,1]^2)$ (see Chapter 1 of van der Vaart and Wellner (1996)), for any frequencies $\omega \neq 0 \mod 2\pi$,

$$\Big(I_{n,R}^{\tau_1,\tau_2}(\omega)\Big)_{(\tau_1,\tau_2)\in[0,1]^2} \rightsquigarrow \Big(\mathbb{I}(\tau_1,\tau_2;\omega)\Big)_{(\tau_1,\tau_2)\in[0,1]^2} \quad \text{as } n \to \infty,$$

where the limiting process $\mathbb{I}$ is such that

$$\mathbb{E}[\mathbb{I}(\tau_1,\tau_2;\omega)] = \mathfrak{f}_{q_{\tau_1},q_{\tau_2}}(\omega) \quad \text{for all } (\tau_1,\tau_2) \in (0,1)^2 \ \text{ and } \omega \neq 0 \mod 2\pi$$

and $\mathbb{I}(\tau_1,\tau_2;\omega_1)$ and $\mathbb{I}(\tau_3,\tau_4;\omega_2)$ are independent for any $\tau_1,\ldots,\tau_4$ as soon as $\omega_1 \neq \omega_2$.

In view of this asymptotic independence at different frequencies, it seems natural to consider smoothed versions of $I_{n,R}^{\tau_1,\tau_2}(\omega)$, namely, for $(\tau_1,\tau_2) \in [0,1]^2$ and $\omega \in \mathbb{R}$, averages of the form

$$\hat{G}_{n,R}(\tau_1,\tau_2;\omega) := \frac{2\pi}{n}\sum_{s=1}^{n-1} W_n\big(\omega - 2\pi s/n\big) I_{n,R}^{\tau_1,\tau_2}(2\pi s/n), \tag{2.4}$$

where W_n denotes a sequence of weighting functions. For the special case $\tau_1 = \tau_2$, the consistency of a closely related estimator is established by Hagemann (2011). However, even for $\tau_1 = \tau_2$, obtaining the asymptotic distribution of *smoothed CR-periodograms* is not trivial, and so far has remained an open problem. Similarly, Dette et al. (2011) do not provide any results on the asymptotic distributions of their (smoothed) *Laplace rank-based periodograms*. Note that even consistency results in Hagemann (2011), as well as in Dette et al. (2011) are only pointwise in τ_1, τ_2.

In the present paper, we fill that gap. Theorem 3.5 below does not only provide pointwise asymptotic distributions for smoothed CR-periodograms, but also describes the asymptotic behavior of a properly centered and rescaled version of the full collection $\{\hat{G}_{n,R}(\tau_1,\tau_2;\omega),\ (\tau_1,\tau_2) \in [0,1]^2\}$ as a sequence of *stochastic processes*. Such convergence results (process convergence rather than pointwise) are of particular interest, as they can be used to obtain the asymptotic distribution of functionals of smoothed CR-periodograms as estimators of functionals of the corresponding copula spectral density kernel. As an example, we derive, in Section 4, the asymptotic distributions of periodograms computed from various rank-based autocorrelation concepts (Spearman, Gini, Blomqvist, ...).

In the process of analyzing the asymptotic behavior of $\{\hat{G}_{n,R}(\tau_1, \tau_2; \omega)\}$, we establish several intermediate results of independent interest. For instance, we prove an asymptotic representation theorem (Theorem 3.6(i)), where we show that, uniformly in $\tau_1, \tau_2 \in [0,1]^2$, $\omega \in \mathbb{R}$, the smoothed periodogram $\hat{G}_{n,R}(\tau_1, \tau_2; \omega)$ can be approximated by

$$\hat{G}_{n,U}(\tau_1, \tau_2; \omega) := \frac{2\pi}{n} \sum_{s=1}^{n-1} W_n\big(\omega - 2\pi s/n\big) I_{n,U}^{\tau_1,\tau_2}(2\pi s/n), \tag{2.5}$$

where

$$I_{n,U}^{\tau_1,\tau_2}(\omega) := \frac{1}{2\pi}\frac{1}{n} d_{n,U}^{\tau_1}(\omega) d_{n,U}^{\tau_2}(-\omega), \tag{2.6}$$

and

$$d_{n,U}^{\tau}(\omega) := \sum_{t=0}^{n-1} I\{U_t \leq \tau\} \mathrm{e}^{-\mathrm{i}\omega t} \quad \text{with} \quad U_t := F(X_t).$$

We conclude this section with two remarks clarifying the relation between the approach considered here, that of Dette et al. (2011, 2013), and some other copula-based approaches in the analysis of time series.

Remark 2.1. The classical L^2-periodogram of a real-valued time series can be represented in two distinct ways, providing two distinct interpretations. First, it can be defined as the Fourier transform of the empirical autocovariance function. More precisely, considering the empirical autocovariance

$$\hat{\gamma}_k := \frac{1}{n-k} \sum_{t=1}^{n-k} (X_{t+k} - \bar{X})(X_t - \bar{X}) \quad k \geq 0, \quad \hat{\gamma}_k := \hat{\gamma}_{-k} \quad k < 0,$$

the classical L^2-periodogram can be defined as

$$I_n(\omega) := \frac{1}{2\pi} \sum_{|k|<n} \frac{n-k}{n} \hat{\gamma}_k \mathrm{e}^{-\mathrm{i}k\omega}. \tag{2.7}$$

However, an alternative definition is

$$I_n(\omega) := \frac{1}{2\pi}\frac{1}{n} \Big| \sum_{t=0}^{n-1} X_t e^{-\mathrm{i}t\omega} \Big|^2 = \frac{n}{4}(\hat{b}_1^2 + \hat{b}_2^2) \tag{2.8}$$

where b_1, b_2 are the coefficients of the projection of the observations $X_0, ..., X_{n-1}$ on the basis $(1, \sin(\omega t), \cos(\omega t))$, that is

$$(2.9)\quad (\hat{a}, \hat{b}_1, \hat{b}_2) = \text{Argmin}_{(a,b_1,b_2)' \in \mathbb{R}^3} \sum_{t=0}^{n-1} \Big(X_t - a - b_1 \cos(\omega t) - b_2 \sin(\omega t) \Big)^2.$$

This suggests two different starting points for generalization. We either can replace autocovariances in (2.7) by alternative measures of dependence such as (empircal) joint distributions or copulas, or consider alternative loss functions in the minimization step (2.8). Replacing the autocovariance function by the pairwise copula with $\tau_1 = \tau_2 = \tau$ yields the τ-quantile periodogram of Hagemann (2011), which we also consider here, under the name of CR-periodogram, for general $(\tau_1, \tau_2) \in [0, 1]^2$. Replacing the quadratic loss in (2.9) was, in a time series context, first considered by Li (2008, 2012) and Dette et al. (2011, 2013), who observed that substituting the check function $\rho_\tau(x) = x(\tau - I\{x < 0\})$ of Koenker and Bassett (1978) for the standard L^2-loss leads to an estimator for the quantity

$$\tilde{\mathfrak{f}}_{\tau,\tau}(\omega) := \frac{1}{2\pi f^2(q_\tau)} \sum_{k \in \mathbb{Z}} \mathrm{e}^{-\mathrm{i}\omega k} \Big(\mathbb{P}(X_0 \le q_\tau, X_{-k} \le q_\tau) - \tau^2 \Big).$$

This latter expression is is a weighted version of the copula spectral density kernel at $\tau_1 = \tau_2 = \tau$ introduced in (2.2). This weighting, which involves $f(q_\tau)$, is undesirable, since it involves the unknown marginal distribution of X_t, which is unrelated with its dynamics. Dette et al. (2011) demonstrate that, by considering ranks instead of the original data, that weighting can be removed. The same authors also proposed a generalization to cross-periodograms associated with distinct quantile levels. See Li (2012), Dette et al. (2011), and Hagemann (2011) for details and discussion.

Remark 2.2. The benefits of considering joint distributions and copulas as a measure of serial dependence in a nonparametric time-domain analysis of time series has been realized by many authors. Skaug and Tjøstheim (1993) and Hong (1999) used joint distribution functions to test for serial independence at given lag. Subsequently, related approaches were taken by many authors, and an overview of related results can be found in Tjøstheim (1996) and Hong (1999). Copula-based tests of serial independence were considered by Genest and Rémillard (2004), among others. Linton and Whang (2007) introduced the so-called *quantilogram*, defined as the autocorrelation of the series of indicators $I\{X_t \le \hat{q}_\tau\}$, $t = 0, \ldots, n-1$, where $\hat{q}_\tau$ denotes the empirical τ-quantile; they discuss the application of this quantilogram (closely related to Hagemann's τ-quantile periodogram) to measuring directional predictability of time series. They do not, however, enter into any spectral considerations. An extension of those concepts to the dependence between several time series was recently considered in Han et al. (2014). Finally, Davis and Mikosch (2009) also considered a related quantity which is based on autocorrelations of indicators of extreme events.

3. Asymptotic properties of rank-based copula periodograms.

The derivation of the asymptotic properties of CR-periodograms requires some assumptions on the underlying process and the weighting functions W_n.

Recall that the *rth order joint cumulant* $\mathrm{cum}(\zeta_1, \ldots, \zeta_r)$ of the random vector $(\zeta_1, \ldots, \zeta_r)$ is defined as

$$\mathrm{cum}(\zeta_1, \ldots, \zeta_r) := \sum_{\{\nu_1,\ldots,\nu_p\}} (-1)^{p-1}(p-1)!(\mathbb{E}\prod_{j\in\nu_1} \zeta_j) \cdots (\mathbb{E}\prod_{j\in\nu_p} \zeta_j),$$

with summation extending over all partitions $\{\nu_1, \ldots, \nu_p\}$, $p = 1, \ldots, r$ of $\{1, \ldots, r\}$ (cf. Brillinger (1975), p. 19).

The assumption we make on the dependence structure of the process $(X_t)_{t\in\mathbb{Z}}$ is as follows. Its relation to more classical assumptions of weak dependence is discussed in Propositions 3.1 and 3.2 below, and in Lemma 3.3.

(C) *There exist constants $\rho \in (0, 1)$ and $K < \infty$ such that, for arbitrary intervals $A_1, ..., A_p \subset \mathbb{R}$ and arbitrary $t_1, ..., t_p \in \mathbb{Z}$,*

$$|\,\mathrm{cum}(I\{X_{t_1} \in A_1\}, \ldots, I\{X_{t_p} \in A_p\})| \leq K\rho^{\max_{i,j}|t_i - t_j|}. \tag{3.1}$$

The crucial point here is that we replace an assumption on the cumulants of the original observations by an assumption on the cumulants of certain indicators. Thus, in contrast to classical assumptions, condition (C) does not require the existence of any moments. Additionally, note that the sets A_j in (3.1) only need to be intervals, not general Borel sets as in classical mixing assumptions.

PROPOSITION 3.1. *Assume that the process $(X_t)_{t\in\mathbb{Z}}$ is strictly stationary and exponentially α-mixing, i. e.,*

$$\alpha(n) := \sup_{\substack{A\in\sigma(X_0,X_{-1},\ldots)\\ B\in\sigma(X_n,X_{n+1},\ldots)}} |\mathbb{P}(A \cap B) - \mathbb{P}(A)P(B)| \leq K\kappa^n\ ,\ n \in \mathbb{N} \tag{3.2}$$

for some $K < \infty$ and $\kappa \in (0, 1)$. Then Assumption (C) holds.

While mixing assumptions are very general and intuitively interpretable, which makes them quite attractive from a probabilistic point of view, verifying conditions such as (3.1) or (3.2) can be difficult in specific applications. An alternative description of dependence that is often easier to check for was recently proposed by Wu and Shao (2004). More precisely, these authors assume that the process $(X_t)_{t\in\mathbb{Z}}$ can be represented as

$$X_t = g(\dots, \varepsilon_{t-2}, \varepsilon_{t-1}, \varepsilon_t) \tag{3.3}$$

where g denotes some measurable function and $(\varepsilon_t)_{t\in\mathbb{Z}}$ is a collection of i.i.d. random variables. Note that the function g is not assumed to be linear, which makes this kind of process very general. To quantify the long-run dependence between $(..., X_{-1}, X_0)$, and $(X_t, X_{t+1}, ...)$, denote by $(\varepsilon_t^*)_{t\le 0}$ an independent copy of $(\varepsilon_t)_{t\le 0}$ and define $X_t^* := g(\dots, \varepsilon_{-1}^*, \varepsilon_0^*, \varepsilon_1, ..., \varepsilon_t)$, $t \in \mathbb{N}$. The process $(X_t)_{t\in\mathbb{Z}}$ satisfies a *geometric moment contraction of order a* property (shortly, GMC(a) throughout this paper) if, for some $K < \infty$ and $\sigma \in (0, 1)$,

$$\mathbb{E}|X_n - X_n^*|^a \le K\sigma^n \ , \ n \in \mathbb{N}; \tag{3.4}$$

see Wu and Shao (2004). Examples of processes that satisfy this condition include, (possibly, under mild additional conditions on the parameters) ARMA, ARCH, GARCH, asymmetric GARCH, random coefficient autoregressive, quantile autoregressive and Markov models, to name just a few. Proofs and additional examples can be found in Shao and Wu (2007) and Shao (2010). The definition in (3.4) still requires the existence of moments, which is quite undesirable in our setting. However, the following result shows that a modified version of (3.4) is sufficient for our purposes.

Proposition 3.2. *Assume that the strictly stationary process $(X_t)_{t\in\mathbb{Z}}$ can be represented as in (3.3), and that X_0 has distribution function F. Let the process $(F(X_t))_{t\in\mathbb{Z}}$ satisfy GMC(a) for some $a > 0$, i.e. assume that there exist $K < \infty$ and $\sigma \in (0, 1)$ such that*

$$\mathbb{E}|F(X_n) - F(X_n^*)|^a \le K\sigma^n \ , \ n \in \mathbb{N}. \tag{3.5}$$

Then Assumption (C) holds.

The important difference between assumptions (3.4) and (3.5) lies in the fact that, in condition (3.5), only the random variables $F(X_t) = U_t$, which possess moments of arbitrary order, appear. This implies that a GMC(a) condition on X_t with arbitrarily small values of a, together with a very mild regularity condition on F, are sufficient to imply Assumption (C). More precisely, we have the following result.

LEMMA 3.3. *Assume that* $(X_t)_{t\in\mathbb{Z}}$ *is strictly stationary. Let* $(X_t)_{t\in\mathbb{Z}}$ *satisfy the* $GMC(b)$ *condition for some* $b > 0$*, and assume that the distribution function* F *of* X_0 *is Hölder-continuous of order* $\gamma > 0$*. Then* (3.5) *holds for any* $a > 0$.

For a proof of Lemma 3.3, note that

$$\mathbb{E}|F(X_t) - F(X_t^*)|^b \leq 2^{b-a/\gamma}\mathbb{E}|F(X_t) - F(X_t^*)|^{a/\gamma} \leq C\mathbb{E}|X_t - X_t^*|^a \leq C\eta^t.$$

REMARK 3.1. Although not very deep at first sight, the above result has some remarkable implications. In particular, we show in the Appendix that, under a very mild regularity condition on F, the copula spectra of a GMC(a) process are analytical functions of the frequency ω. This is in sharp contrast with classical spectral density analysis, where higher-order moments are required to obtain smoothness of the spectral density.

We now are ready to state a first result on the asymptotic properties of the CR-periodogram $I_{n,R}^{\tau_1,\tau_2}$ defined in (2.3).

PROPOSITION 3.4. *Assume that* F *is continuous and that* $(X_t)_{t\in\mathbb{Z}}$ *is strictly stationary and satisfies Assumption (C). Then, for every* $\omega \neq 0 \mod 2\pi$,

$$\Big(I_{n,R}^{\tau_1,\tau_2}(\omega)\Big)_{(\tau_1,\tau_2)\in[0,1]^2} \rightsquigarrow \Big(\mathbb{I}(\tau_1,\tau_2;\omega)\Big)_{(\tau_1,\tau_2)\in[0,1]^2} \quad \text{in } \ell^\infty([0,1]^2).$$

The (complex-valued) limiting processes $\mathbb{I}$ *are of the form*

$$\mathbb{I}(\tau_1,\tau_2;\omega) = \frac{1}{2\pi}\mathbb{D}(\tau_1;\omega)\overline{\mathbb{D}(\tau_2;\omega)}$$

with $\mathbb{D}(\tau;\omega) = \mathbb{C}(\tau;\omega) + \mathrm{i}\mathbb{S}(\tau;\omega)$ *where* $\mathbb{C}$ *and* $\mathbb{S}$ *denote two centered jointly Gaussian processes. For* $\omega_1 \neq \omega_2$ *with* $\omega_1,\omega_2 \neq 0 \mod 2\pi$*, the processes* $\mathbb{D}(\cdot;\omega_1)$ *and* $\mathbb{D}(\cdot;\omega_2)$ *are mutually independent; for* $\omega = \omega_1 = \omega_2 \neq 0 \mod 2\pi$*, their covariance structure takes the form*

$$\mathbb{E}\Big[(\mathbb{C}(\tau_1;\omega),\mathbb{S}(\tau_1;\omega))'(\mathbb{C}(\tau_2;\omega),\mathbb{S}(\tau_2;\omega)))\Big] = \pi\begin{pmatrix} \Re\mathfrak{f}_{q_{\tau_1},q_{\tau_2}}(\omega) & -\Im\mathfrak{f}_{q_{\tau_1},q_{\tau_2}}(\omega) \\ \Im\mathfrak{f}_{q_{\tau_1},q_{\tau_2}}(\omega) & \Re\mathfrak{f}_{q_{\tau_1},q_{\tau_2}}(\omega) \end{pmatrix}.$$

As stated in Section 2, smoothed versions of the CR-periodogram kernel yield consistent estimators of the copula spectral density kernel $\mathfrak{f}$. In order to establish the convergence of the smoothed CR-periodogram process (2.4), we require the weights W_n in (2.4) to satisfy the following assumption, which is quite standard in classical time series analysis [see, for example, p. 147 of Brillinger (1975)].

(W) The weight function W is real-valued and even, with support $[-\pi, \pi]$; moreover, it has bounded variation, and satisfies $\int_{-\pi}^{\pi} W(u)\mathrm{d}u = 1$.

Denoting by $b_n > 0$, $n = 1, 2, \ldots$, a sequence of scale parameters such that $b_n \to 0$ and $nb_n \to \infty$ as $n \to \infty$, define

$$W_n(u) := \sum_{j=-\infty}^{\infty} b_n^{-1} W(b_n^{-1}[u + 2\pi j]).$$

We now are ready to state our main result.

THEOREM 3.5. *Let Assumptions (C) and (W) hold. Assume that X_0 has a continuous distribution function F and that there exist constants $\kappa_1 > 0$ and $k \in \mathbb{N}$, such that*

$$b_n = o(n^{-1/(2k+1)}) \quad \textit{and} \quad b_n n^{1-\kappa_1} \to \infty.$$

Then, for any fixed $\omega \in \mathbb{R}$, the process

$$\mathbb{G}_n(\cdot,\cdot;\omega) := \sqrt{nb_n}\Big(\hat{G}_{n,R}(\tau_1,\tau_2;\omega) - \mathfrak{f}_{q_{\tau_1},q_{\tau_2}}(\omega) - B_n^{(k)}(\tau_1,\tau_2;\omega)\Big)_{\tau_1,\tau_2\in[0,1]}$$

satisfies

$$\mathbb{G}_n(\cdot,\cdot;\omega) \rightsquigarrow H(\cdot,\cdot;\omega) \tag{3.6}$$

in $\ell^\infty([0,1]^2)$, where the bias $B_n^{(k)}$ is given by

$$B_n^{(k)}(\tau_1,\tau_2;\omega) := \begin{cases} \displaystyle\sum_{j=2}^{k} \frac{b_n^j}{j!} \int_{-\pi}^{\pi} v^j W(v)dv \frac{d^j}{d\omega^j} \mathfrak{f}_{q_{\tau_1},q_{\tau_2}}(\omega) & \omega \neq 0 \mod 2\pi, \\ n(2\pi)^{-1}\tau_1\tau_2 & \omega = 0 \mod 2\pi \end{cases} \tag{3.7}$$

and $\mathfrak{f}_{q_{\tau_1},q_{\tau_2}}$ is defined in (2.2). The process $H(\cdot,\cdot;\omega)$ in (3.6) is a centered Gaussian process characterized by

$$\begin{aligned}\mathrm{Cov}\,\big(H(u_1,v_1;\omega),H(u_2,v_2;\omega)\big) &= 2\pi\Big(\int_{-\pi}^{\pi} W^2(w)dw\Big)\\ &\times\Big(\mathfrak{f}_{q_{u_1},q_{u_2}}(\omega)\mathfrak{f}_{q_{v_2},q_{v_1}}(\omega) + \mathfrak{f}_{q_{u_1},q_{v_2}}(\omega)\mathfrak{f}_{q_{v_1},q_{u_2}}(\omega) I\{\omega = 0 \mod \pi\}\Big).\end{aligned}$$

Moreover, $H(\omega+\pi) = \overline{H(\omega)}$, $H(\omega + 2\pi) = H(\omega)$, and the family $\{H(\omega), \omega \in [0,\pi]\}$ is a collection of independent processes. In particular, the weak convergence (3.6) holds jointly for any finite fixed collection of frequencies ω.

For fixed quantile levels τ_1, τ_2, the asymptotic distribution of $\mathbb{G}_n(\tau_1, \tau_2; \omega)$ is the same as the distribution of the smoothed L^2 cross-periodogram [see Chapter 7 of Brillinger (1975)] corresponding to the (unobservable) bivariate time series $(I\{F(X_t) \le \tau_1\}, I\{F(X_t) \le \tau_2\})_{0 \le t \le n-1}$. In particular, the estimation of the marginal quantiles has no impact on the asymptotic distribution of $\mathbb{G}_n$. Intuitively, this can be explained by the fact that $(\hat{q}_{\tau_1}, \hat{q}_{\tau_2})$ converge at $n^{-1/2}$ rate while the normalization $\sqrt{nb_n}$ appearing in $\mathbb{G}_n$ is strictly slower.

One aspect of Theorem 3.5 that does not appear in the context of classical spectral density estimation is the convergence of $\mathbb{G}_n$ as a process. Establishing this result is challenging, and it requires the development of new tools. On the other hand, once convergence has been established at process level, it can be applied to derive the asymptotic distributions of various related statistics: see Section 4.

REMARK 3.2. In the derivation of Theorem 3.5, it would be natural to show that $d^{\tau}_{n,R}(\omega)$ and $d^{\tau}_{n,U}(\omega)$ are sufficiently close uniformly with respect to τ and ω as $n \to \infty$. Indeed, using modifications of standard arguments from empirical process theory, it is possible to establish that

$$n^{-1/2} \sup_{\substack{\omega \in \mathbb{R} \\ \tau \in [0,1]}} |d^{\tau}_{n,R}(\omega) - d^{\tau}_{n,U}(\omega)| = o_P(r_n) \tag{3.8}$$

for some rate $r_n \to 0$ depending on the underlying dependence structure. Unfortunately, the best rate r_n that can theoretically be obtained *must* be slower than $o(n^{-1/4})$, and this makes the approximation (3.8) useless for establishing Theorem 3.5 for practically relevant choices of the bandwidth parameter.

REMARK 3.3. Another type of process convergence is frequently discussed in the literature on classic L^2-based spectral analysis, which is dealing with *empirical spectral processes* of the form

$$\left(\int_{-\pi}^{\pi} g(\omega) I_n(\omega) d\omega \right)_{g \in \mathcal{G}}$$

with $\mathcal{G}$ denoting a suitable class of functions. For more details, see Dahlhaus (1988), Dahlhaus and Polonik (2009), and the references therein. Those processes are completely different from the processes considered above, and the mathematical tools that need to be developed for their analysis also differ substantially. It would be very interesting to extend our results to classes of integrated periodograms that are indexed by classes of functions. Such an extension, however, is beyond of the scope of the present paper.

Remark 3.4. At first glance, it seems surprising that the asymptotic theory developed here does not require the marginal distribution function F to have a continuous Lebesgue density, although the CR-periodograms in (2.3) are based on marginal quantiles. The reason is that the estimators which are constructed from $X_0, ..., X_{n-1}$ are almost surely equal to estimators based on the (unobserved) transformed variables $F(X_0), ..., F(X_{n-1})$. A similar phenomenon can be observed in the estimation of copulas, see e.g. Fermanian, Radulović and Wegkamp (2004).

In order to establish Theorem 3.5, we prove (an *asymptotic representation* result) that the estimator $\hat{G}_{n,R}$ can be approximated by $\hat{G}_{n,U}$ in a suitable uniform sense. Theorem 3.5 then follows from the asymptotic properties of $\hat{G}_{n,U}$, which we state now.

Theorem 3.6. *Let Assumptions (C) and (W) hold, and assume that the distribution function F of X_0 is continuous. Let b_n be such that, for some $k \in \mathbb{N}, \kappa_2 > 0$, $b_n = o(n^{-1/(2k+1)})$ and $(nb_n)^{-1} = o(n^{-\kappa_2})$. Then,*

(i) for any $\omega \in \mathbb{R}$, as $n \to \infty$,

$$\sqrt{nb_n}\big(\hat{G}_{n,U}(\tau_1, \tau_2; \omega) - \mathbb{E}\hat{G}_{n,U}(\tau_1, \tau_2; \omega)\big)_{\tau_1,\tau_2 \in [0,1]} \rightsquigarrow H(\cdot, \cdot; \omega)$$

in $\ell^\infty([0,1]^2)$, where the process $H(\cdot, \cdot; \omega)$ is defined in Theorem 3.5;

(ii) still as $n \to \infty$,

$$\sup_{\substack{\tau_1,\tau_2 \in [0,1] \\ \omega \in \mathbb{R}}} \Big|\mathbb{E}\hat{G}_{n,U}(\tau_1, \tau_2; \omega) - \mathfrak{f}_{q_{\tau_1}, q_{\tau_2}}(\omega) - B_n^{(k)}(\tau_1, \tau_2, \omega)\Big| = O((nb_n)^{-1}) + o(b_n^k).$$

where $B_n^{(k)}$ is defined in (3.7);

(iii) for any $\omega \in \mathbb{R}$,

$$\sup_{\tau_1,\tau_2 \in [0,1]} |\hat{G}_{n,R}(\tau_1, \tau_2; \omega) - \hat{G}_{n,U}(\tau_1, \tau_2; \omega)| = o_P\big((nb_n)^{-1/2} + b_n^k\big);$$

if moreover the kernel W is uniformly Lipschitz-continuous, this bound is uniform with respect to $\omega \in \mathbb{R}$.

4. Spearman, Blomqvist and Gini spectra. In the past decades, considerable effort has been put into replacing empirical autocovariances by alternative (scalar) measures of dependence – see for example Kendall (1938), Blomqvist (1950), Cifarelli, Conti and Regazzini (1996), Ferguson, Genest and Hallin (2000), and Schmid et al. (2010) for a recent survey. Such measures of association provide a good compromise between the limited information contained in autocovariances on one hand, and the fully nonparametric nature of joint distributions and copulas on the other.

A particularly appealing class of such dependence measures is given by general rank-based autocorrelations [see Hallin and Puri (1992) or Hallin (2012) for a survey]. The idea of using ranks in a time-series concept is not new, and rank-based measures of serial dependence actually can be traced back to the early developments of rank-based inference; runs statistics, or the serial version of Spearman's rho [see Wald and Wolfowitz (1943)] are such early examples. The asymptotics of rank-based autocorrelations are well studied under assumptions of white noise or, at least, *exchangeability*, and under contiguous alternatives of serial dependence. An alternative approach to deriving the asymptotic distribution of rank-based autocorrelations, which is applicable under general kinds of dependence, is based on their representation as functionals of empirical copula processes and was considered, for instance, in Fermanian, Radulović and Wegkamp (2004).

Despite the great success of the L^2-periodogram in time series analysis, the only attempt to consider Fourier transforms of rank-based autocorrelations (or any other rank-based scalar measures of dependence), to the best of our knowledge, is that of Ahdesmäki et al. (2005). The latter paper is of a more empirical nature, and no theoretic foundation is provided. The aim of the present section is to introduce a general class of frequency domain methods and discuss their connection to rank-based extensions of autocovariances.

4.1. *The Spearman periodogram.* To illustrate our purpose, first consider in detail the classical example of Spearman's rank autocorrelation coefficients (more precisely, a version of it – see Remark 4.1); at lag k, that coefficient can be defined as

$$\hat{\rho}_n^k := \frac{12}{n^3} \sum_{t=0}^{n-|k|-1} \Big(R_{n;t} - \frac{n+1}{2}\Big)\Big(R_{n;t+|k|} - \frac{n+1}{2}\Big).$$

Letting $\mathcal{F}_n := \{2\pi j/n | \; j = 1, \dots, \lfloor \frac{n-1}{2} \rfloor - 1, \lfloor \frac{n-1}{2} \rfloor\}$, define the *Spearman* and *smoothed Spearman periodograms* as

$$I_{n,\rho}(\omega) := \frac{1}{2\pi} \sum_{|k|<n} e^{-\mathrm{i}\omega k} \hat{\rho}_n^k, \quad \omega \in \mathcal{F}_n$$

and

$$\hat{G}_{n,\rho}(\omega) := \frac{2\pi}{n} \sum_{s=1}^{n-1} W_n\big(\omega - 2\pi s/n\big) I_{n,\rho}(2\pi s/n), \quad \omega \in \mathbb{R},$$

respectively. Intuition suggests that the (smoothed) rank-based periodogram $\hat{G}_{n,\rho}$ should be an estimator for the Fourier transform

$$\mathfrak{f}_\rho(\omega) := \frac{1}{2\pi}\frac{1}{12}\sum_{k\in\mathbb{Z}} \mathrm{e}^{-\mathrm{i}\omega k}\rho_k$$

of the population counterpart

$$\rho_k = \rho(C_k) = 12\int_{[0,1]^2}(C_k(u,v) - uv)dudv, \tag{4.9}$$

of $\hat{\rho}_n^k$, where C_k is the copula associated with (X_t, X_{t+k}) [see e.g. Schmid et al. (2010)]. Due to the presence of ranks, the investigation of the asymptotic properties of the Spearman periodogram under non-exchangeable observations seems highly non-trivial. However, as we shall demonstrate now, those properties can be obtained via Theorem 3.5 by establishing a connection between the Spearman periodogram and the CR-periodogram.

PROPOSITION 4.1. *For any $\omega \in \mathcal{F}_n$,*

$$I_{n,\rho}(\omega) = 12\int_{[0,1]^2} I_{n,R}^{u,v}(\omega)dudv, \tag{4.10}$$

where $I_{n,R}^{u,v}$ is defined in (2.3) Moreover, for any $\omega \in \mathbb{R}$,

$$\hat{G}_{n,\rho}(\omega) = 12\int_{[0,1]^2}\hat{G}_{n,R}(u,v;\omega)dudv$$

where $\hat{G}_{n,R}$ is defined in (2.4).

Proof of Proposition 4.1 Simple algebra yields

$$I_{n,\rho}(\omega) = \frac{12}{2\pi}\frac{1}{n}d_{n,\rho}(\omega)d_{n,\rho}(-\omega) \quad \text{with} \quad d_{n,\rho}(\omega) := \frac{1}{n}\sum_{t=0}^{n-1} R_{n;t}\mathrm{e}^{-\mathrm{i}\omega t}.$$

Observe that

$$I_{n,\rho}(\omega) = \frac{12}{2\pi}\frac{1}{n^3}\sum_{s,t=0}^{n-1} R_{n;t}R_{n;s}\mathrm{e}^{-\mathrm{i}\omega t}e^{i\omega s}.$$

On the other hand,

$$\begin{aligned}
\int_{[0,1]^2} I_{n,R}^{u,v}(\omega) du dv &= \frac{12}{2\pi}\frac{1}{n}\sum_{s,t=0}^{n-1} e^{-\mathrm{i}\omega t} e^{\mathrm{i}\omega s} \int_{[0,1]^2} I\{R_{n;t} \le nu, R_{n;s} \le nv\} du dv \\
&= \frac{12}{2\pi}\frac{1}{n}\sum_{s,t=0}^{n-1} e^{-\mathrm{i}\omega t} e^{\mathrm{i}\omega s}(1-n^{-1}R_{n;t})(1-n^{-1}R_{n;s}) \\
&= I_{n,\rho}(\omega) + \frac{12}{2\pi}\frac{1}{n^2}\sum_{s,t=0}^{n-1} e^{-\mathrm{i}\omega t} e^{\mathrm{i}\omega s}(n - R_{n;t} - R_{n;s}). \qquad (4.11)
\end{aligned}$$

For $\omega \in \mathcal{F}_n$, $\sum_{t=0}^{n-1} e^{\mathrm{i}\omega t} = 0$, so that the second term in (4.11) vanishes. The claim follows. $\square$

This result is useful in several ways. On one hand, it allows to easily derive the asymptotic distribution of the smoothed Spearman periodogram by applying the continuous mapping theorem in combination with Theorem 3.5—see Proposition 4.2 below. On the other hand, it motivates the definition of a general class of rank-based spectra, to be discussed in the next section.

PROPOSITION 4.2. *Under the assumptions of Theorem 3.5, for any fixed frequency $\omega \in (0,\pi)$,*

$$I_{n,\rho}(\omega) \rightsquigarrow 12\int_0^1\int_0^1 \mathbb{I}(\tau_1,\tau_2;\omega) d\tau_1 d\tau_2$$

and

$$\sqrt{nb_n}\big(\hat{G}_{n,\rho}(\omega) - \mathfrak{f}_\rho(\omega) - B_{n,\rho}^{(k)}(\omega)\big) \xrightarrow{\mathcal{D}} Z_\rho(\omega) \sim \mathcal{N}\Big(0, 2\pi\mathfrak{f}_\rho^2(\omega)\int W^2(w)dw\Big)$$

where

$$B_{n,\rho}^{(k)}(\omega) := \sum_{j=2}^{k} \frac{b_n^j}{j!}\int v^j W(v) dv \frac{d^j}{d\omega^j}\mathfrak{f}_\rho(\omega).$$

Moreover $\{Z_\rho(\omega)\}_{\omega\in(0,\pi)}$ is a collection of mutually independent random variables. The weak convergence above holds jointly for any finite, fixed collection of frequencies ω.

This result is a direct consequence of the more general Proposition 4.3, which we establish in the next section.

REMARK 4.1. A closely related version of the Spearman periodogram was recently considered by Ahdesmäki et al. (2005). The main difference with our approach is that these authors use a slightly different version of the lag-k Spearman coefficient, namely

$$\tilde{\rho}_k := \frac{1}{n}\frac{12}{(n-k)^2-1}\sum_{t=0}^{n-k-1}\Big(R^k_{n;t}-\frac{n-k+1}{2}\Big)\Big(\bar{R}^k_{n;t+k}-\frac{n-k+1}{2}\Big)$$

where $R^k_{n;t}$ denotes the rank of X_t among $X_0, ..., X_{n-k-1}$ and $\bar{R}^k_{n;t}$ the rank of X_t among $X_{k-1}, ..., X_{n-1}$, respectively. Letting $\tilde{\rho}_k := \tilde{\rho}_{-k}$ for $k < 0$, Ahdesmäki et al. (2005) then consider a statistic of the form $\sum_{|k|<n} e^{ik\omega}\tilde{\rho}_k$. Note that these authors investigate their method by means of a simulation study and do not provide any asymptotic theory.

4.2. *A general class of rank-based spectra.* The findings in the previous section suggest considering a general class of rank-based periodograms which are defined in terms of the CR-periodogram as

$$I_{n,\mu}(\omega) := \int_{[0,1]^2} I^{u,v}_{n,R}(\omega)\mathrm{d}\mu(u,v), \quad \omega \in \mathcal{F}_n \tag{4.12}$$

where μ denotes an arbitrary finite measure on $[0,1]^2$. A smoothed version of $I_{n,\mu}$ is defined through

$$\hat{G}_{n,\mu}(\omega) := \frac{2\pi}{n}\sum_{s=1}^{n-1} W_n\big(\omega - 2\pi s/n\big) I_{n,\mu}(2\pi s/n), \quad \omega \in \mathbb{R}.$$

As discussed in the previous section, taking μ as 12 times the uniform distribution on $[0,1]^2$ yields the Fourier transform of Spearman autocorrelations.

The general results in Theorem 3.5 combined with the continuous mapping theorem imply that the smoothed periodogram $\hat{G}_{n,\mu}$ is a consistent and asymptotically normal estimator of a spectrum of the form

$$\mathfrak{f}_\mu(\omega) := \frac{1}{2\pi}\frac{1}{12}\sum_{k\in\mathbb{Z}} \mathrm{e}^{-\mathrm{i}\omega k}\int_{[0,1]^2} C_k(u,v)\mathrm{d}\mu(u,v),$$

where C_k denotes the copula of the pair (X_0, X_k).

PROPOSITION 4.3. *Under the assumptions of Theorem 3.5, for any fixed frequency $\omega \in (0,\pi)$,*

$$\sqrt{nb_n}\big(\hat{G}_{n,\mu}(\omega) - \mathfrak{f}_\mu(\omega) - B^{(k)}_{n,\mu}(\omega)\big) \xrightarrow{\mathcal{D}} Z_\mu(\omega) \sim \mathcal{N}\Big(0, \sigma^2_\mu\Big)$$

where the variance σ_μ^2 takes the form

$$\sigma_\mu^2 = 2\pi \int_{-\pi}^{\pi} W^2(w)dw \int_{[0,1]^2} \int_{[0,1]^2} \mathfrak{f}_{q_u,q_{u'}}(\omega)\mathfrak{f}_{q_v,q_{v'}}(\omega)d\mu(u,v)d\mu(u',v')$$

and the bias is given by

$$B_{n,\mu}^{(k)}(\omega) := \sum_{j=2}^{k} \frac{b_n^j}{j!} \int v^j W(w)dw \frac{d^j}{d\omega^j}\mathfrak{f}_\mu(\omega).$$

Moreover, $\{Z_\mu(\omega)\}_{\omega\in(0,\pi)}$ is a collection of independent random variables. The weak convergence above holds jointly for any finite, fixed collection of frequencies ω.

Proof Assumption (C) entails

$$\mathfrak{f}_\mu(\omega) - B_{n,\mu}^{(k)}(\omega) = \int_{[0,1]^2} \mathfrak{f}_{q_u,q_v}(\omega) - B_n^{(k)}(u,v;\omega)d\mu(u,v).$$

This yields

$$\hat{G}_{n,\mu}(\omega) - \mathfrak{f}_\mu(\omega) + B_{n,\mu}^{(k)}(\omega) = \int_{[0,1]^2} \mathbb{G}_n(u,v;\omega)d\mu(u,v)$$

where $\mathbb{G}_n$ was defined in Theorem 3.5. An application of the Continuous Mapping Theorem implies

$$\sqrt{nb_n}\big(\hat{G}_{n,\mu}(\omega) - \mathfrak{f}_\mu(\omega) - B_{n,\mu}^{(k)}(\omega)\big) \xrightarrow{\mathcal{D}} \int_{[0,1]^2} H(u,v;\omega)d\mu(u,v).$$

Since $H(\cdot,\cdot;\omega)$ is a centered Gaussian process, the integral $\int_{[0,1]^2} H(u,v;\omega)dudv$ follows a normal distribution with mean zero and variance

$$\begin{aligned}
&\int_{[0,1]^2} \int_{[0,1]^2} \mathrm{Cov}(H(u,v;\omega), H(u',v';\omega))d\mu(u,v)d\mu(u',v') \\
&= 2\pi \int W^2(w)dw \int_{[0,1]^2} \int_{[0,1]^2} \mathfrak{f}_{q_u,q_{u'}}(\omega)\mathfrak{f}_{q_v,q_{v'}}(\omega)d\mu(u,v)d\mu(u',v')
\end{aligned}$$

This completes the proof. □

4.3. *The Blomqvist and Gini periodograms.* In this section, we identify two measures μ that correspond to two classical measures of serial dependence, Blomqvist's beta [see Blomqvist (1950); Schmid et al. (2010); Genest, Carabarín-Aguirre and Harvey (2013)] and Gini's gamma [see Nelsen (1998)] coefficients, which lead to the definition of the *Blomqvist* and *Gini spectra*, respectively.

Let C_k denote the copula of the pair (X_0, X_k) and assume that it is continuous. The corresponding *Blomqvist beta coefficient at lag k* is

$$\beta_k := 4C_k(1/2, 1/2) - 1. \tag{4.13}$$

Similarly, Gini's gamma, also known as *Gini's lag k rank association coefficient* [see Nelsen (1998)] is the copula-based quantity

$$\begin{aligned}\Gamma_k &:= 2\int_{[0,1]^2} (|u+v-1| - |v-u|) dC_k(u,v) \\ &= 4\Big(\int_{[0,1]} C_k(u,u) - u^2 du + \int_{[0,1]} C_k(u,1-u) - u(1-u) du\Big). \end{aligned} \tag{4.14}$$

This motivates the definition of the *Blomqvist spectrum*

$$\mathfrak{f}_\beta(\omega) := \frac{1}{2\pi} \sum_{k\in\mathbb{Z}} \mathrm{e}^{-\mathrm{i}\omega k} \beta_k$$

and the *Gini spectrum*

$$\mathfrak{f}_\Gamma(\omega) := \frac{1}{2\pi} \sum_{k\in\mathbb{Z}} \mathrm{e}^{-\mathrm{i}\omega k} \Gamma_k.$$

Sample versions of the Blomqvist and Gini coefficients are

$$\hat\beta_n^k := \frac{1}{n-|k|} \sum_{t=1}^{n-|k|-1} \Big(4I\{R_{n;t} \le 1/2, R_{n;t+|k|} \le 1/2\} - 1\Big),$$

and

$$\hat\Gamma_n^k := \frac{2}{n(n-|k|)} \sum_{t=0}^{n-|k|-1} \Big(|R_{n;t} + R_{n;t+|k|} - n| - |R_{n;t} - R_{n;t+|k|}|\Big),$$

respectively. To establish the connection with the general periodogram defined in the previous section, consider the measures μ_β which puts mass 4 in the point $(1/2, 1/2)$ and μ_Γ which puts mass 4 on the sets $\{(u,u) : u \in [0,1]\}$ and $\{(u, 1-u) : u \in [0,1]\}$, respectively.

PROPOSITION 4.4. *For any $\omega \in \mathcal{F}_n$,*

$$I_{n,\beta}(\omega) := \int_{[0,1]^2} I_{n,R}^{u,v}(\omega) d\mu_\beta(u,v) = \frac{1}{2\pi} \sum_{|k|<n} \frac{n-k}{n} e^{\mathrm{i}\omega k} \hat{\beta}_n^k$$

and

$$I_{n,\Gamma}(\omega) := \int_{[0,1]^2} I_{n,R}^{u,v}(\omega) d\mu_\Gamma(u,v) = \frac{1}{2\pi} \sum_{|k|<n} \frac{n-k}{n} e^{\mathrm{i}\omega k} \hat{\Gamma}_n^k.$$

Proof on Proposition 4.4 Observing that

$$|n - R_{n;t} - R_{n;t+k}| = 2\max(n - R_{n;t} - R_{n;t+k}, 0) - (n - R_{n;t} - R_{n;t+k})$$

and

$$|R_{n;t} - R_{n;t+k}| = 2\max(R_{n;t}, R_{n;t+k}) - (R_{n;t} + R_{n;t+k})$$

yields

$$\begin{aligned} &|R_{n;t} + R_{n;t+k} - n| - |R_{n;t} - R_{n;t+k}| \\ &\quad = 2\max(n - R_{n;t} - R_{n;t+k}, 0) - 2\max(R_{n;t}, R_{n;t+k}) + 2(R_{n;t} + R_{n;t+k}) - n. \end{aligned}$$

On the other hand,

$$\begin{aligned} \int_0^1 I_{n,R}^{u,u}(\omega)\mathrm{d}u &= \frac{1}{2\pi}\frac{1}{n}\sum_{s,t=0}^{n-1} e^{-\mathrm{i}\omega t}e^{\mathrm{i}\omega s}\int_0^1 I\{R_{n;t} \le nu, R_{n;s} \le nu\}\mathrm{d}u \\ &= \frac{1}{2\pi}\frac{1}{n}\sum_{s,t=0}^{n-1} e^{-\mathrm{i}\omega t}e^{\mathrm{i}\omega s}(1 - n^{-1}\max(R_{n;t}, R_{n;s})) \\ &= -\frac{1}{2\pi}\frac{1}{n^2}\sum_{s,t=0}^{n-1} e^{-\mathrm{i}\omega t}e^{\mathrm{i}\omega s}\max(R_{n;t}, R_{n;s}) \end{aligned}$$

and

$$\begin{aligned} \int_0^1 I_{n,R}^{u,1-u}(\omega)\mathrm{d}u &= \frac{1}{2\pi}\frac{1}{n}\sum_{s,t=0}^{n-1} e^{-\mathrm{i}\omega t}e^{\mathrm{i}\omega s}\int_0^1 I\{R_{n;t} \le nu, R_{n;s} \le n(1-u)\}\mathrm{d}u \\ &= \frac{1}{2\pi}\frac{1}{n}\sum_{s,t=0}^{n-1} e^{-\mathrm{i}\omega t}e^{\mathrm{i}\omega s}\max(1 - n^{-1}R_{n;t} - n^{-1}R_{n;s}, 0). \end{aligned}$$

Elementary algebra yields, for arbitrary functions a from $\mathbb{Z}^2$ to $\mathbb{Z}$ such that $a(j,k) = a(k,j)$ for all j,k,

$$\sum_{|k|<n} \sum_{t=0}^{n-1-|k|} e^{\mathrm{i}\omega k} a(t, t+|k|) = \sum_{s=0}^{n-1} \sum_{t=0}^{n-1} e^{-\mathrm{i}\omega t} e^{\mathrm{i}\omega s} a(t,s).$$

This implies (recall that $\omega \in \mathcal{F}_n$)

$$\begin{aligned} I_{n,\Gamma}(\omega) &= \frac{1}{2\pi}\frac{2}{n} \sum_{|k|<n} \sum_{t=0}^{n-1-|k|} e^{\mathrm{i}\omega k} \Big(|R_{n;t} + R_{n;t+k} - n| - |R_{n;t} - R_{n;t+k}| \Big) \\ &= \frac{1}{2\pi}\frac{4}{n^2} \sum_{s=0}^{n-1} \sum_{t=0}^{n-1} e^{-\mathrm{i}\omega t} e^{\mathrm{i}\omega s} \Big(\max(n - R_{n;t} - R_{n;s}, 0) - \max(R_{n;t}, R_{n;s}) \Big) \\ &\quad + \frac{1}{2\pi}\frac{2}{n^2} \sum_{s=0}^{n-1} \sum_{t=0}^{n-1} e^{-\mathrm{i}\omega t} e^{\mathrm{i}\omega s} \Big(2(R_{n;t} + R_{n;s}) - n \Big) \\ &= 4 \Big(\int_{[0,1]} I_{n,R}^{u,u}(\omega) du + \int_{[0,1]} I_{n,R}^{u,1-u}(\omega) du \Big). \end{aligned}$$

The representation for $I_{n,\beta}$ can be derived similarly; details are omitted for the sake of brevity. □

Smoothed versions of the Blomqvist and Gini periodograms can be defined accordingly, and their asymptotic distributions follow from Proposition 4.3. In particular, this yields consistent estimators of the Blomqvist and Gini spectra defined above.

We conclude this section with some general remarks. First, note that the approach above can be applied to any scalar dependence measure that can be represented as a continuous linear functional of the copula. For instance, Cifarelli, Conti and Regazzini (1996) consider a general measure of monotone dependence of the form

$$\int_{[0,1]^2} g(|u+v-1|) - g(|u-v|) dC(u,v) \tag{4.15}$$

where $g : [0,1] \to \mathbb{R}$ is strictly increasing and convex. Choosing $g(x) = x$ and $g(x) = x^2$ yields (up to constants) the Gini and Spearman rank correlations, respectively. Under suitable assumptions on g, the monotone dependence measure in (4.15) can be written (by applying integration-by-parts) in the form of equation (4.12), and the results from section 4.2 apply.

Other measures of serial dependence such as Kendall's τ (see Ferguson, Genest and Hallin (2000)) only can be represented as non-linear functionals of the copula. More general rank-based autocorrelation coefficients also have been introduced in the context of inference for ARMA models (see Hallin and Puri (1992) or Hallin (2012)); they involve score functions, typically are not time-revertible, and lead to possibly unbounded measures μ. We expect that the general results presented here can be extended to the periodograms associated with such coefficients, but leave this question to future research.

5. Simulation study. In this section, we show how the result on the asymptotic distribution of the smoothed CR-periodogram defined in (2.4) can be used to construct asymptotic confidence intervals for the copula spectra. We consider three different models:

(a) the QAR(1) process

$$Y_t = 0.1\Phi^{-1}(U_t) + 1.9(U_t - 0.5)Y_{t-1} \tag{5.16}$$

[cf. Koenker and Xiao (2006)], where (U_t) is a sequence of i. i. d. standard uniform random variables, and Φ denotes the distribution function of the standard normal distribution;

(b) the AR(2) process

$$Y_t = -0.36Y_{t-2} + \varepsilon_t, \tag{5.17}$$

where (ε_t) is standard normal white noise [cf. Li (2012)];

(c) the ARCH(1) process

$$Y_t = \big(1/1.9 + 0.9Y_{t-1}^2\big)^{1/2}\varepsilon_t \tag{5.18}$$

where (ε_t) is standard normal white noise [cf. Lee and Rao (2012)].

For each model, 10,000 independent copies of length $n \in \{2^8, 2^9, 2^{10}, 2^{11}\}$ were generated. For each of them, the smoothed CR-periodograms

$$\tilde{G}_{n,R}(\tau_1, \tau_2; \omega_{jn}) := \hat{G}_{n,R}(\tau_1, \tau_2; \omega_{jn})/W_n^j, \quad W_n^j := \sum_{0=s\neq j}^{n-1} W_n\big(\omega_{jn} - \omega_{sn}\big), \tag{5.19}$$

were computed for $\omega_{jn} := 2\pi j/n$, $j = 1, \ldots, n/2-1)$ and $\tau_1, \tau_2 \in \{0.1, 0.5, 0.9\}$, where we used the kernel of order 4

$$W(u) := \frac{15}{32}\frac{1}{\pi}\Big(7(u/\pi)^4 - 10(u/\pi)^2 + 3\Big)I\{|u| \leq \pi\}$$

minimizing the asymptotic IMSE (see Gasser, Muller and Mammitzsch (1985)). The bandwidth was chosen as $b_n = 0.4n^{-1/4}$ which is of lower order than the IMSE-optimal bandwidth $n^{-1/9}$ to reduce bias and the factor $(W_n^j)^{-1}$ ensures that the weights in (5.19) sum up to one for every n.

Based on Theorem 3.6, we then computed pointwise asymptotic $(1-\alpha)$-level confidence bands for the real and imaginary parts of the spectrum, namely,

$$IC_{1,n}(\tau_1, \tau_2; \omega) := \Re \tilde{G}_{n,R}(\tau_1, \tau_2; \omega) \pm \Re\sigma(\tau_1, \tau_2; \omega)\Phi^{-1}(1-\alpha/2), \tag{5.20}$$

for the real part, and

$$IC_{2,n}(\tau_1, \tau_2; \omega) := \Im \tilde{G}_{n,R}(\tau_1, \tau_2; \omega) \pm \Im\sigma(\tau_1, \tau_2; \omega)\Phi^{-1}(1-\alpha/2), \tag{5.21}$$

for the imaginary part of the copula spectrum. As usual, Φ stands for the standard normal distribution function, and

$$\big(\Re\sigma(\tau_1, \tau_2; \omega)\big)^2 := 0 \vee \begin{cases} c(\tau_1, \tau_2; \omega, \omega) & \text{if } \tau_1 = \tau_2, \\ \frac{1}{2}\big(c(\tau_1, \tau_2; \omega, \omega) + c(\tau_1, \tau_2; \omega, -\omega)\big) & \text{if } \tau_1 \neq \tau_2, \end{cases}$$

and

$$\big(\Im\sigma(\tau_1, \tau_2; \omega)\big)^2 := 0 \vee \begin{cases} 0 & \text{if } \tau_1 = \tau_2, \\ \frac{1}{2}\big(c(\tau_1, \tau_2; \omega, \omega) - c(\tau_1, \tau_2; \omega, -\omega)\big) & \text{if } \tau_1 \neq \tau_2 \end{cases}$$

are estimators for $\operatorname{Var}\big(\Re\hat{G}_{n,R}(\tau_1, \tau_2; \omega)\big)$ and $\operatorname{Var}\big(\Im\hat{G}_{n,R}(\tau_1, \tau_2; \omega)\big)$, respectively. Here

$$\begin{aligned} c(\tau_1, \tau_2; \omega, \omega') &:= \Big(\frac{2\pi}{n} W_n^j\Big)^2 \\ \times \Bigg[\sum_{s=1}^{n-1} & W_n\big(\omega - 2\pi s/n\big) W_n\big(\omega' - 2\pi s/n\big) \hat{G}_{n,R}(\tau_1, \tau_1; 2\pi s/n) \hat{G}_{n,R}(\tau_2, \tau_2; 2\pi s/n) \\ &+ \sum_{s=1}^{n-1} W_n\big(\omega - 2\pi s/n\big) W_n\big(\omega' + 2\pi s/n\big) \big|\hat{G}_{n,R}(\tau_1, \tau_1; 2\pi s/n)\big|^2 \Bigg], \end{aligned}$$

is an estimator for the covariance of $\tilde{G}_{n,R}(\tau_1, \tau_2; \omega)$ and $\tilde{G}_{n,R}(\tau_1, \tau_2; \omega')$; this follows from the representation in Theorem 3.6(iii) and Theorem 7.4.3 in Brillinger (1975). To motivate this approach, recall that, for any complex-valued random variable Z with complex conjugate $\bar{Z}$,

$$\operatorname{Var}(\Re Z) = \frac{1}{2}\big(\operatorname{Var}(Z) + \Re\operatorname{Cov}(Z, \bar{Z})\big); \quad \operatorname{Var}(\Im Z) = \frac{1}{2}\big(\operatorname{Var}(Z) - \Re\operatorname{Cov}(Z, \bar{Z})\big).$$

TABLE 1
Coverage frequencies for the confidence intervals $IC_n(\tau_1, \tau_2, \omega)$,
$n = 2^8$, $b_n = 0.4n^{-1/4}$, $1 - \alpha = 0.95$

Model	(τ_1, τ_2) ω/π	$(0.1, 0.1)$ $(\Re)$	$(0.1, 0.9)$ $(\Im)$	$(0.5, 0.5)$ $(\Re)$	$(0.1, 0.9)$ $(\Re)$	$(0.9, 0.9)$ $(\Re)$
(a) QAR(1) (5.16)	1/8	0.911	0.921	0.906	0.987	0.899
	1/4	0.934	0.917	0.920	0.979	0.910
	1/2	0.947	0.919	0.932	0.976	0.915
	3/4	0.946	0.918	0.927	0.979	0.916
	7/8	0.941	0.915	0.931	0.979	0.921
(b) AR(2) (5.17)	1/8	0.913	0.926	0.900	0.975	0.916
	1/4	0.935	0.925	0.917	0.967	0.940
	1/2	0.940	0.927	0.929	0.966	0.949
	3/4	0.939	0.924	0.928	0.969	0.947
	7/8	0.937	0.920	0.928	0.972	0.945
(c) ARCH(1) (5.18)	1/8	0.860	0.910	0.906	0.902	0.878
	1/4	0.872	0.905	0.922	0.909	0.887
	1/2	0.902	0.897	0.937	0.946	0.914
	3/4	0.906	0.894	0.934	0.959	0.924
	7/8	0.906	0.891	0.935	0.962	0.920

TABLE 2
Coverage frequencies for the confidence intervals $IC_n(\tau_1, \tau_2, \omega)$,
$n = 2^9$, $b_n = 0.4n^{-1/4}$, $1 - \alpha = 0.95$

Model	(τ_1, τ_2) ω/π	$(0.1, 0.1)$ $(\Re)$	$(0.1, 0.9)$ $(\Im)$	$(0.5, 0.5)$ $(\Re)$	$(0.1, 0.9)$ $(\Re)$	$(0.9, 0.9)$ $(\Re)$
(a) QAR(1)(5.16)	1/8	0.934	0.932	0.915	0.974	0.916
	1/4	0.953	0.933	0.931	0.968	0.925
	1/2	0.954	0.932	0.940	0.968	0.934
	3/4	0.952	0.926	0.939	0.973	0.932
	7/8	0.953	0.923	0.941	0.975	0.934
(b) AR(2) (5.17)	1/8	0.930	0.934	0.913	0.962	0.932
	1/4	0.950	0.932	0.928	0.956	0.951
	1/2	0.948	0.935	0.933	0.957	0.949
	3/4	0.951	0.932	0.936	0.964	0.952
	7/8	0.949	0.931	0.937	0.965	0.955
(c) ARCH(1) (5.18)	1/8	0.890	0.932	0.918	0.913	0.892
	1/4	0.900	0.924	0.938	0.917	0.903
	1/2	0.922	0.912	0.939	0.948	0.928
	3/4	0.926	0.913	0.944	0.957	0.934
	7/8	0.928	0.908	0.943	0.958	0.937

TABLE 3
Coverage frequencies for the confidence intervals $IC_n(\tau_1, \tau_2, \omega)$,
$n = 2^{10}$, $b_n = 0.4n^{-1/4}$, $1 - \alpha = 0.95$

Model	(τ_1, τ_2) ω/π	$(0.1, 0.1)$ $(\Re)$	$(0.1, 0.9)$ $(\Im)$	$(0.5, 0.5)$ $(\Re)$	$(0.1, 0.9)$ $(\Re)$	$(0.9, 0.9)$ $(\Re)$
(a) QAR(1) (5.16)	1/8	0.942	0.943	0.933	0.961	0.924
	1/4	0.959	0.938	0.941	0.963	0.929
	1/2	0.953	0.938	0.941	0.962	0.934
	3/4	0.954	0.935	0.941	0.967	0.933
	7/8	0.956	0.935	0.943	0.969	0.936
(b) AR(2) (5.17)	1/8	0.939	0.943	0.931	0.953	0.940
	1/4	0.954	0.939	0.942	0.954	0.952
	1/2	0.954	0.944	0.945	0.953	0.955
	3/4	0.950	0.937	0.942	0.956	0.954
	7/8	0.954	0.937	0.940	0.959	0.952
(c) ARCH(1) (5.18)	1/8	0.900	0.935	0.933	0.911	0.906
	1/4	0.901	0.930	0.945	0.916	0.908
	1/2	0.929	0.928	0.945	0.942	0.928
	3/4	0.941	0.916	0.948	0.954	0.937
	7/8	0.940	0.918	0.948	0.953	0.936

TABLE 4
Coverage frequencies for the confidence intervals $IC_n(\tau_1, \tau_2, \omega)$,
$n = 2^{11}$, $b_n = 0.4n^{-1/4}$, $1 - \alpha = 0.95$

Model	(τ_1, τ_2) ω/π	$(0.1, 0.1)$ $(\Re)$	$(0.1, 0.9)$ $(\Im)$	$(0.5, 0.5)$ $(\Re)$	$(0.1, 0.9)$ $(\Re)$	$(0.9, 0.9)$ $(\Re)$
(a) QAR(1) (5.16)	1/8	0.953	0.945	0.944	0.957	0.933
	1/4	0.957	0.943	0.945	0.961	0.932
	1/2	0.955	0.938	0.949	0.960	0.938
	3/4	0.952	0.938	0.946	0.963	0.939
	7/8	0.954	0.936	0.945	0.964	0.945
(b) AR(2) (5.17)	1/8	0.953	0.944	0.943	0.954	0.947
	1/4	0.954	0.944	0.945	0.953	0.956
	1/2	0.955	0.946	0.945	0.951	0.954
	3/4	0.954	0.947	0.940	0.954	0.957
	7/8	0.952	0.945	0.943	0.956	0.951
(c) ARCH(1) (5.18)	1/8	0.911	0.942	0.944	0.918	0.908
	1/4	0.918	0.937	0.950	0.926	0.917
	1/2	0.934	0.931	0.947	0.946	0.937
	3/4	0.944	0.931	0.949	0.954	0.943
	7/8	0.944	0.928	0.950	0.958	0.945

For $n \to \infty$, the estimated variances above converge to the asymptotic variance in Theorem 3.5. However, in small samples the more elaborate version considered here typically leads to better coverage probabilities.

In Tables 1–4, we report the simulated coverage frequencies associated with

$$\mathbb{P}\Big(\Re\mathfrak{f}_{q_{\tau_1},q_{\tau_2}}(\omega) \in IC_{1,n}(\tau_1,\tau_2,\omega)\Big) \text{ and } \mathbb{P}\Big(\Im\mathfrak{f}_{q_{\tau_1},q_{\tau_2}}(\omega) \in IC_{2,n}(\tau_1,\tau_2,\omega)\Big).$$

Inspection of Tables 1–4 reveals that, as n gets larger, the coverage frequencies converge to the confidence level $1-\alpha$. For models (5.16)–(5.17), those frequencies are quite close to $1-\alpha$ even for moderately large values of n. Due to boundary effects, the coverage frequencies for ω close to multiples of π are too low in all three models, but, as noted earlier, they improve as n increases. Finally, in models (5.16) and (5.18) and when n is not large, the confidence intervals involving extreme quantiles tend to cover less frequently. Again, the accuracy improves with increasing sample size.

6. Conclusions. Spectral analysis for the past fifty years has been a major tool in the analysis of time series. Being essentially covariance-based, however, classical L^2 spectral methods have obvious limitations—for instance (see Figures 1 and 2), they cannot discriminate between QAR or ARCH and white noise processes. Quantile-related spectral concepts have been proposed, which palliate those limitations. No asymptotic distributional results, however, have been available in the literature for the estimation of such concepts, which so far has precluded most practical applications.

In this paper, we provide (Theorem 3.5), in the very strong form of convergence to a Gaussian process of the smoothed copula rank-based periodogram process, such asymptotic results for the generalization (Dette et al. 2011, 2013) of the copula rank periodograms proposed by Hagemann (2011).

Being copula- or rank-based, our spectral concepts furthermore are invariant under monotone increasing continuous marginal transformations of the data, and are likely to enjoy appealing robustness features their traditional L^2 counterparts are severely lacking. Another application is in the asymptotic behavior of the spectra associated with more classical rank-based autocorrelation coefficients, such as the Spearman, Gini, or Blomqvist spectra.

Copula rank-based periodogram methods are improving over the classical ones both from the point of view of efficiency (detection of nonlinear features) and from the point of view of robustness. They are likely to be ideal tools for a large variety of problems of practical interest, such as change-point analysis, model diagnostics, or local stationary procedures—essentially, all problems covered in the traditional spectral context can be extended here, with the huge advantage that nonlinear features that cannot be accounted for by traditional methods can be analyzed via the new ones. This seems to offer most promising perspectives for future research.

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7. Proof of Theorem 3.6. The proof of Theorem 3.6 relies on a series of technical Lemmas; for the readers' convenience, we begin by giving a general overview of the main steps and the corresponding lemmas.

For all $n \in \mathbb{N}$, consider the stochastic process

$$\hat{H}_{n,U}(\tau_1, \tau_2; \omega) := \sqrt{nb_n}\big(\hat{G}_{n,U}(\tau_1, \tau_2; \omega) - \mathbb{E}\hat{G}_{n,U}(\tau_1, \tau_2; \omega)\big), \tag{7.22}$$

indexed by $(\tau_1, \tau_2) \in [0, 1]^2$ and $\omega \in \mathbb{R}$; for $a = (a_1, a_2) \in [0, 1]^2$, write $\hat{H}_n(a; \omega)$ for $\hat{H}_{n,U}(a_1, a_2; \omega)$.

The key step in the process of establishing parts (i) and (iii) of Theorem 3.6 is a uniform bound on the increments of the process $\hat{H}_{n,U}$. That bound is required, for example, when showing the stochastic equicontinuity of $\hat{H}_n(a; \omega) - \hat{H}_n(b; \omega)$. We derive such a bound by a restricted chaining technique, which is described in Lemma 7.1. The application of Lemma 7.1 requires two ingredients. First, we need a general bound, uniform in a and b, on the moments of $\hat{H}_n(a; \omega) - \hat{H}_n(b; \omega)$. Such a bound is derived in Lemma 7.2. Second, we need a sharper bound on the increments $\hat{H}_n(a; \omega) - \hat{H}_n(b; \omega)$ when a and b are "sufficiently close". We provide this result in Lemma 7.7.

Lemma 7.2 is a very general result, relying on an abstract condition on the cumulants of discrete Fourier transforms of certain indicator functions, see (7.26). The link between assumption (C) and (7.26) is established in Lemma 7.4.

Finally, the proof of part (ii) of Theorem 3.6 follows by a series of uniform generalizations of results from Brillinger (1975), the details of which are provided in the Online Appendix [Lemmas 8.1-8.5].

7.1. *Proof of Part (i) of Theorem 3.6.* In view of Theorems 1.5.4 and 1.5.7 in van der Vaart and Wellner (1996), it is sufficient to prove the following two claims:

(i1) convergence of the finite-dimensional distributions of the process (7.22), that is,

$$\big(\hat{H}_n(a_{1j}, a_{2j}; \omega_j)\big)_{j=1,\dots,k} \xrightarrow{d} \big(H(a_{1j}, a_{2j}; \omega_j)\big)_{j=1,\dots,k} \tag{7.23}$$

for any $(a_{1j}, a_{2j}, \omega_j) \in [0, 1]^2 \times \mathbb{R}$, $j = 1, \dots, k$ and $k \in \mathbb{N}$;

(i2) stochastic equicontinuity: for any $x > 0$ and any $\omega \in \mathbb{R}$,

$$\lim_{\delta \downarrow 0} \limsup_{n\to\infty} \mathbb{P}\Big(\sup_{\substack{a,b\in[0,1]^2\\ \|a-b\|_1\le\delta}} |\hat{H}_n(a; \omega) - \hat{H}_n(b; \omega)| > x\Big) = 0. \tag{7.24}$$

Note indeed that (7.24) implies stochastic equicontinuity of both the real part $\big(\Re\hat{H}_n(a;\omega)\big)_{a\in[0,1]^2}$ and the imaginary part $\big(\Im\hat{H}_n(a;\omega)\big)_{a\in[0,1]^2}$ of $\hat{H}_n$.

First consider (i1). Observe that $\hat{G}_{n,U}(\tau_1,\tau_2;\omega)$ is the traditional smoothed periodogram estimator [see Chapter 7.1 in Brillinger (1975)] of the cross-spectrum of the *clipped processes* $(I\{F(X_t)\le\tau_1\})_{t\in\mathbb{Z}}$ and $(I\{F(X_t)\le\tau_2\})_{t\in\mathbb{Z}}$. Thus, (7.23) is an immediate corollary of Theorem 7.4.4 in Brillinger (1975). The limiting first and second moment structures are given by Theorem 7.4.1 and Corollary 7.4.3 in Brillinger (1975). This implies the desired convergence (7.23) of finite-dimensional distributions. Note that, by condition (C), the summability condition required for the three theorems to hold [Assumption 2.6.2(ℓ), for every ℓ; cf. Brillinger (1975)] is implied by Assumption (A2).

Turning to (i2), in the notation from van der Vaart and Wellner (1996), p. 95, put $\Psi(x):=x^6$: the Orlicz norm $\|X\|_\Psi=\inf\{C>0:\mathbb{E}\Psi(|X|/C)\le 1\}$ coincides with the L_6 norm $\|X\|_6=(\mathbb{E}|X|^6)^{1/6}$. Therefore, by Lemma 7.2 and Lemma 7.4, we have, for any $\kappa>0$ and sufficiently small $\|a-b\|_1$,

$$\|\hat{H}_n(a;\omega)-\hat{H}_n(b;\omega)\|_\Psi\le K\Big(\frac{\|a-b\|_1^\kappa}{(nb_n)^2}+\frac{\|a-b\|_1^{2\kappa}}{nb_n}+\|a-b\|_1^{3\kappa}\Big)^{1/6}.$$

It follows that, for all a,b with $\|a-b\|_1$ sufficiently small and $\|a-b\|_1\ge(nb_n)^{-1/\gamma}$ and all $\gamma\in(0,1)$ such that $\gamma<\kappa$,

$$\begin{aligned}\|\hat{H}_n(a;\omega)-\hat{H}_n(b;\omega)\|_\Psi &\le K\Big(\|a-b\|_1^{\kappa+2\gamma}+\|a-b\|_1^{2\kappa+\gamma}+\|a-b\|_1^{3\kappa}\Big)^{1/6}\\ &\le \bar{K}\|a-b\|_1^{\gamma/2}.\end{aligned}$$

Note that $\|a-b\|_1\ge(nb_n)^{-1/\gamma}$ iff $d(a,b):=\|a-b\|_1^{\gamma/2}\ge(nb_n)^{-1/2}=:\bar{\eta}_n/2$.

Denoting by $D(\varepsilon,d)$ the *packing number* of $([0,1]^2,d)$ [cf. van der Vaart and Wellner (1996), p. 98], we have $D(\varepsilon,d)\asymp\varepsilon^{-4/\gamma}$. Therefore, by Lemma 7.1, for all $x,\delta>0$ and $\eta\ge\bar{\eta}_n$,

$$\begin{aligned}&\mathbb{P}\Big(\sup_{\|a-b\|_1\le\delta^{2/\gamma}}|\hat{H}_n(a;\omega)-\hat{H}_n(b;\omega)|>x\Big)\\ &=\mathbb{P}\Big(\sup_{d(a,b)\le\delta}|\hat{H}_n(a;\omega)-\hat{H}_n(b;\omega)|>x\Big)\\ &\le\left[\frac{8\tilde{K}}{x}\left(\int_{\bar{\eta}_n/2}^{\eta}\epsilon^{-2/(3\gamma)}\mathrm{d}\epsilon+(\delta+2\bar{\eta}_n)\eta^{-4/(3\gamma)}\right)\right]^6\\ &\qquad\qquad+\mathbb{P}\Big(\sup_{d(a,b)\le\bar{\eta}_n}|\hat{H}_n(a;\omega)-\hat{H}_n(b;\omega)|>x/4\Big).\end{aligned}$$

Now choose $1 > \gamma > 2/3$. Letting n tend to infinity, the second term tends to zero by Lemma 7.7 since, by construction, $1/\gamma > 1$ and

$$d(a,b) \leq \bar{\eta}_n \quad \text{iff} \quad \|a-b\|_1 \leq 2^{2/\gamma}(nb_n)^{-1/\gamma}.$$

All together, this implies

$$\lim_{\delta\downarrow 0}\limsup_{n\to\infty}\mathbb{P}\Big(\sup_{d(a,b)\leq\delta}|\hat{H}_n(a;\omega)-\hat{H}_n(b;\omega)|>x\Big) \leq \left[\frac{8\tilde{K}}{x}\int_0^{\eta}\epsilon^{-2/(3\gamma)}\mathrm{d}\epsilon\right]^6,$$

for every $x, \eta > 0$; the claim follows, since the integral in the right-hand side can be made arbitrarily small by choosing η accordingly.

7.2. *Proof of Part (ii) of Theorem 3.6.* Essentially, this part of Theorem 3.6 is a uniform version of Theorem 5.6.2 in Brillinger (1975) in the present setting of Laplace spectra. The proof is based on a series of uniform versions of results from Brillinger (1975); details are provided in the online supplement [see in particular Lemma 8.5].

7.3. *Proof of Part (iii) of Theorem 3.6.* It follows from the continuity of F that the ranks of the random variables $X_0, ..., X_{n-1}$ and $F(X_0), ..., F(X_{n-1})$ coincide almost surely. Thus, without loss of generality, we can assume that the estimator is computed from the unobservable data $F(X_0), ..., F(X_{n-1})$. In particular, this implies that we can assume the marginals to be uniform.

Denote by $\hat{F}_n^{-1}(\tau) := \inf\{x : F(x) \geq \tau\}$ the generalized inverse of $\hat{F}_n$ and let $\inf\emptyset := 0$. Elementary computation shows that, for any $k \in \mathbb{N}$,
(7.25)

$$\sup_{\omega\in\mathbb{R}}\sup_{\tau\in[0,1]}\left|d_{n,R}^{\tau}(\omega)-d_{n,U}^{\hat{F}_n^{-1}(\tau)}(\omega)\right| \leq n\sup_{\tau\in[0,1]}|\hat{F}_n(\tau)-\hat{F}_n(\tau-)| = O_P(n^{1/2k})$$

where $\hat{F}_n(\tau-) := \lim_{\xi\uparrow 0}\hat{F}_n(\tau-\xi)$ and the O_P-bound in the above equation follows from Lemma 8.6. By the definition of $\hat{G}_{n,R}$ and arguments similar to the ones used in the proof of Lemma 7.7, it follows that

$$\sup_{\omega\in\mathbb{R}}\sup_{\tau_1,\tau_2\in[0,1]}|\hat{G}_{n,R}(\tau_1,\tau_2;\omega)-\hat{G}_{n,U}(\hat{F}_n^{-1}(\tau_1),\hat{F}_n^{-1}(\tau_2);\omega)| = o_P(1).$$

It therefore suffices to bound the differences

$$\sup_{\tau_1,\tau_2\in[0,1]}|\hat{G}_{n,U}(\tau_1,\tau_2;\omega)-\hat{G}_{n,U}(\hat{F}_n^{-1}(\tau_1),\hat{F}_n^{-1}(\tau_2);\omega)|$$

pointwise and uniformly in ω.

In what follows, we give a detailed proof of the statement for fixed $\omega \in \mathbb{R}$ and sketch the arguments needed for the proof of the uniform result.

By (7.22) we have, for any $x > 0$ and δ_n with

$$n^{-1/2} \ll \delta_n = o(n^{-1/2} b_n^{-1/2} (\log n)^{-d}),$$

where d is the constant from Lemma 7.3 corresponding to $j = k$,

$$\begin{aligned}
P^n(\omega) :=& \mathbb{P}\Big(\sup_{\tau_1,\tau_2 \in [0,1]} |\hat{G}_{n,U}(\hat{F}_n^{-1}(\tau_1), \hat{F}_n^{-1}(\tau_2); \omega) - \hat{G}_{n,U}(\tau_1,\tau_2;\omega)| > x((nb_n)^{-1/2} + b_n^k)\Big) \\
\leq& \mathbb{P}\Big(\sup_{\tau_1,\tau_2 \in [0,1]} \sup_{\substack{\|(u,v)-(\tau_1,\tau_2)\|_\infty \\ \leq \sup_{\tau\in[0,1]} |\hat{F}_n^{-1}(\tau)-\tau|}} |\hat{G}_{n,U}(u,v;\omega) - \hat{G}_{n,U}(\tau_1,\tau_2;\omega)| > x((nb_n)^{-1/2} + b_n^k)\Big) \\
\leq& \mathbb{P}\Big(\sup_{\tau_1,\tau_2 \in [0,1]} \sup_{\substack{|u-\tau_1|\leq\delta_n \\ |v-\tau_2|\leq\delta_n}} |\hat{G}_{n,U}(u,v;\omega) - \hat{G}_{n,U}(\tau_1,\tau_2;\omega)| > x((nb_n)^{-1/2} + b_n^k), \\
& \qquad \sup_{\tau\in[0,1]} |\hat{F}_n^{-1}(\tau) - \tau| \leq \delta_n \Big) + \mathbb{P}\Big(\sup_{\tau\in[0,1]} |\hat{F}_n^{-1}(\tau) - \tau| > \delta_n \Big) \\
=& P_1^n + P_2^n, \quad \text{say.}
\end{aligned}$$

It follows from Lemma 7.5 that P_2^n is $o(1)$. As for P_1^n, it is bounded by

$$\begin{aligned}
&\mathbb{P}\Big(\sup_{\tau_1,\tau_2 \in [0,1]} \sup_{\substack{|u-\tau_1|\leq\delta_n \\ |v-\tau_2|\leq\delta_n}} |\hat{H}_{n,U}(u,v;\omega) - \hat{H}_{n,U}(\tau_1,\tau_2;\omega)| > (1 + (nb_n)^{1/2} b_n^k) x/2 \Big) \\
&\quad + I\Big\{ \sup_{\tau_1,\tau_2 \in [0,1]} \sup_{\substack{|u-\tau_1|\leq\delta_n \\ |v-\tau_2|\leq\delta_n}} |\mathbb{E}\hat{G}_{n,U}(u,v;\omega) - \mathbb{E}\hat{G}_{n,U}(\tau_1,\tau_2;\omega)| > ((nb_n)^{-1/2} + b_n^k) x/2 \Big\}
\end{aligned}$$

where the first term tends to zero in view of (7.24). To see that the indicator in the second term also is $o(1)$, note that

$$\begin{aligned}
& \sup_{\tau_1,\tau_2 \in [0,1]} \sup_{\substack{|u-\tau_1|\leq\delta_n \\ |v-\tau_2|\leq\delta_n}} |\mathbb{E}\hat{G}_{n,U}(u,v;\omega) - \mathbb{E}\hat{G}_{n,U}(\tau_1,\tau_2;\omega)| \\
\leq & \sup_{\tau_1,\tau_2 \in [0,1]} \sup_{\substack{|u-\tau_1|\leq\delta_n \\ |v-\tau_2|\leq\delta_n}} |\mathbb{E}\hat{G}_{n,U}(u,v;\omega) - \mathfrak{f}_{q_u,q_v}(\omega) - B_n^{(k)}(u,v,\omega)| \\
& + \sup_{\tau_1,\tau_2 \in [0,1]} \sup_{\substack{|u-\tau_1|\leq\delta_n \\ |v-\tau_2|\leq\delta_n}} |B_n^{(k)}(\tau_1,\tau_2,\omega) + \mathfrak{f}_{q_{\tau_1},q_{\tau_2}}(\omega) - \mathbb{E}\hat{G}_{n,U}(\tau_1,\tau_2;\omega)| \\
& + \sup_{\tau_1,\tau_2 \in [0,1]} \sup_{\substack{|u-\tau_1|\leq\delta_n \\ |v-\tau_2|\leq\delta_n}} |\mathfrak{f}_{q_u,q_v}(\omega) + B_n^{(k)}(u,v,\omega) - \mathfrak{f}_{q_{\tau_1},q_{\tau_2}}(\omega) - B_n^{(k)}(\tau_1,\tau_2,\omega)| \\
= & o(n^{-1/2} b_n^{-1/2} + b_n^k) + O(\delta_n (1 + |\log \delta_n|)^d),
\end{aligned}$$

where d still is the constant from Lemma 7.3 corresponding to $j = k$. Here, we have applied part (ii) of Theorem 3.6 to bound the first two terms and Lemma 7.3 for the third one. For any ω, thus, $P^n(\omega) = o(1)$, which establishes the pointwise version of the claim.

We now turn to the uniformity (with respect to ω) issue. For an arbitrary $y_n > 0$, similar arguments as above yield, with the same δ_n,

$$\begin{aligned}
&\mathbb{P}\Big(\sup_{\omega\in\mathbb{R}}\sup_{\tau_1,\tau_2\in[0,1]}|\hat{G}_{n,R}(\tau_1,\tau_2;\omega)-\hat{G}_{n,U}(\tau_1,\tau_2;\omega)|>y_n\Big)\\
&\le \mathbb{P}\Big(\sup_{\omega\in\mathbb{R}}\sup_{\tau_1,\tau_2\in[0,1]}\sup_{\substack{|u-\tau_1|\le\delta_n\\|v-\tau_2|\le\delta_n}}|\hat{H}_{n,U}(u,v;\omega)-\hat{H}_{n,U}(\tau_1,\tau_2;\omega)|>(nb_n)^{1/2}y_n/2\Big)\\
&+ I\Big\{\sup_{\omega\in\mathbb{R}}\sup_{\tau_1,\tau_2\in[0,1]}\sup_{\substack{|u-\tau_1|\le\delta_n\\|v-\tau_2|\le\delta_n}}|\mathbb{E}\hat{G}_{n,U}(u,v;\omega)-\mathbb{E}\hat{G}_{n,U}(\tau_1,\tau_2;\omega)|>y_n/2\Big\}+o(1).
\end{aligned}$$

That the indicator in the latter expression is $o(1)$ follows by the same arguments as above [note that Lemma 7.3 and the statement of part (ii) both hold uniformly in $\omega \in \mathbb{R}$]. To bound the probability term, observe that by Lemma 7.6, $\sup_{\tau_1,\tau_2}\sup_{j=1,\dots,n}|I_{n,U}^{\tau_1,\tau_2}(2\pi j/n)|$ is $O_P(n^{2/K})$ for any $K > 0$. Moreover, the uniform Lipschitz continuity of W implies that W_n also is uniformly Lipschitz continuous with constant of order $O(b_n^{-2})$. Combining those facts with Lemma 7.3 and the assumptions on b_n, we obtain

$$\sup_{\substack{\omega_1,\omega_2\in\mathbb{R}\\|\omega_1-\omega_2|\le n^{-3}}}\sup_{\tau_1,\tau_2\in[0,1]}|\hat{H}_{n,U}(\tau_1,\tau_2;\omega_1)-\hat{H}_{n,U}(\tau_1,\tau_2;\omega_2)|=o_P(1).$$

By periodicity of $\hat{H}_{n,U}$ in the argument ω, it thus remains to show that

$$\max_{\omega=0,2\pi n^{-3},\dots,2\pi}\sup_{\tau_1,\tau_2\in[0,1]}\sup_{\substack{|u-\tau_1|\le\delta_n\\|v-\tau_2|\le\delta_n}}|\hat{H}_{n,U}(u,v;\omega)-\hat{H}_{n,U}(\tau_1,\tau_2;\omega)|=o_P(1).$$

Lemmas 7.1 and 7.7 entail the existence of a random variable $S(\omega)$ such that, for any $\omega \in \mathbb{R}$,

$$\sup_{\tau_1,\tau_2\in[0,1]}\sup_{\substack{|u-\tau_1|\le\delta_n\\|v-\tau_2|\le\delta_n}}|\hat{H}_{n,U}(u,v;\omega)-\hat{H}_{n,U}(\tau_1,\tau_2;\omega)|\le|S(\omega)|+R_n(\omega)$$

where $\sup_{\omega\in\mathbb{R}}|R_n(\omega)| = o_P(1)$ and

$$\max_{\omega=0,2\pi n^{-3}\dots,2\pi}\mathbb{E}[|S^{2L}(\omega)|]\le K_L^{2L}\bigg(\int_0^\eta\epsilon^{-4/(2L\gamma)}\mathrm{d}\epsilon+(\delta_n^{\gamma/2}+2(nb_n)^{-1/2})\eta^{-8/(2L\gamma)}\bigg)^{2L}$$

for any $0 < \gamma < 1, L \in \mathbb{N}$, $0 < \eta < \delta_n$, and a constant K_L depending on L only. For appropriate choice of L and γ, this latter bound is $o(n^{-3})$; since the maximum is over a set with $O(n^3)$ elements, this completes the proof of part (iii). $\square$

7.4. *Details for the proof of Parts (i) and (iii) of Theorem 3.6.* This section contains the main Lemmas used in Sections 7.1 and 7.3 above. We use the notation introduced at the beginning of the proof of Theorem 3.6. The proofs of the results presented here can be found in the online appendix [Section 8.3].

For the statement of the first result, recall that, for any non-decreasing, convex function $\Psi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\Psi(0) = 0$ the *Orlicz norm* of a real-valued random variable Z is defined as [see e.g. van der Vaart and Wellner (1996), Chapter 2.2]

$$\|Z\|_\Psi = \inf\Big\{C > 0 : \mathbb{E}\Psi\Big(|Z|/C\Big) \le 1\Big\}.$$

LEMMA 7.1. *Let $\{\mathbb{G}_t : t \in T\}$ be a separable stochastic process with $\|\mathbb{G}_s - \mathbb{G}_t\|_\Psi \le Cd(s,t)$ for all s,t with $d(s,t) \ge \bar\eta/2 \ge 0$. Denote by $D(\epsilon, d)$ the packing number of the metric space (T,d). Then, for any $\delta > 0$, $\eta \ge \bar\eta$, there exists a random variable S_1 and a constant $K < \infty$ such that*

$$\begin{aligned} \sup_{d(s,t)\le\delta} |\mathbb{G}_s - \mathbb{G}_t| &\le S_1 + 2 \sup_{d(s,t)\le\bar\eta, t\in\tilde T} |\mathbb{G}_s - \mathbb{G}_t| \quad \textit{and} \\ \|S_1\|_\Psi &\le K\Big[\int_{\bar\eta/2}^{\eta} \Psi^{-1}\big(D(\epsilon,d)\big)\,d\epsilon + (\delta + 2\bar\eta)\Psi^{-1}\big(D^2(\eta,d)\big)\Big], \end{aligned}$$

where the set $\tilde T$ contains at most $D(\bar\eta, d)$ points. In particular, by Markov's inequality [cf. van der Vaart and Wellner (1996), p. 96],

$$\mathbb{P}\Big(|S_1| > x\Big) \le \Big(\Psi\Big(x\big[8K\big(\int_{\bar\eta/2}^{\eta} \Psi^{-1}\big(D(\epsilon,d)\big)\,d\epsilon + (\delta+2\bar\eta)\Psi^{-1}\big(D^2(\eta,d)\big)\big)\big]^{-1}\Big)\Big)^{-1}.$$

for any $x > 0$.

LEMMA 7.2. *Let $X_0, ..., X_{n-1}$ be the finite realization of a strictly stationary process with $X_0 \sim U[0,1]$, and let (W) hold. For $x = (x_1, x_2)$ let $\hat H_n(x;\omega) := \sqrt{nb_n}(\hat G_n(x_1,x_2;\omega) - \mathbb{E}[\hat G_n(x_1,x_2;\omega)])$. For any Borel set A, define*

$$d_n^A(\omega) := \sum_{t=0}^{n-1} I\{X_t \in A\}\mathrm{e}^{-\mathrm{i}t\omega}.$$

Assume that, for $p = 1, \ldots, P$*, there exist a constant* C *and a function* $g : \mathbb{R}^+ \to \mathbb{R}^+$*, both independent of* $\omega_1, ..., \omega_p \in \mathbb{R}, n$ *and* $A_1, ..., A_p$*, such that*

$$\Big|\mathrm{cum}(d_n^{A_1}(\omega_1), \ldots, d_n^{A_p}(\omega_p))\Big| \le C\Big(\Big|\Delta_n\Big(\sum_{i=1}^{p} \omega_i\Big)\Big| + 1\Big) g(\varepsilon) \tag{7.26}$$

for any Borel sets $A_1, \ldots, A_p$ *with* $\min_j \mathbb{P}(X_0 \in A_j) \le \varepsilon$*. Then, there exists a constant* K *(depending on* C, L, g *only) such that*

$$\sup_{\omega \in \mathbb{R}} \sup_{\|a-b\|_1 \le \varepsilon} \mathbb{E}|\hat{H}_n(a;\omega) - \hat{H}_n(b;\omega)|^{2L} \le K \sum_{\ell=0}^{L-1} \frac{g^{L-\ell}(\varepsilon)}{(nb_n)^\ell}$$

for all ε *with* $g(\varepsilon) < 1$ *and all* $L = 1, \ldots, P$*.*

LEMMA 7.3. *Under the assumptions of Theorem 3.5, the derivative* $(\tau_1, \tau_2) \mapsto \dfrac{d^j}{d\omega^j} \mathfrak{f}_{q_{\tau_1}, q_{\tau_2}}(\omega)$ *exists and satisfies, for any* $j \in \mathbb{N}_0$ *and some constants* C, d *that are independent of* $a = (a_1, a_2), b = (b_1, b_2)$ *but may depend on* j*,*

$$\sup_{\omega \in \mathbb{R}} \Big| \frac{d^j}{d\omega^j} \mathfrak{f}_{q_{a_1}, q_{a_2}}(\omega) - \frac{d^j}{d\omega^j} \mathfrak{f}_{q_{b_1}, q_{b_2}}(\omega) \Big| \le C\|a-b\|_1 (1 + |\log \|a-b\|_1|)^d.$$

LEMMA 7.4. *Let the strictly stationary process* $(X_t)_{t \in \mathbb{Z}}$ *satisfy Assumption (C). For any Borel set* A*, define*

$$d_n^A(\omega) := \sum_{t=0}^{n-1} I\{X_t \in A\} \mathrm{e}^{-\mathrm{i}t\omega}.$$

Let $A_1, \ldots, A_p \subset [0,1]$ *be intervals, and let* $\varepsilon := \min_{j=1,\ldots,p} \mathbb{P}(X_0 \in A_j)$*. Then, for any* p*-tuple* $\omega_1, ..., \omega_p \in \mathbb{R}$*,*

$$\Big|\mathrm{cum}(d_n^{A_1}(\omega_1), \ldots, d_n^{A_p}(\omega_p))\Big| \le C\Big(\Big|\Delta_n\Big(\sum_{i=1}^{p} \omega_i\Big)\Big| + 1\Big) \varepsilon(|\log \varepsilon| + 1)^d,$$

where $\Delta_n(\lambda) := \sum_{t=0}^{n-1} \mathrm{e}^{\mathrm{i}t\lambda}$ *and the constants* C, d *depend only on* K, p*, and* ρ *[with* ρ *from condition (C)].*

LEMMA 7.5. *Let* $X_0, \ldots, X_{n-1}$ *be the finite realization of a strictly stationary process satisfying (C) and such that* $X_0 \sim U[0,1]$*. Then,*

$$\sup_{\tau \in [0,1]} |\hat{F}_n^{-1}(\tau) - \tau| = O_P(n^{-1/2}).$$

LEMMA 7.6. *Let the strictly stationary process $(X_t)_{t\in\mathbb{Z}}$ satisfy Assumption (C); assume moreover that $X_0 \sim U[0,1]$. For any $y \in [0,1]$, define*

$$d_n^y(\omega) := \sum_{t=0}^{n-1} I\{X_t \leq y\} e^{-\mathrm{i}\omega t}.$$

Then, for any $k \in \mathbb{N}$,

$$\sup_{\omega\in\mathcal{F}_n} \sup_{y\in[0,1]} |d_n^y(\omega)| = O_P(n^{1/2+1/k}).$$

LEMMA 7.7. *Under the assumptions of Theorem 3.6, let δ_n be a sequence of non-negative real numbers. Assume that there exists $\gamma \in (0,1)$, such that $\delta_n = O((nb_n)^{-1/\gamma})$. Then,*

$$\sup_{\omega\in\mathbb{R}} \sup_{\substack{u,v\in[0,1]^2 \\ \|u-v\|_1\leq\delta_n}} |\hat{H}_n(u;\omega) - \hat{H}_n(v;\omega)| = o_P(1).$$

8. Online Appendix.

8.1. *Details for the Proof of Part (ii) of Theorem 3.6.* For $p \geq 2$, $k_1, \dots, k_{p-1} \in \mathbb{Z}$ and $x_1, \dots, x_p \in \mathbb{R}$, consider the *Laplace cumulant* of order p

$$\begin{aligned}\gamma^{x_1,\dots,x_p}_{k_1,\dots,k_{p-1}} &:= \operatorname{cum}\big[I\{X_{k_1} \leq x_1\}, I\{X_{k_2} \leq x_2\}, \dots, I\{X_0 \leq x_p\}\big] \\ &= \sum_{\{\nu_1,\dots,\nu_R\}} (-1)^{R-1}(R-1)! \prod_{j=1}^{R} \mathbb{P}\Big(X_{k_i} \leq x_i \,:\, i \in \nu_j\Big), \quad k_p := 0,\end{aligned}$$

where the summation runs over all partitions $\{\nu_1, \dots, \nu_R\}$ of $\{1, \dots, p\}$. All results in this part of the Appendix are established under the following condition on Laplace cumulants:

(CS) Let $p \geq 2, \delta > 0$. There exists a non-increasing function $a_p : \mathbb{N} \to \mathbb{R}^+$ such that

$$\sup_{x_1,\dots,x_p} |\gamma^{x_1,\dots,x_p}_{k_1,\dots,k_{p-1}}| \leq a_p\big(\max_j |k_j|\big) \quad \text{and} \quad \sum_{k \in \mathbb{N}} k^\delta a_p(k) < \infty.$$

This condition follows from Assumption (C) but is in fact somewhat weaker.

Note that under assumption (CS) the following quantity, which we call *Laplace spectrum of order* p, exists as soon as $p - 1 < \delta$

$$\mathfrak{f}_{x_1,\dots,x_p}(\omega_1, \dots, \omega_{p-1}) := \frac{1}{(2\pi)^{p-1}} \sum_{k_1,\dots,k_{p-1}=-\infty}^{\infty} \gamma^{x_1,\dots,x_p}_{k_1,\dots,k_{p-1}} \mathrm{e}^{-\mathrm{i}(\omega_1 k_1 + \dots + \omega_{p-1}k_{p-1})}.$$

The existence of $\mathfrak{f}_{x_1,\dots,x_p}(\omega_1, \dots, \omega_{p-1})$ follows, since under (CS)

$$\begin{aligned}\Big|\sum_{k_1,\dots,k_{p-1}=-\infty}^{\infty} \gamma^{x_1,\dots,x_p}_{k_1,\dots,k_{p-1}} \mathrm{e}^{-\mathrm{i}(\omega_1 k_1 + \dots + \omega_{p-1}k_{p-1})}\Big| &\leq \sum_{k_1,\dots,k_{p-1}=-\infty}^{\infty} a_p\big(\max_j |k_j|\big) \\ \leq \quad a_p(0) + \sum_{m=1}^{\infty} a_p(m) \big|\{k_1, ..., k_{p-1} : \max_j |k_j| = m\}\big| &< \infty,\end{aligned}$$

since $\big|\{k_1, ..., k_{p-1} : \max_j |k_j| = m\}\big| = O(m^{p-2})$.

The main result in this section is Lemma 8.5, giving an asymptotic expansion of the expectation $\mathbb{E}[\hat{G}_{n,U}(\tau_1, \tau_2; \omega)]$ that holds uniformly in τ_1, τ_2, and ω. Essentially, it is a uniform version, for Laplace spectra, of Theorem 5.6.2 in Brillinger (1975). The proof is based on a series of uniform reinforcements of results from Brillinger (1975).

We first prove the following version of Lemma P4.1 in Brillinger (1975) in the special case where no tapering is applied, so that the constant can be chosen as 2.

LEMMA 8.1. *Let* $h_n(u) := I\{0 \leq u < n\}$ *and* $\Delta_n(\lambda) := \sum_{t=0}^{n-1} \mathrm{e}^{-\mathrm{i}\lambda t}$. *Then, for any* $K \in \mathbb{N}, K \geq 2$, $u_1, \ldots, u_{K-1} \in \mathbb{Z}$ *and* $\lambda \in \mathbb{R}$,

$$\Big|\sum_{t=0}^{n-1} h_n(t+u_1)\cdots h_n(t+u_{K-1})\mathrm{e}^{-\mathrm{i}\lambda t} - \Delta_n(\lambda)\Big| \leq 2(|u_1|+\ldots+|u_{K-1}|). \tag{8.1}$$

Proof. The left-hand side in (8.1) is bounded by

$$\sum_{j=1}^{K-1}\sum_{t=0}^{n-1} |h_n(t+u_j) - h_n(t)| \leq 2\sum_{j=1}^{K-1} |u_j|. \qquad \square$$

Next, we extend Lemma P4.2, still from Brillinger (1975). Define

$$\begin{aligned} E_n(\tau_1,\ldots,\tau_K,\lambda_1,\ldots,\lambda_K) &:= \mathrm{cum}(d_{n,U}^{\tau_1}(\lambda_1),\ldots,d_{n,U}^{\tau_K}(\lambda_K)) \\ &\quad - \Delta_n\Big(\sum_{j=1}^K \lambda_j\Big) \sum_{|u_1|<n} \cdots \sum_{|u_{K-1}|<n} \gamma_{u_1,\ldots,u_{K-1}}^{q_{\tau_1},\ldots,q_{\tau_K}} \exp\Big(-\mathrm{i}\sum_{j=1}^{K-1}\lambda_j u_j\Big). \end{aligned}$$

LEMMA 8.2. *Under (CS) with* $p = K$ *and* $\delta > K$,

$$\begin{aligned} &\Big|E_n(\tau_1,\ldots,\tau_K,\lambda_1,\ldots,\lambda_K)\Big| \\ &\quad \leq 2\sum_{|u_1|<n}\cdots\sum_{|u_{K-1}|<n} (|u_1|+\ldots+|u_{K-1}|)\Big|\gamma_{u_1,\ldots,u_{K-1}}^{q_{\tau_1},\ldots,q_{\tau_K}}\Big| \leq 2(K-1)C_K, \end{aligned}$$

for all $\tau_1,\ldots,\tau_K \in [0,1]$ *and* $\lambda_1,\ldots,\lambda_K \in \mathbb{R}$, *where* C_K *does not depend on* λ_i, q_{τ_i}.

Proof. By multilinearity of the cumulants, we have

$$\begin{aligned} &\mathrm{cum}(d_{n,U}^{\tau_1}(\lambda_1),\ldots,d_{n,U}^{\tau_K}(\lambda_K)) \\ &= \sum_{t_1=0}^{n-1}\cdots\sum_{t_K=0}^{n-1} h_n(t_1)\ldots h_n(t_K)\exp\Big(-\mathrm{i}\sum_{j=1}^K \lambda_j t_j\Big)\gamma_{t_1-t_K,\ldots,t_{K-1}-t_K}^{q_{\tau_1},\ldots,q_{\tau_K}} \\ &= \sum_{|u_1|<n}\cdots\sum_{|u_{K-1}|<n} \exp\Big(-\mathrm{i}\sum_{j=1}^K \lambda_j u_j\Big)\gamma_{u_1,\ldots,u_{K-1}}^{q_{\tau_1},\ldots,q_{\tau_K}} \\ &\qquad \times \sum_{t=0}^{n-1} h_n(t+u_1)\cdots h_n(t+u_{K-1})h_n(t)\exp\Big(-\mathrm{i}\sum_{j=1}^K \lambda_j t\Big). \end{aligned}$$

Therefore,

$$E_n(\tau_1,\dots,\tau_K,\lambda_1,\dots,\lambda_K) = \sum_{|u_1|<n}\cdots\sum_{|u_{K-1}|<n} \exp\Big(-\mathrm{i}\sum_{j=1}^{K}\lambda_j u_j\Big)\gamma^{q_{\tau_1},\dots,q_{\tau_K}}_{u_1,\dots,u_{K-1}}$$
$$\times\Bigg(\sum_{t=0}^{n-1} h_n(t+u_1)\cdots h_n(t+u_{K-1})h_n(t)\exp\Big(-\mathrm{i}\sum_{j=1}^{K}\lambda_j t\Big) - \Delta_n\Big(\sum_{j=1}^{K}\lambda_j\Big)\Bigg).$$

Applying the triangle inequality and Lemma 8.1, and taking condition (CS) into account, completes the proof. □

Finally, we establish a uniform version of Theorem 4.3.2 in Brillinger (1975). Recalling the definition of $d^{\tau}_{n,U}$ given in (2.6), let

$$\begin{aligned}\varepsilon_n(\tau_1,\dots,\tau_K,\lambda_1,\dots,\lambda_K) &:= \mathrm{cum}(d^{\tau_1}_{n,U}(\lambda_1),\dots,d^{\tau_K}_{n,U}(\lambda_K))\\ &\quad - (2\pi)^{K-1}\Delta_n\Big(\sum_{j=1}^{k}\lambda_j\Big)\mathfrak{f}_{q_{\tau_1},\dots,q_{\tau_K}}(\lambda_1,\dots,\lambda_K).\end{aligned}$$

THEOREM 8.3. *If (CS) holds with $p = K$ and $\delta > K+1$, then*

$$\sup_n \sup_{\substack{\tau_1,\dots,\tau_K\in[0,1]\\ \lambda_1,\dots,\lambda_K\in\mathbb{R}}} \Big|\varepsilon_n(\tau_1,\dots,\tau_K,\lambda_1,\dots,\lambda_K)\Big| < \infty.$$

Proof. By the definition of $\mathfrak{f}_{q_{\tau_1},\dots,q_{\tau_K}}$, we have

$$\begin{aligned}&\mathrm{cum}(d^{\tau_1}_{n,U}(\lambda_1),\dots,d^{\tau_K}_{n,U}(\lambda_K))\\ &= \Delta_n\Big(\sum_{j=1}^{K}\lambda_j\Big)(2\pi)^{K-1}\mathfrak{f}_{q_{\tau_1},\dots,q_{\tau_K}}(\lambda_1,\dots,\lambda_{K-1})\\ &\quad - \Delta_n\Big(\sum_{j=1}^{K}\lambda_j\Big)\sum_{|u_1|\vee\dots\vee|u_{K-1}|\ge n}\gamma^{q_{\tau_1},\dots,q_{\tau_K}}_{u_1,\dots,u_{K-1}}\exp\Big(-\mathrm{i}\sum_{j=1}^{K-1}\lambda_j u_j\Big)\\ &\quad + E_n(\tau_1,\dots,\tau_K,\lambda_1,\dots,\lambda_K).\end{aligned}$$

Noting that $|\Delta_n(\lambda)| \le n$, we have by condition condition (CS),

$$\begin{aligned}&\sup_{\substack{\tau_1,\dots,\tau_K\in[0,1]\\ \lambda_1,\dots,\lambda_K\in\mathbb{R}}}\Big|\sum_{|u_1|\vee\dots\vee|u_{K-1}|\ge n}\gamma^{q_{\tau_1},\dots,q_{\tau_K}}_{u_1,\dots,u_{K-1}}\exp\Big(-\mathrm{i}\sum_{j=1}^{K-1}\lambda_j u_j\Big)\Big|\\ &\le \sup_{\substack{\tau_1,\dots,\tau_K\in[0,1]\\ \lambda_1,\dots,\lambda_K\in\mathbb{R}}}\sum_{m=n}^{\infty}\sum_{|u_1|\vee\dots\vee|u_{K-1}|=m}|\gamma^{q_{\tau_1},\dots,q_{\tau_K}}_{u_1,\dots,u_{K-1}}| \le \sum_{m=n}^{\infty}O(m^{K-2})a(m) = O(1/n).\end{aligned}$$

The claim follows by applying Lemma 8.2 to E_n. $\square$

In analogy to Theorem 5.2.2 in Brillinger (1975), we also have

LEMMA 8.4. *Under (CS) with $K = 2, \delta > 3$,*

(8.2)
$$\mathbb{E}I_{n,U}^{\tau_1,\tau_2}(\omega) = \begin{cases} \mathfrak{f}_{q_{\tau_1},q_{\tau_2}}(\omega) + \frac{1}{2\pi n}\left[\frac{\sin(n\omega/2)}{\sin(\omega/2)}\right]^2 \tau_1\tau_2 + \varepsilon_n^{\tau_1,\tau_2}(\omega) & \omega \neq 0 \mod 2\pi \\ \mathfrak{f}_{q_{\tau_1},q_{\tau_2}}(\omega) + \frac{n}{2\pi}\tau_1\tau_2 + \varepsilon_n^{\tau_1,\tau_2}(\omega) & \omega \neq 0 \mod 2\pi \end{cases}$$

with $\sup_{\tau_1,\tau_2\in[0,1],\omega\in\mathbb{R}} |\varepsilon_n^{\tau_1,\tau_2}(\omega)| = O(1/n)$.

Remark: For the Fourier frequencies $\omega = \frac{2\pi j}{n}$, $j \in \mathbb{Z}$, the second term in the right-hand side of (8.2) vanishes, leading to the useful simple result

$$\mathbb{E}I_n^{\tau_1,\tau_2}(\omega) = \mathfrak{f}_{q_{\tau_1},q_{\tau_2}}(\omega) + \frac{n}{2\pi}\tau_1\tau_2 I\{\omega = 0 \mod 2\pi\} + \varepsilon_n^{\tau_1,\tau_2}(\omega).$$

Proof. First note that, by definition,

$$\mathbb{E}I_n^{\tau_1,\tau_2}(\omega) = \frac{1}{2\pi n}\Big(\mathrm{cum}(d_{n,U}^{\tau_1}(\omega), d_{n,U}^{\tau_2}(-\omega)) + (\mathbb{E}d_{n,U}^{\tau_1}(\omega))(\mathbb{E}d_{n,U}^{\tau_2}(-\omega))\Big)$$

The result follows from applying Theorem 8.3 and noting that

$$\mathbb{E}d_{n,U}^{\tau}(\omega) = \tau\sum_{t=0}^{n-1} e^{-\mathrm{i}\omega t} = \tau\frac{e^{-\mathrm{i}\omega n} - 1}{e^{-\mathrm{i}\omega} - 1},$$

for $\omega \neq 0 \mod 2\pi$, while, for $\omega = 0 \mod 2\pi$, obviously, $\mathbb{E}d_{n,U}^{\tau}(\omega) = n\tau$. $\square$

LEMMA 8.5. *Assume that (CS), with $p = 2$ and $\delta > k + 1$, and (W) hold. Then, with the notation of Theorem 3.5,*

$$\sup_{\tau_1,\tau_2\in[0,1],\omega\in\mathbb{R}} \Big|\mathbb{E}\hat{G}_n(\tau_1,\tau_2;\omega) - \mathfrak{f}_{q_{\tau_1},q_{\tau_2}}(\omega) - B_n^{(k)}(\tau_1,\tau_2;\omega)\Big| = O((nb_n)^{-1}) + o(b_n^k).$$

Proof. By definition of $\hat{G}_n$ and Lemma 8.4, following the proof of Theorem 5.6.1 in Brillinger (1975), we have, uniformly in τ_1, τ_2 and ω,

$$
\begin{aligned}
&\mathbb{E}\hat{G}_n(\tau_1, \tau_2; \omega) \\
&= \frac{1}{n}\sum_{s=1}^{n-1} W_n\big(\omega - 2\pi s/n\big)\mathfrak{f}_{q_{\tau_1}, q_{\tau_2}}(2\pi s/n) + \frac{2\pi}{n}\sum_{s=1}^{n-1} W_n\big(\omega - 2\pi s/n\big)\varepsilon_n^{\tau_1,\tau_2}(2\pi s/n) \\
&= \int_0^{2\pi} W_n(\omega - \alpha)\mathfrak{f}_{q_{\tau_1}, q_{\tau_2}}(\alpha)\mathrm{d}\alpha + O(b_n^{-1}n^{-1}) \\
&= b_n^{-1}\int_{-\infty}^{\infty} W(b_n^{-1}[\omega - \alpha])\mathfrak{f}_{q_{\tau_1}, q_{\tau_2}}(\alpha)\mathrm{d}\alpha + O(b_n^{-1}n^{-1}) \\
&= \mathfrak{f}_{q_{\tau_1}, q_{\tau_2}}(\omega) + B_n^{(k)}(\tau_1, \tau_2; \omega) + o(b_n^k) + O(b_n^{-1}n^{-1}),
\end{aligned}
$$

where the last equality follows from the fact that (CS) implies that the function $\omega \mapsto \mathfrak{f}_{q_{\tau_1}, q_{\tau_2}}(\omega)$ is k times continuously differentiable with derivatives that are bounded uniformly in τ_1, τ_2. $\square$

8.2. *Proofs for Section 3.* In this appendix, we give the proofs for Propositions 3.1, 3.2 and 3.4.

8.2.1. *Proof of Proposition 3.1.* Recall that, by the definition of cumulants,

$$
\begin{aligned}
&|\operatorname{cum}(I\{X_{t_1} \in A_1\}, \dots, I\{X_{t_p} \in A_p\})| \\
&= \Big|\sum_{\{\nu_1,\dots,\nu_R\}} (-1)^{R-1}(R-1)!\mathbb{P}\Big(\bigcap_{i\in\nu_1}\{X_{t_i} \in A_i\}\Big)\cdots\mathbb{P}\Big(\bigcap_{i\in\nu_R}\{X_{t_i} \in A_i\}\Big)\Big|
\end{aligned}
\tag{8.3}
$$

where the summation is performed with respect to all partitions $\{\nu_1, \dots, \nu_R\}$ of the set $\{1, \dots, p\}$. It suffices to establish that

$$
\Big|\operatorname{cum}(I\{X_{t_1} \in A_1\}, \dots, I\{X_{t_p} \in A_p\})\Big| \le K_p\alpha\Big(\big\lfloor p^{-1}\max_{i,j}|t_i - t_j|\big\rfloor\Big).
$$

In the case $t_1 = \ldots = t_p$ this is obviously true. If at least two indices are distinct, choose j with $\max_{i=1,\dots,p-1}(t_{i+1} - t_i) = t_{j+1} - t_j > 0$ and let $(Y_{t_{j+1}}, \dots, Y_{t_p})$ be a random vector that is independent of $(X_{t_1}, \dots, X_{t_j})$ and possesses the same joint distribution as $(X_{t_{j+1}}, \dots, X_{t_p})$. By an elementary property of the cumulants [cf. Theorem 2.3.1 (iii) in Brillinger (1975)] we have

$$
\operatorname{cum}(I\{X_{t_1} \in A_1\}, \dots, I\{X_{t_j} \in A_j\}, I\{Y_{t_{j+1}} \in A_{j+1}\}, \dots, I\{Y_{t_p} \in A_p\}) = 0.
$$

Therefore, we can write the cumulant of interest as

$$\begin{aligned}
&\Big| \operatorname{cum}(I\{X_{t_1} \in A_1\}, \dots, I\{X_{t_p} \in A_p\}) \\
&\quad - \operatorname{cum}(I\{X_{t_1} \in A_1\}, \dots, I\{X_{t_j} \in A_j\}, I\{Y_{t_{j+1}} \in A_{j+1}\}, \dots, I\{Y_{t_p} \in A_p\}) \Big| \\
&= \Big| \sum_{\{\nu_1,\dots,\nu_R\}} (-1)^{R-1}(R-1)! [P_{\nu_1} \cdots P_{\nu_R} - Q_{\nu_1} \cdots Q_{\nu_R}] \Big|,
\end{aligned}$$

where the sum runs over all partitions $\{\nu_1, \dots, \nu_R\}$ of $\{1, \dots, p\}$,

$$P_{\nu_r} := \mathbb{P}\Big(\bigcap_{i \in \nu_r} \{X_{t_i} \in A_i\} \Big) \text{ and } Q_{\nu_r} := \mathbb{P}\Big(\bigcap_{\substack{i \in \nu_r \\ i \le j}} \{X_{t_i} \in A_i\} \Big) \mathbb{P}\Big(\bigcap_{\substack{i \in \nu_r \\ i > j}} \{X_{t_i} \in A_i\} \Big),$$

$r = 1, \dots, R$, with $\mathbb{P}(\bigcap_{i \in \emptyset} \{X_{t_i} \in A_i\}) := 1$ by convention. Since X_t is α-mixing, it follows that, for any partition $\nu_1, ..., \nu_R$ and any $r = 1, ..., R$, we have $|P_{\nu_r} - Q_{\nu_r}| \le \alpha(t_{j+1} - t_j)$. Thus, for every partition $\nu_1, ..., \nu_R$,

$$|P_{\nu_1} \cdots P_{\nu_R} - Q_{\nu_1} \cdots Q_{\nu_R}| \le \sum_{r=1}^{R} |P_{\nu_r} - Q_{\nu_r}| \le R\alpha(t_{j+1} - t_j).$$

All together, this yields

$$|\operatorname{cum}(I\{X_{t_1} \in A_1\}, \dots, I\{X_{t_p} \in A_p\})| \le \alpha(t_{j+1} - t_j) \sum_{\{\nu_1,\dots,\nu_R\}} R!.$$

Noting that $p(t_{j+1} - t_j) \ge \max_{i_1,i_2} |t_{i_1} - t_{i_2}|$ and observing that α is a monotone function, we obtain

$$|\operatorname{cum}(I\{X_{t_1} \in A_1\}, \dots, I\{X_{t_p} \in A_p\})| \le K\alpha(\max |t_i - t_j|).$$

Now, additionally assume that $\alpha(n) \le C\kappa^n$. Then,

$$\begin{aligned}
\alpha(\lfloor p^{-1} \max |t_i - t_j| \rfloor) &\le C\kappa^{\lfloor p^{-1} \max |t_i - t_j| \rfloor} \le C\kappa^{-1} (\kappa^{1/p})^{p \lfloor p^{-1} \max |t_i - t_j| \rfloor + 1} \\
&\le C\kappa^{-1} (\kappa^{1/p})^{\max |t_i - t_j|}.
\end{aligned}$$

Setting $\rho = \kappa^{1/p} \in (0, 1)$ completes the proof. $\square$

8.2.2. *Proof of Proposition 3.2.* We follow the ideas of the proof of Proposition 2 in Wu and Shao (2004). Let $p \geq 2$ and assume without loss of generality that $t_1 \leq t_2 \leq ... \leq t_p$. For $t > 0$, define the coupled random variables $X'_t := g(..., \varepsilon^*_{-1}, \varepsilon^*_0, \varepsilon_1, ...\varepsilon_t)$. Choose an arbitrary $j \in \{1, ..., p-1\}$ that satisfies $t_{j+1} - t_j = \max_i(t_{i+1} - t_i)$. In the case $\max_i(t_{i+1} - t_i) = 0$, there is nothing to prove. So, assume that $\max_i(t_{i+1} - t_i) > 0$. Define $V_i := I\{X_{t_i - t_j} \in A_i\}$ and $V'_i := I\{X'_{t_i - t_j} \in A_i\}$. Strict stationarity implies

$$
\begin{aligned}
&\mathrm{cum}(I\{X_{t_1} \in A_1\}, ..., I\{X_{t_j} \in A_j\}, I\{X_{t_{j+1}} \in A_j\}, ..., I\{X_{t_p} \in A_p\}) \\
&\quad = \mathrm{cum}(V_1, ..., V_p) \\
&\quad = \mathrm{cum}(V_1, ..., V_j, V_{j+1} - V'_{j+1}, V_{j+2}, ..., V_p) \\
&\qquad + \sum_{m=1}^{p-j-1} \mathrm{cum}(V_1, ..., V_j, V'_{j+1}, ..., V'_{j+m}, V_{j+1+m} - V'_{j+1+m}, ..., V_p) \\
&\qquad + \mathrm{cum}(V_1, ..., V_j, V'_{j+1}, ..., V'_p). \qquad (8.4)
\end{aligned}
$$

By an elementary property of cumulants, the last term in (8.4) is zero since the groups of random variables $(V_t)_{t<0}$ and $(V'_t)_{t\geq 0}$ are independent by definition of the V'_t. Additionaly, by the definition of cumulants, uniform boundedness of indicators, and Assumption (G), we obtain the bounds

$$
\begin{aligned}
&\big|\,\mathrm{cum}(V_1, ..., V_j, V_{j+1} - V'_{j+1}, V_{j+2}, ..., V_p)| \leq C\mathbb{E}|V_{j+1} - V'_{j+1}\big| \leq C\sigma^{t_{j+1}-t_j}, \\
&\big|\,\mathrm{cum}(V_1, ..., V_j, V'_{j+1}, ..., V'_{j+m}, V_{j+1+m} - V'_{j+1+m}, ..., V_p)\big| \leq C\sigma^{t_{j+m+1}-t_j}.
\end{aligned}
$$

Observe that $\max_{i\neq l}|t_i - t_l| \geq p \max_i(t_{i+1} - t_i)$. The bound

$$
|\,\mathrm{cum}(I\{X_{t_1} \in A_1\}, ..., I\{X_{t_p} \in A_p\})| \leq C(\sigma^{1/p})^{\max|t_i - t_j|}
$$

follows from the fact that the number of summands in the sum is at most p. Setting $\rho := \sigma^{1/p}$ completes the proof. □

8.2.3. *Proof of Proposition 3.4.* It suffices to prove that

$$
\Big(n^{-1/2} d^{\tau}_{n,R}(\omega)\Big)_{\tau\in[0,1]} \rightsquigarrow \Big(\mathbb{D}(\tau;\omega)\Big)_{\tau\in[0,1]} \quad \text{in } \ell^\infty([0,1]). \tag{8.5}
$$

Now, for (8.5) to hold, it is sufficient that $\big(n^{-1/2} d^{\tau}_{n,U}(\omega)\big)_{\tau\in[0,1]}$ satisfies the following two conditions:

(i1) convergence of the finite-dimensional distributions, that is,

$$
\big(n^{-1/2} d^{\tau_j}_{n,U}(\omega_j)\big)_{j=1,\ldots,k} \xrightarrow{d} \big(\mathbb{D}(\tau_j;\omega_j)\big)_{j=1,\ldots,k}, \tag{8.6}
$$

for any $\tau_j \in [0,1]$ and $\omega_j \neq 0(\mod 2\pi)$, $j = 1, \ldots, k$ and $k \in \mathbb{N}$;

(i2) stochastic equicontinuity: for any $x > 0$ and any $\omega \neq 0 \mod 2\pi$,

$$\lim_{\delta \downarrow 0} \limsup_{n\to\infty} \mathbb{P}\Big(\sup_{\substack{\tau_1,\tau_2\in[0,1] \\ |\tau_1-\tau_2|\le\delta}} |n^{-1/2}(d_{n,U}^{\tau_1}(\omega) - d_{n,U}^{\tau_2}(\omega))| > x \Big) = 0. \tag{8.7}$$

Indeed, under (i1) and (i2), an application of Theorems 1.5.4 and 1.5.7 in van der Vaart and Wellner (1996) yields

$$\Big(n^{-1/2} d_{n,U}^{\tau}(\omega) \Big)_{\tau\in[0,1]} \rightsquigarrow \Big(\mathbb{D}(\tau;\omega) \Big)_{\tau\in[0,1]} \quad \text{in } \ell^\infty([0,1]), \tag{8.8}$$

which, in combination with

$$\sup_{\tau\in[0,1]} |n^{-1/2}(d_{n,R}^{\tau}(\omega) - d_{n,U}^{\tau}(\omega))| = o_P(1), \quad \text{for } \omega \neq 0 \mod 2\pi, \tag{8.9}$$

which we prove below, yields the desired result that (8.5) holds. To prove (8.9), observe that, by (7.25), it suffices to bound the term

$$\sup_{\tau\in[0,1]} n^{-1/2} |d_{n,U}^{\hat{F}_n^{-1}(\tau)}(\omega) - d_{n,U}^{\tau}(\omega))|.$$

Now, for any $x > 0$ and $\delta_n = o(1)$ such that $n^{1/2}\delta_n \to \infty$,

$$\begin{aligned}
&\mathbb{P}\Big(\sup_{\tau\in[0,1]} n^{-1/2} |d_{n,U}^{\hat{F}_n^{-1}(\tau)}(\omega) - d_{n,U}^{\tau}(\omega))| > x \Big) \\
&\le \mathbb{P}\Big(\sup_{\tau\in[0,1]} \sup_{|u-\tau|\le\delta_n} |d_{n,U}^{u}(\omega) - d_{n,U}^{\tau}(\omega)| > x n^{1/2}, \sup_{\tau\in[0,1]} |\hat{F}_n^{-1}(\tau) - \tau| \le \delta_n \Big) \\
&\quad + \mathbb{P}\Big(\sup_{\tau\in[0,1]} |\hat{F}_n^{-1}(\tau) - \tau| > \delta_n \Big) = o(1) + o(1),
\end{aligned}$$

where the first $o(1)$ follows from (8.7), and the second one is a consequence of Lemma 7.5.

It thus remains to establish (8.6) and (8.7). First consider (8.7). Letting $T := (\tau_1 \wedge \tau_2, \tau_1 \vee \tau_2]$, we use the following moment inequality, which holds for $\omega \neq 0 \mod 2\pi$ and $\kappa \in (0,1)$, if $|\tau_1 - \tau_2|$ is small enough:

$$\mathbb{E}\big|n^{-1/2}(d_n^{\tau_1}(\omega) - d_n^{\tau_2}(\omega))\big|^{2L} = n^{-L}\mathbb{E}\prod_{m=1}^{2L} d_n^T\big((-1)^{m-1}\omega\big)\big)$$

$$= n^{-L}\sum_{\{\nu_1,\dots,\nu_R\}}\prod_{r=1}^{R}\mathrm{cum}\left(d_n^T\big((-1)^{m-1}\omega\big)\ :\ m \in \nu_r\right) \tag{8.10}$$

$$\leq n^{-L}\sum_{\{\nu_1,\dots,\nu_R\}}\prod_{r=1}^{R}\Big[\tilde{C}\Big(\big|\Delta_n\big(\omega\sum_{m\in\nu_r}(-1)^{m-1}\big)\big|+1\Big)|\tau_1-\tau_2|^{\kappa}\Big] \tag{8.11}$$

$$\leq Cn^{-L}\sum_{R=1}^{2L} n^{R\wedge(2L-R)}|\tau_1-\tau_2|^{\kappa R} = C\sum_{R=1}^{2L} n^{-|R-L|}|\tau_1-\tau_2|^{\kappa R}. \tag{8.12}$$

Equality in (8.10) (summation is with respect to all partitions $\{\nu_1,\dots,\nu_R\}$ of the set $\{1,\dots,2L\}$) follows from Theorem 2.3.2 in Brillinger (1975). Inequality (8.11) follows from Lemma 7.4, and holds for arbitrary $\kappa \in (0,1)$ as long as $|\tau_1 - \tau_2|$ is small enough.

As for (8.12), note that the fact that

$$\Delta_n(\omega) = \begin{cases} n & \omega = 0 \mod 2\pi, \\ \sin\big(\omega(n+1/2)\big)/\sin(\omega/2) & \omega \neq 0 \mod 2\pi, \end{cases}$$

implies $|\Delta_n(\omega)| \leq |\sin(\omega/2)|^{-1}$ if $\omega \neq 0 \mod 2\pi$. Therefore, (8.12) follows if we show that

$$|\{j = 1,\dots,R\ :\ |\nu_j| \geq 2\}| \leq R \wedge (2L - R) \tag{8.13}$$

for any partition $\{\nu_1,\dots,\nu_R\}$ of the set $\{1,\dots,2L\}$. If $R \leq L$, the bound obviously holds true. For any $R > L$, let us show that

$$|\{j = 1,\dots,R\ :\ |\nu_j| = 1\}| \geq 2(R - L) \tag{8.14}$$

holds for all $\{\nu_1,\dots,\nu_R\}$. Denote by S the number of "singles" [sets ν_j with $|\nu_j| = 1$] in the given partition $\{\nu_1,\dots,\nu_R\}$: the number of sets containing two or more elements is thus $R - S$, which implies that there are more than $2(R-S) + S = 2R - S$ elements in total. Inequality (8.14) follows, because if S were strictly smaller than $2(R - L)$, this would imply that the total number $2R - S$ of elements were strictly larger than $2L$.

Inequality (8.14) implies that the number of elements in sets with two or more elements is bounded by $2L - 2(R - L) = 4L - 2R$, which in turn implies that there are no more than $2L - R$ such sets, since each of them contains at least two elements. Inequality (8.13), hence also (8.12), follow.

We now use the moment inequality (8.12) and Lemma 7.1 for establishing (8.7). Define $\Psi(x) := x^{2L}$, and note that, for $\omega \neq 0 \mod 2\pi$, $\gamma \in (0, \kappa)$ and $\tau_1, \tau_2 \in [0,1]$ with $|\tau_1 - \tau_2| > n^{-1/\gamma}$, we have

$$\begin{aligned}
(8.15)\quad & \|n^{-1/2}(d_n^{\tau_1}(\omega) - d_n^{\tau_2}(\omega))\|_\Psi = (\mathbb{E}\big|n^{-1/2}(d_n^{\tau_1}(\omega) - d_n^{\tau_2}(\omega))\big|^{2L})^{1/(2L)} \\
&\leq \Big(\bar{C}\sum_{R=1}^{2L} n^{-|R-L|}|\tau_1-\tau_2|^{\kappa R}\Big)^{1/(2L)} \leq \tilde{C}\sum_{R=1}^{2L} n^{-|R-L|/(2L)}|\tau_1-\tau_2|^{\kappa R/(2L)} \\
&\qquad \leq \tilde{C}\sum_{R=1}^{2L}|\tau_1-\tau_2|^{(\kappa R+\gamma|R-L|)/(2L)} \leq C|\tau_1-\tau_2|^{\gamma/2} =: Cd(\tau_1,\tau_2).
\end{aligned}$$

Letting $\bar{\eta}_n := 2n^{-1/2}$ and choosing γ and L such that $\gamma L > 1$, Lemma 7.1 entails, for any $\eta \geq \bar{\eta}_n$,

$$\begin{aligned}
(8.16)\quad & \mathbb{P}\Big(\sup_{\substack{\tau_1,\tau_2\in[0,1]\\ d(\tau_1,\tau_2)\leq\delta}} n^{-1/2}|d_{n,U}^{\tau_1}(\omega) - d_{n,U}^{\tau_2}(\omega)| > 2x\Big) \\
&\qquad \leq \Big(\frac{8K}{x}\Big[\int_{\bar{\eta}_n/2}^{\eta}\varepsilon^{-1/(\gamma L)}\mathrm{d}\varepsilon + (\delta+2\bar{\eta}_n)\eta^{-2/(\gamma L)}\Big]\Big)^{2L} \\
&\qquad\qquad + \mathbb{P}\Big(\sup_{\substack{\tau_1,\tau_2\in[0,1]\\ d(\tau_1,\tau_2)\leq\bar{\eta}_n}} n^{-1/2}|d_{n,U}^{\tau_1}(\omega) - d_{n,U}^{\tau_2}(\omega)| > x/2\Big).
\end{aligned}$$

Furthermore,

(8.17)

$$\begin{aligned}
&\sup_{\substack{\tau_1,\tau_2\in[0,1]\\ d(\tau_1,\tau_2)\leq\bar{\eta}_n}} n^{-1/2}|d_{n,U}^{\tau_1}(\omega) - d_{n,U}^{\tau_2}(\omega)| \leq \sup_{\substack{\tau_1,\tau_2\in[0,1]\\ d(\tau_1,\tau_2)\leq\bar{\eta}_n}} n^{-1/2}\sum_{t=0}^{n-1} I\{X_t \in (a\wedge b, a\vee b]\} \\
&\leq \sup_{|x-y|\leq 2^{2/\gamma}n^{-1/\gamma}} n^{1/2}|\hat{F}_n(x) - \hat{F}_n(y) - (x-y)| + \sup_{|x-y|\leq 2^{2/\gamma}n^{-1/\gamma}} n^{1/2}|x-y| \\
&= O_P\Big((n^{2-1/\gamma}+n)^{1/(2k)}\big[n^{-1/\gamma}\big(|\log n|/\gamma\big)^{d_k} + n^{-1}\big]^{1/2} + n^{1/2-1/\gamma}\Big) \\
&= o_P(1).
\end{aligned}$$

Together, (8.16) and (8.17) imply

$$\lim_{\delta\downarrow 0}\limsup_{n\to\infty}\mathbb{P}\Big(\sup_{\substack{\tau_1,\tau_2\in[0,1]\\|\tau_1-\tau_2|\le\delta}}|n^{-1/2}(d^{\tau_1}_{n,U}(\omega)-d^{\tau_2}_{n,U}(\omega))|>x\Big)$$
$$\le\left[\frac{8\tilde{K}}{x}\frac{\gamma L}{\gamma L-1}\eta^{(\gamma L-1)/(\gamma L)}\right]^{2L}+o(1)$$

for every $x,\eta>0$. Condition (8.7) follows, since the integral in the right-hand side can be made arbitrarily small by choosing η accordingly.

Turning to (8.6), we employ Lemma 7.4 in combination with Lemma P4.5 and Theorem 4.3.2 from Brillinger (1975). More precisely, we have to verify that, for any $\tau_1,\ldots,\tau_k\in[0,1]$, $k\in\mathbb{N}$, and $\omega_1,\ldots,\omega_k\ne 0 \mod 2\pi$, all cumulants of the vector

$$n^{-1/2}\big(d^{\tau_1}_{n,U}(\omega_1),d^{\tau_1}_{n,U}(-\omega_1),\ldots,d^{\tau_k}_{n,U}(\omega_k),d^{\tau_k}_{n,U}(-\omega_k)\big)$$

converge to the corresponding cumulants of the vector

$$\big(\mathbb{D}(\tau_1;\omega_1),\mathbb{D}(\tau_1;-\omega_1),\ldots,\mathbb{D}(\tau_k;\omega_k),\mathbb{D}(\tau_k;-\omega_k)\big).$$

It is easy to see that the cumulants of order one converge as desired:

$$|\mathbb{E}(n^{-1/2}d^{\tau}_{n,U}(\omega))|=n^{-1/2}|\Delta_n(\omega)|\tau\le n^{-1/2}\tau|\sin(\omega/2)|^{-1}=o(1),$$

for any $\tau\in[0,1]$ and $\omega\ne 0 \mod 2\pi$. Furthermore, for the cumulants of order two, applying Theorem 4.3.1 in Brillinger (1975) to the bivariate process $(I\{X_t\le q_{\mu_1}\},I\{X_t\le q_{\mu_2}\})$, we obtain

$$\mathrm{cum}(n^{-1/2}d^{\mu_1}_{n,U}(\lambda_1),n^{-1/2}d^{\mu_2}_{n,U}(\lambda_2))=2\pi n^{-1}\Delta_n(\lambda_1+\lambda_2)\mathfrak{f}_{q_{\mu_1},q_{\mu_2}}(\lambda_1)+o(1)$$

for any $(\lambda_1,\mu_1),(\lambda_2,\mu_2)\in\bigcup_{i=1}^k\{(\omega_i,\tau_i),(-\omega_i,\tau_i)\}$. which yields the correct second moment structure. Finally, the cumulants of order J, with $J\in\mathbb{N}$ and $J\ge 3$, all tend to zero as, in view of Lemma 7.4,

$$\begin{aligned}&\mathrm{cum}(n^{-1/2}d^{\mu_1}_{n,U}(\lambda_1),\ldots,n^{-1/2}d^{\mu_J}_{n,U}(\lambda_J))\\&\le Cn^{-J/2}(|\Delta_n(\sum_{j=1}^J\lambda_j)|+1)g_p(\max_{j=1}^J\mu_j)=O(n^{-(J-2)/2})=o(1)\end{aligned}$$

for $(\lambda_1,\mu_1),\ldots,(\lambda_J,\mu_J)\in\bigcup_{i=1}^k\{(\omega_i,\tau_i),(-\omega_i,\tau_i)\}$. This implies that the limit $\mathbb{D}(\tau;\omega)$ is Gaussian, and completes the proof of (8.6). Proposition 3.4 follows. $\square$

8.3. *Proofs of the results from Section 7.4.* We begin this section by stating an auxiliary technical result that is used in the proofs of Lemmas 7.5, 7.6 and 7.7. Its proof relies on Lemma 7.4.

LEMMA 8.6. *Assume that $(X_t)_{t\in\mathbb{Z}}$ is a strictly stationary process satisfying (C) and such that $X_0 \sim U[0,1]$. Denote by $\hat{F}_n$ the empirical distribution function of $X_0, ..., X_{n-1}$. Then, for any $k \in \mathbb{N}$, there exists a constant d_k depending on k only such that*

$$\sup_{x,y\in[0,1],|x-y|\le\delta_n} \sqrt{n}|\hat{F}_n(x) - \hat{F}_n(y) - (x-y)|$$
$$= O_P\Big((n^2\delta_n + n)^{1/2k}(\delta_n|\log\delta_n|^{d_k} + n^{-1})^{1/2}\Big)$$

as $\delta_n \to 0$.

Proof of Lemma 8.6. Observe the decomposition

$$\begin{aligned}|\hat{F}_n(x) - \hat{F}_n(y) - (x-y)| \le\; & \Big|\hat{F}_n(x) - \hat{F}_n\Big(\frac{\lfloor nx\rfloor}{n}\Big) - \Big(x - \frac{\lfloor nx\rfloor}{n}\Big)\Big| \\ & + \Big|\hat{F}_n(y) - \hat{F}_n\Big(\frac{\lfloor ny\rfloor}{n}\Big) - \Big(y - \frac{\lfloor ny\rfloor}{n}\Big)\Big| \\ & + \Big|\hat{F}_n\Big(\frac{\lfloor nx\rfloor}{n}\Big) - \hat{F}_n\Big(\frac{\lfloor ny\rfloor}{n}\Big) - \Big(\frac{\lfloor nx\rfloor}{n} - \frac{\lfloor ny\rfloor}{n}\Big)\Big|.\end{aligned}$$

Since $|\lfloor ny\rfloor/n - y| \le 1/n$, and by monotonicity of $\hat{F}_n$,

$$\begin{aligned}\Big|\hat{F}_n(y) - \hat{F}_n\Big(\frac{\lfloor ny\rfloor}{n}\Big) - \Big(y - \frac{\lfloor ny\rfloor}{n}\Big)\Big| &\le \hat{F}_n\Big(\frac{1+\lfloor ny\rfloor}{n}\Big) - \hat{F}_n\Big(\frac{\lfloor ny\rfloor}{n}\Big) + \frac{1}{n} \\ &\le \Big|\hat{F}_n\Big(\frac{1+\lfloor ny\rfloor}{n}\Big) - \hat{F}_n\Big(\frac{\lfloor ny\rfloor}{n}\Big) - \frac{1}{n}\Big| + \frac{2}{n}.\end{aligned}$$

A similar bound holds with x substituting y, so that, letting $M_n := \{j/n | j = 0, ..., n\}$,

$$\sup_{x,y\in[0,1],|x-y|\le\delta_n} |\hat{F}_n(x) - \hat{F}_n(y) - (x-y)|$$
$$\le \max_{x,y\in M_n,|x-y|\le\delta_n+2n^{-1}} |\hat{F}_n(x) - \hat{F}_n(y) - (x-y)| + 4/n.$$

The cardinality of the set $\{x, y \in M_n : |x-y| \le \delta_n + 2n^{-1}\}$ is of the order $O(n^2(\delta_n + n^{-1}))$. Recalling that $\max_{j=1,\dots,N}|Z_j| = O_P(N^{1/m})$ as $N \to \infty$ for any sequence $(Z_j)_{j\in\mathbb{Z}}$ of random variables with uniformly bounded moments of order m, the claim follows if we can show that, for any $k \in \mathbb{N}$,

$$\sup_{x,y\in[0,1],|x-y|\le\delta} \mathbb{E}\Big(\frac{n^{1/2}|\hat{F}_n(x) - \hat{F}_n(y) - (x-y)|}{((\delta(1+|\log\delta|)^d)\vee n^{-1})^{1/2}}\Big)^{2k} \le C_k.$$

Now, this latter inequality is a consequence of the fact that, for all $y > x$,

$$\begin{aligned} &\mathbb{E}(\hat{F}_n(x) - \hat{F}_n(y) - (x-y))^{2k} \\ (8.18) \qquad &= n^{-2k} \sum_{\nu_1,\dots,\nu_R, |\nu_j|\ge 2} \prod_{r=1}^{R} \mathrm{cum}(d_n^{(x,y]}(0), \dots, d_n^{(x,y]}(0)) \end{aligned}$$

where $d := \max(d_1, \dots, d_k)$ [recall the notation $d_n^{(x,y]}(\omega)$ from Lemma 7.4] and the sum runs over all partitions of $\{1, \dots, 2k\}$; in view of Lemma 7.4, this latter quantity in turn is bounded by

$$\tilde{C}_k n^{-2k} \sum_{j=1}^{k} n^j |x-y|^j (1 + |\log(y-x)|)^{jd},$$

where the constant $\tilde{C}_k$ only depends on k, ρ, and K. This completes the proof of Lemma 8.6. □

8.3.1. *Proof of Lemma 7.1.* As in the proof of Theorem 2.2.4 in van der Vaart and Wellner (1996), we construct nested sets $T_0 \subset T_1 \subset T_2 \subset \dots \subset T_k \subset T$ such that every T_j is a maximal set of points with $d(s,t) > \eta 2^{-j}$, for all $s, t \in T_j$. Here maximal means that no point can be added without destroying the validity of the inequality. Stop adding subsets when k is such that $\Delta_k := \eta/2^k < \bar{\eta} \le \eta/2^{k-1}$.

For $s, t \in T$ with $d(s,t) \le \delta$, denote by $s', t' \in T_k$ the points closest to s and t, respectively. Then, since by construction $d(s,t) \ge \Delta_k \ge \bar{\eta}/2$ for any $s \ne t$, $s, t \in T_k$,

$$\begin{aligned} \sup_{d(s,t)\le\delta} |\mathbb{G}_s - \mathbb{G}_t| &= \sup_{d(s,t)\le\delta} |\mathbb{G}_s - \mathbb{G}_{s'} - (\mathbb{G}_t - \mathbb{G}_{t'}) - (\mathbb{G}_{t'} - \mathbb{G}_{s'})| \\ &\le \sup_{\substack{d(s',t')\le\delta+2\Delta_k \\ s',t'\in T_k}} |\mathbb{G}_{t'} - \mathbb{G}_{s'}| + 2 \sup_{t'\in T_k} \sup_{t: d(t,t')\le\Delta_k} |\mathbb{G}_t - \mathbb{G}_{t'}| \\ &\le \sup_{\substack{d(s',t')\le\delta+2\bar{\eta} \\ s',t'\in T_k}} |\mathbb{G}_{t'} - \mathbb{G}_{s'}| + 2 \sup_{t'\in T_k} \sup_{t: d(t,t')\le\bar{\eta}} |\mathbb{G}_t - \mathbb{G}_{t'}|. \end{aligned}$$

Adapting the proof of Theorem 2.2.4 in van der Vaart and Wellner (1996), let us show that

(8.19)
$$\Big\|\sup_{\substack{d(s,t)\leq\delta+2\bar\eta\\ s,t\in T_k}}|\mathbb{G}_t-\mathbb{G}_s|\Big\|_\Psi \leq 4K\Big[\int_{\bar\eta/2}^{\eta}\Psi^{-1}\big(D(\epsilon,d)\big)\mathrm{d}\epsilon+(\delta+2\bar\eta)\Psi^{-1}\big(D^2(\eta,d)\big)\Big].$$

By the definition of packing numbers, we have $|T_j|\leq D(\eta 2^{-j},d)$. Let every point $t_j\in T_j$ be linked to a unique $t_{j-1}\in T_{j-1}$ such that $d(t_j,t_{j-1})\leq\eta 2^{-j}$. This yields, for every t_k a chain $t_k,t_{k-1},\dots,t_0$ connecting t_k to a point $t_0\in T_0$. For two arbitrary points $s_k,t_k\in T_k$, the difference of increments along their respective chains is bounded by

$$\begin{aligned}|(\mathbb{G}_{s_k}-\mathbb{G}_{s_0})-(\mathbb{G}_{t_k}-\mathbb{G}_{t_0})| &= |\sum_{j=0}^{k-1}(\mathbb{G}_{s_{j+1}}-\mathbb{G}_{s_j})-\sum_{j=0}^{k-1}(\mathbb{G}_{t_{j+1}}-\mathbb{G}_{t_j})|\\ &\leq 2\sum_{j=0}^{k-1}\max_{(u,v)\in L_j}|\mathbb{G}_u-\mathbb{G}_v|,\end{aligned}$$

where L_j denotes the set of all links (u,v) from points $u\in T_{j+1}$ to points $v\in T_j$. Because the links were constructed by connecting any point in T_{j+1} to a unique point in T_j, we have $|L_j|=|T_{j+1}|$. By assumption,

$$\|\mathbb{G}_u-\mathbb{G}_v\|_\Psi\leq Cd(u,v)\leq C\eta 2^{-j}\quad\text{for all}(u,v)\in L_j.$$

Therefore, it follows from Lemma 2.2.2 in van der Vaart and Wellner (1996) that

(8.20)
$$\begin{aligned}&\Big\|\max_{s,t\in T_k}|(\mathbb{G}_s-\mathbb{G}_{s_0})-(\mathbb{G}_t-\mathbb{G}_{t_0})|\Big\|_\Psi\leq 2\sum_{j=0}^{k-1}\tilde K\Psi^{-1}\big(D(\eta 2^{-(j+1)},d)\big)C\eta 2^{-j}\\ &\leq K\sum_{j=0}^{k-1}\Psi^{-1}\big(D(\eta 2^{-j-1},d)\big)4\eta(2^{-j}-2^{-j-1})\leq 4K\int_{\bar\eta/2}^{\eta}\Psi^{-1}\big(D(\epsilon,d)\big)\mathrm{d}\epsilon\end{aligned}$$

for some constant K only depending on Ψ and C.

In (8.20), $s_0=s_0(s)$ and $t_0=t_0(t)$ are the endpoints of the chains starting at s and t, respectively. We therefore have

(8.21)
$$\Big\|\max_{\substack{d(s,t)\leq\delta+2\bar\eta\\ s,t\in T_k}}|\mathbb{G}_t-\mathbb{G}_s|\Big\|_\Psi\leq 4K\int_{\bar\eta/2}^{\eta}\Psi^{-1}\big(D(\epsilon,d)\big)\mathrm{d}\epsilon+\Big\|\max_{\substack{d(s,t)\leq\delta+2\bar\eta\\ s,t\in T_k}}|\mathbb{G}_{s_0(s)}-\mathbb{G}_{t_0(t)}|\Big\|_\Psi.$$

To complete the proof, we use the same arguments as in van der Vaart and Wellner (1996). For every pair of endpoints $s_0(s), t_0(t)$ of chains starting at $s, t \in T_k$ with distance $d(s,t) \le \delta$, choose exactly one pair $s_k^0, t_k^0 \in T_k$, with $d(s_k^0, t_k^0) < \delta + 2\bar{\eta}$, whose chains end at s_0, t_0. Because $|T_0| = D(\eta 2^{-0}, d)$, there are at most $D^2(\eta, d)$ such (s_k^0, t_k^0) pairs. Therefore, we have the following bound for the second term in the right-hand side in (8.21):

$$\begin{aligned}
\Big\| \max_{\substack{d(s,t)\le\delta+2\bar{\eta}\\ s,t\in T_k}} |\mathbb{G}_{s_0(s)} - \mathbb{G}_{t_0(t)}| \Big\|_\Psi \quad &\le \quad \Big\| \max_{\substack{d(s,t)\le\delta+2\bar{\eta}\\ s,t\in T_k}} |(\mathbb{G}_{s_0(s)} - \mathbb{G}_{s_k^0}) - (\mathbb{G}_{t_0(t)} - \mathbb{G}_{t_k^0})| \Big\|_\Psi \\
&\quad + \Big\| \max_{\substack{d(s,t)\le\delta+2\bar{\eta}\\ s,t\in T_k}} |(\mathbb{G}_{s_k^0} - \mathbb{G}_{t_k^0})| \Big\|_\Psi \\
&= \quad S_1^n + S_2^n, \quad \text{say.}
\end{aligned}$$

Noting that S_1^n is bounded by the right-hand side in (8.20), while S_2^n can be bounded by employing Lemma 2.2.2 in van der Vaart and Wellner (1996) again, we obtain the desired inequality

$$\Big\| \max_{\substack{d(s,t)\le\delta+2\bar{\eta}\\ s,t\in T_k}} |(\mathbb{G}_{s_k^0} - \mathbb{G}_{s_k^0})| \Big\|_\Psi \le (\delta + 2\bar{\eta})\Psi^{-1}\big(D^2(\eta, d)\big)K.$$

This completes the proof of Lemma 7.1. $\square$

8.3.2. *Proof of Lemma 7.2.* The proof consists of two steps. In the first step, we derive the representation

$$\mathbb{E}|\hat{H}_n(a;\omega) - \hat{H}_n(b;\omega)|^{2L} = \sum_{\substack{\{\nu_1,\dots,\nu_R\}\\ |\nu_j|\ge 2,\ j=1,\dots,R}} \prod_{r=1}^{R} \mathcal{D}_{a,b}(\nu_r) \tag{8.22}$$

where the summation runs over all partitions $\{\nu_1, \dots, \nu_R\}$ of $\{1, \dots, 2L\}$ such that each set ν_j contains at least two elements, and

$$\begin{aligned}
\mathcal{D}_{a,b}(\xi) := &\sum_{\ell_{\xi_1},\dots,\ell_{\xi_q}\in\{1,2\}} n^{-3q/2} b_n^{q/2} \Big(\prod_{m\in\xi} \sigma_{\ell_m}\Big) \\
&\times \sum_{s_{\xi_1},\dots,s_{\xi_q}=1}^{n-1} \Big(\prod_{m\in\xi} W_n(\omega - 2\pi s_m/n)\Big) \operatorname{cum}(D_{\ell_m,(-1)^{m-1}s_m} : m \in \xi),
\end{aligned}$$

for any set $\xi := \{\xi_1, \dots, \xi_q\} \subset \{1, \dots, 2L\}$, where $q := |\xi|$ and

$$D_{\ell,s} := d_n^{M_1(\ell)}(2\pi s/n) d_n^{M_2(\ell)}(-2\pi s/n), \quad \ell = 1, 2, \quad s = 1, \dots, n-1,$$

with the sets $M_1(1)$, $M_2(2)$, $M_2(1)$, $M_1(2)$ and the signs $\sigma_\ell \in \{-1,1\}$ defined in (8.24) below.

In step two of the proof we employ assumption (7.26) to prove

$$\sup_{\substack{\xi\subset\{1,\dots,2L\}\\ |\xi|=q}} \sup_{\|a-b\|_1\le\varepsilon} |\mathcal{D}_{a,b}(\xi)| \le C(nb_n)^{1-q/2}g(\varepsilon), \quad 2\le q\le 2L. \tag{8.23}$$

To conclude the proof of the lemma, it is sufficient to observe that, for any partition in (8.22),

$$\Big|\prod_{r=1}^{R}\mathcal{D}_{a,b}(\nu_r)\Big| \le Cg^R(\varepsilon)(nb_n)^{R-L}$$

[note that $\sum_{r=1}^{R}|\nu_r| = 2L$].

Step 1. For the proof of (8.22), let $a=(a_1,a_2)$ and $b=(b_1,b_2)$. Then,

$$\mathbb{E}|\hat{H}_n(a;\omega)-\hat{H}_n(b;\omega)|^{2L} = n^{-3L}b_n^L \sum_{s_1,\dots,s_{2L}=1}^{n-1}\Big(\prod_{m=1}^{2L}W_n(\omega-2\pi s_m/n)\Big)$$
$$\times \sum_{j_1,\dots,j_{2L}=0}^{n-1}\sum_{k_1,\dots,k_{2L}=0}^{n-1}\mathbb{E}\Big[\prod_{m=1}^{2L}A_{j_m k_m}(a,b)\Big]\exp\Big(-\frac{2\pi}{n}\mathrm{i}\sum_{m=1}^{2L}(-1)^{m-1}s_m(j_m-k_m)\Big),$$

where

$$\begin{aligned}A_{jk}(a,b) &:= B_{jk}(a,b)-\mathbb{E}B_{jk}(a,b)\\ B_{jk}(a,b) &:= I\{X_j\le a_1\}I\{X_k\le a_2\} - I\{X_j\le b_1\}I\{X_k\le b_2\}\\ &= \sigma_1 I\{X_j\in M_1(1)\}I\{X_k\in M_2(1)\} + \sigma_2 I\{X_j\in M_1(2)\}I\{X_k\in M_2(2)\}\end{aligned}$$

with

$$\begin{aligned} \sigma_1 &:= 2I\{a_1>b_1\}-1, & \sigma_2 &:= 2I\{a_2>b_2\}-1,\\ M_1(1) &:= (a_1\wedge b_1, a_1\vee b_1], & M_2(2) &:= (a_2\wedge b_2, a_2\vee b_2],\\ M_2(1) &:= \begin{cases}[0,a_2] & b_2\ge a_2\\ [0,b_2] & a_2>b_2,\end{cases} & M_1(2) &:= \begin{cases}[0,b_1] & b_2\ge a_2\\ [0,a_1] & a_2>b_2.\end{cases}\end{aligned} \tag{8.24}$$

Note that, for each $\ell=1,2$, $\mathbb{P}(X_0\in M_\ell(\ell)) = \lambda(M_\ell(\ell)) \le \|a-b\|_1 \le \varepsilon$. The product theorem (Theorem 2.3.2 of Brillinger (1975)) entails

$$\mathbb{E}\Big[\prod_{\ell=1}^{2L}A_{j_\ell k_\ell}(a,b)\Big] = \sum_{\substack{\{\nu_1,\dots,\nu_R\}\\ |\nu_j|\ge 2,\ j=1,\dots,R}}\prod_{r=1}^{R}\mathrm{cum}(B_{j_i k_i}(a,b)\ :\ i\in\nu_r)$$

where the sum runs over all partitions $\{\nu_1, \dots, \nu_R\}$ of $\{1, \dots, 2L\}$. Note that $\mathbb{E}A_{jk}(a,b) = 0$; consequently, a summand is vanishing for any partition which has some ν_j with $|\nu_j| = 1$. Therefore, it suffices to consider summation over the partitions for which $|\nu_j| \geq 2$ for all $j = 1, \dots, R$.

Furthermore,

$$
\begin{aligned}
&\sum_{j=0}^{n-1}\sum_{k=0}^{n-1} B_{jk}(a,b)\exp(-\mathrm{i}(2\pi/n)[s(j-k)]) \\
&= \sum_{j=0}^{n-1}\sum_{k=0}^{n-1}\Big(\sigma_1 I\{X_j \in M_1(1)\}I\{X_k \in M_2(1)\} + \sigma_2 I\{X_j \in M_1(2)\}I\{X_k \in M_2(2)\}\Big) \\
&\qquad\qquad \times \exp(-\mathrm{i}(2\pi/n)[s(j-k)]) \\
&= \sigma_1 D_{1,s} + \sigma_2 D_{2,s},
\end{aligned}
$$

which yields

$$
\begin{aligned}
\mathbb{E}|\hat{H}_n(a;\omega) - \hat{H}_n(b;\omega)|^{2L} &= n^{-3L} b_n^L \sum_{s_1,\dots,s_{2L}=1}^{n-1} \Big(\prod_{m=1}^{2L} W_n(\omega - 2\pi s_m/n)\Big) \\
&\quad \times \sum_{\substack{\{\nu_1,\dots,\nu_R\} \\ |\nu_j|\geq 2,\ j=1,\dots,R}} \prod_{r=1}^{R} \mathrm{cum}(\sigma_1 D_{1,(-1)^{m-1}s_m} + \sigma_2 D_{2,(-1)^{m-1}s_m} \,:\, m \in \nu_r) \\
&= \sum_{\substack{\{\nu_1,\dots,\nu_R\} \\ |\nu_j|\geq 2,\ j=1,\dots,R}} \prod_{r=1}^{R} \mathcal{D}_{a,b}(\nu_r),
\end{aligned}
$$

and concludes the proof of (8.22).

Step 2. Still by the product theorem, letting $q = |\xi|$,

$$
\begin{aligned}
\mathcal{D}_{a,b}(\xi) &= \sum_{\ell_{\xi_1},\dots,\ell_{\xi_q} \in \{1,2\}} n^{-3q/2} b_n^{q/2} \sum_{s_{\xi_1},\dots,s_{\xi_q}=1}^{n-1} \Big(\prod_{m\in\xi} W_n(\omega - 2\pi s_m/n)\Big) \\
&\quad \times \Big(\prod_{m\in\xi} \sigma_{\ell_m}\Big) \sum_{\{\mu_1,\dots,\mu_N\}} \prod_{j=1}^{N} \mathrm{cum}(d_n^{M_k(\ell_m)}(2\pi(-1)^{k+m} s_m/n) : (m,k) \in \mu_j)
\end{aligned}
$$

where the summation runs over all *indecomposable partitions* $\{\mu_1, \dots, \mu_N\}$

(see Brillinger (1975), p. 20) of the scheme

$$\begin{array}{cc} (\xi_1, 1) & (\xi_1, 2) \\ \vdots & \vdots \\ (\xi_q, 1) & (\xi_q, 2). \end{array} \tag{8.25}$$

Note that for each $m \in \xi \subset \{1, \dots, 2L\}$, there exists a $j \in \{1, 2\}$ such that $\mathbb{P}(X_0 \in M_j(\ell_m)) = \lambda(M_j(\ell_m)) \le \|a - b\|_1 \le \varepsilon$.

Now, by assumption (7.26),

$$\begin{aligned} |\mathcal{D}_{a,b}(\xi)| \quad \le \quad & K n^{-3q/2} b_n^{q/2} 2^q \sum_{\{\mu_1,\dots,\mu_N\}} \sum_{s_{\xi_1},\dots,s_{\xi_q}=1}^{n-1} \Big(\prod_{m\in\xi} \Big| W_n(\omega - 2\pi s_m/n) \Big| \Big) \\ & \times \Big(\Big| \Delta_n \Big(\frac{2\pi}{n} \sum_{(m,k)\in\mu_1} (-1)^{m+k} s_m \Big) \Big| + 1 \Big) \\ & \times \cdots \times \Big(\Big| \Delta_n \Big(\frac{2\pi}{n} \sum_{(m,k)\in\mu_N} (-1)^{m+k} s_m \Big) \Big| + 1 \Big) g(\varepsilon). \end{aligned}$$

An indecomposable partition $\{\mu_1, \dots, \mu_N\}$ of the scheme (8.25) consists of at most $N \le q+1$ sets, because any partition with $N \ge q+2$ is necessarily decomposable. To see this, note that there is only one partition with $N = 2q$ and that this partition is decomposable. Any partition with $N = 2q - i < 2q$ sets can be obtained by i steps of agglomeration (i. e., iteratively merging sets from the partition, where each step reduces the number of sets by one unit). Obviously, it requires at least $q-1$ steps to obtain an indecomposable partition. Therefore, any partition that is the result of a sequence of $q-2$ steps (or less) is decomposable. Any partition with at least $2q-(q-2) = q+2$ sets thus is decomposable.

We now follow an argument from Brillinger (cf. the proof of his Theorem 7.4.4) to complete the proof. As sketched there, we have, with the common convention that $\prod_{i\in\emptyset} a_i := 1$,

$$\begin{aligned} & \prod_{j=1}^{N} \Big(\Big| \Delta_n \Big(\frac{2\pi}{n} \sum_{(m,k)\in\mu_j} (-1)^{m+k} s_m \Big) \Big| + 1 \Big) \\ & \qquad = \sum_{I\subset\{1,\dots,N\}} \prod_{j\in I} \Delta_n \Big(\frac{2\pi}{n} \sum_{(m,k)\in\mu_j} (-1)^{m+k} s_m \Big), \end{aligned}$$

by using the fact that

$$0 \le \Delta_n \Big(\frac{2\pi}{n} k \Big) = \begin{cases} n & k \in n\mathbb{Z} \\ 0 & k \notin n\mathbb{Z}. \end{cases}$$

As explained by Brillinger, the functions Δ_n introduce linear constraints on summation with respect to s_m, $m \in \xi$. First note that the case $|I| = q+1 = N$ is irrelevant. Indeed, we then have that

$$\sum_{s_{\xi_1},\dots,s_{\xi_q}=1}^{n-1} \Big(\prod_{m\in\xi} \Big|W_n(\omega - 2\pi s_m/n)\Big|\Big) \prod_{j\in I} \Delta_n\Big(\frac{2\pi}{n} \sum_{(m,k)\in\mu_j} (-1)^{k+m} s_m\Big) = 0,$$

because $|I| > q$ implies that there exists an index $j \in I$ with $|\mu_j| = |\{(\bar{m},\bar{k})\}| = 1$, which in turn implies $\sum_{(m,k)\in\mu_j} (-1)^{m+k} s_m = (-1)^{\bar{m}+\bar{k}} s_{\bar{m}} \notin n\mathbb{Z}$ for all $s_{\bar{m}} = 1,\dots,n-1$.

Next consider the case $|I| \le q$. We have

$$\begin{aligned}&\sum_{s_{\xi_1},\dots,s_{\xi_q}=1}^{n-1} \Big(\prod_{m\in\xi} \Big|W_n(\omega - 2\pi s_m/n)\Big|\Big) \prod_{j\in I} \Delta_n\Big(\frac{2\pi}{n} \sum_{(m,k)\in\mu_j} (-1)^{k+m} s_m\Big)\\ &\qquad = \sum_{(s_{\xi_1},\dots,s_{\xi_q})\in S_n(\mu,I)} \Big(\prod_{m\in\xi} \Big|W_n(\omega - 2\pi s_m/n)\Big|\Big) n^{|I|},\end{aligned}$$

where

$$\begin{aligned}S_n(\mu,I) := \Big\{(s_{\xi_1},\dots,s_{\xi_q}) \in \{1,\dots,n-1\}^q \,\Big|\, \\ \sum_{(m,k)\in\mu_j} (-1)^{k+m} s_m \in n\mathbb{Z},\ \forall \mu_j \in \mu,\ j \in I\Big\}.\end{aligned}$$

Elementary linear algebra implies that there are $|I|$ linear constraints if $|I| < N$ and $|I|-1$ linear constraints if $|I| = N$. More precisely, for every element μ_j of the partition $\{\mu_1,\dots,\mu_N\}$, define a vector

$$w_m^{(j)} := (-1)^{m+1} I\{(m,1)\in\mu_j\} + (-1)^{m+2} I\{(m,2)\in\mu_j\} \in \{-1,0,1\}^L$$

for $m = 1,\dots,L$. Observe that the linear constraint introduced by the equality $\sum_{(m,k)\in\mu_j} (-1)^{k+m} s_m \in n\mathbb{Z}$ can be written as $(s_1,\dots,s_m)' w^{(j)} \in n\mathbb{Z}$. In particular, the linear constraints corresponding to $\mu_{j_1},\dots,\mu_{j_\ell}$ are linearly dependent if and only if $\sum_{k=1}^{\ell} w^{(j_k)} = 0$, which follows from the special structure of the vectors $w^{(j)}$ [note, in particular, that at each position $k = 1,\dots,2L$, at most two vectors $w^{(1)},\dots,w^{(N)}$ can have non-zero entries, and that in this case the entry in one vector is 1 and the entry in the other vector is -1]. However, for non-decomposable partitions $\sum_{k=1}^{\ell} w^{(j_k)} = 0$ if and only if $\{j_1,\dots,j_\ell\} = \{1,\dots,N\}$.

To complete the proof of (8.23) it is therefore sufficient to show that
(8.26)

$$\sum_{(s_{\xi_1},\ldots,s_{\xi_q})\in S_n(\mu,I)} \left(\prod_{m\in\xi} \Big|W_n(\omega-2\pi s_m/n)\Big|\right) = O\Big((b_n^{-1})^{|I|-\lfloor |I|/N\rfloor} n^{q-(|I|-\lfloor |I|/N\rfloor)}\Big),$$

because this implies that $\mathcal{D}_{a,b}(\xi)$ is of the order

$$n^{-3q/2} b_n^{q/2} \max_{N\le q} \max_{|I|\le N} (b_n^{-1})^{|I|-\lfloor |I|/N\rfloor} n^{q-(|I|-\lfloor |I|/N\rfloor)} n^{|I|} g(\varepsilon) \asymp (nb_n)^{1-q/2} g(\varepsilon).$$

As for the proof of (8.26), it suffices to point out that $|I| - \lfloor |I|/N\rfloor$ of the s-indices can be expressed via the independent linear constraints and will take only a number of values which is less or equal to q. Then, (8.26) follows from the fact that

$$n\frac{1}{n}\sum_{s=1}^{n-1} \Big|W_n(\omega-2\pi s/n)\Big| \le n\left(\int_{\mathbb{R}} b_n^{-1} \Big|W\big(b_n^{-1}(\omega-\beta)\big)\Big| \mathrm{d}\beta + o(1)\right) = O(n),$$

and $|W_n(\omega)| \le \|W\|_\infty b_n^{-1} = O(b_n^{-1})$. The proof is thus complete. $\square$

8.3.3. *Proof of Lemma 7.3.* Observe that

$$\begin{aligned}
&\mathrm{cum}(I\{X_0 \le q_{a_1}\}, I\{X_k \le q_{a_2}\}) - \mathrm{cum}(I\{X_0 \le q_{b_1}\}, I\{X_k \le q_{b_2}\})\\
&\quad = \mathrm{cum}(I\{F(X_0) \le a_1\}, I\{F(X_k) \le a_2\})\\
&\qquad - \mathrm{cum}(I\{F(X_0) \le b_1\}, I\{F(X_k) \le b_2\})\\
&\quad = \sigma_1 \,\mathrm{cum}(I\{F(X_0) \in M_1(1)\}, I\{F(X_k) \in M_2(1)\})\\
&\qquad + \sigma_2 \,\mathrm{cum}(I\{F(X_0) \in M_1(2)\}, I\{F(X_k) \in M_2(2)\})
\end{aligned}$$

where

$$\begin{aligned}
\sigma_1 &:= 2I\{a_1 > b_1\} - 1, & \sigma_2 &:= 2I\{a_2 > b_2\} - 1,\\
M_1(1) &:= (a_1 \wedge b_1, a_1 \vee b_1], & M_2(2) &:= (a_2 \wedge b_2, a_2 \vee b_2],\\
M_2(1) &:= \begin{cases} [0, a_2] & b_2 \ge a_2 \\ [0, b_2] & a_2 > b_2, \end{cases} & M_1(2) &:= \begin{cases} [0, b_1] & b_2 \ge a_2 \\ [0, a_1] & a_2 > b_2. \end{cases}
\end{aligned}$$

In particular, observe that $\lambda(M_j(j)) \leq \|a-b\|_1$ for $j=1,2$. We thus have

$$\begin{aligned}
&\Big|\frac{\mathrm{d}^j}{\mathrm{d}\omega^j}\mathfrak{f}_{q_{a_1},q_{a_2}}(\omega) - \frac{\mathrm{d}^j}{\mathrm{d}\omega^j}\mathfrak{f}_{q_{b_1},q_{b_2}}(\omega)\Big| \\
&\quad \leq \sum_{k\in\mathbb{Z}} |k|^j |\operatorname{cum}(I\{F(X_0)\in M_1(1)\}, I\{F(X_k)\in M_2(1)\})| \\
&\qquad + \sum_{k\in\mathbb{Z}} |k|^j |\operatorname{cum}(I\{F(X_0)\in M_1(2)\}, I\{F(X_k)\in M_2(2)\}| \\
&\quad \leq 4\sum_{k=0}^{\infty} k^j \Big((K\rho^k)\wedge \|a-b\|_1\Big),
\end{aligned}$$

and the assertion follows by simple algebraic manipulations similar to those in the proof of Proposition 3.1. □

8.3.4. *Proof of Lemma 7.4.* By the definition of cumulants and strict stationarity, we have

$$\begin{aligned}
&\operatorname{cum}(d_n^{A_1}(\omega_1),\ldots,d_n^{A_p}(\omega_p)) \\
&= \sum_{t_1=0}^{n-1}\cdots\sum_{t_p=0}^{n-1} \operatorname{cum}(I_{\{X_{t_1}\in A_1\}},\ldots,I_{\{X_{t_p}\in A_p\}})\exp\Big(-\mathrm{i}\sum_{j=1}^p t_j\omega_j\Big) \\
&= \sum_{t_1=0}^{n-1}\exp\Big(-\mathrm{i}t_1\sum_{j=1}^p\omega_j\Big)\sum_{t_2,\ldots,t_p=0}^{n-1}\exp\Big(-\mathrm{i}\sum_{j=2}^p\omega_j(t_j-t_1)\Big) \\
&\qquad \times \operatorname{cum}(I_{\{X_0\in A_1\}}, I_{\{X_{t_2-t_1}\in A_2\}}\ldots, I_{\{X_{t_p-t_1}\in A_p\}}) \\
&= \sum_{u_2,\ldots,u_p=-n}^{n} \operatorname{cum}(I_{\{X_0\in A_1\}}, I_{\{X_{u_2}\in A_2\}}\ldots, I_{\{X_{u_p}\in A_p\}})\exp\Big(-\mathrm{i}\sum_{j=2}^p\omega_j u_j\Big)
\end{aligned}$$

$$\times\sum_{t_1=0}^{n-1}\exp\Big(-\mathrm{i}t_1\sum_{j=1}^p\omega_j\Big)I_{\{0\leq t_1+u_2<n\}}\cdots I_{\{0\leq t_1+u_p<n\}}. \tag{8.27}$$

Lemma 8.1 implies that

$$\Big|\Delta_n(\sum_{j=1}^p\omega_j) - \sum_{t_1=0}^{n-1}\exp\Big(-\mathrm{i}t_1\sum_{j=1}^p\omega_j\Big)I_{\{0\leq t_1+u_2<n\}}\cdots I_{\{0\leq t_1+u_p<n\}}\Big| \leq 2\sum_{j=2}^p|u_j|. \tag{8.28}$$

Let us show that, for any $p+1$ intervals $A_0,\ldots,A_p\subset\mathbb{R}$ and any p-tuple $\kappa := (\kappa_1,\ldots,\kappa_p)\in\mathbb{R}_+^p$, $p\geq 2$

$$\sum_{k_1,\ldots,k_p=-\infty}^{\infty}\Big(1+\sum_{j=1}^p|k_j|^{\kappa_j}\Big)\big|\operatorname{cum}\big(I_{\{X_{k_1}\in A_1\}},\ldots,I_{\{X_{k_p}\in A_p\}},I_{\{X_0\in A_0\}}\big)\big|$$

$$\leq C\varepsilon(|\log\varepsilon|+1)^d. \tag{8.29}$$

To this end, define $k_0 = 0$ and consider the set

$$T_m := \Big\{(k_1,...,k_p) \in \mathbb{Z}^p \big| \max_{i,j=0,\dots,p} |k_i - k_j| = m\Big\}$$

and note that $|T_m| \leq c_p m^{p-1}$ for some constant c_p. With this notation, it follows from condition (C) and the bound

$$|\operatorname{cum}(I\{X_{t_1} \in A_1\},...,I\{X_{t_p} \in A_p\})| \leq C \min_{i=1,\dots,p} P(X_0 \in A_i),$$

which follows from the definition of cumulants and some simple algebra, that

$$\sum_{k_1,\dots,k_p=-\infty}^{\infty} \Big(1 + \sum_{j=1}^{p} |k_j|^{\kappa_j}\Big) \big|\operatorname{cum}\big(I_{\{X_{k_1}\in A_1\}},\dots,I_{\{X_{k_p}\in A_p\}}, I_{\{X_0\in A_0\}}\big)\big|$$

$$= \sum_{m=0}^{\infty} \sum_{(k_1,\dots,k_p)\in T_m} \Big(1 + \sum_{j=1}^{p} |k_j|^{\kappa_j}\Big) \big|\operatorname{cum}\big(I_{\{X_{k_1}\in A_1\}},\dots,I_{\{X_{k_p}\in A_p\}}, I_{\{X_0\in A_0\}}\big)\big|$$

$$\leq \sum_{m=0}^{\infty} \sum_{(k_1,\dots,k_p)\in T_m} \Big(1 + p m^{\max_j \kappa_j}\Big)\Big(\rho^m \wedge \varepsilon\Big) K_p$$

$$\leq C_p \sum_{m=0}^{\infty} \Big(\rho^m \wedge \varepsilon\Big) |T_m| m^{\max_j \kappa_j}.$$

For $\varepsilon \geq \rho$, (8.29) follows trivially. For $\varepsilon < \rho$, set $m_\varepsilon := \log\varepsilon/\log\rho$ and note that $\rho^m \leq \varepsilon$ if and only if $m \geq m_\varepsilon$. Thus

$$\begin{aligned}
\sum_{m=0}^{\infty} \Big(\rho^m \wedge \varepsilon\Big) m^u &\leq \sum_{m\leq m_\varepsilon} m^u \varepsilon + \sum_{m>m_\varepsilon} m^u \rho^m \\
&\leq C\Big(\varepsilon m_\varepsilon^{u+1} + \rho^{m_\varepsilon} \sum_{m=0}^{\infty} (m+m_\varepsilon)^u \rho^m\Big).
\end{aligned}$$

Observing that $\rho^{m_\varepsilon} = \varepsilon$ completes the proof of the desired inequality (8.29). The lemma then follows from (8.27), (8.28), (8.29) and the triangular inequality. □

8.3.5. *Proof of Lemma 7.5.* By the functional delta method applied to the map $F \mapsto F^{-1}$ [see Theorem 3.9.4 and Lemma 3.9.23(ii) in van der Vaart and Wellner (1996)], it suffices to show that $\sqrt{n}(\hat{F}_n(\tau) - \tau)$ converges to a tight Gaussian limit with continuous sample paths. This can be done by proving convergence of finite-dimensional distributions together with stochastic equicontinuity [see the discussion in the proof of Theorem 3.6(iii)]. The stochastic equicontinuity follows by an application of Lemma 7.1, Lemma 8.6 and (8.18). For the convergence of the finite-dimensional distributions, apply the cumulant central limit theorem [Lemma P4.5 in Brillinger (1975)] in combination with Lemma 7.4. □

8.3.6. *Proof of Lemma 7.6.* Let $T := [0,1]$, $T_n := \{j/n : j = 0, \dots, n\}$, and note that, for n large enough,

(8.30)

$$\sup_{\omega\in\mathcal{F}_n}\sup_{\tau\in T}|d_n^\tau(\omega)| \le \max_{\omega\in\mathcal{F}_n}\max_{\tau\in T_n}|d_n^\tau(\omega)| + \max_{\omega\in\mathcal{F}_n}\max_{\tau\in T_n}\sup_{|\eta-\tau|\le 1/n}|d_n^\tau(\omega) - d_n^\eta(\omega)|.$$

Expressing moments in terms of cumulants, straightforward arguments and Lemma 7.4 yield

$$\max_{\omega\in\mathcal{F}_n}\max_{\tau\in T_n}\mathbb{E}|d_n^\tau(\omega)|^{2k} \le C_k n^k.$$

Thus $n^{-1/2}d_n^\tau(\omega)$ has uniformly bounded moments of order $2k$. Recall that an arbitrary sequence $(Z_j)_{j\in\mathbb{Z}}$ of random variables with uniformly bounded moments of order m is such that $\max_{j=1,\dots,N}|Z_j| = O_P(N^{1/m})$. Thus

$$\max_{\omega\in\mathcal{F}_n}\max_{\tau\in T_n} n^{-1/2}|d_n^\tau(\omega)| = O_P((n^2)^{1/2k}) = O_P(n^{1/k})$$

since the maximum is taken over $O(n^2)$ values. For the second term in the right-hand side of (8.30), note that

$$\max_{\omega\in\mathcal{F}_n}\left|d_n^\tau(\omega) - d_n^\eta(\omega)\right| \le \sum_{t=0}^{n-1} I\{X_t \le \tau\vee\eta\} - I\{X_t \le \tau\wedge\eta\}.$$

Thus, by Lemma 8.6, we have

$$\begin{aligned}
&\max_{\omega\in\mathcal{F}_n}\max_{\tau\in T_n}\sup_{|\eta-\tau|\le 1/n}|d_n^\tau(\omega) - d_n^\eta(\omega)| \\
&\le n\max_{\tau\in T_n}\sup_{|\eta-\tau|\le 1/n}|\hat{F}_n(\tau\vee\eta) - \hat{F}_n(\tau\wedge\eta) - \tau\vee\eta + \tau\wedge\eta| + C \\
&= O_P(n^{1/2+1/2k}(\log n)^{d_k}).
\end{aligned}$$

for some constant d_k. This completes the proof. □

8.3.7. *Proof of Lemma 7.7.* Without loss of generality, we can assume that $n^{-1} = o(\delta_n)$ [otherwise, enlarge the supremum by considering $\tilde{\delta}_n := \max(n^{-1}, \delta_n)$]. Letting $u = (u_1, u_2)$ and $v = (v_1, v_2)$,

$$\hat{H}_n(u;\omega) - \hat{H}_n(v;\omega) = b_n^{1/2}n^{-1/2}\sum_{s=1}^{n-1} W_n(\omega - 2\pi s/n)(K_{s,n}(u,v) - \mathbb{E}K_{s,n}(u,v))$$

where [with $d_{n,U}$ defined in (2.6)]

$$\begin{aligned} K_{s,n}(u,v) &:= n^{-1}\big(d_{n,U}^{u_1}(2\pi s/n)d_{n,U}^{u_2}(-2\pi s/n) - d_{n,U}^{v_1}(2\pi s/n)d_{n,U}^{v_2}(-2\pi s/n)\big) \\ &= d_{n,U}^{u_1}(2\pi s/n)n^{-1}\big[d_{n,U}^{u_2}(-2\pi s/n) - d_{n,U}^{v_2}(-2\pi s/n)\big] \\ &\quad + d_{n,U}^{v_2}(-2\pi s/n)n^{-1}\big[d_{n,U}^{u_1}(2\pi s/n) - d_{n,U}^{v_1}(2\pi s/n)\big]. \end{aligned}$$

Note that, by Lemma 7.6, we have, for any $k \in \mathbb{N}$,

$$\sup_{y\in[0,1]}\sup_{\omega\in\mathcal{F}_n}|d_{n,U}^{y}(\omega)| = O_P\Big(n^{1/2+1/k}\Big). \tag{8.31}$$

Furthermore, by Lemma 8.6, for any $\ell \in \mathbb{N}$,

$$\begin{aligned} &\sup_{\omega\in\mathbb{R}}\sup_{y\in[0,1]}\sup_{x:|x-y|\le\delta_n} n^{-1}|d_{n,U}^{x}(\omega) - d_{n,U}^{y}(\omega)| \\ &\le \sup_{y\in[0,1]}\sup_{x:|x-y|\le\delta_n} n^{-1}\sum_{t=0}^{n-1}|I\{Y_t\le x\} - I\{Y_t\le y\}| \\ &\le \sup_{y\in[0,1]}\sup_{x:|x-y|\le\delta_n} |\hat{F}_n(x\vee y) - \hat{F}_n(x\wedge y) - F(x\vee y) + F(x\wedge y)| + C\delta_n \\ &= O_P\big(\rho_n(\delta_n,\ell) + \delta_n\big), \end{aligned}$$

with $\rho_n(\delta_n,\ell) := n^{-1/2}(n^2\delta_n + n)^{1/2\ell}(\delta_n|\log\delta_n|^{d_\ell} + n^{-1})^{1/2}$, where d_ℓ is a constant depending only on ℓ. Combining these arguments and observing that $\sup_{\omega\in\mathbb{R}}\sum_{s=1}^{n-1}\Big|W_n(\omega - 2\pi s/n)\Big| = O(n)$ yields

$$\sup_{\omega\in\mathbb{R}}\sup_{\substack{u,v\in[0,1]^2\\ \|u-v\|_1\le\delta_n}}\Big|\sum_{s=1}^{n-1}W_n(\omega-2\pi s/n)K_{s,n}(u,v)\Big| = O_P\big(n^{3/2+1/k}(\rho(\delta_n,\ell)+\delta_n)\big). \tag{8.32}$$

Next, define the intervals

$$M_1(1) := (u_1\wedge v_1, u_1\vee v_1], \qquad M_2(2) := (u_2\wedge v_2, u_2\vee v_2],$$

$$M_2(1) := \begin{cases}[0,u_2] & v_2\ge u_2\\ [0,v_2] & u_2 > v_2,\end{cases} \qquad M_1(2) := \begin{cases}[0,v_1] & v_2\ge u_2\\ [0,u_1] & v_2 > v_2.\end{cases}$$

With this notation, observe that

$$\begin{aligned} &\sup_{\|u-v\|_1\le\delta_n}\sup_{s=1,\dots,n-1}|\mathbb{E}K_{s,n}(u,v)| \\ &\quad\le n^{-1}\sup_{\|u-v\|_1\le\delta_n}\sup_{s=1,\dots,n-1}\big|\operatorname{cum}(d_{n,U}^{M_1(1)}(2\pi s/n), d_{n,U}^{M_2(1)}(-2\pi s/n))\big| \\ &\quad\quad + n^{-1}\sup_{\|u-v\|_1\le\delta_n}\sup_{s=1,\dots,n-1}\big|\operatorname{cum}(d_{n,U}^{M_1(2)}(2\pi s/n), d_{n,U}^{M_2(2)}(-2\pi s/n))\big| \end{aligned} \tag{8.33}$$

where we have used the fact that $\mathbb{E}d_{n,U}^{M}(2\pi s/n) = 0$. Lemma 7.4 and the fact that $\lambda(M_j(j)) \leq \delta_n$ (with λ denoting the Lebesgue measure over $\mathbb{R}$) for $j = 1, 2$ yield

$$\sup_{\|u-v\|_1\leq\delta_n} \sup_{s=1,\ldots,n-1} |\operatorname{cum}(d_n^{M_1(j)}(2\pi s/n), d_n^{M_2(j)}(-2\pi s/n))| \leq C(n+1)\delta_n(1+|\log\delta_n|)^d,$$

It follows that the right-hand side in (8.33) is $O(\delta_n|\log\delta_n|^d)$. Therefore, since $\sup_{\omega\in\mathbb{R}}\sum_{s=1}^{n-1}\Big|W_n(\omega-2\pi s/n)\Big| = O(n)$, we obtain

$$\sup_{\omega\in\mathbb{R}} \sup_{\|u-v\|_1\leq\delta_n} \Big|b_n^{1/2}n^{-1/2}\sum_{s=1}^{n-1}W_n(\omega-2\pi s/n)\mathbb{E}K_{s,n}(u,v)\Big| = O\big((nb_n)^{1/2}\delta_n|\log n|^d\big).$$

Observe that, in view of the assumption that $n^{-1} = o(\delta_n)$, we have $\delta_n = O(n^{1/2}\rho_n(\delta_n,\ell))$, which, in combination with (8.32), yields

$$\begin{aligned}
&\sup_{\omega\in\mathbb{R}} \sup_{\|u-v\|_1\leq\delta_n} |\hat{H}_n(u;\omega) - \hat{H}_n(v;\omega)| \\
&= O_P\Big((nb_n)^{1/2}[n^{1/2+1/k}(\rho_n(\delta_n,\ell)+\delta_n) + \delta_n|\log\delta_n|^d]\Big) \\
&= O_P\Big((nb_n)^{1/2}n^{1/2+1/k}\rho_n(\delta_n,\ell)\Big) \\
&= O_P\Big((nb_n)^{1/2}n^{1/k+1/\ell}(n^{-1}\vee\delta_n(\log n)^{d_\ell})^{1/2}\Big).
\end{aligned}$$

This latter quantity is $o_P(1)$: indeed, for arbitrary k and ℓ,

$$O((nb_n)^{1/2}n^{1/k+1/\ell}\delta_n^{1/2}(\log n)^{d_\ell/2}) = O((nb_n)^{1/2-1/2\gamma}n^{1/k+1/\ell}(\log n)^{d_\ell/2});$$

in view of the assumptions on b_n, which imply $(nb_n)^{1/2-1/2\gamma} = o(n^{-\kappa})$ for some $\kappa > 0$, this latter quantity is $o(1)$ for k, ℓ sufficiently large. The term $(nb_n)^{1/2}n^{1/k+1/\ell}n^{-1/2}$ can be handled in a similar fashion. This concludes the proof. □

Address of the First author
Fakultät für Mathematik
Ruhr-Universität Bochum
44780 Bochum, Germany
E-mail: tobias.kley@ruhr-uni-bochum.de

Address of the Second author
Fakultät für Mathematik
Ruhr-Universität Bochum
44780 Bochum, Germany
E-mail: stanislav.volgushev@ruhr-uni-bochum.de

Address of the Third author
Fakultät für Mathematik
Ruhr-Universität Bochum
44780 Bochum, Germany
E-mail: holger.dette@rub.de

Address of the Fourth author
Marc Hallin
ECARES
Université libre de Bruxelles CP 114/04
50, avenue F.D. Roosevelt
B-1050 Bruxelles, Belgium
E-mail: mhallin@ulb.ac.be