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On polarization interface conditions for time-harmonic Maxwell's equations

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Abstract: We consider the time-harmonic Maxwell's equations with a polarization interface condition. The interface condition demands that one component of the electric field vanishes at the interface and that the corresponding component of the magnetic field has no jump across the interface. These conditions have been derived in the literature as a homogenization limit for thin wire inclusion. We analyze the limit equations and provide an existence result and a Fredholm-alternative.

1. Introduction

Polarization filters for electromagnetic waves are interesting for many technical applications such as, e.g., LED monitors. The filters have the property that waves with a certain polarization can pass the filter, waves with the oppposite polarization cannot pass the filter, see Figure 1. Polarization filters are also a very interesting mathematical object. Maxwell's equations describe the two electromagnetic fields E and H on the two sides of the filter, the filter is modelled by some interface conditions. These conditions demand continuity for certain components of the fields and homogeneous Dirichlet conditions for other components. The transmission conditions have been derived recently with mathematical rigor. In this article, we are concerned with the analysis of the limit model and clarify the well-posedness. We treat the time-dependent system; well-posedness of the time-dependent system could be concluded with a Fourier transform in time.

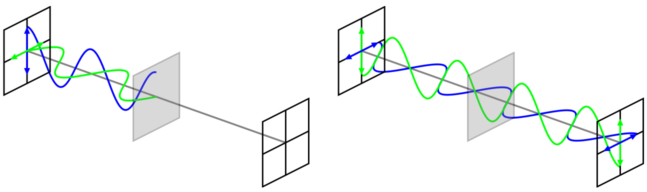

Figure 1. Polarization: Electromagnetic waves (electric field in green and magnetic field in blue) propagating from the left-hand side and interacting with a polarization interface.

The system of interest is (1.4), it has been derived with homogenization techniques in [9] and [18]. Let us describe these results briefly, using the notation of the latter work. Maxwell's equations are studied in a complex domain Ω_η, accordingly, the solutions E^η and H^η depend on the parameter $\eta > 0$. The domain is obtained by removing a small scale structure from an underlying domain Ω, an open cuboid in $\mathbb{R}^3$. The cuboid contains a flat interface Γ. A microstructure Σ_η is

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defined (in the most relevant of three examples) by rods with cross-sectional diameter of order η, rods that are parallel and distributed along Γ with periodicity η. The complex domain is obtained by removing the union of rods from the underlying domain: $\Omega_\eta = \Omega \setminus \Sigma_\eta$. The geometry is visualized in Figure 2. Maxwell's equations in Ω_η with boundary conditions that model perfectly conducting material in the rods provides solutions E^η, H^η. In [9] and [18], the effective limit system (1.4) is derived. In [9], asymptotic expansions are used. In [18], for rods parallel to the x_1-direction, it is shown with oscillating test-functions that, whenever there is a weak L^2-convergence $(E^\eta, H^\eta) \to (E, H)$, the limit (E, H) satisfies system (1.4).

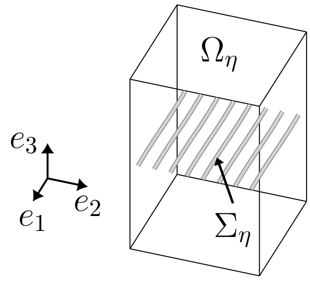

FIGURE 2. Domains Ω_η and Σ_η.

The articles [9] and [18] are concerned with the derivation of an effective system, not with the analysis of that system. To the best of our knowledge, an important question is open: Can we have results on existence and uniqueness for the limit system (1.4) (in dependence of ω). The article at hand answers this question. Let us stress the importance of this step: Without this analysis, we cannot be sure that the system (1.4) is complete in the sense that all equations are found. Additionally, we cannot be sure if (1.4) is over-determined (such that it cannot be solved in general). With the analysis of the paper at hand we clarify that the limit system is well-posed.

On the technical side, we mention that the derivation in [18] assumes that the solution sequence (E^η, H^η) is bounded in L^2. Since this is not verified, [18] does not provide the existence of solutions to (1.4).

The existence and uniqueness results (or, more precisely, the Fredholm alternative) for system (1.4) turns out to be quite tricky. There is a natural approach that we sketch in Section 1.2: One defines variants of $H(\text{curl})$-spaces that incorporate some of the interface conditions. With a compactness result for these $H(\text{curl})$-spaces one derives a Fredholm alternative; this is the classical approach, see [14]. We did not succeed in showing the desired compactness.

Since we are lacking a compactness result, we must use another approach. Our idea is to use an equivalent formulation of the Maxwell system with a family of Helmholtz-type systems. The analysis of the latter is quite simple so that we can determine the spectrum of the Maxwell system.

1.1. **Maxwell equations with a polarization interface condition.** To simplify the setting, we consider cuboids $\Omega \subset \mathbb{R}^3$, aligned with the coordinate axes, and an aligned interface. We consider length parameters $l_1, l_2, l_3^+, l_3^- > 0$ and $l_3 = l_3^+ + l_3^-$ to define the intervals

$$I_1 := (0, l_1)\,,\ I_2 := (0, l_2)\,,\ I_3 := (-l_3, l_3)\,,\ I_3^+ := (0, l_3^+)\,,\ I_3^- := (-l_3^-, 0)\,. \tag{1.1}$$

The domain Ω and the interface Γ are

$$\Omega := I_1 \times I_2 \times I_3\,, \qquad \Gamma := I_1 \times I_2 \times \{0\}\,. \tag{1.2}$$

The upper part of the domain is $\Omega_+ := I_1 \times I_2 \times I_3^+$ and the lower part is $\Omega_- := I_1 \times I_2 \times I_3^-$. For the top and the bottom boundary of Ω we use the notation

$$\partial_{\text{top}}\Omega := I_1 \times I_2 \times \{l_3^+\}\,, \quad \partial_{\text{bot}}\Omega := I_1 \times I_2 \times \{-l_3^-\}\,, \quad \partial_{\text{hor}}\Omega := \partial_{\text{top}}\Omega \cup \partial_{\text{bot}}\Omega\,. \tag{1.3}$$

We treat time-harmonic Maxwell's equations, solutions are pairs (E, H) with $E, H : \Omega \to \mathbb{C}^3$. The polarization interface is given by Γ. The reader should think of this interface as an effective description of long and thin conductive objects that are elongated in the first coordinate direction e_1, as visualized in Figure 2. The effective behavior of this geometry is given by the interface conditions (1.4c)–(1.4d). For notational convenience, we use periodicity boundary conditions along the lateral boundaries. In the following system, the frequency $\omega > 0$, the permittivity $\varepsilon > 0$, the permeability $\mu > 0$ and source terms f_h, f_e are given. The system has to be solved for E and H.

$$\text{(1.4a)} \qquad \operatorname{curl} E = i\omega\mu H + f_h \qquad \text{in } \Omega\,,$$
$$\text{(1.4b)} \qquad \operatorname{curl} H = -i\omega\varepsilon E + f_e \qquad \text{in } \Omega \setminus \Gamma\,,$$
$$\text{(1.4c)} \qquad E_1|_\Gamma = 0 \qquad \text{on } \Gamma\,,$$
$$\text{(1.4d)} \qquad [\![H_1]\!]_\Gamma = 0 \qquad \text{on } \Gamma\,,$$
$$\text{(1.4e)} \qquad E \times e_3 = 0 \qquad \text{on } \partial_{\text{top}}\Omega \cup \partial_{\text{bot}}\Omega\,,$$
$$\text{(1.4f)} \qquad x \mapsto (E, H)(x) \qquad \text{is } x_1\text{- and } x_2\text{-periodic}\,.$$

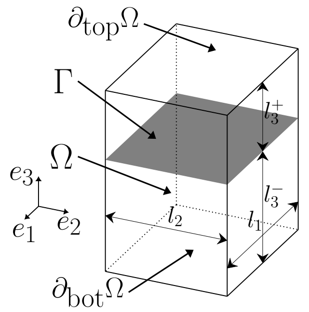

Figure 3. Geometry

Equation (1.4a) is imposed on the entire domain Ω; this implies that the tangential components E_1 and E_2 of E cannot jump across Γ. This implicit interface condition is supplemented by the interface conditions (1.4c) and (1.4d), which impose that E_1 vanishes at the interface and that H_1 does not jump across the interface. At the upper and lower boundaries of Ω, (1.4e) models a perfect conductor.

The weak solution concept for system (1.4) is made precise in Definition 2.3. This definition uses function spaces $X \subset H_0(\operatorname{curl}, \Omega)$ and $Y \subset H(\operatorname{curl}, \Omega \setminus \Gamma)$, see (2.2b) and (2.2a), respectively. The spaces include boundary conditions and interface conditions. Essentially, we will find solutions of system (1.4) with $E \in X$ and $H \in Y$. Technically, in the definition, the weak solution concept is slightly different: it uses Y as a space of test-functions for the E-equation and X as a space of test-functions for the H-equation. In the following, when we speak of solutions to (1.4), we always mean weak solutions in the sense of Definition 2.3.

Due to the simple geometry, we can determine the spectrum of the Maxwell operator. For lengths $L_1, L_2, L_3 > 0$ we set

$$\text{(1.5)} \qquad \sigma(L_1, L_2, L_3) := \left\{ \frac{4\pi^2}{\varepsilon\mu} \left(\frac{k_1^2}{L_1^2} + \frac{k_2^2}{L_2^2} + \frac{k_3^2}{4L_3^2} \right) \middle|\, k_1, k_2, k_3 \in \mathbb{N}_0 \right\},$$

with $\mathbb{N}_0 = \{0, 1, 2, ...\}$. These are actually the eigenvalues of the operator $-(\varepsilon\mu)^{-1}\Delta$ on the cuboid with side-lengths L_1, L_2, L_3 with periodicity conditions in the first two directions and with a Neumann condition in the third direction. The eigenfunctions for this operator $-(\varepsilon\mu)^{-1}\Delta$ are, when $(0, 0, 0)$ is a corner of the cuboid, $u(x_1, x_2, x_3) = \cos(2\pi k_1 x_1/L_1)\cos(2\pi k_2 x_2/L_2)\cos(\pi k_3 x_3/L_3)$.

We will see that the spectrum of the Maxwell operator in the geometry of (1.2) is given by

$$\text{(1.6)} \qquad \sigma_M := \sigma(l_1, l_2, l_3) \cup \sigma(l_1, l_2, l_3^+) \cup \sigma(l_1, l_2, l_3^-)\,.$$

It is the union of the spectra of $(\varepsilon\mu)^{-1}\Delta$ on the total domain, the upper part and the lower part. Our main result determines the spectrum of the Maxwell system.

Theorem 1.1 (Spectrum of the Maxwell system with a polarization interface)**.** *Let Ω and Γ be as in (1.2), we consider the Maxwell system (1.4) with parameters $\varepsilon, \mu > 0$. The spectrum of the Maxwell system (1.4) is σ_M of (1.6) in the following sense: (i) For $\omega^2 \notin \sigma_M$, system (1.4) has a unique solution for arbitrary $(f_h, f_e) \in L^2(\Omega, \mathbb{C}^3)^2$. (ii) For $\omega^2 \in \sigma_M$, system (1.4) has a non-trivial solution for $(f_h, f_e) = 0$.*

Theorem 1.1 implies the statement of the subsequent remark, which is less precise in the description of the spectrum. Remark 1.2 formulates a statement that we expect to be valid not only in the cuboid geometry with a flat interface, but also for more general domains Ω and more general interfaces Γ.

Remark 1.2 (Spectrum and Fredholm alternative)**.** *For parameters $\varepsilon, \mu > 0$ and in the geometry of (1.2), there holds for some discrete set $\sigma \subset \mathbb{R}$: For every $0 \neq \omega^2 \notin \sigma$, there exists a unique solution $(E, H) \in X \times Y$ of (1.4) for every $f_h, f_e \in L^2(\Omega, \mathbb{C}^3)$. For every $0 \neq \omega^2 \in \sigma$, the homogeneous system (1.4) has a non-trivial solution.*

In the above statements, it is no loss of generality to assume that the right-hand sides $f_h, f_e \in L^2(\Omega, \mathbb{C}^3)$ are divergence free in the distributional sense. Indeed, general f_h, f_e can be decomposed with a Helmholtz decomposition to reduce the Maxwell system to divergence-free right-hand sides. We sketch the well-known argument in Lemma 2.5 and Remark 2.6.

1.2. **On a variational formulation.** The Maxwell system (1.4) possesses a natural variational formulation. Let us describe this formulation even though our proof on existence and uniqueness follows a different route and uses Helmholtz equations.

We start with the "coercive Maxwell system", which is formally obtained by setting $\omega = i$. For notational convenience, we assume in this overview $f_h = 0$. Using E as the only unknown, the coercive Maxwell system reads $\operatorname{curl}(\operatorname{curl} E) + \varepsilon\mu\, E = -\mu f_e$. We use the function space X, defined in (2.2b), which consists of functions in $H(\operatorname{curl}, \Omega)$ which are periodic in the first and second direction, have vanishing tangential components at $\partial_{\text{hor}}\Omega$ and a vanishing first component at Γ. We define the sesquilinear form $B\colon X \times X \to \mathbb{C}$,

$$B(E, \phi) := \int_\Omega \left\{ \operatorname{curl} E \cdot \operatorname{curl} \bar{\phi} + \varepsilon\mu\, E \cdot \bar{\phi} \right\} . \tag{1.7}$$

The form B is coercive on X since $B(E, E)$ controls the L^2-norm of E and the L^2-norm of $\operatorname{curl} E$. By the Lax–Milgram lemma, for every $f_e \in L^2(\Omega)$, we can solve uniquely the problem $B(E, \phi) = -\mu \int f_e \cdot \bar{\phi}\ \forall \phi \in X$, which implies $\operatorname{curl}(\operatorname{curl} E) + \varepsilon\mu\, E = -\mu f_e$ in $\Omega \setminus \Gamma$. By setting $H := -\mu^{-1} \operatorname{curl} E$, due to our choice of function spaces, we see that all equations of (1.4) are satisfied with $\omega = i$ and $f_h = 0$ — the only open point being (1.4d) concerning the jump of H_1 across Γ.

The fact that we have solved $B(E, \phi) = -\mu \int f_e \cdot \bar{\phi}$ implies that

$$-\int_\Omega H \cdot \operatorname{curl} \bar{\phi} + \int_\Omega \varepsilon\, E \cdot \bar{\phi} = -\int_\Omega f_e \cdot \bar{\phi}$$

holds for all $\phi \in X$. This encodes not only (1.4b), but also, since $\operatorname{curl} H|_{\Omega_\pm}$ is an $L^2(\Omega_\pm)$-function, that the interface integral over the jump vanishes,

$$0 = \int_\Gamma (e_3 \times [\![H]\!]_\Gamma) \cdot \bar{\phi} = \int_\Gamma [\![H_1]\!]_\Gamma\ \bar{\phi}_2$$

for all $\phi \in X$. Since ϕ_2 can take arbitrary values on the inner interface Γ, the function H_1 cannot have a jump across Γ (in the sense of traces). With this argument, also

(1.4d) is verified. We conclude that, for the "coercive Maxwell system", the weak form can be solved uniquely and it encodes all equations.

The next goal is to transfer this existence and uniqueness result for $\omega = i$ to arbitrary $\omega \in \mathbb{R}$, or, more precisely, to all $\omega \in \mathbb{R} \setminus \{0\}$ except for the countable set of eigenvalues. The underlying idea is to perform the following Steps: (i) Repeat the arguments on some smaller space $X_{\rm div} \subset X$, which is defined by introducing an additional condition on the divergence of the functions E. (ii) Show that the space $X_{\rm div}$ is a compact subset of $L^2(\Omega)$. (iii) The difference between the original Maxwell system and the coercive system consists in the multiplication operator $(\omega^2 + 1)\,\mathrm{id}$. This operator is a bounded operator on $L^2(\Omega)$ and, hence, defines a compact perturbation of the system by the compactness of (ii). The conclusion of (i)–(iii) is that the Maxwell system defines a Fredholm operator.

Without the interface Γ, this program works. The space $X_{\rm div} \subset X$ is defined by imposing $\nabla \cdot E = 0$ in the distributional sense. The compactness is classical, see, e.g., [14]. We obtain that the Maxwell system is Fredholm.

With the interface Γ, we did not succeed to perform a proof along these lines. The main obstacle regards Steps (i) and (ii): We did not find a suitable divergence condition that allows to derive the compactness of $X_{\rm div}$ and to repeat the remaining arguments.

1.3. **Literature.** This contribution is related to compactness of $H(\mathrm{curl})$-spaces, for the classical theory see [14] and [11], extensions to mixed boundary conditions are made in [3], other methods are used [6], a very general treatment on the related Helmholtz decompositions is given in [2]. We also refer to the references in these works and to a short summary given in [17]. A Fredholm alternative for Maxwell's equations in a quasiperiodic setting is the basis of the analysis in [12]. We emphasize that all of these works do not treat polarization interfaces.

The homogenization of bulk heterogeneities is very classical, often treated with two-scale convergence, see [1]. Interesting limit equations can occur when resonances are exploited, see [4] and [5]. In [15], a closely related setting is investigated: Maxwell's equations in a domain with thin structures, possibly wires; in contrast to the present article, the inclusions are filling a subdomain and a bulk homogenization is performed. The work contains numerical aspects, more in this direction can be found in [10] and the references therein.

Small inclusions along a hypersurface require different methods, but they are also a classical topic, see [7]. A detailed study of inclusions along a hypersurface is given for Helmholtz equations in [8] and [16]. These results concern the critical scaling and treat η^0 and η^1 approximations of solutions. In this sense, those results are much more detailed than those of [9] and [18], which are the homogenization counterpart of the work at hand.

1.4. **Reformulation as a Helmholtz-type system.** We consider generalized divergence-free right-hand sides f_e and f_h in the sense that $f_e \in X_0^\perp$ and $f_h \in Y_0^\perp$ for X_0 and Y_0 given by (2.5). These properties imply, written in a strong form,

$$\operatorname{div} f_e = 0 \ \text{ in } \Omega \setminus \Gamma\,, \qquad \operatorname{div} f_h = 0 \ \text{ in } \Omega\,, \qquad (f_h)_3|_{\partial_{\rm hor}\Omega} = 0\,. \tag{1.8}$$

Our results are based on the observation that for such right-hand sides the Maxwell system (1.4) is equivalent to the Helmholtz-type system (1.9) below. We only have to demand a non-resonance condition on the frequency, namely $\omega^2 \notin \sigma(l_1) \subset \sigma_M$ for

$$\sigma(L_1) := \left\{ \frac{4\pi^2}{\varepsilon\mu} \frac{k_1^2}{L_1^2} \,\middle|\, k_1 \in \mathbb{N}_0 \right\} .$$

The six equations of (1.9) provide a strong formulation of the Helmholtz-type system. We emphasize that this strong formulation cannot be used for right-hand sides f_h and f_e of class $L^2(\Omega)$; this regards, in particular, the boundary condition (1.10b), since, in general, f_e does not have a trace. It is therefore necessary to understand system (1.9) in the weak sense, provided in Definition 3.1. The strong formulation is given here only for convenience of the reader.

$$\Delta E_1 + \omega^2 \varepsilon\mu E_1 = -i\omega\mu (f_e)_1 - (\operatorname{curl} f_h)_1 \,, \tag{1.9a}$$

$$\Delta H_1 + \omega^2 \varepsilon\mu H_1 = i\omega\varepsilon (f_h)_1 - (\operatorname{curl} f_e)_1 \,, \tag{1.9b}$$

$$\partial_1^2 E_2 + \omega^2 \varepsilon\mu E_2 = \partial_2\partial_1 E_1 + i\omega\mu\partial_3 H_1 + \partial_1 (f_h)_3 - i\omega\mu (f_e)_2 \,, \tag{1.9c}$$

$$\partial_1^2 H_2 + \omega^2 \varepsilon\mu H_2 = \partial_2\partial_1 H_1 - i\omega\varepsilon\partial_3 E_1 - \partial_1 (f_e)_3 + i\omega\varepsilon (f_h)_2 \,, \tag{1.9d}$$

$$\partial_1^2 E_3 + \omega^2 \varepsilon\mu E_3 = \partial_3\partial_1 E_1 - i\omega\mu\partial_2 H_1 - \partial_1 (f_h)_2 - i\omega\mu (f_e)_3 \,, \tag{1.9e}$$

$$\partial_1^2 H_3 + \omega^2 \varepsilon\mu H_3 = \partial_3\partial_1 H_1 + i\omega\varepsilon\partial_2 E_1 - \partial_1 (f_e)_2 + i\omega\varepsilon (f_h)_3 \,. \tag{1.9f}$$

The equations are complemented with the boundary and interface conditions

$$E_1 = 0 \qquad \text{on } \Gamma \cup \partial_{\text{top}}\Omega \cup \partial_{\text{bot}}\Omega \,, \tag{1.10a}$$

$$\partial_3 H_1 = (f_e)_2 \qquad \text{on } \partial_{\text{top}}\Omega \cup \partial_{\text{bot}}\Omega \,, \tag{1.10b}$$

$$x \mapsto (E, H)(x) \qquad \text{is } x_1\text{- and } x_2\text{-periodic} \,. \tag{1.10c}$$

We remark that (1.10b) is the natural boundary condition when (1.9b) is written in a weak form.

System (1.9), complemented with boundary conditions (1.10), for $\omega^2 \notin \sigma_M$, can be solved in three steps: In the first step, E_1 is found as the solution to the Helmholtz problem (1.9a) with the boundary conditions (1.10a) and (1.10c); one has to solve two uncoupled Helmholtz problems, posed in Ω_+ and Ω_-. In the second step, H_1 is found as the solution to the Helmholtz problem (1.9b) with the boundary conditions (1.10b) and (1.10c); one has to solve one Helmholtz problems in Ω. In the third step, the four unknowns E_2, E_3, H_2, H_3 are found by solving, for almost every $(x_2, x_3) \in I_2 \times I_3$, one dimensional Helmholtz problem on I_1 with periodicity boundary conditions.

In the above procedure, steps one and two can be interchanged. Once E_1 and H_1 are found, the last four equation can be solved independently of each other.

Based on the three step procedure, it is easy to show that the Helmholtz-type system (1.9) can be solved uniquely for every $\omega^2 \notin \sigma_M$. The existence statement of Theorem 1.1 then follows from the fact that every solution to the Helmholtz-type system is a solution to the Maxwell system if $\omega^2 \notin \sigma(l_1) \subset \sigma_M$. This latter fact is derived in Lemma 4.2.

The uniqueness statement of part (i) of Theorem 1.1 is a consequence of the fact that the two systems of equations are equivalent (we show this even for all $\omega^2 \notin \sigma(l_1)$). The uniqueness statement for solutions of the Helmholtz system, which holds for $\omega^2 \notin \sigma_M$, yields a uniqueness statement for the solutions of the Maxwell system under the same frequency assumption.

In all of these arguments, one has to be careful in the choice of weak solution concepts. Furthermore, one must reduce the problem to right-hand sides

$f_h, f_e \in L^2(\Omega, \mathbb{C}^3)$ that are divergence-free; more precisely, we demand $f_h \in Y_0^\perp$ and $f_e \in X_0^\perp$, for the definitions of the spaces see (2.5) below. As announced, a Helmholtz decomposition argument provides that it is no restriction to consider only divergence-free right-hand sides, see Lemma 2.5.

The following proposition gives a precise description of the fact that, loosely speaking, systems (1.4) and (1.9) are equivalent.

Proposition 1.3 (Equivalence of the Maxwell and the Helmholtz-type system). *Let the geometry and the coefficients be as above. Then, the Maxwell system* (1.4) *and the Helmholtz-type system are equivalent in the following sense:*

(i) For $\omega^2 \notin \sigma(l_1)$ and a right-hand side given by $(f_e, f_h) \in X_0^\perp \times Y_0^\perp$, (E, H) is a weak solution to the Maxwell system (1.4) *in the sense of Definition 2.3 if, and only if, it is a weak solution to the Helmholtz-type system* (1.9) *in the sense of Definition 3.1.*

(ii) For $\omega^2 \in \sigma(l_1)$ and $f_e = f_h = 0$, there exists a non-trivial weak solution (E_M, H_M) to the Maxwell system (1.4) *and a non-trivial weak solution (E_H, H_H) to the Helmholtz-type system* (1.9). *Moreover, every solution (E_M, H_M) to the Maxwell system is a solution to the Helmholtz-type system. It is not true that every solution (E_H, H_H) to the Helmholtz-type system is a solution to the Maxwell system.*

Proof. We show part (i) of the proposition in two lemmas. Lemma 4.1 provides that every solution of the Maxwell system is a solution of the Helmholtz system. Lemma 4.2, which is only valid for $\omega^2 \notin \sigma(l_1)$, provides that every solution of the Helmholtz system is a solution of the Maxwell system.

Regarding part (ii), we note that $\sigma(l_1) \subset \sigma_M$. Thus, we can apply Lemma 1.5, which provides explicit solutions for the homogeneous Maxwell system, for $\omega^2 \in \sigma_M$. Since Lemma 4.1 is valid for arbitrary $\omega^2 > 0$, these solutions are also solution to the homogeneous Helmholtz-type system.

Regarding the last sentence, we give an example in Remark 1.4. □

Remark 1.4 (On the failure of equivalence for special frequencies). *In the proof of Proposition 1.3 (ii), we did not need the equivalence of the systems; instead, we constructed solutions for the homogeneous Maxwell-type system in Lemma 1.5. Lemma 4.1 guarantees that they are also solutions to the Helmholtz-type system.*

In the case $\omega^2 \in \sigma(l_1)$, we consider the following example: For $\omega^2 = 4\pi^2 k_1^2/(\varepsilon\mu l_1^2)$ with some $0 < k_1 \in \mathbb{N}$, we set $E_1 = E_3 = 0$, $E_2 = \sin(2\pi k_1 x_1/l_1)$ and $H = 0$. Then (E, H) is a solution to the homogeneous Helmholtz-type system. On the other hand, because of $\operatorname{curl} E = 2\pi k_1/l_1 \cos(2\pi k_1 x_1/l_1) e_3 \neq 0 = i\omega\mu H$ the function (E, H) is not a solution to the homogeneous Maxwell-type system.

1.5. **Formal equivalence of the Maxwell system and the Helmholtz-type system.** In this motivational section, we present calculations for smooth solutions. We consider only smooth and divergence-free source functions that satisfy (1.8).

From the Maxwell system to the Helmholtz-type system. Since the source terms have vanishing divergence, the same is true for E in $\Omega \setminus \Gamma$ and for H in Ω, see the first two equations of (1.4). Taking the curl of these two equations and using the identity $-\Delta = \operatorname{curl}\operatorname{curl} - \nabla \operatorname{div}$, we obtain the Helmholtz equations (1.9a) and (1.9b) for the components E_1 and H_1. Since (1.4b) is imposed only in $\Omega \setminus \Gamma$, the Helmholtz equations hold only on $\Omega \setminus \Gamma$. Let us sketch why (1.9b) holds also across the interface.

Firstly, we note that (1.4a) is posed on Ω so that

$$[\![H_3]\!]_\Gamma = \frac{1}{i\omega\mu}(-[\![\partial_2 E_1]\!]_\Gamma + [\![\partial_1 E_2]\!]_\Gamma - [\![(f_h)_3]\!]_\Gamma) = 0\,. \tag{1.11}$$

Therefore, the third component of H has no jump. From (1.4a) we therefore obtain, since E_2 has no jump,

$$[\![\partial_3 H_1]\!]_\Gamma = [\![\partial_1 H_3]\!]_\Gamma + [\![(f_e)_2]\!]_\Gamma = 0\,.$$

Relation (1.4d) implies that H_1 has no jump across the interface. Since neither H_1 nor $\partial_3 H_1$ have a jump across the interface, (1.9b) holds in all of Ω.

The boundary condition (1.10a) was demanded with (1.4c) and (1.4e). The component H_3 vanishes on the horizontal boundaries by (1.4a) and the assumption that $(f_h)_3|_{\partial_{\rm hor}\Omega} = 0$. The Neumann boundary condition (1.10b) follows from (1.4b).

Regarding the other equations, we present the calculation for (1.9c). We look at two components of (1.4), namely

$$\begin{aligned} \partial_1 E_2 - \partial_2 E_1 &= i\omega\mu H_3 + (f_h)_3\,,\\ \partial_3 H_1 - \partial_1 H_3 &= -i\omega\varepsilon E_2 + (f_e)_2\,. \end{aligned}$$

Applying ∂_1 to the first equation and inserting $\partial_1 H_3$ from the second equation, we find

$$\begin{aligned} \partial_1^2 E_2 - \partial_1\partial_2 E_1 &= i\omega\mu\partial_1 H_3 + \partial_1 (f_h)_3\\ &= i\omega\mu[\partial_3 H_1 + i\omega\varepsilon E_2 - (f_e)_2] + \partial_1 (f_h)_3\\ &= -\omega^2\mu\varepsilon E_2 + i\omega\mu[\partial_3 H_1 - (f_e)_2] + \partial_1 (f_h)_3\,. \end{aligned}$$

This is exactly (1.9c). The other equations are derived accordingly.

From the Helmholtz-type system to the Maxwell system. As in the above implication, we use different calculations for the first components and for the other components. It is common to all calculations that we use the invertibility of the operator $(\partial_1^2 + \varepsilon\mu\omega^2)$ on the interval $(0, l_1)$. We present the calculations for the first two components of (1.4b), the other relations are derived analogously.

To show the first component of (1.4b), we multiply (1.9a) by $-i\omega\varepsilon$, take the derivative of (1.9d) with respect to the third component and the derivative of (1.9f) with respect to the second component. Summing the first two resulting equations and subtracting the third yields:

$$\begin{aligned} (\partial_1^2 + \varepsilon\mu\omega^2)(\partial_3 H_2 - \partial_2 H_3) &= -(\Delta + \varepsilon\mu\omega^2)(-i\omega\varepsilon E_1) - \varepsilon\mu\omega^2 (f_e)_1 + i\omega\varepsilon(\operatorname{curl} f_h)_1\\ &\quad + \partial_3(\partial_2\partial_1 H_1 - i\omega\varepsilon\partial_3 E_1 - \partial_1 (f_e)_3 + i\omega\varepsilon (f_h)_2)\\ &\quad - \partial_2(\partial_3\partial_1 H_1 + i\omega\varepsilon\partial_2 E_1 - \partial_1 (f_e)_2 + i\omega\varepsilon (f_h)_3)\\ &= (\partial_1^2 + \varepsilon\mu\omega^2)(i\omega\varepsilon E_1 - (f_e)_1)\,, \end{aligned}$$

where we used $\operatorname{div} f_e = 0$. Inverting the operator $(\partial_1^2 + \varepsilon\mu\omega^2)$ yields the first component of (1.4b).

To show the second component of (1.4b), we multiply (1.9c) by $i\omega\varepsilon$ and take the derivative of (1.9f) with respect to the first component. Subtracting the second relation from the first yields

$$(\partial_1^2 + \varepsilon\mu\omega^2)(\partial_3 H_1 - \partial_1 H_3) = (\partial_1^2 + \varepsilon\mu\omega^2)(-i\omega\varepsilon E_2 + (f_e)_2)\,.$$

Inverting the operator $(\partial_1^2 + \varepsilon\mu\omega^2)$ yields the second component of (1.4b).

It remains to verify the interface and boundary conditions (1.4c)–(1.4f) and, in order to obtain that (1.4a) holds across Γ, that E_1 and E_2 have no jump across Γ. Equation (1.10a) shows that E_1 vanishes at Γ, which gives (1.4c) and shows that E_1 is continuous across Γ. Equation (1.9b) implies that both H_1 and $\partial_3 H_1$ have no jump across Γ, which shows, in particular, (1.4d). Since $\partial_3 H_1$ does not jump across Γ, the whole right-hand side of (1.9c) and, thus the left-hand side does not jump across Γ. By inverting the operator $(\partial_1^2 + \varepsilon\mu\omega^2)$, we obtain the continuity of E_2 across Γ.

In order to derive (1.4e), it remains to show that E_2 vanishes at $\partial_{\mathrm{hor}}\Omega$. For this, we note that the right-hand side of (1.9c) vanishes at $\partial_{\mathrm{hor}}\Omega$ since E_1 vanishes at $\partial_{\mathrm{hor}}\Omega$ and the remaining terms cancel by the boundary condition (1.10b) and the assumption $(f_h)_3|_{\partial_{\mathrm{hor}}\Omega} = 0$ given by (1.8). Inverting the operator $(\partial_1^2 + \varepsilon\mu\omega^2)$ in (1.9c) gives $E_2|_{\partial_{\mathrm{hor}}\Omega} = 0$.

1.6. **Eigenfunctions for the Maxwell system.** The next lemma provides non-trivial solutions of the homogeneous Maxwell system for $\omega^2 \in \sigma_M$.

Lemma 1.5 (Eigenfunctions of the Maxwell system)**.** *For frequencies $\omega > 0$ with $\omega^2 \in \sigma_M$, the homogeneous Maxwell system has a non-trivial solution. We provide an explicit solution by distinguishing two cases. In the case $\omega^2 \in \sigma(l_1, l_2, l_3)$, we find eigenfunctions that do not vanish along Γ. In the case $\omega^2 \in \sigma_M \setminus \sigma(l_1, l_2, l_3)$, we find eigenfunctions that live in the upper or in the lower domain.*

Case $\omega^2 \in \sigma(l_1, l_2, l_3)$: We use the following function on Ω,

$$w(x_1, x_2, x_3) = \cos(2\pi k_1 x_1/l_1)\cos(2\pi k_2 x_2/l_2)\cos(\pi k_3(x_3 + l_3^-)/l_3)\,, \tag{1.12}$$

and set

$$E(x) = \begin{pmatrix} 0 \\ \partial_3 w \\ -\partial_2 w \end{pmatrix}, \qquad H(x) = (i\omega\mu)^{-1} \begin{pmatrix} -\partial_2^2 w - \partial_3^2 w \\ \partial_1\partial_2 w \\ \partial_1\partial_3 w \end{pmatrix}. \tag{1.13}$$

We note that $E_1 = 0$ on Γ is satisfied and $E_1 = E_2 = 0$ holds along horizontal boundaries. Furthermore, H_1 has no jump across Γ.

Case $\omega^2 \in \sigma_M \setminus \sigma(l_1, l_2, l_3)$: In this case, for either the symbol "+" or the symbol "−" (fixed from now on), there holds $\omega^2 \in \sigma(l_1, l_2, l_3^\pm) \setminus \sigma(l_1, l_2, l_3)$. We note that $k_3 \neq 0$ since otherwise $\omega^2 \in \sigma(l_1, l_2, l_3)$. We use

$$w(x_1, x_2, x_3) = \cos(2\pi k_1 x_1/l_1)\cos(2\pi k_2 x_2/l_2)\sin(\pi k_3 x_3/l_3^\pm)\,, \tag{1.14}$$

which is not vanishing identically because of $k_3 \neq 0$, and set

$$H(x) = \mathbf{1}_{\Omega_\pm} \begin{pmatrix} 0 \\ \partial_3 w \\ -\partial_2 w \end{pmatrix}, \qquad E(x) = \mathbf{1}_{\Omega_\pm}(-i\omega\varepsilon)^{-1} \begin{pmatrix} -\partial_2^2 w - \partial_3^2 w \\ \partial_1\partial_2 w \\ \partial_1\partial_3 w \end{pmatrix}. \tag{1.15}$$

Note that $E_1 = 0$ on Γ is satisfied and $E_1 = E_2 = 0$ holds along the horizontal boundary $\{x_3 = \pm l_3^\pm\}$ by $\sin(\pi k_3(\pm l_3^\pm)/l_3^\pm) = 0$. Furthermore, H_1 has no jump across Γ.

Remark 1.6 (On the spectrum of the polarization interface problem)**.** *The spectrum of the Maxwell system (1.4) without interface (obtained by demanding $[\![H_2]\!]_\Gamma = 0$ instead of (1.4c)), is given by $\sigma(l_1, l_2, l_3)$. The spectrum of the Maxwell system (1.4) with a full reflection interface condition (obtained by demanding $E_2|_\Gamma = 0$ instead* *of* (1.4d)*), is* $\sigma(l_1, l_2, l_3^+) \cup \sigma(l_1, l_2, l_3^-)$*. Thus, for this simple geometry, the spectrum for the polarization interface condition is the union of these spectra.*

We include the following warning: It is not true that every eigenfunction for the full reflection condition is also an eigenfunctions for the polarization condition. An example is constructed with w *from* (1.14) *(choosing either* $+$ *or* $-$*) by setting*

$$H(x) = \mathbf{1}_{\Omega_\pm} \begin{pmatrix} -\partial_3 w \\ 0 \\ \partial_1 w \end{pmatrix}, \qquad E(x) = \mathbf{1}_{\Omega_\pm} (-i\omega\varepsilon)^{-1} \begin{pmatrix} \partial_1\partial_2 w \\ -\partial_1^2 w - \partial_3^2 w \\ \partial_2\partial_3 w \end{pmatrix}. \tag{1.16}$$

This pair satisfies the Maxwell equations in $\Omega \setminus \Gamma$*,* E_1 *and* E_2 *vanish along* Γ*, it is therefore a solution of the full reflection problem. On the other hand,* H_1 *has a jump across* Γ*, the above fields are therefore no solution to the polarization problem.*

The remainder of this article is structured as follows: We provide solution concepts for the Maxwell system in Section 2, discuss two different solution concepts for the Helmholtz-type system in Section 3, show the equivalence of the two systems in Section 4, and provide existence and uniqueness results for the Helmholtz system in Section 5.

2. Maxwell system

In this section, we introduce and analyze the weak solution concept for the Maxwell system (1.4). We consider always the geometry of (1.1)–(1.2).

2.1. **Weak form of the Maxwell system.**

We define the periodic extension of the sets Ω and Γ in the (e_1, e_2)-plane by

$$\Omega_\# := \mathbb{R}^2 \times \{-l_3^-, l_3^+\}, \qquad \Gamma_\# := \mathbb{R}^2 \times \{0\}.$$

Every function u on Ω is identified with its periodic extension $\tilde{u}$ on $\Omega_\#$, defined with the rule

$$\tilde{u}(k_1 l_1 + k_2 l_2 + x) = u(x) \qquad \forall k_1, k_2 \in \mathbb{Z},\ x \in \Omega.$$

In order to impose the periodicity condition (1.4f), we introduce the function spaces

$$H_\#(\mathrm{curl}, \Omega) := \{u \in H(\mathrm{curl}, \Omega) \mid \tilde{u} \in H_{\mathrm{loc}}(\mathrm{curl}, \Omega_\#)\},$$
$$H_\#(\mathrm{curl}, \Omega \setminus \Gamma) := \{u \in H(\mathrm{curl}, \Omega \setminus \Gamma) \mid \tilde{u} \in H_{\mathrm{loc}}(\mathrm{curl}, \Omega_\# \setminus \Gamma_\#)\}.$$

These two spaces impose the periodicity conditions in x_1- and x_2-direction. The second space is larger than the first space, functions in $H_\#(\mathrm{curl}, \Omega \setminus \Gamma)$ can have a jump across Γ (in all components).

Using trace theorems, one can conclude that the traces of tangential components are periodic, e.g.: $E_3(x_1 = l_1, x_2, x_3) = E_3(x_1 = 0, x_2, x_3)$ for almost every x_2, x_3 in the sense of traces. As a warning, we note that the two properties $u \in H_{\mathrm{loc}}(\mathrm{curl}, \Omega_\#)$ and periodicity together do not imply $u|_\Omega \in H(\mathrm{curl}, \Omega)$; the reason is that the $H(\mathrm{curl}, \Omega)$-norm could still be unbounded. Thus, our requirement $u \in H(\mathrm{curl}, \Omega)$ is an additional assumption.

In order to incorporate the interface condition (1.4c) and the boundary conditions (1.4e) for the electric field E, we define a subspace $X \subset H_\#(\mathrm{curl}, \Omega)$. To introduce the interface condition (1.4d) for the magnetic field, we define a subspace $Y \in H_\#(\mathrm{curl}, \Omega \setminus \Gamma)$. In order to define the spaces X and Y, we use spaces of smooth functions. In the subsequent table, the second column indicates whether or not it is demanded that functions vanish in a neighborhood of the horizontal boundaries

$\partial_{\text{hor}}\Omega = \partial_{\text{top}}\Omega \cup \partial_{\text{bot}}\Omega$. The last column indicates whether or not the functions can have a jump across Γ ("jump" or "no j."); furthermore, it can be demanded that the function vanishes in a neighborhood of Γ ("=0").

Space	$\partial_{\text{hor}}\Omega$	Γ
$D_\#(\overline{\Omega}) \coloneqq \left\{u \in C^\infty\left(\overline{\Omega}\right) \,\middle|\, \tilde{u} \in C^\infty\left(\overline{\Omega_\#}\right)\right\}$	$\neq 0$	no j.
$D_\#(\Omega) \coloneqq \left\{u \in C^\infty(\overline{\Omega}) \,\middle|\, \operatorname{supp}(\tilde{u}) \cap \bar{\Omega} \subset \Omega_\# \text{ is compact}\right\}$	$= 0$	no j.
$D_\#(\Omega \setminus \Gamma) \coloneqq \left\{u \in C^\infty(\overline{\Omega}) \,\middle|\, \operatorname{supp}(\tilde{u}) \cap \bar{\Omega} \subset \Omega_\# \setminus \Gamma_\# \text{ is compact}\right\}$	$= 0$	$= 0$
$D_\#(\overline{\Omega}; \Gamma) \coloneqq \left\{u \colon \overline{\Omega} \to \mathbb{C} \,\middle|\, \tilde{u}\vert_{\mathbb{R}^2 \times I_3^\pm} \in C^\infty\left(\mathbb{R}^2 \times \overline{I_3^\pm}\right)\right\}$	$\neq 0$	jump

With these spaces of smooth functions, we can now define the solution spaces X and Y. Essentially, the space Y contains functions $u \in H_\#(\operatorname{curl}, \Omega \setminus \Gamma)$ such that u_1 does not jump across Γ; the component u_2 might jump across Γ. The space X contains functions $u \in H_\#(\operatorname{curl}, \Omega)$ such that u_1 and u_2 vanish at $\partial_{\text{hor}}\Omega$ and the component u_1 vanishes also along Γ.

$$\text{(2.2a)} \quad Y \coloneqq \left\{ u \in H_\#(\operatorname{curl}, \Omega \setminus \Gamma) \,\middle|\, \int_{\Omega\setminus\Gamma} \operatorname{curl} u \cdot \phi = \int_{\Omega\setminus\Gamma} u \cdot \operatorname{curl} \phi \right.$$
$$\left. \forall\, \phi = (\phi_1, \phi_2, \phi_3)\,,\ \phi_1 \in D_\#(\Omega \setminus \Gamma)\,, \phi_2 \in D_\#(\Omega)\,, \phi_3 \in D_\#(\Omega) \right\},$$

$$\text{(2.2b)} \quad X \coloneqq \left\{ u \in H_\#(\operatorname{curl}, \Omega) \,\middle|\, \int_{\Omega\setminus\Gamma} \operatorname{curl} u \cdot \psi = \int_{\Omega\setminus\Gamma} u \cdot \operatorname{curl} \psi \ \forall\, \psi \in Y \right\}.$$

We formulated the definition in such a way that X is characterized with the help of Y. Indeed, one can also characterize Y in terms of X. This illustrates the duality of the two spaces.

Lemma 2.1 (Characterization of Y in terms of X)**.** *The two spaces X and Y of (2.2) satisfy*

$$\text{(2.3)} \qquad Y = \left\{ u \in H_\#(\operatorname{curl}, \Omega \setminus \Gamma) \,\middle|\, \int_{\Omega\setminus\Gamma} \operatorname{curl} u \cdot \phi = \int_{\Omega\setminus\Gamma} u \cdot \operatorname{curl} \phi \ \forall\, \phi \in X \right\}.$$

The proof of the lemma is given in the appendix. It is based on a density argument.

Remark 2.2. *Tangential traces of functions $E \in X$ and $H \in Y$ are well defined, see Theorem 3.29 in [14]. In particular, in the sense of traces, we may write $E \times n = 0$ on $\partial_{hor}\Omega$ and $E_1 = 0$ on Γ. Similarly, H satisfies $[\![H_1]\!]_\Gamma = 0$.*

With the help of X and Y we can now formulate the weak solution concept for the Maxwell system (1.4). The motivation of our definition is: We use test-functions $\psi \in Y$ in (1.4a) and test-functions $\phi \in X$ in (1.4b).

Definition 2.3 (Weak solutions of the Maxwell system)**.** *Let $(f_h, f_e) \in L^2(\Omega, \mathbb{C}^3)^2$. We say $(E, H) \in L^2(\Omega, \mathbb{C}^3) \times L^2(\Omega, \mathbb{C}^3)$ is a solution to the Maxwell system (1.4) if*

$$\text{(2.4a)} \qquad \int_{\Omega\setminus\Gamma} E \cdot \operatorname{curl} \psi = \int_\Omega (i\omega\mu H + f_h) \cdot \psi \qquad \forall \psi \in Y\,,$$

$$\text{(2.4b)} \qquad \int_{\Omega} H \cdot \operatorname{curl} \phi = \int_\Omega (-i\omega\varepsilon E + f_e) \cdot \phi \qquad \forall \phi \in X\,.$$

We obtain in the next subsection that every solution $(E, H) \in L^2(\Omega, \mathbb{C}^3)^2$ of (2.4) satisfies indeed $E \in X$ and $H \in Y$, see Lemma 2.4. This shows: We could have demanded $E \in X$ and $H \in Y$ in the above definition without changing the solution concept.

2.2. **Regularity of solutions to the Maxwell system.** In this section, it is relevant to consider also divergence-free right-hand sides. We use the following construction. The spaces X_0 and Y_0 of gradients are given as

$$X_0 := \{\nabla\varphi \in X \mid \varphi \in H^1_{0,\#}(\Omega)\}\,, \qquad Y_0 := \{\nabla\varphi \in Y \mid \varphi \in H^1_{\#}(\Omega\setminus\Gamma)\}\,, \tag{2.5}$$

where $H^1_{0,\#}(\Omega)$ and $H^1_{\#}(\Omega\setminus\Gamma)$ are the spaces of scalar-valued periodic functions, the first with homogeneous Dirichlet condition on the horizontal boundaries, the second contains functions that jump across Γ, for precise definitions see Section 3. The index 0 of X_0 and Y_0 indicates, that the are the subsets of X and Y, respectively, with vanishing curl. The $L^2(\Omega)$-orthogonal complements are $X_0^\perp$ and $Y_0^\perp$. In the subsequent result, we make a statement on divergence-free right-hand sides; more precisely, we will demand $f_h \in Y_0^\perp$ and $f_e \in X_0^\perp$. This ensures that $E \in X_0^\perp$ and $H \in Y_0^\perp$, i.e.

$$\int_\Omega E\cdot g = 0 \qquad \forall g \in X_0\,, \tag{2.6}$$

$$\int_\Omega H\cdot g = 0 \qquad \forall g \in Y_0\,. \tag{2.7}$$

Lemma 2.4 (Regularity of solutions to the Maxwell system)**.** *Let $f_h, f_e \in L^2(\Omega, \mathbb{C}^3)$ define a right-hand side. Let $(E, H) \in L^2(\Omega, \mathbb{C}^3) \times L^2(\Omega, \mathbb{C}^3)$ be a solution to the Maxwell system in the sense of Definition 2.3. Then, the solution has the property $(E, H) \in X \times Y$. Moreover, if the right-hand side satisfies $(f_h, f_e) \in Y_0^\perp \times X_0^\perp$, there additionally holds $(E, H) \in X_0^\perp \times Y_0^\perp$.*

Proof. By the definition of weak solutions, (E, H) satisfies (2.4). Choosing the test-functions $\psi \in D_\#(\Omega, \mathbb{C}^3)$ and $\phi \in D_\#(\Omega\setminus\Gamma)$ in (2.4), we see that (1.4a) and (1.4b) are satisfied in the sense of distributions. We therefore know the distributional curl of E and H and know that they are given by $L^2(\Omega)$-functions, namely, $\operatorname{curl} E = i\omega\mu H + f_h \in L^2(\Omega, \mathbb{C}^3)$ and $\operatorname{curl} H = -i\omega\varepsilon E + f_e \in L^2(\Omega, \mathbb{C}^3)$. Thus, $E \in H_\#(\operatorname{curl}, \Omega)$ and $H \in H_\#(\operatorname{curl}, \Omega\setminus\Gamma)$.

The fact that $\operatorname{curl} E$ and $\operatorname{curl} H$ are L^2-functions allows us to insert (1.4) into (2.4), which yields

$$\int_{\Omega\setminus\Gamma} E\cdot\operatorname{curl}\psi = \int_{\Omega\setminus\Gamma}\operatorname{curl} E\cdot\psi \qquad \forall\,\psi \in Y\,,$$

$$\int_{\Omega\setminus\Gamma} H\cdot\operatorname{curl}\phi = \int_{\Omega\setminus\Gamma}\operatorname{curl} H\cdot\phi \qquad \forall\,\phi \in X\,.$$

The first equation verifies $E \in X$, see (2.2b). The second equation implies $H \in Y$, since the definition of Y in (2.2a) considers test-functions ϕ only in a subset of X.

We consider now $(f_h, f_e) \in Y_0^\perp \times X_0^\perp$. In order to show $(E, H) \in X_0^\perp \times Y_0^\perp$, we choose test-functions of the form $\phi = \nabla\varphi \in X_0$ and $\psi = \nabla\eta \in Y_0$ in (2.4). Since the curl of gradients vanishes, the left-hand sides of (2.4) vanish. Moreover, due to the orthogonality of $Y_0^\perp \ni f_h \perp \nabla\varphi \in Y_0$ and $X_0^\perp \ni f_e \perp \nabla\eta \in X_0$ the last terms on the right-hand sides vanish and we obtain

$$0 = \int_\Omega i\omega\mu H \cdot \psi \;\; \forall \psi \in Y_0 \,, \qquad 0 = \int_\Omega -i\omega\varepsilon E \cdot \phi \;\; \forall \phi \in X_0 \,,$$

which shows that $(\varepsilon E, \mu H) \in X_0^\perp \times Y_0^\perp$ and, thus, $(E, H) \in X_0^\perp \times Y_0^\perp$. □

We note that, for arbitrary right-hand sides $f_h, f_e \in L^2(\Omega, \mathbb{C}^3)$, the Maxwell-system can be always replaced by a system with $(f_h, f_e) \in Y_0^\perp \times X_0^\perp$.

Lemma 2.5 (Non-divergence-free right-hand sides)**.** *Let $f_h, f_e \in L^2(\Omega, \mathbb{C}^3)$ define a right-hand side for the Maxwell system, ε and μ independent of x. We consider the Helmholtz decompositions*

$$f_h = \tilde f_h + \nabla\Psi \qquad \text{with } \tilde f_h \in Y_0^\perp \text{ and } \nabla\Psi \in Y_0 \,,$$
$$f_e = \tilde f_e + \nabla\Phi \qquad \text{with } \tilde f_e \in X_0^\perp \text{ and } \nabla\Phi \in X_0 \,.$$

We consider arbitrary functions $E, H \in L^2(\Omega, \mathbb{C}^3)$ and modified functions $\tilde E = E - (i\omega\varepsilon)^{-1}\nabla\Phi$ and $\tilde H = H + (i\omega\mu)^{-1}\nabla\Psi$. Then there holds: (E, H) is a weak solution to the Maxwell system for the right-hand side (f_h, f_e) if and only if $(\tilde E, \tilde H)$ is weak a solution to the Maxwell system for the right-hand sides $(\tilde f_h, \tilde f_e)$.

Proof. Lemma 2.5 is a direct consequence of the fact that the curl of a gradient vanishes, $\operatorname{curl} \nabla\Phi = \operatorname{curl} \nabla\Psi = 0$. □

Remark 2.6 (Non-divergence-free right-hand sides and general ε, μ)**.** *Lemma 2.5 deals with the case of coefficients ε and μ that are independent of x. For the general case, i.e. $\mu, \varepsilon \in L^\infty(\Omega, \mathbb{C}^{3\times 3})$, one has to choose different constructions. The well-known approach in this general case is to define scalar products by $\langle u, v\rangle_\mu := \int_\Omega \mu u \cdot \overline{v}$ and $\langle u, v\rangle_\varepsilon = \int_\Omega \varepsilon u \cdot \overline{v}$, and to consider with $Y_0^{\perp_\mu}$ and $X_0^{\perp_\varepsilon}$ the orthogonal complements of Y_0 and X_0 with respect to these scalar products. Decompositions of the form $(i\omega\mu)^{-1} f_h = (i\omega\mu)^{-1} \tilde f_h + \nabla\Psi$ with the choice $\tilde H = H + \nabla\Psi$ show that it is sufficient to consider $f_h \in Y_0^{\perp_\mu}$ and decompositions of the form $(i\omega\varepsilon)^{-1} f_e = (i\omega\varepsilon)^{-1} \tilde f_e + \nabla\Phi$ with the choice $\tilde E = E - \nabla\Phi$ show that it is sufficient to consider $f_e \in X_0^{\perp_\varepsilon}$.*

3. Helmholtz-type system

In this section, we analyze the Helmholtz-type system (1.9). We introduce two different solution concepts, weak solutions and very weak solutions. Actually, we will see that the two concepts coincide if we restrict the solution space for the very weak solutions to $X \times Y$. Nevertheless, the distinction is useful in the result regarding equivalence of Maxwell- and Helmholtz-system, see Section 4. The solvability properties of the Helmholtz-system are easy to obtain, see Section 5.

3.1. **Weak form of the Helmholtz-type system.** In order to define the weak solution concept for (1.9), we use spaces of periodic functions. As before, for an arbitrary function $u\colon \Omega \to \mathbb{C}$, we write $\tilde u$ for the periodic extension of u. We define, in this order: periodic functions (no restrictions regarding interface or horizontal boundaries), periodic functions that vanish along $\partial_{\mathrm{hor}}\Omega$, periodic functions that can jump across Γ, periodic functions that vanish along $\partial_{\mathrm{hor}}\Omega$ and along Γ:

$$H^1_\#(\Omega) := \{u \in H^1(\Omega) \mid \tilde u \in H^1_{\mathrm{loc}}(\Omega_\#)\} \,,$$
$$H^1_{0,\#}(\Omega) := \left\{ u \in H^1_\#(\Omega) \middle| \int_\Omega \nabla u \cdot \psi = -\int_\Omega u \operatorname{div}(\psi) \;\forall\, \psi \in H^1_\#(\Omega, \mathbb{C}^3) \right\} ,$$

$$H^1_\#(\Omega \setminus \Gamma) := \{u \in H^1(\Omega \setminus \Gamma) \mid \tilde u \in H^1_{\mathrm{loc}}(\Omega_\# \setminus \Gamma_\#)\}\,,$$
$$H^1_{0,\#}(\Omega \setminus \Gamma) := \left\{u \in H^1_\#(\Omega) \middle| \int_\Omega \nabla u \cdot \psi = -\int_\Omega u \operatorname{div}(\psi)\ \forall\, \psi \in H^1_\#(\Omega \setminus \Gamma, \mathbb{C}^3)\right\}.$$

In the Helmholtz-type system (1.9), the last four equations contain only derivatives in the x_1-direction. These equations are ordinary differential equations with solutions that depend on x_1, for every parameter input $(x_2, x_3) \in I_2 \times I_3$. Accordingly, for the solution concept, we need spaces of functions that have some x_1-regularity, but not necessarily regularity in the other coordinates.

Similar to the above constructions, we associate, to every function $u\colon I_1 \to \mathbb{C}$, its periodic extension $\tilde u\colon \mathbb{R} \to \mathbb{C}$, defined through

$$\tilde u(k_1 l_1 + x_1) = u(x_1) \qquad \forall k_1 \in \mathbb{Z}\,,\ x_1 \in I_1\,.$$

We can then define

$$D_\#(I_1) := \{u \in C^\infty(\overline{I_1}) \mid \tilde u \in C^\infty(\mathbb{R})\}\,,$$
$$H^1_\#(I_1) := \{u \in H^1(I_1) \mid \tilde u \in H^1_{\mathrm{loc}}(\mathbb{R})\}\,.$$

The cross sections of Ω, $\Omega \setminus \Gamma$ and Γ are denoted as

$$U := I_2 \times I_3\,, \qquad \Gamma_U := I_2 \times \{0\}\,. \tag{3.1}$$

As solution spaces for the Helmholtz-type system we use

$$W := L^2(U, H^1_\#(I_1)) \tag{3.2a}$$
$$X_H := \left\{E \in L^2(\Omega, \mathbb{C}^3) \,\middle|\, E_1 \in H^1_{0,\#}(\Omega \setminus \Gamma)\,,\ E_2, E_3 \in W\right\}\,, \tag{3.2b}$$
$$Y_H := \left\{H \in L^2(\Omega, \mathbb{C}^3) \,\middle|\, H_1 \in H^1_\#(\Omega)\,,\ H_2, H_3 \in W\right\}\,. \tag{3.2c}$$

We will consider right-hand sides $(f_h, f_e) \in L^2(\Omega, \mathbb{C}^3)^2$ in the Helmholtz-type system (when, we show the equivalence to the Maxwell system, we consider only $(f_h, f_e) \in Y_0^\perp \times X_0^\perp$). Given such functions, we define linear forms $F_{E_1} \in H^1_{0,\#}(\Omega\setminus\Gamma)'$, $F_{H_1} \in H^1_\#(\Omega)'$ and $F_{E_2}, F_{H_2}, F_{E_3}, F_{H_3} \in W'$ as follows:

$$\langle F_{E_1}, \phi_1\rangle := \int_\Omega i\omega\mu (f_e)_1\phi_1 + f_h \cdot \operatorname{curl}(e_1\phi_1) \qquad \forall \phi_1 \in H^1_{0,\#}(\Omega \setminus \Gamma)\,, \tag{3.3a}$$
$$\langle F_{H_1}, \psi_1\rangle := \int_\Omega -i\omega\varepsilon (f_h)_1\psi_1 + f_e \cdot \operatorname{curl}(e_1\psi_1) \qquad \forall \psi_1 \in H^1_\#(\Omega)\,, \tag{3.3b}$$
$$\langle F_{E_2}, \phi_2\rangle := \int_\Omega (f_h)_3\partial_1\phi_2 + i\omega\mu (f_e)_2\phi_2 \qquad \forall \phi_2 \in W\,, \tag{3.3c}$$
$$\langle F_{H_2}, \psi_2\rangle := \int_\Omega (f_e)_3\partial_1\psi_2 - i\omega\varepsilon (f_h)_2\psi_2 \qquad \forall \psi_2 \in W\,, \tag{3.3d}$$
$$\langle F_{E_3}, \phi_3\rangle := \int_\Omega -(f_h)_2\partial_1\phi_3 + i\omega\mu (f_e)_3\phi_3 \qquad \forall \phi_3 \in W\,, \tag{3.3e}$$
$$\langle F_{H_3}, \psi_3\rangle := \int_\Omega -(f_e)_2\partial_1\psi_3 - i\omega\varepsilon (f_h)_3\psi_3 \qquad \forall \psi_3 \in W\,. \tag{3.3f}$$

Definition 3.1 (Weak solution of the Helmholtz-type system)**.** *We consider a right-hand side $(f_h, f_e) \in L^2(\Omega, \mathbb{C}^3)^2$. We say $(E, H) \in X_H \times Y_H$ is a weak solution of the Helmholtz-type system (1.9) if, for all $\phi \in X_H$ and all $\psi \in Y_H$:*

$$\int_\Omega \left\{\nabla E_1 \cdot \nabla \phi_1 - \omega^2 \varepsilon\mu E_1 \phi_1\right\} = \langle F_{E_1}, \phi_1\rangle\,, \tag{3.4a}$$

$$\text{(3.4b)} \quad \int_\Omega \left\{\nabla H_1 \cdot \nabla \psi_1 - \omega^2 \varepsilon\mu H_1 \psi_1\right\} = \langle F_{H_1}, \psi_1\rangle\,,$$

$$\text{(3.4c)} \quad \int_\Omega \left\{\partial_1 E_2 \partial_1 \phi_2 - \omega^2 \varepsilon\mu E_2 \phi_2\right\} = \int_\Omega \{\partial_2 E_1 \partial_1 \phi_2 - i\omega\mu \partial_3 H_1 \phi_2\} + \langle F_{E_2}, \phi_2\rangle\,,$$

$$\text{(3.4d)} \quad \int_\Omega \left\{\partial_1 H_2 \partial_1 \psi_2 - \omega^2 \varepsilon\mu H_2 \psi_2\right\} = \int_\Omega \{\partial_2 H_1 \partial_1 \psi_2 - i\omega\varepsilon \partial_3 E_1 \psi_2\} + \langle F_{H_2}, \psi_2\rangle\,,$$

$$\text{(3.4e)} \quad \int_\Omega \left\{\partial_1 E_3 \partial_1 \phi_3 - \omega^2 \varepsilon\mu E_3 \phi_3\right\} = \int_\Omega \{\partial_3 E_1 \partial_1 \phi_3 + i\omega\mu \partial_2 H_1 \phi_3\} + \langle F_{E_3}, \phi_3\rangle\,,$$

$$\text{(3.4f)} \quad \int_\Omega \left\{\partial_1 H_3 \partial_1 \psi_3 - \omega^2 \varepsilon\mu H_3 \psi_3\right\} = \int_\Omega \{\partial_3 H_1 \partial_1 \psi_3 - i\omega\varepsilon \partial_2 E_1 \psi_3\} + \langle F_{H_3}, \psi_3\rangle\,.$$

3.2. **Very weak solutions of the Helmholtz-type system.** Additionally to the weak solution concept, we introduce the concept of very weak solutions. Essentially, in the very weak concept, all derivatives are moved to the test-functions, it therefore has the character of a distributional concept. We do not call it a distributional solution since test-functions do not necessarily have a compact support. Indeed, we encode some boundary and interface conditions with test-functions that are not vanishing at boundaries.

Definition 3.2 (Very weak solution of the Helmholtz-type problem)**.** *Let* $(f_h, f_e) \in L^2(\Omega, \mathbb{C}^3)^2$ *and* $F_{E_1}, F_{H_1}, F_{E_2}, F_{H_2}, F_{E_3}, F_{H_3}$ *be given by* (3.3)*. We say that* $(E, H) \in L^2(\Omega, \mathbb{C}^3) \times L^2(\Omega, \mathbb{C}^3)$ *is a very weak solution to the Helmholtz-type problem* (1.9) *if*

$$\text{(3.5a)} \quad \int_\Omega \left\{-E_1 \Delta \phi_1 - \omega^2 \varepsilon\mu E_1 \phi_1\right\} = \langle F_{E_1}, \phi_1\rangle\,,$$

$$\text{(3.5b)} \quad \int_\Omega \left\{-H_1 \Delta \psi_1 - \omega^2 \varepsilon\mu H_1 \psi_1\right\} = \langle F_{H_1}, \psi_1\rangle\,,$$

$$\text{(3.5c)} \quad \int_\Omega \left\{-E_2 \partial_1^2 \phi_2 - \omega^2 \varepsilon\mu E_2 \phi_2\right\} = \int_\Omega \{-E_1 \partial_2 \partial_1 \phi_2 + i\omega\mu H_1 \partial_3 \phi_2\} + \langle F_{E_2}, \phi_2\rangle\,,$$

$$\text{(3.5d)} \quad \int_\Omega \left\{-H_2 \partial_1^2 \psi_2 - \omega^2 \varepsilon\mu H_2 \psi_2\right\} = \int_\Omega \{-H_1 \partial_2 \partial_1 \psi_2 - i\omega\varepsilon E_1 \partial_3 \psi_2\} + \langle F_{H_2}, \psi_2\rangle\,,$$

$$\text{(3.5e)} \quad \int_\Omega \left\{-E_3 \partial_1^2 \phi_3 - \omega^2 \varepsilon\mu E_3 \phi_3\right\} = \int_\Omega \{-E_1 \partial_3 \partial_1 \phi_3 - i\omega\mu H_1 \partial_2 \phi_3\} + \langle F_{E_3}, \phi_3\rangle\,,$$

$$\text{(3.5f)} \quad \int_\Omega \left\{-H_3 \partial_1^2 \psi_3 - \omega^2 \varepsilon\mu H_3 \psi_3\right\} = \int_\Omega \{-H_1 \partial_3 \partial_1 \psi_3 + i\omega\varepsilon E_1 \partial_2 \psi_3\} + \langle F_{H_3}, \psi_3\rangle\,,$$

for all test-functions

$$\text{(3.5g)} \quad \phi_1 \in D_\#(\Omega \setminus \Gamma)\,, \quad \phi_2 \in D_\#(\Omega)\,, \quad \phi_3 \in D_\#(\overline{\Omega}; \Gamma)\,,$$

$$\text{(3.5h)} \quad \psi_1 \in \{\psi_1 \in D_\#(\overline{\Omega}) \mid \partial_3 \psi_1 \in D_\#(\Omega)\}\,, \quad \psi_2 \in D_\#(\overline{\Omega}; \Gamma)\,, \quad \psi_3 \in D_\#(\Omega)\,.$$

3.3. **Equivalence of the weak and very weak Helmholtz solution concept.** In this section, we show the equivalence of the solution concepts. Lemma 3.3 provides that every very weak solution of the Helmholtz-type system (Definition 3.2), which has the additional regularity $(E, H) \in X \times Y$, is also a weak solution of the Helmholtz-type system (Definition 3.1). Lemma 3.4 provides that every weak solution is also a very weak solution.

For the proof of the following lemma, we use another space of periodic functions. In the spirit of previous constructions, every function u on $I_1 \times I_2$ is identified with its periodic extension $\tilde{u}$ on $\mathbb{R}^2$, defined as

$$\tilde{u}(k_1 l_1 + k_2 l_2 + x) = u(x) \qquad \forall k_1, k_2 \in \mathbb{Z}\,,\ x \in I_1 \times I_2\,,$$

and the corresponding space is

$$D_\#(I_1 \times I_2) \coloneqq \{u \in C^\infty(\overline{I_1} \times \overline{I_2}) \mid \tilde{u} \in C^\infty(\mathbb{R}^2)\}\,.$$

Moreover, we use the spaces

$$D(I_3 \setminus \{0\}) \coloneqq \{u \in C_c^\infty(I_3 \setminus \{0\})\}\,,$$
$$D(\overline{I_3}; \{0\}) \coloneqq \left\{u \colon \overline{I}_3 \to \mathbb{C} \,\middle|\, u|_{I_3^\pm} \in C^\infty\left(\overline{I_3^\pm}\right)\right\}\,.$$

Lemma 3.3 (Very weak implies weak). *We consider right-hand sides $(f_h, f_e) \in L^2(\Omega, \mathbb{C}^3)^2$. Let $(E, H) \in L^2(\Omega, \mathbb{C}^3)^2$ be a very weak solution to the Helmholtz-type system of Definition 3.2 with the regularity $(E, H) \in X \times Y$. Then, (E, H) is also a weak solution to the Helmholtz-type system, see Definition 3.1. In particular, $(E, H) \in X_H \times Y_H$.*

Proof. Let $(E, H) \in X \times Y$ be a very weak solution to the Helmholtz-type system, i.e., a solution to (3.5). We have to show the regularities $E_1 \in H^1_{0,\#}(\Omega \setminus \Gamma)$, $H_1 \in H^1_\#(\Omega)$, E_2, H_2, E_3, $H_3 \in W$ and that (E, H) satisfies (3.4). The systems (3.5) and (3.4) differ essentially by some integration by parts and we have to show that these integrations by parts are admissible and do not come with boundary terms.

$H^1_{0,\#}(\Omega \setminus \Gamma)$**-regularity of** E_1: Let $\zeta \in D_\#(I_1 \times I_2)$ and $\varphi \in D(\overline{I_3}; \{0\})$ be arbitrary. Noting that $E \in X$, we choose $\psi = (0, \zeta\varphi, 0) \in Y$ in the definition of X, see (2.2b), which gives

$$\int_\Omega (\operatorname{curl} E)_2 \zeta\varphi = \int_\Omega \{-E_1 \zeta \partial_3 \varphi + E_3 \partial_1 \zeta \varphi\}\,.$$

We rewrite this with Fubini's theorem as

$$\int_{I_3 \setminus \{0\}} \left(-\int_{I_1 \times I_2} E_1 \zeta\right) \partial_3 \varphi = \int_{I_3 \setminus \{0\}} \left(\int_{I_1 \times I_2} \{-E_3 \partial_1 \zeta + (\operatorname{curl} E)_2 \zeta\}\right) \varphi\,.$$

This relation allows to conclude that the expression $\int_{I_1 \times I_2} E_1 \zeta$ has a distributional derivative in direction x_3 and that the expression is vanishing at the points $x_3 = 0$ and $x_3 = \pm l_3^\pm$. Since ζ was arbitrary, this fact encodes that E_1 vanishes at the top boundary and along Γ.

This shows that E_1 solves the Poisson-problem (3.5a) with periodicity and Dirichlet boundary conditions. A Weyl-type lemma (see Lemma A.2) implies the regularity and the boundary conditions that are encoded with $E_1 \in H^1_{0,\#}(\Omega \setminus \Gamma)$. We note that we include Lemma A.2 to have this text self-contained; it might also be possible to conclude with results of Chapter 3 of [13].

An integration by parts in (3.5a) is permitted and we obtain (3.4a).

$H^1_\#(\Omega)$-regularity of H_1: Let $\zeta \in D_\#(I_1 \times I_2)$ and $\varphi \in C_c^\infty(I_3)$. Noting that $H \in Y$, we choose $\phi = (0, \zeta\varphi, 0)$ in the definition of Y, see (2.2a), and find

$$\int_\Omega (\operatorname{curl} H)_2 \zeta\varphi = \int_\Omega \{-H_1\zeta\partial_3\varphi + H_3\partial_1\zeta\varphi\}\,.$$

Since $\varphi \in C_c^\infty(I_3)$ was arbitrary, this equation implies

$$\partial_3\left(\int_{I_1\times I_2} H_1\zeta\right) = \int_{I_1\times I_2} \{-H_3\partial_1\zeta + (\operatorname{curl} H)_2\zeta\} \quad \text{in } L^2(I_3)\,,$$

and, in particular, that the expression $\int_{I_1\times I_2} H_1\zeta$ has no jump at $x_3 = 0$. Since $\zeta \in D_\#(I_1 \times I_2)$ was arbitrary, we conclude that H_1 has no jump across Γ. Together with the fact that H_1 solves (3.5b), we deduce $H_1 \in H^1_\#(\Omega)$, see Lemma A.3. Integration by parts is allowed and equation (3.5b) implies (3.4b).

***W*-regularity of E_2, H_2, E_3, H_3**: We present the argument for E_2. We choose ϕ_2 of the form $\phi_2(x_1, x_2, x_3) = \varphi(x_1)\zeta(x_2, x_3)$ for arbitrary $\zeta \in C_c^\infty(U \setminus \Gamma_U)$ (we recall $U = I_2 \times I_3$ and $\Gamma_U = I_2 \times \{0\}$) and $\varphi \in D_\#(I_1)$. Due to the compact support of ζ and the $H^1(\Omega\setminus\Gamma)$ regularity of E_1 and H_1, we can integrate by parts all terms on the right-hand sides of (3.5c) that contain E_1 and H_1 and derivatives in the direction e_2 or e_3. We get

$$\int_\Omega \{-E_2\partial_1^2\phi_2 - \omega^2\varepsilon\mu E_2\phi_2\} = \int_\Omega \{\partial_2 E_1\partial_1\phi_2 - i\omega\mu\partial_3 H_1\phi_2\} + \langle F_{E_2}, \phi_2\rangle\,. \tag{3.6}$$

Since $\zeta \in C_c^\infty(U\setminus\Gamma)$ was arbitrary, and since the functions in the subsequent formula are all of class $L^2(\Omega)$, we obtain the desired equations pointwise: For almost every $(x_2, x_3) \in U$, there holds

$$\begin{aligned}
&\int_{I_1} \{-E_2(\cdot, x_2, x_3)\partial_1^2\varphi - \omega^2\varepsilon\mu E_2(\cdot, x_2, x_3)\varphi\} \\
&= \int_{I_1} \{\partial_2 E_1(\cdot, x_2, x_3)\partial_1\varphi - i\omega\mu\partial_3 H_1(\cdot, x_2, x_3)\varphi\} \\
&\quad + \int_{I_1} \{(f_h)_3(\cdot, x_2, x_3)\partial_1\varphi + i\omega\mu(f_e)_2(\cdot, x_2, x_3)\varphi\}\,.
\end{aligned}$$

The Weyl-Lemma implies, for a.e. $(x_2, x_3) \in U$, there holds $E_2(\cdot, x_2, x_3) \in H^1_\#(I_1)$. This can be also obtain from the fact that $E_2(\cdot, x_2, x_3)$ solves an ordinary differential equation. Thus, there exists $C > 0$ (independent of x_1 and x_2), such that

$$\begin{aligned}
\|E_2(\cdot, x_2, x_3)\|_{H^1(I_1)} \le C \big(&\|(f_h)_3(\cdot, x_2, x_3)\|_{L^2(I_1)} + \|(f_e)_2(\cdot, x_2, x_3)\|_{L^2(I_1)} \\
&+\|\partial_2 E_1(\cdot, x_2, x_3)\|_{L^2(I_1)} + \|\partial_3 H_1(\cdot, x_2, x_3)\|_{L^2(I_1)}\big)\,.
\end{aligned}$$

Integrating this inequality (or, more precisely, the squared inequality) over $I_2 \times I_3^\pm$ yields, by $L^2(U)$-boundedness of the right-hand side:

$$\begin{aligned}
\int_{I_2\times I_3^\pm} \|E_2(\cdot, x_2, x_3)\|^2_{H^1(I_1)}\,dx_2\,dx_3 \le C \Big(&\|(f_h)_3\|^2_{L^2(\Omega^\pm)} + \|(f_e)_2\|^2_{L^2(\Omega^\pm)} \\
&+\|E_1\|^2_{H^1(\Omega^\pm)} + \|H_1\|^2_{H^1(\Omega^\pm)}\Big)\,,
\end{aligned}$$

which implies $E_2 \in L^2(U, H^1_\#(I_1))$. We can therefore integrate the first term of the left-hand side of (3.6) by parts and obtain (3.4c) for ϕ_2 of the form $\phi_2(x_1, x_2, x_3) = \varphi(x_1)\zeta(x_2, x_3)$ for $\zeta \in C_c^\infty(U \setminus \Gamma_U)$, $\varphi \in D_\#(I_1)$. By the density of the span of such functions in W, (3.4c) holds for arbitrary $\phi_2 \in W$.

For the equations of H_2, E_3, H_3, we can use the same argument. Thus, $(E, H) \in X_H \times Y_H$ is a weak solution of the Helmholtz-type problem (3.4). □

Lemma 3.4 (Weak implies very weak)**.** *Let $(f_h, f_e) \in L^2(\Omega, \mathbb{C}^3)^2$ and $(E, H) \in X_H \times Y_H$ be a weak solution to the Helmholtz-type system, i.e., a solution to (3.4). Then, (E, H) is a very weak solution to the Helmholtz-type system, i.e., a solution to (3.5).*

Proof. The solutions of (3.5) require less regularity than the solutions of (3.4) and the equations itself differ only by some integration by parts. Thus, let $(E, H) \in X_H \times Y_H$ be a weak solution to the Helmholtz-type system, i.e., a solution to (3.4). We have to show that all integrations by parts can be performed without boundary terms.

The property $E_1 \in H^1_{0,\#}(\Omega \setminus \Gamma)$ allows to integrate by parts without boundary terms; we exploit the periodicity in e_1 and e_2 direction and the vanishing trace of E_1 at Γ and $\partial_{\mathrm{hor}}\Omega$.

In (3.4b), the integration by parts for the left-hand side is not producing boundary terms since $\partial_3\psi_1 \in D_\#(\Omega)$ and, thus, vanishes at $\partial_{\mathrm{hor}}\Omega$ and is periodic in the e_1 and e_2 directions. For the left-hand sides of (3.4c)–(3.4f), there do not arise boundary terms due to the periodicity of the functions in the e_1-direction.

For the right-hand sides of (3.5c) and (3.5f), there is no boundary term for H_1 because of $\phi_2 \in D_\#(\Omega)$ and $\psi_3 \in D_\#(\Omega)$. For the right-hand sides of (3.5d) and (3.5e), the boundary term for the H_1-term vanishes due to the periodicity in the e_2-direction of the test-functions and H_1. □

4. Equivalence of the Maxwell and the Helmholtz-type system

Lemma 4.1 (Maxwell implies Helmholtz)**.** *We consider right-hand sides $(f_h, f_e) \in Y_0^\perp \times X_0^\perp$. Let $(E, H) \in X \times Y$ be a solution to the Maxwell system, see Definition 2.3. Then, (E, H) is a weak solution of the Helmholtz-type system, see Definition 3.1. In particular, $(E, H) \in X_H \times Y_H$.*

Proof. Let $(E, H) \in L^2(\Omega, \mathbb{C}^3) \times L^2(\Omega, \mathbb{C}^3)$ be a solution to the Maxwell system according to Definition 2.3, i.e., system (2.4) is satisfied. Our aim is to show that (E, H) is a very weak solution to the Helmholtz-type system, i.e., a solution to (3.5). Lemma 3.3 then yields that (E, H) is also a weak solution to the Helmholtz-type system.

Equation for E_1: Let $\varphi_1 \in D(\Omega \setminus \Gamma)$ be arbitrary. We choose $\phi = i\omega\mu(\varphi_1, 0, 0)$ in (2.4b), noting that the condition $\phi \in X$ is satisfied since ϕ is smooth and its first and second component vanish at $\partial_{\mathrm{hor}}\Omega$ and its first component vanishes on Γ. We obtain

$$\int_\Omega i\omega\mu H \cdot \operatorname{curl}(\varphi_1, 0, 0) = \int_\Omega (\omega^2 \varepsilon\mu E + i\omega\mu f_e) \cdot (\varphi_1, 0, 0)\,.$$

We choose $\psi = \operatorname{curl}(\varphi_1, 0, 0) = (0, \partial_3\varphi_1, -\partial_2\varphi_1)$ in (2.4a), noting that $\psi \in Y$ is satisfied since the first component of ψ does not jump along Γ. We find

$$\int_\Omega E \cdot \operatorname{curl}\operatorname{curl}(\varphi_1, 0, 0) = \int_\Omega (i\omega\mu H + f_h) \cdot \operatorname{curl}(\varphi_1, 0, 0)\,.$$

Summing up the two equations yields

$$\int_\Omega \left\{ E \cdot \operatorname{curl}\operatorname{curl}(e_1\varphi_1) - \omega^2 \varepsilon\mu E_1\varphi_1 \right\} = \int_\Omega \left\{ i\omega\mu (f_e)_1\varphi_1 + f_h \cdot \operatorname{curl}(e_1\varphi_1) \right\}\,. \tag{4.1}$$

Since we have divergence-free data, Lemma 2.4 implies that $E \in X_0^\perp$. Therefore, taking $g = \nabla\partial_1\varphi_1 \in X_0$ in (2.6), we obtain

$$\int_\Omega E \cdot \nabla \operatorname{div}(e_1\varphi_1) = 0\,. \tag{4.2}$$

Using the identity curl curl $= -\Delta + \nabla$ div and subtracting (4.2) from (4.1) gives

$$\int_\Omega \left\{E \cdot \Delta(e_1\varphi_1) - \omega^2\varepsilon\mu E_1\varphi_1\right\} = \int_\Omega \left\{i\omega\mu(f_e)_1\varphi_1 + f_h \cdot \operatorname{curl}(e_1\varphi_1)\right\}\,.$$

This provides (3.5a) for $\phi_1 = \varphi_1$.

Equation for H_1: The derivation of (3.5b) is very similar to the derivation of (3.5a). Let $\eta_1 \in D_\#(\overline{\Omega})$ with $\partial_3\eta_1 \in D_\#(\Omega)$ be arbitrary. We choose $\psi = -i\omega\varepsilon(\eta_1, 0, 0) \in Y$ in (2.4a), where $\psi \in Y$ since η_1 does not jump across Γ. We choose $\phi = \operatorname{curl}(e_1\eta_1) = (0, \partial_3\eta_1, -\partial_2\eta_1) \in X$ in (2.4b), where $\phi \in X$ since $\partial_3\eta_1 \in D_\#(\Omega)$ and, thus, the second component of ϕ vanishes at $\partial_{\text{hor}}\Omega$. Moreover, we choose $g = -\nabla\partial_1\eta_1 \in Y_0$ in (2.7). Adding the resulting three equations and using the identity curl curl $= -\Delta + \nabla$ div gives (3.5b) for $\psi_1 = \eta_1$.

Equation for E_2: Let $\varphi_2 \in D_\#(\Omega)$ be arbitrary. We choose $\phi = i\omega\mu(0, \varphi_2, 0) \in X$ in (2.4b) and obtain:

$$i\omega\mu \int_\Omega \{-H_1\partial_3\varphi_2 + H_3\partial_1\varphi_2\} = \int_\Omega \left\{\omega^2\varepsilon\mu E_2\varphi_2 + i\omega\mu(f_e)_2\varphi_2\right\}\,.$$

Choosing $\psi = (0, 0, \partial_1\varphi_2) \in Y$ in (2.4b) gives:

$$\int_\Omega \{E_1\partial_2\partial_1\varphi_2 - E_2\partial_1\partial_1\varphi_2\} = \int_\Omega \{i\omega\mu H_3\partial_1\varphi_2 + (f_h)_3\partial_1\varphi_2\}\,.$$

Summing these equations gives (3.5c) for $\phi_2 = \varphi_2$.

Equations for H_2, E_3, H_3: These equations are derived as the equation for E_2. We only sketch the necessary steps.

Equation for H_2: Let $\eta_2 \in D_\#(\overline{\Omega}; \Gamma)$ be arbitrary. We choose $\psi = i\omega\varepsilon(0, \eta_2, 0) \in Y$ in (2.4a) and $\phi = (0, 0, \partial_1\eta_2) \in X$ in (2.4b). Summing the resulting equations gives (3.5d) for $\psi_2 = \eta_2$.

Equation for E_3: Let $\varphi_3 \in D_\#(\overline{\Omega}; \Gamma)$ be arbitrary. We choose $\phi = i\omega\mu(0, 0, \varphi_3) \in X$ in (2.4b) and $\psi = (0, \partial_1\varphi_3, 0) \in Y$ in (2.4a). Summing the resulting equations gives (3.5e) for $\phi_3 = \varphi_3$.

Equation for H_3: Let $\eta_3 \in D_\#(\Omega)$ be arbitrary. We choose $\psi = i\omega\varepsilon(0, 0, \eta_3) \in Y$ in (2.4a) and $\phi = (0, \partial_1\eta_3, 0) \in X$ in (2.4b). Summing the resulting equations gives (3.5f) for $\psi_3 = \eta_3$. □

With the above result, we know that Maxwell solutions are also Helmholtz solutions. We next want to show the opposite implication for the case $\omega^2 \notin \sigma(l_1)$: Weak solution of the Helmholtz-type system are also a solution to the Maxwell system.

In this derivation, we use spaces of smooth functions on the cross section $U = I_2 \times I_3$ of Ω. These spaces corresponds to the subsets of $D_\#(\overline{\Omega})$, $D_\#(\Omega)$, $D_\#(\Omega \setminus \Gamma)$ and $D_\#(\overline{\Omega}; \Gamma)$ of functions that are independent of x_1. In the spirit of previous constructions, every function u on U is identified with its periodic extension $\tilde{u}$ on $\mathbb{R} \times I_3$, defined as

$$\tilde{u}(k_1 l_2 + x) = u(x) \qquad \forall k_1 \in \mathbb{Z}\,,\; x \in I_2 \times I_3\,,$$

In the subsequent table, the second column indicates whether or not it is demanded that functions vanish in a neighborhood of the horizontal boundaries $\partial_{\text{hor}} U$. The last column indicates whether or not the functions can have a jump across Γ_U, or if it is demanded that the function vanishes in a neighborhood of Γ_U.

Space	$\partial_{\text{hor}} U$	Γ_U
$D_\#(\overline{U}) \coloneqq \left\{u \in C^\infty\left(\overline{U}\right) \,\middle|\, \tilde{u} \in C^\infty\left(\overline{U_\#}\right)\right\}$	$\neq 0$	no j.
$D_\#(U) \coloneqq \left\{u \in C^\infty(\overline{U}) \,\middle|\, \operatorname{supp}(\tilde{u}) \cap \bar{U} \subset U_\# \text{ is compact}\right\}$	$= 0$	no j.
$D_\#(U \setminus \Gamma_U) \coloneqq \left\{u \in C^\infty(\overline{U}) \,\middle|\, \operatorname{supp}(\tilde{u}) \cap \bar{U} \subset U_\# \setminus \Gamma_{U,\#} \text{ is compact}\right\}$	$= 0$	$= 0$
$D_\#(\overline{U}; \Gamma_U) \coloneqq \left\{u \colon \overline{U} \to \mathbb{C} \,\middle|\, \tilde{u}\|_{\mathbb{R}\times I_3^\pm} \in C^\infty\left(\mathbb{R} \times \overline{I_3^\pm}\right)\right\}$	$\neq 0$	jump

Lemma 4.2 (Helmholtz implies Maxwell)**.** *We consider right-hand sides $(f_h, f_e) \in Y_0^\perp \times X_0^\perp$ and $\omega^2 \notin \sigma(l_1)$. Let $(E, H) \in X_H \times Y_H$ be a solution to the Helmholtz-type system, see Definition 3.1. Then, (E, H) is a weak solution of the Maxwell system, see Definition 2.3.*

Proof. Let $(E, H) \in X_H \times Y_H$ be a weak solution to the Helmholtz-type system, by Lemma 3.4, (E, H) is also a very weak solution of the Helmholtz-type system, see Definiton 3.2. In the following, we consider test-functions ϕ_i and ψ_i for $i \in \{1, 2, 3\}$, which allow a separation of the x_1 variable from the x_2 and x_3 variables, i.e. functions of the form

$$\phi_i(x_1, x_2, x_3) = \varphi(x_1)\zeta_i(x_2, x_3)\,, \quad \psi_i(x_1, x_2, x_3) = \varphi(x_1)\xi_i(x_2, x_3) \quad \forall i \in \{1, 2, 3\}$$

for arbitrary $\varphi \in D_\#(I_1)$ and arbitrary

$$\begin{aligned} &\zeta_1 \in D_\#(U \setminus \Gamma_U)\,, && \zeta_2 \in D_\#(U)\,, && \zeta_3 \in D_\#(\overline{U}; \Gamma_U)\,, \\ &\xi_1 \in \{\xi_1 \in D_\#(\overline{U}) \mid \partial_2\xi_1 \in D_\#(U)\}\,, && \xi_2 \in D_\#(\overline{U}; \Gamma_U)\,, && \xi_3 \in D_\#(U)\,. \end{aligned}$$

We show that E and H solve (2.4) for test-functions of the form $\phi = \phi_i e_i$ and $\psi = \psi_i e_i$. By this procedure, we recover Maxwell's equations componentwise. A density argument then yields that (2.4) is satisfied for arbitrary $\psi \in Y$ and $\phi \in X$.

In the following we consider arbitrary functions $\varphi \in D_\#(I_1)$. In the construction of test-functions, we use the solution $\tilde{\varphi} \in D_\#(I_1)$ of the equation

$$-\partial_1^2\tilde{\varphi} - \omega^2\varepsilon\mu\tilde{\varphi} = \varphi\,. \tag{4.4}$$

By the condition on the frequency ω, this problem is uniquely solvable.

Test-functions ϕ_1: Let $\zeta_1 \in D_\#(U \setminus \Gamma_U)$ be arbitrary. We define $\tilde{\phi}_1(x_1, x_2, x_3) \coloneqq \tilde{\varphi}(x_1)\zeta_1(x_2, x_3)$ and, thus, $\tilde{\phi}_1 \in D_\#(\Omega \setminus \Gamma)$. We choose $\phi_1 = i\omega\varepsilon\tilde{\phi}_1$ in (3.5a), $\psi_2 = \partial_3\tilde{\phi}_1$ in (3.5d) and $\psi_3 = -\partial_2\tilde{\phi}_1$ in (3.5f), which gives

$$\begin{aligned} i\omega\varepsilon \int_\Omega \left\{-E_1\Delta\tilde{\phi}_1 - \omega^2\varepsilon\mu E_1\tilde{\phi}_1\right\} &= \langle F_{E_1}, i\omega\varepsilon\tilde{\phi}_1\rangle\,, \\ \int_\Omega \left\{H_2\partial_1^2\partial_3\tilde{\phi}_1 + \omega^2\varepsilon\mu H_2\partial_3\tilde{\phi}_1\right\} &= \int_\Omega \left\{H_1\partial_2\partial_1\partial_3\tilde{\phi}_1 + i\omega\varepsilon E_1\partial_3\partial_3\tilde{\phi}_1\right\} - \langle F_{H_2}, \partial_3\tilde{\phi}_1\rangle\,, \\ \int_\Omega \left\{H_3\partial_1^2\partial_2\tilde{\phi}_1 + \omega^2\varepsilon\mu H_3\partial_2\tilde{\phi}_1\right\} &= \int_\Omega \left\{H_1\partial_3\partial_1\partial_2\tilde{\phi}_1 - i\omega\varepsilon E_1\partial_2\partial_2\tilde{\phi}_1\right\} - \langle F_{H_3}, \partial_2\tilde{\phi}_1\rangle\,. \end{aligned}$$

We subtract the third equation from the second equation, add and subtract the expression $i\omega\varepsilon E_1\partial_1^2\tilde\phi_1$ and replace the term $E_1\Delta\tilde\phi_1$ with the first equation to obtain

$$
\begin{aligned}
&\int_\Omega \left\{ H_2\partial_3(\partial_1^2\tilde\phi_1 + \omega^2\varepsilon\mu\tilde\phi_1) - H_3\partial_2(\partial_1^2\tilde\phi_1 + \omega^2\varepsilon\mu\tilde\phi_1)\right\}\\
&= \int_\Omega \left\{ -i\omega\varepsilon E_1(\partial_1^2\tilde\phi_1 + \omega^2\varepsilon\mu\tilde\phi_1)\right\} - \langle F_{H_2}, \partial_3\tilde\phi_1\rangle + \langle F_{H_3}, \partial_2\tilde\phi_1\rangle - \langle F_{E_1}, i\omega\varepsilon\tilde\phi_1\rangle\,.
\end{aligned}
\tag{4.5}
$$

In order to simplify the source term, we exploit that the function satisfies $f_e \in X_0^\perp$, and, thus, is orthogonal to gradients (the product with $\nabla\partial_1\tilde\phi_1$ vanishes):

$$
\begin{aligned}
&- \langle F_{H_2}, \partial_3\tilde\phi_1\rangle + \langle F_{H_3}, \partial_2\tilde\phi_1\rangle - \langle F_{E_1}, i\omega\varepsilon\tilde\phi_1\rangle\\
&\quad= \int_\Omega \big\{ -(f_e)_3\partial_1\partial_3\tilde\phi_1 + i\omega\varepsilon(f_h)_2\partial_3\tilde\phi_1 - (f_e)_2\partial_1\partial_2\tilde\phi_1 - i\omega\varepsilon(f_h)_3\partial_2\tilde\phi_1\\
&\qquad\qquad + \omega^2\varepsilon\mu(f_e)_1\tilde\phi_1 - i\omega\varepsilon f_h\cdot \operatorname{curl}(e_1\tilde\phi_1)\big\}\\
&\quad= \int_\Omega \big\{ -f_e\cdot\nabla\partial_1\tilde\phi_1 + (f_e)_1\partial_1^2\tilde\phi_1 + \omega^2\varepsilon\mu(f_e)_1\tilde\phi_1\big\}\\
&\quad= \int_\Omega \big\{ (f_e)_1\partial_1^2\tilde\phi_1 + \omega^2\varepsilon\mu(f_e)_1\tilde\phi_1\big\}\,.
\end{aligned}
\tag{4.6}
$$

Replacing the source terms in (4.5) with (4.6), we obtain

$$
\begin{aligned}
&\int_\Omega \left\{ H_2\partial_3(\partial_1^2\tilde\phi_1 + \omega^2\varepsilon\mu\tilde\phi_1) - H_3\partial_2(\partial_1^2\tilde\phi_1 + \omega^2\varepsilon\mu\tilde\phi_1)\right\}\\
&\quad= \int_\Omega \left\{ -i\omega\varepsilon E_1(\partial_1^2\tilde\phi_1 + \omega^2\varepsilon\mu\tilde\phi_1) + (f_e)_1(\partial_1^2\tilde\phi_1 + \omega^2\varepsilon\mu\tilde\phi_1)\right\}\,,
\end{aligned}
\tag{4.7}
$$

which is a weak formulation of

$$(\partial_1^2 + \omega^2\varepsilon\mu)(\operatorname{curl} H)_1 = (\partial_1^2 + \omega^2\varepsilon\mu)(-i\omega\varepsilon E_1 + (f_e)_1)\,.$$

Because of (4.4), the function $\phi_1(x_1,x_2,x_3) := \varphi(x_1)\zeta_1(x_2,x_3)$ satisfies $\phi_1 = -\partial_1^2\tilde\phi_1 - \omega^2\varepsilon\mu\tilde\phi_1$. This allows to simplify (4.7) to

$$\int_\Omega \{H_2\partial_3\phi_1 - H_3\partial_2\phi_1 + i\omega\varepsilon E_1\phi_1\} = \int_\Omega (f_e)_1\phi_1\,.$$

This is relation (2.4b) for $\phi = e_1\phi_1$. We recall that we have chosen $\phi_1(x_1,x_2,x_3) = \varphi(x_1)\zeta_1(x_2,x_3)$ with arbitrary $\varphi \in D_\#(I_1)$ and $\zeta_1 \in D_\#(U\setminus\Gamma_U)$.

Test-functions ψ_1: We proceed as with ϕ_1. Let $\xi_1 \in D_\#(\overline{U})$ with $\partial_2\xi_1 \in D_\#(U)$. We define $\tilde\psi_1(x_1,x_2,x_3) := \tilde\varphi(x_1)\xi_1(x_2,x_3)$ such that $\tilde\psi_1 \in D_\#(\overline\Omega)$ and $\partial_3\tilde\psi(x_1,x_2,x_3) = \tilde\varphi(x_1)\partial_2\xi_1(x_2,x_3)$ and $\partial_3\tilde\psi \in D_\#(\Omega)$. We choose $\psi_1 = -i\omega\mu\tilde\psi_1$ in (3.5b), $\phi_2 = \partial_3\tilde\psi_1 \in D_\#(\Omega)$ in (3.5c) and $\phi_3 = -\partial_2\tilde\psi_1$ in (3.5e). Adding the resulting equations and exploiting that the function f_h is orthogonal to gradients shows (2.4a) for $\psi = e_1\psi_1$ with $\psi_1(x_1,x_2,x_3) = \varphi(x_1)\xi_1(x_2,x_3)$.

Test-functions ϕ_2: For $\zeta_2 \in D_\#(U)$, we use $\tilde\phi_2(x_1,x_2,x_3) = \tilde\varphi(x_1)\zeta_2(x_2,x_3)$, which satisfies $\tilde\phi_2 \in D_\#(\Omega)$. We can choose $\phi_2 = i\omega\varepsilon\tilde\phi_2$ in (3.5c) and $\psi_3 = \partial_1\tilde\phi_2 \in$

$D_\#(\Omega)$ in (3.5f). Adding the resulting equations yields

$$\int_\Omega \left\{-H_1\partial_3(-\partial_1\partial_1\tilde{\phi}_2 - \omega^2\varepsilon\mu\tilde{\phi}_2) + H_3\partial_1(-\partial_1\partial_1\tilde{\phi}_2 - \omega^2\varepsilon\mu\tilde{\phi}_2)\right\} = \int_\Omega (-i\omega\varepsilon E_2 + (f_e)_2)\,(-\partial_1\partial_1\tilde{\phi}_2 - \omega^2\varepsilon\mu\tilde{\phi}_2)\,. \tag{4.8}$$

We simplify again with $\phi_2(x_1,x_2,x_3) = \varphi(x_1)\zeta_2(x_2,x_3) = -\partial_1^2\tilde{\phi}_2 - \omega^2\varepsilon\mu\tilde{\phi}_2$ to

$$\int_\Omega \{-H_1\partial_3\phi_2 + H_3\partial_1\phi_2\} = \int_\Omega (-i\omega\varepsilon E_2 + (f_e)_2)\,\phi_2\,.$$

This is (2.4b) for $\phi = e_2\phi_2$.

Test-functions ψ_2: For a function $\xi_2 \in D_\#(\overline{U};\Gamma_U)$ we use $\tilde{\psi}_2(x_1,x_2,x_3) := \tilde{\varphi}(x_1)\xi_2(x_2,x_3)$, which satisfies $\tilde{\psi}_2 \in D_\#(\overline{\Omega};\Gamma)$. We can choose $\psi_2 = -i\omega\mu\tilde{\psi}_2$ in (3.5d) and $\phi_3 = \partial_1\tilde{\psi}_2$ in (3.5e). Adding the resulting equations yields (2.4a) for $\psi = e_2\psi_2$ with $\psi_2(x_1,x_2,x_3) = \varphi(x_1)\xi_2(x_2,x_3)$.

Test-functions ϕ_3: For a function $\zeta_3 \in D_\#(\overline{U};\Gamma_U)$ we use $\tilde{\phi}_3(x_1,x_2,x_3) := \tilde{\varphi}(x_1)\zeta_3(x_2,x_3)$, which satisfies $\tilde{\phi} \in D_\#(\overline{\Omega},\Gamma)$. We can choose $\phi_3 = i\omega\varepsilon\tilde{\phi}_3$ in (3.5e) and $\psi_2 = \partial_1\tilde{\phi}_3$ in (3.5d). Adding the resulting equations yields (2.4b) for $\phi = e_3\phi_3$ with $\phi_3(x_1,x_2,x_3) = \varphi(x_1)\zeta_3(x_2,x_3)$.

Test-functions ψ_3: For $\xi_3 \in D_\#(U)$ we use $\tilde{\psi}_3(x_1,x_2,x_3) = \tilde{\varphi}(x_1)\xi_3(x_2,x_3)$, which satisfies $\tilde{\psi}_3 \in D_\#(\Omega)$. We can choose $\psi_3 = -i\omega\mu\tilde{\psi}_3$ in (3.5f) and $\phi_2 = \partial_1\tilde{\psi}_3$ in (3.5c). Adding the resulting equations yields (2.4a) for $\psi = e_3\psi_3$ with $\psi_3(x_1,x_2,x_3) = \varphi(x_1)\xi_3(x_2,x_3)$.

Density of test-functions: Taking finite linear combinations, we obtain that (E,H) solves (2.4) for arbitrary $\phi \in \mathcal{X}$ and $\psi \in \mathcal{Y}$, where $\mathcal{X}$ and $\mathcal{Y}$ are given as vector spaces of smooth functions. In the subsequent formula, the index $i \in \{1,2,3\}$ stands for the components of the vector fields $\phi = (\phi_1,\phi_2,\phi_3)$ and $\psi = (\psi_1,\psi_2,\psi_3)$:

(4.9)

$$\mathcal{X} := \left\{\phi\colon \Omega \to \mathbb{C}^3 \,\middle|\, \phi_i(x_1,x_2,x_3) = \sum_{j=1}^N \varphi_i^j(x_1)\zeta_i^j(x_2,x_3) \text{ for some } N \in \mathbb{N}, \right.$$
$$\left. \varphi_i^j \in D_\#(I_1)\,\forall i \le 3\,, \zeta_1^j \in D_\#(U\setminus\Gamma_U), \zeta_2^j \in D_\#(U), \zeta_3^j \in D_\#(\overline{U};\Gamma_U)\ \forall j\right\},$$

$$\mathcal{Y} := \left\{\psi\colon \Omega \to \mathbb{C}^3 \,\middle|\, \psi_i(x_1,x_2,x_3) = \sum_{j=1}^N \varphi_i^j(x_1)\xi_i^j(x_2,x_3) \text{ for some } N \in \mathbb{N}, \right.$$
$$\left. \varphi_i^j \in D_\#(I_1)\,\forall i\,, \xi_1^j \in D_\#(\overline{U})\,, \partial_{x_3}\xi_1^j \in D_\#(U), \xi_2^j \in D_\#(\overline{U},\Gamma_U), \xi_3^j \in D_\#(U)\forall j\right\}.$$

By Lemma 4.4 below, the set $\mathcal{X}$ is dense in X (in the topology of $H(\mathrm{curl},\Omega)$), and $\mathcal{Y}$ is dense in Y (in the topology of $H(\mathrm{curl},\Omega\setminus\Gamma)$). We therefore obtain (2.4) for all $\psi \in Y$ and $\phi \in X$. This verifies that (E,H) is a solution to the Maxwell system. □

It remains to prove the density result of Lemma 4.4. As a preparation, we show a density property for scalar-valued functions, formulated as Lemma 4.3. We mention that a similar argument is also used in the proof of Lemma A.3.

Lemma 4.3 (H^1-density of functions with vanishing normal derivative)**.** *We define a vector space of scalar-valued smooth functions as*

$$\mathcal{Z} := \Big\{ u \in D_\#(\overline{\Omega}) \,\Big|\, u(x) = \sum_{j=1}^{N} \phi_1^j(x_1)\phi_2^j(x_2)\zeta_3^j(x_3) \text{ for some } N \in \mathbb{N}, \\ \phi_1^j \in D_\#(I_1)\,,\, \phi_2^j \in D_\#(I_2)\,,\, \zeta^j \in C^\infty(\overline{I_3}) \text{ with } \partial_{x_3}\zeta^j \in C_c^\infty(I_3)\,\forall j \Big\}\,.$$

This space is dense in $H^1_\#(\Omega)$, there holds $\overline{\mathcal{Z}}^{H^1(\Omega)} = H^1_\#(\Omega)$

Proof. Let $u \in H^1_\#(\Omega)$ be an arbitrary function that we wish to approximate and let $\delta > 0$ be arbitrary. We choose an approximation $u_\delta(x) = \sum_{j=1}^{N_\delta} \phi_\delta^j(x_1, x_2)\zeta_\delta^j(x_3)$ for some $N_\delta \in \mathbb{N}$ with $\|u - u_\delta\|_{H^1(\Omega)} < \delta$ with the properties: $\phi_\delta^j(x_1, x_2) = \phi_{\delta,1}^j(x_1)\phi_{\delta,2}^j$ for $\phi_{\delta,1}^j \in D_\#(I_1)$, $\phi_{\delta,2}^j \in D_\#(I_2)$ and $\zeta_\delta^j \in C^\infty(\overline{I_3})$ for all $j \le N_\delta$. The approximation u_δ is not necessarily in $\mathcal{Z}$, since $\partial_{x_3}\zeta^j \in C_c^\infty(I_3)$ is not guaranteed.

We therefore construct a second approximation: For fixed δ, we can approximate, for every $j \le N_\delta$, the derivative $\partial_{x_3}\zeta_\delta^j$ by a function $f_\delta^j \in C_c^\infty(I_3)$ such that $\|\partial_{x_3}\zeta_\delta^j - f_\delta^j\|_{L^2(I_3)} < \delta N_\delta^{-1} C_\delta^{-1}$ for $C_\delta := \max_{j\in\{1,\dots N_\delta\}} \|\phi_\delta^j\|_{C^1(\overline{\Omega})}$. We define an improved approximating function ζ as the integral over f as follows:

$$\tilde{\zeta}_\delta^j(x_3) := \tilde{\zeta}_\delta^j(l_3) + \int_{l_3}^{x_3} f_\delta^j(t)\mathrm{d}t\,.$$

This approximation satisfies, for some constant C that is independent of δ, j, N_δ and C_δ, an estimate $\|\zeta_\delta^j - \tilde{\zeta}_\delta^j\|_{H^1(I_3)} < \delta C N_\delta^{-1} C_\delta^{-1}$. We can now define the approximation $\tilde{u}_\delta \in \mathcal{Z}$ as

$$\tilde{u}_\delta(x) := \sum_{j=1}^{N_\delta} \phi_\delta^j(x_1, x_2)\tilde{\zeta}_\delta^j(x_3)\,.$$

We obtain that $\tilde{u}_\delta$ is a good approximation of u from

$$\begin{aligned}\|u_\delta - \tilde{u}_\delta\|_{L^2(\Omega)} &\le \sum_{j=1}^{N_\delta} \|\phi_\delta^j(x_1, x_2)(\zeta_\delta^j - \tilde{\zeta}_\delta^j)\|_{L^2(\Omega)} \\ &\le N_\delta \max_{j\le N_\delta} \|\phi_\delta^j\|_{C^0(\overline{\Omega})} \max_{j\le N_\delta} \|\zeta_\delta^j - \tilde{\zeta}_\delta^j\|_{L^2(\Omega)} \le C\delta\,.\end{aligned}$$

Derivative such as $\|\partial_i(u_\delta - \tilde{u}_\delta)\|_{L^2(\Omega)}$ for $i \in \{1, 2, 3\}$ are also of order δ, as can be shown with the same calculation, replacing functions that depend on x_i by their derivative with respect to x_i. □

Lemma 4.4 (Density of functions in X and Y)**.** *The vector spaces $\mathcal{X}, \mathcal{Y}$ of* (4.9) *have the density properties $\overline{\mathcal{X}}^{H(\mathrm{curl},\Omega)} = X$ and $\overline{\mathcal{Y}}^{H(\mathrm{curl},\Omega\setminus\Gamma)} = Y$.*

Before we start the slightly technical proof, let us sketch the overall idea. Let us assume that we want to approximate a function $H \in Y$ up to an error of order δ. We construct an approximation $\tilde{H}_\delta$ in two steps. Step 1: With a convolution, we mollify H in the horizontal variables x_1 and x_2; this defines the first approximation, H_δ. The approximation keeps relevant properties of H and satisfies, additionally, smoothness, $H_{\delta,1}, H_{\delta,2} \in H^1(\Omega\setminus\Gamma)$, $H_{\delta,3} \in L^2(I_3, H^1_\#(I_1 \times I_2, \mathbb{C}^3))$ (we recall that, for the original function H, we have only a control of the curl). Step 2: We approximate H_δ to satisfy all desired properties.

Proof. Approximation of $H \in Y$: Let $H \in Y$ be an arbitrary function that we wish to approximate. *Step 1: Horizontal smoothing.* For a compactly supported function η on $\mathbb{R}^2$ with integral 1, we define the standard mollifier as $\eta_\delta := \delta^{-2}\eta(\cdot/\delta)$ for $\delta > 0$. We choose $H_\delta := H *_{1,2} \eta_\delta$, where $*_{1,2}$ denotes the convolution with respect to the first and second argument; with the notation $y = (\hat{y}, 0)$ for $\hat{y} \in \mathbb{R}^2$, we set

$$H_\delta(x) := \int_{\mathbb{R}^2} \operatorname{curl} H(x-y)\, \eta_\delta(y)\, \mathrm{d}\hat{y}\,.$$

The approximation is smooth in tangetial directions, in particular, there holds $H_\delta \in L^2(I_3, H^1_\#(I_1 \times I_2, \mathbb{C}^3))$. Since the convolution commutes with derivatives, we also obtain $\operatorname{curl} H_\delta = \operatorname{curl}(H *_{1,2} \eta_\delta) = \operatorname{curl} H *_{1,2} \eta_\delta \in L^2(\Omega_\pm)$ and, thus, $H_\delta \to H$ in $H(\operatorname{curl}, \Omega \setminus \Gamma)$. A standard calculation yields, for arbitrary $\phi = (\phi_1, \phi_2, \phi_3)$ with $\phi_1 \in D_\#(\Omega \setminus \Gamma)\,, \phi_2 \in D_\#(\Omega)\,, \phi_3 \in D_\#(\Omega)$, that

$$\int_{\Omega\setminus\Gamma} \operatorname{curl} H_\delta \cdot \phi = \int_{\Omega\setminus\Gamma} H_\delta \cdot \operatorname{curl} \phi\,. \tag{4.10}$$

This shows that the first component of H_δ does not jump across Γ, the approximation satisfies $H_\delta \in Y$. The main point of this proof is that the control of the curl now allows to control all derivatives, at least for the first two components: The distributional derivatives in $\Omega \setminus \Gamma$ are

$$\partial_3 H_{\delta,1} = (\operatorname{curl} H_\delta)_2 + \partial_1 H_{\delta,3}\,, \tag{4.11}$$

$$\partial_3 H_{\delta,2} = -(\operatorname{curl} H_\delta)_1 - \partial_2 H_{\delta,3}\,, \tag{4.12}$$

and are thus elements of $L^2(\Omega)$. This yields $H_{\delta,1}, H_{\delta,2} \in H^1_\#(\Omega \setminus \Gamma)$. Since the first component of H_δ does not jump across Γ, we even have $H_{\delta,1} \in H^1_\#(\Omega)$.

Step 2: Second approximation. By classical approximation arguments and by Lemma 4.3, we can choose a smooth approximation $\tilde{H}_\delta \in \mathcal{X}$ of H_δ such that

$$\|\tilde{H}_{\delta,1} - H_{\delta,1}\|_{H^1(\Omega)} + \|\tilde{H}_{\delta,2} - H_{\delta,2}\|_{H^1(\Omega\setminus\Gamma)} + \|\tilde{H}_{\delta,3} - H_{\delta,3}\|_{L^2(I_3, H^1_\#(I_1\times I_2)} \le \delta\,. \tag{4.13}$$

The proof is complete when we show that this approximation also satisfies that $\|\tilde{H}_\delta - H_\delta\|_{H(\operatorname{curl},\Omega\setminus\Gamma)}$ is of order δ. For the $L^2(\Omega \setminus \Gamma)$-norm, this is obvious from (4.13). We estimate the curl as

$$\|\operatorname{curl}(\tilde{H}_\delta - H_\delta)\|_{L^2(\Omega\setminus\Gamma)} = \left\| \begin{pmatrix} \partial_2(\tilde{H}_{\delta,3} - H_{\delta,3}) - \partial_3(\tilde{H}_{\delta,2} - H_{\delta,2}) \\ \partial_3(\tilde{H}_{\delta,1} - H_{\delta,1}) - \partial_1(\tilde{H}_{\delta,3} - H_{\delta,3}) \\ \partial_1(\tilde{H}_{\delta,2} - H_{\delta,2}) - \partial_2(\tilde{H}_{\delta,1} - H_{\delta,1}) \end{pmatrix} \right\|_{L^2(\Omega\setminus\Gamma)} \le 6\delta\,, \tag{4.14}$$

where we used the $L^2(\Omega)$-smallness (4.13) of the entries.

Approximation of $E \in X$: We approximate $E \in X$ analogously to $H \in Y$. We define $E_\delta := E *_{1,2} \eta_\delta \in H(\operatorname{curl}, \Omega)$. One obtains the relation of (4.10) with H_δ replaced E_δ, and with ϕ replaced by $\psi \in Y$. This verifies $E_\delta \in X$. We identify the distributional derivatives of E by means of the curl similarly to (4.11)–(4.12) and obtain $E_{\delta,1}, E_{\delta,2} \in H^1_\#(\Omega)$. Moreover, because of $E \in X$, we can deduce further $E_{\delta,1} \in H^1_{0,\#}(\Omega \setminus \Gamma)$ and $E_{\delta,2} \in H^1_{0,\#}(\Omega)$. Finally, we approximate E_δ by $\tilde{E}_\delta \in \mathcal{X}$ with respect to the $H^1(\Omega) \times H^1(\Omega \setminus \Gamma) \times L^2(I_3, H^1_\#(I_1 \times I_2))$-norm by classical approximation arguments. The convergence in $H(\operatorname{curl}, \Omega)$ follows as in (4.14). ☐

5. Existence and uniqueness of the Helmholtz-type system

Proposition 5.1 (Existence and uniqueness for Helmholtz). *Let Ω and Γ be as in (1.2), we consider the Helmholtz system (1.9) with parameters $\varepsilon, \mu > 0$ and a frequency ω with $\omega^2 \notin \sigma_M$ of (1.6). Then, for arbitrary $(f_h, f_e) \in L^2(\Omega, \mathbb{C}^3)^2$, system (1.9) has a unique weak solution $(E, H) \in X_H \times Y_H$.*

Proof. Because of $\omega^2 \notin \sigma_M$, the two Helmholtz problems (1.9a) for E_1 in Ω_+ and in Ω_- can be solved uniquely. Furthermore, the Helmholtz problem (1.9b) for H_1 in Ω can be solved uniquely.

The operator $(-\partial_1^2 - \omega^2 \varepsilon\mu \,\mathrm{id})$ can be inverted. More precisely, the equation

$$-\partial_1^2 \varphi - \omega^2 \varepsilon\mu\varphi = g + \partial_1 h \tag{5.1}$$

has a unique solution $\varphi \in H^1_\#(I_1)$ for every $g, h \in L^2(I_1)$, as can be shown, e.g., with Fourier series. This observation allows us to solve the remaining four equations of (1.9) uniquely for E_2, H_2, E_3, H_3. □

The next lemma shows that the spectrum $\sigma(H)$ of (3.4) is σ_M.

Lemma 5.2 (Spectrum of the Helmholtz-type system). *In the situation of Proposition 5.1, but with $\omega^2 \in \sigma_M$, the homogeneous Helmholtz-type system has a non-trivial solution.*

Proof. In Lemma 1.5, we obtained a non-trivial solution (E, H) to the homogeneous Maxwell system. By Lemma 4.1 this solution is also a non-trivial solution (E, H) to the homogeneous Helmholtz-type system. □

Proof of Theorem 1.1. For $\omega^2 \notin \sigma_M$, Proposition 5.1 yields the existence of a solution to the Helmholtz system. By Lemma 4.2, this solution solves also the Maxwell system. Regarding uniqueness: When (E, H) is a solution to the homogeneous Maxwell system, then, by Lemma 4.1, it is also a solution to the homogeneous Helmholtz system, and hence vanishes by the uniqueness statement of Proposition 5.1.

For $\omega^2 \in \sigma_M$, non-trivial solutions to the homogeneous Maxwell system are provided in Lemma 1.5. □

Appendix A. Regularity of distributional solutions

In order to prove that very weak solutions are also weak solutions, one has to show that very weak solutions have an H^1-type regularity. A classical result of this type is known as Weyl's lemma:

Lemma A.1 (Lemma of Weyl for H^{-1}-right-hand sides). *Let $U \subset \mathbb{R}^n$ be open and bounded, $n \in \mathbb{N}$. Let $u \in L^1_{\mathrm{loc}}(U, \mathbb{C})$ be a distributional solution of $-\Delta u = f - \nabla \cdot g$ for data $f \in L^2(U, \mathbb{C})$ and $g \in L^2(U, \mathbb{C}^n)$ in the sense that*

$$-\int_U u\Delta\phi = \int_U f\phi + g \cdot \nabla\phi \qquad \forall \phi \in C_c^\infty(U)\,. \tag{A.1}$$

Then, u has the regularity $u \in H^1_{\mathrm{loc}}(U)$.

Proof. Let $u \in L^1_{\mathrm{loc}}(U)$ be a solution to (A.1). We consider the weak solution $\tilde{u} \in H^1_0(U, \mathbb{C})$ of $\int_U \nabla\tilde{u} \cdot \nabla\phi = \int_U f\phi + g \cdot \nabla\phi \;\forall \phi \in H^1_0(U)$. The difference $(\tilde{u} - u)$ is harmonic in the sense that $\int_U (\tilde{u} - u)\Delta\phi = 0$ for all $\phi \in C_c^\infty(U)$. Thus, by the classical Weyl lemma, $(\tilde{u} - u) \in C^\infty(U)$, see [19]. This implies $u \in H^1_{\mathrm{loc}}(U)$. □

We note that Lemma A.1 provides only local H^1-regularity. In order to obtain H^1-regularity on the entire domain, we extend the function in all directions and use the local H^1-regularity of the extended function to conclude. Extensions over the periodic boundary are directly given with the periodic extension of the function (compare the definition of periodic functions). Across Dirichlet boundaries, we use an odd extension, see Lemma A.2. Across Neumann boundaries, we use an even extension, see Lemma A.3. The subsequent lemma is used in the text for the domains Ω_+ and Ω_-.

Lemma A.2 (Regularity for vanishing distributional Dirichlet boundary data)**.** *Let the domain be a product of three open intervals, $\Omega = I_1 \times I_2 \times I_3 \subset \mathbb{R}^3$. For right-hand sides $f \in L^2(\Omega, \mathbb{C})$ and $g \in L^2(\Omega, \mathbb{C}^3)$, let $u \in L^2(\Omega, \mathbb{C})$ satisfy*

$$-\int_\Omega u \Delta\phi = \int_\Omega f\phi + g \cdot \nabla\phi \qquad \forall \phi \in D_\#(\Omega)\,. \tag{A.2}$$

On the upper and lower boundary, let u satisfy a Dirichlet condition in the following sense: For every $\varphi \in C^\infty_\#(I_1 \times I_2)$ there exists a function $h = h_\varphi \in L^2(I_3)$ such that

$$\int_{I_3} \left\{ \left(\int_{I_1\times I_2} u\,\varphi \right) \partial_3\psi + h\psi \right\} = 0 \qquad \forall \psi \in C^\infty(\overline{I_3})\,. \tag{A.3}$$

Then, the solution u of (A.2) is of class $u \in H^1_{0,\#}(\Omega)$ and is a weak solution in the sense that

$$\int_\Omega \nabla u \cdot \nabla\phi = \int_\Omega f\phi + g \cdot \nabla\phi \qquad \forall \phi \in H^1_{0,\#}(\Omega)\,. \tag{A.4}$$

Regarding the boundary condition (A.3): The condition implies that, for every test-function φ, the expression $G_\varphi \colon x_3 \mapsto \left(\int_{I_1\times I_2} u\varphi \right)$ has a weak derivative in $L^2(I_3)$, namely $h = \partial_3 G_\varphi$. This implies not only $G_\varphi \in H^1(I_3)$, but it also yields that the boundary values vanish, $G_\varphi \in H^1_0(I_3)$.

Proof. As announced, we extend u in all directions and conclude by applying the Lemma of Weyl to the extended function. Without loss of generality, we assume $I_3 = (0, l_3)$. In directions x_1 and x_2, we extend periodically; to simplify notation, we keep the notation u for the extended function, $u = \tilde{u} \in H^1_{\mathrm{loc}}(\Omega_\#)$. In order to show the H^1-regularity up to the boundaries $x_3 = l_3$ and $x_3 = 0$. We perform the proof for the lower boundary $x_3 = 0$, the other proof is analogous. We set $\check{\Omega} \colon = I_1 \times I_2 \times (-l_3, l_3)$. We reflect u odd across the boundary $x_3 = 0$ to define the extended function $\check{u}$. When we show that the distributional Laplace of the extended function is of class $H^{-1}(\check{\Omega})$, we can use the Lemma of Weyl on the extended domain to conclude the H^1-regularity of the original function up to the boundary.

For $x \in \Omega$ with $(x_1, x_2) =: \hat{x}$, we define $\check{u}(\hat{x}, x_3) \in L^2(\Omega)$ by

$$\check{u}(\hat{x}, x_3) := \begin{cases} u(\hat{x}, x_3) & \text{if } x_3 > 0\,, \\ -u(\hat{x}, -x_3) & \text{if } x_3 < 0\,, \end{cases}$$

the functions $\check{f}$ and $\check{g}$ in the same way as odd extensions.

We consider test-functions as follows: The horizontal dependence is given by $\varphi \in D_\#(I_1 \times I_2)$. Vertical dependences are given by $\psi \in C^\infty_c((-l_3, l_3))$. We define

$\check{\psi}(x_3) = \psi(x_3) - \psi(-x_3)$, which is a smooth odd function with, in particular, $\check{\psi}(0) = 0$. This allows to find a sequence $\check{\psi}_n \in C_c^\infty(I_3)$ such that $\check{\psi}_n \to \check{\psi}|_{I_3}$ in $H^1(I_3)$.

With the aim to show that $\check{u}$ solves an elliptic equation on $\check{\Omega}$, we compute, using the notation $\Delta_{1,2} = \partial_1^2 + \partial_2^2$,

$$\begin{aligned}
\int_{\check{\Omega}} \check{u}\Delta(\varphi\psi) &= \int_{\Omega} \check{u}\Delta(\varphi\psi) + \int_{\check{\Omega}\setminus\Omega} \check{u}\Delta(\varphi\psi) \\
&= \int_{\Omega} \check{u}\Delta(\varphi\check{\psi}) = \int_{\Omega} u\,(\Delta_{1,2}\varphi\,\check{\psi} + \varphi\,\partial_3^2\check{\psi}) \\
&\overset{(A.3)}{=} \int_{\Omega} u\,\Delta_{1,2}\varphi\,\check{\psi} - \int_{I_3} \partial_3\left(\int_{I_1\times I_2} u\,\varphi\right)\partial_{x_3}\check{\psi} \\
&= \lim_{n\to\infty} \int_{\Omega} u\,\Delta_{1,2}\varphi\,\check{\psi}_n - \int_{I_3} \partial_3\left(\int_{I_1\times I_2} u\varphi\right)\partial_{x_3}\check{\psi}_n \\
&= \lim_{n\to\infty} \int_{\Omega} u\,\Delta_{1,2}\varphi\,\check{\psi}_n + \int_{I_3} \left(\int_{I_1\times I_2} u\varphi\right)\partial_{x_3}^2\check{\psi}_n \\
&= \lim_{n\to\infty} \int_{\Omega} u\,\Delta(\varphi\check{\psi}_n) \\
&\overset{(A.2)}{=} \lim_{n\to\infty} \int_{\Omega} f\,\varphi\,\check{\psi}_n + g\cdot\nabla\varphi\,\check{\psi}_n \\
&= \int_{\Omega} f\,\varphi\,\check{\psi} + g\cdot\nabla\varphi\,\check{\psi} = \int_{\check{\Omega}} \check{f}\,\varphi\,\psi + \check{g}\cdot\nabla\varphi\,\psi\,.
\end{aligned}$$

Since test-functions of the form $\phi = \varphi(x_1,x_2)\psi(x_3)$ span a dense (dense with respect to the norm of $H^2(\check{\Omega})$) subset of all smooth functions, we obtain

$$-\int_{\check{\Omega}} \check{u}\,\Delta\phi = \int_{\check{\Omega}} \check{f}\,\phi + \check{g}\cdot\nabla\phi\,. \tag{A.5}$$

for all $\phi \in D_\#(\check{\Omega})$. As announced, Lemma A.1 allows us to conclude $u \in H^1_{\#,\mathrm{loc}}(\check{\Omega})$. Using an analogous reflection argument at the top boundary, we can deduce the H^1-regularity up to the boundary, i.e. $u \in H^1_\#(\Omega)$.

It remains to check the Dirichlet condition in the sense of traces, $u \in H^1_{0,\#}(\Omega)$. Because of $u \in H^1_\#(\Omega)$, we can consider the traces on, e.g., the boundary $x_3 = 0$. Relation (A.3) yields, for every function $\varphi \in C^\infty_\#(I_1 \times I_2)$, that $\int_{I_1\times I_2} u\varphi$ vanishes along $x_3 = 0$. This implies that the trace of u along this boundary vanishes, we obtain $u \in H^1_{0,\#}(\Omega)$. □

Lemma A.3 (Regularity for vanishing distributional Neumann boundary data)**.** *Let the domain be a product of three open intervals, $\Omega = I_1 \times I_2 \times I_3 \subset \mathbb{R}^3$. For* *right-hand sides* $f \in L^2(\Omega, \mathbb{C})$ *and* $g \in L^2(\Omega, \mathbb{C}^3)$*, let* $u \in L^2(\Omega, \mathbb{C})$ *satisfy*

$$\int_\Omega -u\Delta\phi = \int_\Omega f\phi + g \cdot \nabla\phi \qquad \forall \phi \in D_\#(\bar{\Omega}) \text{ with } \partial_3\phi \in D_\#(\Omega)\,. \tag{A.6}$$

We assume that vertical derivatives of averages vanish along the boundaries in the following sense: For every $\varphi \in C^\infty_\#(I_1 \times I_2)$ *there exists a function* $h = h_\varphi \in L^2(I_3)$ *such that*

$$\int_{I_3} \left\{ \left(\int_{I_1 \times I_2} u\,\varphi \right) \partial_3\psi + h\psi \right\} = 0 \qquad \forall \psi \in D(I_3)\,. \tag{A.7}$$

Then, the solution u *of* (A.2) *is of class* $u \in H^1_\#(\Omega)$ *and is a weak solution in the sense that*

$$\int_\Omega \nabla u \cdot \nabla\phi = \int_\Omega f\phi + g \cdot \nabla\phi \qquad \forall \phi \in H^1_\#(\Omega)\,. \tag{A.8}$$

Let us highlight the differences between Lemma A.2 and Lemma A.3. (i) In relation (A.6), the test-functions do not have to vanish at the horizontal boundaries; on the other hand, we demand that the vertical derivative vanishes in a neighborhood of the boundary. (ii) Relation (A.7) implies that $\left(\int_{I_1 \times I_2} u\,\varphi \right) \in H^1(I_3)$, but there is no information on the boundary values.

Proof. As in the last proof, it suffices to show that a suitable extension of u satisfies an equation on an extended domain; an application of the lemma of Weyl then yields the H^1-regularity up to the boundary. The proof is much like the proof of Lemma A.2. We restrict ourselves to a sketch of the differences and to the arguments regarding the lower boundary.

We assume again $I_3 = (0, l_3)$ and consider the extended domain $\check{\Omega} := I_1 \times I_2 \times (-l_3, l_3)$. In this proof, we extend u to an even function and set $\check{u}(\hat{x}, x_3) \in L^2(\Omega)$ by

$$\check{u}(\hat{x}, x_3) := \begin{cases} u(\hat{x}, x_3) & \text{if } x_3 > 0\,, \\ u(\hat{x}, -x_3) & \text{if } x_3 < 0\,, \end{cases}$$

and define the functions $\check{f}$ and $\check{g}$ in the same way as even extensions.

Test-functions are chosen with $\varphi \in D_\#(I_1 \times I_2)$ and $\psi \in C^\infty([-l_3, l_3])$. The symmetrized variant is defined as $\check{\psi}(x_3) = \psi(x_3) + \psi(-x_3)$. We note that $\check{\psi}$ is even and continuous across $x_3 = 0$, the value for $x_3 = 0$ is arbitrary, the derivative vanishes in $x_3 = 0$. Of course, $\partial_{x_3}\psi$ is not vanishing in a neighborhood of $x_3 = 0$. We choose an approximation as follows: $\check{\psi}_n \in C^\infty(\overline{I_3})$ with $\partial_{x_3}\check{\psi}_n \in C^\infty_c(I_3)$ such that $\check{\psi}_n \to \check{\psi}|_{I_3}$ in $H^1(I_3)$.

At this point, one performs the same lengthy computation as in the proof of Lemma A.2. The first three equalities are identical. The fourth equality remains true since $\partial_{x_3}\psi$ vanishes for $x_3 = 0$ (and a reference to (A.3) is not necessary). For the equality that was using (A.2), we now use (A.6). The result of the calculation is

$$-\int_{\check{\Omega}} \check{u}\,\Delta\phi = \int_{\check{\Omega}} \check{f}\,\phi + \check{g} \cdot \nabla\phi \tag{A.9}$$

for test-functions of the form $\phi = \varphi(x_1, x_2)\psi(x_3)$. An arbitrary function $\phi \in D_\#(\check{\Omega})$ can be approximated in $H^2(\check{\Omega})$ with linear combinations of such functions. We obtain that (A.9) holds for all $\phi \in D_\#(\check{\Omega})$. This concludes the argument for the lower boundary $x_3 = 0$, the upper boundary is treated analogously. Lemma A.1 of Weyl allows to conclude $u \in H^1_\#(\Omega)$.

The weak form (A.8) is obtained from (A.9) as follows: An arbitrary $\phi \in H^1_\#(\Omega)$ is approximated with $\phi_n \in C^\infty_\#(\overline{\Omega})$ with $\partial_{x_3}\phi_n = 0$ along horizontal boundaries (approximation in $H^1(\Omega)$ as $n \to \infty$ as in Lemma 4.4). Relation (A.9) is used with $\phi = \phi_n$. An integration by parts yields the equality of (A.8) for ϕ_n. In the limit $n \to \infty$, we obtain (A.8). □

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