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Promoting Understanding of the Role of Assumptions in Mathematics

In this keynote, I examined the educational potential of foregrounding the notion of *assumptions* in mathematics instruction—an often-overlooked but fundamental aspect of mathematical activity. I argued that students and teachers alike must develop an understanding of how assumptions underlie mathematical claims and how different legitimate assumptions may lead to valid but distinct conclusions. Drawing on two intervention studies—one in teacher education and another in school mathematics—I demonstrated how productive ambiguity and task design can be strategically used to make assumptions an explicit focus of learning. I highlighted the conceptual coherence between the two interventions and advocated for instructional continuity between school and teacher education levels as a means to support a unified and consistent approach to the instructional treatment of fundamental topics like assumptions. Such coherence is especially important in the preparation of prospective teachers, as it can foster a powerful synergy between their epistemological stance toward the nature of mathematical knowledge and their instructional orientation in the classroom.

Introduction

What does it mean to learn deeply in mathematics? This question has guided much of my research and resonates with broader inquiries into deep learning across disciplines. A central theme in the literature is that deep disciplinary learning involves more than acquiring facts or procedures—it requires an understanding of how knowledge is produced, justified, and communicated within a field (e.g., Bruner, 1960; Goldman, 2023; Schwab, 1978; Stylianides et al., 2022). In mathematics, such understanding is closely connected to practices like proving, problem posing, and problem solving, all of which rest upon the often-overlooked concept of *assumptions*.

By "assumptions" I refer to the statements that doers of mathematics use or accept—often without explicitly stating them or realizing they use them—as the foundation for their mathematical assertions or arguments (Fawcett, 1938). This broad definition should make it clear that, although often kept in the background in mathematical work at school, assumptions are ubiquitous in mathematical activity and at all levels—from elementary school to university-level mathematics. For example, when we answer 6+6=? with "12," we are assuming the use of the decimal (base 10) system. However, if the question were posed, say, in base 7, the answer would be "15." I am offering

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this example for illustration purposes only, as in school mathematics we do not normally need to specify the base, with base 10 being the only one known to students. That said, this specification may be important in certain mathematical contexts.

In school mathematics and other subjects, instruction typically prioritizes the acquisition of disciplinary content—the "what" of knowledge production—while paying comparatively little attention to the epistemic practices by which such content is generated—the "how" of knowledge production (Goldman, 2023). Although content acquisition is undeniably important, the processes through which knowledge is constructed, justified, and communicated are equally critical (e.g., Kitcher, 1984; Lampert, 1992). These epistemic practices are not only essential but also inherently challenging to teach and learn. As such, without deliberate and systematic instructional attention, they are unlikely to be developed effectively (Stylianides & Stylianides, 2013). The notion of assumptions is no exception: unless it becomes an explicit object of study in school mathematics and teacher education, students and teachers are unlikely to develop an understanding of the role of assumptions in mathematical activity.

My keynote focused on the educational potential of foregrounding assumptions in mathematics instruction. Although assumptions underpin virtually all mathematical activity, they receive scant attention in school curricula and teacher education. For example, assumptions were found to receive virtually no attention in textbooks used in mathematics courses for prospective elementary teachers (McCrory & Stylianides, 2014). This gap in practice is concerning, as research has shown that both students and teachers struggle with identifying and using assumptions in meaningful ways in areas such as proving and modelling (e.g., Blum & Borromeo Ferri, 2009; Stylianides et al., 2017; Suh et al., 2021). For example, making assumptions and defining variables were reported by elementary teachers to be the most challenging part of problem formulation when orchestrating mathematical modelling activities in their classrooms (Suh et al., 2021).

Given these challenges, there is a clear need for interventions to help both students and teachers develop a deeper understanding of the notion of assumptions. In response to this need, we developed two interventions to promote an understanding of the role of assumptions in mathematics—one at the teacher education level with prospective elementary teachers (Stylianides & Stylianides, 2024) and another at the school level with elementary and secondary students (Komatsu et al., 2024).

The two interventions

As the interventions have already been published, I provide here only a synopsis of each, and I invite interested readers to refer to the full papers for further information. Chronologically, the intervention in teacher education happened first and served as the foundation for the school-based intervention. For this reason, I begin with the teacher education study.

In Stylianides and Stylianides (2024), we reported a design-based research intervention aimed at helping prospective elementary teachers develop essential *mathematical knowledge for teaching* (MKT; Ball et al., 2008) related to the notion of assumptions. Developed over multiple research cycles of our design experiment in an undergraduate mathematics course for prospective elementary teachers in the United States, the intervention strategically leveraged *productive ambiguity* (Foster, 2011) through a deliberately ambiguous task in which the role of assumptions surfaced and was reflected upon in purposefully organized ways. The task and its deliberate implementation prompted varied solutions depending on participants' assumptions, allowing for the development of three key elements of MKT related to the notion of assumptions: recognizing that conclusions depend on assumptions, understanding that different assumptions can lead to valid but differing conclusions, and discerning how task ambiguity can be pedagogically productive. The intervention intertwined mathematical content with pedagogical reflection using specifically designed prompts, called *conceptual awareness pillars* (*CAP*s; Stylianides & Stylianides, 2009), to deepen participants' epistemological and instructional awareness. Overall, our approach to promoting MKT illustrated by the intervention offers a paradigmatic case of how teacher educators can use productive ambiguity and CAPs to design learning opportunities for prospective teachers to intertwine mathematical learning with pedagogical awareness, thus developing "pedagogically functional mathematical knowledge" (Ball & Bass, 2000, p. 95).

Building upon the foundation laid by Stylianides and Stylianides (2024), in Komatsu et al. (2024) we extended the focus from teacher education to school classrooms by aiming to identify task design principles for producing or modifying existing mathematics tasks to create learning opportunities for students to recognize that conclusions hinge on underlying assumptions and appreciate the importance of making assumptions explicit to reach shared understanding about a situation or consensus on a conclusion. Through a three-year design-based research project conducted in Japanese elementary and secondary schools, we first used existing literature to construct an initial version of task design principles, which we then empirically tested and refined by designing and implementing, with the help of collaborating teachers, two tasks in Japanese school classrooms. One of the tasks dealt with functional relationships (in secondary school), while the other addressed geometric interpretations of "sameness" (in elementary school). The study shows how these tasks, along with purposeful teacher facilitation, enabled students to grapple with ambiguity productively and reflect on how different assumptions shape mathematical conclusions. The task design principles that we derived for creating these learning opportunities for students are potentially scalable and could be used by teachers to adapt existing tasks, or design new ones, to achieve similar goals. In the absence of appropriate opportunities in textbooks or other curriculum materials for students to develop an understanding of the role of assumptions in mathematical activity, the role of teachers "as curriculum designers" (Pepin et al., 2019) gains significance and places demands on teacher education to prepare teachers to productively engage with task design principles such as ours.

Concluding Remarks

The two intervention studies I discussed above, and in more detail during the keynote, present another example in the area of assumptions, similar to the example in the area of proof discussed in Stylianides and Stylianides (2022), of how we, as a field, can design interventionist research programs in teacher education that align with the pedagogical approaches we would expect teachers themselves to adopt when promoting student learning in school settings. In particular, the school-based intervention (Komatsu et al., 2024) helped establish conceptual coherence with the earlier teacher education intervention (Stylianides & Stylianides, 2024), particularly with regard to the way it utilized the notion of productive ambiguity, thus demonstrating a vertically aligned approach to fostering epistemic awareness about assumptions across the two educational levels. When the learning experiences of students and prospective teachers around a particular topic are conceptually aligned, this can support a unified and consistent approach to the instructional treatment of that topic across educational levels. Such coherence is especially important in the preparation of prospective teachers, as it can foster a powerful synergy between their epistemological stance toward the nature of mathematical knowledge and their instructional orientation in the classroom.

Reflecting more generally on the state of research in mathematics education, this has historically placed greater emphasis on issues related to learning and curriculum than on the design and theorization of teaching practices that effectively support learning (Bishop, 1998; Stylianides & Stylianides, 2013, 2017). In particular, research concerning student learning has developed a robust empirical and theoretical foundation for characterizing students' cognitive and affective domains, and it has been especially effective in identifying and documenting learning difficulties, including misconceptions, common errors, counterproductive beliefs, and negative attitudes. However, comparatively less attention has been devoted to developing theory-based and empirically tested instructional approaches for addressing these persistent learning challenges (Stylianides & Stylianides, 2013, 2017; Stylianides et al., 2017, 2024). To advance the field, there is a pressing need to redress this imbalance—specifically, to increase the number of classroom-based studies that not only investigate but also actively intervene in problems of learning. Furthermore, as in the two intervention studies I discussed herein, efforts must be made to promote coherence between instructional messages delivered in school and teacher education to support continuity between student learning and teacher development.

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