An Algorithm for Constructing Hadamard Matrices

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Abstract

We define a new class of Hadamard matrices and present some examples obtained by an algorithm using a computer program.

Introduction

A Hadamard matrix $\mathbf{H}_n$ is an orthogonal $(n \times n)$-matrix in which all elements are either $+1$ or -1. If $n > 2$ then $n \equiv 0 \ mod\ 4$ is a necessary condition that a $\mathbf{H}_n$ exists. In this paper we define a class of Hadamard matrices and give a list of examples obtained by a computer program.

Hadamard matrices play an important in various field of statistics and mathematics. E.g. in statistics: incomplete balanced block designs, orthogonal arrays, factorial designs, Taguchi methods and Hotellings weighing problem; e.g. in mathematics: geometry, combinatorics, approximation theory and coding theory. There exists a wide-spread literature on Hadamard matrices, especially on the existence problem. A survey may be found in various text books of combinatorial theory.

1 Hadamard Schemes

To $m \in \mathbb{N}$ ($\mathbb{N}$ includes 0) we write

$$\begin{aligned}\mathbb{N}_m &= \{0, 1, \ldots, m-1, m\},\\ \mathbb{Z}_m &= \{-m, -(m-1), \ldots, -1, 0, 1, \ldots, m-1, m\}.\end{aligned}$$

To $u \in \mathbb{Z}$ let $\rho(u)$ denote the additive residual class of u modulo $\mathbb{Z}_m$. (So, to $u \in \mathbb{Z}$ there exists exactly one $z \in \mathbb{Z}$ with $\rho(u) = u + z(2m+1) \in \mathbb{Z}_m$.) Let

$$M := \{1, i, -1, -i\}.$$

Here i is the imaginary unit. M is a multiplicative cyclic group of order 4 generated by i. For the elements of M we write $'+'$ instead of 1, $'-'$ instead of -1 and $'j'$ instead of $-i$. We use the following notation: Small boldface letters denote vectors, vectors in $M^{\mathbb{Z}_m}$ carry an asterix; matrices are denoted by boldface capital letters.

If $\hat{\mathbf{A}}$ is an $(n \times m)$-matrix whose elements $A_{i,j}$ are $(p \times q)$-matrices, then we call $\hat{\mathbf{A}}$ a *block-matrix.* Block matrices are marked by carrying a hat accent. k-fold iterated block matrices are also marked with a hat accent, (but with one hat only). If $\hat{\mathbf{A}}$ is an (iterated) block-matrix, then the matrix obtained from $\hat{\mathbf{A}}$ by stripping off all the block boundaries is called the *deblocked* matrix of $\hat{\mathbf{A}}$. The deblocked matrix of $\hat{\mathbf{A}}$ is denoted by deleting the hat accent. E.g. if $\hat{\mathbf{A}} = [A_{i,j}]_{i=1\ldots n, j=1\ldots m}$ is an $(n \times m)$-block-matrix whose elements $\mathbf{A}_{i,j} = [a_{i,j;u,v}]_{u=1,\ldots p, v=1,\ldots q}, i = 1, \ldots n, j = 1, \ldots m$ are $(p \times q)$- matrices, then

$$\mathbf{A} = [a_{(i-1)p+u,(j-1)q+v}]_{\substack{i=1,\ldots,n,\, u=1,\ldots,p,\\ j=1,\ldots,m,\, v=1,\ldots,q}}$$

is the deblocked matrix of $\hat{\mathbf{A}}$.

If $z = z_1 + iz_2 \in \mathbb{C}$ is a complex number then $\overline{z} = z_1 - iz_2$ is its conjugate.

The mapping $\sigma : M \times M \longrightarrow \{-1, 0, 1\}$ is defined by

$$\sigma(u, v) := \begin{cases} u \cdot v & if \;\; u = v,\\ -u \cdot v & if \;\; u = -v,\\ 0 & if \;\; u \neq \pm v.\end{cases} \tag{1}$$

To $k \in \mathbb{Z}$, $\mathbf{a}^*, \mathbf{c}^* \in M^{\mathbb{Z}_m}$ we define

$$\chi_k(\mathbf{a}^*, \mathbf{c}^*) := \sum_{u \in \mathbb{Z}_m} \sigma(a_u, c_{\rho(u+k)}). \tag{2}$$

Then

$$\begin{aligned} \chi_0(\mathbf{a}^*, \mathbf{a}^*) &= \sum_{u \in \mathbb{N}_m} \sigma(a_u, a_u) = 2m + 1, & (3)\\ \chi_k(\mathbf{a}^*, \mathbf{c}^*) &= \chi_{\rho(k)}(\mathbf{a}^*, \mathbf{c}^*), & (4)\\ \chi_k(\mathbf{a}^*, \mathbf{c}^*) &= \chi_{-k}(\mathbf{c}^*, \mathbf{a}^*). & (5) \end{aligned}$$

So $\chi_k(\mathbf{a}^*, \mathbf{c}^*)$, $k \in \mathbb{Z}$ is completely determined by the indices $k = 1, \ldots, m$. To $\mathbf{a}^*, \mathbf{c}^*$ we define the *χ-vector* of $\mathbf{a}^*, \mathbf{c}^*$:

$$\chi(\mathbf{a}^*, \mathbf{c}^*) := [\chi_k(\mathbf{a}^*, \mathbf{c}^*)]_{k \in \mathbb{N}_m}.$$

Definition 1.

a) $\mathbf{a}^* \in M^{\mathbb{Z}_m}$ is *symmetric*, if $a_u = a_{-u}$ for every $u \in \mathbb{N}_m$.

b) If $\mathbf{a} \in M^{\mathbb{N}_m}$ then

$$\epsilon(\mathbf{a}) := [a_m, a_{m-1}, \ldots, a_{-1}, a_0, a_1, \ldots, a_{m-1}, a_m]$$

is the *symmetric extension of* $\mathbf{a}$.

c) If $\mathbf{a}, \mathbf{b} \in M^{\mathbb{N}_m}$ then

$$\chi(\mathbf{a}, \mathbf{b}) := \chi(\epsilon(\mathbf{a}), \epsilon(\mathbf{b})). \qquad \blacksquare$$

Definition 2. Suppose $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \in M^{\mathbb{N}_m}$.

a) $[\mathbf{a}, \mathbf{b}]$ is a *Hadamard pair (of size m)*, if

$$\chi(\mathbf{a}, \mathbf{a}) + \chi(\mathbf{b}, \mathbf{b}) = [1, 0, \ldots, 0]. \tag{6}$$

b) Suppose that $[\mathbf{a}, \mathbf{b}], [\mathbf{c}, \mathbf{d}]$ are Hadamard pairs.

$$\begin{bmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{bmatrix} \tag{7}$$

is a *Hadamard scheme (of size m)* , if

$$\chi(\mathbf{a}, \mathbf{c}) + \chi(\mathbf{b}, \mathbf{d}) = \mathbf{0}. \qquad \blacksquare$$

$\mathbf{HP}(m)$ symbolizes a Hadamard pair, $\mathbf{HS}(m)$ a Hadamard scheme (of size m) and $\mathbf{HM}(n)$ an $(n \times n)$-Hadamard matrix.

Definition 3. If $k \in \mathbb{Z}, \mathbf{x}^*, \mathbf{y}^* \in M^{\mathbb{Z}_m}, d \in M$ then

$$I(k, \mathbf{x}^*, \mathbf{y}^*; d) := \{u \in \mathbb{Z}_m : \sigma(x_u, y_{\rho(u+k)}) = d\} \ .$$

If $k, \mathbf{x}^*, \mathbf{y}^*$ are fixed then $I(d) := I(k, \mathbf{x}^*, \mathbf{y}^*; d)$.

To $\mathbf{x}, \mathbf{y} \in M^{\mathbb{N}_m}, d \in M$ define $I(k, \mathbf{x}, \mathbf{y}; d) := I(k, \epsilon(\mathbf{x}), \epsilon(\mathbf{y}); d)$. $\blacksquare$

So for every fixed $k, \mathbf{x}^*, \mathbf{y}^*$

$$\mathbb{Z}_m = I(+) + I(i) + I(-) + I(j) \tag{8}$$

is a partition of $\mathbb{Z}_m$.

Lemma 1. Using the notation of definition 2 we obtain

a) $[\mathbf{a}, \mathbf{b}]$ is a $\mathbf{HP}(m)$ if and only if

$$\begin{array}{lllllll} \#I(k,\mathbf{a},\mathbf{a};+) & + & \#I(k,\mathbf{b},\mathbf{b};+) & = & \#I(k,\mathbf{a},\mathbf{a};-) & + & \#I(k,\mathbf{b},\mathbf{b};-), \\ \#I(k,\mathbf{a},\mathbf{a};i) & + & \#I(k,\mathbf{b},\mathbf{b};i) & = & \#I(k,\mathbf{a},\mathbf{a};j) & + & \#I(k,\mathbf{b},\mathbf{b};j) \end{array} \tag{9}$$

for $k = 1, \ldots, m$.

b) $\begin{bmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{bmatrix}$ is a $\mathbf{HS}(m)$ if and only if $[\mathbf{a}, \mathbf{b}]$ and $[\mathbf{c}, \mathbf{d}]$ are $\mathbf{HP}(m)$'s and

$$\begin{array}{lllllll} \#I(k,\mathbf{a},\mathbf{c};+) & + & \#I(k,\mathbf{b},\mathbf{d};+) & = & \#I(k,\mathbf{a},\mathbf{c};-) & + & \#I(k,\mathbf{b},\mathbf{d};-), \\ \#I(k,\mathbf{a},\mathbf{c};i) & + & \#I(k,\mathbf{b},\mathbf{d};i) & = & \#I(k,\mathbf{a},\mathbf{c};j) & + & \#I(k,\mathbf{b},\mathbf{d};j). \end{array} \tag{10}$$

for $k = 0, 1, \ldots, m$.

Proof.

$$\chi_k(\mathbf{a},\mathbf{a}) + \chi_k(\mathbf{b},\mathbf{b}) = 0$$

$$\Longleftrightarrow$$

$$\sum_{u\in\mathbb{Z}_m} \sigma(a_u, a_{\rho(u+k)}) + \sigma(b_u, b_{\rho(u+k)}) = 0$$

$$\Longleftrightarrow$$

$$\sum_{d\in M}\Big(\sum_{u\in I(k,\mathbf{a},\mathbf{a};d)} \sigma(a_u, a_{\rho(u+k)}) + \sum_{u\in I(k,\mathbf{b},\mathbf{b};d)} \sigma(b_u, b_{\rho(u+k)}) = 0$$

$$\Longleftrightarrow$$

$$\begin{array}{rlllllll}
 & (+) & \cdot & \#I(k,\mathbf{a},\mathbf{a};+) & + & (-) & \cdot & \#I(k,\mathbf{a},\mathbf{a};-) \\
+ & (i) & \cdot & \#I(k,\mathbf{a},\mathbf{a};i) & + & (j) & \cdot & \#I(k,\mathbf{a},\mathbf{a};j) \\
+ & (+) & \cdot & \#I(k,\mathbf{b},\mathbf{b};+) & + & (-) & \cdot & \#I(k,\mathbf{b},\mathbf{b};-) \\
+ & (i) & \cdot & \#I(k,\mathbf{b},\mathbf{b};i) & + & (j) & \cdot & \#I(k,\mathbf{b},\mathbf{b};j) \quad = 0 \\
\Longleftrightarrow & & & & & & & \\
 & (+) & \cdot & \#I(k,\mathbf{a},\mathbf{a};+) & + & (-) & \cdot & \#I(k,\mathbf{a},\mathbf{a};-) \\
+ & (+) & \cdot & \#I(k,\mathbf{b},\mathbf{b};+) & + & (-) & \cdot & \#I(k,\mathbf{b},\mathbf{b};-) \quad = 0 \\
\textit{and} & & & & & & & \\
+ & (i) & \cdot & \#I(k,\mathbf{a},\mathbf{a};i) & + & (j) & \cdot & \#I(k,\mathbf{a},\mathbf{a};j) \\
+ & (i) & \cdot & \#I(k,\mathbf{b},\mathbf{b};i) & + & (j) & \cdot & \#I(k,\mathbf{b},\mathbf{b};j) \quad = 0.
\end{array}$$

b) Part b follows by a completely analogous calculation. ∎

The following lemma gives a representation of $\chi(\mathbf{a})$, using indices in $\mathbb{N}_m$ only.

Lemma 2. Suppose $\mathbf{a} \in M^{\mathbb{N}_m}$ and $k \in 1, \ldots, m$. Then

$$\chi_k(\mathbf{a}) = 1 \quad + \quad 2 \cdot \sum_{u=1}^{[\frac{k+1}{2}]-1} \sigma(a_{-(u-[\frac{k+1}{2}])}, a_{u+[\frac{k}{2}]})$$

$$
\begin{aligned}
&+ \quad 2 \cdot \sum_{u=[\frac{k+1}{2}]}^{m-[\frac{k}{2}]} \quad \sigma(a_{u-[\frac{k+1}{2}]}, a_{u+[\frac{k}{2}]}) \\
&+ \quad 2 \cdot \sum_{u=m-[\frac{k}{2}]+1}^{m} \quad \sigma(a_{u-[\frac{k+1}{2}]}, a_{-(u+[\frac{k}{2}])+(2m+1)}).
\end{aligned}
$$

Proof. The mapping $k \to \rho(2k)$ is bijectiv on $\mathbb{Z}_m$. We denote its inverse mapping by $\cdot//2$, i.e. $k \to k//2 \;\; for \;\; every \;\; k \in \mathbb{Z}_m$. Then

$$
k//2 = \begin{cases} k/2 & : k \in \mathbb{N}_m, \;\; k \;\; even, \\ (k-1)/2 - m & : k \in \mathbb{N}_m, \;\; k \;\; odd. \end{cases} \tag{11}
$$

It follows

$$
\begin{aligned}
\chi_k(\mathbf{a}) &= \chi_k(\epsilon(\mathbf{a})) \\
&= \sum_{u=-m}^{m} \sigma(a_u, a_{\rho(u+k)}) \\
&= \sum_{u=-m}^{m} \sigma(a_u, a_{\rho(u+k//2+k//2)}) \\
&= \sum_{u=-m}^{m} \sigma(a_{\rho(u-k//2)}, a_{\rho(u+k//2)}) \\
&= \sigma(a_{-k//2}, a_{+k//2}) + 2 \cdot \sum_{u=1}^{m} \sigma(a_{\rho(u-|k//2|)}, a_{\rho(u+|k//2|)}) \\
&= 1 + 2 \cdot \sum_{u=1}^{m} \sigma(a_{|\rho(u-|k//2|)|}, a_{|\rho(u+|k//2|)|}). \qquad (12)
\end{aligned}
$$

If k is even, then $\rho(k//2) > 0$, if k odd, then $\rho(k//2) < 0$. Using this and (11) we determine $|\rho(u - |k//2|)|$ and $|\rho(u + |k//2|)|$.

$$
|\rho(u - |k//2|)| = \begin{cases} -(u - k/2) & k \;\; even, \quad u \leq k/2 - 1, \\ \quad u - k/2 & k \;\; even, \quad u \geq k/2, \\ -(u - (m - (k-1)/2)) & k \;\; odd, \quad u \leq m - (k-1)/2, \\ \quad u - (m - (k-1)/2) & k \;\; odd, \quad u \geq m - (k+1)/2; \end{cases}
$$

$$|\rho(u+|k//2|)| = \begin{cases} u+k/2 & k \ \textit{even}, \quad u \le m-k/2, \\ -(u+k/2)+(2m+1) & k \ \textit{even}, \quad u \ge m-k/2+1, \\ u+m-(k-1)/2 & k \ \textit{odd}, \quad u \le (k-1)/2, \\ -u+m+(k+1)/2) & k \ \textit{odd}, \quad u \ge (k+1)/2. \end{cases}$$

Inserting these values (7) and using Gauss-brackets leads to the equation of the lemma. ■

To $\mathbf{a} \in M^{\mathbb{N}_m}$, $k \in \{1, \ldots, m\}$ define

$$\psi_k(\mathbf{a}) := \frac{1}{2}(\chi_k(a) - 1). \tag{13}$$

and $\psi(\mathbf{a}) := [\psi_{\mathbf{k}}(\mathbf{a})]_{\mathbf{k=1,\ldots,m}}$. Then we obtain the following version of Lemma 1:

Lemma 3. Suppose $\mathbf{a} \in M^{\mathbb{N}_m}$ and $k \in 1, \ldots, m$. Then

$$\psi_k(\mathbf{a}) = \sum_{\substack{u+v=k \\ 1 \le u < v}} \sigma(a_u, a_v) + \sum_{u=0}^{m-k} \sigma(a_u, a_{u+k}) + \sum_{\substack{u+v=2m+1-k \\ u<v\le m}} \sigma(a_u, a_v). \quad ■$$

From (6) follows

Lemma 4. $[\mathbf{a}, \mathbf{b}]$ is a $\mathbf{HP}(m)$ if and only if

$$\psi(\mathbf{a}) + \psi(\mathbf{b}) := -\mathbf{1} \tag{14}$$

(where $-\mathbf{1} = -[1, \ldots, 1]$. ■

To obtain a Hadamard scheme one needs two Hadamard pairs $[\mathbf{a}, \mathbf{b}]$, $[\mathbf{c}, \mathbf{d}]$ satisfying $\chi(\mathbf{a}, \mathbf{c}) + \chi(\mathbf{b}, \mathbf{d}) = \mathbf{0}$. The following lemma is a crucial support for the constructíon of Hadamard schemes, for it delivers to a given Hadamard pair a second one so that they together form a Hadamard scheme.

Lemma 5. If $[\mathbf{a}, \mathbf{b}]$ is a $\mathbf{HP}(m)$ then

$$\begin{bmatrix} \mathbf{a}, & \mathbf{b} \\ \overline{\mathbf{b}}, & -\overline{\mathbf{a}} \end{bmatrix}$$

is a $\mathbf{HS}$(m).

Proof. a) We show first, that $[\overline{\mathbf{b}}, -\overline{\mathbf{a}}]$ is a Hadamard pair:

$$\begin{aligned}
&\chi_k(\overline{\mathbf{b}}, \overline{\mathbf{b}}) + \chi_k((-\overline{\mathbf{a}}), (-\overline{\mathbf{a}})) \\
&= \sum_{u \in \mathbb{Z}_m} (\sigma(\overline{b}_u, \overline{b}_{\rho(u+k)}) + \sigma(-\overline{a}_u, -\overline{a}_{\rho(u+k)})) \\
&= \overline{\sum_{u \in \mathbb{Z}_m} (\sigma(b_u, b_{\rho(u+k)} + a_u, a_{\rho(u+k)}))} \\
&= \overline{\chi_k(\sigma(\mathbf{b}, \mathbf{b})) + \chi_k(\sigma(\mathbf{a}, \mathbf{a}))} \\
&= \overline{0} = 0.
\end{aligned}$$

b) Now we verify the condition b of definition 3.

$$\begin{aligned}
&\chi_k(\mathbf{a}, \overline{\mathbf{b}}) + \chi_k(\mathbf{b}, -\overline{\mathbf{a}}) \\
&= \sum_{u \in \mathbb{Z}_m} (\sigma(a_u, \overline{b}_{\rho(u+k)}) + \sigma(b_u, -\overline{a}_{\rho(u+k)})) \\
&= \sum_{u \in \mathbb{Z}_m} (\sigma(a_u, b_{\rho(u+k)}) - \sigma(b_u, a_{\rho(u+k)})) \\
&= \sum_{u \in \mathbb{Z}_m} \sigma(a_u, b_{\rho(u+k)}) - \sum_{u \in \mathbb{Z}_m} \sigma(b_{\rho(-u+k)}, a_{-u}) \\
&= \sum_{u \in \mathbb{Z}_m} \sigma(a_u, b_{\rho(u+k)}) - \sum_{v \in \mathbb{Z}_m} \sigma(b_{\rho(v+k)}, a_v) \\
&= 0.
\end{aligned}$$

■

To obtain a $\mathbf{HS}$(m) it is therefore sufficient to construct a $\mathbf{HP}$(m).

It is sometimes desirable to confine the investigation of $\mathbf{HP}$(m) to special normalized cases. To define them requires the following preparations: On M introduce the linear order relation $+ > i > - > j$. Then $>$ introduces canonically the lexicographical order relation $>$ on $M^{\mathbb{N}_m}$: $\mathbf{a} = [a_0, \ldots, a_m] > \mathbf{b} = [b_0, \ldots, b_m]$ if there exists $r \in \mathbb{N}_m$ with $a_r > b_r$ and $a_s = b_s$, $s = 0, \ldots, r-1$.

$x \in \mathbb{N}_m$ is *i-leading*, if x has the following property: If $D := \{u \in \mathbb{N}_m : x_u \in \{i, j\}\} \neq \emptyset$, and *min* is the now well defined mimimal index with

$x_{min} \in \{i, j\}$, then $x_{min} = i$.

Definition 4. $[\mathbf{a}, \mathbf{b}],\ \ \mathbf{a}, \mathbf{b} \in M^{\mathbb{Z}_m}$ is a *normalized pair* if:

a) $\mathbf{a} \geq \mathbf{b}$,

b) $a_0 = b_0 = +$,

c) $\mathbf{a}$ and $\mathbf{b}$ are i-leading. ■

The following lemma is trivial:

Lemma 6. To every pair $[\tilde{\mathbf{a}}, \tilde{\mathbf{b}}],\ \ \tilde{\mathbf{a}} \leq \tilde{\mathbf{b}} \in \mathbb{Z}_m$ there exists a normalized pair $[\mathbf{a}, \mathbf{b}],\ \ \mathbf{a}, \mathbf{b} \in \mathbb{Z}_m$ with $\chi(\mathbf{a}, \mathbf{a}) = \chi(\tilde{\mathbf{a}}, \tilde{\mathbf{a}})$. ■

2 The Construction of Hadamard Matrices

Define

$$\mathbf{H}_+ = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad \mathbf{H}_i = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}, \quad \mathbf{H}_- = -\mathbf{H}_+, \quad \mathbf{H}_j = -\mathbf{H}_i.$$

The matrices $\mathbf{H}_u, u \in M$ are symmetric $\mathbf{HM}(2)$. To

$$\mathbf{K} = [k_{p,q}]_{\substack{p=0,\ldots,a-1, \\ q=0,\ldots,b-1}}, \quad k_{p,q} \in M$$

define

$$\hat{\mathbf{H}}_{\mathbf{K}} := [\mathbf{H}_{k_{p,q}}]_{\substack{p=0,\ldots,a-1, \\ q=0,\ldots,b-1}}.$$

So the blocked matrix $\hat{\mathbf{H}}_{\mathbf{K}}$ is obtained by replacing each element $k_{p,q} \in M$ of $\mathbf{K}$ by the corresponding (2×2)- matrix $\mathbf{H}_{k_{p,q}}$.

Definition 5. If $\mathbf{a}^* \in M^{\mathbb{Z}_m}$, then the $((2m+1) \times (2m+1))$-matrix

$$\mathbf{Z}(\mathbf{a}^*) := [a_{\rho(j-i)}]_{i,j \in M^{\mathbb{Z}_m}}$$

is the *circulant matrix* of $\mathbf{a}^*$. If $\mathbf{a} \in M^{\mathbb{N}_m}$ then $\mathbf{Z}(\mathbf{a}) := \mathbf{Z}(\epsilon(\mathbf{a}))$. ■

Lemma 7. Suppose $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \in M^{\mathbb{N}_m}$. If

$$\left[\begin{array}{cc} \mathbf{a}, & \mathbf{b} \\ \mathbf{c}, & \mathbf{d} \end{array}\right]$$

is a $\mathbf{HS}$(m), then define the two-fold blocked (2×2)-matrix

$$\hat{\mathbf{H}} = \left[\begin{array}{cc} \hat{\mathbf{H}}_{\mathbf{Z}(\mathbf{a})} & \hat{\mathbf{H}}_{\mathbf{Z}(\mathbf{b})} \\ \hat{\mathbf{H}}_{\mathbf{Z}(\mathbf{c})} & \hat{\mathbf{H}}_{\mathbf{Z}(\mathbf{d})} \end{array}\right].$$

Then the $(4(2m+1) \times 4(2m+1))$ deblocked matrix $\mathbf{H}$ of $\hat{\mathbf{H}}$ is a $\mathbf{HM}(8m+4)$.

Proof. In a matrix $\mathbf{R} = [r_{i,j}]_{i=0,\ldots,s-1,\, j=0,\ldots,t-1}$ the submatrix consisting of the rows $i_1, \ldots, i_w$, $0 \le i_1 < \ldots < i_w t \le s-1$ is denoted by $\mathbf{R}(i_1, \ldots, i_w)$. If $0 \le g \le 4m$ is even, then for $\mathbf{x} \in \{\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}\}$ we obtain

$$\hat{\mathbf{H}}_{\mathbf{Z}(\mathbf{x})}(g, g+1) = [\mathbf{H}_{x_{\rho(-m-g)}}, \ldots, \mathbf{H}_{x_{\rho(0-g)}}, \ldots, \mathbf{H}_{x_{\rho(m-g)}}].$$

To $0 \le p < 8m+4$ define $g(p) = 2 \cdot [p/2]$. Then $\mathbf{H}(p)$ is a row of the deblocked $(2 \times (8m+4))$-submatrix $\mathbf{H}(g(p), g(p)+1)$ of

$$\begin{aligned} \hat{\mathbf{H}}(g(p), g(p)+1) \quad = \\ & [\mathbf{H}_{x_{\rho(-m-g(p))}}, \ldots, \mathbf{H}_{x_{\rho(0-g(p))}}, \ldots, \mathbf{H}_{x_{\rho(m-g(p))}}, \\ & \ \mathbf{H}_{y_{\rho(-m-g(p))}}, \ldots, \mathbf{H}_{y_{\rho(0-g(p))}}, \ldots, \mathbf{H}_{y_{\rho(m-g(p))}}] \end{aligned}$$

where $[\mathbf{x}, \mathbf{y}] = [\mathbf{a}, \mathbf{b}]$, if $0 \le p < 4m+2$ and $[\mathbf{x}, \mathbf{y}] = [\mathbf{c}, \mathbf{d}]$ if $4m+2 \le p < 8m+4$.

Now suppose $0 \le p < q < 8m+4$. We show that the rows number p and q of $\mathbf{H}$ are orthogonal, thereby proving the theorem. If $g(p) = g(q)$, then p is even, $q = p+1$ and so $\mathbf{H}(g(p), g(p)+1)$ consists of these two rows. From

$$\mathbf{H}(g(p), g(p)+1) \cdot \mathbf{H}(g(p), g(p)+1)' = (8m+4) \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right]$$

we trivially obtain their orthogonality.

Now if $g(p) \neq g(q)$, then the two $(2 \times (4m+2))$-submatrices $\mathbf{H}(g(p), g(p)+1)$ and $\mathbf{H}(g(q), g(q)+1)$ of $\mathbf{H}$ have no common row. Define

$$\mathbf{A}(p,q) = \mathbf{H}(g(p), g(p)+1) \cdot \mathbf{H}(g(q), g(q)+1)'$$

We show

$$\mathbf{A}(p,q) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

thereby proving the lemma.

Define

$$[\mathbf{w}, \mathbf{x}] = \begin{cases} [\mathbf{a}, \mathbf{b}] & : 0 \leq p < 4m+2, \\ [\mathbf{c}, \mathbf{d}] & : 4m+2 \leq p < 8m+4, \end{cases}$$

and likewise

$$[\mathbf{y}, \mathbf{z}] = \begin{cases} [\mathbf{a}, \mathbf{b}] & : 0 \leq q < 4m+2, \\ [\mathbf{c}, \mathbf{d}] & : 4m+2 \leq q < 8m+4. \end{cases}$$

We use the partition (8) to obtain canonically a partition of $\mathbf{A}(p,q)$ as follows:

$$\begin{aligned} \mathbf{A}(p,q) &= \sum_{d \in M} \sum_{v \in I(k,\mathbf{w},\mathbf{x};d)} \mathbf{H}_{\mathbf{w}_v} \times \mathbf{H}'_{\mathbf{x}_{\rho(v+k)}} \\ &+ \sum_{d \in M} \sum_{v \in I(k,\mathbf{y},\mathbf{z};d)} \mathbf{H}_{\mathbf{y}_v} \times \mathbf{H}'_{\mathbf{z}_{\rho(v+k)}} \\ &= \sum_{v \in I(k,\mathbf{w},\mathbf{x};+)} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} + \sum_{v \in I(k,\mathbf{y},\mathbf{z};+)} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \\ &+ \sum_{v \in I(k,\mathbf{w},\mathbf{x};-)} \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} + \sum_{v \in I(k,\mathbf{y},\mathbf{z};-)} \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} \\ &+ \sum_{v \in I(k,\mathbf{w},\mathbf{x};i)} \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} + \sum_{v \in I(k,\mathbf{y},\mathbf{z};i)} \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} \end{aligned}$$

$$
\begin{aligned}
&+ \sum_{v\in I(k,\mathbf{w},\mathbf{x};j)} \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix} + \sum_{v\in I(k,\mathbf{y},\mathbf{z};j)} \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix} \\
&= (\#I(k,\mathbf{w},\mathbf{x};+) + \#I(k,\mathbf{y},\mathbf{z};+)) \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \\
&+ (\#I(k,\mathbf{w},\mathbf{x};-) + \#I(k,\mathbf{y},\mathbf{z};-)) \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} \\
&+ (\#I(k,\mathbf{w},\mathbf{x};i) + \#I(k,\mathbf{y},\mathbf{z};i)) \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} \\
&+ (\#I(k,\mathbf{w},\mathbf{x};j) + \#I(k,\mathbf{y},\mathbf{z};j)) \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.
\end{aligned}
$$

where zeros in the last equation follow from lemma 1. ■

3 Examples

3.1 An Algorithmic Construction of Hadamard Matrices

If a $\mathbf{HP}(m) = [\mathbf{a}, \mathbf{b}]$ exists, a $\mathbf{HM}(8m+4)$ can be obtained by the following steps:

Step 1. Build the $\mathbf{HS}$(m)

$$
\mathbf{HS}(m) = \begin{bmatrix} \mathbf{a}, & \mathbf{b} \\ \overline{\mathbf{b}}, & -\overline{\mathbf{a}} \end{bmatrix}
$$

Step 2. Build the cyclic extension

$$
\mathbf{HS}^*(m) = \begin{bmatrix} \epsilon(\mathbf{a}), & \epsilon(\mathbf{b}) \\ \epsilon(\overline{\mathbf{b}}), & \epsilon(-\overline{\mathbf{a}}) \end{bmatrix}
$$

Step 3. Build the cyclic $((2m+1) \times (2m+1))$-matrices $\mathbf{Z}(\mathbf{a}), \mathbf{Z}(\mathbf{b}), \mathbf{Z}(\overline{\mathbf{b}}), \mathbf{Z}(-\overline{\mathbf{a}})$.

Step 4. Using the four cyclic $(2m+1)\times(2m+1)$-block matrices $\hat{\mathbf{H}}_{\mathbf{Z(a)}}, \hat{\mathbf{H}}_{\mathbf{Z(b)}}, \hat{\mathbf{H}}_{\mathbf{Z(\overline{b})}}, \hat{\mathbf{H}}_{\mathbf{Z(-\overline{a})}}$, having as elements (blocks) (2×2)-matrices $\mathbf{H}_u,\ u\in M$ build the twofold blocked matrix

$$\hat{\mathbf{H}} = \begin{bmatrix} \hat{\mathbf{H}}_{\mathbf{Z(a)}}, & \hat{\mathbf{H}}_{\mathbf{Z(b)}} \\ \hat{\mathbf{H}}_{\mathbf{Z(\overline{b})}}, & \hat{\mathbf{H}}_{\mathbf{Z(-\overline{a})}}. \end{bmatrix}$$

Step 5. Build the $((8m+4)\times(8m+4))$-matrix deblocked $\{+1,-1\}$ matrix $\mathbf{H}$ of $\hat{\mathbf{H}}$. Then $\mathbf{H}$ is a $\mathbf{HM}(8m+4)$. ■

3.2 Two Examples

We apply these steps to the sizes $m=0$ and $m=1$.

$$m=0.$$

The only normalized pair is $[[+],[+]]$, obtained from $\mathbf{a}=[a_0]=[+]$ and $\mathbf{b}=[b_0]=[+]$ is a $\mathbf{HP}(0)$.

Step 1. Obviously,

$$\begin{bmatrix} + & + \\ + & - \end{bmatrix},$$

is a **HS(0)**, (which after deblocking accidentally is itself a $\mathbf{HM}(2)$.)

Step 2. Here $\epsilon(\mathbf{a})=\mathbf{a}, \epsilon(\mathbf{b})=\mathbf{b}$. So $\mathbf{HS}^*(0)=\mathbf{HS}(0)$.

Step 3. $\mathbf{Z}([\mathbf{a}])=\mathbf{Z}([\mathbf{b}])=\mathbf{Z}([\overline{\mathbf{b}}])=\mathbf{Z}([+]=[+], \mathbf{Z}([-\overline{\mathbf{a}}])=\mathbf{Z}([-])=[-]$

Step 4.

$$\hat{\mathbf{H}} = \begin{bmatrix} \left[\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\right] & \left[\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\right] \\ & \\ \left[\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\right] & \left[\begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix}\right] \end{bmatrix}$$

Step 5. Writing "+" instead of 1 and "-" instead of -1 we obtain the matrix

$$\mathbf{H} = \begin{bmatrix} + & + & + & + \\ + & - & + & - \\ + & + & - & - \\ + & - & - & +. \end{bmatrix}$$

This is a $\mathbf{HM}(8 \cdot 0 + 4)$. ■

$$m = 1.$$

Define $\mathbf{a} = [+, -]$, $\mathbf{b} = [+, i]$. Then according to section 2 $[+, -], [+, i]$ is a normalized $\mathbf{HS}(1)$.

Following again steps 1 to 6 gives us a $\mathbf{HM}(12)$.

(Indeed, from the symmetric extensions $\epsilon(\mathbf{a}) = [-, +, -], \epsilon(\mathbf{b}) = [i, +, i]$ of $\mathbf{a}$ and $\mathbf{b}$ we obtain $\chi_1(\mathbf{a}) = \chi_1(\epsilon(\mathbf{a})) = (-1) + 1 + (-1) = -1$, $\chi_1(\mathbf{b}) = \chi_1(\epsilon(\mathbf{b})) = 0 + 1 + 0 = 1$. This implies $\psi_1(a) + \psi_1(b) = \frac{1}{2}((\chi_k(a) - 1) + (\chi_k(a) - 1)) = -1$. So, as it be checked also easily directly $[[1, -], [1, i]]$ is a $\mathbf{HP}(1)$.)

Step 1. The correponding $\mathbf{HS}(1)$ is

$$\mathbf{HS}(1) = \begin{bmatrix} [& + & - &] & [& + & i &] \\ [& + & j &] & [& - & + &] \end{bmatrix}.$$

Step 2.

$$\mathbf{HS}^*(1) = \begin{bmatrix} [& - & + & - &] & [& i & + & i &] \\ [& j & + & j &] & [& + & - & + &] \end{bmatrix}.$$

Step 3.

$$\mathbf{Z}(\mathbf{a}) = \begin{bmatrix} - & + & - \\ - & - & + \\ + & - & - \end{bmatrix}, \quad \mathbf{Z}(\mathbf{b}) = \begin{bmatrix} i & + & i \\ i & i & + \\ + & i & i \end{bmatrix},$$

$$\mathbf{Z}(\overline{\mathbf{b}}) = \begin{bmatrix} j & + & j \\ j & j & + \\ + & j & j \end{bmatrix}, \quad \mathbf{Z}(-\overline{\mathbf{a}}) = \begin{bmatrix} + & - & + \\ + & + & - \\ - & + & + \end{bmatrix}.$$

Step 4.

$$
\mathbf{H_K} = \begin{bmatrix}
\begin{bmatrix}
\begin{bmatrix} - & - \\ - & + \end{bmatrix} & \begin{bmatrix} + & + \\ + & - \end{bmatrix} & \begin{bmatrix} - & - \\ - & + \end{bmatrix} \\
\begin{bmatrix} - & - \\ - & + \end{bmatrix} & \begin{bmatrix} - & - \\ - & + \end{bmatrix} & \begin{bmatrix} + & + \\ + & - \end{bmatrix} \\
\begin{bmatrix} + & + \\ + & - \end{bmatrix} & \begin{bmatrix} - & - \\ - & + \end{bmatrix} & \begin{bmatrix} - & - \\ - & + \end{bmatrix}
\end{bmatrix} &
\begin{bmatrix}
\begin{bmatrix} - & + \\ + & + \end{bmatrix} & \begin{bmatrix} + & + \\ + & - \end{bmatrix} & \begin{bmatrix} - & + \\ + & + \end{bmatrix} \\
\begin{bmatrix} - & + \\ + & + \end{bmatrix} & \begin{bmatrix} - & + \\ + & + \end{bmatrix} & \begin{bmatrix} + & + \\ + & - \end{bmatrix} \\
\begin{bmatrix} + & + \\ + & - \end{bmatrix} & \begin{bmatrix} - & + \\ + & + \end{bmatrix} & \begin{bmatrix} - & + \\ + & + \end{bmatrix}
\end{bmatrix} \\
\begin{bmatrix}
\begin{bmatrix} + & - \\ - & - \end{bmatrix} & \begin{bmatrix} + & + \\ + & - \end{bmatrix} & \begin{bmatrix} + & - \\ - & - \end{bmatrix} \\
\begin{bmatrix} + & - \\ - & - \end{bmatrix} & \begin{bmatrix} + & - \\ - & - \end{bmatrix} & \begin{bmatrix} + & + \\ + & - \end{bmatrix} \\
\begin{bmatrix} + & + \\ + & - \end{bmatrix} & \begin{bmatrix} + & - \\ - & - \end{bmatrix} & \begin{bmatrix} + & - \\ - & - \end{bmatrix}
\end{bmatrix} &
\begin{bmatrix}
\begin{bmatrix} + & + \\ + & - \end{bmatrix} & \begin{bmatrix} - & - \\ - & + \end{bmatrix} & \begin{bmatrix} + & + \\ + & - \end{bmatrix} \\
\begin{bmatrix} + & + \\ + & - \end{bmatrix} & \begin{bmatrix} + & + \\ + & - \end{bmatrix} & \begin{bmatrix} - & - \\ - & + \end{bmatrix} \\
\begin{bmatrix} - & - \\ - & + \end{bmatrix} & \begin{bmatrix} + & + \\ + & - \end{bmatrix} & \begin{bmatrix} + & + \\ + & - \end{bmatrix}
\end{bmatrix}
\end{bmatrix}
$$

Step 5.

$$
\mathbf{H_K} = \begin{bmatrix}
- & - & + & + & - & - & - & + & + & + & - & + \\
- & + & + & - & - & + & + & + & + & - & + & + \\
- & - & - & - & + & + & - & + & - & + & + & + \\
- & + & - & + & + & - & + & + & + & + & + & - \\
+ & + & - & - & - & - & + & + & - & + & - & + \\
+ & - & - & + & - & + & + & - & + & + & + & + \\
+ & - & + & + & + & - & + & + & - & - & + & + \\
- & - & + & - & - & - & + & - & - & + & + & - \\
+ & - & + & - & + & + & + & + & + & + & - & - \\
- & - & - & - & + & - & + & - & + & - & - & + \\
+ & + & + & - & + & - & - & - & + & + & + & + \\
+ & - & - & - & - & - & - & + & + & - & + & -
\end{bmatrix}
$$

3.3 Tables of all normalized $\mathbf{HP(m)}$ of sizes $\mathbf{m}$ for $\mathbf{0 \leq m \leq 7}$

In the following tables on the left entry the Hadamard-pairs are listed. The right entry contains the two corresponding Ψ-vectors. Every $\mathbf{HP}(m)$ generates a $\mathbf{HM}(8m+4)$.

m=0

[ ]	[ ]
[ ]	[ ]

m=1

[i]	[0]
[-]	[-1]

m=2

[i j]	[-1 -1]
[- -]	[0 0]
[- i]	[-1 0]
[i -]	[0 -1]

m=3

[i j j]	[0 0 -2]
[+ - +]	[-1 -1 1]
[i i j]	[0 -2 0]
[- + +]	[-1 1 -1]
[- i j]	[-2 -1 0]
[i - -]	[1 0 -1]
[+ i j]	[0 -1 0]
[i + -]	[-1 0 -1]
[i - j]	[0 -2 -1]
[- i -]	[-1 1 0]
[i + j]	[0 0 -1]
[- i +]	[-1 -1 0]
[i j i]	[-2 0 0]
[+ + -]	[1 -1 -1]
[- - i]	[0 -1 1]
[i j -]	[-1 0 -2]
[+ - i]	[0 -1 -1]
[i j +]	[-1 0 0]

m = 4

[i j j j]	[1 1 -1 -1]
[- + + -]	[-2 -2 0 0]
[- i j j]	[-1 0 -1 -1]
[i + + -]	[0 -1 0 0]
[- i j j]	[-1 0 -1 -1]
[i - + +]	[0 -1 0 0]
[i i i j]	[1 -1 -1 1]
[+ - + -]	[-2 0 0 -2]
[i - i j]	[-1 -1 -1 0]
[+ i + -]	[0 0 0 -1]
[i - i j]	[-1 -1 -1 0]
[- i + +]	[0 0 0 -1]
[- - i j]	[-1 -2 1 0]
[i j - -]	[0 1 -2 -1]
[- i - j]	[-1 0 -2 1]
[i - j -]	[0 -1 1 -2]
[i - - j]	[1 -1 -2 0]
[- i j -]	[-2 0 1 -1]
[i j + j]	[-1 1 0 -1]
[- - i +]	[0 -2 -1 0]
[i i + j]	[1 -1 0 -1]
[- + i -]	[-2 0 -1 0]
[- i + j]	[-1 -2 0 -1]
[i + i -]	[0 1 -1 0]
[i j i i]	[-1 1 -1 1]
[- - + +]	[0 -2 0 -2]
[+ - i i]	[1 0 -1 0]
[i j + -]	[-2 -1 0 -1]
[i j + i]	[-1 -1 0 1]
[+ - i -]	[0 0 -1 -2]
[+ - + i]	[-1 0 0 0]
[i j j -]	[0 -1 -1 -1]
[- + + i]	[-1 0 0 0]
[i j j -]	[0 -1 -1 -1]

m=5

[+ - i j j]	[0 -1 -2 -1 0]
[i i - + -]	[-1 0 1 0 -1]
[- i - j j]	[0 1 -2 0 -1]
[i + j - +]	[-1 -2 1 -1 0]
[- + i i j]	[-2 -1 -2 1 0]
[i i + - -]	[1 0 1 -2 -1]
[+ - + i j]	[-2 -1 0 1 -1]
[i j j - -]	[1 0 -1 -2 0]
[i j - + j]	[-2 0 -1 0 -1]
[- - i i +]	[1 -1 0 -1 0]
[- i j - i]	[-2 -1 1 0 0]
[i + + j -]	[1 0 -2 -1 -1]
[+ - i - i]	[0 1 -1 -1 0]
[i j + i -]	[-1 -2 0 0 -1]
[- + i - i]	[-2 1 1 -1 0]
[i i - j +]	[1 -2 -2 0 -1]
[i - - + i]	[0 -2 -1 1 1]
[+ i j i -]	[-1 1 0 -2 -2]

m=6

[+ - i j j j]	[1 0 -1 0 -1 -1]
[i j - - + -]	[-2 -1 0 -1 0 0]
[- i + i j j]	[-1 0 -1 -3 -1 0]
[i - i + - -]	[0 -1 0 2 0 -1]
[i - + i j j]	[-1 -1 1 -2 -1 -2]
[- i i + - -]	[0 0 -2 1 0 1]
[i - - j i j]	[-1 -1 -1 0 -1 0]
[+ i i - + -]	[0 0 0 -1 0 -1]
[i + i - i j]	[-1 1 -1 0 0 -1]
[- i + j - -]	[0 -2 0 -1 -1 0]
[- i i j - j]	[-1 0 0 -1 -1 -1]
[i - - + i +]	[0 -1 -1 0 0 0]
[i i - i - j]	[1 1 0 -1 -1 0]
[- + i - j +]	[-2 -2 -1 0 0 -1]
[- + + i - j]	[-1 -2 -2 0 -1 0]
[i i j + j +]	[0 1 1 -1 0 -1]
[i - - + - j]	[-1 -1 0 0 0 -2]
[- i i i j +]	[0 0 -1 -1 -1 1]
[i + - - + j]	[-1 -1 0 -2 -2 0]
[- i j j j -]	[0 0 -1 1 1 -1]
[+ + i - j i]	[1 -2 1 0 -2 -1]
[i j + j + -]	[-2 1 -2 -1 1 0]
[i j - + j i]	[-3 -1 -1 -1 0 0]
[+ - i i - -]	[2 0 0 0 -1 -1]
[- i j j - i]	[-1 -2 -2 1 -1 1]
[i + + - i -]	[0 1 1 -2 0 -2]
[+ - - i - i]	[1 0 0 -2 1 0]
[i j i + j -]	[-2 -1 -1 1 -2 -1]
[- + + i - i]	[-1 0 0 0 -1 0]
[i j j - i -]	[0 -1 -1 -1 0 -1]
[- - - i + i]	[1 0 0 0 1 -2]
[i j i - j +]	[-2 -1 -1 -1 -2 1]
[+ - - i + i]	[1 -2 -2 0 1 0]
[i j i - j -]	[-2 1 1 -1 -2 -1]

m=7

[i j + i j j j] [- + i - - + +]	[0 0 2 0 0 -3 0] [-1 -1 -3 -1 -1 2 -1]
[+ - i + j j j] [i j - j + - +]	[2 -1 0 1 0 -2 -1] [-3 0 -1 -2 -1 1 0]
[i i j j i j j] [- + - - - - +]	[0 -2 2 0 -2 -2 -2] [-1 1 -3 -1 1 1 1]
[i + - i i j j] [- i i + + - +]	[0 0 -2 -1 -2 1 -2] [-1 -1 1 0 1 -2 1]
[i + - - i j j] [+ i i j + - +]	[0 0 -2 -1 -2 -1 -1] [-1 -1 1 0 1 0 0]
[+ i j + i j j] [i - + j + - -]	[0 -1 2 0 0 0 -1] [-1 0 -3 -1 -1 -1 0]
[i j i i - j j] [+ - + + i + -]	[0 0 -2 0 -3 -2 0] [-1 -1 1 -1 2 1 -1]
[i j i i - j j] [- + + - i + +]	[0 0 -2 0 -3 -2 0] [-1 -1 1 -1 2 1 -1]
[i j + i - j j] [- + i - i + +]	[0 -2 0 0 -1 -1 -2] [-1 1 -1 -1 0 0 1]
[i i i j j i j] [- + - - - - +]	[0 -2 2 0 -2 -2 -2] [-1 1 -3 -1 1 1 1]
[+ i i + j i j] [i - + j + - -]	[0 -1 2 0 0 0 -1] [-1 0 -3 -1 -1 -1 0]
[i + i - - i j] [- i - j j - +]	[0 0 0 -1 -1 0 -1] [-1 -1 -1 0 0 -1 0]
[+ + - i + i j] [i j i + j - -]	[0 -1 0 -1 1 1 0] [-1 0 -1 0 -2 -2 -1]
[+ - i i j - j] [i j - + + i +]	[0 -1 -1 -1 -2 -2 0] [-1 0 0 0 1 1 -1]
[+ - - i j - j] [i j j + - j -]	[0 -1 -1 -1 0 -1 0] [-1 0 0 0 -1 0 -1]
[i j i + j - j] [+ - - i + i +]	[-2 0 1 0 -2 -2 -1] [1 -1 -2 -1 1 1 0]
[i + - - i - j] [- i j j - j +]	[0 0 -1 -1 0 -1 -1] [-1 -1 0 0 -1 0 0]
[+ - i j + - j] [i i + + j i -]	[-2 -1 -1 1 -1 0 -2] [1 0 0 -2 0 -1 1]
[+ i - j + - j] [i + i - i j +]	[0 -3 1 0 -1 -1 -2] [-1 2 -2 -1 0 0 1]

m=7 (continued)

[i + j + + - j]	[0 0 1 -1 1 0 -1]
[- i - j j i +]	[-1 -1 -2 0 -2 -1 0]
[i i - + + - j]	[0 -2 1 0 -1 -1 -3]
[- + i i i j +]	[-1 1 -2 -1 0 0 2]
[i j i i i + j]	[0 0 -3 0 0 2 0]
[+ - + + - i -]	[-1 -1 2 -1 -1 -3 -1]
[i j i i - + j]	[-2 0 -1 -2 -1 0 0]
[+ - + + i i -]	[1 -1 0 1 0 -1 -1]
[- i - - + + j]	[0 -1 -1 0 -1 -3 -1]
[i + j j i j +]	[-1 0 0 -1 0 2 0]
[- i - - + + j]	[0 -1 -1 0 -1 -3 -1]
[i + i j j i +]	[-1 0 0 -1 0 2 0]
[i j j j i j i]	[-2 0 -2 -2 -2 2 0]
[+ - - + - - -]	[1 -1 1 1 1 -3 -1]
[- + - i + j i]	[-4 -1 0 1 -1 -1 0]
[i i i - j + +]	[3 0 -1 -2 0 0 -1]
[i + - - + j i]	[-2 -2 0 -1 -1 -1 1]
[- i i i j + +]	[1 1 -1 0 0 0 -2]
[i j i j j i i]	[-2 0 -2 -2 -2 2 0]
[+ - - + - - -]	[1 -1 1 1 1 -3 -1]
[+ - + i j i i]	[-2 1 0 1 0 -1 0]
[i i j + + - -]	[1 -2 -1 -2 -1 0 -1]
[i - + + j i i]	[0 -2 0 -1 0 -1 3]
[+ i j i + - -]	[-1 1 -1 0 -1 0 -4]
[+ - i j - i i]	[0 -1 0 -1 -1 0 0]
[i j - + j - -]	[-1 0 -1 0 0 -1 -1]
[+ i j j i - i]	[0 -1 1 -2 -2 -2 0]
[i + - + + i -]	[-1 0 -2 1 1 1 -1]
[i + - j + - i]	[-2 0 -1 -3 -1 1 0]
[+ i j + i i -]	[1 -1 0 2 0 -2 -1]
[+ + i - + - i]	[0 1 1 -1 1 -2 -1]
[i j - i i j +]	[-1 -2 -2 0 -2 1 0]
[i - j + + - i]	[0 -4 -1 -1 -1 0 1]
[+ i + i j i -]	[-1 3 0 0 0 -1 -2]
[- + i i j + i]	[-2 -1 -1 3 0 0 0]
[i i - - + j +]	[1 0 0 -4 -1 -1 -1]
[i + i - j + i]	[0 -2 -1 1 0 0 1]
[- i - i + j +]	[-1 1 0 -2 -1 -1 -2]

3.4 One normalized HP(8)

m=8

[- - + - i - j j]	[-1 0 1 -2 0 0 1 0]
[i j i i + j - -]	[0 -1 -2 1 -1 -1 -2 -1]
[i i - j - j i j]	[-1 1 -1 -1 -1 -2 -2 1]
[- - i - i + + -]	[0 -2 0 0 0 1 1 -2]

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