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On Median Quantifiers

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On Median Quantifiers*

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We denote the set of all real numbers r with $0 \leq r \leq 1$ by $\langle 0, 1\rangle$. Let μ be a function with

$$\mu : \langle 0, 1\rangle \times \langle 0, 1\rangle \to \langle 0, 1\rangle.$$

The starting point of the present paper is the following well-known theorem [1, 2] characterizing so-called median functions.

Theorem 1
If 1. $\forall r \forall s(r, s \in \langle 0, 1\rangle \to \min(r, s) \leq \mu(r, s) \leq \max(r, s))$

2. *μ is monotone, commutative, associative and continuous*

then there exists exactly one $c \in \langle 0, 1\rangle$ *such that for every* $r, s \in \langle 0, 1\rangle$,

1. *if* $c \leq \min(r, s)$ *then* $\mu(r, s) = \min(r, s)$
2. *if* $\max(r, s) \leq c$ *then* $\mu(r, s) = \max(r, s)$
3. *if* $\min(r, s) < c < \max(r, s)$ *then* $\mu(r, s) = c$.

The aim of the present paper is to generalize this theorem to fuzzy quantifiers.

First, we have to introduce the following notations and definitions.

Let U be a non-empty set called universe. A fuzzy set F on U is a mapping

$$F : U \to \langle 0, 1\rangle,$$

i. e. we do not distinguish between a fuzzy set F and its membership function μ_F. By $\boldsymbol{U}$, $\boldsymbol{\emptyset}$, and C_r we denote the universal, the empty, and a constant fuzzy set on U, respectively, for every $x \in U$ defined by

$$\boldsymbol{U}(x) =_{def} 1$$
$$\boldsymbol{\emptyset}(x) =_{def} 0$$
$$C_r(x) =_{def} r$$

where r is a fixed real number from $\langle 0, 1\rangle$.

In the following we shall very often use the notation $F\langle x := r\rangle$ where $F \in \mathcal{F}(U)$, $x \in U$, and $r \in \langle 0, 1\rangle$. For every $y \in U$ we define this notation as follows

$$(F\langle x := r\rangle)(y) =_{def} \begin{cases} r & \text{if } y = x \\ F(y) & \text{if } y \neq x \end{cases}$$

In the field of two-valued logic A. Mostowski has introduced the concept of a general quantifier [3]. Following this approach by a general fuzzy quantifier Q on U [4–7] we understand a mapping of the form

$$Q : \mathcal{F}(U) \to \langle 0, 1\rangle.$$

For formulating the following lemma 2 and theorem 3 we shall use the special fuzzy quantifiers *ALL* and *EX* defined for arbitrary $F \in \mathcal{F}(U)$ by

$$ALL(F) =_{def} \operatorname{Inf}\{F(x) | x \in U\}$$
$$\text{and} \qquad EX(F) =_{def} \operatorname{Sup}\{F(x) | x \in U\}.$$

Furthermore, for arbitrary general fuzzy quantifiers Q on U we define:

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Definition 1

1. *Q is said to be monotone on $\mathcal{F}(U)$*
   $=_{def}$ $\forall F \forall G (F, G \in \mathcal{F}(U) \wedge F \subseteq G \rightarrow Q(F) \leq Q(G))$
2. *Q is said to be strongly associative on U and $\mathcal{F}(U)$*
   $$=_{def} \forall x \forall y \forall F \forall G \begin{pmatrix} x, y \in U \wedge F, G \in \mathcal{F}(U) \\ \rightarrow Q(F \langle x := Q(G) \rangle) = Q(G \langle y := Q(F \langle x := G(y) \rangle) \rangle) \end{pmatrix}$$
3. *Q is said to be weakly continuous with respect to U*
   $=_{def}$ *For every fixed $x \in U$ the functions φ and ψ, for every $r \in \langle 0, 1 \rangle$ defined by*
   $$\varphi(r) =_{def} Q(\boldsymbol{U} \langle x := r \rangle)$$
   $$\psi(r) =_{def} Q(C_r \langle x := 0 \rangle)$$
   are continuous on $\langle 0, 1 \rangle$.
4. *Q is said to be weakly commutative with respect to U*
   $=_{def}$ $\forall x \forall y (x, y \in U \rightarrow Q(\boldsymbol{U} \langle x := 0 \rangle) = Q(\boldsymbol{U} \langle y := 0 \rangle))$

Lemma 2

If 1. $\forall F (F \in \mathcal{F}(U) \rightarrow ALL(F) \leq Q(F) \leq EX(F))$

2. *Q is monotone on $\mathcal{F}(U)$*
3. *Q is strongly associative on U and $\mathcal{F}(U)$*
4. *Q is weakly continuous with respect to U*

then

1. $\forall x \forall F (x \in U \wedge F \in \mathcal{F}(U) \wedge Q(\boldsymbol{U} \langle x := 0 \rangle) \leq F(x) \rightarrow Q(F) \leq F(x))$
2. $\forall x \forall F (x \in U \wedge F \in \mathcal{F}(U) \wedge F(x) \leq Q(\boldsymbol{U} \langle x := 0 \rangle) \rightarrow F(x) \leq Q(F))$

Proof

Case 1 $\operatorname{card} U = 1$.

In this case we assume

(1) $$U = \{x\}.$$

Obviously, for arbitrary $F \in \mathcal{F}(U)$ we get

(2) $$ALL(F) = EX(F) = F(x),$$

hence by assumption 1 of lemma 2

(3) $$Q(F) = F(x),$$

i. e. in case 1 lemma 2 holds trivially.

Case 2 $\operatorname{card} U \geq 2$.

ad 1 For arbitrary $x \in U$ and $F \in \mathcal{F}(U)$ we have to prove

(4) $$Q(F) \leq F(x).$$

We assume

(5) $$Q(\boldsymbol{U} \langle x := 0 \rangle) \leq F(x).$$

Because of $\boldsymbol{U} \langle x := 1 \rangle = \boldsymbol{U}$ and $ALL(\boldsymbol{U}) = 1$ by assumption 1 of lemma 2 we have

(6) $$Q(\boldsymbol{U} \langle x := 1 \rangle) = 1,$$

hence

(7) $$Q(\boldsymbol{U} \langle x := 0 \rangle) \leq F(x) \leq Q(\boldsymbol{U} \langle x := 1 \rangle),$$

i. e. for the function φ defined on $\langle 0, 1\rangle$ for the fixed $x \in U$ by

$$\varphi(r) =_{def} Q(\boldsymbol{U}\langle x := r\rangle) \qquad (r \in \langle 0, 1\rangle)$$

we have

(8) $$\varphi(0) \leq F(x) \leq \varphi(1).$$

Now, by assumption 4 of lemma 2, the function φ is continuous on $\langle 0, 1\rangle$, hence by the intermediate value theorem we obtain

(9) $$\exists s\,(s \in \langle 0, 1\rangle \wedge \varphi(s) = F(x)),$$

i. e.

(10) $$\exists s\,(s \in \langle 0, 1\rangle \wedge Q(\boldsymbol{U}\langle x := s\rangle) = F(x)).$$

Because of

(11) $$F = F\langle x := F(x)\rangle$$

by (10) we get

(12) $$F = F\langle x := Q(\boldsymbol{U}\langle x := s\rangle)\rangle,$$

consequently,

(13) $$Q(F) = Q\,(F\langle x := Q(\boldsymbol{U}\langle x := s\rangle)\rangle).$$

Obviously, we have

(14) $$\forall r\,(r \in \langle 0, 1\rangle \rightarrow F\langle x := r\rangle \subseteq \boldsymbol{U}\langle x := r\rangle),$$

hence for $r =_{def} Q(\boldsymbol{U}\langle x := s\rangle)$

(15) $$F\langle x := Q(\boldsymbol{U}\langle x := s\rangle)\rangle \subseteq \boldsymbol{U}\langle x := Q(\boldsymbol{U}\langle x := s\rangle)\rangle,$$

consequently by assumption 2 of lemma 2, i. e. by the monotonicity of Q on $\mathcal{F}(U)$,

(16) $$Q\,(F\langle x := Q(\boldsymbol{U}\langle x := s\rangle)\rangle) \leq Q\,(\boldsymbol{U}\langle x := Q(\boldsymbol{U}\langle x := s\rangle)\rangle).$$

Now, we apply the strong associativity of Q on U and on $\mathcal{F}(U)$, i. e. assumption 3 of lemma 2, to

(17) $$Q\,(\boldsymbol{U}\langle x := Q(\boldsymbol{U}\langle x := s\rangle)\rangle).$$

by putting

(18) $$F =_{def} \boldsymbol{U} \text{ and } G =_{def} \boldsymbol{U}\langle x := s\rangle.$$

Hence we obtain

(19)
$$Q\,(\boldsymbol{U}\langle x := Q(\boldsymbol{U}\langle x := s\rangle)\rangle) = Q\,((\boldsymbol{U}\langle x := s\rangle)\langle y := Q\,(\boldsymbol{U}\langle x := (\boldsymbol{U}\langle x := s\rangle)(y)\rangle)\rangle).$$

Because in case 2 we assumed that $\operatorname{card} U \geq 2$, we can suppose that $y \neq x$, hence we get

(20) $$(\boldsymbol{U}\langle x := s\rangle)(y) = 1,$$

consequently

(21) $$\boldsymbol{U}\langle x := (\boldsymbol{U}\langle x := s\rangle)(y)\rangle = \boldsymbol{U}\langle x := 1\rangle = \boldsymbol{U},$$

hence

(22) $$Q\,(\boldsymbol{U}\langle x := (\boldsymbol{U}\langle x := s\rangle)(y)\rangle) = Q(\boldsymbol{U}) = 1.$$

Furthermore, because of $y \neq x$ we get

(23) $$(\boldsymbol{U}\langle x := s\rangle)\langle y := 1\rangle = \boldsymbol{U}\langle x := s\rangle,$$

hence by (10), (22), and (23) we obtain

(24) $$\begin{aligned} Q\,((\boldsymbol{U}\langle x := s\rangle)\langle y := Q\,(\boldsymbol{U}\langle x := (\boldsymbol{U}\langle x := s\rangle)(y)\rangle)\rangle) &= Q\,((\boldsymbol{U}\langle x := s\rangle)\langle y := 1\rangle) \\ &= Q\,(\boldsymbol{U}\langle x := s\rangle) = F(x). \end{aligned}$$

By combining (13), (16), (19), and (24) we obtain (4), i. e.

$$Q(F) \leq F(x).$$

ad 2 For arbitrary $x \in U$ and $F \in \mathcal{F}(U)$ we have to prove

(25) $$F(x) \leq Q(F).$$

We assume

(26) $$F(x) \leq Q(\boldsymbol{U}\langle x := 0\rangle),$$

hence we have

(27) $$0 \leq F(x) \leq Q(\boldsymbol{U}\langle x := 0\rangle).$$

Because of

(28) $$C_0\langle x := 0\rangle = \emptyset$$

and

(29) $$\boldsymbol{U}\langle x := 0\rangle = C_1\langle x := 0\rangle$$

we obtain

(30) $$\begin{aligned} Q(C_0\langle x := 0\rangle) &= Q(\emptyset) \\ &\leq EX(\emptyset) \\ &= 0 \\ &\leq F(x) \\ &\leq Q(\boldsymbol{U}\langle x := 0\rangle) \\ &= Q(C_1\langle x := 0\rangle) \end{aligned}$$

Now, we consider the function ψ defined on $\langle 0, 1\rangle$ for the fixed $x \in U$ by

$$\psi(r) =_{def} Q(C_r\langle x := 0\rangle) \qquad (r \in \langle 0, 1\rangle).$$

By (30) we have

(31) $$\psi(0) \leq F(x) \leq \psi(1).$$

By assumption 4 of lemma 2, the function ψ is continuous on $\langle 0, 1\rangle$, hence by the intermediate value theorem we get

(32) $$\exists s\,(s \in \langle 0, 1\rangle \wedge \psi(s) = F(x)),$$

i. e.

(33) $$\exists s\,(s \in \langle 0, 1\rangle \wedge Q(C_s\langle x := 0\rangle) = F(x)).$$

We put

(34) $$G =_{def} C_s\langle x := 0\rangle,$$

hence by (11) we obtain

(35) $$Q(F) = Q(F\langle x := Q(G)\rangle).$$

Now, we have

(36) $$F\langle x := Q(G)\rangle \supseteq \emptyset\langle x := Q(G)\rangle,$$

hence by assumption 2 of lemma 2, i. e. by the monotonicity of Q on U and $\mathcal{F}(U)$,

(37) $$Q(F\langle x := Q(G)\rangle) \geq Q(\emptyset\langle x := Q(G)\rangle),$$

hence by (35) and (37)

(38) $$Q(F) \geq Q(\emptyset\langle x := Q(G)\rangle).$$

Now, by the associativity of Q on U and on $\mathcal{F}(U)$ for $F =_{def} \emptyset$ and $G =_{def} C_s\langle x := 0\rangle$ we obtain for $y = x$

(39) $$Q(\emptyset\langle x := Q(G)\rangle) = Q(G\langle x := Q(\emptyset\langle x := G(x)\rangle)\rangle).$$

By (34), i. e. by definition of G, we have

(40) $$G(x) = (C_s \langle x := 0 \rangle)(x) = 0.$$

Furthermore, we get

(41) $$\emptyset \langle x := G(x) \rangle = \emptyset \langle x := 0 \rangle = \emptyset,$$

hence

(42) $$Q(\emptyset \langle x := G(x) \rangle) = Q(\emptyset \langle x := 0 \rangle) = Q(\emptyset) = 0,$$

thus

(43) $$\begin{aligned} G\langle x := Q(\emptyset \langle x := G(x) \rangle)\rangle &= G\langle x := 0 \rangle \\ &= (C_s \langle x := 0 \rangle)\langle x := 0 \rangle \\ &= C_s \langle x := 0 \rangle, \end{aligned}$$

hence by (33) and (43)

(44) $$\begin{aligned} Q(G\langle x := Q(\emptyset \langle x := G(x) \rangle)\rangle) &= Q(C_s \langle x := 0 \rangle) \\ &= F(x). \end{aligned}$$

By (38), (39), and (44) we obtain

(45) $$Q(F) \geq F(x),$$

i. e. (25) holds.

■

Theorem 3

If 1. $\forall F (F \in \mathcal{F}(U) \rightarrow ALL(F) \leq Q(F) \leq EX(F))$

2. *Q is monotone on $\mathcal{F}(U)$*
3. *Q is strongly associative on U and $\mathcal{F}(U)$*
4. *Q is weakly continuous on U and $\mathcal{F}(U)$*
5. *Q is weakly commutative on U, i. e.*

$$\forall x \forall y (x, y \in U \rightarrow Q(\boldsymbol{U} \langle x := 0 \rangle) = Q(\boldsymbol{U} \langle y := 0 \rangle))$$

then there exists exactly one $c \in \langle 0, 1 \rangle$ such that for every $F \in \mathcal{F}(U)$,

1. $c \leq ALL(F) \rightarrow Q(F) = ALL(F)$
2. $EX(F) \leq c \rightarrow Q(F) = EX(F)$
3. $ALL(F) < c < EX(F) \rightarrow Q(F) = c.$

Proof

In order to prove the existence of c we fix an $x_0 \in U$ and define $c \in \langle 0, 1 \rangle$ by

(1) $$c =_{def} Q(\boldsymbol{U} \langle x_0 := 0 \rangle).$$

Because of assumption 5 we have

(2) $$\forall x \forall y (x, y \in U \rightarrow Q(\boldsymbol{U} \langle x := 0 \rangle) = Q(\boldsymbol{U} \langle y := 0 \rangle)),$$

hence c does not depend on the chosen $x_0 \in U$.

ad 1 We assume

(3) $$c \leq ALL(F).$$

By the definition of ALL and (2) we get

(4) $$\forall x (x \in U \rightarrow Q(\boldsymbol{U} \langle x := 0 \rangle) \leq F(x)),$$

hence by assertion 1 of lemma 2,

(5) $$\forall x(x \in U \to Q(F) \le F(x)),$$

thus by definition of *ALL*,

(6) $$Q(F) \le ALL(F),$$

hence by assumption 1,

(7) $$Q(F) = ALL(F).$$

ad 2 We assume

(8) $$EX(F) \le c.$$

By (2) and the definition of *EX* we get

(9) $$\forall x(x \in U \to F(x) \le Q(\boldsymbol{U}\langle x := 0\rangle)),$$

hence by conclusion 2 of lemma 2

(10) $$\forall x(x \in U \to F(x) \le Q(F)),$$

thus by definition of *EX*,

(11) $$EX(F) \le Q(F),$$

hence by assumption 1,

(12) $$Q(F) = EX(F).$$

ad 3 By assumption we have

(13) $$ALL(F) < c < EX(F),$$

hence by definition of *ALL* and of *EX*

(14) there exists an $x_0 \in U$ and a $y_0 \in U$ such that $F(x_0) < c < F(y_0)$.

Now, we define a fuzzy set F' on U as follows:

(15) $$F'(x) =_{def} \begin{cases} F(x) & \text{if } F(x) \le c \\ c & \text{if } F(x) > c \end{cases}.$$

Then we get

(16) $$\forall x(x \in U \to F'(x) \le F(x) \wedge F'(x) \le c).$$

Hence from (16) we get

(17) $$Q(F') \le Q(F)$$

and

(18) $$EX(F') \le c.$$

From (18) by assumption 2 of the theorem we obtain

(19) $$Q(F') = EX(F').$$

Furthermore, by (14) and (15) we have

(20) $$F'(y_0) = c,$$

hence by (18)

(21) $$EX(F') = c,$$

thus by (17), (19), and (21)

(22) $$c \le Q(F).$$

Furthermore, we define a second fuzzy set F'' on U as follows:

(23) $$F''(x) =_{def} \begin{cases} F(x) & \text{if } F(x) \ge c \\ c & \text{if } F(x) < c \end{cases}.$$

Then we get

(24) $$\forall x (x \in U \to F(x) \le F''(x) \wedge c \le F''(x)).$$

From (24) by monotonicity of Q we obtain

(25) $$Q(F) \le Q(F''),$$

furthermore, by definition of ALL

(26) $$c \le ALL(F'').$$

From (26) by assumption 1 of the theorem we obtain

(27) $$Q(F'') = ALL(F'').$$

Furthermore, by (14) and (23) we have

(28) $$F''(x_0) = c,$$

hence by (26)

(29) $$ALL(F'') = c,$$

thus by (25), (27), and (29)

(30) $$Q(F) \le c,$$

hence by (22) and (30)

(31) $$Q(F) = c.$$

In order to prove the uniqueness of c we can assume that $\mathrm{card}\, U \ge 2$ because the case of $\mathrm{card}\, U = 1$ is trivial. Now assume we have $c, c' \in \langle 0, 1\rangle$ which fulfill the conclusion of theorem 3. Without loss of generality we can assume that $c \le c'$. Then we fix $x_0 \in U$ and define a fuzzy set F for arbitrary $x \in U$ by

(32) $$F(x) =_{def} \begin{cases} c & \text{if } x = x_0 \\ c' & \text{if } x \ne x_0 \end{cases}.$$

Then we get

(33) $$c = ALL(F)$$

and

(34) $$EX(F) = c'.$$

By conclusion 1 for c and conclusion 2 for c' we get

(35) $$Q(F) = ALL(F) = c$$

and

(36) $$Q(F) = EX(F) = c',$$

hence

(37) $$c = c'.$$

■

Remark The strong associativity of a quantifier Q includes a certain version of commutativity of Q, but not its "full" commutativity. This fact will discussed in detail in an extended version of this paper.

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