p Values and Alternative Boundaries for CUSUM Tests

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Abstract

Firstly rather accurate approximations to the p value functions of the common Standard CUSUM test and the OLS-based CUSUM test for structural change are derived. Secondly alternative boundaries for both tests are suggested and their properties are examined by simulation of expected p values. It turns out that the power of the OLS-based CUSUM test for early and late structural changes can be improved, whereas this weakness of the Standard CUSUM test cannot be repaired by the new boundaries.

Keywords: CUSUM test, structural change.

JEL Classification Number: C22, C12

1 Introduction and summary

One of the first tests on structural change with unknown break point was the Standard CUSUM test, introduced by Brown, Durbin and Evans (1975), henceforth BDE. Whereas this test is based on recursive residuals, which are independently distributed under the null hypothesis, Ploberger and Krämer (1992), henceforth PK, suggested a test based on the ordinary least squares residuals. Both are suitable to test the constancy of regression coefficients in linear regression relationships, although no version is uniformly superior to the other. For both tests approximations to the asymptotic p value functions are derived, which are closely linked to the crossing probabilities of (tied down) Brownian motions, the limiting distributions of the tests. It is much more convenient to have p values instead of the common critical values for fixed confidence levels like given in BDE (1975) or in Kuan and Hornik (1995), who put CUSUM tests in a more general context of strutctural change tests. Hansen (1997) gives approximate asymptotic p values for another class of structural change tests based on F statistics and here similar results will be obtained for CUSUM tests.

Afterwards alternative boundaries, that are proportional to the standard deviation of (tied down) Brownian motions, are suggested for both tests in order to repair their weakness in detecting structural shifts early and late in the sample period. Although this cannot be

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accomplished for the Standard CUSUM test due to the properties of the recursive residuals under the alternative, the OLS-based CUSUM test can indeed be improved. This OLS-based CUSUM test with the alternative boundaries has rather evenly distributed rejection properties for structural changes early, midway and late in the sample period. [2]

2 The model and the tests

The standard linear regression model

$$y_t = x_t^\top \beta + u_t \qquad (t = 1, \ldots, n) \tag{1}$$

is considered, where at time t, y_t is the observation of the dependent variable, $x_t = (1, x_{t2}, \ldots, x_{tk})^\top$ is a $k \times 1$ vector of observations of the independent variables, with the first component equal to unity, u_t are iid(0, σ^2), and β is the $k \times 1$ vector of regression coefficients. The CUSUM tests are concerned with testing against the alternative that this unkown coefficient vector varies over time. Like in PK (1992) and in Krämer, Ploberger, Alt (1988), henceforth KPA, it is assumed that the regressors x_t and the disturbances u_t are defined on a common probability space, such that

$$\limsup_{n \to \infty} \frac{1}{n} \sum_{t=1}^{n} ||x_t||^{2+\delta} \; < \; \infty \quad \text{a.s.} \tag{2}$$

for some $\delta > 0$ ($||\cdot||$ the Euclidean norm), and that

$$\frac{1}{n} \sum_{t=1}^{n} x_t x_t^\top \quad \longrightarrow \quad Q. \tag{3}$$

for some finite regular matrix Q. Furthermore it is assumed that the disturbances u_t are stationary and ergodic, with

$$\mathrm{E}[u_t|\mathcal{A}_t] = 0, \quad \mathrm{E}[u_t^2|\mathcal{A}_t] = \sigma^2, \tag{4}$$

where $\mathcal{A}_t$ is the σ-field generated by $\{y_{t-s}, x_{t-s}, u_{t-s} | s \geq 1\}$. These assumptions allow in particular for dynamic models, in which case they imply stability.
The Standard CUSUM test is based on the cumulative sum of the recursive residuals

$$\tilde{u}_t \;=\; \frac{y_t - x_t^\top \hat{\beta}^{(t-1)}}{\sqrt{1 + x_t^\top \left(X^{(t-1)\top} X^{(t-1)}\right)^{-1} x_t}} \qquad (t = k+1, \ldots, n), \tag{5}$$

which have zero mean and variance σ^2 under the null hypothesis. $\hat{\beta}^{(t-1)}$ is the ordinary least squares estimation of the regression coefficients based on the observations up to $t-1$. The path of the CUSUM quantity is defined as

$$W_n(t) \;=\; \frac{1}{\tilde{\sigma}\sqrt{n-k}} \sum_{i=k+1}^{\lfloor k+t(n-k) \rfloor} \tilde{u}_i \qquad (0 \leq t \leq 1), \tag{6}$$

[2]R code for all tests and procedures proposed is available from the auther upon request.

where $\tilde{\sigma} = \sqrt{\frac{1}{n-k}\sum_{t=k+1}^{n}(\tilde{u}_t - \bar{\tilde{u}})^2}$. The meaning of the variable t changes slightly, it is standardized to the interval [0,1].
If there is just a single structural change at fixed time $t_0 < 1$ the mean of the recursive residuals will be zero only up to t_0 and differing afterwards. Hence the CUSUM path $W_n(t)$ will start to leave its zero mean at t_0. H_0 is rejected whenever $W_n(t)$ crosses either $c(t)$ or $-c(t)$ with $c(t) = \lambda + 2\lambda t$, which is equivalent to rejecting the null hypothesis when the test statistic

$$S = \sup_{0\le t\le 1}\left|\frac{W_n(t)}{1+2t}\right| \tag{7}$$

is larger than λ, which depends on the significance level of the test. KPA show that for $n \to \infty$

$$W_n(t) \quad \xrightarrow{\mathrm{d}} \quad B(t), \tag{8}$$

where $\xrightarrow{\mathrm{d}}$ denotes convergence in distribution and where $B(t)$ is the standard Brownian motion.
The OLS-based CUSUM test is defined analogously using the OLS residuals $\hat{u}_t = y_t - x_t^\top \hat{\beta}$ instead of the recursive residuals. The OLS-based CUSUM quantity is defined for t in [0,1] as

$$W_n^0(t) \quad = \quad \frac{1}{\hat{\sigma}\sqrt{n}}\sum_{i=1}^{\lfloor nt\rfloor}\hat{u}_i, \tag{9}$$

where $\hat{\sigma} = \sqrt{\frac{1}{n-k}\sum_{t=1}^{n}\hat{u}_t^2}$. This path will always not only start in zero but also return to zero, but if there is structural change at t_0 it should have a peak close to the break point t_0. H_0 is rejected if the path crosses either λ or $-\lambda$, which is equivalent to rejecting when the test statistic

$$S^0 \quad = \quad \sup_{0\le t\le 1}\left|W_n^0(t)\right| \tag{10}$$

is larger than λ, which determines the significance level of the test. PK show that for $n \to \infty$

$$W_n^0(t) \quad \xrightarrow{\mathrm{d}} \quad B^0(t),$$

where $B^0(t)$ is the standard Brownian bridge or tied down Brownian motion.

3 p values of the CUSUM tests

In the context of these two CUSUM tests usually just three critical values, for the confidence levels 1%, 5% and 10%, are indicated, although it is much more convenient, especially for implementation in a statistical software package, to have an explicit formula to calculate p values. In this section an approximation to the p value function of both tests will be derived.
If S is the test statistic and given an observation s the p value is by definition:

$$\mathrm{P}_{H_0}(S \ge s). \tag{11}$$

Thus the p value is the confidence level of the test with critical value s.
Like already suggested in the last section p values of CUSUM tests are closely related to crossing probabilities of (tied-down) Brownian motions. According to (11) the p value for the Standard CUSUM test is

$$\mathrm{P}_{H_0}(S \geq s) \;=\; \mathrm{P}_{H_0}\Big(|W_n(t)| \;\geq\; s + 2st \quad \text{for some } 0 \leq t \leq 1\Big). \tag{12}$$

KPA show that this probability converges for $n \to \infty$ to the corresponding crossing probability of a Brownian motion. Hence the asymptotic p value function $p(s)$ is

$$p(s) \;=\; \mathrm{P}\Big(|B(t)| \;\geq\; s + 2st \quad \text{for some } 0 \leq t \leq 1\Big). \tag{13}$$

Therefore $p(s)$ is the level of the test with critical value s. Now an upper and a lower bound for this function will be derived by applying results of Durbin (1971), who examined the crossing probabilities of Brownian motions for linear boundaries. In lemma 3 he shows that

$$\begin{aligned} \mathrm{P}\Big(B(t) \geq at + b \quad \text{for some } 0 \leq t \leq 1\Big) \;&= \\ 1 - \Phi(a+b) + \exp(-2ab)\,\Phi(a-b), \end{aligned} \tag{14}$$

where $\Phi(\cdot)$ is the standard normal distribution function. Neglecting the probability that a single path of a Brownian motion crosses both lines, which is sufficiently small for large values of s, an upper bound is just twice the value of (14). But this can take values up to 2, whereas the real p value is of course not larger than 1, so that a proper upper bound is

$$p_U(s) \quad = \quad \min\Big\{1,\; 2\Big(1 - \Phi(3s) + \exp(-4s^2)\,\Phi(s)\Big)\Big\}. \tag{15}$$

To get an exact formula the probability that $B(t)$ crosses both lines has to be subtracted. But also for that probability Durbin's lemma 7 gives just an upper bound, which neglects three or more crossings:

$$\begin{aligned} \mathrm{P}\Big(B(t_1) \;\geq\; s + 2st_1 \;\wedge\; B(t_2) \;\leq\; -s - 2st_2 \quad \text{for some } 0 \leq t_1, t_2 \leq 1\Big) \;\leq \\ 2\Big(\exp(-4s^2)(1 - \Phi(5s)) + \exp(-16s^2)\,(1 - \Phi(s))\Big). \end{aligned} \tag{16}$$

Subtracting (16) from twice the value of (14) gives a lower bound for $p(s)$:

$$\begin{aligned} p_L(s) \;\;=\;\; 2\Big(&1 - \Phi(3s) + \exp(-4s^2)\,(\Phi(s) + \Phi(5s) - 1) \\ &- \exp(-16s^2)\,(1 - \Phi(s))\Big). \end{aligned} \tag{17}$$

The major drawback of this approximation is that it takes the value 0 for $s = 0$, although $p(0)$ obviously equals 1. But figure 1 shows clearly that both functions are rather good approximations to (13) for sufficiently large values of s. It can also be seen that up to its maximum, which is around $p_L(0.3) = 0.956$, $p_L(s)$ is closer to the simulated p values,

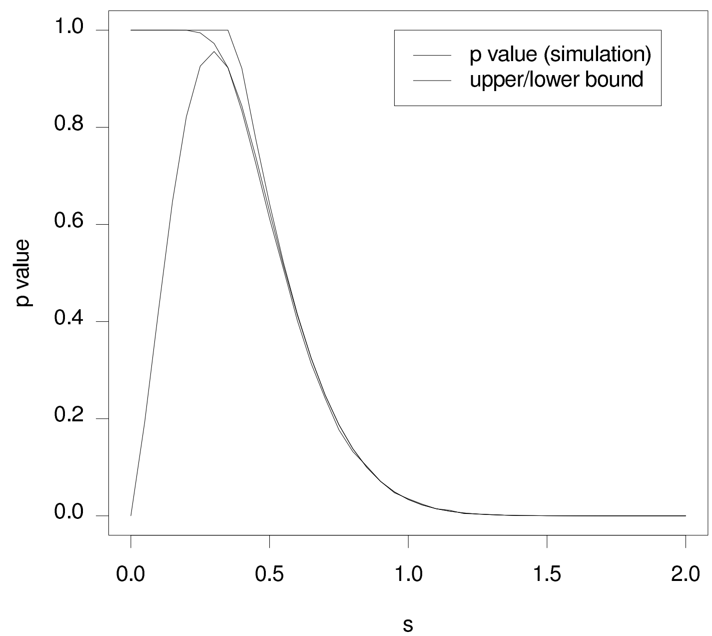


Figure 1: p value of the Standard CUSUM test

so that it approximates p values smaller than 0.96 very well. Larger p values can be approximated by linear interpolation, but they are not of great interest anyway. The practically useful small p values can be calculated equivalently well by both functions, which differ in 0.85, the critical value to the confidence level 0.1, just by $5 \cdot 10^{-6}$.
As both functions are easily calculated by any statistical software package, the lower boundary with linear interpolation for small s should be recommended for implementation:

$$\hat{p}(s) \;=\; \left\{ \begin{array}{cc} p_L(s) & s \geq 0.3 \\ 1 - 0.1465s & s < 0.3 \end{array} \right. . \tag{18}$$

Due to the similarity of the test statistics of the OLS-based and the Standard CUSUM test and the close relation between their asymptotic distributions the determination of the p value function $p^0(s)$ is analogous to that in the previous section. Hence

$$\begin{aligned} p^0(s) &= \mathrm{P}_{H_0}\Big(|W_n^0(t)| \;\geq\; s \quad \text{for some } 0 \leq t \leq 1\Big) \\ &\overset{\text{asy}}{=} \mathrm{P}\Big(|B^0(t)| \;\geq\; s \quad \text{for some } 0 \leq t \leq 1\Big) \\ &= \mathrm{P}\Big(|B(t)| \;\geq\; s \quad \text{for some } 0 \leq t \leq 1 \;\mid\; B(1) = 0\Big), \end{aligned} \tag{19}$$

where $\overset{\text{asy}}{=}$ denotes asymptotic equality. This is again approximated by twice the probability that a tied down Brownian motion crosses the line s parallel to the x-axis, which is given by lemma 4 in Durbin (1971). Cutting this function again at 1 gives the upper bound

$$p^0_U(s) \quad = \quad \min\Big\{1,\ 2\exp(-2s^2)\Big\}. \tag{20}$$

To derive the lower bound the probability that the path of a Brownian bridge crosses both

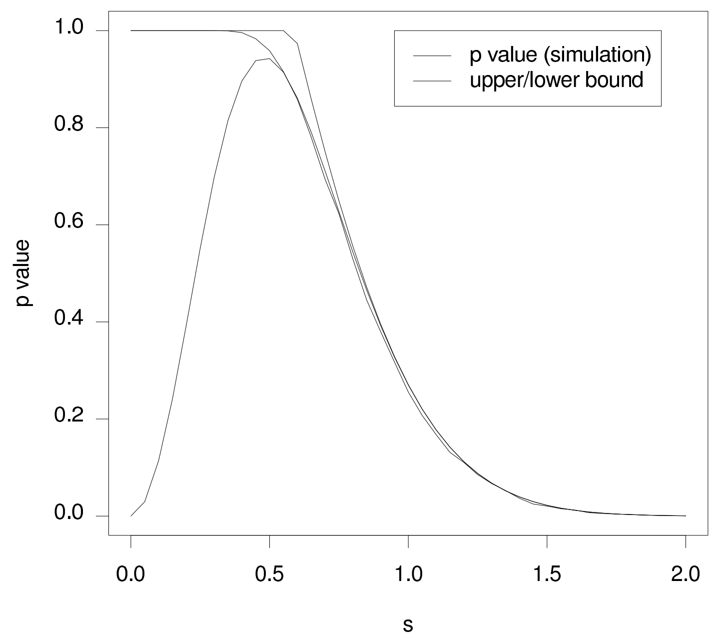


Figure 2: p value of the OLS-based CUSUM test

lines is needed, for which Durbin's lemma 6 provides the following inequality

$$\mathrm{P}\Big(B^0(t_1)\ \geq\ s\ \wedge\ B^0(t_2)\ \leq\ -s \quad \text{for some } 0 \leq t_1, t_2 \leq 1\Big)\ \leq$$

$$2\exp(-8s^2). \tag{21}$$

For the same reasons as for the Standard CUSUM test a lower bound for $p^0(s)$ is given by

$$p^0_L(s) \quad = \quad 2\Big(\exp(-2s^2) - \exp(-8s^2)\Big). \tag{22}$$

Figure 2 shows that the properties of the approximations are rather similar to those of the Standard CUSUM test. Again the upper bound has to be "cut" at the p value 1 and the lower bound provides the better approximation to the simulated p values, although it decreases again for small values of s. It takes its maximum 0.94 at $\sqrt{-\ln(0.25)/6} = 0.48$ and larger values can again be approximated by linear interpolation. Practically relevant p values can be calculated equivalently well by both functions as they differ in 1.22, the critical value for the level 0.1, just by $1.3 \cdot 10^{-5}$. Most suitable for implementation in a statistical software package is the combination of the lower bound and a linear interpolation for small values of s:

$$\hat{p^0}(s) = \begin{cases} p_L^0(s) & s \geq 0.48 \\ 1 - 0.1147s & s < 0.48 \end{cases} . \tag{23}$$

4 Alternative boundaries for CUSUM tests

One of the major drawbacks of both CUSUM tests is that they have poor power for early and late structural changes. To have similar properties over the whole time interval it seems natural to consider boundaries that are proportional to the standard deviation of the limiting distribution, so that the rejection probability is spread evenly. Thus the alternative boundary for the Standard CUSUM test is

$$b(t) = \lambda\sqrt{t}, \tag{24}$$

as the variance of a Standard Brownian motion (starting in 0) is $\mathrm{VAR}[B(t)] = t$. Similarly the variance of a Brownian Bridge is $\mathrm{VAR}[B^0(t)] = t(1-t)$, so that the alternative boundary for the OLS-based CUSUM test is

$$d(t) = \lambda\sqrt{t(1-t)}. \tag{25}$$

The parameter λ depends on the confidence level of the test, which is hard to evaluate, because the crossing probabilities of (tied-down) Brownian motions are not calculated as easily as for straight lines. For this reason BDE chose the boundary $c(t)$, which is tangential to $b(t)$ in $t = 0.5$ and the same argument holds for the linear boundary of the OLS-based CUSUM test. Here the critical values will be assessed by simulation, but firstly the alternative test statistics will be defined.
Rejecting the null hypothesis when the trajectory $W_n(t)$ either crosses $b(t)$ from (24) or $-b(t)$ is equivalent to rejecting if the alternative test statistic

$$S_A = \sup_{\varepsilon \leq t \leq 1} \left| \frac{W_n(t)}{\sqrt{t}} \right|, \tag{26}$$

with $\varepsilon > 0$ exceeds λ. The level α of the test is linked (asymptotically) to the critical value λ by the following equation:

$$\begin{aligned} \alpha &= \mathrm{P}(S_A \geq \lambda) \\ &\overset{\text{asy}}{=} \mathrm{P}\Big(|B(t)| \geq b(t) \quad \text{for some } \varepsilon \leq t \leq 1\Big). \end{aligned} \tag{27}$$

The point $t = 0$ has to be excluded as the rejection condition $B(0) \geq b(0) = 0$ would be satisfied trivially. It is also not possible to evaluate the supremum on (0,1], because even

then the rejection probability would converge against 1. Hence a compact interval $[\varepsilon, 1]$ with $\varepsilon > 0$ is needed, here $\varepsilon = 0.001$ is used.
To evaluate the pairs of values of λ and α which solve (27) two methods are used:
Method 1: A Brownian motion is simulated by the cumulative sum of $n = 5000$ normally distributed random numbers; then it is checked whether the absolute value of this simulated Brownian motion crosses $\pm b(t)$ (with fixed parameter λ). This is repeated $k = 5000$ times and the resulting percentage of crossings is an estimator for level α corresponding to λ. The results have been smoothed by a third order polynomial but have just a mean absolute difference of $2 \cdot 10^{-3}$ from the original data.
Method 2: This method is an application of the algorithm of Wang and Pötzelberger (1997) for crossing probabilites of Brownian motions for arbitrary boundaries. The boundary $b(t)$ is approximated by a piecewise linear function by simple interpolation in 128 subintervals of the same size. The formula that Wang and Pötzelberger provide is evaluated 200,000 times, which gives an estimation of $\alpha/2$ as only the crossing of a single boundary is being considered. Hence the approximation is poor for small λ and can take values larger than 1. The advantage however is that an estimation of the standard deviation is provided as well, which was smaller than 10^{-6} for all simulated values.

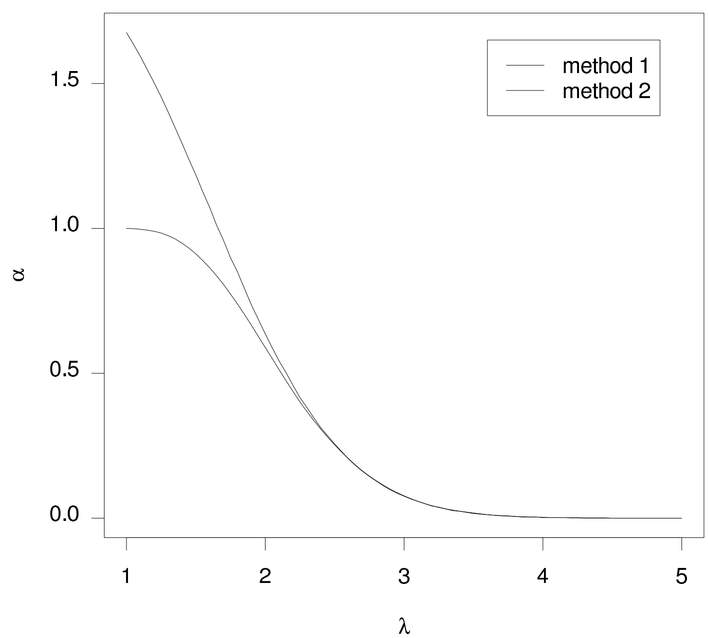


Figure 3: Results of method 1 and 2

A graphical comparison of both methods in figure 3 shows that the results are rather similar for large values of λ; in particular the critical values for the common confidence levels 1%, 5% and 10% are identical:

$$\begin{aligned} \alpha &= 0.10, & \lambda &= 2.90, \\ \alpha &= 0.05, & \lambda &= 3.15, \\ \alpha &= 0.01, & \lambda &= 3.65. \end{aligned} \tag{28}$$

To compare the shape of the linear and the alternative rejection area both boundaries for the confidence level $\alpha = 0.01$ are plotted in figure 4, the shape for the other confidence levels is rather similar. It can be seen that the the alternative boundary offers advantages

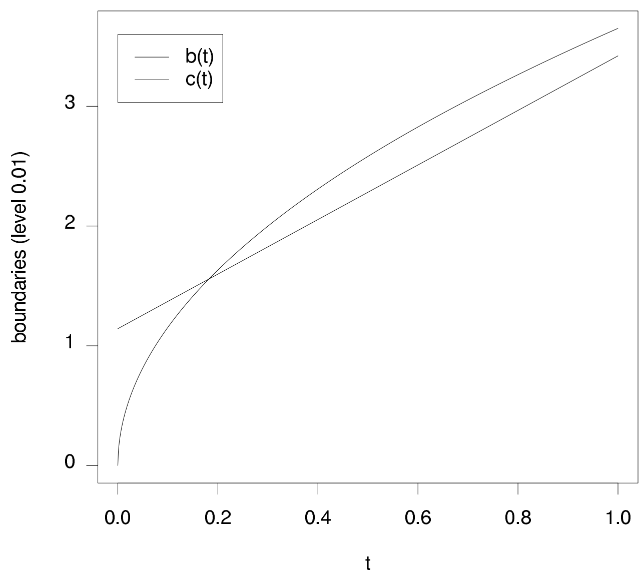


Figure 4: Boundaries of the Standard CUSUM test

only for $t \leq 0.2$. Therefore structural changes that occur late in the sample can be detected more easily with the linear boundaries; even for early shifts there is little hope that the advantage of the alternative boundary can be used as figure 5 illustrates: Although the break point is at 10% of the 1000 observations in this simulated data set, the CUSUM path actually crosses the boundary much later, where the alternative boundary lies above the linear one, i.e. the usual Standard CUSUM test would have rejected the null hypothesis anyway.

Now the alternative boundaries for the OLS-based CUSUM test will be investigated in the same way as it was done for the Standard CUSUM test. To reject H_0 if the OLS-based CUSUM trajectory $W_n^0(t)$ crosses the alternative boundary $\pm d(t)$ from (25) is equivalent

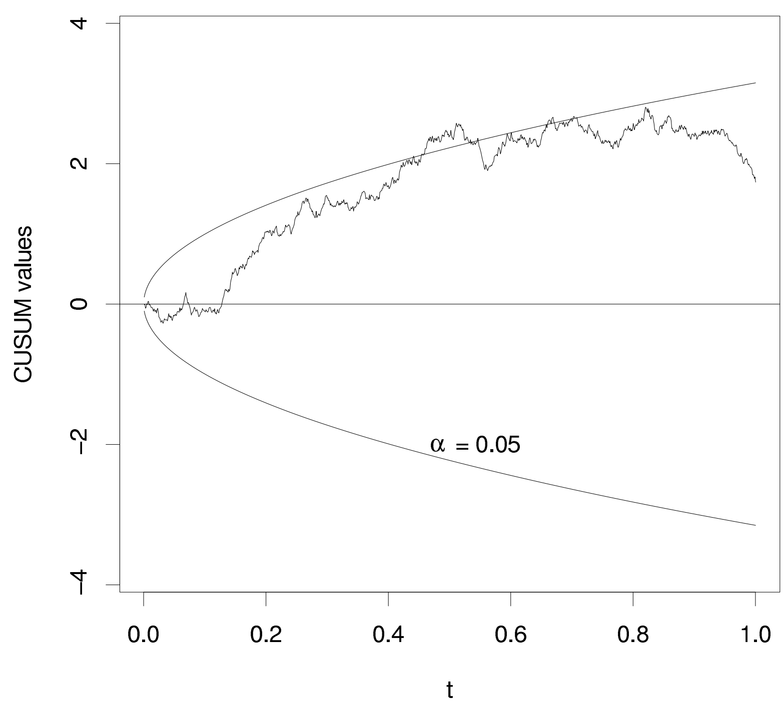


Figure 5: Standard CUSUM test with alternative boundaries

to rejecting if the alternative test statistic

$$S_A^0 \quad = \quad \sup_{\varepsilon \le t \le 1-\varepsilon} \left| \frac{W_n^0(t)}{\sqrt{t(1-t)}} \right| . \tag{29}$$

exceeds λ. The critical values λ are again linked to the confidence level α by

$$\begin{aligned} \alpha &= \mathrm{P}(S_A^0 \ \geq \ \lambda) \\ &\overset{\text{asy}}{=} \mathrm{P}\Big(|B^0(t)| \ \geq \ d(t) \quad \text{for some } \varepsilon \le t \le 1-\varepsilon\Big). \end{aligned} \tag{30}$$

In this case both limits of the interval have to be excluded and with the same arguments as above the compact interval $[\varepsilon, 1-\varepsilon]$ (with $\varepsilon = 0.001$) will be considered.
Analogously to method 1 from the previous section the corresponding pairs of λ and α are evaluated by simulation with the following result for the common confidence levels 1%, 5% and 10%:

$$\begin{aligned} \alpha &= 0.10, & \lambda &= 3.13, \\ \alpha &= 0.05, & \lambda &= 3.37, \\ \alpha &= 0.01, & \lambda &= 3.83. \end{aligned} \tag{31}$$

Using these results the rejection regions for the OLS-based CUSUM test can be compared. Figure 6 shows both boundaries for the level $\alpha = 0.01$. In contrast to the boundaries of

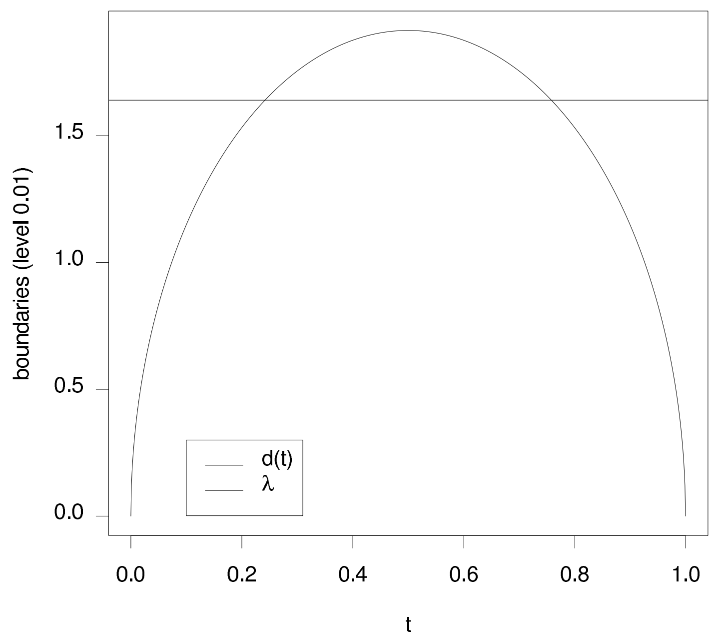
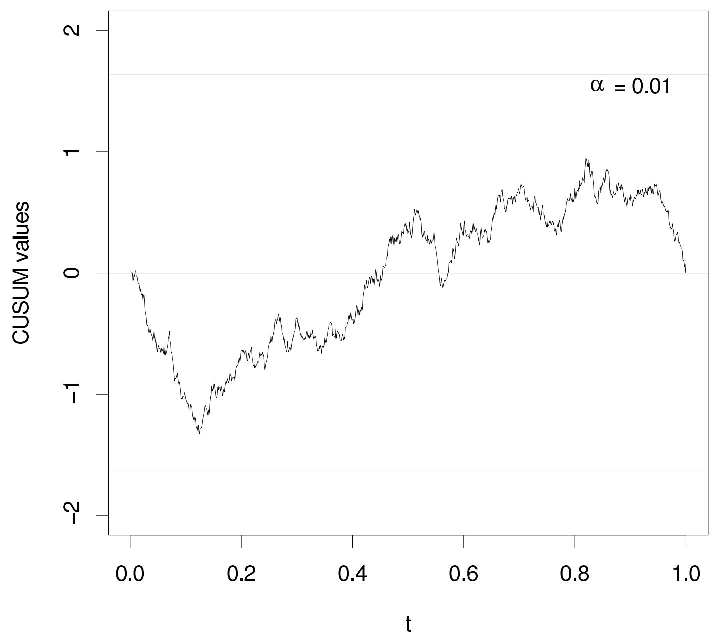


Figure 6: Boundaries of the OLS-based CUSUM test

the Standard CUSUM test the new boundary lies under the linear one at the beginning as well as at the end. These advantages can be worth the disadvantage in the middle as figure 7 indicates, which shows the OLS-based CUSUM trajectory for the same simulated data as above with a structural shift after 10% of the 1000 observations. Whereas the linear boundaries fail to detect the structural change at level $\alpha = 0.01$, the new boundaries are able to find evidence for a structural shift at the same level. The reason for that is the behaviour of the CUSUM values under the alternative: the path has its peak around the break point so that the advantages of the alternative boundaries can be used for early and late structural changes.
To emphasize this simulation of expected p values will be used in the next section.



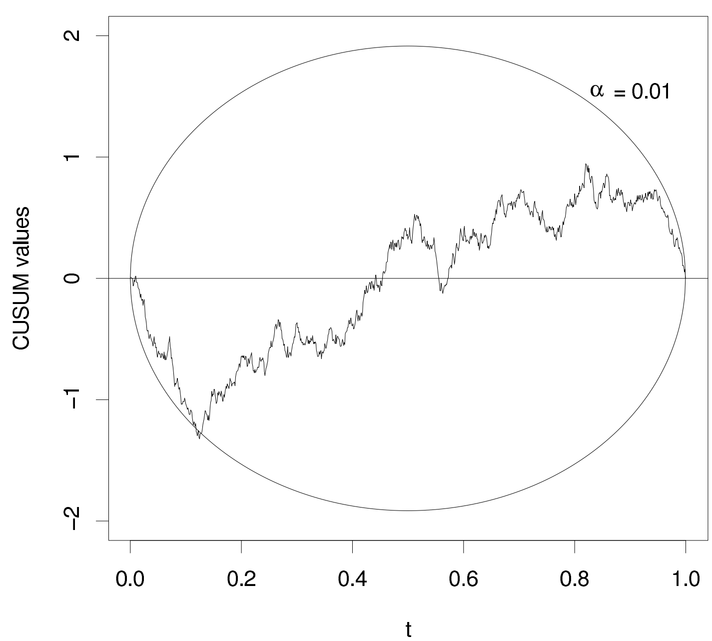


Figure 7: OLS-based CUSUM test with linear and alternative boundaries

5 Simulation of expected p values

Firstly expected p values will be defined according to Sackrowitz and Samuel-Cahn (1999). If T_0 is the test statistic distributed according to the null distribution F_0 and T, the test statistic under some specified alternative F_θ, takes the value t the usual p value is given by

$$\mathrm{P}(T_0 \geq t | T = t). \tag{32}$$

Thus the expected p value results if in (32) is the unconditional probability is considered

$$EPV(\theta) \quad = \quad \mathrm{P}(T_0 \geq T), \tag{33}$$

which is 1 - expected power (over all possible levels). Under H_0 the expected p value is obviously 0.5; a small expected p value indicates good chances to reject the null hypothesis.

Expected p values seem to be convenient to compare the power of the two OLS-based CUSUM tests as they don't depend on the confidence level; the power of the two Standard CUSUM tests won't be compared because the alternative boundaries could just offer disadvantages.

To compare the OLS-based CUSUM tests a simple model is chosen like in PK (1992), where $k = 2$, $x_t = (1, (-1))^\top$ and $u_t \sim$ nid(0,1). Then the timing, the intensity and the angle of a single shift are varied in the following way:

$$\beta_t \quad = \quad \begin{cases} \beta & \text{for } t \leq \lfloor qn \rfloor \\ \beta + \Delta\beta & \text{for } t > \lfloor qn \rfloor \end{cases}, \tag{34}$$

and the shift $\Delta\beta$ is given by

$$\Delta\beta \quad = \quad \frac{g}{\sqrt{n}} \begin{pmatrix} \cos\psi \\ \sin\psi \end{pmatrix}, \tag{35}$$

where ψ is the angle between the shift and the mean regressor $(1, 0)^\top$. Including the angle is necessary as neither the Standard nor the OLS-based CUSUM test are able to pick up shifts with an angle of 90°. The intensity of the shift is $||\Delta\beta|| = |g|\sqrt{n}$, which occurs at time $t = \lfloor qn \rfloor$ with $n = 500$. With q taking values 0.1, 0.3, 0.5, 0.7, 0.9 structural changes early, midway and late in the sample period are covered. In 1000 runs one test statistic under H_0 and one under the specified alternative are simulated and it is checked whether the null test statistic is larger. The empirical probabilities are reported in table 1 and it can be seen that the linear boundaries cause some weaknesses for early and late structural changes, whereas the properties of the test are rather good for q between 0.3 and 0.7. The alternative boundaries can solve the weakness for early and late changes and they spread the rejection probability more evenly over the whole sample period.

q	g	ψ 0°	18°	36°	54°	72°	90°
OLS-based CUSUM test with linear boundaries							
0.1	4.8	0.395	0.415	0.419	0.489	0.458	0.505
	7.2	0.293	0.329	0.374	0.425	0.509	0.483
	9.6	0.201	0.207	0.267	0.389	0.484	0.526
	12.0	0.124	0.133	0.166	0.331	0.442	0.545
0.3	4.8	0.151	0.198	0.255	0.350	0.449	0.536
	7.2	0.045	0.060	0.121	0.221	0.435	0.513
	9.6	0.005	0.015	0.029	0.127	0.332	0.544
	12.0	0.002	0.001	0.008	0.055	0.308	0.519
0.5	4.8	0.118	0.144	0.224	0.338	0.468	0.509
	7.2	0.020	0.024	0.067	0.171	0.381	0.500
	9.6	0.004	0.006	0.016	0.065	0.294	0.508
	12.0	0.000	0.000	0.003	0.039	0.247	0.504
0.7	4.8	0.179	0.185	0.252	0.364	0.464	0.484
	7.2	0.038	0.051	0.119	0.226	0.398	0.512
	9.6	0.005	0.010	0.031	0.121	0.377	0.521
	12.0	0.000	0.000	0.005	0.056	0.299	0.512
0.9	4.8	0.399	0.431	0.421	0.443	0.488	0.482
	7.2	0.334	0.339	0.392	0.412	0.480	0.487
	9.6	0.211	0.204	0.268	0.384	0.488	0.507
	12.0	0.109	0.136	0.190	0.308	0.458	0.508
OLS-based CUSUM test with alternative boundaries							
0.1	4.8	0.367	0.382	0.396	0.436	0.477	0.503
	7.2	0.256	0.246	0.308	0.409	0.457	0.494
	9.6	0.138	0.145	0.231	0.309	0.455	0.500
	12.0	0.054	0.071	0.118	0.260	0.395	0.494
0.3	4.8	0.227	0.243	0.300	0.393	0.456	0.507
	7.2	0.066	0.087	0.138	0.293	0.438	0.533
	9.6	0.023	0.012	0.037	0.156	0.390	0.539
	12.0	0.001	0.003	0.010	0.106	0.346	0.514
0.5	4.8	0.197	0.207	0.262	0.359	0.465	0.505
	7.2	0.049	0.075	0.114	0.271	0.422	0.513
	9.6	0.004	0.011	0.038	0.141	0.395	0.536
	12.0	0.001	0.001	0.007	0.045	0.338	0.520
0.7	4.8	0.224	0.250	0.303	0.415	0.485	0.522
	7.2	0.080	0.077	0.170	0.276	0.441	0.523
	9.6	0.009	0.016	0.039	0.184	0.394	0.527
	12.0	0.002	0.006	0.012	0.095	0.347	0.544
0.9	4.8	0.383	0.376	0.426	0.453	0.482	0.488
	7.2	0.229	0.264	0.291	0.397	0.480	0.488
	9.6	0.137	0.154	0.203	0.328	0.447	0.500
	12.0	0.060	0.074	0.129	0.229	0.404	0.494

Table 1: Simulation of expected p values of the alternative OLS-based CUSUM test

6 Conclusion

Firstly explicit formulae for approximating the (asymptotic) p values for the common Standard and the OLS-based CUSUM test are derived, which are rather useful for computation and implementation. Secondly alternative boundaries that are proportional to the standard deviation of the limiting distributions are suggested. They fail to improve that properties of the Standard CUSUM test, but they can solve the weakness of the OLS-based CUSUM test for early and late structural changes. If a CUSUM test should be applied to data where the potential break point is not known, the alternative OLS-based CUSUM test is probably the most recommendable.

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