Lyapunov exponent for Stochastic Time Series

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Abstract

This paper deals with the problem of the discrimination between stable and unstable time series. One criterion for the separation is given by the size of the Lyapunov exponent, which was originally defined for deterministic systems. However, this paper will show, that the Lyapunov exponent can also be analyzed and used for ergodic stochastic time series. Experimental results illustrate the classification by the Lyapunov exponent.

Although the Lyapunov exponent is a discriminatory parameter of the asymptotic behavior and can be interpreted as a parameter of the asymptotic distribution in the stochastic case, it has to be estimated from a given time series, where the process might still be in the transient state. Experimental results will show that in special cases the estimation leads to misclassifications of the time series and the underlying process due to the uncertainty of estimators for the Lyapunov exponent.

1 Introduction

In connection with the description and the analysis of time series the Lyapunov exponent can be used for the determination of the predictability of

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time series (Busse et al. 2001). Another possible field of application is to discriminate between ergodic stochastic processes with stationary distributions and processes, like chaotic systems, with local "instability" of the asymptotic distribution.

A formal discrimination between stable and unstable time series can be achieved by analyzing the Lyapunov exponent, which was suggested in connection with the predictability of deterministic processes and stochastic time series with additive noise (Busse et al. 2001).

Although the Lyapunov exponent is a discriminatory parameter of the asymptotic distribution, it has to be estimated from a given finite time series. Consequently the estimation causes the main problem, if the Lyapunov exponent should distinguish between the different kinds of time series. If the sample size does not suffice or the time series is strongly disturbed by noise the estimation may be biased. In the literature various approaches for the separation criterion estimation have been suggested (Kantz and Schreiber (1997), Gencay (1996), Sano and Sawada (1985), Eckmann et al. (1986). However, all these methods are susceptible to interference. Noisy time series, missing numerical stability or limited number of data often yield a bad estimator. Consequently the classification into a stable or unstable system could be possibly incorrect.

The remainder of this paper is organized as follows. Necessary notation is described in Sec. 2. After an introduction of the Lyapunov exponent (Sec. 3) we will show the connection between stable and unstable processes and the value of the Lyapunov exponent (Sec. 4). Because of this, it is possible to analyze deterministic or stochastic processes with non-necessarily additive noise.

Experimental results with stable and unstable processes demonstrate the separation by the Lyapunov exponent (see Sec. 5). Additional experimental results illustrate the method of discrimination by the Lyapunov exponent and the possible misclassification, if the sample size is not large enough. A conclusion is drawn in Sec. 6.

2 Deterministic and stochastic processes

An intuitive separation between deterministic and stochastic processes includes the aspects of functional relationships with and without random errors

(Tong 1993). The dynamics of deterministic processes is defined by

$$x_{t+1} = f_t(x_0) = f(x_t) \quad , \tag{1}$$

with initial point or initial state $x_0 \in \mathbb{R}^k$, x_t describes the state at time t. The functional relationship is described by f and it is assumed that f is differentiable everywhere.

A chaotic process can be represented by a deterministic process the asymptotic behavior of which is locally unstable in contrast to a regular deterministic or to an ergodic stochastic system (Abarbanel 1996), (Tong 1993), (Eckmann and Ruelle 1982).

As a formal definition of stability of a series x_t we use:
A series x_t is defined to be asymptotically stable if all fixed points or periodical orbits are asymptotically stable in the following sense (Jetschke (1989), p. 59-60):

a) A fixed point $x^0 = f(x^0)$ is said to be asymptotically stable if

$$\begin{aligned} \exists\, \delta > 0, \text{ so that } \forall x_0 \text{ with } \parallel x_0 - x^0 \parallel < \delta : \\ \lim_{N\to\infty} \parallel x_N - x^0 \parallel = 0. \end{aligned} \tag{2}$$

b) A trajectory $(x_i)_{i=0}^{N-1}$ of a process (X_t) is said to be a periodical orbit, if $f^N(x_0) = x_0$ and $f^i(x_0) \neq x_0$ for $i = 1, \ldots, (N-1)$ and $f^i = \underbrace{f \circ \cdots \circ f}_{i\ times}$.
A periodical orbit C is asymptotically stable, if a point $x \in C$ is an asymptotically stable fixed point of f^N.

We call a deterministic process stable, if all fixed points and periodical orbits are asymptotically stable and the process is unstable if there is a fixed point or the periodical orbit which is not asymptotically stable or no fixed point exists.

In contrast to deterministic processes a stochastic process is a functional relationship with random noise, which reads

$$X_{t+1} = f_t(X_0, \epsilon) = f(X_t, \epsilon). \tag{3}$$

It is assumed that such a process is a sequence of random variables, where X_0 is the random variable realized in the initial point x_0. The random variable X_t describes the state at time t, the realization or observation of which is denoted by x_t. The functional relationship f is stochastically disturbed with non-necessarily additive noise ϵ. The asymptotic behavior of a stochastic process should ideally be independent of the initial state.

We transmit the definition of a deterministic stable process to stochastic processes. The proper asymptotical stability for a fixed point or periodical orbit is not demanded. We use the asymptotical stochastical stability for the definition of a stable stochastic process. Starting from equation (2) we define asymptotical stochastical stability in the following sense:

a) A fixed point $x^0 = f(x^0)$ is said to be asymptotically stochastically stable if

$$\begin{aligned} &\exists\, \delta > 0, \text{ so that } \forall x_0 \text{ mit } \parallel x_0 - x^0 \parallel < \delta : \\ &P\left(\lim_{N\to\infty} \frac{1}{N}\sum_{t=0}^{N-1} X_t - x^0 = 0\right) = 1. \end{aligned} \tag{4}$$

b) A trajectory $(x_i)_{i=0}^{N-1}$ of a stochastic process $(X_t)_{t\in T}$ is said to be a stochastically periodical orbit, if $||f^N(x_0) - x_0|| < \delta$ and $||f^i(x_0) - x_0|| \geq \delta$ for $i = 1, \ldots, (N-1)$ and $f^i = \underbrace{f \circ \cdots \circ f}_{i\ times}$.

A periodical orbit C is asymptotically stochastically stable, if a point $x \in C$ is an asymptotically stochastically stable fixed point of f^N.

The definition of asymptotical stochastical stability (eq. (4)) is related to the definition for deterministic processes(eq. 2).

Again, we call a stochastic process stable, if all fixed points and periodical orbits are asymptotical stochastical stable and the process is unstable if there is a fixed point or the periodical orbit which is not asymptotically stable or no fixed point exists.

A specific stochastic process is an ergodic stochastic process, the asymptotic behavior of which is uniform and stable and independent of the initial state. In this context a mean stationary discrete random process X_t with mean $E_P(X)$ is called ergodic (Schlittgen and Streitberg (1994)), if

$$P\left(\lim_{N\to\infty} \frac{1}{N}\sum_{t=0}^{N-1} X_t = E_P(X)\right) = 1. \tag{5}$$

The definition of ergodic stochastic processes is the same like the definition of ergodic deterministic processes, if the process average $E_P(X)$ is inserted for the ensemble average.

For ergodic processes X_t it is true that

$$P\left(\lim_{N\to\infty}\frac{1}{N}\sum_{t=0}^{N-1} g(X_t) = E(g(X))\right) = 1 \tag{6}$$

for any measurable function g (Stout (1974), pp. 167, p. 182).

It is conceptually possible to transfer a deterministic observation series into a stochastic time series by assuming a functional relationship and a noise ϵ with a one-point distribution (Busse 2003).

3 The Lyapunov exponent in a stochastic context

One possibility to distinguish between stable and unstable time series is given by the computation of the largest Lyapunov exponent (here as often briefly called the Lyapunov exponent). The Lyapunov exponent $\lambda(x_0)$ of a deterministic process is formally defined by Eckmann and Ruelle (1982):

$$\lambda(x_0) := \lim_{N\to\infty}\frac{1}{N}\sum_{i=0}^{N-1} \ln|f'(x_i)|. \tag{7}$$

This characteristic feature measures an average logarithmic expansion rate along two different trajectories of the same underlying process. In Busse et al. (2001) it was been used for classification of predictable time series.

For the separation it is necessary to analyze the Lyapunov exponent in a stochastic framework. Note that the random effect has not to be necessarily additive in the functional expression of the dynamics of stochastic processes. It will be shown that the Lyapunov exponent can be interpreted as the expected value of the asymptotic distribution of an ergodic process.

Let X_0 be the random variable realized in the initial point x_0. Let X_t be the random variable, which describes the state at time t, the realization of which is denoted by x_t.

The functional relationship of the time series is denoted by $f(x,\epsilon)$, the time series is defined by $x_{t+1} = f(x_t,\epsilon)$ and the random effect is not-necessarily additive (see Sec. 2).

Now, the Lyapunov exponent can be naturally generalized as the asymptotic expectation (if existing) of a transformation of the given stochastic process

$$\tilde{\lambda}(x_0) := \lim_{t\to\infty} E[\ln |f'(X_t(x_0))|]. \tag{8}$$

However, this expected value is mostly unknown and has to be estimated. One obvious possibility is the calculation of the long time average, which is defined for discrete time processes by

$$\bar{g}(x_0) := \lim_{N\to\infty} \frac{1}{N} \sum_{t=0}^{N-1} g(X_t(x_0)),$$

where g is any arbitrary, measurable function. In the case of the Lyapunov exponent estimation $g(x) = \ln |f'(x)|$. Note, this $\bar{g}(x_0)$ is the definition of the Lyapunov exponent for stochastic processes in Busse et al. (2001).

This long time average is allowed to be dependent on initial state x_0. Because of (6) following from ergodicity (Stout (1974),p. 181), however, in the case of ergodic processes, this long time average is independent of x_0 and

$$\bar{g}(x_0) \overset{\text{a.s.}}{=} E(g(x)). \tag{9}$$

Thus for ergodic processes the Lyapunov exponent $\tilde{\lambda}(x_0)$ in (8) is independent of the initial state, and is the same as defined in Busse et al. (2001) for stochastic processes with an additive noise. It can be written as

$$\tilde{\lambda} = \int \ln |f'(x)| p(x) dx, \tag{10}$$

using $g(x) = \ln |f'(x)|$ and $p(x)$ as the density of the underlying process.

The ensemble average can thus obviously be estimated by means of

$$\hat{\lambda} = \frac{1}{N} \sum_{i=0}^{N-1} \ln |f'(x_i)|, \tag{11}$$

which is also an estimator of the Lyapunov exponent in (7).

Consequently, the Lyapunov exponent can be used for any given deterministic or stochastic time series. The problem of starting time dependence vanishes due to the equality of the long time average and the ensemble average under the condition of ergodicity.

4 Stable and unstable processes

Based on sections 2 and 3 it is possible to use the Lyapunov exponent for the separation between stable and unstable processes, because the Lyapunov exponent can be regarded as convergence or divergence criterion. Thus a negative Lyapunov exponent suggests a stable process, because of the independence of the initial state the same asymptotic behavior is achieved. Whereas in the case of a positive Lyapunov exponent the long time behavior is sensitive with regard to the initial state. In that case we have an unstable process. Because we have no information about the true functional relationship, we suggest a default modeling in each case. This first modeling can be regarded as starting point for a more detailed analysis of the underlying process. More precisely:

- Given a stable process, then $\lambda(x_0) < 0$ (see Appendix A).

  A stable process like a mean stationary and average ergodic process could be used for a default modeling.

- Given a process like a random walk, then $\lambda(x_0) \approx 0$ (because of $\lim_{N\to\infty} \frac{1}{N}\sum_{i=0}^{N-1} \ln|1| = 0$).

  In this case the random walk is a good choice for the default modeling.

- Given an unstable process, then $\lambda(x_0) > 0$.

  The trajectories of two different, nearby initial points diverge exponentially on average by a factor of $e^{N\lambda}$ after N iterations. In this case, the limiting behavior is not uniform, it is unstable and in literature it is denoted by *strange attractor* (Eckmann and Ruelle 1982), (Grassberger and Procaccia 1983a) (Grassberger and Procaccia 1983b).

  An unstable process lika a chaotic process could be used for a default modeling (see details about chaotic time series in (Tong 1993)).

5 Experimental results

The Lyapunov exponent gives a clue for the classification between stable and unstable processes. Its computation is manageable, if the functional relationship of the time series is given. However, the underlying process is generally unknown in case of real-world problems, i.e. the derivative f' in equation 7 of the function f is often unknown. Consequently, it is necessary to evaluate a proper estimator from the given time series. On the one hand f' can be numerically evaluated, on the other hand the divergence of two nearby trajectories can be graphically considered. Various approaches are suggested in the literature (for more details see Abarbanel (1996), Eckmann and Ruelle (1982), Sano and Sawada (1985), Gencay (1996) Wolf, Swinney, and Vastan (1985)), Kantz and Schreiber (1997)). The method, which was implemented by Kantz and Schreiber (1997), is applied to the estimation of the Lyapunov exponent in the following examples.

Note, the Lyapunov exponent is a characteristic of the asymptotic behavior. This property implies that the observations should not be taken in the so-called *transient status*. The series has to be in the asymptotic state for the data to be used for the evaluation of f'. An inadequate evaluation of f' may be due to the observation series still lasting in transient state. Therefore, it is possible that the estimation leads to misclassifications between stable and unstable time series, if noisy data or short time series are given.

5.1 Experiments with stable and unstable data sets

We applied the method of Lyapunov exponent estimation to different functions (see equations (12) and (13)). These functions have the advantage that the exact Lyapunov exponent can be evaluated analytically.

To generate an ergodic, mean stationary stochastic process as an example of a stable process a uniformly distributed noise term, $U[0, 1]$, is added to the functional relationship as follows:

$$x_t = (0.9x_{t-1} + 0.05\epsilon), \epsilon \sim U[0, 1]. \tag{12}$$

A deterministic chaotic time series as an example of an unstable process is created by

$$x_t = (2.5x_{t-1}) \mod 1. \tag{13}$$

In both cases the initial point $x_0 = 0.699$ is used and a sample size of 1024.

The estimation of the Lyapunov exponent yields good results with respect to separation. For the ergodic stochastic process it was estimated $\hat{\lambda} = -0.0945$ with a real Lyapunov exponent of $\lambda = -0.11$, i.e. $\hat{\lambda} < 0$, which describes stable stochastic behavior. For the chaotic process the estimation of the Lyapunov exponent leads to $\hat{\lambda} = 0.92$ ($\lambda = 0.92$), i.e. the property of $\hat{\lambda} \geq 0$ is fulfilled. These examples show, that the Lyapunov exponent estimations can correctly classify the different processes.

5.2 Experiments with short time series

In order to study the influence of different lengths of time series, various lengths were generated from the functions (12) and (13). The aim is to verify the separation in dependence on the length of the data sets. The evaluation of the Lyapunov exponent can yield bad estimators, if the time series is too short, since in this case the processes are likely to be in transient states. Thus, for short time series it is to decide, whether a classification is still possible.

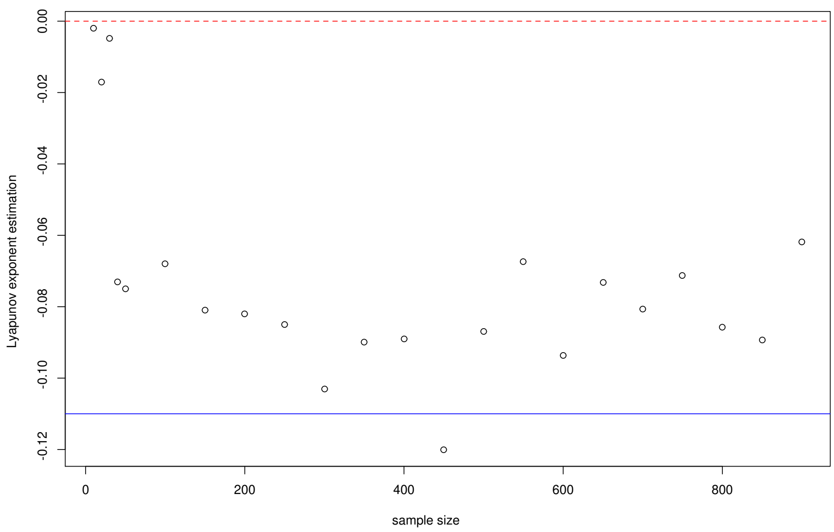

Figure 1: Classifications with respect to the estimation of the Lyapunov exponent for the well-predictable process (12) are correct for all sample sizes.

We applied the method of Lyapunov exponent estimation to sample sizes 10, 20, 30, 40, 50, 100, 150, $200, \ldots, 900$ independent realizations of the stochastic process. Fig. 1 illustrates the estimated Lyapunov exponents in dependence on the sample sizes. The dashed line indicates the classification criterion. Estimates above this line lead to misclassification, estimations below classify correctly. The solid line labels the true Lyapunov exponent.

It is shown, that every estimation leads to the correct classification, even if the values differ from the true Lyapunov exponent. However, the sample sizes of $10, 20$ and 30 are too small for a reliable classification.

For the chaotic time series (13), again sample sizes of 10, 20, 30, 40,

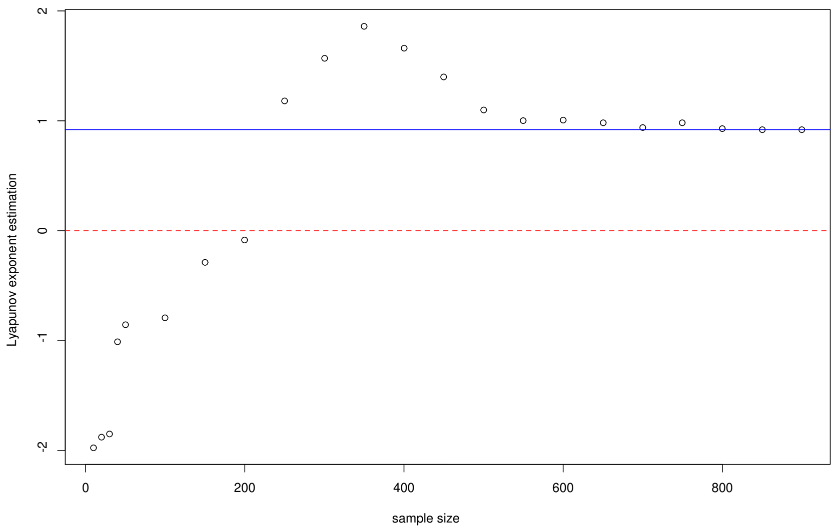

Figure 2: The results of the Lyapunov exponent estimation for the chaotic time series (13) show correct classification only for sample sizes greater than 200.

50, 100, 150, 200, $\ldots$, 900 are used. Again, the dashed line describes the classification line. Estimations below this line yield misclassification. The solid line indicates the true Lyapunov exponent.

It is shown, that short time series lead to misclassifications of the underlying process. The sample sizes of 10 to 200 yield non-correct classifications. However, sample sizes of 250 to 900 characterize the time series correctly. Good and nearly exact estimators are generated by sample sizes of 500 and more.

6 Conclusion

The Lyapunov exponent was been analyzed for stochastic processes with non-necessarily additive noise in the context of a separation between stable and unstable time series. This criterion characterizes the asymptotic behavior of a process. It was shown that the statistical definition of the Lyapunov exponent can be interpreted as an asymptotic characterization of the given stochastic process. Under the condition of ergodicity the ensemble average, is equal to the long time average, and can be used for the Lyapunov exponent estimation.

In this article examples of stochastic and chaotic functions were inspected with respect to separation. It was shown that the estimation yields correct classifications both for ergodic stochastic and chaotic processes. In case of unstable time series the estimator was even evaluated exactly with respect to the true Lyapunov exponent.

Several sample sizes were studied in order to analyze the effect of short time series. It could not be expected due to the transient state that the separation would be always correctly evaluated. In fact short chaotic time series yield misclassifications. However, in the case of ergodic stochastic time series the estimations never lead to misclassifications.

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A Appendix

The Lyapunov-Exponent of a stable deterministic or stochastic process is smaller than 0 (see Sec. 4).

Extended Proof: We distinguish between deterministic and stochastic processes and between a fixed point and a periodical orbit.

a) Deterministic process, fixed point:

   Given a deterministic process $x_{t+1} = f_t(x_0) = f(x_t)$ (cp. equation (1)). The initial point x_0 belongs to the attraction zone of the fixed point x^0 implies

$$
\begin{aligned}
\lambda(x_0) &= \lambda(x^0). \\
\lambda(x^0) &= \lim_{N\to\infty} \frac{1}{N} \sum_{i=0}^{N-1} \ln|f'(x^0)| \qquad \text{(Lyapunov exponent,} \\
&\qquad \text{started in the fixed point)} \\
&= \lim_{N\to\infty} \frac{1}{N} N \ln|f'(x^0)| \qquad \text{(sum is independent of } i\text{)} \\
&= \lim_{N\to\infty} \ln|f'(x^0)| \\
&= \ln|f'(x^0)| \qquad \text{(independence of } N\text{)} \\
&= \ln\left|\lim_{x\to x^0} \frac{f(x)-f(x^0)}{x-x^0}\right| \\
&< \ln\frac{\epsilon}{\delta} \qquad \text{(see equation (2) and the comment that} \\
&\qquad \text{all observations of the trajectory} \\
&\qquad \text{lie in a } \epsilon\text{-neighborhood} \\
&\qquad \text{of the fixed point for any } \delta > \epsilon > 0) \\
&\leq \ln 1 \qquad (\epsilon \text{ is smaller than } \delta \text{, see eq. (2))} \\
&= 0
\end{aligned}
\tag{14}
$$

See Section 2 and Jetschke (1989), p.117.

b) Stochastic process, fixed point:

Given a stochastic process $X_{t+1} = f_t(X_0, \epsilon) = f(X_t, \epsilon)$ (cp. equation (3)). Let the initial point x_0 belong to the attraction zone of the fixed point x^0.

Primarily we have to show that $\lambda(x_0) \leq \lambda(x^0)$ in terms of unbiased estimations.

$$
\begin{aligned}
\lambda(x_0) = & \lim_{N\to\infty} \frac{1}{N} \sum_{i=0}^{N-1} \ln|f'(x_i)| \qquad \text{(Lyapunov exponent,} \\
& \text{started in the initial point)} \\
= & \lim_{N\to\infty} \frac{1}{N} \sum_{i=0}^{N-1} g(x_i) \qquad \text{(with transformation } g(x) = \ln|f'(x)|\text{)} \\
= & \, E[g(X)] \qquad \text{(cp. equation 6)} \\
\leq & \, g[E(X)] \qquad \text{(Jensen inequality with a concave function)} \\
= & \, g(x^0) \qquad \text{(the expected value of } X \text{ is set to be } x^0, E(X) = x^0\text{)} \\
= & \ln|f'(x^0)| \qquad \text{(backward transformation)} \\
= & \, \lambda(x^0) \qquad \text{(cp. deterministic case)}
\end{aligned} \tag{15}
$$

The further proof is analogical to the deterministic case. The Lyapunov exponent started in the initial point is smaller than 0.

c) Deterministic process, periodical orbit:

Given a deterministic process $x_{t+1} = f_t(x_0) = f(x_t)$ (cp. equation (1)). The initial point x_0 belongs to the attraction zone of a periodical orbit with period K, $\overline{x^1}, ..., \overline{x^K}$. For a periodical orbit holds true:

$\overline{x^k} = \overline{x^{k+K}}$ and $\overline{x^{k+1}} = \overline{x^{k+1+K}}$ and $\overline{x^{k+2}} = \overline{x^{k+2+K}}$ and so on.

In addition, this means that:

$f'(\overline{x^k}) = (f^{K+1})'(\overline{x^k})$ and $f^{K+1}(\overline{x^k}) = f^K(\overline{x^{k+1}})$.

It holds true:

$$
\begin{aligned}
|(f^K)'(\overline{x^{k+1}})| = & \left| \lim_{\Delta x\to 0} \frac{f^K(\overline{x^{k+1}} + \Delta x) - f^K(\overline{x^{k+1}})}{\Delta x} \right| < 1 \qquad \text{(periodical orbit,} \\
& \text{cp. deterministic process, fixed point)} \\
|(f^{K+1})'(\overline{x^k})| < 1 & \qquad \text{(because of } f^{K+1}(\overline{x^k}) = f^K(\overline{x^{k+1}})\text{)} \\
& \text{and the independence of k)} \\
|f'(\overline{x^k})| < 1 & \qquad \text{(because of } f'(\overline{x^k}) = (f^{K+1})'(\overline{x^k})\text{)}.
\end{aligned} \tag{16}
$$

For the Lyapunov exponent follows:

$$
\begin{aligned}
\lambda(x_0) &= \frac{1}{K}\sum_{k=1}^{K} \ln|f'\overline{(x^k)}| \qquad \text{(attraction zone of the periodical orbit)} \\
&< \frac{1}{K}\cdot K \cdot \ln 1 \qquad \text{(cp. deterministic case, fixed point)} \\
&< 0.
\end{aligned}
\tag{17}
$$

d) Stochastic process, periodical orbit:

Given a stochastic process $X_{t+1} = f_t(X_0, \epsilon) = f(X_t, \epsilon)$ (cp. equation 3). The initial point x_0 belongs to the attraction zone of a periodical orbit with the period K, $\overline{(x^1)}, ...\overline{(x^K)}$.

Without loss of generality holds true
$\lambda(x_0) \leq \lambda(\overline{x^k})$ (cp. stochastic case, fixed point in a local view).

Consequently, for the Lyapunov exponent of a stochastic process with a periodical orbit follows:

$$
\begin{aligned}
\lambda(x_0) &\leq \frac{1}{K}\sum_{i=1}^{K} \ln|f'\overline{(x^i)}| \qquad \text{(cp. deterministic case, periodical orbit)} \\
&< \frac{1}{K}\cdot K \cdot \ln 1 \qquad \text{(cp. deterministic case, fixed point)} \\
&< 0.
\end{aligned}
\tag{18}
$$