On the existence of moments - With an application to German stock returns

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Abstract

Stock returns are often modeled as having infinite second or fourth moments, with consequences for test statistics which have not yet been fully explored. Conclusions on the existence of moments are usually drawn from a generalized Pareto or simple Pareto tail index estimate. In a recent study McCulloch (1997) demonstrated that this estimator indicates distributions with even finite fourth moments, although the samples were drawn from infinite-variance stable laws, which points out the doubtful role of the tail index estimate as evidence for the finiteness of moments. Based on an fQ-System for continuous unimodal distributions, introduced by Scheffner (1998), we derive an alternative condition for the existence of moments. An estimation algorithm for the fQ-parameters is proposed and an application to the 30 most busy German stocks shows that daily returns can be modeled as being at least approximately fQ-distributed with finite second moments.

Key words: Tail estimation; fQ-System; Distribution of stock returns

JEL classification: C13

1 The problem

Let

$$X_i := \ln(P_i/P_{i-1}) \quad , i = 2, \ldots, n \ , \tag{1}$$

be the returns of some common stock, where $\ln(\cdot)$ is the natural logarithm and P_i is the price in period i, adjusted for dividends, stock splits etc.. Since returns are the cumulative outcome of a large number of individual decisions arriving continuously in time, they can be regarded as the sum of iid random variables. Following the Central Limit Theorem the limiting distribution of the returns, after suitable shifting and scaling, must be a member of the stable class (Zolotarev, 1986, chap.1). Since the sum of returns as defined in (1) should belong to the same class of distributions as the returns themselves, it is reasonable to assume that stock returns are at least approximately governed by a stable law.

Most statistical analyses of stock returns assume a normal distribution, which is the most familiar stable distribution. However, distributions of observed returns are much more leptokurtic than is consistent with normality, and since Mandelbrot (1963) and Fama (1965) stock returns have often been modelled as having a tail behaviour of the asymptotic Pareto-Lévy form, i.e.

$$\lim_{x \to -\infty} \mathrm{P}(X_i < x) = c_1 \eta^\alpha |x|^{-\alpha} [1 + o(1)] \quad , x < 0 \ , \tag{2}$$

$$\lim_{x \to \infty} \mathrm{P}(X_i > x) = c_2 \eta^\alpha |x|^{-\alpha} [1 + o(1)] \quad , x > 0 \ , \tag{3}$$

as $|x| \to \infty$, where η is a scale parameter, and the symmetry parameters c_1 and c_2 satisfy $c_1, c_2 \geq 0$, $c_1 + c_2 > 0$. The most important parameter is the tail index or shape parameter α, which is the maximal moment exponent of the distribution, i.e.

$$\alpha = \sup\{\, \delta > 0 : \; \mathrm{E}|X_i|^{\delta} < \infty \,\} \,. \tag{4}$$

Note that when $0 < \alpha < 2$, (2) and (3) are necessary and sufficient conditions for X_i to belong to the normal domain of attraction of a stable law with characteristic exponent α (Ibragimov and Linnik, 1971, p.76ff). Of course stable random variables themselves follow (2) and (3) and therefore they lie in their own domain of attraction. When $\alpha \geq 2$, X_i is in the domain of attraction of a normal distribution, and when $2 < \alpha < 4$ it is important to note that X_i^2 lies in the normal domain of attraction of a stable law with characteristic exponent $\alpha/2 < 2$, so that the partial sums of X_i^2, appropriately standardized, no longer converge weakly to a normal distribution (Phillips and Loretan, 1991). For further information about domains of attraction, normal domains of attraction, and stable distributions see e.g. Ibragimov and Linnik (1971, chap.2) or Zolotarev (1986).

However, the nonexistence of moments of returns or squared returns of order $\delta \geq \alpha$ in (4) is of crucial relevance for the asymptotic distribution of test statistics, of which the asymptotic theory in the standard case relies on finite second or fourth moments. Based on fundamental results by Davis and Resnick (1985a, 1985b, 1986), who provide a general theory for sample covariances when the tail behaviour of the underlying distribution satisfies (2) and (3) with $0 < \alpha < 2$ and $2 \leq \alpha < 4$, respectively, asymptotic null distributions of several tests have been dicussed in recent years: Phillips and Loretan (1991) consider the Durbin-Watson and the von Neumann ratio, Krämer and Runde (1991) the autocorrelation coefficient, and Phillips and Hajivassiliou (1987) and Krämer and Runde (1992) focus on the t-statistic under these nonstandard assumptions. The asymptotic null distribution of the F-statistic and the Box-

Pierce Q-statistic is derived by Runde (1993, 1997), and that of the χ^2-statistic by Mittnik et al. (1996). Loretan and Phillips (1994) suggest a procedure to test for covariance stationarity, and tests for cointegration and Dickey-Fuller unit root tests are investigated by Caner (1995) and Mittnik and Kim (1996), respectively, to mention only some of the results.

From practical point of view all these methods are adaptive tests, where the value of the tail index α has to be specified from a given series of returns on the first stage of the procedure. Concluding from this result on the existence of moments, either the standard tests or the modified procedures have to be applied on the second stage, depending on whether the second or fourth moments are finite or not.

A well established method in empirical work is to estimate the tail index α directly from the data and to conclude from (4) that all moments of order $\delta < \hat{\alpha}$ exist. A convenient, easy to implement, and therefore most favoured estimator in empirical finance is

$$\hat{\alpha}_{k,u} = \left[\frac{1}{k} \sum_{j=1}^{k} \Big(\ln(X_{(n-j+1)}) - \ln(X_{(n-k)}) \Big) \right]^{-1} , \tag{5}$$

where k is some integer and $X_{(1)} \leq X_{(2)} \leq \ldots \leq X_{(n)}$ are the ascending order statistics corresponding to the sample of n consecutive stock returns. This estimator was originally proposed by Hill (1975) as conditional maximum likelihood estimator of the maximum moment exponent α. The index 'u' in $\hat{\alpha}_{k,u}$ points out that (5) only estimates the upper tail shape. For asymmetric distributions the lower tail must be considered separately, and it is usually concluded that all moments of order $\delta < \min(\,\hat{\alpha}_{k,u},\ \hat{\alpha}_{k,l}\,)$ are finite, where $\hat{\alpha}_{k,l}$ is the analogue to (5), applied to the absolute values of the first k order statistics. Note that $\hat{\alpha}_{GP} = 1/\hat{\alpha}_{k,u}$ is an estimator for the upper tail index of a generalized Pareto distribution (for details see DuMouchel, 1983). For symmetric or approximately symmetric distributions α_{GP} is estimated by merging the upper sample fraction with the absolute values of the lower sample fraction.

Applying the Hill estimator for tail index estimation leads to the question how to choose the cut-off value k in (5). Some empirical applications use the 10% upper observations (DuMouchel, 1983; Akgiray and Booth, 1988), while others just try different sample fractions (Hols and de Vries, 1991; Loretan and Phillips, 1994). A recent study by Lux (1997) reports an almost steady decrease of $\hat{\alpha}_{k,u}$ with increasing values of k. Tail shape estimates from 3.71 at the 0.5% tail size to 2.08 at the 15% tail size indicate that $\hat{\alpha}_{k,u} < 2$ could be obtained using larger sample fractions, which may lead to considerable consequences for the test procedures mentioned above. However, this element of uncertainty seems to be eleminated by Beirland et al. (1996), who propose an algorithm to find an optimal sample fraction k^*.

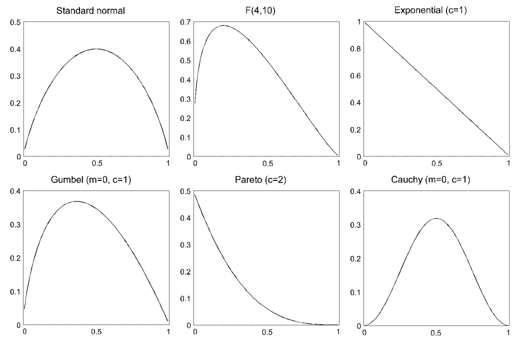
Dewachter and Gielens (1994) showed that the Hill estimator (5) provides a biased estimator of the degrees of freedom when the true distribution is Student's t, and in a recent study McCulloch (1997) demonstrated that tail index estimates in excess of 2 are to be expected for iid symmetric stable samples with α as low as 1.65, i.e. $\hat{\alpha}_{k,u}$ indicates a finite second moment although the true distribution is stable with infinite variance. McCulloch stated that the results of the works which estimate the maximal moment exponent above 2 are therefore in no way inconsistent with a stable distribution for asset returns and recommended the maximum likelihood estimator if the distribution is truly stable.

Given any return series, these findings now result in an uncertainty in deciding whether the second or fourth moment of the underlying distribution exists: The apriori assumption of a stable law is too restrictive, while a tail index estimate above 2 is far from evidence for a finite variance. Assuming returns only to be continuous and unimodal, we show a way out of this dilemma: Based on an fQ-System which is introduced in Section 2 we derive a nessecary and sufficient condition for the existence of moments in Section 3. An algorithm to estimate the fQ-parameters is proposed in Section 4 and Section 5 reports an application to the 30 most busy German stocks.

2 The fQ-System

Let X be a continuous unimodal distributed random variable with density f and distribution function F, where unimodality means that there exists a point x_{mod}, such that the density function $f(x)$ is monotone increasing for $x \leq x_{mod}$ and monotone decreasing for $x \geq x_{mod}$. Note that this definition of unimodality also includes e.g. the exponential and the uniform distributions. Let μ be the location parameter and σ the scale parameter of X, which are not nessecarily the expectation and the variance, respectively.

Figure 1: f-F-plots of various well-known distributions



Considering an f-F-plot, where the density $f(x)$ is plotted against the distribution function $F(x)$, one always obtains curves similar to those in figure 1, which shows f-F-plots of some selected distributions. All curves in figure 1 look similar to density functions of Beta distributions (see e.g. Johnson et al., 1994a, p. 220), so we suggest to fit a common parametric function to these curves of the form

$$F'(x) = f(x) = \sigma F(x)^{\beta}[1 - F(x)]^{\gamma} \quad , \; x \in \mathbb{R} , \tag{6}$$

where $\sigma > 0$, $\beta, \gamma \geq 0$, and F' denotes the derivative of F. This differs from the beta distribution in that we have to take the additional parameter σ into account, since $F(x)^{\beta}[1 - F(x)]^{\gamma}$ is not nessecarily a density. Replacing the unknown parameters by their estimates and solving the differential equation (6) numerically provides a nonparametric density estimator, which can be shown to be in some sense superior to kernel density estimates (Scheffner and Runde, 1998). However, substituting $x = Q(p) := F^{-1}(p)$, we get

$$fQ(p) := f(Q(p)) = \sigma p^{\beta}(1 - p)^{\gamma} \quad , \; 0 \leq p \leq 1 . \tag{7}$$

This generalisation of Parzen's (1979) fQ-tail-representation is called the fQ-function in what follows.

Let $Q_X(p)$ and $Q_Y(p)$ be the quantile functions of two continuous random variables X and $Y := a + X$, $a \in \mathbb{R}$. Then

$$Q_Y(p) = a + Q_X(p) ,$$

which implies

$$Q'_Y(p) = Q'_X(p) .$$

Therefore,

$$Q'(p) = \frac{1}{fQ(p)} \;, \tag{8}$$

and hence

$$fQ_Y(p) = fQ_X(p) \;, \quad \forall a \in \mathbb{R} \;,$$

which shows that the fQ-function is independent of the location parameter. Without loss of generality we therefore set $\mu = Q(1/2)$, which will later turn out to be an appropriate choice for the representation of the quantile function.

DEFINITION 2.1. *Let $X \in \mathbb{R}$ be a continuous unimodal random variable. X is said to be <u>fQ-distributed</u> with parameters $\mu \in \mathbb{R}$, $\sigma > 0$, $\beta, \gamma \geq 0$, in short $X \sim \mathcal{F}(\mu, \sigma, \beta, \gamma)$, if the fQ-function of X is given by*

$$fQ(p) = \sigma p^{\beta}(1-p)^{\gamma} \quad , \; 0 \leq p \leq 1 \;, \quad \textit{and} \quad Q(1/2) = \mu \;. \tag{9}$$

DEFINITION 2.2. *The set of all fQ-distributions*

$$\mathcal{FQ} := \{X \in \mathbb{R} : \; X \sim \mathcal{F}(\mu, \sigma, \beta, \gamma), \; \mu \in \mathbb{R}, \; \sigma > 0, \; \beta, \gamma \geq 0\}$$

is called <u>fQ-System</u>.

Table 1 gives the density functions $f(x)$ and the fQ-functions $fQ^*(p)$ of some well-known distributions (for details see e.g. Johnson et al., 1994a, 1994b). Only four distributions in table 1 fit exactly into the fQ-System: $\sigma = 1$, $\beta = \gamma = 0$ results in the uniform distribution, $\sigma = 1$, $\beta = 0$ and $\gamma = 1$ in the exponential distribution, $\sigma = \beta = \gamma = 1$ in the logistic distribution, and $\sigma = c^{-1}$, $\beta = 0$ and $\gamma = c + 1$, $c > 0$, in the Pareto distribution.

Table 1: Densities and fQ-functions for various distributions

Distribution	$f(x)$	$fQ^*(p)\ ,\ p \in [0,1]$
Uniform	$1\ ,\quad x \in [0,1]$	1
Exponential	$\mathrm{e}^{-x}\ ,\quad x > 0$	$1-p$
Logistic	$\mathrm{e}^{x}(1+\mathrm{e}^{x})^{-2}\ ,\quad x \in \mathbb{R}$	$p(1-p)$
Weibull	$cx^{c-1}\mathrm{e}^{-x^c}\ ,\quad x > 0,\ c > 0$	$c(1-p)\ln(\frac{1}{1-p})^{1-1/c}$
Gumbel	$\mathrm{e}^{-x}\mathrm{e}^{-\mathrm{e}^{-x}}\ ,\quad x \in \mathbb{R}$	$(1-p)\ln(\frac{1}{1-p})$
Normal	$(2\pi)^{-1/2}\mathrm{e}^{-x^2/2}\ ,\quad x \in \mathbb{R}$	$(2\pi)^{-1/2}\mathrm{e}^{-1/2\vert\Phi^{-1}(p)\vert}$
Cauchy	$(\pi(1+x^2))^{-1}\ ,\quad x \in \mathbb{R}$	$\pi^{-1}(\cos(\pi(p-1/2)))^2$
Pareto	$(cx^{1+1/c})^{-1}\ ,\quad x > 1,\ c > 0$	$c^{-1}(1-p)^{1+c}$

From the last column in table 1 it is obvious that the fQ-functions of most of the well-known distributions can only be approximated by the three-parametric function given in (7). For a short-tailed, a skewed and a heavy-tailed distribution, figure 2 shows the fQ-functions and their approximations, obtained by minimizing the distance

$$d(fQ, fQ^*) := \left(\int_0^1 \left(fQ(p) - fQ^*(p)\right)^2 dp\right)^{1/2} \tag{10}$$

over all fQ-functions of the form (7) with respect to σ, β and γ (for details see Scheffner, 1998). For the distributions considered here the exact and approximated fQ-functions are hardly distinguishable. Figure 3 gives the differences between these two curves.

Figure 2: fQ-functions (solid line) and their approximations (dashed line) for various distributions

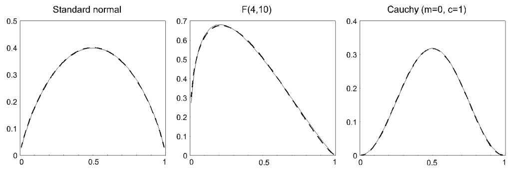
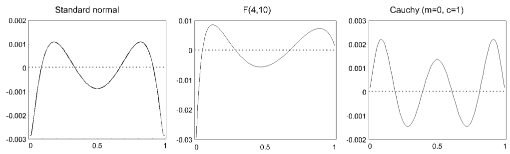


Figure 3: Differences between fQ-functions and their approximations



Since any returns as defined in (1) can safely be assumed continuous and unimodal distributed, their distribution can be approximated by an fQ-distribution as given in definition 1.1. In fact in section 5 it is shown that the daily returns considered here seem to follow fQ-distributions. Even if returns are not exactly fQ-distributed, the fQ-System provides an approximation of the true underlying distribution. This approximation generally produces a bias, but as figure 3 points out, this bias is negligible compared to the gain of a given parametric fQ-function. In particular, the fQ-parameters β and γ can be used to decide upon the existence of moments.

3 A condition for the existence of moments

Let $X \sim \mathcal{F}(\mu, \sigma, \beta, \gamma)$ with quantile function $Q(p)$. The relations (7) and (8) imply that

$$Q'(p) = \frac{1}{fQ(p)} = \sigma^{-1} p^{-\beta} (1-p)^{-\gamma} , \tag{11}$$

from which the quantile function of an fQ-distribution is obtained by numerical integration as

$$Q(p) = \mu - \frac{1}{\sigma} \int_p^{1/2} x^{-\beta} (1-x)^{-\gamma} dx , \quad p \in (0, 1/2] , \tag{12}$$

$$Q(p) = \mu + \frac{1}{\sigma} \int_{1/2}^{p} x^{-\beta} (1-x)^{-\gamma} dx , \quad p \in (1/2, 1) . \tag{13}$$

The following lemma gives lower and upper bounds for the quantile function in (12) and (13) which are needed for the central theorem in this section.

LEMMA 3.1. *Let $X \sim \mathcal{F}(\mu, \sigma, \beta, \gamma)$ with quantile function $Q(p)$. Then for $p > \frac{1}{2}$ and*

(a) $\gamma \neq 1$:

$$\frac{(1-p)^{1-\gamma} - 2^{\gamma-1}}{\sigma(\gamma-1)} \; \leq \; Q(p) - \mu \; \leq \; \frac{2^{\beta}[(1-p)^{1-\gamma} - 2^{\gamma-1}]}{\sigma(\gamma-1)} , \tag{14}$$

(b) $\gamma = 1$:

$$\frac{1}{\sigma} [\ln(1/2) - \ln(1-p)] \; \leq \; Q(p) - \mu \; \leq \; \frac{2^{\beta}}{\sigma} [\ln(1/2) - \ln(1-p)] . \tag{15}$$

For $p < \frac{1}{2}$ and

(c) $\beta \neq 1$:

$$\frac{2^{\beta-1} - p^{1-\beta}}{\sigma(\beta-1)} \; \geq \; Q(p) - \mu \; \geq \; \frac{2^{\gamma}[2^{\beta-1} - p^{1-\beta}]}{\sigma(\beta-1)} , \tag{16}$$

(d) $\beta = 1$*:*

$$\frac{1}{\sigma}\left[\ln(p) - \ln(1/2)\right] \;\geq\; Q(p) - \mu \;\geq\; \frac{2^\gamma}{\sigma}\left[\ln(p) - \ln(1/2)\right] . \tag{17}$$

PROOF. We only prove the first part of the lemma. The case $p < 1/2$ is proved along the same lines. For $p > 1/2$ the quantile function is given by (13):

$$Q(p) = \mu + \frac{1}{\sigma}\int_{1/2}^{p} x^{-\beta}(1-x)^{-\gamma}dx .$$

Since $x^{-\beta}$ is monotone decreasing if $x \to 1$, we have for $x > 1/2$

$$1 \;\leq\; x^{-\beta} \;\leq\; 2^{\beta} ,$$

which implies

$$\frac{1}{\sigma}\int_{1/2}^{p}(1-x)^{-\gamma}dx \;\leq\; Q(p) - \mu \;\leq\; \frac{2^{\beta}}{\sigma}\int_{1/2}^{p}(1-x)^{-\gamma}dx .$$

With

$$\int_{1/2}^{p}(1-x)^{-\gamma}dx \;=\; \begin{cases} \frac{(1-p)^{1-\gamma}-2^{\gamma-1}}{\sigma(\gamma-1)} & , \gamma \neq 1 \\ \ln(1/2) - \ln(1-p) & , \gamma = 1 \end{cases}$$

the first part of the lemma is proved. □

Let X be a continuous random variable with density $f(x)$ and $g(x)$ is a continuous function on $\mathbb{R}$. Substituting $x = Q(p)$ and using the relation (8), we have

$$\begin{aligned} \mathrm{E}[g(X)] \;&=\; \int_{-\infty}^{\infty} g(x)f(x)\,dx \\ &=\; \int_{0}^{1} g(Q(p))\,dp , \end{aligned}$$

which implies that moments of X are easily expressed in terms of the quantile function

$$\mathrm{E}(X^\rho) = \int_0^1 [Q(p)]^\rho dp \,. \tag{18}$$

Together with lemma 3.1 this leads to a nessecary and sufficient condition for the existence of moments of fQ-distributed random variables.

THEOREM 3.2. *Let $X \sim \mathcal{F}(\mu, \sigma, \beta, \gamma)$, then*

$$\mathrm{E}(X^\rho) < \infty \;\Leftrightarrow\; \max(\beta, \gamma) < \frac{\rho + 1}{\rho} \,, \quad \rho \in \mathbb{N} \,. \tag{19}$$

PROOF. Since we are only interested in the tail behaviour we set without loss of generality $\mu = 0$ and $\sigma = 1$. From (18) we have

$$\begin{aligned} \mathrm{E}|X^\rho| < \infty \;&\Leftrightarrow\; \int_0^1 |[Q(p)]^\rho| \, dp < \infty \\ &\Leftrightarrow\; \int_0^{1/2} |[Q(p)]^\rho| \, dp + \int_{1/2}^1 |[Q(p)]^\rho| \, dp < \infty \\ &\Leftrightarrow\; \int_{1/2}^1 |[Q(1-p)]^\rho| \, dp + \int_{1/2}^1 |[Q(p)]^\rho| \, dp < \infty \,. \end{aligned} \tag{20}$$

Since $Q(p) = \int_{1/2}^p x^{-\beta}(1-x)^{-\gamma} dx$ for $p > 1/2$, it is sufficient to consider the second integral in (20) and then transfering the conditions obtained for (β, γ) to (γ, β). We have to distinguish three cases:

1. $\gamma < 1$: From (14) we have

$$\begin{aligned} \int_{1/2}^1 |[Q(p)]^\rho| \, dp \;&\le\; \frac{2^\beta}{\gamma - 1} \int_{1/2}^1 |[(1-p)^{1-\gamma} - (1/2)^{1-\gamma}]^\rho| \, dp \\ &=\; \frac{2^\beta}{\gamma - 1} \int_0^{1/2} |[q^{1-\gamma} - (1/2)^{1-\gamma}]^\rho| \, dq \\ &=\; \frac{2^\beta}{\gamma - 1} \int_0^{1/2} |\sum_{j=0}^{\rho} (-1)^j \binom{\rho}{j} \underbrace{q^{(1-\gamma)(\rho-j)}}_{\le 1} (1/2)^{(1-\gamma)j}| \, dq \\ &\le\; \frac{2^{\beta-1}}{\gamma - 1} \sum_{j=0}^{\rho} \binom{\rho}{j} (1/2)^{(1-\gamma)j} \, dq \\ &<\; \infty \qquad \forall\, \beta > 0, \, \rho \in \mathbb{N} \,. \end{aligned}$$

2. $\gamma = 1$: From (15) we have

$$\begin{aligned}
\int_{1/2}^{1} |[Q(p)]^\rho|\, dp &\le 2^\beta \int_{1/2}^{1} [\ln(1/2) - \ln(1-p)]^\rho \, dp \\
&= 2^{\beta-1} \int_0^\infty u^\rho \mathrm{e}^{-u}\, du \\
&= 2^{\beta-1} \rho! \\
&< \infty \qquad \forall\, \beta > 0,\ \rho \in \mathbb{N}\ .
\end{aligned}$$

3. $\gamma > 1$: From (14) we have

$$\begin{aligned}
\int_{1/2}^{1} |[Q(p)]^\rho|\, dp &\le \lim_{t\to 1} \frac{2^\beta}{\gamma-1} \int_{1/2}^{t} |[(1-p)^{1-\gamma} - (1/2)^{1-\gamma}]^\rho|\, dp \\
&= \lim_{t\to 0} \frac{2^\beta}{\gamma-1} \int_{t}^{1/2} |\sum_{j=0}^{\rho} (-1)^j \binom{\rho}{j} \underbrace{q^{(1-\gamma)(\rho-j)}}_{\le q^{(1-\gamma)\rho}} (1/2)^{(1-\gamma)j}|\, dq \\
&\le \underbrace{\frac{2^\beta}{\gamma-1} [\sum_{j=0}^{\rho} \binom{\rho}{j} (1/2)^{(1-\gamma)j}]}_{<\infty \quad \forall\, \beta>0,\, \gamma>1,\, \rho\in\mathbb{N}} \lim_{t\to 0} \int_t^{1/2} q^{(1-\gamma)\rho}\, dq\ .
\end{aligned}$$

With

$$\begin{aligned}
\lim_{t\to 0} \int_t^{1/2} q^{(1-\gamma)\rho}\, dq < \infty \quad &\Leftrightarrow \quad (1-\gamma)\rho > -1 \\
&\Leftrightarrow \quad \gamma < 1 + \frac{1}{\rho}
\end{aligned}$$

the theorem is proved, since with (16) and (17) the first integral in (20) can be computed similar to the second, which gives the condition $\beta < 1 + 1/\rho$. □

EXAMPLE 3.3. Consider the distance $d(fQ, fQ_C)$ in (10), where fQ_C is the fQ-function of the Cauchy distribution given in table 1. Minimizing $d(fQ, fQ_C)$ over all fQ-functions of the form $fQ(p) = \sigma p^\beta (1-p)^\gamma$ with respect to σ, β and γ, i.e

$$(\sigma_C, \beta_C, \gamma_C) = \arg\min_{(\sigma,\beta,\gamma)} \left(\int_0^1 [\sigma p^\beta (1-p)^\gamma - \pi^{-1}(\cos(\pi(p-1/2)))^2]^2\, dp \right)^{1/2},$$

results in $\beta_C = \gamma_C = 2.368$. This leads to $\rho_{max} = 0$, where ρ_{max} is the maximal finite moment, i.e.

$$\rho_{max} = \sup\{\rho \in \mathbb{N} \,:\, \max(\beta, \gamma) < \frac{\rho + 1}{\rho}\} \,. \tag{21}$$

Minimizing $d(fQ, fQ_N)$, where fQ_N is the fQ-function of the standard normal distribution, provides $\beta_N = \gamma_N = 0.807$ with $\rho_{max} = \infty$.

4 Parameter estimation

Since the particular definition of the fQ-distribution renders the maximum likelihood and the method of moments estimation techniques impracticable, an iteration algorithm is used to estimate the parameters of the fQ-System. Starting with some initial values $\beta^{(0)}$ and $\gamma^{(0)}$ we obtain estimates for μ and σ based on the Least-Squares-Method. Minimizing the sum of the weighted squared residuals over β and γ provides $\beta^{(1)}$ and $\gamma^{(1)}$ with which the next iteration step is started. Finally, estimates $\hat{\mu}$, $\hat{\sigma}$, $\hat{\beta}$ and $\hat{\gamma}$ are obtained when the iteration algorithm terminates.

4.1 Moments of order statistics

The Least-Squares estimation of μ and σ requires the expectation and the variance of the i-th order statistic and the covariance of the i-th and j-th order statistics of the underlying distribution. Let $X_{(i)}$ denote the i-th order statistic of a sample X_i, $i = 1, \ldots, n$. To obtain representations of $\mathrm{E}(X_{(i)})$ and $\mathrm{Cov}(X_{(i)}, X_{(j)})$ in the case of fQ-distributions, we refer to Blom (1958).

THEOREM 4.1. *Let* $X_i \sim \mathcal{F}(\mu, \sigma, \beta, \gamma),\ i = 1, \ldots, n$. *Then for* $\beta/2 < i < n + 1 - \gamma/2$ we have

$$\mathrm{E}(X_{(i)}) = Q(\pi_i) + O(n^{-3/2}) \ , \tag{22}$$

where

$$\pi_i := \frac{i - \frac{\beta}{2}}{n + 1 - \frac{\beta+\gamma}{2}} \ . \tag{23}$$

PROOF. From Blom (1958, p.64), the expectation of the i-th order statistic is given by

$$\mathrm{E}(X_{(i)}) = Q(\pi_i) + n^{-1} R(\pi_i, a_i, b_i) + O(n^{-3/2}) \ ,$$

where

$$R(\pi_i, a_i, b_i) := \frac{1}{2}\pi_i(1 - \pi_i)Q''(\pi_i) + (a_i(1 - \pi_i) - b_i\pi_i)Q'(\pi_i) \ , \tag{24}$$

$$\pi_i := \frac{i - a_i}{n + 1 - a_i - b_i} \ , \tag{25}$$

and where a_i, b_i are some scalars depending on i. From (11) we have

$$\begin{aligned} Q'(p) &= \frac{1}{fQ(p)} = \sigma^{-1} p^{-\beta}(1 - p)^{-\gamma} \\ \Rightarrow \quad Q''(p) &= -\frac{fQ'(p)}{fQ^2(p)} \ . \end{aligned}$$

In view of

$$fQ'(p) = fQ(p)\left(\frac{\beta}{p} - \frac{\gamma}{1 - p}\right)$$

it follows immediately that

$$\frac{Q''(p)}{Q'(p)} = -\frac{fQ'(p)}{fQ(p)} = \frac{\gamma}{1-p} - \frac{\beta}{p} \,. \tag{26}$$

Substituting (26) into (24), we have

$$\begin{aligned} R(\pi_i, a_i, b_i) &= \frac{1}{2}\pi_i(1-\pi_i)Q'(\pi_i)\left(\frac{\gamma}{1-\pi_i} - \frac{\beta}{\pi_i}\right) + (a_i(1-\pi_i) - b_i\pi_i)Q'(\pi_i) \\ &= Q'(\pi_i)\left((1-\pi_i)(a_i - \frac{\beta}{2}) + \pi_i(\frac{\gamma}{2} - b_i)\right) \,. \end{aligned}$$

Therefore

$$\begin{aligned} & R(\pi_i, a_i, b_i) = 0 \\ \Leftrightarrow \quad & a_i = \frac{\beta}{2} \quad \text{and} \quad b_i = \frac{\gamma}{2} \,. \end{aligned}$$

Substituting these coefficients into (25) proves the theorem. The restriction on the index i is nessecary to ensure $\pi_i \in (0,1)$. □

A similar result is obtained for the covariances of order statistics.

THEOREM 4.2. *Let $X_i \sim \mathcal{F}(\mu, \sigma, \beta, \gamma)$, $i = 1, \ldots, n$. Then for $\beta/2 < i < n + 1 - \gamma/2$ we have*

$$\mathrm{Cov}(X_{(i)}, X_{(j)}) = \frac{\pi_i(1-\pi_j)}{n + 2 - \frac{\beta+\gamma}{2}} Q'(\pi_i)Q'(\pi_j) + O(n^{-2}) \quad , i < j \,, \tag{27}$$

with π_i given in (25).

PROOF. With the reprensentation of $\mathrm{Cov}(X_{(i)}, X_{(j)})$ given in Blom (1958, p.62), the proof is analoguous to that of theorem 4.1. □

The representations of $\mathrm{E}(X_{(i)})$ and $\mathrm{Cov}(X_{(i)}, X_{(j)})$ in theorem 4.1 and theorem 4.2, respectively, have the advantage that the coefficients a_i and b_i in (25) are given as a function of the fQ-parameters, independently of i and n. Based on the good approximations by fQ-functions, (22) and (27) therefore produce very good approximations of the moments of order statistics for any continuous unimodal distribution (for details see Scheffner, 1998).

4.2 ABLU estimates for μ and σ

To robustify the estimation procedure, we use a $k\%$ trimmed sample, which remains after deleting the $k_1\%$ smallest and $k_2\%$ largest values in the sample, where $k_1 + k_2 = k$ and k_1 and k_2 depend on the initial values $\beta^{(0)}$ and $\gamma^{(0)}$. The following definition simplifies the notation.

DEFINITION 4.3. *Let* $\tilde{X}_i \sim \mathcal{F}(\mu, \sigma, \beta^{(0)}, \gamma^{(0)})$, $i = 1, \ldots, \tilde{n}$ *and let*

$$i_1 := [\beta^{(0)}/2] + 1 \quad \textit{and} \quad i_{\tilde{n}} := [\tilde{n} + 1 - \gamma^{(0)}/2] \ , \tag{28}$$

then $X_1, \ldots, X_n$ *with* $X_{(1)} := \tilde{X}_{(i_1)}$, $X_{(2)} := \tilde{X}_{(i_1+1)}, \ldots, X_{(n)} := \tilde{X}_{(i_{\tilde{n}})}$ *is called the* $\underline{(\beta^{(0)}, \gamma^{(0)})\textit{-trimmed sample.}}$

Suppose $X_i \sim \mathcal{F}(\mu, \sigma, \beta^{(0)}, \gamma^{(0)})$, $i = 1, \ldots, n$ be an $(\beta^{(0)}, \gamma^{(0)})$-trimmed sample of iid random variables with any given $\beta^{(0)}$ and $\gamma^{(0)}$, and let $X_{(i)}$ be the i-th order statistic of the subsample. Consider the standardized variables $Y_i := (X_i - \mu)/\sigma$, i.e. $Y_i \sim \mathcal{F}(0, 1, \beta^{(0)}, \gamma^{(0)})$, then $Y_{(i)} = (X_{(i)} - \mu)/\sigma$, $i = 1, \ldots, n$. Loyd (1952) suggests to estimate μ and σ via Least-Squares based on order statistics. For that purpose let $\mathbf{Y} := (Y_{(1)}, \ldots, Y_{(n)})'$, $\mathbf{a} := \mathrm{E}(\mathbf{Y})$ and $\mathbf{B} := \mathrm{Cov}(\mathbf{Y})$, i.e.

$$a_i = \mathrm{E}(Y_{(i)}) \qquad \text{and}$$

$$b_{ij} := \mathbf{B}_{(i,j)} = \mathrm{Cov}(Y_{(i)}, Y_{(j)}) \ ,$$

where $\mathbf{B}_{(i,j)}$ denotes the element in the i-th row and j-th column of the covariance matrix $\mathbf{B}$.

This implies for the order statistic $X_{(i)}$ that

$$\mathrm{E}(X_{(i)}) = \mu + \sigma a_i \ ,$$

$$\mathrm{Cov}(X_{(i)}, X_{(j)}) = \sigma^2 b_{ij} \ ,$$

and the parameters μ and σ can be estimated using the generalized linear model

$$\mathbf{X} = \mathbf{A}\boldsymbol{\vartheta} + \mathbf{e} \ , \tag{29}$$

where $\mathbf{X} = (X_{(1)}, \ldots, X_{(n)})'$, $\mathbf{A} = (\mathbf{1_n}, \mathbf{a})$, $\boldsymbol{\vartheta} = (\mu, \sigma)'$ and $\mathbf{e} \sim (0, \sigma^2\mathbf{B})$. The best linear unbiased estimator (BLUE) for $\boldsymbol{\vartheta}$ is given by

$$\hat{\boldsymbol{\vartheta}} = (\mathbf{A}'\mathbf{B}^{-1}\mathbf{A})^{-1}\mathbf{A}'\mathbf{B}^{-1}\mathbf{X} \ . \tag{30}$$

With

$$\boldsymbol{\Delta} := \frac{\mathbf{B}^{-1}(\mathbf{1_n a'} - \mathbf{a 1_n'})\mathbf{B}^{-1}}{(\mathbf{a'B}^{-1}\mathbf{a})(\mathbf{1_n' B}^{-1}\mathbf{1_n}) - (\mathbf{a'B}^{-1}\mathbf{1_n})^2} \tag{31}$$

the estimates for μ and σ can be given directly.

LEMMA 4.4. *The best linear unbiased estimators for μ and σ in the generalized linear model (29) are given by*

$$\hat{\mu} \ = \ -\mathbf{a}'\boldsymbol{\Delta}\mathbf{X} \ , \tag{32}$$

$$\hat{\sigma} \ = \ \mathbf{1_n}\boldsymbol{\Delta}\mathbf{X} \ . \tag{33}$$

PROOF. Starting with (30) the representations (32) and (33) for $\hat{\mu}$ and $\hat{\sigma}$ are obtained by simple matrix operations. For details see e.g. Balakrishnan and Cohen (1991, p.80ff.). □

To estimate μ and σ by (32) and (33), both $\mathbf{a}$ and $\mathbf{B}$ have to be computed, which generally raises non-trivial numerical problems (see e.g. Arnold et al., 1992, chapter 4.10). However, for fQ-distributions, $\mathbf{a}$ and $\mathbf{B}$ can be approximated with theorem 4.1 and theorem 4.2 by

$$\mathrm{E}(Y_{(i)}) \;\approx\; Q^{(0)}(\pi_i^{(0)}) \;=:\; \tilde{a}_i \qquad \text{and} \tag{34}$$

$$\begin{aligned} \mathrm{Cov}(Y_{(i)}, Y_{(j)}) \quad &\approx \quad \frac{\pi_i^{(0)}(1-\pi_j^{(0)})}{n+2-(\beta^{(0)}+\gamma^{(0)})/2}\,(Q^{(0)})'(\pi_i^{(0)})(Q^{(0)})'(\pi_j^{(0)}) \\ &=: \quad \tilde{b}_{ij} \quad , i<j \ , \end{aligned} \tag{35}$$

where $Q^{(0)}$ and $(Q^{(0)})'$ is the quantile function and its derivative, respectively, when β and γ are replaced by $\beta^{(0)}$ and $\gamma^{(0)}$, and

$$\pi_i^{(0)} := \frac{i-\beta^{(0)}/2}{n+1-(\beta^{(0)}+\gamma^{(0)})/2} \ . \tag{36}$$

Remember that we consider the $(\beta^{(0)}, \gamma^{(0)})$-trimmed sample and not the original sample, so that the choice of the smallest and largest value in (28) ensures that $\pi_i^{(0)} \in (0,1)$. The approximately best linear unbiased estimators (ABLUE) for μ and σ are now given by lemma 4.4, when a_i and b_{ij} are replaced by $\tilde{a}_i$ and $\tilde{b}_{ij}$, respectively.

THEOREM 4.5. *Let $X_i \sim \mathcal{F}(\mu, \sigma, \beta^{(0)}, \gamma^{(0)})$, $i = 1, \ldots, n$ be an $(\beta^{(0)}, \gamma^{(0)})$-trimmed sample of iid random variables with any given $\beta^{(0)}$ and $\gamma^{(0)}$. Let further*

$$V_1 := \sum_{i=1}^{n} f_i^{(0)}(c_i - c_{i+1}) \ , \qquad V_2 := \sum_{i=1}^{n} \tilde{a}_i f_i^{(0)}(d_i - d_{i+1}) \ ,$$

$$V_3 := \sum_{i=1}^{n} f_i^{(0)}(d_i - d_{i+1}) \ ,$$

$$Z_1 := \sum_{i=1}^{n} f_i^{(0)}(c_i - c_{i+1})x_{(i)} \ , \qquad Z_2 := \sum_{i=1}^{n} f_i^{(0)}(d_i - d_{i+1})x_{(i)} \ ,$$

$$c_i := f_i^{(0)} - f_{i-1}^{(0)} \quad and \quad d_i := \tilde{a}_i f_i^{(0)} - \tilde{a}_{i-1} f_{i-1}^{(0)} \quad , \; i = 1, \ldots, n+1 \, ,$$

where $\tilde{a}_i$ is given by (34) and

$$f_i^{(0)} := fQ^{(0)}(\pi_i^{(0)}) = (\pi_i^{(0)})^{\beta^{(0)}}(1 - \pi_i^{(0)})^{\gamma^{(0)}} \, ,$$

with $\pi_i^{(0)}$ given by (36) and $f_0^{(0)} = f_{n+1}^{(0)} = \tilde{a}_0 f_0^{(0)} = \tilde{a}_{n+1} f_{n+1}^{(0)} := 0$. Then the approximately best linear unbiased estimators for μ and σ are given by

$$\hat{\hat{\mu}} = \frac{V_2 Z_1 - V_3 Z_2}{V_1 V_2 - V_3^2} \, , \tag{37}$$

$$\hat{\hat{\sigma}} = \frac{V_1 Z_2 - V_3 Z_1}{V_1 V_2 - V_3^2} \, . \tag{38}$$

PROOF. Since the element in the i-th row and j-th column of the covariance matrix $\mathbf{B}$ is approximated by

$$\tilde{\mathbf{B}}_{(i,j)} := \tilde{b}_{ij} = \frac{\pi_i^{(0)}(1 - \pi_j^{(0)})}{(n^{(0)} + 2) f_i^{(0)} f_j^{(0)}} \, , \quad i < j \, ,$$

where $n^{(0)} := n - (\beta^{(0)} + \gamma^{(0)})/2$, the inverse $\tilde{\mathbf{B}}^{-1}$ of $\tilde{\mathbf{B}}$ is a tri-diagonal matrix given by

$$\tilde{\mathbf{B}}^{-1}_{(i,j)} = \begin{cases} 2(n^{(0)}+1)(n^{(0)}+2)(f_i^{(0)})^2 & , \; j = i \\ -(n^{(0)}+1)(n^{(0)}+2) f_i^{(0)} f_j^{(0)} & , \; j = i+1 \text{ or } j = i-1 \\ 0 & , \; \text{otherwise} \end{cases} \, . \tag{39}$$

With Lemma 4.4, straightforward calculations lead to the ABLU estimators given by (37) and (38). □

Note that in view of the special structure of the tri-diagonal matrix $\tilde{\mathbf{B}}^{-1}$ the very time expensive computation of the inverse of the $n \times n$ matrix $\tilde{\mathbf{B}}$ is not nessecary.

4.3 The iteration algorithm

Given some initial values $(\mu^{(0)}, \sigma^{(0)}, \beta^{(0)}, \gamma^{(0)}) =: \theta^{(0)}$, the parameters $\theta^{(1)} = (\mu^{(1)}, \sigma^{(1)}, \beta^{(1)}, \gamma^{(1)})$ as starting values for the next step of the iteration algorithm are obtained by applying the Gauss-Newton method to minimize $\hat{\mathbf{e}}'(\widetilde{\mathbf{B}}^{(0)})^{-1}\hat{\mathbf{e}}$ with respect to β and γ, i.e.

$$(\beta^{(1)}, \gamma^{(1)}) = \arg\min_{\beta,\gamma} \; \hat{\mathbf{e}}'(\widetilde{\mathbf{B}}^{(0)})^{-1}\hat{\mathbf{e}} \,, \tag{40}$$

with $\hat{e}_i = (x_{(i)} - \hat{\hat{\mu}} - \hat{\hat{\sigma}}\tilde{a}_i)$, $1 = 1, \ldots, n$, where the ABLU estimators $\hat{\hat{\mu}}$ and $\hat{\hat{\sigma}}$, depending on $\beta^{(0)}$ and $\gamma^{(0)}$, are given by (37) and (38), respectively. The values $\mu^{(1)}$ and $\sigma^{(1)}$ are provided in the last step of the minimization procedure.

Note that the elements $\widetilde{\mathbf{B}}^{-1}_{(i,j)}$ of $\widetilde{\mathbf{B}}^{-1}$ in (39), which depend on β and γ, change in each step of the Gauss-Newton minimization algorithm, while $(\widetilde{\mathbf{B}}^{(0)})^{-1}$ in (40) is kept fixed until the next iteration.

The next step of the iteration starts with $\theta^{(1)}$, and $\theta^{(2)}$ is the solution of (40), when $(\widetilde{\mathbf{B}}^{(0)})^{-1}$ is replaced by $(\widetilde{\mathbf{B}}^{(1)})^{-1}$, i.e. in the s-th step $\theta^{(s)}$ is obtained by minimizing $\hat{\mathbf{e}}'(\widetilde{\mathbf{B}}^{(s-1)})^{-1}\hat{\mathbf{e}}$ with respect to β and γ. The procedure terminates on the r-th step if $||\theta^{(r)} - \theta^{(r-1)}||_2 < \epsilon$, where ϵ can be chosen arbitrarily small. Finally, $(\hat{\mu}, \hat{\sigma}, \hat{\beta}, \hat{\gamma})$ is given by $\theta^{(r)}$.

4.4 Choosing initial values

To start the iteration, initial values $\theta^{(0)} = (\mu^{(0)}, \sigma^{(0)}, \beta^{(0)}, \gamma^{(0)})$ have to be determined to compute the approximations (34) and (35) in the first step of the algorithm. For extreme parameter constellations, for instance skewed distributions with large variances, the Gauss-Newton method is very susceptible to badly chosen initial values. We therefore suggest a data-driven choice for $\theta^{(0)}$.

Since μ and σ are not involved in the minimization algorithm, we set $\mu^{(0)}$ to the median m, $\sigma^{(0)}$ to the empirical standard deviation s and consider the standardized data $y_i = (x_i - m)/s$, $i = 1, \ldots, n$.

With $g_j = j/(l+1)$, $j = 1, \ldots, l$, a simple kernel estimation is applied to estimate the density function of the underlying distribution at the points $Q(g_j) = y_{([ng_j]+1)}$, i.e. to obtain

$$\widehat{fQ}(g_j) = \hat{f}(y_{([ng_j]+1)}) = \frac{1}{nh}\sum_{i=1}^{n} K(\frac{y_{([ng_j]+1)} - y_i}{h}) \quad , j = 1, \ldots, l \; .$$

We use the Gauss kernel $K(u) = (2\pi)^{-1/2}\exp(-u^2/2)$, but any other kernel will do as well. The bandwidth h has to be chosen such that at least three points $Q(g_j)$ are covered. For details concerning kernel estimation see e.g. Silverman (1986). In practice it has been shown that the number of points $l = \min(n, 20)$ is sufficient to obtain good initial values.

Since $\widehat{fQ}(g_j)$ can be regarded as the density estimation of a Beta(a,b)-distribution at points g_j, its empirical moments are

$$m_1 = \sum_{j=1}^{l} g_j \widetilde{fQ}(g_j) \qquad \text{and} \qquad m_2 = \sum_{j=1}^{l} (g_j - m_1)^2 \widetilde{fQ}(g_j) \; ,$$

where $\widetilde{fQ}(g_j) := \widehat{fQ}(g_j) \, / \, \sum_{j=1}^{l} \widehat{fQ}(g_j)$ to obtain a density. Following Johnson et al. (1994a, p.228), the moment estimators of a Beta(a,b)-distribution are given by

$$\hat{a} = \frac{m_1^2(1-m_1)}{m_2} - m_1 \qquad \text{and} \qquad \hat{b} = \frac{\hat{a}(1-m_1)}{m_1} \; ,$$

which finally leads to the initial values

$$\beta^{(0)} = \hat{a} - 1 \qquad \text{and} \qquad \gamma^{(0)} = \hat{b} - 1 \; ,$$

where the different parametrizations of the fQ-function and the Beta distribution are taken into account.

5 Empirical application

Tail shape estimation in empirical finance can generally be divided into two groups: The first assumes a priori Paretion stable distributions or distributions in the domain of attraction of stable laws, and finds characteristic exponents $\alpha < 2$ for stock returns (Buckle, 1995), excess bond returns (McCulloch, 1985), foreign-exchange-rate changes (So, 1987), commodity-price movements (Liu and Borsen, 1995), and real-estate returns (Young and Graff, 1995), to mention some recent studies. Since the prior commitment to a tail index $\alpha < 2$ is too restrictive, the second group assumes a generalized Pareto distribution which also permits Paretian tail behaviour with $\alpha > 2$: Shape parameters have been estimated along the lines of (5) for stock returns (Jansen and de Vries, 1991), the Canadian/U.S. dollar exchange rate (Hols and de Vries, 1991), U.S. stock returns and exchange-rate returns for different currencies (Loretan and Phillips, 1994), and high-frequency data of the German share price index DAX (Lux, 1997). All the latter studies provide tail index estimates above 2, which has been cited as evidence against infinite-variance laws, so that Loretan and Phillips (1994) state that stock returns are better modeled by finite variance distributions.

In this section we report an empirical application of our results to German stock returns. We include all stocks that make up the DAX, except Lufthansa, Henkel, Veba and Viag, for which no uninterrupted series of returns could be obtained. Time ranges from January 4th, 1960 until September 29th, 1995, comprising $n = 8916$ trading days on the Frankfurt stock exchange. The data was provided by the Deutsche Finanzdatenbank (DFDB) in Karlsruhe.

Figure 4 shows the fQ-density estimations of the BASF and BMW returns, which are obtained by solving the differential equation (6) numerically when the parameters are replaced by their estimators. For comparison, normal densities with expectations and variances estimated from the returns are superimposed. As for all stocks the well-known leptocurtic behaviour of return distributions is obvious, not illustrated here.

Figure 4: fQ-density estimation (solid line) and normal density (dotted)

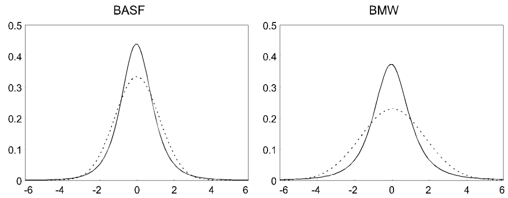


Figure 5: Q-Q-plots for various returns

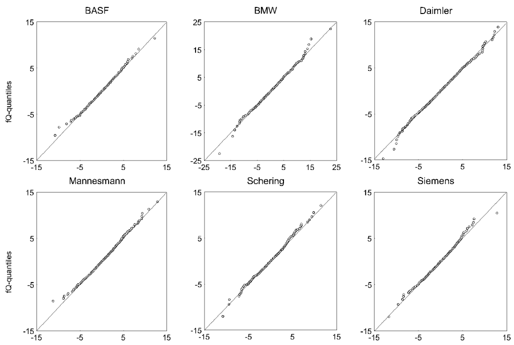


Figure 5 shows some Q-Q-plots, where the quantiles of the fQ-distributions with parameters estimated along the lines of section 4 with $\epsilon = 0.01$ are plotted against the returns. For all returns the estimation algorithm terminated after less than four iterations. Figure 5 shows that the fQ-distribution provides very good approximations of the true return distributions.

Table 2: Empirical results for German stock returns

Company	$\hat{\beta}$	$\hat{\gamma}$	ρ_{max}
BASF	1.136	1.168	5
BMW	1.276	1.330	3
Continental	1.214	1.285	3
Daimler	1.274	1.327	3
Deutsche Babcock	1.300	1.337	2
Degussa	1.331	1.376	2
Bayer	1.127	1.174	5
Hoechst	1.149	1.188	5
MAN	1.260	1.314	3
Karstadt	1.256	1.291	3
Linde	1.234	1.289	3
Mannesmann	1.161	1.217	4
Metallgesellschaft	1.382	1.436	2
Preussag	1.237	1.337	2
RWE	1.228	1.290	3
Schering	1.193	1.238	4
Siemens	1.198	1.220	4
Thyssen	1.118	1.170	5
VW	1.205	1.217	4
Kaufhof	1.326	1.331	3
Bayr. Hypo.	1.268	1.315	3
Bayr. Vereinsbank	1.225	1.271	3
Commerzbank	1.223	1.265	3
Deutsche Bank	1.189	1.232	4
Dresdner Bank	1.172	1.225	4
Allianz	1.350	1.377	2

The theoretical results applied to daily stock returns are summerized in table 2, which reveals several things: The second and third row give the estimates of the fQ-parameters β and γ, respectively. For all stocks, $\hat{\gamma}$ is greater than $\hat{\beta}$ which shows that the underlying distributions are slightly positive skewed. The last row shows the maximal finite moments $\rho_{max} \in \mathbb{N}$ given in (21). All stock returns can be modeled with finite variance and in most of the cases, even the third moments exist. The largest values of ρ_{max} are obtained for BASF, Bayer and Hoechst, which are known to be less volatile than the remaining stocks that make up the DAX.

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