# Large dimensional limits of Jacobi processes of compact and non-compact type and related Pearson diffusions

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# Contents

# Chapter 1

# Introduction

In this thesis, we study certain dynamical models which admit the following informal description: Take $N \in \mathbb{N}$ identical particles and place them on a line or a line segment, i.e. on an interval $I \subset \mathbb{R}$. The particles are subject to repulsion forces which increase the closer the particles are to each other and the closer the particles get to the two boundary points of the interval $I$, at least if the boundary points are finite. The movement of the particles might also be influenced by some random noise, the size of which we will interpret to be the temperature of the system. The repulsion forces are hereby large enough to enforce that no collisions between particles occur, and that the particles stay confined to $I$. Last but not least, the system might be subject to damping or positive feedback. In these cases, the total energy of the system is not constant.
The prime example of such a setting is the *Dyson Brownian motion* (DBM), which is named after Freeman J. Dyson who published his findings on the DBM model in the paper [43] in 1962. He was the first to consider time dependent random matrices by studying the eigenvalues of hermitian matrices whose entries consist of Brownian motions which are independent up to the hermitian constraint. He discovered that these eigenvalues actually behave like a dynamical model as sketched in the first paragraph, and was able to quantitatively describe the underlying dynamic. Mathematically, the dynamic of DBM falls into the framework of systems of stochastic differential equations (SDEs) of the form

$$X_{i,t} = X_{i,0} + \int_0^t \sigma(X_{i,s})\, dB_{i,s} + \int_0^t \left[ b(X_{i,s}) + \sum_{j\colon j\neq i} \frac{H(X_{i,s}, X_{j,s})}{X_{i,s} - X_{j,s}} \right] ds\,, \tag{1.0.1}$$

$i \in \{1, \dots, N\}$, $X_0 = \hat{x}_0$. Here $(X_t = (X_{1,t}, \dots, X_{N,t}))_{t\geq 0}$ is the vector of particle positions where the numbering is in increasing order, i.e. it holds $X_{1,t} \leq \cdots \leq X_{N,t}$, and $(B_t = (B_{1,t}, \dots, B_{N,t}))_{t\geq 0}$ is an $N$-dimensional standard Brownian motion. For an introduction to stochastic integrals w.r.t. $(B_t)_{t\geq 0}$ and to stochastic differential equations, see e.g. one of the books [67, 84, 95]. The quantities appearing in the SDE (1.0.1) have the following interpretation:

- $\hat{x}_0 \in \{x \in I^N \colon x_1 \leq \cdots \leq x_N\}$ is the deterministic vector of initial particle positions;

- $\sigma\colon \mathbb{R} \to \mathbb{R}$ quantifies the size of the random noise in the system depending on the current particle position; in particular in the case $\sigma \equiv 0$ we have a non-random dynamic governed by ordinary differential equations (ODEs);
- $b\colon \mathbb{R} \to \mathbb{R}$ quantifies the damping/positive feedback and the size of the repulsion from the boundary of $I$;
- $H\colon \mathbb{R}^2 \to \mathbb{R}$ with $H(x,y) = H(y,x)$ quantifies the size of the repulsion between the particles due to the $\frac{1}{X_{i,t}-X_{j,t}}$ term, thereby depending on the current positions of $X_{i,t}$ and $X_{j,t}$.

In the case $\hat{x}_0 = 0, \sigma \equiv 1$, $b \equiv 0$, $H \equiv 1$, we get the eigenvalue dynamic of the hermitian matrix model which Dyson studied. One of Dyson's key findings was the mutual repulsion of the eigenvalues of the considered hermitian matrix model. In particular, the random variable $X_t = (X_{1,t}, \dots, X_{N,t})$ satisfies

$$X_{1,t} < \cdots < X_{N,t} \text{ for each } t > 0 \text{ almost surely.} \tag{1.0.2}$$

Another important property of the Dyson Brownian motion is the following behavior for large particle number $N$: One can show that under a proper $N$-dependent rescaling of the system, the empirical distribution of particle positions tends to a non-random and non-trivial probability measure. This kind of behavior of eigenvalues of random matrices has first been observed by Eugene P. Wigner in 1955, where he showed that the empirical distribution of eigenvalues of certain symmetric random matrices tends to the semicircle law ([111]). The latter distribution is thereby named after the graph of its probability density function which is given by $x \mapsto \frac{1}{2\pi}\sqrt{4-x^2}$, $x \in [-2,2]$. In [112], Wigner was able to extend his results to more general symmetric random matrices with centered entries having normalized variance and which are independent up to the symmetry constraint; such matrices are nowadays simply called *Wigner matrices*. Later on, it was shown that the empirical eigenvalue distributions of random hermitian and symmetric quaternion matrices with analogous normalization also converge to the semicircle law. By the construction of DBM and the fact that $B_t$ has the same distribution as $\sqrt{t}B_1$ for each $t > 0$, one can thus infer that the empirical distributions of DBM at time $t$ and with starting configurations equal to zero tend to a semicircle distribution which gets rescaled by the factor $\sqrt{t}$. However, the description of the limit measure is more complicated if the vector of initial particle positions $\hat{x}_0$ of the DBM is chosen in such a way that the limit of empirical measures of the starting configurations is not a Dirac measure. The major breakthrough to tackle this problem came several decades after Wigner's and Dyson's seminal papers: In 1991 Dan V. Voiculescu showed that his prior work on freeness of non-commutative operators gives the correct framework to address this type of question ([107]). This theory later on became known as *free probability theory*; for an introduction see Section 2.4 and references therein. In particular, Voiculescu's findings showed that for each $t > 0$ the limit distribution of DBM for non-trivial initial limit distribution at time $t$ is given by the *free additive convolution* of the initial limit distribution and a semicircle

distribution, see Theorem 5.1.1.

One of the aims of this thesis is to find descriptions of the large $N$ asymptotics of empirical distributions of more general stochastic processes of the type as in (1.0.1) and, if possible, also to describe the dependence of the limit measure at time $t > 0$ on the initial limit distribution. While the existence of such limits has been shown under quite general conditions in [73], the actual identification of a limit point can be complicated. A good example of this is the Jacobi process, which corresponds to the choice

$$\sigma(x) = \sqrt{2x(1-x)}\,, \quad b(x) = \frac{\beta}{2}\left(p - (p+q)x\right) \,, \quad H(x,y) = x + y - 2xy \text{ and } I = [0,1]\,, \tag{1.0.3}$$

where $\beta, p, q$ are parameters. One can show that, if one chooses the real parameters $\beta, p, q$ large enough, then the repulsion of particles is big enough so that particles do not collide, i.e. (1.0.2) holds (see Lemma 3.5.2). Note that, when studying the large $N$ limit of empirical measures of the Jacobi processes, we now have more degrees of freedom than in the DBM case since we can vary the dependence of $\beta, p, q$ on $N$. In the case that $\beta$ is constant in $N$ and $p = p(N), q = q(N)$ grow linearly in $N$, the limit of corresponding empirical measures has been first described in [30]. In contrast to the semicircle limit described above, so far, an explicit description of the probability density function is out of reach except for one special case corresponding to $\lim_{N\to\infty} p(N)/N = \lim_{N\to\infty} q(N)/N = 1$. One can, however, describe these limits by ODEs for the corresponding limit moments and by free stochastic calculus. We briefly summarize these results in Section 5.4. Apart from this limit regime, one can use the degrees of freedom in the choice of the parameters $\beta, p, q$ to obtain other non-trivial limits. One such limit was obtained in [104] when letting $\beta \downarrow 0$ and letting $p(N), q(N)$ grow linearly in $N$ as before. Such a limit is called *high-temperature limit* as the parameter $\beta$ is commonly interpreted as inverse temperature. We will only consider particle systems which are non-colliding, i.e. such that the condition (1.0.2) is fulfilled. Thus, we do not discuss high-temperature limits in this work since in this setting collisions happen with positive probability (see the introduction to Chapter 3 and references therein). Nevertheless, we will prove that one still has enough freedom in the choice of $p$ and $q$ to obtain further limits by letting $\liminf_{N\to\infty} p(N)/N = \infty$ or $\liminf_{N\to\infty} q(N)/N = \infty$, see Section 5.7.

Overall, this thesis is structured in the following way: In Chapter 2, we first introduce frequently used notation and terminology regarding probability measures and elementary symmetric polynomials. Then, we briefly discuss Markov processes studied in the context of the theories of Dunkl and Heckman & Opdam on multivariate special functions related to reflection groups as these give important examples of processes of the form (1.0.1). We then move on to the subject of free probability theory with the primary focus lying on the definition of the free additive and multiplicative convolutions, which will help us to describe large $N$ limits of empirical measures of processes of the form (1.0.1) in Section 5. With the same intention in mind, we also give a concise introduction to free stochastic

calculus, which can be thought of as an $N = \infty$ variant of Itô calculus for continuous $N \times N$ matrix valued semimartingales.

In Chapter 3, we consider a subclass of processes of the form (1.0.1) named *non-colliding $\beta$-Pearson diffusions*. In the case that the particle number $N$ is equal to 1, these processes have first been studied by Eugene Wong ([113]). Under certain constraints on the parameters of the model, he observed that the stationary distributions of these processes are members of the class of *Pearson distributions*. A Pearson distribution is hereby characterized by the fact that it is a probability measure on $\mathbb{R}$ which is absolutely continuous with respect to Lebesgue measure, and that its probability density function $f(x)$ satisfies a differential equation of the form

$$\frac{d}{dx} f(x) = \frac{ax + b}{cx^2 + dx + e} f(x) \,, \tag{1.0.4}$$

see e.g. [29] Section 19.4. Pearson distributions in turn include important probability distributions known from the field of statistics such as normal-, $\chi^2$-, Student's $t$-, and $F$-(Fisher-Snedecor) distributions. In [10], Theodoros Assiotis observed that many important examples of eigenvalue processes of dynamical hermitian random matrix models fall into the framework of a multivariate generalization of these processes where additionally the non-colliding property (1.0.2) is satisfied. Hence, Assiotis appropriately calls this class of processes 'non-collding Pearson diffusions'. This class can be divided into 6 disjoint subclasses, two of which correspond to the DBMs and the Jacobi processes discussed above. We will add an additional parameter $\beta \geq 1$ to the class studied by Assiotis, who focused on the case $\beta = 2$. This addition is natural as in the literature $\beta$-DBMs and $\beta$-Jacobi processes are also considered ([15, 31]).
So far, only five of the six subclasses of non-colliding $\beta$-Pearson diffusions have been studied in depth before. Thus, we primarily focus on the treatment of the remaining sixth subclass consisting of non-compact Jacobi/Fisher-Snedecor processes, see Section 3.6. The term 'Fisher-Snedecor' refers to the fact that for some parameters these are multivariate, non-colliding versions of the processes studied by Wong which have an $F$-(Fisher-Snedecor) distribution as stationary probability measure. The multivariate non-compact Jacobi/Fisher-Snedecor processes hereby exhibit a strong resemblance to the Jacobi processes as the coefficient functions $\sigma, b, H$ in (1.0.1) and the interval $I$ are given by

$$\sigma(x) = \sqrt{2x(x-1)} \,,\ b(x) = \frac{\beta}{2}\left(-p + (p+q)x\right) \,,\ H(x,y) = 2xy - (x+y) \text{ and } I = [1, \infty) \,,$$

compare (1.0.3). In this setting, we provide answers to the following three questions:

- Under what conditions on the parameters $\beta, p, q$ exist unique non-colliding solutions to the SDEs (1.0.1)?

- Under what conditions on the parameters $\beta, p, q$ does an invariant probability measure of the dynamic (1.0.1) exist, and, in case of existence, can one identify this measure?

- Does a (concrete) random matrix model exist such that the corresponding eigenvalue process is a non-compact Jacobi process?

In turn, the answers to these questions will reveal both similarities and differences of Jacobi processes and non-compact Jacobi processes. Moreover, we will also show that one can approximate non-colliding geometric Brownian motions (see Section 3.4 for a definition) by non-compact Jacobi processes.
To complete the picture, we also state conditions such that non-colliding unique solutions and invariant probability measures exist in the cases of the other five subclasses of non-colliding $\beta$-Pearson diffusions, thereby citing already known results.

We then move on to study the large $N$ asymptotics of empirical measures of non-colliding $\beta$-Pearson diffusions in Chapter 4. While in [73] the same problem was considered for more general processes arising as solutions to SDEs of the form (1.0.1), we show that, when specializing to non-colliding $\beta$-Pearson diffusions, one can improve the type of convergence from weak-convergence in probability to almost sure weak convergence, see Theorem 4.2.2. This improvement is obtained by showing that the empirical moments converge locally uniformly in $t$ almost surely to deterministic functions which arise as solutions to ODEs with a recursive structure. Further, we prove that the Cauchy-transforms of the limit measures are solutions to some quasi-linear PDEs. Theorem 4.2.2 is a generalization of corresponding results published by the present author and coauthors in [13, 12].

Building on the previous, in Chapter 5, we turn to the task of describing the limit measures obtained in Chapter 4 using free probability theory. In Sections 5.1-5.6 we discuss the limits corresponding to a 'natural' large $N$ scaling of the 6 subclasses of non-colliding $\beta$-Pearson diffusions. To give a complete picture, we first recall prior results on this topic. New contributions are then obtained by the study of limits of non-compact Jacobi processes and Hua-Pickrell diffusions, where connections to the free positive multiplicative Brownian motion, studied by Phillipe Biane in [19, 20], are shown. As already mentioned above, we also use the degrees of freedom in the parameters $p, q$ of the Jacobi and non-compact Jacobi processes to obtain limit measures which differ from those obtained in the 'natural scaling' in Section 5.7. Limit theorems of this type have previously been published by the present author and coauthors in [13]. To obtain these limits, we will need to consider higher order space-time rescalings of the empirical measures. Whereas we carry this out in full detail only for the two types of Jacobi processes, similar results also hold for the other subclasses of non-colliding $\beta$-Pearson diffusions.

Lastly, in Chapter 6 we give a new self-contained proof of a formula relating the uniform

distribution on an interval, the semicircle distribution, and the distribution of free positive multiplicative Brownian motion via the free additive convolution and the exponential map, see Theorem 6.0.1. This formula has previously been proven by Michael Voit and the present author using different methods in [12]. The result is of special interest in the context of this thesis as it corresponds to formulas for the limits of empirical measures of non-colliding $\beta$-Pearson diffusions in four special cases, see Theorem 5.3.4, Theorem 5.5.7, Corollary 5.6.5 and Corollary 5.7.3. By proving this identity, we also obtain new integral formulas for the distribution of free positive multiplicative Brownian motion, which generalize the already known moment formulas involving Laguerre polynomials.

# Chapter 2

# Preliminaries

## 2.1 Some notation concerning probability measures

We start by introducing notation for some elementary operations on and elements of the set of real probability measures. Let $A$ be a topological space. We denote by $\mathcal{B}(A)$ the Borel-$\sigma$-algebra of $A$ and by $M^1(A)$ the set of Borel probability measures on $A$. For $a \in A$ we denote by $\delta_a \in M^1(A)$ the *Dirac measure at* $a$, i.e. the probability measure which assigns mass 1 to the set $\{a\}$. Let $(A, \mathcal{A}, \mu)$ be a measure space and $(B, \mathcal{B})$ be a measurable space. For $f\colon A \to B$ measurable we write $f(\mu)$ for the image measure of $\mu$ under $f$. An elementary case of such an image measure is the following: For $a \in \mathbb{R}$ we denote by $D_a\colon \mathbb{R} \to \mathbb{R}$ the linear map $x \mapsto ax$. Then $D_a(\mu)$ denotes the linear rescaling of $\mu \in M^1(\mathbb{R})$ by the factor $a$.
The following three probability measures on $\mathbb{R}$ will be of special interest in Chapters 5 & 6:
The *semicircle distribution* $\mu_{\mathrm{sc},R} \in M^1(\mathbb{R})$ *with radius* $R$ is the probability measure which is supported on $[-R, R]$ and with density on its support given by

$$\mu_{\mathrm{sc},R}(dx) = \frac{2}{\pi R^2}\sqrt{R^2 - x^2}\, dx\,.$$

Moreover we set $\mu_{\mathrm{sc},0} := \delta_0$. One can easily see that $D_a(\mu_{\mathrm{sc},R}) = \mu_{\mathrm{sc},aR}$ for all $a \geq 0, R \geq 0$.
The *Marchenko-Pastur distribution with parameters* $\alpha, \beta > 0$ is given by

$$\mu_{\mathrm{MP},\alpha,\beta} := \begin{cases} (1-\alpha)\delta_0 + \tilde{\mu}\,, & \alpha \in (0,1) \\ \tilde{\mu}\,, & \alpha \geq 1\,, \end{cases}$$

where $\tilde{\mu}$ is the measure on $[x_-, x_+]$, $x_\pm := \beta\left(\sqrt{\alpha} \pm 1\right)^2$, with density

$$\tilde{\mu}(dx) = \frac{1}{2\pi\beta x}\sqrt{(x_+ - x)(x - x_-)}\, dx\,.$$

This agrees with the notion of Marchenko-Pastur distributions in [3] Exercise 5.3.27. However, we point out that in the given reference there is a typo concerning the mass of

$\tilde{\mu}$ for $\alpha \in (0,1)$ which is $\alpha$ and not 1. The same typo in turn also appears in the papers [109] & [13]. Moreover, we point out that there are also other parametrizations of the Marchenko-Pastur distributions in the literature. Further, note that in our notation the second parameter is just a scaling parameter, i.e. $D_a(\mu_{\mathrm{MP},\alpha,\beta}) = \mu_{\mathrm{MP},\alpha,a\beta}$ for all $a > 0$. For $A \in \mathcal{B}(\mathbb{R})$ with finite positive Lebesgue measure we denote by $\mathrm{Unif}_A \in M^1(\mathbb{R})$ the *uniform distribution on* $A$.

## 2.2 Elementary symmetric polynomials

To study the properties of non-colliding $\beta$-Pearson diffusions in Chapter 3, transformations by the following multivariate polynomials will be very useful: For $m \in \mathbb{N}$, $n \in \{0, 1, \ldots, m\}$ we denote by $e_n^m$ the $n$-th *elementary symmetric polynomial* in $m$ variables which is defined by the equation

$$\prod_{j=1}^{m}(z - x_j) = \sum_{j=0}^{m}(-1)^{m-j} e_{m-j}^m(x) z^j \,,\ z \in \mathbb{C},\, x \in \mathbb{C}^m \,. \tag{2.2.1}$$

We can also write the elementary symmetric polynomials more explicitly:

$$e_n^m(x) = \sum_{1 \le i_1 < \cdots < i_n \le m} x_{i_1} \cdots x_{i_n} \,.$$

The following theorem shows that elementary symmetric polynomials can be seen as the building blocks for all symmetric polynomials (see Theorem 6.1 in Chapter IV of [70]):

**Theorem 2.2.1.** *Let $f$ be a symmetric polynomial in $m$ variables. Then there exists a polynomial $g$ in $m$ variables such that $f(x) = g(e_1^m(x), \ldots, e_m^m(x))$, $x \in \mathbb{C}^m$.*

Another important property of the elementary symmetric polynomials is the following (here int denotes the interior of a set):

**Lemma 2.2.2.** *Let $C_N^A := \{x \in \mathbb{R}^N \colon x_1 \le \cdots \le x_N\}$. Consider the map $f : C_N^A \to \mathbb{R}^N$, $f(x) = (e_1^N(x), \ldots, e_N^N(x))$. Then $f : C_N^A \to f(C_N^A)$ is a homeomorphism, and $f : \operatorname{int} C_N^A \to f(\operatorname{int} C_N^A)$ is a $C^\infty$-diffeomorphism.*

*Proof.* The map $f \colon C_N^A \to \mathbb{R}^N$ is one to one by (2.2.1) and the fact that a polynomial is, up to a constant, uniquely characterized by its set of roots counted with multiplicity. Moreover, the fact that $f \colon C_N^A \to \mathbb{R}^N$ is a $C^\infty$-function is clear. $f^{-1}|_{f(\operatorname{int} C_N^A)}$ is a $C^\infty$-function by the Theorem of Section 2 in [23]. For the fact that $f^{-1} \colon f(C_N^A) \to C_N^A$ is continuous see e.g. [105]. □

For later reference we will also write several symmetric polynomials in terms of elementary symmetric polynomials. For these formulas we introduce the following notation:

Let $x \in \mathbb{R}^N$ and let $S \subseteq \{1, \ldots, N\}$ be a nonempty set. Denote by $x_S \in \mathbb{R}^{|S|}$ the vector with coordinates $x_i$, $i \in S$, in the natural ordering on $S$. With this convention we have

$$\begin{aligned} \sum_{i=1}^{N} e_{k-1}^{N-1}(x_{\{1,\ldots,N\}\setminus\{i\}}) &= (N-k+1)e_{k-1}^{N}(x)\,, \\ \sum_{i=1}^{N} e_{k-1}^{N-1}(x_{\{1,\ldots,N\}\setminus\{i\}})x_i &= ke_k^N(x), \end{aligned} \tag{2.2.2}$$

and

$$e_{k-1}^{N-1}(x_{\{1,\ldots,N\}\setminus\{i\}}) - e_{k-1}^{N-1}(x_{\{1,\ldots,N\}\setminus\{j\}}) = -(x_i - x_j)e_{k-2}^{N-2}(x_{\{1,\ldots,N\}\setminus\{i,j\}})\,.$$

Hence

$$\begin{aligned} \sum_{\substack{i,j=1\\ i\neq j}}^{N} \frac{e_{k-1}^{N-1}(x_{\{1,\ldots,N\}\setminus\{i\}})}{x_i - x_j} &= \sum_{\substack{i,j=1\\ i<j}}^{N} \frac{e_{k-1}^{N-1}(x_{\{1,\ldots,N\}\setminus\{i\}}) - e_{k-1}^{N-1}(x_{\{1,\ldots,N\}\setminus\{j\}})}{x_i - x_j} \\ &= -\sum_{\substack{i,j=1\\ i<j}}^{N} e_{k-2}^{N-2}(x_{\{1,\ldots,N\}\setminus\{i,j\}}) = -\frac{(N-k+2)(N-k+1)}{2}e_{k-2}^{N}(x)\,, \end{aligned} \tag{2.2.3}$$

$$\begin{aligned} \sum_{\substack{i,j=1\\ i\neq j}}^{N} \frac{e_{k-1}^{N-1}(x_{\{1,\ldots,N\}\setminus\{i\}})x_i x_j}{x_i - x_j} &= \sum_{\substack{i,j=1\\ i<j}}^{N} \frac{e_{k-1}^{N-1}(x_{\{1,\ldots,N\}\setminus\{i\}}) - e_{k-1}^{N-1}(x_{\{1,\ldots,N\}\setminus\{j\}})}{x_i - x_j}x_i x_j \\ &= -\sum_{\substack{i,j=1\\ i<j}}^{N} e_{k-2}^{N-2}(x_{\{1,\ldots,N\}\setminus\{i,j\}})x_i x_j = -\frac{k(k-1)}{2}e_k^N(x) \end{aligned} \tag{2.2.4}$$

and

$$\sum_{\substack{i,j=1\\ i<j}}^{N} (x_i + x_j)e_{k-2}^{N}(x_{\{1,\ldots,N\}\setminus\{i,j\}}) = (k-1)(N-k+1)e_{k-1}^{N}(x)\,. \tag{2.2.5}$$

## 2.3 Markov processes associated to the Dunkl- and Heckman-Opdam heat kernels

The multivariate Jacobi processes of compact and non-compact type are, at least for certain parameters $p_N, q_N$ and written in suitable coordinates, examples of Markov processes whose generators are well studied in the context of the theory of Dunkl and Heckman-Opdam on generalizations of spherical functions on symmetric spaces. The aim of this chapter is to give a short introduction to the differential-difference operators studied in these theories and to the related Markov processes. We point out that the knowledge of

the facts presented in this section is not required for the understanding of the results presented in Chapters 3-6. In several places in Chapter 3, however, we will indicate the close connection between non-colliding $\beta$-Pearson diffusions and Markov processes associated to Weyl group invariant Dunkl and Heckman-Opdam heat kernels.

In both the Dunkl and Heckman-Opdam setting one is interested in harmonic analysis and spherical functions on symmetric spaces, [5] provides a good general overview over the topic. Symmetric spaces can be classified by so called root systems (cf. [56] Chapter X). In turn, these root systems also play an integral role in the Dunkl and Heckman-Opdam theories, hence we start by giving a formal definition. For a broader introduction to root systems see [59] Chapter III.

Let $\mathfrak{a}$ be an Euclidean space with positive definite symmetric bilinear form $\langle \cdot, \cdot \rangle$. For non-zero $\alpha \in \mathfrak{a}$ we denote by $\sigma_\alpha \colon \mathfrak{a} \to \mathfrak{a}$ the reflection on the hyperplane orthogonal to $\alpha$. This reflection is given explicitly by

$$\sigma_\alpha(x) = x - \langle x, \alpha \rangle \alpha^\wedge$$

where $\alpha^\wedge := \frac{2\alpha}{\langle \alpha, \alpha \rangle}$.

**Definition 2.3.1.** A finite subset $\Sigma \subset \mathfrak{a} \setminus \{0\}$ is called a *(crystallographic) root system in* $\mathfrak{a}$ if the following hold:

(i) $\sigma_\alpha(\Sigma) = \Sigma$ for all $\alpha \in \Sigma$,

(ii) $\langle \alpha, \beta^\wedge \rangle \in \mathbb{Z}$ for all $\alpha, \beta \in \Sigma$.

The dimension of $\mathrm{span}_{\mathbb{R}}(\Sigma)$ is called the *rank of* $\Sigma$. We call a root system $\Sigma$ *reduced* if $\mathbb{R}\alpha \cap \Sigma = \{\pm\alpha\}$ for all $\alpha \in \Sigma$.
For a root system $\Sigma$ denote by $W(\Sigma)$ the group generated by the reflections $\{\sigma_\alpha \colon \alpha \in \Sigma\}$. We call $W(\Sigma)$ the *Weyl group of* $\Sigma$. It holds that $W(\Sigma)$ is finite.
A map $k \colon \Sigma \to [0, \infty)$ which is $W(\Sigma)$ invariant, i.e. $k \circ f|_\Sigma = k$ for all $f \in W(\Sigma)$, is called a *multiplicity function*.

Further, we define a useful subset of a root system $\Sigma$: Fix some half-space $V$ such that no root $\alpha \in \Sigma$ lies on the boundary of $V$. Then we call $\Sigma^+ := \Sigma \cap V$ the set of *positive roots* (the wording *positive* is explained in [59] Chapter III Section 10). Note that this definition depends on the choice of $V$.
Let $H_\alpha := \{x \in \mathfrak{a} \colon \langle x, \alpha \rangle = 0\}$ be the hyperplane orthogonal to $\alpha \in \mathfrak{a}$. We call any connected component $C$ of $\mathfrak{a} \setminus \bigcup_{\alpha \in R} H_\alpha$ a *Weyl chamber*.

**Example 2.3.2.** Throughout let $\mathfrak{a} = \mathbb{R}^N$ and denote by $\{e_1, \ldots, e_N\}$ the standard orthornormal basis corresponding to the standard inner product.

- Let $N \geq 2$. The *root system of type* $A_{N-1}$ is given by

$$\Sigma = \{\pm(e_j - e_i) \colon 1 \leq i < j \leq N\}\,.$$

$\Sigma$ is a reduced root system and its rank is $N-1$ as $\operatorname{span}_{\mathbb{R}}(\Sigma) = \{x \in \mathbb{R}^N \colon \sum_{i=1}^N x_i = 0\}$. A possible choice for the set of positive roots is $\Sigma^+ = \{e_j - e_i \colon 1 \leq i < j \leq N\}$. For instance, a possible choice of Weyl chamber of $\Sigma$ is given by $C = \{x \in \mathbb{R}^N \colon x_1 < \cdots < x_N\}$. The Weyl group $W(\Sigma)$ restriced to $\Sigma$ has one orbit, thus any multiplicity function is constantly equal to some $k \geq 0$.

- The *root system of type* $B_N$ is given by

$$\Sigma = \{a(e_j \pm e_i) \colon 1 \leq i < j \leq N, a \in \{-1, 1\}\} \cup \{\pm e_i \colon 1 \leq i \leq N\}\,.$$

$\Sigma$ is a reduced root system of rank $N$. A possible choice for the set of positive roots is

$$\Sigma^+ = \{e_j \pm e_i \colon 1 \leq i < j \leq N\} \cup \{e_i \colon 1 \leq i \leq N\}\,.$$

For instance, a possible choice of Weyl chamber of $\Sigma$ is given by $C = \{x \in \mathbb{R}^N \colon 0 < x_1 < \cdots < x_N\}$. The Weyl group $W(\Sigma)$ of $\Sigma$ has two orbits, thus any multiplicity function is determined by the two values $k_1, k_2 \in [0, \infty)$ on the respective orbits.

- The *root system of type* $BC_N$ is given by

$$\Sigma = \{a(e_j \pm e_i) \colon 1 \leq i < j \leq N, a \in \{-1, 1\}\} \cup \{ae_i \colon 1 \leq i \leq N, a \in \{-2, -1, 1, 2\}\}\,.$$

$\Sigma$ is not reduced and its rank is $N$. A possible choice for the set of positive roots is

$$\Sigma^+ = \{e_j \pm e_i \colon 1 \leq i < j \leq N\} \cup \{ae_i \colon 1 \leq i \leq N, a \in \{1, 2\}\}\,.$$

The Weyl chambers of $\Sigma$ are the same as for the root system of type $B_N$. The Weyl group $W(\Sigma)$ of $\Sigma$ has three orbits, thus any multiplicity function is determined by the three values $k_1, k_2, k_3 \in [0, \infty)$ on the respective orbits.

Next, we define the differential-difference operators which are central objects in the Dunkl and Heckman-Opdam theories.

### 2.3.1 Rational Dunkl theory

For a reduced root system $\Sigma$ on $\mathfrak{a}$ with some subset of positive roots $\Sigma^+$ and some fixed multiplicity function $k$, the *Dunkl operators* are defined by

$$D_\xi f(x) := \partial_\xi f(x) + \sum_{\alpha \in \Sigma^+} k(\alpha)\langle \alpha, \xi \rangle \frac{f(x) - f(\sigma_\alpha(x))}{\langle \alpha, x \rangle}\,, \quad f \in C^1(\mathfrak{a})\,, \quad \xi \in \mathfrak{a}\,,$$

where $\partial_\xi$ denotes the usual partial derivative. Note that in this definition no actual singularities appear since

$$\frac{f(x)-f(\sigma_\alpha(x))}{\langle \alpha, x\rangle} = -\frac{1}{\langle \alpha, x\rangle}\int_0^1 \frac{\partial}{\partial t} f(x - t\langle x,\alpha\rangle \alpha^\wedge)\, dt = \int_0^1 \langle \alpha^\wedge, \nabla f(x - t\langle x,\alpha\rangle\alpha^\wedge)\rangle\, dt\,.$$

Further, $D_\xi$ maps polynomials of degree $n$ to polynomials of degree $n-1$ and $C^n$-functions to $C^{n-1}$-functions. A remarkable property of these operators is that they commute, i.e. $D_\xi D_\eta = D_\eta D_\xi$ for all $\xi, \eta \in \mathfrak{a}$; for further properties of these operators see e.g. [42] Section 6.4. The *Dunkl-Laplacian* is defined by

$$\Delta^{\mathrm{rat}}_{\Sigma,k} f(x) := \sum_{i=1}^N D^2_{\xi_i} f(x)\,, \quad f \in C^2(\mathfrak{a})\,,$$

where $\{\xi_1, \dots, \xi_N\}$ is some orthonormal basis of $\mathfrak{a}$. One can show that this definition is independent of the choice of basis. For $f \in C^2(\mathfrak{a})$ which is $W(\Sigma)$ invariant, i.e. $f \circ g = f$ for all $g \in W(\Sigma)$, we have the formula

$$\Delta^{\mathrm{rat}}_{\Sigma,k} f(x) = \Delta f(x) + \sum_{\alpha \in R^+} \frac{2k(\alpha)}{\langle \alpha, x\rangle} \partial_\alpha f(x)\,, \tag{2.3.1}$$

where $\Delta$ denotes the usual Laplacian on $\mathfrak{a}$. Let $C$ be a Weyl chamber associated to $\Sigma$. Then $\Delta^{\mathrm{rat}}_{\Sigma,k}$ is the generator of an almost surely continuous Feller process with values in $\overline{C}$ by [90] Proposition 4.5 & Theorem 4.10. Such a process is usually called *radial Dunkl process*.

In this thesis we consider the radial Dunkl processes corresponding to root systems of type $A_{N-1}$ and $B_N$ in Sections 3.2 & 3.3. In the $A_{N-1}$ case the generator (2.3.1) can be written as

$$\Delta f(x) + 2k \sum_{\substack{1 \le i,j \le N \\ i \ne j}} \frac{1}{x_i - x_j} \partial_{x_i} f(x)$$

for some multiplicity parameter $k \ge 0$, and the radial Dunkl process takes values in the closed Weyl chamber of type $A_{N-1}$ given by $\{x \in \mathbb{R}^N \colon x_1 \le \cdots \le x_N\}$. For $k = 1$ this process is the Dyson Brownian motion referred to in the introductory Chapter 1, and for general $k \ge 0$ this process is often called $\beta$-Dyson Brownian motion where $\beta = 2k$ (see e.g. [3] Chapter 4.3).

In the $B_N$ case the generator (2.3.1) can be written as

$$\Delta f(x) + 2\sum_{i=1}^N \left( k_1 \frac{1}{x_i} + k_2 \sum_{\substack{j=1 \\ j \ne i}}^N \left( \frac{1}{x_i - x_j} + \frac{1}{x_i + x_j} \right) \right) \partial_{x_i} f(x)$$

for some multiplicity parameters $k_1, k_2 \ge 0$, and the radial Dunkl process takes values in the closed Weyl chamber of type $B_N$ given by $\{x \in \mathbb{R}^N \colon 0 \le x_1 \le \cdots \le x_N\}$. These

processes are often called multivariate Bessel processes (of type $B_N$), see e.g. [4]. After the coordinate transformation $x_i \mapsto x_i^2$, $i \in \{1, \ldots, N\}$, one obtains the squared Bessel/$\beta$-Laguerre processes where $\beta = 2k_2$, $k_1 = \nu/\beta$ for some parameter $\nu$ (see e.g. [8, 103] and references therein for an introduction to these processes).

### 2.3.2 Trigonometric Dunkl theory

For a root system $\Sigma$ on $\mathfrak{a}$ with subset of positive roots $\Sigma^+$ and some fixed multiplicity function $k$ the *trigonometric Dunkl operator* or *Cherednik operator* is given by

$$D_\xi f(x) := \partial_\xi f(x) + \sum_{\alpha \in \Sigma^+} \frac{k(\alpha)\langle \alpha, \xi \rangle}{1 - e^{-\langle \alpha, x \rangle}} \left( f(x) - f(\sigma_\alpha(x)) \right) - \langle \rho(k), \xi \rangle f(x)\,, \quad f \in C^1(\mathfrak{a})\,, \quad \xi \in \mathfrak{a}\,,$$

where $\rho(k) := \frac{1}{2} \sum_{\alpha \in R^+} k(\alpha)\alpha$ is the half-sum of positive roots. As in the rational Dunkl setting, in this definition no actual singularities are present. In particular, the trigonometric Dunkl operators map $C^n$-functions to $C^{n-1}$-functions and they map polynomials in $e^{\langle \lambda, \cdot \rangle}$ to themselves, where the $\lambda$ are elements of the weight lattice of $\Sigma$ (see [59] Chapter III Section 13 for a definition). Moreover, these operators commute. For further properties see Theorem 4.2 in [5]. The associated *Heckman-Opdam Laplacian* is defined by

$$\Delta^{\mathrm{trig}}_{\Sigma,k} f(x) := \sum_{i=1}^{N} D^2_{\xi_i} f(x) - |\rho(k)|^2 f(x)\,, \quad f \in C^2(\mathfrak{a})\,,$$

where $\xi_1, \ldots, \xi_N$ is an orthonormal basis of $\mathfrak{a}$ and $|x| = \sqrt{\langle x, x \rangle}$. For $W(\Sigma)$-invariant functions $f \in C^2(\mathfrak{a})$ it takes the form

$$\Delta^{\mathrm{trig}}_{\Sigma,k} f(x) = \Delta f(x) + \sum_{\alpha \in \Sigma^+} k(\alpha) \coth\left( \frac{\langle \alpha, x \rangle}{2} \right) \partial_\alpha f(x)\,. \tag{2.3.2}$$

Let $C$ be a Weyl chamber associated to $\Sigma$. Then $\Delta^{\mathrm{trig}}_{\Sigma,k}$ is the generator of an almost surely continuous Feller process with values in $\overline{C}$ (see [93] introduction to Section 3 and Proposition 4.1). We will call instances of these processes *(non-compact) radial Heckman-Opdam processes* where 'non-compact' refers to the unbounded state space. In the case that $\Sigma$ is of type $A_{N-1}$ the generator (2.3.2) can be written as

$$\Delta f(x) + k \sum_{\substack{1 \le i,j \le N \\ i \ne j}} \coth\left( \frac{x_i - x_j}{2} \right) \partial_{x_i} f(x) \tag{2.3.3}$$

for some multiplicity parameter $k \ge 0$, and the corresponding process takes values in $\{x \in \mathbb{R}^N \colon x_1 \le \cdots \le x_N\}$. After the transformation $x_i \mapsto \exp(x_i)$, $i \in \{1, \ldots, N\}$, and in the case $\beta := 2k = 2$, these processes are called non-colliding geometric Brownian motions by T. Assiotis ([10] Section 2). For more information on these processes see Section 3.4

and references therein.
In the $BC_N$-case the generator (2.3.2) can be written as

$$\Delta f(x) + \sum_{i=1}^{N} \Bigg( k_1 \coth(x_i/2) + 2k_2 \coth(x_i) + k_3 \sum_{\substack{j=1 \\ j\neq i}}^{N} \left( \coth\left(\frac{x_i - x_j}{2}\right) + \coth\left(\frac{x_i + x_j}{2}\right)\right)\Bigg) \partial_{x_i} f(x) \tag{2.3.4}$$

for some multiplicity parameters $k_1, k_2, k_3 \geq 0$, and the corresponding process takes values in $\{x \in \mathbb{R}^N \colon 0 \leq x_1 \leq \cdots \leq x_N\}$. After the coordinate change $x_i \mapsto (1 + \cosh(x_i))/2$, $i \in \{1, \ldots, N\}$, the instances of the corresponding processes are called non-compact Jacobi processes ([13] Section 5) or Fisher-Snedecor processes ([10] Section 2), see Section 3.6 for further information.

Lastly, there is a closely related class of processes: For Weyl group invariant $f \in C^2(\mathfrak{a})$ set

$$\widetilde{\Delta}^{\mathrm{trig}}_{\Sigma,k} f(x) := \Delta f(x) + \sum_{\alpha \in \Sigma^+} k(\alpha) \cot\left(\frac{\langle \alpha, x\rangle}{2}\right) \partial_\alpha f(x)\,. \tag{2.3.5}$$

Note that formally $\widetilde{\Delta}^{\mathrm{trig}}_{\Sigma,k} g(x) = -\left(\Delta^{\mathrm{trig}}_{\Sigma,k} f\right)(ix)$ for $g(x) = f(ix)$. It turns out that also $\widetilde{\Delta}^{\mathrm{trig}}_{\Sigma,k}$ is the generator of an almost surely continuous Feller process, in this case the process takes values in the *Weyl alcove* $A_0 := \{x \in \mathfrak{a} \colon 0 \leq \langle \alpha, x\rangle \leq \pi \ \forall\ \alpha \in \Sigma^+\}$. This fact can be inferred from [86] Theorem 3.9 using the arguments leading to Theorem 4.10 in [90]. We will call instances of these processes *(compact) radial Heckman-Opdam processes*.
In the $A_{N-1}$ case the generator has the same form as in (2.3.3) where one simply has to replace coth by cot. The state space of the corresponding process here is given by $\{x \in \mathbb{R}^N \colon x_1 \leq \cdots \leq x_N \leq x_1 + \pi\}$. In the case $k = 1$, transferring this process to the Torus $\mathbb{T} := \{z \in \mathbb{C} \colon |z| = 1\}$ by $x_j \mapsto \exp(ix_j)$, $j \in \{1, \ldots, N\}$, yields the eigenvalue process of the matrix valued unitary Brownian motion which first was considered in [43] This particular process will not be discussed in this work as it is not a member of the class of non-colliding $\beta$-Pearson diffusions which will be studied in Chapters 3-5. For more information on these processes see e.g. [26].
In the $BC_N$ case the generator (2.3.5) has the same form as in (2.3.3) where, again, one has to replace coth by cot. The state space of the corresponding processes here is given by the compact set $\{x \in \mathbb{R}^N \colon 0 \leq x_1 \leq \cdots \leq x_N \leq \pi\}$. These processes were first considered in [31]. After the coordinate change $x_i \mapsto (1 + \cos(x_i))/2$, $i \in \{1, \ldots, N\}$, the latter processes are called (multivariate) $\beta$-Jacobi processes and first appeared in the context of dynamical random matrices in [39]. They will be introduced more broadly in Section 3.5.

## 2.4 Free probability theory

In this section we collect some of the main definitions of free probability theory which will be used to formulate most of the results in Chapters 5 & 6. The following references might be useful for a broader introduction to the topics discussed below: For an introductory book on free probability theory with a strong emphasis on its connection to large random matrices see [77]. For basics about $W^*$-probability spaces see D.V. Voiculescu's lecture notes in [45]. For an introduction to $C^*$- and von Neumann algebras see [98]. Our exposition is mainly based on the first two references.

As already mentioned in the introduction, free probability theory provides effective tools to analyze the spectrum of random matrices when the matrix dimension goes to infinity. The prime example of this is the following:

**Example 2.4.1.** Denote by $M_N(\mathbb{C})$ the complex $N \times N$ matrices. For a hermitian matrix $A \in M_N(\mathbb{C})$ denote by $\lambda_{1,N}(A), \ldots, \lambda_{N,N}(A)$ its eigenvalues. For each $N$ let $A_N \in M_N(\mathbb{C})$ be a deterministic hermitian matrix and assume that the empirical spectral measures $\frac{1}{N}\sum_{i=1}^N \delta_{\lambda_{i,N}(A_N)}$ converge for $N \to \infty$ weakly to some compactly supported probability measure $\mu \in M^1(\mathbb{R})$. For each $N$ let $B_N$ be an $N$-dimensional hermitian Gaussian matrix, i.e. the entries $(B_N)_{i,j}$, $1 \leq i \leq j \leq N$, are independent, the diagonal entries have real centered normal distribution with variance $1/N$ and the offdiagonal entries have complex centered normal distribution with variance $1/N$. In other words, the distribution of $B_N$ is given by the (renormalized) *Gaussian unitary ensemble (GUE)*. By the results of Wigner, it holds that $\frac{1}{N}\sum_{i=1}^N \delta_{\lambda_{i,N}(B_N)}$ weakly converges for $N \to \infty$ to the *standard semicircle distribution* $\mu_{\mathrm{sc},2}$ a.s. Note that for each $N$ the random eigenvalues of $A_N + B_N$ might be quite difficult to describe in general. But in the large $N$ limit the picture in a way simplifies: It holds that also $\frac{1}{N}\sum_{i=1}^N \delta_{\lambda_{i,N}(A_N+B_N)}$ converges weakly to some compactly supported probability measure $\rho$, and that $\rho$ can be described by a universal mechanism called *free additive convolution* which only takes $\mu$ and $\mu_{\mathrm{sc},2}$ as input; see Section 2.4.1.

One of the aims of this section to give a precise definition of the 'universal mechanism' mentioned in the previous example. Another goal is to fix definitions which are relevant to free stochastic calculus which is the topic of Section 2.5.
A short disclaimer is in order: As its name suggests many aspects of free probability theory are analogous to classical probability theory. However, we stress that free probability is not an extension of the latter.

Our exposition will focus primarily on the analytic aspects of free probability theory although some parts of the theory can be formulated in a purely algebraic context. Further, we will solely treat the parts of the theory which are relevant to the comprehension of Chapters 5 & 6. In particular, the combinatorial aspects of free probability will not be treated here.

We start by introducing some notation and definitions regarding (operator) algebras and linear functionals thereon:

**Definition 2.4.2.**

- We call an algebra $\mathcal{A}$ with unit element a *unital algebra* and denote its unit element by $\mathbf{1}$ and its zero element by $\mathbf{0}$.

- Let $H$ be a Hilbert space over $\mathbb{C}$ with inner product $\langle \cdot, \cdot \rangle$. Denote by $\mathcal{B}(H)$ the Banach algebra of bounded linear operators on $H$ and by $||\cdot||$ the operator norm on $\mathcal{B}(H)$. Further, we denote by $a^* \in \mathcal{A}$ the adjoint of $a \in \mathcal{B}(H)$ and by $\mathcal{A}_{\text{sa}} := \{a \in \mathcal{A} \colon a = a^*\}$ the set of self-adjoint elements.
  We call $\mathcal{A} \subset \mathcal{B}(H)$ a *$C^*$-algebra* if it is a unital algebra over $\mathbb{C}$ which is closed in operator norm and is stable under taking adjoints.

- Let $\mathcal{A}$ be a $C^*$-algebra. We call a linear functional $\varphi \colon \mathcal{A} \to \mathbb{C}$

  - a *state*, if $\varphi(\mathbf{1}) = 1$ and $\phi(a^*a) \geq 0$ for all $a \in \mathcal{A}$,
  - *faithful*, if for all $a \in \mathcal{A}$ it holds that $\varphi(a^*a) = 0$ implies $a = \mathbf{0}$,
  - *tracial*, if $\varphi(ab) = \varphi(ba)$ for all $a, b \in \mathcal{A}$.

  For an operator $a \in \mathcal{A}$ denote its spectrum by

  $$\sigma(a) := \{\lambda \in \mathbb{C} \colon a - \lambda \mathbf{1} \text{ does not have an inverse in } \mathcal{B}(H)\}.$$

  We denote the set of *non-negative elements of* $\mathcal{A}$ by

  $$\mathcal{A}_+ := \{a \in \mathcal{A} \colon a \text{ is self-adjoint and } \sigma(a) \subseteq [0, \infty)\}.$$

  Let $a \in \mathcal{A}$ be *normal*, i.e. $a^*a = aa^*$, and $f \colon \sigma(a) \to \mathbb{C}$ be a bounded Borel measurable function. Then we can define $f(a) \in \mathcal{A}$ by functional calculus: Let $P^a \colon \mathcal{B}(\sigma(a)) \to \mathcal{A}$ be the *projection-valued spectral measure of* $a$, i.e. the unique projection-valued measure such that $a = \int_{\sigma(a)} \lambda \, dP^a(\lambda)$ (see e.g. Theorem 12.23 in [92] where $P^a$ is called a *resolution of the identity*). We then define $f(a) := \int_{\sigma(a)} f \, dP^a \in \mathcal{A}$.

- A *von Neumann algebra* $\mathcal{A} \subset \mathcal{B}(H)$ is a $C^*$-algebra which is closed in the *weak operator topology*, i.e. if $(T_i)_{i \in I} \subset \mathcal{A}$ is a net such that there exists $T \in \mathcal{B}(H)$ with $\lim_i \langle T_i \xi, \eta \rangle = \langle T\xi, \eta \rangle$ for all $\xi, \eta \in H$ then $T \in \mathcal{A}$. A linear functional $\varphi \colon \mathcal{A} \to \mathbb{C}$ is called *normal* if it is continuous w.r.t. the weak-$^*$ topology.

Building on the previous, we define the main structures of this section:

**Definition 2.4.3.** A *non-commutative probability space* is a pair $(\mathcal{A}, \varphi)$ where $\mathcal{A}$ is a unital algebra and $\varphi\colon \mathcal{A} \to \mathbb{C}$ is a linear functional satisfying $\varphi(\mathbf{1}) = 1$. Elements of $\mathcal{A}$ will be called *non-commutative random variables.*
A *$C^*$-probability space* is a pair $(\mathcal{A}, \varphi)$ such that $\mathcal{A}$ is a $C^*$-algebra and $\varphi\colon \mathcal{A} \to \mathbb{C}$ is a state.
A *(tracial) $W^*$-probability space* is a pair $(\mathcal{A}, \tau)$ such that $\mathcal{A}$ is a von Neumann algebra and $\tau\colon \mathcal{A} \to \mathbb{C}$ is a state on $\mathcal{A}$ which is faithful, normal and tracial.

Note that by definition every $W^*$-probability space is a $C^*$-probability space and every $C^*$-probability space is a non-commutative probability space. Further, the term 'non-commutative' here needs to be understood as 'possibly non-commutative' since commutative algebras are not excluded from the definition above. Free probability is essentially a non-commutative theory, however, in view of the central notion of *freeness*:

**Definition 2.4.4.** Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space and let $\mathcal{A}_1, \ldots, \mathcal{A}_n \subset \mathcal{A}$ be unital subalgebras. We call $\mathcal{A}_1, \ldots, \mathcal{A}_n$ *freely independent (w.r.t. $\varphi$)* or simply *free* if for each $m \geq 2$ and $a_1, \ldots, a_m \in \mathcal{A}$ with

- $\varphi(a_i) = 0$ for all $i \in \{1, \ldots, m\}$,
- $a_i \in \mathcal{A}_{j_i}$ for some $j_i \in \{1, \ldots, n\}$ for all $i \in \{1, \ldots, m\}$ and
- $j_1 \neq j_2$, $j_2 \neq j_3$, $\ldots$, $j_{m-1} \neq j_m$,

we have: $\varphi(a_1 \cdots a_m) = 0$.
Let $n, m \in \mathbb{N}_0$ with $n + m \geq 2$. Let $a_1, \ldots, a_n \in \mathcal{A}$ and let $\mathcal{A}_1, \ldots, \mathcal{A}_m \subset \mathcal{A}$ be unital subalgebras. We call $a_1, \ldots, a_n, \mathcal{A}_1, \ldots, \mathcal{A}_m$ *freely independent (w.r.t. $\varphi$)* or simply *free* if the unital algebras $\mathrm{alg}(\mathbf{1}, a_1), \ldots, \mathrm{alg}(\mathbf{1}, a_n), \mathcal{A}_1, \ldots, \mathcal{A}_m$ are free. Here $\mathrm{alg}(x_1, \ldots, x_k)$ denotes the algebra generated by the elements $x_1, \ldots, x_k \in \mathcal{A}$.
Likewise, free independence between sets of non-commutative random variables is defined via the free independence of the corresponding generated unital subalgebras.

Coming back to Example 2.4.1, it turns out that freeness of the (random) matrices $A_N$ and $B_N$ is precisely what can be observed asymptotically in the large $N$ limit: Denote by $\mathrm{Tr}^N\colon M_N(\mathbb{C}) \to \mathbb{C}$ the matrix trace and by $\mathrm{tr}_E^N$ the normalized trace-expectation, i.e.

$$\mathrm{tr}_E^N\colon \left\{M_N(\mathbb{C})\text{-valued random variable } A\colon E(|\mathrm{Tr}^N(A)|) < \infty)\right\} \to \mathbb{C}\,,$$
$$\mathrm{tr}_E^N(A) := E\left(\frac{1}{N}\,\mathrm{Tr}^N(A)\right)\,.$$

Let $I_N \in M_N(\mathbb{C})$ be the identity matrix. Then one can show that e.g. the normalized trace expectation of products of the form

$$\left(A_N^{q_1} - \mathrm{tr}_E^N(A_N^{q_1})I_N\right) \cdot \left(B_N^{p_1} - \mathrm{tr}_E^N(B_N^{p_1})I_N\right) \cdots \left(A_N^{q_m} - \mathrm{tr}_E^N(A_N^{q_m})I_N\right) \cdot \left(B_N^{p_m} - \mathrm{tr}_E^N(B_N^{p_1})I_N\right)$$

vanishes for large $N$ whenever $p_1, q_1, \ldots, p_m, q_m \in \mathbb{N}$. More on this phenomenon which is called *asymptotic freeness* of random matrices can be found in [77] Chapter 4.

We now focus on the analytic aspects of free probability. So in the following we assume that $(\mathcal{A}, \varphi)$ is a $C^*$-probability space. Let $a \in \mathcal{A}$ be a normal operator and recall that by definition $f(a) = \int_{\sigma(a)} f \, dP^a$ for bounded and measurable $f\colon \mathbb{C} \to \mathbb{C}$. We denote by $\mu_a \in M^1(\mathbb{C})$ the *$*$-distribution (or spectral distribution) of $a$ (w.r.t. $\varphi$)*, i.e. the measure defined by

$$\varphi(f(a)) = \int_{\mathbb{R}} f \, d\mu_a \,, \quad f\colon \mathbb{C} \to \mathbb{C} \text{ bounded and measurable}\,.$$

Clearly the support of $\mu_a$ is a subset of $\sigma(a)$. There are two important special cases: If $a \in \mathcal{A}$ is self-adjoint, then $\sigma(a) \subset \mathbb{R}$ and we can, and will, view $\mu_a$ as an element of $M^1(\mathbb{R})$. Since $\mu_a$ is compactly supported and is in turn characterized by its moments, it also holds that $\mu_a$ is the unique element of $M^1(\mathbb{R})$ such that

$$\varphi(a^n) = \int_{\mathbb{R}} x^n \, d\mu_a(x) \text{ for all } n \in \mathbb{N}\,.$$

If $a \in \mathcal{A}$ is unitary, then $\sigma(a) \subset \mathbb{T} := \{z \in \mathbb{C}\colon |z| = 1\}$. Hence we can view $\mu_a$ as an element of $M^1(\mathbb{T})$. Again, $\mu_a$ is characterized by its moments (see e.g. [96] Theorem 11.3) which are given by $\varphi(a^n)$, $n \in \mathbb{N}$.

### 2.4.1 Free additive convolution

We define the free additive convolution mentioned in Example 2.4.1 in two steps: First, we define this map for compactly supported probability measures via free operators in a non-commutative probability space and then, in a second step, for more general probability measures having second moments via analytic tools. The first approach here relies on the following Theorem:

**Theorem & Definition 2.4.5.** *Let $\mu_1, \mu_2 \in M^1(\mathbb{R})$ be two compactly supported probability measures. Then there exists a $C^*$-probability space inhibiting two self-adjoint non-commutative random variables $a_1$ and $a_2$ such that $a_1$ and $a_2$ are free and such that $\mu_{a_1} = \mu_1$, $\mu_{a_2} = \mu_2$. Moreover, the moments of $\mu_{a_1+a_2}$ are uniquely determined by the moments of $\mu_1$ and $\mu_2$.*
*We call $\mu_1 \boxplus \mu_2 := \mu_{a_1+a_2}$ the* free additive convolution *of $\mu_1$ and $\mu_2$.*

*Proof.* For a proof of the first fact concerning the existence of $a_1, a_2$ cf. the free product construction described in [3] Section 5.3.1. For the second statement concerning the moments of $\mu_{a_1+a_2}$ see [77] Chapter 1 Proposition 13. □

Next, we extend the definition of free additive convolution to unbounded probability measures on $\mathbb{R}$. This extension does not use any $C^*$-algebra methods but only properties

of the probability measures themselves. While one can do this in full generality (see [77] Section 3.6), for our purposes it suffices to consider the situation that the measures involved all have a second finite moment. For the desired extension we will use a measure transform which linearizes under free additive convolution; for the classical convolution of measures such a measure transform is given by the logarithm of the characteristic function:

$$\log\left(E(e^{ia(X+Y)})\right) = \log\left(E(e^{iaX})\right) + \log\left(E(e^{iaY})\right)\ , \quad a \in \mathbb{R}\,,$$

where $X$ and $Y$ are (classically) independent real random variables. In the free probability setting the details are a bit more involved: We start with the following measure transform which is very useful in its own right:

**Definition 2.4.6.** The *Cauchy-transform* $G_\mu\colon \mathbb{C}\setminus\mathbb{R} \to \mathbb{C}$ *of* $\mu \in M^1(\mathbb{R})$ is defined by

$$G_\mu(z) = \int_{\mathbb{R}} \frac{1}{z-x}\, d\mu(x)\,.$$

We collect important properties of the Cauchy-transform. For proofs, further properties and examples see [77] Chapter 3. Denote the upper half plane by $\mathbb{C}^+ := \{z \in \mathbb{C}\colon \operatorname{Im}(z) > 0\}$ and the lower half plane by $\mathbb{C}^- := \{z \in \mathbb{C}\colon \operatorname{Im}(z) < 0\}$. Then we have:

**Theorem 2.4.7.** *Let* $\nu, \mu, \mu_1, \mu_2, \cdots \in M^1(\mathbb{R})$. *Then:*

*(a)* $G_\mu(\mathbb{C}^+) \subset \mathbb{C}^-$ *and* $G_\mu\colon \mathbb{C}^+ \to \mathbb{C}^-$ *is analytic.*

*(b) If* $G_\mu = G_\nu$, *then* $\mu = \nu$.

*(c) If there exists a function* $G\colon \mathbb{C}\setminus\mathbb{R}$ *with* $G(\overline{z}) = \overline{G(z)}$, $G_{\mu_n}(z) \xrightarrow{n\to\infty} G(z)$ *for all* $z \in \mathbb{C}^+$ *and it holds* $\lim_{y\to\infty} iyG(iy) = 1$, *then there exists* $\rho \in M^1(\mathbb{R})$ *such that* $\mu_n \xrightarrow{n\to\infty} \rho$ *weakly and* $G_\rho = G$.

*(d) Assume that* $\mu$ *has finite variance* $\sigma^2$. *Then on the open disc with centre* $-i(4\sigma)^{-1}$ *and radius* $(4\sigma)^{-1}$ *there is a unique analytic function* $R_\mu$ *such that*

$$G_\mu\left(R_\mu(z) + \frac{1}{z}\right) = z \ \text{ for all } \ |z + i(4\sigma)^{-1}| < (4\sigma)^{-1}\,.$$

*If* $\sigma^2 = 0$, *i.e.* $\mu = \delta_x$ *for some* $x \in \mathbb{R}$, *the domain of* $R_\mu$ *should be interpreted as* $\mathbb{C}^-$.

**Definition 2.4.8.** For $\mu \in M^1(\mathbb{R})$ with finite variance we call $R_\mu$ in Theorem 2.4.7 (d) the *R-transform of* $\mu$.

Using the $R$-transform we can extend Definition 2.4.5 (that this actually is an extension see [77] Theorem 18 in Chapter 2, and for a proof of the statement see Theorem 28 in Chapter 3 therein):

**Theorem & Definition 2.4.9.** *Let* $\nu_1, \nu_2 \in M^1(\mathbb{R})$ *have finite variances. Then there exists a unique probability measure, denoted by* $\nu_1 \boxplus \nu_2$, *such that* $R_{\nu_1} + R_{\nu_2} = R_{\nu_1 \boxplus \nu_2}$ *on the common domain of the functions involved.*

### 2.4.2 Free multiplicative convolution

Similarly to the free additive convolution we can also treat the multiplicative case. We first motivate this convolution by the distribution of products of free operators. To do so, we assume that $(\mathcal{A}, \tau)$ is a $C^*$-probability space such that the state $\tau$ is tracial. There are actually two variants to consider: The first variant corresponds to the multiplication of two free self-adjoint non-commutative random variables $a, b \in \mathcal{A}$, one of which is non-negative. Again, the statement of the theorem follows from the results referenced in the proof of Theorem & Definition 2.4.5:

**Theorem & Definition 2.4.10.** *Let $\mu_1 \in M^1(\mathbb{R})$, $\mu_2 \in M^1([0,\infty))$ be two compactly supported probability measures. Then there exists a tracial $C^*$-probability space inhibiting two self-adjoint non-commutative random variables $a_1$ and $a_2$ such that $a_2$ is non-negative, $a_1$ and $a_2$ are free, and $\mu_{a_1} = \mu_1$, $\mu_{a_2} = \mu_2$. Moreover, the moments of $\mu_{a_2^{1/2} a_1 a_2^{1/2}}$ are uniquely determined by the moments of $\mu_1$ and $\mu_2$.*
*We call $\mu_1 \boxtimes \mu_2 := \mu_2 \boxtimes \mu_1 := \mu_{a_2^{1/2} a_1 a_2^{1/2}}$ the* free multiplicative convolution *of $\mu_1$ and $\mu_2$.*

One can extend this definition to arbitrary probability measures on $\mathbb{R}$, one of which is supported on $[0,\infty)$. However, for our purposes it suffices to only treat the case that both measures are supported on $[0,\infty)$. We will consider this extension further below.

The second variant of free multiplicative convolution corresponds to the multiplication of two unitary non-commutative random variables $a, b \in \mathcal{A}$. Here we define

$$\mu_a \boxtimes \mu_b := \mu_{ab} \in M^1(\mathbb{T})\,.$$

Again, using the free product construction mentioned in the proof of Theorem & Definition 2.4.5, one sees that this actually defines a free multiplicative convolution $\boxtimes\colon M^1(\mathbb{T}) \times M^1(\mathbb{T}) \to M^1(\mathbb{T})$. In contrast to the free additive convolution on $\mathbb{R}$ and the multiplicative convolution on $[0,\infty)$, we here do not need to extend this definition further since $\mathbb{T}$ is compact.
We will see below that in the $[0,\infty)$-case one can define an analog of the $R$-transform, called the $S$-transform, for the free multiplicative convolution. In principal one has such an $S$-transform for the $\mathbb{T}$-case as well, the details are more subtle when one considers measures with vanishing first moment, however. As we will not make use of this transform, we skip the details, for a treatment of this topic see [114] Chapter 3.

We now come back to the $[0,\infty)$-case. Our exposition is based on [16]. First, for $\mu \in M^1([0,\infty))$ we define its *$\psi$-transform* by

$$\psi_\mu(z) := \int_{[0,\infty)} \frac{xz}{1-xz}\, d\mu(x)\,, \quad z \in \mathbb{C} \setminus [0,\infty)\,.$$

Note that $z(1+\psi_\mu(z)) = G_\mu(1/z)$ and hence $\mu$ is determined by $\psi_\mu$.

**Proposition 2.4.11.** *Let $\mu \in M^1([0,\infty)) \setminus \{\delta_0\}$. Then $\psi_\mu$ is univalent in the left half plane $iC^+$ and $\psi_\mu(iC^+)$ is a domain contained in the circle with center $(\mu(\{0\})-1)/2$ and radius $(1-\mu(\{0\}))/2$. Moreover, we have $\psi_\mu(iC^+) \cap \mathbb{R} = (\mu(\{0\})-1, 0)$.*

Hence we can define the following measure transform:

**Definition 2.4.12.** Let $\mu \in M^1([0,\infty)) \setminus \{\delta_0\}$. Then we define its *$S$-transform* by

$$S_\mu(z) := \frac{1+z}{z}\chi_\mu(z)\,, \quad z \in \psi_\mu(iC^+)\,,$$

where $\chi_\mu$ is the inverse function of $\psi_\mu|_{iC^+}$.

Note that by Theorem & Definition 2.4.10 we have $\mu \boxtimes \delta_0 = \delta_0 \boxtimes \mu = \delta_0$ for all compactly supported $\mu \in M^1([0,\infty))$. When extending the free multiplicative convolution to measures which are not compactly supported we will enforce that this property is preserved. In this extension the $S$-transform plays an analogous role for free multiplicative convolution as the $R$-transform does for free additive convolution:

**Theorem & Definition 2.4.13.** *Let $\nu_1, \nu_2 \in M^1([0,\infty))$. If $\nu_i = \delta_0$ for some $i \in \{1,2\}$ then we set $\nu_1 \boxtimes \nu_2 := \delta_0$. So now assume that $m := \max(\nu_1\{0\}, \nu_2\{0\}) - 1 < 0$. Then there exists a unique $\mu \in M^1([0,\infty)) \setminus \{\delta_0\}$ such that*

$$S_\mu(z) = S_{\nu_1}(z) S_{\nu_2}(z) \tag{2.4.1}$$

*for all $z \in \psi_\mu(iC^+) \cap \psi_{\nu_1}(iC^+) \cap \psi_{\nu_2}(iC^+)$. Moreover, it holds that*

$$\mu(\{0\}) = \max\{\nu_1(\{0\}), \nu_2(\{0\}))\,.$$

*In particular,* (2.4.1) *holds on some domain containing $(m,0)$.*
*We define the* free multiplicative convolution *on $(M^1([0,\infty)) \setminus \{\delta_0\}))^2$ by $\nu_1 \boxtimes \nu_2 := \mu$.*

That this definition is actually an extension of Theorem & Definition 2.4.10 see [106] Theorem 2.6. For a proof of the statement of Theorem & Definition 2.4.13 see [16] Section 6.

### 2.4.3 A proposition for the joint distribution of freely projected operators

In this section we present a consequence of a result of [79] which already has been pointed out implicitly in [54] Section 2.1. The setting here is as follows: Let $(\mathcal{A}, \tau)$ be a tracial non-commutative probability space. Recall that this means that $\mathcal{A}$ is a unital complex algebra and $\tau\colon \mathcal{A} \to \mathbb{C}$ is a linear functional normalized by $\tau(\mathbf{1}) = 1$ such that $\tau(ab) = \tau(ba)$ for all $a, b \in \mathcal{A}$. Denote by $\mathbb{C}\langle X_1, \ldots, X_n\rangle$ the algebra of polynomials with non-commuting

indeterminates $X_1, \ldots, X_n$ with complex coefficients. For $a_1, \ldots, a_n \in \mathcal{A}$ we will call the linear functional $\mu_{a_1,\ldots,a_n} \colon \mathbb{C}\langle X_1, \ldots, X_n\rangle \to \mathbb{C}$ which is determined by

$$\mu_{a_1,\ldots,a_n}(\mathbf{1}) = 1 \text{ and}$$
$$\mu_{a_1,\ldots,a_n}(X_{i_1} \cdots X_{i_k}) = \tau(a_{i_1} \cdots a_{i_k}), \quad i_j \in \{1, \ldots, n\},\ j \in \{1, \ldots, k\},\ k \in \mathbb{N},$$

*the joint distribution of* $a_1, \ldots, a_n$. Further, we will call a self-adjoint, idempotent element $P \in \mathcal{A}$ a *projection*. We can now formulate the result:

**Proposition 2.4.14.** *Let* $(\mathcal{A}, \tau)$ *and* $(\mathcal{A}', \tau')$ *be two tracial non-commutative probability spaces. Let* $P \in \mathcal{A}$ *be a projection such that* $\tau(P) = 1/k$ *for some* $k \in \mathbb{N}$. *Let* $a_1, \ldots, a_n \in \mathcal{A}$ *such that* $\{a_1, \ldots, a_n\}$ *and* $P$ *are freely independent. Let* $a_1^i, \ldots, a_n^i \in \mathcal{A}'$, $i \in \{1, \ldots, k\}$, *such that the algebras*

$$\mathrm{alg}(\mathbf{1}, a_1^1, \ldots, a_n^1), \ldots, \mathrm{alg}(\mathbf{1}, a_1^k, \ldots, a_n^k)$$

*are freely independent and for all* $i \in \{1, \ldots, k\}$ *the joint distribution of* $a_1^i, \ldots, a_n^i$ *equals the joint distribution of* $\{a_1, \ldots, a_n\}$.
*Then the joint distribution of* $Pa_1P, \ldots, Pa_nP$ *in the non-commutative probability space* $(P\mathcal{A}P, k\tau(\cdot))$ *equals the joint distribution of*

$$\frac{1}{k}\sum_{i=1}^{k} a_1^i, \ldots, \frac{1}{k}\sum_{i=1}^{k} a_n^i\,.$$

*Proof.* For a linear functional $\mu \colon \mathbb{C}\langle X_1, \ldots, X_n\rangle$ with $\mu(\mathbf{1}) = 1$ we denote by $R(\mu)$ its $n$-dimensional $R$-transform, see [79] Section 1.2 and Definition 3.9. One important property of this multivariate $R$-transform is that, like in the one-dimensional case, it linearizes under the addition of free random variables ([79] equation (1.6)), in particular we have

$$R\left(\mu^{(\mathcal{A}')}_{\frac{1}{k}\sum_{i=1}^k a_1^i, \ldots, \frac{1}{k}\sum_{i=1}^k a_n^i}\right) = \sum_{i=1}^{k} R\left(\mu^{(\mathcal{A}')}_{\frac{1}{k}a_1^i, \ldots, \frac{1}{k}a_n^i}\right) = kR\left(\mu^{(\mathcal{A})}_{\frac{1}{k}a_1, \ldots, \frac{1}{k}a_n}\right).$$

Here the superscripts $(\mathcal{A})$, $(\mathcal{A}')$, as well as $(P\mathcal{A}P)$ below, indicate in which non-commutative probability space the non-commutative random variables in the subscripts are viewed.
On the other hand, by Application 1.11 in [79] we have

$$kR\left(\mu^{(\mathcal{A})}_{\frac{1}{k}a_1, \ldots, \frac{1}{k}a_n}\right) = R\left(\mu^{(P\mathcal{A}P)}_{Pa_1P, \ldots, Pa_nP}\right).$$

As the $R$-transform uniquely characterizes the joint distribution (see [79] equation (3.13)), the claim follows. □

## 2.5 Free stochastic calculus

In this section we give a concise introduction to free stochastic calculus. We will use this theory to describe limits of empirical measures of non-colliding $\beta$-Pearson diffusions in Section 5. Roughly speaking, free stochastic calculus can be thought of as an $N = \infty$ analogue of classical Itô calculus for $N \times N$ matrix valued Itô processes. In this theory the latter random matrix processes are replaced by norm-continuous functions $t \mapsto x_t$ where the operators $x_t$ are elements of a common von Neumann algebra.
Our exposition is based on [21] and [80] where we adopt definitions and notation of the latter reference.

Throughout this section let $(\mathcal{A}, \tau)$ be a $W^*$-probability space. Throughout we implicitly assume that $\mathcal{A}$ is large enough to inhibit the operators which we will consider below. We start by observing that, as in classical probability theory, we have a notion of conditional expectation (see [62] Theorem 7):

**Theorem & Definition 2.5.1.** *Let $\mathcal{B} \subseteq \mathcal{A}$ be a von Neumann subalgebra. Then there exists a unique linear map $f\colon \mathcal{A} \to \mathcal{B}$ with the following properties:*

*(i)* $f(\mathbf{1}) = \mathbf{1}$,

*(ii)* $f(b_1 a b_2) = b_1 f(a) b_2$ *for all* $a \in \mathcal{A}$ *and* $b_1, b_2 \in \mathcal{B}$,

*(iii)* $f(a) \in \mathcal{A}_+$ *for all* $a \in \mathcal{A}_+$.

*We call $f$ the* (non-commutative) expectation onto $\mathcal{B}$ *and write* $\tau[\cdot \mid \mathcal{B}] := f(\cdot)$.

One can show that it holds $\tau[a \mid \mathcal{B}] \neq \mathbf{0}$ if $a \in \mathcal{A}_+ \setminus \{\mathbf{0}\}$ ([62] Theorem 7). Further, analogously to classical probability theory, in the case that $a \in \mathcal{A}$ and $\mathcal{B}$ are freely independent we have the formula $\tau[a \mid \mathcal{B}] = \tau(a)\mathbf{1}$ ([80] Section 2.1).
As in the theory of stochastic calculus we moreover have the following notion of filtrations and martingales:

**Definition 2.5.2.** For $(\mathcal{A}_t)_{t\geq 0}$ a family of von Neumann subalgebras of $\mathcal{A}$ with $\mathcal{A}_s \subseteq \mathcal{A}_t$ for all $0 \leq s \leq t$ we call $(\mathcal{A}_t)_{t\geq 0}$ a *filtration of* $(\mathcal{A}, \tau)$ and $(\mathcal{A}, (\mathcal{A}_t)_{t\geq 0}, \tau)$ a *filtered $W^*$-probability space.*
Let $(\mathcal{A}_t)_{t\geq 0}$ be as above. We call a collection $(x_t)_{t\geq 0}$ with $x_t \in \mathcal{A}_t$ for all $t \geq 0$ a *free (adapted) process.* A free process $(x_t)_{t\geq 0}$ is called a *free martingale* if $\tau[x_t \mid \mathcal{A}_s] = x_s$ for all $0 \leq s \leq t$.

We can now define the non-commutative analogue of $\mathbb{R}^n$- and $\mathbb{C}^n$-valued Brownian motions (see [81] Definition 2):

**Definition 2.5.3.** We say that $(x_t = (x_{1,t}, \ldots, x_{n,t}))_{t\geq 0}$ is an *$n$-dimensional semicircular Brownian motion (w.r.t. $(\mathcal{A}_t)_{t\geq 0}$)* if the following four conditions are met:

- (self-adjoint & adapted) $x_{i,t} \in \mathcal{A}_t$ is self-adjoint for all $i \in \{1, \ldots, n\}$ and $t \geq 0$.
- (start in $\mathbf{0}$) $x_{i,0} = \mathbf{0}$ for all $i \in \{1, \ldots, n\}$.
- (jointly free increments) $\{x_{1,t} - x_{1,s}, \ldots, x_{n,t} - x_{n,s}\}$ is freely independent of $\mathcal{A}_s$ for all $0 \leq s \leq t$.
- (semicircular distribution) $x_{1,t} - x_{1,s}, \ldots, x_{n,t} - x_{n,s}$ are freely independent operators, each having distribution $\mu_{\mathrm{sc},2\sqrt{t-s}}$ for all $0 \leq s \leq t$.

We say that $(z_t = (z_{1,t}, \ldots, z_{n,t}))_{t \geq 0}$ is an *n-dimensional circular Brownian motion* if

$$\Big(\sqrt{2} \cdot (\mathrm{Re}(z_{1,t}), \mathrm{Im}(z_{1,t}), \ldots, \mathrm{Re}(z_{n,t}), \mathrm{Im}(z_{n,t}))\Big)_{t \geq 0}$$

is a $2n$-dimensional semicircular Brownian motion, where $\mathrm{Re}(a) := \frac{1}{2}(a + a^*)$ and $\mathrm{Im}(a) := \frac{1}{2i}(a - a^*)$ for $a \in \mathcal{A}$.

In the literature, the semicircular Brownian motion is sometimes simply called 'free Brownian motion' ([19, 20]) and circular Brownian motion 'complex free Brownian motion' ([25]). As in the classical case one can readily show that each component of a (semi-)circular Brownian motion is a free martingale.

The following Theorem is the free probability analogue of Lévy's characterization of Brownian motion ([18] Theorem 6.2):

**Theorem 2.5.4.** *Let $(\mathcal{A}, \tau)$ be a $W^*$-probability space and let $(\mathcal{A}_t)_{t \geq 0} \subset \mathcal{A}$ be a filtration. If*

$$(x_t = (x_{1,t}, \ldots, x_{n,t}))_{t \geq 0}$$

*is an $n$-tuple of self-adjoint processes such that $(x_{i,t})_{t \geq 0}$ is an $(\mathcal{A}_t)_{t \geq 0}$ free martingale with $x_{i,0} = \mathbf{0}$ for all $i \in \{1, \ldots, n\}$, and for all $0 \leq s < t$ one has*

*(i) $\tau(|x_{i,t} - x_{i,s}|^4) \leq K(t-s)^2$ for some constant $K > 0$ and all $i \in \{1, \ldots, n\}$,*

*(ii) for any $i, j \in \{1, \ldots, n\}$ and all $A, B \in \mathcal{A}_s$ one has*

$$\tau(A x_{i,t} B x_{j,t}) = \tau(A x_{i,s} B x_{j,s}) + \tau(A)\tau(B)(t-s)\mathbb{1}_{i=j} + o(t-s)\,,$$

*then $(x_t)_{t \geq 0}$ is an $n$-dimensional semicircular Brownian motion.*

*Proof.* Note that by the Hölder inequality for traces (see e.g. [97] Theorem 2.1.4) we have

$$\tau(|x_{i,t} - x_{i,s}|^2 |x_{j,t} - x_{j,s}|^2) \leq \tau(|x_{i,t} - x_{i,s}|^4)^{1/2} \tau(|x_{j,t} - x_{j,s}|^4)^{1/2}\,.$$

Hence the statement is equivalent to Theorem 6.2 in [18] when the time parameter $t$ is restricted to $[0, 1]$. It is easy to see that the proof given there also applies in the case of general $t \geq 0$. □

Next, we want to define the free version of the classical Itô integral w.r.t. Brownian motion. To specify the space of simple integrands we need some additional definitions. In particular, the notion of tensor products of Hilbert spaces and bounded operators thereon will be important, we refer to [63] Section 2.6 for a proper treatment of these topics.
Let $H_1, H_2$ be two Hilbert spaces and denote by $H_1 \otimes H_2$ their vector space tensor product. There is a unique inner product $\langle \cdot, \cdot \rangle_{H_1 \otimes_2 H_2}$ on $H_1 \otimes H_2$ such that $\langle h_1 \otimes h_2, g_1 \otimes g_2 \rangle_{H_1 \otimes_2 H_2} = \langle h_1, g_1 \rangle \cdot \langle h_2, g_2 \rangle$ for all $h_1, g_1 \in H_1$, $h_2, g_2 \in H_2$. We denote by $H_1 \otimes_2 H_2$ the completion of $H_1 \otimes H_2$ w.r.t. the norm induced by $\langle \cdot, \cdot \rangle_{H_1 \otimes_2 H_2}$. One can show that there is an injective $*$-homomorphism $\mathcal{B}(H_1) \otimes \mathcal{B}(H_2) \to \mathcal{B}(H_1 \otimes_2 H_2)$ which allows us to view $\mathcal{B}(H_1) \otimes \mathcal{B}(H_2)$ (and $*$-subalgebras thereof) as a $*$-subalgebra of $\mathcal{B}(H_1 \otimes_2 H_2)$.
Now fix some Hilbert space $H$. For a von Neumann algebra $\mathcal{A} \subset \mathcal{B}(H)$ we denote by $\mathcal{A}^{op}$ the *opposite von Neumann algebra of* $\mathcal{A}$, i.e. $\mathcal{A}^{op} = \mathcal{A}$ as sets and $\mathcal{A}^{op}$ has the same addition, involution $\cdot^*$ and topological structure as $\mathcal{A}$ but the opposing multiplication $a \cdot^{op} b := ba$ where the latter product is in $\mathcal{A}$. If $\tau$ is a tracial state on $\mathcal{A}$, we write $\tau^{op}(a) := \tau(a)$ for the induced tracial state on $\mathcal{A}^{op}$ to stress the algebra structure which we are considering. Now consider the tensor product $\mathcal{A} \otimes \mathcal{A}^{op}$ which by the aforementioned is a $^*$-subalgebra of $\mathcal{B}(H \otimes_2 H)$ with the small subtlety that the multiplication is here given by

$$\left( \sum_{i=1}^n a_i \otimes b_i \right) \left( \sum_{j=1}^m c_i \otimes d_i \right) = \sum_{i=1}^n \sum_{j=1}^m a_i c_i \otimes d_i b_i \quad \text{for all} \quad a_i, b_i, c_i, d_i \in \mathcal{A}\,.$$

We will need the following operations on $\mathcal{A} \otimes \mathcal{A}^{op}$: For $u = \sum_{j=1}^n a_j \otimes b_j \in \mathcal{A} \otimes \mathcal{A}^{op}$ and $x \in \mathcal{A}$ we set $u \# x := \sum_{j=1}^n a_j x b_j$ where the products on the r.h.s. are in $\mathcal{A}$. We write $\cdot^{\text{flip}} \colon \mathcal{A} \otimes \mathcal{A}^{op} \to \mathcal{A} \otimes \mathcal{A}^{op}$ for the linear map which is determined by $(a \otimes b)^{\text{flip}} = b \otimes a$. Building on this, we write $u^\star := (u^*)^{\text{flip}}$ for $u \in \mathcal{A} \otimes \mathcal{A}^{op}$ where $\cdot^*$ is the involution of $\mathcal{B}(H \otimes_2 H)$.

**Definition 2.5.5** ([80] 3.1.3-3.1.5)**.** Let $(\mathcal{A}, (\mathcal{A}_t)_{t \geq 0}, \tau)$ be a filtered $W^*$-probability space. We call a map $u_\cdot \colon [0, \infty) \to \mathcal{A} \otimes \mathcal{A}^{op}$ a *simple adapted biprocess* if there exists times $0 = t_0 < \cdots < t_n$ such that we have $u_{t_i} \in \mathcal{A}_{t_i} \otimes \mathcal{A}_{t_i}^{op}$ for all $i \in \{0, \ldots, n-1\}$ and

$$u_t = \sum_{i=0}^{n-1} u_{t_i} \mathbb{1}_{[t_i, t_{i+1})}(t) \quad \text{for all} \quad t \geq 0\,. \tag{2.5.1}$$

We denote the set of all simple adapted biprocesses by $\mathbb{S}_a$.
Let $u \in \mathbb{S}_a$ and $m_\cdot \colon [0, \infty) \to \mathcal{A}$. Let $0 = t_0 < \cdots < t_n$ be times such that (2.5.1) holds for $u$. Then we define the integral of $u$ w.r.t. $m$ by

$$\int_0^\infty u_t \# dm_t := \int_0^\infty u \# dm := \sum_{i=0}^{n-1} u_{t_i} \# (m_{t_{i+1}} - m_{t_i})\,.$$

Further we set for $0 \leq s \leq t$

$$\int_s^t u_r \# dm_r := \int_s^t u \# dm := \int_0^\infty \left( \mathbb{1}_{[s,t)}(r) u_r \right) \# dm_r.$$

As in classical stochastic calculus (see e.g. [95] Chapter 14 problem 2) it is easy to verify that these integrals are well-defined, i.e. they do not depend on the chosen representation of $u$ in (2.5.1). Moreover, note that the integral of simple adapted processes is a sum of terms where one multiplies the increments of the integrator from the left and from the right by elements of $\mathcal{A}$ which is natural in the non-commutative context. In particular, one has the same structure in classical matrix Itô calculus albeit the construction of the Itô integral in this particular context usually is done on the level of individual matrix components and not via matrix biprocesses.
We list immediate properties of the integral:

**Lemma 2.5.6.** *Let $u \in \mathbb{S}_a$ and $m_\cdot \colon [0,\infty) \to \mathcal{A}$. Then:*

*(a)* $\mathbb{S}_a \to \mathcal{A}$, $u \mapsto \int_0^\infty u \# dm$ *is linear.*

*(b)* $\left(\int_0^\infty u \# dm\right)^* = \int_0^\infty u^\star \# dm^*$.

*(c)* $\int_r^s u_t \# dm_t = \int_0^s u_t \# dm_t - \int_0^r u_t \# dm_t$ *for all* $0 \leq r \leq s$.

*(d) If $m$ is adapted then $t \mapsto \int_0^t u \# dm$ is adapted.*

*(e) If $m$ is a one-dimensional $(\mathcal{A}_t)_{t\geq 0}$ semicircular or circular Brownian motion then $t \mapsto \int_0^t u \# dm$ is an $(\mathcal{A}_t)_{t\geq 0}$ free martingale.*

In order to extend this integral from simple adapted biprocesses to more general processes we need a suitable closure of $\mathbb{S}_a$. To this end we first define a suitable closure of $\mathcal{A} \otimes \mathcal{A}^{op}$: The *minimal (or spatial) $C^*$-algebra tensor product* $\mathcal{A} \otimes_{\min} \mathcal{A}^{op}$ is defined as the norm closure of $\mathcal{A} \otimes \mathcal{A}^{op}$ in $\mathcal{B}(H \otimes_2 H)$. For details on the construction and properties of the minimal $C^*$-algebra tensor product see e.g. [64] Section 11.1 or [98] Section 6.1.
We write $\tau \otimes \tau^{op}$ for the unique state on $\mathcal{A} \otimes_{\min} \mathcal{A}^{op}$ which is determined by $(\tau \otimes \tau^{op})(a \otimes b) = \tau(a)\tau^{op}(b)$, see [64] Proposition 11.1.1. Note that the linear isometry $\cdot^{\text{flip}} \colon \mathcal{A} \otimes \mathcal{A}^{op} \to \mathcal{A} \otimes \mathcal{A}^{op}$ (and hence also $\cdot^\star$) can be uniquely extended to a linear isometry on $\mathcal{A} \otimes_{\min} \mathcal{A}^{op}$ which we denote by the same decoration.
Next, consider the Fréchet space of locally square Bochner-integrable functions

$$L^2_{\text{loc}}\left([0,\infty), \mathcal{A} \otimes_{\min} \mathcal{A}^{op}\right) := \Bigg\{ u_\cdot \colon [0,\infty) \to \mathcal{A} \otimes_{\min} \mathcal{A}^{op} \colon u \text{ is strongly measurable and } \int_0^t ||u_s||^2_{\mathcal{B}(H \otimes_2 H)}\, ds < \infty \text{ for all } t \geq 0 \Bigg\}$$

with the topology induced by the seminorms $u \mapsto \left(\int_0^t ||u_s||^2_{\mathcal{B}(H\otimes_2 H)}\, ds\right)^{1/2}$, $t \geq 0$; see e.g. [27] Appendix E for the notions of strongly measurable and Bochner integrable functions. Denote by

$$\Lambda^2 := \overline{\mathbb{S}_a}^{L^2_{\text{loc}}([0,\infty), \mathcal{A} \otimes_{\min} \mathcal{A}^{op})}$$

the closure of $\mathbb{S}_a$ in $L^2_{\mathrm{loc}}([0,\infty), \mathcal{A} \otimes_{\min} \mathcal{A}^{op})$. We will now extend the stochastic integral from $\mathbb{S}_a$ to $\Lambda^2$: It holds the free Burkholder-Davis-Gundy inequality (Theorem 3.2.1 in [21] and Theorem 3.1.8 in [80]):

$$\left|\left|\int_0^\infty u \# dx\right|\right| \leq 2\sqrt{2} \left(\int_0^\infty ||u_t||^2_{\mathcal{B}(H \otimes_2 H)}\, dt\right)^{1/2} \quad \text{and} \tag{2.5.2}$$

$$\left|\left|\int_0^\infty u \# dz\right|\right| \leq 4 \left(\int_0^\infty ||u_t||^2_{\mathcal{B}(H \otimes_2 H)}\, dt\right)^{1/2} \tag{2.5.3}$$

for all $u \in \mathbb{S}_a$ and $x.\colon [0,\infty) \to \mathcal{A}$, respectively $z.\colon [0,\infty) \to \mathcal{A}$, a one-dimensional semicircular, respectively circular, Brownian motion. If $m$ is a (semi-)circular Brownian motion, it follows that for fixed $s \geq 0$ the map

$$\mathbb{S}_a \to C([s,\infty), \mathcal{A})\,,\ u \mapsto \left(t \mapsto \int_s^t u_r \# dm_r\right) \tag{2.5.4}$$

is linear and continuous, where the topology on $\mathbb{S}_a$ is the one of $L^2_{\mathrm{loc}}([0,\infty), \mathcal{A} \otimes_{\min} \mathcal{A}^{op})$ and the topology on $C([s,\infty), \mathcal{A})$ is the one which is induced by the seminorms $x \mapsto \sup_{r \in [s,t]} ||x_r||$, $t \geq s$. By the continuous linear extension theorem for topological vector spaces (see e.g. [102] Theorem 5.1) we hence can extend the integral from $\mathbb{S}_a$ to $\Lambda^2$:

**Definition 2.5.7.** Let $m$ be a one-dimensional semicircular or circular Brownian motion and $u \in \Lambda^2$. Then we write

$$t \mapsto \int_s^t u_r \# dm_r$$

for the evaluation of the unique continuous linear extension (from $\mathbb{S}_a$ to $\Lambda^2$) of the map (2.5.4) at the point $u$ .

We mention that one can also use $L^p$ theory to extend the integral, in particular one also has an Itô isometry in the $p = 2$ case as in the classical setting, see [21] Section 3.1.

We now list some fundamental properties of the extended non-commutative stochastic integral:

**Lemma 2.5.8.** *Let $(\mathcal{A}_t)_{t \geq 0} \subset \mathcal{A}$. Let $m$ be either a one-dimensional semicircular or circular $(\mathcal{A}_t)_{t \geq 0}$ Brownian motion. Let $u \in \Lambda^2$. Then it holds:*

*(a) $t \mapsto \int_0^t u_r \# dm_r$ is a free martingale w.r.t. $(\mathcal{A}_t)_{t \geq 0}$.*

*(b) $\int_s^t u_r \# dm_r = \int_0^t u_r \# dm_r - \int_0^s u_r \# dm_r$ and $\left(\int_s^t u_r \# dm_r\right)^* = \int_s^t u_r^\star \# dm_r^*$ for all $0 \leq s \leq t$.*

*(c) It holds (2.5.2), if $m$ is a semicircular Brownian motion, and (2.5.3), if $m$ is a circular Brownian motion.*

As a natural next step towards free Itô calculus we define free Itô processes (semi-martingales):

**Definition 2.5.9.** We call a map $k_\cdot\colon [0,\infty) \to \mathcal{A}$ *locally Bochner integrable* if

$$k|_{[0,t]}\colon ([0,t], \mathcal{B}([0,t]), \mathrm{Leb}) \to (\mathcal{A}, ||\cdot||)$$

is Bochner integrable for all $t > 0$; here Leb denotes the Lebesgue measure. For such a $k$ we denote by $\int_0^t k_s\, ds$ the corresponding *Bochner integral* over $[0,t]$.
Let $(x_t)_{t\geq 0}$ be an $n$-dimensional $(\mathcal{A}_t)_{t\geq 0}$ semicircular Brownian motion for some filtration $(\mathcal{A}_t)_{t\geq 0} \subset \mathcal{A}$. We call $m_\cdot\colon [0,\infty) \to \mathcal{A}$ a *free Itô process* if $m_0 \in \mathcal{A}_0$ and if there exists $u_i \in \Lambda^2$, $i \in \{1,\ldots,n\}$, and $k_\cdot\colon [0,\infty) \to \mathcal{A}$ locally Bochner integrable and adapted such that

$$dm_t = \sum_{i=1}^n u_{i,t}\#dx_{i,t} + k_t\, dt\,, \text{ i.e.}$$
$$m_t = m_0 + \sum_{i=1}^n \int_0^t u_{i,s}\#dx_{i,s} + \int_0^t k_s\, ds \text{ for all } t \geq 0\,.$$

If $w_\cdot\colon [0,\infty) \to \mathcal{A}\otimes\mathcal{A}^{op}$ is continuous and adapted we write

$$w_t\#dm_t := \sum_{i=1}^n (w_t u_{i,t})\#dx_{i,t} + w_t\#k_t\, dt\,, \text{ i.e.}$$
$$\int_0^t w_s\#dm_s := w_0\#m_0 + \sum_{i=1}^n \int_0^t (w_s u_{i,s})\#dx_{i,s} + \int_0^t w_s\#k_s\, ds \text{ for all } t \geq 0\,.$$

If $w_t = a_t \otimes b_t$ for some continuous and adapted $(a_t)_{t\geq 0} \subset \mathcal{A}$,$(b_t)_{t\geq 0} \subset \mathcal{A}^{op}$, then we also write $a_t\, dm_t\, b_t := (a_t \otimes b_t)\#dm_t$.

One can readily check that a free Itô process is continuous in operator norm and adapted. Further, $w_t\#dm_t$ is well defined as $t \mapsto w_t u_t$ is in $\Lambda^2$ ([80] Corollary 3.1.12) and $t \mapsto w_t\#k_t$ is locally Bochner integrable. Moreover, we will use analogous notations as above for integrals w.r.t. an $n$-dimensional circular Brownian motion as these can be regarded as free Itô processes w.r.t. an $2n$-dimensional semicircular Brownian motion.
To state the Itô product rule in our non-commutative context we define the bilinear map $Q_\tau\colon (\mathcal{A}\otimes\mathcal{A}^{op}) \times (\mathcal{A}\otimes\mathcal{A}^{op}) \to \mathcal{A}$ which is determined by

$$Q_\tau(a\otimes b, c\otimes d) = \tau(bc)ad\,.$$

In [21] this map is denoted by $\langle\langle\cdot,\cdot\rangle\rangle$. Using the norm estimate

$$||Q_\tau(a,b)|| \leq ||a||_{\mathcal{B}(H\otimes_2 H)}||b||_{\mathcal{B}(H\otimes_2 H)}$$

one can uniquely extend $Q_\tau$ to a bilinear map on $(\mathcal{A}\otimes_{\min}\mathcal{A}^{op}) \times (\mathcal{A}\otimes_{\min}\mathcal{A}^{op})$ which we will denote by the same symbol. We now can formulate the free Itô product rule:

**Theorem 2.5.10** (Free Itô product rule ([80] Theorem 3.2.5)).

*(a) Let $(x_t)_{t\geq 0}$ be an $n$-dimensional semicircular Brownian motion, $u_{j,i} \in \Lambda^2$ and $k_{j,\cdot}\colon [0,\infty) \to \mathcal{A}$ locally Bochner integrable for all $i \in \{1,\dots,n\}$ and $j \in \{1,2\}$. Consider free Itô processes $dm_{j,t} := \sum_{i=1}^n u_{j,i,t}\#dx_{i,t} + k_{j,t}\,dt$, $j \in \{1,2\}$. Then it holds*

$$d(m_{1,t}m_{2,t}) = m_{1,t}\,dm_{2,t} + dm_{1,t}\,m_{2,t} + \sum_{i=1}^n Q_\tau(u_{1,i,t}, u_{2,i,t})\,dt\,.$$

*(b) Let $(z_t)_{t\geq 0}$ be an $n$-dimensional circular Brownian motion, $u_{j,i}, v_{j,i} \in \Lambda^2$ and $k_{j,\cdot}\colon [0,\infty) \to \mathcal{A}$ locally Bochner integrable for all $i \in \{1,\dots,n\}$ and $j \in \{1,2\}$. Consider free Itô processes $dm_{j,t} := \sum_{i=1}^n \left(u_{j,i,t}\#dz_{i,t} + v_{j,i,t}\#dz_{i,t}^*\right) + k_{j,t}\,dt$, $j \in \{1,2\}$. Then it holds*

$$d(m_{1,t}m_{2,t}) = m_{1,t}\,dm_{2,t} + dm_{1,t}\,m_{2,t} + \sum_{i=1}^n \left(Q_\tau(u_{1,i,t}, v_{2,i,t}) + Q_\tau(v_{1,i,t}, u_{2,i,t})\right) dt\,.$$

One also has Itô formulas for polynomials and so called non-commutative $C^2$-functions of free Itô processes, see [80] Theorems 3.4.4 & 4.3.4. We however do not need this level of generality in this work and thus only state a formula for the dynamic of traces of powers of free Itô processes. Our motivation is the following: In Section 5 we will be interested in the distributions of certain free Itô processes. As these distributions are compactly supported we can characterize them by their moments. Hence the following traced free Itô formula will prove quite useful (Theorem 3.5.3 in [80] where also more general $C^2$-functions are considered):

**Theorem 2.5.11** (Traced Itô formula).

*(a) Let $(x_t)_{t\geq 0}$ be an $n$-dimensional circular Brownian motion, $u_i \in \Lambda^2$ for all $i \in \{1,\dots,n\}$ and $k_\cdot\colon [0,\infty) \to \mathcal{A}$ locally Bochner integrable and adapted. Let $(m_t)_{t\geq 0}$ be a free Itô process with $dm_t = \sum_{i=1}^n u_{i,t}\#dx_{i,t} + k_t\,dt$. Then for all $t \geq 0$ it holds*

$$\begin{aligned}&\tau(m_t^n) - \tau(m_0^n)\\ =&n\int_0^t \left(\tau(k_s m_s^{n-1}) + \frac{1}{2}\sum_{i=1}^n \sum_{k=0}^{n-2} (\tau\otimes\tau^{op})\left(u_{i,s}^{\text{flip}}\cdot(m_s^k\otimes m_s^{n-2-k})\cdot u_{i,s}\right)\right) ds\,.\end{aligned}$$

*(b) Let $(z_t)_{t\geq 0}$ be an $n$-dimensional circular Brownian motion, $u_i, v_i \in \Lambda^2$ for all $i \in \{1,\dots,n\}$ and $k_\cdot\colon [0,\infty) \to \mathcal{A}$ locally Bochner integrable and adapted. Let $(m_t)_{t\geq 0}$ be a free Itô process with $dm_t = \sum_{i=1}^n \left(u_{i,t}\#dz_{i,t} + v_{i,t}\#dz_{i,t}^*\right) + k_t\,dt$. Then for all $t \geq 0$ it holds*

$$\begin{aligned}&\tau(m_t^n) - \tau(m_0^n)\\ =&n\int_0^t \left(\tau(k_s m_s^{n-1}) + \sum_{i=1}^n \sum_{k=0}^{n-2} (\tau\otimes\tau^{op})\left(v_{i,s}^{\text{flip}}\cdot(m_s^k\otimes m_s^{n-2-k})\cdot u_{i,s}\right)\right) ds\,.\end{aligned}$$

**Remark 2.5.12.** The inclined reader might have noticed the following: In the construction of the free Itô integral in [80] the closure of $\mathbb{S}_a$ is not taken in $L^2_{\mathrm{loc}}([0,\infty), \mathcal{A} \otimes_{\min} \mathcal{A}^{op})$ but in $L^2_{\mathrm{loc}}([0,\infty), \mathcal{A} \overline{\otimes} \mathcal{A}^{op})$ where $\mathcal{A} \overline{\otimes} \mathcal{A}^{op}$ is the von Neumann algebra tensor product (see [64] Section 11.2 for the latter notion). As pointed out in [80] Remark 3.1.7, both closures yield the same space, however, since $\mathcal{A} \otimes_{\min} \mathcal{A}^{op} \subset \mathcal{A} \overline{\otimes} \mathcal{A}^{op}$ and the operator norm topology on $\mathcal{A} \otimes_{\min} \mathcal{A}^{op}$ is coarser than the weak operator topology on $\mathcal{A} \overline{\otimes} \mathcal{A}^{op}$. This also explains why in Theorem 2.5.11 one can use $\tau \otimes \tau^{op}$ instead of $\tau \bar{\otimes} \tau^{op}$ which is, in the notation of [80], the normal state on the von Neumann algebra tensor product $\mathcal{A} \overline{\otimes} \mathcal{A}^{op}$ which is determined by $a \otimes b \mapsto \tau(a)\tau^{op}(b)$ (see [64] Prop. 11.2.7 for this latter notion).

# Chapter 3

# The class of non-colliding $\beta$-Pearson diffusions

In this chapter, we study a class of multivariate stochastic processes which includes all three classical eigenvalue processes from random matrix theory, i.e. Dyson's Brownian motion, Wishart & Laguerre processes and the Jacobi process. This class, called non-colliding Pearson diffusions, has been introduced by Theodoros Assiotis in [10] Section 2 in the case that the underlying matrix models are hermitian. Recall that in random matrix theory one often encodes the dimension over the reals of the underlying (skew-)field of the random matrix model by $\beta$ where $\beta = 1$ corresponds to real matrices, $\beta = 2$ to complex matrices and $\beta = 4$ to matrices over the quaternion numbers. With this convention, the non-colliding Pearson diffusions in [10] correspond to the case $\beta = 2$. On the other hand, it is known that the three classical dynamical random eigenvalue models referred to above can be extended to general $\beta > 0$ by stochastic calculus. Thus we will also consider general parameter $\beta$ for the other instances of Pearson-diffusions. It is known, however, that for Dyson-Brownian motions, squared Bessel processes, Jacobi processes and radial Heckman-Opdam processes of type $A$ a parameter $\beta < 1$ leads to collisions of particles a.s. ([33] Proposition 1, [31] Proposition 3.1, [26] Theorem 7.1). As we are interested in non-colliding solutions, we will not cover the case $0 < \beta < 1$.

The content of this chapter is as follows: First, we define non-colliding $\beta$-Pearson diffusions in Section 3.1. Then, we recapitulate known results from the literature regarding Dyson Brownian motions, squared Bessel processes, Jacobi processes and radial Heckman-Opdam processes of type $A$ in Sections 3.2, 3.3, 3.5 and parts of 3.4. What is more, in Section 3.4 we state new results regarding non-colliding solutions and invariant probability measures/stationary solutions of non-colliding inhomogeneous geometric Brownian motions in the case that one of the involved parameters is positive and for general $\beta \geq 1$. Similarly, we will state new results regarding non-colliding solutions and invariant probability measures/stationary solutions of non-compact Jacobi/Fisher-Snedecor processes and Hua-Pickrell diffusions in Sections 3.6 & 3.7 for general $\beta \geq 1$. So far, some of these

results are only known in the case $\beta = 2$ by [9, 10]. Moreover, in Section 3.6 we will prove new results regarding non-compact Jacobi/Fisher-Snedecor processes. Concretely, we show that for some parameters these are eigenvalue processes of certain dynamical random matrices built from the Brownian motion on the positive definite matrices (Subsection 3.6.2), and that, using the right rescaling, one can approximate non-colliding geometric Brownian motions by these processes (Subsection 3.6.3).

## 3.1 General form of the model

Denote by $C_N^A := \{x \in \mathbb{R}^N \colon x_1 \leq \cdots \leq x_N\}$ the closed Weyl-chamber of type $A_{N-1}$ (see Section 2.3 for this naming convention) and let $a(x) := a_2x^2 + a_1x + a_0$, $a_1, a_2, a_3 \in \mathbb{R}$, be a polynomial of degree $\leq 2$, such that the set $I_a := \{x \in \mathbb{R} \colon a(x) \geq 0\}$ has non empty interior. Let $b(x) = b_1x + b_0$ for some $b_1, b_0 \in \mathbb{R}$ be an affine linear function. In this chapter we will study solutions $(X_t = (X_{1,t}, \ldots, X_{N,t}))_{t\geq 0}$ to SDEs of the following form:

$$dX_{i,t} = \sqrt{2a(X_{i,t})}\, dB_{i,t} + \beta \left[\frac{1}{2} b(X_{i,t}) + a(X_{i,t}) \sum_{j\colon j\neq i} \frac{1}{X_{i,t} - X_{j,t}}\right] dt\,, \tag{3.1.1}$$

$i \in \{1, \ldots, N\}$, $X_0 = \hat{x}_0$ for some deterministic $\hat{x}_0 \in A$, where the state space $A$ of the solution $(X_t)_{t\geq 0}$ is some connected subset of $C_N^A \cap (I_a)^N$ with non-empty interior. We will call a solution $(X_t)_{t\geq 0}$ to this SDE which satisfies

$$X_{1,t} < \cdots < X_{N,t} \text{ for all } t > 0 \text{ a.s.}$$

a *non-colliding $\beta$-Pearson diffusion* with parameters $a_0, a_1, a_2, b_0, b_1$. By the results of [51], for $\beta \in \{1, 2\}$ these SDEs are solved by the eigenvalue processes of matrix-valued Itô diffusions $(M_t)_{t\geq 0}$ satisfying certain matrix-SDEs. In [51], more complicated diffusion and drift coefficients than the ones given here are studied. However, we will see that our particular choice of diffusion and drift coefficient in (3.1.1) will enable us to find explicit parameter ranges for $b_1, b_0$ in dependence on $N, a, \beta$ under which unique solutions to this system of SDEs exist. Moreover, we will be able to explicitly state the corresponding stationary invariant probability measures whenever they exist.
In [10] it is shown that in the case $\beta = 2$ the transition probability densities of solutions to such SDEs can be written in terms of the transition probability densities of the corresponding $N = 1$ processes via determinantal formulas. The $N = 1$ case, in turn, was studied via the corresponding Fokker-Planck equations for the transition probability densities by Eugene Wong in [113]. As mentioned in the introductory Chapter 1, Wong showed that, in case of existence, the stationary distributions of such processes are Pearson distributions, i.e. probability measures on $\mathbb{R}$ which are absolutely continuous w.r.t. Lebesgue measure and whose probability density function $f$ satisfies the differential equation

$$\frac{d}{dx} f(x) = \frac{ax + b}{cx^2 + dx + e} f(x)\,,$$

for some $a, b, c, d, e \in \mathbb{R}$. For more information on these measures see e.g. [29] Section 19.4 and references therein. This connection to Pearson distributions explains the name 'Pearson' in the term 'non-colliding $\beta$-Pearson diffusions'. We will see that for $N > 1$ and certain choices of coefficient functions $a(x)$ and $b(x)$, invariant probability measures of the dynamic (3.1.1) include (image measures of) Gaussian-$\beta$-Ensembles, $\beta$-Laguerre ensembles, $\beta$-Jacobi ensembles and Hua-Pickrell measures (Cauchy-$\beta$-ensembles); see [48] for an introduction to these measures and their connection to random matrix theory.
We will also study the following system of ODEs which is closely related to the system of SDEs (3.1.1):

$$\frac{d}{dt}x_{i,t} = b(x_{i,t}) + 2a(x_{i,t}) \sum_{j\colon j\neq i} \frac{1}{x_{i,t} - x_{j,t}}\,, \tag{3.1.2}$$

$i \in \{1, \dots, N\}$, $x_0 = \hat{x}_0 \in A$, where, again, $A$ is some connected subset of $C_N^A \cap (I_a)^N$ with non-empty interior. Formally, solutions $(x_t = (x_{1,t}, \dots, x_{N,t}))_{t\geq 0}$ to (3.1.2) arise as limits $x_t = \lim_{\beta\to\infty} X^\beta_{2t/\beta}$, where $(X^\beta_t)_{t\geq 0}$ is a solution to (3.1.1). We will thus understand (3.1.2) to be a $\beta = \infty$ variant of (3.1.1). In mathematical physics contexts the parameter $\beta$ is often interpreted as inverse temperature of the system, hence we will also refer to solutions to (3.1.2) as *frozen non-colliding Pearson diffusions*.

Because of the singularities appearing in (3.1.1) and (3.1.2), we now need to specify what we actually mean by solutions to these equations. For equation (3.1.2) we consider the following notion where we denote by $\operatorname{int} C_N^A$ the interior $\{x \in \mathbb{R}^N \colon x_1 < \cdots < x_N\}$ of the set $C_N^A$:

**Definition 3.1.1.** A *non-colliding solution* $(x_t)_{t\geq 0}$ *to* (3.1.2) *with start in* $\hat{x}_0$ is a continuous function $x_\cdot \colon [0, \infty) \to A$ with $x_0 = \hat{x}_0$ such that for $t > 0$ it holds $x_t \in (\operatorname{int} C_N^A) \cap (I_a)^N$ and $x_t$ satisfies (3.1.2). We will call such a solution *unique* if there exists not another such solution.

We will use the same notion of unique and strong solutions to (3.1.1) as in [87] Chapter IX §1, i.e.:

**Definition 3.1.2.** A pair $(X, B) = ((X_t)_{t\geq 0}, (B_t)_{t\geq 0})$ of $N$-dimensional real stochastic processes will be called a *non-colliding solution to* (3.1.1) *with start in* $\hat{x}_0$ if the following four conditions are met:

(i) The processes $(X_t)_{t\geq 0}$ and $(B_t)_{t\geq 0}$ are defined on the same filtered probability space $(\Omega, (\mathcal{F}_t)_{t\geq 0}, P)$ and both are adapted to $(\mathcal{F}_t)_{t\geq 0}$.

(ii) $(B_t)_{t\geq 0}$ is a standard real $N$-dimensional Brownian motion with respect to $(\mathcal{F}_t)_{t\geq 0}$.

(iii) $X_0 = \hat{x}_0$ a.s. and $X_t \in (\operatorname{int} C_N^A) \cap (I_a)^N$ for all $t > 0$ a.s.

(iv)

$$X_{i,t} - X_{i,0} = \int_0^t \sqrt{2a(X_{i,s})}\, dB_{i,s} + \beta \int_0^t \left[ \frac{1}{2} b(X_{i,s}) + a(X_{i,s}) \sum_{j\colon j\neq i} \frac{1}{X_{i,s} - X_{j,s}} \right] ds\,,$$

for all $t > 0$ a.s., $i \in \{1, \ldots, N\}$, where the Riemann integral is understood, if needed, as an improper Riemann integral.

We call a non-colliding solution $(X, B)$ to (3.1.1) *strong*, if $X$ is adapted to the completion of the natural filtration $\mathcal{F}_t^B := \sigma\{B_s \colon 0 \leq s \leq t\}$ of $B$.
We call a non-colliding solution $(X, B)$ to (3.1.1) *(pathwise) unique*, if whenever $(X', B')$ is another non-colliding solution to (3.1.1) defined on the same probability space $(\Omega, (\mathcal{F}_t)_{t\geq 0}, P)$ and $B = B'$ a.s. then $X$ and $X'$ are indistinguishable.
If it is clear from the context which Brownian motion $B$ we are considering, we simply say that $X$ is a non-colliding solution (possibly strong or unique) to (3.1.1) if $(X, B)$ is such a solution.

Since we are only considering deterministic initial conditions $\hat{x}_0$, every pathwise unique non-colliding solution to (3.1.1) is automatically a strong solution by Theorem 1.7 in Chapter IX of [87].

In order to study the non-colliding property of $(X_t)_{t\geq 0}$, it is often convenient to write the drift parts in the system of SDEs (3.1.1) differently. To do so, note that for $x \in \operatorname{int} C_N^A$ we have

$$\sum_{j\colon j\neq i} \frac{x_i^2}{x_i - x_j} = \sum_{j\colon j\neq i} \frac{x_i(x_i - x_j + x_j)}{x_i - x_j} = (N-1)x_i + \sum_{j\colon j\neq i} \frac{x_i x_j}{x_i - x_j}\,,$$
$$\sum_{j\colon j\neq i} \frac{x_i}{x_i - x_j} = \frac{1}{2} \sum_{j\colon j\neq i} \frac{x_i + x_j + x_i - x_j}{x_i - x_j} = \frac{N-1}{2} + \frac{1}{2} \sum_{j\colon j\neq i} \frac{x_i + x_j}{x_i - x_j}\,.$$

Thus we can rewrite (3.1.1) as

$$dX_{i,t} = \sqrt{2a(X_{i,t})}\, dB_{i,t} + \frac{\beta}{2} \left[ \tilde{b}(X_{i,t}) + \sum_{j\colon j\neq i} \frac{G(X_{i,t}, X_{j,t})}{X_{i,t} - X_{j,t}} \right] dt\,, \tag{3.1.3}$$

where

$$\tilde{b}(x) := (2a_2(N-1) + b_1)\, x + a_1(N-1) + b_0\,, \quad G(x,y) := 2a_2 xy + a_1(x+y) + 2a_0\,.$$

Note that consequently non-colliding $\beta$-Pearson diffusions are special cases of solutions to the SDEs (1.0.1) as mentioned in the introduction. In the following subsections we are listing all possible variants of non-colliding $\beta$-Pearson diffusions where we distinguish cases in dependence of the nature of the roots of the polynomial $a$. Each such case then gives rise to a family of stochastic processes depending on the parameters $N, \beta, b_1, b_0$. The

name of each such family is thereby usually inspired by the corresponding one-dimensional ($N = 1$) versions which especially makes sense in view of the following: As mentioned in [10] p. 2649, in the case $\beta = 2$ one obtains the $N$-dimensional variant of a non-colliding $\beta$-Pearson diffusion by considering the Doob-$h$-transform of $N$ i.i.d. one-dimensional Pearson diffusions where the function $h$ is the Vandermonde determinant. In the literature one often finds the notion '$N$ independent processes conditioned to never collide' for this construction. The only case where we differ from the naming convention described above is the case of Fisher-Snedecor diffusions corresponding to a quadratic polynomial $a$ with two distinct real roots and positive leading coefficient. These multivariate processes can be seen as extensions of the univariate Fisher-Snedecor processes which are mainly studied in statistical contexts due to the fact that for certain parameters their invariant probability measure is given by a Fisher-Snedecor/$F$-distribution ([46, 14]). As these processes structurally have a lot of similarities to the Jacobi processes (e.g. here $a$ is a quadratic polynomial with two distinct real roots and negative leading coefficient), we will also call these processes 'non-compact Jacobi processes' where 'non-compact' here refers to the unbounded state space of these processes.
Within each family of processes we will give sufficient criteria for the parameters $\beta, b_1, b_0$ which ensure the existence and uniqueness of non-colliding solutions. We point out, however, that we might write these results down in terms of slightly different parametrizations of the coefficient function $b$ than by the numbers $b_1, b_0$ which is mostly due to historic reasons. Moreover, the densities of invariant probability measures of such solutions, if they exist, will be listed. We also mention dynamical random matrix models such that the corresponding eigenvalue processes are non-colliding $\beta$-Pearson diffusions. Note that, as was shown in [51], in general one can always find such a random matrix model in the case $\beta \in \{1, 2\}$ by considering the solution to certain matrix SDEs. We thus primarily focus on models which are more 'explicit' in the sense that they are defined using elementary operations on well known matrix processes such as the hermitian Brownian motion or the Brownian motion on the general linear group $\mathrm{Gl}(N, \mathbb{C})$.

**Remark 3.1.3.** For existence and uniqueness of solutions to (3.1.1) we rely heavily on Piotr Graczyk and Jacek Małecki's seminal paper [52] on the non-colliding property of particle systems on the real line. We point out that in that paper there is a minor error in Section 4 in formula (4.3) which results in a missing term in formula (4.6) therein; see Subsection 3.7.0.1 at the end of Chapter 3 for the necessary correction. However, the validity of all main statements of the referenced paper are not affected by this correction.

In what follows, we omit the trivial non-random case $a \equiv 0$.

## 3.2 (Ornstein-Uhlenbeck) Dyson Brownian motions

We start by considering the simplest case of non-colliding $\beta$-Pearson diffusions corresponding to $a_2 = a_1 = 0$ and $a_0 > 0$. By transitioning to the time-rescaled process $(X_{t/a_0})_{t \geq 0}$,

we can assume w.l.o.g. that $a_0 = 1/2$. Thus we get the following system of SDEs

$$dX_{i,t} = dB_{i,t} + \frac{\beta}{2}\left[b_1 X_{i,t} + b_0 + \sum_{j:\, j\neq i} \frac{1}{X_{i,t} - X_{j,t}}\right] dt\,, \tag{3.2.1}$$

$i \in \{1, \ldots, N\}$, $X_0 = \hat{x}_0 \in C_N^A$. By Corollary 6.2 in [52], for all $b_1, b_0 \in \mathbb{R}$, $\beta \geq 1$ and any starting configuration $X_0 = \hat{x}_0 \in C_N^A$, the process $(X_t)_{t\geq 0}$ is a process with values in $C_N^A$ which is non-colliding for $t > 0$.
Note that in the case $b_1 = 0$, by considering the process $\left(X_{1,t} - \frac{\beta}{2}b_0 t, \ldots, X_{N,t} - \frac{\beta}{2}b_0 t\right)_{t\geq 0}$, we can assume w.l.o.g. that $b_0 = 0$. The resulting process is the well known $\beta$-Dyson Brownian motion. It is the first eigenvalue model in random matrix theory based on Brownian motion which has been considered in the literature and has been conceived as a time parameter variant of the Gaussian symmetric, hermitian and symplectic ensembles in [43]. As the aforementioned matrix ensembles are quite well understood, the same can be said about Dyson Brownian motion in the case $\beta \in \{1, 2, 4\}$. In particular, in the case $\beta = 2$ a concrete random matrix model such that the corresponding eigenvalue process is a Dyson Brownian motion is given by $A + B_t$, $t \geq 0$, where $A \in M_N(\mathbb{C})$ is some deterministic hermitian matrix such that its vector of ordered eigenvalues equals $\hat{x}_0$ and $(B_t)_{t\geq 0}$ is a hermitian Brownian motion. Similarly one also has an associated random matrix model if $\beta = 1$, see [3] Section 4.3.1.
For general $\beta$, the Dyson Brownian motion is also known as radial Dunkl process of type $A_{N-1}$, see Section 2.3 for this point of view.
For $b_1 \neq 0$ (for some authors only for $b_1 < 0$) the solution to (3.2.1) is known as *non-colliding Ornstein-Uhlenbeck (OU) Dyson Brownian motion*. Note that, by going over to the shifted process $(X_{1,t} + b_0/b_1, \ldots, X_{N,t} + b_0/b_1)_{t\geq 0}$, we, again, can assume w.l.o.g. that $b_0 = 0$. If one assumes $b_0 = 0$, one obtains the $b_1 \neq 0$ case from the $b_1 = 0$ case by considering

$$e^{\beta b_1 t/2} X_{i,\frac{1-e^{-\beta b_1 t}}{\beta b_1}}\,,\quad i \in \{1, \ldots, N\},\quad t \geq 0\,, \tag{3.2.2}$$

where $(X_t)_{t\geq 0}$ is a solution of (3.2.1) for $b_1 = b_0 = 0$; we carry out a similar transformation in more detail in Proposition 3.3.2 below. One can show that if $b_1 < 0$ holds, the OU Dyson Brownian motion converges for $t \to \infty$ in distribution to its unique invariant probability distribution which is the image measure under $\lambda_i \mapsto \lambda_i/\sqrt{-\beta b_1}$, $i \in \{1, \ldots, N\}$, of the *Gaussian $\beta$-ensemble* with density on $C_N^A$ given by

$$C(N, \beta) \cdot \prod_{i=1}^{N} e^{-\lambda_i^2/2} \cdot \prod_{1\leq j<k\leq N} |\lambda_j - \lambda_k|^\beta$$

where the normalization constant $C(N, \beta)$ is explicitly known (see e.g. [47] Proposition 1.9.4 for this constant).
The deterministic $\beta = \infty$ variants of (OU) Dyson Brownian motions, i.e. non-colliding

solutions to the ODEs

$$\frac{d}{dt}x_{i,t} = b_1 x_{i,t} + b_0 + \sum_{j\colon j\neq i} \frac{1}{x_{i,t} - x_{j,t}} \quad i \in \{1,\dots,N\}\,,$$

have been considered in [109, 110, 108] in the case that $b_1 = b_0 = 0$. By the same space-time transformations as in (3.2.2) this is no actual restriction. Moreover, as shown in the latter references, there is a close connection of these ODEs and the zeros of (univariate) Hermite polynomials.

## 3.3 Squared (Ornstein-Uhlenbeck) Bessel processes

This case arises for $a_2 = 0, a_1 \neq 0$. Using the rescaling

$$(a_1 X_{1,t/a_1^2} + a_0, \dots, a_1 X_{N,t/a_1^2} + a_0)\,,$$

we can assume w.l.o.g. that $a_1 = 1$ and $a_0 = 0$. The SDE (3.1.3) here then reads

$$dX_{i,t} = \sqrt{2X_{i,t}}\, dB_{i,t} + \frac{\beta}{2}\left[b_1 X_{i,t} + b_0 + N - 1 + \sum_{j\colon j\neq i} \frac{X_{i,t} + X_{j,t}}{X_{i,t} - X_{j,t}}\right] dt\,, \tag{3.3.1}$$

$i \in \{1,\dots,N\}$, $X_0 = \hat{x}_0 \in C_N^B := \{x \in \mathbb{R}^N\colon 0 \leq x_1 \leq \cdots \leq x_N\}$, for some $b_1, b_0 \in \mathbb{R}$. We will call solutions to (3.3.1) *squared Bessel processes* and more specifically in the case $b_1 \neq 0$ *squared Ornstein-Uhlenbeck (OU) Bessel processes*. To distinguish these processes from their univariate variants one might add the term 'non-colliding' to these names, which for the sake of brevity we omit. The non-colliding property holds in the following cases:

**Lemma 3.3.1.** *For $b_1 \in \mathbb{R}$, $b_0 \geq 0$, $\beta \geq 1$ and starting configuration $X_0 = \hat{x}_0 \in C_N^B$ the SDE* (3.3.1) *has a unique non-colliding solution.*

*Proof.* By Corollary 6.5 in [52], the Lemma holds in the case $b_1 = 0$. The following Proposition asserts that in order to study the non-colliding property of this process, it suffices to study this case. □

Analogous to Dyson Brownian motion, there is a space-time transformation from which one obtains the $b_1 \neq 0$, i.e. the OU-case, from the $b_1 = 0$ case:

**Proposition 3.3.2.** *Let $(X_t)_{t\geq 0}$ be the unique non-colliding solution of* (3.3.1) *for parameters $\beta \geq 1$, $b_1 = 0$, $b_0 \geq 0$ and starting condition $X_0 = \hat{x}_0 \in C_N^B$. Let $\tilde{b}_1 \in \mathbb{R} \setminus \{0\}$ and set*

$$f(t) := e^{\beta \tilde{b}_1 t/2} \quad \text{and} \quad g(t) := \frac{1 - e^{-\beta \tilde{b}_1 t/2}}{\beta \tilde{b}_1/2}\,.$$

*Then the process* $(Y_t := f(t)X_{g(t)})_{t\geq 0}$ *solves*

$$dY_{i,t} = \sqrt{2Y_{i,t}}\, d\tilde{B}_{i,t} + \frac{\beta}{2}\left[\tilde{b}_1 Y_{i,t} + b_0 + N - 1 + \sum_{j\colon j\neq i} \frac{Y_{i,t}+Y_{j,t}}{Y_{i,t}-Y_{j,t}}\right] dt\,, \quad Y_0 = \hat{x}_0\,,$$

*where* $\tilde{B}_{i,t} := \int_0^t \sqrt{f(s)} dB_{i,g(s)}$, $i \in \{1,\dots,N\}$, $t \geq 0$, *is an* $N$*-dimensional standard real Brownian motion for the filtration* $(\mathcal{G}_t := \mathcal{F}_{g(t)})_{t\geq 0}$.

*Proof.* The Proposition can be deduced from the statements in Section 10 of [90]. For the convenience of the reader we give a direct argument: By Theorem 18.24 in [65] we have

$$dX_{i,g(t)} = \sqrt{2X_{i,g(t)}}\, dB_{i,g(t)} + \frac{\beta}{2} g'(t)\left[b_0 + N - 1 + \sum_{j\colon j\neq i} \frac{X_{i,g(t)}+X_{j,g(t)}}{X_{i,g(t)}-X_{j,g(t)}}\right] dt\,.$$

Now, a simple application of Itô's formula shows that $(Y_t)_{t\geq 0}$ satisfies the claimed SDE as $f(t)g'(t) \equiv 1$. It remains to show that $(\tilde{B}_t)_{t\geq 0}$ is a standard Brownian motion. By using Theorem 18.24 in [65] again, in conjunction with the polarisation identity $4[A,B]_t = [A+B]_t - [A-B]_t$ for quadratic variation, we almost surely have

$$\begin{aligned}\left[\tilde{B}_{i,\cdot}, \tilde{B}_{j,\cdot}\right]_t &= \int_0^t f(s)\, d\left[B_{i,g(\cdot)}, B_{j,g(\cdot)}\right]_s = \int_0^t f(s)\, d\left[B_{i,\cdot}, B_{j,\cdot}\right]_{g(s)} = \delta_{ij}\int_0^t f(s)g'(s)\, ds \\ &= t\delta_{ij}\,,\end{aligned}$$

where $\delta_{ij}$ is the Kronecker delta. Moreover, again by the same theorem referenced above, we know that $\tilde{B}_{i,t} = \int_0^{g(t)} \sqrt{f(g^{-1}(s))}\, dB_{i,s}$ a.s., so $(\tilde{B}_t)_{t\geq 0}$ is a continuous local martingale for the filtration $(\mathcal{G}_t)_{t\geq 0}$. The claim now follows by Lévy's characterization of Brownian motion. □

For $\beta = 1$ a concrete random matrix model such that the corresponding vector of ordered eigenvalues is a squared Bessel process is the following: Let $M \geq N-1$. Let $C_t := (A+B_t)^T\cdot(A+B_t)$, $t \geq 0$, where $A$ is a deterministic $M\times N$ matrix over $\mathbb{R}$ such that its vector of squared ordered singular values equals $\hat{x}_0 \in \{x \in \mathbb{R}^N \colon 0 \leq x_1 < \cdots < x_N\}$, and where $B_t$ is a $M \times N$ random matrix whose entries are independent real Brownian motions. Then the ordered eigenvalues of $(C_t)_{t\geq 0}$ form a squared Bessel process; see [24]. $(C_t)_{t\geq 0}$ is called a Wishart process, and its complex sibling, the Laguerre process, analogously gives a dynamical random matrix model for the case $\beta = 2$, see [69].
Analogously to OU Dyson Brownian motion, in the case that $b_1 < 0$ and $b_0 > 0$ one can infer from Proposition 3.3.2 that $X_t$ converges for $t \to \infty$ in distribution to its invariant probability distribution which is given by the image measure under $\lambda_i \mapsto \lambda_i/(-\beta b_1)$, $i \in \{1,\dots,N\}$, of the *Laguerre* $\beta$*-ensemble* with parameters $\beta, b_0$ whose density on $C_N^B$ is given by

$$C(N,\beta,b_0)\cdot\prod_{i=1}^N \left(\lambda_i^{\beta b_0/2-1} e^{-\lambda_i/2}\right)\cdot \prod_{1\leq j<k\leq N} |\lambda_j - \lambda_k|^\beta \tag{3.3.2}$$

where the normalization constant $C(N, \beta, b_0)$ is explicitly known, see e.g. [47] Proposition 3.10.1.
We also mention that in the case $b_1 = 0$ and $b_0 \geq 1$ the squared Bessel processes are radial Dunkl processes of type $B_N$ after the coordinate change $y_i := 2\sqrt{x_i}$, $i \in \{1, \ldots, N\}$. Likewise in square root coordinates, the $\beta = \infty$ variants of solutions to (3.3.1), i.e. non-colliding solutions to the ODEs

$$\frac{d}{dt}x_{i,t} = b_1 x_{i,t} + b_0 + N - 1 + \sum_{j\colon j\neq i} \frac{x_{i,t} + x_{j,t}}{x_{i,t} - x_{j,t}}\,, \quad i \in \{1, \ldots, N\}\,, \tag{3.3.3}$$

have been considered in [109, 110, 108] for the case $b_1 = 0$. Again, using the space-time transformation of Proposition 3.3.2, one sees that this is not an actual restriction. Moreover, the authors show a close connection of these ODEs to the zeros of (univariate) Laguerre Polynomials.

## 3.4 Non-colliding (inhomogeneous) geometric Brownian motions

Here we are covering the case that $a$ is a quadratic polynomial which has a real root of order 2 and positive leading coefficient. By a simple shift of the coordinate system we can assume w.l.o.g. that $a(x) = a_2 x^2$ for $a_2 > 0$. By going over to the process $(X_{t/a_2})_{t\geq 0}$, we can moreover assume that $a_2 = 1/2$. Thus it suffices to study the following system of SDEs:

$$dX_{i,t} = X_{i,t}\, dB_{i,t} + \frac{\beta}{2}\left[\gamma_1 X_{i,t} + \gamma_0 + \sum_{j\colon j\neq i} \frac{X_{i,t}X_{j,t}}{X_{i,t} - X_{j,t}}\right] dt\,, \tag{3.4.1}$$

$i \in \{1, \ldots, N\}$, for some deterministic start $X_0 = \hat{x}_0$ and some parameters $\gamma_1, \gamma_0 \in \mathbb{R}$. Likewise, we also consider the frozen counterpart of (3.4.1), i.e. the ODEs

$$\frac{d}{dt}x_{i,t} = \gamma_1 x_{i,t} + \gamma_0 + \sum_{j\colon j\neq i} \frac{x_{i,t}x_{j,t}}{x_{i,t} - x_{j,t}}\,, \tag{3.4.2}$$

$i \in \{1, \ldots, N\}$, $x_0 = \hat{x}_0$. If $\gamma_0 = 0$, we will call non-colliding solutions to (3.4.1) *non-colliding geometric Brownian motions* and use the same term preceded by 'frozen' for solutions to (3.4.2). Similarly, if $\gamma_0 \neq 0$, we use the same names except that we add the term 'inhomeogeneous' after 'non-colliding'. This naming convention agrees with the one used in [10] Section 2 and is motivated by the fact that in the case $\beta = 2, \gamma_1 \in \mathbb{R}$ and $\gamma_0 \geq 0$ one can construct solutions to (3.4.1) by considering the Doob $h$-transform of $N$ independent (inhomogeneous) geometric Brownian motions where $h$ is the Vandermonde determinant $h(x) = \prod_{1\leq i<j\leq N}(x_j - x_i)$ (see [10] p. 2649 and references therein).
We can say the following about non-colliding solutions to (3.4.1):

**Lemma 3.4.1.** *Let $N \in \mathbb{N}$, $\beta \geq 1$ and $\gamma_1 \in \mathbb{R}$.*

*(a) For $\gamma_0 \geq 0$ and starting configuration*

$$X_0 = \hat{x}_0 \in \{x \in \mathbb{R}^N \colon 0 < x_1 \leq \cdots \leq x_N\}$$

*both the SDE* (3.4.1) *and the ODE* (3.4.2) *have a unique non-colliding solution. For positive times $t$ these solutions take values in $\{x \in \mathbb{R}^N \colon 0 < x_1 < \cdots < x_N\}$ a.s.*

*(b) For $\gamma_0 > 0$ and starting configuration $X_0 = \hat{x}_0 \in C_N^B$ with $\hat{x}_{1,0} = 0$ both the SDE* (3.4.1) *and the ODE* (3.4.2) *have a unique non-colliding solution. The state space of these solutions is $\big(\operatorname{int} C_N^A\big) \cap [0,\infty)^N$.*

**Remark 3.4.2.** In the case $\gamma_0 = 0$ one can loosen the restriction of the positivity of the starting configuration in (a) in the following way: If the starting configuration $\hat{x}_0 \in C_N^B$ satisfies

$$\hat{x}_{1,0} = \cdots = \hat{x}_{m,0} = 0 < \hat{x}_{m+1,0} \leq \cdots \leq \hat{x}_{N,0}$$

for some $m \in \{1, \ldots, N\}$, then, in a certain sense, there is a unique solution $(X_t)_{t\geq 0}$ to (3.4.1) and it satisfies $X_{1,t} = \cdots = X_{m,t} = 0 < X_{m+1,t} < \cdots < X_{N,t}$ for all $t > 0$ a.s. In other words the first $m$ 'particles' stay glued to 0 and the other $N - m$ 'particles' are non-colliding. See Proposition 3.4.6 below.

*Proof of Lemma 3.4.1.* First consider the time change $t \mapsto 2t/\beta$ applied to solutions to (3.4.1). The resulting processes are now solutions to the SDEs

$$dX_{i,t} = \sqrt{\frac{2}{\beta}} X_{i,t}\, dB_{i,t} + \left[\gamma_1 X_{i,t} + \gamma_0 + \sum_{j\colon j\neq i} \frac{X_{i,t}X_{j,t}}{X_{i,t} - X_{j,t}}\right] dt\,, \tag{3.4.3}$$

$i \in \{1, \ldots, N\}$. To treat the cases $\beta \in [1,\infty)$ and $\beta = \infty$ simultaneously, we here interpret $1/\beta$ as 0 if $\beta = \infty$. In a next step, we will study (3.4.3) in logarithmic coordinates. To this end consider solutions $(Y_t)_{t\geq 0}$ to the system of SDEs

$$dY_{i,t} = \sqrt{\frac{2}{\beta}}\, dB_{i,t} + \left[2\gamma_1 - \frac{2}{\beta} - N + 1 + 2\gamma_0 e^{-2Y_{i,t}} + \sum_{j\colon j\neq i} \coth(Y_{i,t} - Y_{j,t})\right] dt\,. \tag{3.4.4}$$

Set $\sigma_i(x) := \sigma(x) := \sqrt{2/\beta}$, $b_i(x) := b(x) := 2\gamma_1 - \frac{2}{\beta} - N + 1 + 2\gamma_0 e^{-2x}$, $\gamma := 1$, $i \in \{1, \ldots, N\}$. Note that $b_i$ is non-increasing, $\sigma_i$ is 1/2-Hölder continuous and there exists some $c = c(N, \beta, \gamma_1, \gamma_0)$ such that the following estimates hold:

$$b_i(x)x \leq c(1 + |x|^2)\,, \quad \sigma_i^2(x) \leq 2\gamma\,, \quad x \in \mathbb{R}\,.$$

By [52] Corollary 6.8, there exists a unique non-colliding solution to (3.4.4) for all $\beta \in [1,\infty]$, $\gamma_1 \in \mathbb{R}$, $\gamma_0 \geq 0$, and all starting configurations $Y_0 = \hat{y}_0 \in C_N^A$. Now set $\tilde{X}_t := (\exp\big(2Y_{1,t/4}\big), \ldots, \exp\big(2Y_{N,t/4}\big))$, $t \geq 0$. Using

$$\coth\left(\frac{\log(x) - \log(y)}{2}\right) = \frac{x+y}{x-y}\,,$$

we see by Itô's formula that $(\tilde{X}_t)_{t\geq 0}$ solves (3.4.3). Thus we have shown the claim in (a). It remains to discuss the case $\gamma_0 > 0$ and $\hat{x}_0 \in C_N^B$ with $\hat{x}_{1,0} = 0$. We here follow the idea of proof of Corollary 6.5 in [52]. To this end, we first look into the modified system of SDEs

$$dY_{i,t} = \sqrt{\frac{2}{\beta}}|Y_{i,t}|\, dB_{i,t} + \left[\gamma_1 Y_{i,t} + \gamma_0 + \sum_{j\colon j\neq i} \frac{|Y_{i,t}Y_{j,t}|}{Y_{i,t} - Y_{j,t}}\right] dt\,, \quad i \in \{1,\dots,N\}\,, \tag{3.4.5}$$

with initial condition $Y_0 = y_0 \in C_N^B$. Ideally we would like to directly apply the existence & uniqueness Theorem 2.2 in [52]. There are seven conditions on the coefficients of the SDE (3.4.5), namely (C1),(C2),(A1)-(A5) in that paper, which need to be fulfilled. However, (A1) demands that we need to verify the inequality

$$|wz|(y-x) \leq |xy|(z-w) \tag{3.4.6}$$

for all $w < x < y < z$. But e.g. the choice $y = 0$ shows that this inequality does not hold in its stated generality. Our strategy of proof is thus the following: It turns out that five conditions, namely (C2),(A2)-(A5), actually are fulfilled, which we will show below. This means that by Theorem 5.1 in [52] we know that a solution to (3.4.5) exists. We can then show that every such solution which starts in $C_N^B$ stays in that set. Thus we will only have to verify (3.4.6) for $0 \leq w < x < y < z$. Using the same argument of proof as in Theorem 5.3 of [52], we can then deduce the uniqueness of solutions of (3.4.3) in the case $\gamma_1 > 0$.
Setting

$$\sigma(x) := \sqrt{\frac{2}{\beta}}|x|\,, \quad b(x) := \gamma_1 x + \gamma_0\,, \quad H(x,y) := |xy|\,,$$

our SDE reads $dY_{i,t} = \sigma(Y_{i,t})\, dB_{i,t} + \left[b(Y_{i,t}) + \sum_{j\colon j\neq i} \frac{H(Y_{i,t},Y_{j,t})}{Y_{i,t}-Y_{j,t}}\right] dt$, $i \in \{1,\dots,N\}$. As mentioned, in a first step we will show existence of a solution. To this end, we verify the following five conditions:
(C2): We need to find a constant $c \geq 0$ such that $\sigma^2(x) + b(x)x \leq c(1+|x|^2)$ and $H(x,y) \leq c(1+|xy|)$. This holds e.g. for $c = \frac{2}{\beta} + |\gamma_1| + |\gamma_0| + 1$.
(A2): We need to find a $c \geq 0$ such that $\sigma^2(x) + \sigma^2(y) \leq c(x-y)^2 + 4H(x,y)$. By the inequality

$$\sigma^2(x) + \sigma^2(y) = \frac{2}{\beta}\left((x-y)^2 + 2xy\right) \leq \frac{2}{\beta}\left((x-y)^2 + 2\beta|xy|\right) = \frac{2}{\beta}(x-y)^2 + 4H(x,y)\,,$$

we can fulfill this for the choice $c = \frac{2}{\beta}$.
(A3): We need to find a $c \geq 0$ such that

$$H(x,y)(y-x) + H(y,z)(z-y) \leq c(z-y)(z-x)(y-x) + H(x,z)(z-x)$$

for all $x < y < z$. To this end, we will adapt the corresponding argument of the proof of Lemma 4.2 in [9]: We need to verify the existence of some $c \geq 0$ such that

$$0 \leq c(z-y)(z-x)(y-x) + |xz|(z-x) - |xy|(y-x) - |yz|(z-y) =: g_c(y)$$

for all $x < y < z$. We claim that this is true for any $c > 1$. Indeed: Let $c > 1$ and $x < y < z$. Note that $g_c$ on $[0, \infty)$ is a quadratic polynomial with zeros $x$ and $z$ and leading coefficient

$$-c(z-x) - |x| + |z| = \begin{cases} (1-c)(z-x)\,, & x \geq 0 \\ (1-c)z + (1+c)x\,, & x < 0\,, \end{cases}$$

which is negative in both cases. Thus $g_c(y) \geq 0$ in the case $x < y < z$ and $y \geq 0$. Similar considerations show $g_c(y) \geq 0$ in the case $x < y < z$ and $y < 0$.
(A4): We need to show that $G := \{x \in \mathbb{R} \colon 2\sigma^2(x) + H(x,x) = 0\}$ consists of isolated points and for all $x \in G$ it holds

$$b(x) + \sum_{j=1}^{N-2} \frac{H(x, y_j)}{x - y_j} \mathbb{1}_{\mathbb{R}\setminus\{x\}}(y_j) \neq 0$$

for all $y_1, \ldots, y_{N-2} \in \mathbb{R}$, with the convention 'undefined $\times$ 0 = 0'. In our case it holds $G = \{0\}$ and

$$b(0) + \sum_{j=1}^{N-2} \frac{H(0, y_j)}{-y_j} \mathbb{1}_{\mathbb{R}\setminus\{0\}}(y_j) = \gamma_0 > 0 \ \text{ for all } \ y_1, \ldots, y_{N-2} \in \mathbb{R}\,.$$

(A5): This condition is empty for our SDE since the affine linear part $b$ of the drift coefficient is the same for each particle $Y_i$.
As mentioned above, Theorem 5.1 in [52] now implies the existence of non-colliding solutions of the SDEs (3.4.5) for any starting configuration $Y_0 = y_0 \in C_N^A$. We will now show that any solution $(Y_t)_{t \geq 0}$ of (3.4.5) which starts in $Y_0 = y_0 \in C_N^B$ must stay in that set. We will adopt the local time argument from the proof of Theorem 7 in [51], which we here reproduce for the convenience of the reader. Set $D_t := b(Y_{1,t}) + \sum_{j\colon j \neq 1} \frac{H(Y_{1,t}, Y_{j,t})}{Y_{1,t} - Y_{j,t}}$ and

$$\vartheta := \inf\{t > 0 \colon Y_{1,t} < 0\}\,, \ \kappa := \inf\{t > \vartheta \colon D_t = 0\}\,.$$

Suppose $P(\vartheta < \infty) > 0$. Note that $Y_{1,\vartheta(\omega)}(\omega) = 0$ and $D_{\vartheta(\omega)}(\omega) = \gamma_0 > 0$ for all $\omega \in \{\vartheta < \infty\}$. Thus we have $\{\vartheta < \infty\} \subseteq \{\vartheta < \kappa\}$. Moreover, setting $\rho(x) := x^2$, we have for $\epsilon > 0$:

$$\int_0^t \mathbb{1}_{\{0 < Y_{1,s} \leq \epsilon\}} \frac{1}{\rho(Y_{1,s})}\, d[Y_1]_s = \frac{2}{\beta} \int_0^t \mathbb{1}_{\{0 < Y_{1,s} \leq \epsilon\}}\, ds < \infty \quad \text{a.s.}$$

By Lemma 3.3 on p. 389 of [87] we thus know that the local time at 0 of $Y_1$ vanishes a.s. Thus, by Tanaka's formula (see e.g. Theorem 1.2 p. 222 in [87]), we have for $T, t > 0$:

$$\begin{aligned}\min\left(0, Y_{1,(\vartheta+t)\wedge\kappa\wedge T}\right) =& \int_0^{(\vartheta+t)\wedge\kappa\wedge T} \mathbb{1}_{\{Y_{1,s}\leq 0\}}\, dY_{1,s} = \int_{\vartheta\wedge T}^{(\vartheta+t)\wedge\kappa\wedge T} \mathbb{1}_{\{Y_{1,s}\leq 0\}}\, dY_{1,s} \\ =& \sqrt{\frac{2}{\beta}} \int_{\vartheta\wedge T}^{(\vartheta+t)\wedge\kappa\wedge T} \mathbb{1}_{\{Y_{1,s}\leq 0\}} |Y_{1,s}|\, dB_{1,s} + \int_{\vartheta\wedge T}^{(\vartheta+t)\wedge\kappa\wedge T} \mathbb{1}_{\{Y_{1,s}\leq 0\}} D_s\, ds\,.\end{aligned}$$

Now fix some $T > 0$ such that $0 < P(\vartheta < T)$. Let $(\tau_n)_{n\in\mathbb{N}}$ be a localizing sequence of stopping times for the local martingale $\left(Z_t := \int_0^t \mathbb{1}_{\{Y_{1,s}\leq 0\}} |Y_{1,s}|\, dB_{1,s}\right)_{t\geq 0}$, i.e. $\tau_n \uparrow \infty$ for $n \to \infty$ a.s. and $(Z_{t\wedge\tau_n})_{t\geq 0}$ is an $L^2$-martingale for each $n$. Fix some $n_0 \in \mathbb{N}$ such that $P(\tau_{n_0} \geq T) > 1 - P(\vartheta < T)$. Then

$$P(\vartheta < T, \tau_{n_0} \geq T) \geq P(\vartheta < T) + P(\tau_{n_0} \geq T) - 1 > 0$$

and

$$E\left[\min\left(0, Y_{1,(\vartheta+t)\wedge\kappa\wedge T\wedge\tau_{n_0}}\right)\right] = E\left[\int_{\vartheta\wedge T\wedge\tau_{n_0}}^{(\vartheta+t)\wedge\kappa\wedge T\wedge\tau_{n_0}} \mathbb{1}_{\{Y_{1,s}\leq 0\}} D_s\, ds\right] \geq 0 \text{ for all } t > 0\,.$$

We get a contradiction as $Y_{1,\vartheta(\omega)+t}(\omega) < 0$ for small $t$ and $\omega \in \{\vartheta < \infty\}$. Thus we must have $P(\vartheta = \infty) = 1$.
So far we thus have shown the existence of a solution to (3.4.3) in the case $\gamma_0 > 0$. It remains to discuss the uniqueness of such a solution. Going through the proof of Theorem 5.3 of [52], we see that it suffices to verify the following two conditions:
(C1): We need to find a function $\rho\colon [0,\infty) \to [0,\infty)$ with $\int_{(0,\infty)} \frac{1}{\rho(x)}\, dx = \infty$ and $|\sigma(x) - \sigma(y)|^2 \leq \rho(|x-y|)$ for all $x, y \in \mathbb{R}$. This can easily be fulfilled by $\rho(x) = \frac{2}{\beta}x^2$, as $||x| - |y||^2 \leq |x-y|^2$.
(A1): We need to show the inequality $f(z) := \frac{wz}{z-w} \leq \frac{xy}{y-x}$ for all $0 \leq w < x < y < z$. Note that $f'(z) = -w^2/(z-w)^2 \leq 0$ for $w \neq z$. Thus

$$f(z) \leq f(y) \leq \frac{xy}{y-x} \text{ for all } 0 \leq w < x < y < z\,.$$

This finishes the proof. □

In the case of $\gamma_0 > 0$, sufficiently small $\gamma_1$, and $\beta \in [1,\infty]$, we have stationary solutions/invariant probability measures to (3.4.2)/(3.4.1) which we identify in the following two Lemmas. These results so far are only known in the cases $\beta \in \{1, 2\}$ by [88, 10].

**Lemma 3.4.3.** *Let $N \in \mathbb{N}$, $\gamma_1 < 0$, $\gamma_0 > 0$ and $\hat{x}_0 \in C_N^B$. Let $(x_t)_{t\geq 0}$ be the unique non-colliding solution to (3.4.2) for these parameters. Then the limit $\lim_{t\to\infty} x_t = (z_1, \ldots, z_N)$ exists, where $(z_N^{-1}, \ldots, z_1^{-1})$ is the vector of ordered zeros of the rescaled Laguerre polynomial $x \mapsto L_N^{(-(2\gamma_1+1))}(cx)$, where $c = 2\gamma_0$ (see [99] Chapter V for a definition of the Laguerre polynomials). Moreover, $(z_1, \ldots, z_N)$ is the only stationary solution of (3.4.2) in this case.*

*Proof.* A stationary solution $x_t = x_0$, $t \geq 0$, to (3.4.2) satisfies

$$0 = \gamma_1 x_{i,0} + \gamma_0 + \sum_{j\colon j\neq i} \frac{1}{1/x_{j,0} - 1/x_{i,0}}\,, \quad i \in \{1,\dots,N\}\,.$$

Setting $y := (x_{N,0}^{-1},\dots,x_{1,0}^{-1})$ we can write this condition as

$$0 = -\gamma_1/y_i - \gamma_0 + \sum_{j\colon j\neq i} \frac{1}{y_i - y_j}\,, \quad i \in \{1,\dots,N\}\,.$$

By the proof of [99] Theorem 6.7.2 it is known that the unique solution in $\operatorname{int} C_N^B$ to these equations is given by the ordered zeros of the rescaled Laguerre polynomial $x \mapsto L_N^{-(2\gamma_1+1)}(cx)$, where $c = 2\gamma_0$.
Now let $(x_t)_{t\geq 0}$ and $z = (z_1,\dots,z_N)$ be as in the Lemma, i.e. we consider the situation of general starting condition $\hat{x}_0 \in C_N^B$. Let $f(x) := (e_1^N(x),\dots,e_N^N(x))$ where $e_i^N$, $i \in \{1,\dots,N\}$ are the elementary symmetric polynomials in $N$ variables defined in Section 2.2. By Lemma 2.2.2 the map $f\colon C_N^B \to f(C_N^B)$ is a homeomorphism. Now set $y_t := f(x_t)$, $t \geq 0$. Using the formula

$$e_m^N(x) = x_i e_{m-1}^{N-1}(x_{\{1,\dots,N\}\setminus\{i\}}) + e_m^{N-1}(x_{\{1,\dots,N\}\setminus\{i\}})\,, \quad x \in \mathbb{R}^N\,, \quad i \in \{1,\dots,N\}\,, \tag{3.4.7}$$

we have (notationally suppressing dependence of $t$):

$$\frac{d}{dt} y_m = \sum_{i=1}^N e_{m-1}^{N-1}(x_{\{1,\dots,N\}\setminus\{i\}}) \frac{d}{dt} x_i\,, \quad m \in \{1,\dots,N\}\,.$$

Plugging in the ODE for $x_i$ and using formulas (2.2.2) & (2.2.4) we get

$$\frac{d}{dt} y_{m,t} = m\,(\gamma_1 - (m-1)/2)\, y_{m,t} + \gamma_0 (N-m+1) y_{m-1,t}\,, \tag{3.4.8}$$

with the convention $y_{0,t} \equiv 1$. Thus

$$y_{m,t} = e^{m(\gamma_1 - (m-1)/2)t} \left( y_{m,0} + \gamma_0 (N-m+1) \int_0^t e^{-m(\gamma_1-(m-1)/2)r} y_{m-1,r}\, dr \right)\,.$$

Since $\gamma_1 < 0$, we have $m\,(\gamma_1 - (m-1)/2) < 0$ for all $m \in \{1,\dots,N\}$. Inductively one sees that the limit $\lim_{t\to\infty} y_t$ exists and does not depend on $(y_{1,0},\dots,y_{N,0})$. Hence this limit equals $f(z)$ where $z$ is the stationary solution considered above. A transfer back via $f^{-1}$ shows the claim. □

**Lemma 3.4.4.** *Let $N \in \mathbb{N}$, $\beta \geq 1$, $\gamma_1 < 1/\beta$, $\gamma_0 > 0$ and*

$$\hat{x}_0 \in \{x \in \mathbb{R}^N\colon 0 < x_1 \leq \cdots \leq x_N\} = C_N^B \cap (0,\infty)^N\,.$$

*Then the dynamic* (3.4.1) *has as unique invariant probability measure* $\rho$ *the image measure of the Laguerre ensemble* (3.3.2) *with parameters* $\tilde{\beta} := \beta, \tilde{b}_0 := 2/\beta - 2\gamma_1$ *under the map* $x \mapsto \frac{1}{2\beta\gamma_0}(x_N^{-1}, \dots, x_1^{-1})$*, i.e. its density on* $C_N^B \cap (0,\infty)^N$ *is given by*

$$\rho(dx) = C(N,\beta,\gamma_1) \prod_{i=1}^{N} \left( x_i^{\beta(\gamma_1 - N + 1) - 2} e^{-\beta\gamma_0/x_i} \right) \prod_{1 \le j < k \le N} (x_k - x_j)^\beta \, dx \,,$$

*where the normalization constant* $C(N,\beta,\gamma_1)$ *can be calculated using [47] Proposition 3.10.1.*
*Moreover, the non-colliding solution* $(X_t)_{t\ge 0}$ *to* (3.4.1) *for parameters* $\beta, \gamma_1, \gamma_0$ *as above and starting condition* $\hat{x}_0 \in C_N^B \cap (0,\infty)^N$ *converges for* $t \to \infty$ *to* $\rho$ *in distribution.*

*Proof.* Let $(X_t)_{t\ge 0}$ be the unique non-colliding solution to (3.4.1) for the parameters as specified in the Lemma. By the calculations in the first part of the proof of Lemma 3.4.1, the processes $Z_{i,t} := \frac{1}{2}\log(X_{i,8t})$ satisfy

$$dZ_{i,t} = \sqrt{2}\, dB_{i,t} + \beta \left[ -\frac{2}{\beta} + 2\gamma_1 - N + 1 + 2\gamma_0 e^{-2Z_{i,t}} + \sum_{j\colon j \ne i} \coth(Z_{i,t} - Z_{j,t}) \right] dt \,, \quad (3.4.9)$$

$i \in \{1, \dots, N\}$. Thus the dynamic (3.4.1) is equivalent to the dynamic (3.4.9). Moreover, the generator of the process $(Z_t)_{t\ge 0}$ is given by

$$\mathcal{L}f(x) := \Delta f(x) - (\nabla V(x)) \cdot \nabla f(x) \,,$$
$$V(x) := -\beta \sum_{i=1}^{N} \left[ \left( -\frac{2}{\beta} + 2\gamma_1 - N + 1 \right) x_i - \gamma_0 e^{-2x_i} \right] - \beta \sum_{1 \le j < k \le N} \log(\sinh(x_k - x_j)) \,.$$

Now, the candidate for the invariant measure of the dynamic (3.4.9) is given by the *Gibbs measure* $\tilde{\rho}$ which has Lebesgue density $\tilde{\rho}(dx) := \frac{1}{Z} e^{-V(x)} dx$ on $\operatorname{int} C_N^A$, where $Z := \int_{\operatorname{int} C_N^A} e^{-V(x)}\, dx$. We verify that $Z$ is indeed finite. Note that it holds

$$e^{-V(x)} = \prod_{i=1}^{N} \left( e^{(\beta(2\gamma_1 - N + 1) - 2)x_i} e^{-\beta\gamma_0 e^{-2x_i}} \right) \prod_{1 \le j < k \le N} \sinh^\beta(x_k - x_j) \,.$$

This density transforms under the map $C_N^A \to C_N^B \cap (0,\infty)^N$, $x_i \mapsto e^{2x_i}$, $i \in \{1, \dots, N\}$, to

$$\begin{aligned} & c \prod_{i=1}^{N} \left( x_i^{-1} x_i^{(\beta(2\gamma_1 - N + 1) - 2)/2} e^{-\beta\gamma_0/x_i} \right) \prod_{1 \le j < k \le N} \left( \frac{x_k - x_j}{\sqrt{x_j x_k}} \right)^\beta \\ = & c \prod_{i=1}^{N} \left( x_i^{\beta(\gamma_1 - N + 1) - 2} e^{-\beta\gamma_0/x_i} \right) \prod_{1 \le j < k \le N} (x_k - x_j)^\beta \,, \end{aligned} \quad (3.4.10)$$

where $c$ is some positive constant, and where we used $\sinh((\log(x)-\log(y))/2) = \frac{x-y}{2\sqrt{xy}}$ and $\prod_{1\le j<k\le N} x_j x_k = \prod_{j=1}^N x_j^{N-1}$. Another change of measure under the map

$$C_N^B \cap (0,\infty)^N \to C_N^B \cap (0,\infty)^N\,,\ \ (x_1,\dots,x_N) \mapsto \frac{1}{2\beta\gamma_0}(x_N^{-1},\dots,x_1^{-1})$$

gives a density proportional to

$$\begin{aligned}&\prod_{i=1}^N \left(x_i^{-2} x_i^{-\beta(\gamma_1-N+1)+2} e^{-x_i/2}\right) \prod_{1\le j<k\le N} \left(x_j^{-1}-x_k^{-1}\right)^\beta \\ =&\prod_{i=1}^N \left(x_i^{-\beta\gamma_1} e^{-x_i/2}\right) \prod_{1\le j<k\le N} (x_k-x_j)^\beta\ .\end{aligned}$$

Up to the normalization constant this is the density of the Laguerre ensemble (3.3.2) with parameters $\tilde\beta := \beta, \tilde b_0 := 2/\beta - 2\gamma_1 > 0$. This shows that $Z < \infty$. Thus $\tilde\rho$ is the unique invariant distribution of the Markov process generated by $\mathcal{L}$, see e.g. Section 2.2.3. of [91] and the beginning of Section 2 of the survey [75].
In a next step we show that the potential $V$ is convex. To this end we note that

$$\partial_{x_i}\partial_{x_j} V(x) = \begin{cases} 4\beta\gamma_0 e^{-2x_i} - \beta\sum_{k:\,k\neq i}(1-\coth^2(x_i-x_k))\,, & i=j \\ \beta(1-\coth^2(x_i-x_j))\,, & i\neq j\,. \end{cases}$$

Thus the Hessian $(\partial_{x_i}\partial_{x_j}V(x))_{1\le i,j\le N}$ of $V$ is the sum of the positive definite diagonal matrix with entries $4\beta\gamma_0 e^{-2x_i}$, $i\in\{1,\dots,N\}$, and another matrix which we claim is positive semi-definite. One can see this using the following argument:
Let $A_{jk}$, $1\le j<k\le N$, be a triangular array of non-negative numbers and consider the corresponding symmetric matrix with vanishing diagonal which, by abuse of notation, we again denote by $A$:

$$A_{jk} = \begin{cases} 0\,, & j=k\,, \\ A_{jk}\,, & j<k\,, \\ A_{kj}\,, & j>k\,. \end{cases}$$

Now set

$$B = \operatorname{diag}\left(\sum_{j:\,j\neq 1} A_{jk},\dots,\sum_{j:\,j\neq N} A_{jk}\right) \pm A\,.$$

Then $B$ is positive semi-definite, as for all $z\in\mathbb{R}^N\setminus\{0\}$ we have

$$z^T B z = \sum_{j=1}^N z_j^2 \sum_{\substack{k=1\\k\neq j}}^N A_{jk} \pm 2\sum_{1\le j<k\le N} z_j z_k A_{jk} = \sum_{1\le j<k\le N} A_{jk}(z_j\pm z_k)^2 \ge 0\,.$$

In turn we know that the Hessian of $V$ is positive semi-definite which means that $V$ is convex. By Theorem 9.4.10 in [1] we have shown that $\tilde{\rho}$ is a log-concave probability measure. Hence, by Theorem 1.3 in [2], $Z_t$ converges to $\tilde{\rho}$ in distribution for $t \to \infty$. The result follows by applying the coordinate change $x_i \mapsto e^{2x_i}$, $i \in \{1, \dots, N\}$, as above. $\square$

The processes arising as solution to (3.4.1) have previously been considered in the following contexts: In the case $\beta = 1$, $\gamma_0 = 0$ and $\gamma_1 = (N+1)/2$ it was shown in [82] that this process is the eigenvalue process of the matrix process $(M_t)_{t\geq 0} := (G_t^T G_t)_{t\geq 0}$ where $(G_t)_{t\geq 0}$ is the (right-invariant) Brownian motion in the general linear group $\mathrm{Gl}(N, \mathbb{R})$ and where $\cdot^T$ denotes the matrix transposition. One can show that $(M_t)_{t\geq 0}$ is a solution to the matrix SDE

$$dM_t = \sqrt{M_t}\, dB_t\, \sqrt{M_t} + \alpha M_t\, dt\,, \tag{3.4.11}$$

for some $\alpha \in \mathbb{R}$ and where $(B_t)_{t\geq 0}$ is a real symmetric Brownian motion (cf. [83] Section 2.9). If one replaces $(B_t)_{t\geq 0}$ by a hermitian Brownian motion then the corresponding eigenvalue process is a solution to (3.4.1) in the case $\beta = 2$, $\gamma_0 = 0$ and $\gamma_1 = \alpha$; see [51] Section III. Clearly, (3.4.11) is a matrix extension of the SDE of real geometric Brownian motion.
A surprisingly simple matrix model for the case $\beta = 2$, $\gamma_0 = 0$ and $\hat{x}_0 = 0$ has been described by the present author and Michael Voit in [12]. There the matrix process

$$(B_t + t\,\mathrm{diag}(\rho))_{t\geq 0}\,, \quad \text{with} \quad \rho := (-N+1, -N+3, \dots, N-3, N-1) \in \mathbb{R}^N\,,$$

is considered where $(B_t)_{t\geq 0}$ is an $N \times N$ hermitian Brownian motion, and where $\mathrm{diag}(\rho)$ is the diagonal matrix with diagonal $\rho$. It is shown that the corresponding exponentiated eigenvalue process is a non-colliding geometric Brownian motion for the mentioned parameters.
We further mention that in [88] a matrix process is considered whose eigenvalue process solves (3.4.1) for $\beta = 1$ and some $\gamma_0 > 0, \gamma_1 \in \mathbb{R}$ (see Lemma 22 therein; also see Remark 5.3.5).
Furthermore, in the case $\gamma_0 = 0$, the non-colliding geometric Brownian motion written in logarithmic coordinates is a (radial) Heckman-Opdam process for the root system $A_{N-1}$; this follows from a comparison of the SDE (3.4.4) and the generator considered in Section 2.3.2.
To our knowledge solutions to the ODEs (3.4.2) have not been considered in the literature before.

In the remainder of this section we elaborate on Remark 3.4.2 and thus only consider the case $\gamma_0 = 0$. We also restrict ourselves to the case $\beta < \infty$ although the following also works for $\beta = \infty$ with the obvious modifications.
We will use the transformation under elementary symmetric polynomials in $N$ variables which we defined in Section 2.2. We denote this transformation by

$$f_N \colon C_N^B \to [0, \infty)^N\,, \quad f_N(x) = (e_1^N(x), \dots, e_N^N(x))\,.$$

Recall that by Lemma 2.2.2 we know that $f_N\colon C_N^B \to f_N(C_N^B)$ is a homeomorphism and $f_N\colon \operatorname{int} C_N^A \cap [0,\infty)^N \to f_N(\operatorname{int} C_N^A \cap [0,\infty)^N)$ is a diffeomorphism. We now first state how the SDE for non-colliding geometric Brownian motion transforms under this change of coordinates:

**Proposition 3.4.5.** *Assume $(X_t)_{t\geq 0}$ satisfies the system of SDEs*

$$dX_{i,t} = X_{i,t}\, dB_{i,t} + \frac{\beta}{2}\left[\gamma_1 X_{i,t} + \sum_{j\colon j\neq i} \frac{X_{i,t}X_{j,t}}{X_{i,t}-X_{j,t}}\right] dt\,,\ i\in\{1,\dots,N\}\,. \tag{3.4.12}$$

*Then it holds*

$$de_k^N(X_t) = \sum_{j=1}^N X_{j,t} e_{k-1}^{N-1}(X_{\{1,\dots,N\}\setminus\{j\},t})\, dB_{j,t} + k\frac{\beta}{2}\left(\gamma_1 - \frac{k-1}{2}\right) e_k^N(X_t)\, dt\,, \tag{3.4.13}$$

$k\in\{1,\dots,N\}$.

*Proof.* (3.4.13) follows from an application of Itô's formula and the identities (3.4.7), (2.2.2) & (2.2.4). □

We now show that the system of SDEs (3.4.13) has a unique solution for more general starting configurations than the ones lying in $f_N(C_N^A \cap (0,\infty)^N)$. Transferring such solutions back via $f_N^{-1}\colon f_N(C_N^B) \to C_N^B$ explains the heuristic mentioned in Remark 3.4.2.

**Proposition 3.4.6.** *Let $\gamma_1\in\mathbb{R}$ and $\beta\geq 1$. Set $(f_N(x))_0 := 1$, $x\in C_N^B$. Then the system of SDEs*

$$dY_{k,t} = \sum_{j=1}^N \left(f_N^{-1}(Y_t)\right)_j \cdot \left(f_{N-1}(f_N^{-1}(Y_t)_{\{1,\dots,N\}\setminus\{j\}})\right)_{k-1}\, dB_{j,t} + k\frac{\beta}{2}\left(\gamma_1 - \frac{k-1}{2}\right) Y_{k,t}\, dt\,, \tag{3.4.14}$$

*$k\in\{1,\dots,N\}$, $Y_0\in f_N(C_N^B)$, has a unique solution. Moreover it holds:*

$$\begin{gathered}\text{If } Y_{k,0}>0\text{ , then } Y_{k,t}>0 \text{ for all } t>0 \text{ a.s., and}\\ \text{if } Y_{k,0}=0\text{ , then } Y_{k,t}=0 \text{ for all } t>0 \text{ a.s.}\end{gathered}$$

*If $Y_{1,0}>0$ then $\left(f_N^{-1}((Y_{1,t},\dots,Y_{l,t}))\right)_{t\geq 0}$ is the non-colliding solution to (3.4.12) for $N$ replaced there by $l$ where $l := \max\{i\in\{1,\dots,N\}\colon Y_{i,0}=0\}$.*

*Proof.* We distinguish two cases:
Case $Y_{N,0}>0$: Here we can infer from [52] Theorem 4.4 that any solution $(Y_t)_{t\geq 0}$ to (3.4.14) satisfies $Y_t\in f_N(\operatorname{int} C_N^B)$ for all $t>0$ a.s. As $f_N|_{\operatorname{int} C_N^B}$ is a diffeomorphism, existence and uniqueness here follows from the existence and uniqueness of non-colliding solutions to the SDEs (3.4.1) and Itô's formula. We moreover know by this correspondence

between the solutions of SDEs (3.4.1) and (3.4.14) that $Y_{k,t} > 0$ for all $t \geq 0$ and $k \in \{1, \dots, N\}$ a.s.
Case $Y_{N,0} = 0$: First note that it holds

$$x_j e_{N-1}^{N-1}(x_{\{1,\dots,N\}\setminus\{j\}}) = e_N(x) \text{ for all } j \in \{1, \dots, N\}\,.$$

Thus we can rewrite (3.4.14) in the case $k = N$ as

$$dY_{N,t} = Y_{N,t} \sum_{j=1}^{N} dB_{j,t} + N\frac{\beta}{2}\left(\gamma_1 - \frac{N-1}{2}\right) Y_{N,t}\, dt\,.$$

The coefficients of this SDE are clearly Lipschitz continuous, so any solution is unique. Thus we know $Y_{N,t} = 0$ for all $t \geq 0$ a.s. In particular also $\left(f_N^{-1}(Y_t)\right)_1 \equiv 0$ a.s. Since

$$e_{N-1}^{N}((0, x_2, \dots, x_N)) = \prod_{i=2}^{N} x_i = x_j e_{N-2}^{N-1}(x_{\{1,\dots,N\}\setminus\{j\}}) \text{ for all } j \in \{2, \dots, N\}\,,$$

we in turn have

$$Y_{N-1,t} = \left(f_N^{-1}(Y_t)\right)_j \cdot \left(f_{N-1}(f_N^{-1}(Y_t)_{\{1,\dots,N\}\setminus\{j\}})\right)_{N-2}\,,\ j \in \{2, \dots, N\}\,,$$

for all $t \geq 0$ a.s. Thus we can rewrite (3.4.14) in the case $k = N-1$ as

$$dY_{N-1,t} = Y_{N-1,t} \sum_{j=2}^{N} dB_{j,t} + (N-1)\frac{\beta}{2}\left(\gamma_1 - \frac{N-2}{2}\right) Y_{N-1,t}\, dt\,.$$

Now if $Y_{N-1,0} = 0$ we can again deduce that $Y_{N-1,t} = 0$ for all $t \geq 0$ a.s. Iterating this argument we either get that in the case $Y_0 = 0$ the unique solution of (3.4.14) is a process constantly equal to 0 a.s., or we stop at the largest $l \in \{1, \dots, N\}$ such that $Y_{l,0} > 0$. In the latter case we thus so far have shown that $Y_{j,t} = 0$ for all $j \in \{l+1, \dots, N\}$ and $t \geq 0$ a.s. Now define $\tilde{Y}_t := (Y_{1,t}, \dots, Y_{l,t})$. This process satisfies the system of SDEs

$$d\tilde{Y}_{k,t} = \sum_{j=1}^{l} \left(f_l^{-1}(\tilde{Y}_t)\right)_j \left(f_{l-1}(f_l^{-1}(\tilde{Y}_t)_{\{1,\dots,l\}\setminus\{j\}})\right)_{k-1} dB_{j+N-l,t} + k\frac{\beta}{2}\left(\gamma_1 - \frac{k-1}{2}\right) Y_{k,t}\, dt\,,$$

$k \in \{1, \dots, l\}$. By the first part of the proof there exists a unique solution to this system of SDEs with the claimed properties. In particular we have thus shown the existence and uniqueness of solutions to (3.4.14). □

## 3.5 (Compact) Jacobi processes

In this section we consider the case that $a$ is a quadratic polynomial with two distinct real roots and negative leading coefficient. By an affine-linear rescaling in space and a linear

rescaling in time, we can assume w.l.o.g. that $a(x) = x(1-x)$. Set $C_N^{BC} := C_N^A \cap [0,1]^N$. We are interested in non-colliding solutions $(X_t)_{t\geq 0}$ of the SDE

$$dX_{i,t} = \sqrt{2X_{i,t}(1-X_{i,t})}\, dB_{i,t} + \frac{\beta}{2}\left[p - (p+q)X_{i,t} + \sum_{j\colon j\neq i} \frac{X_{i,t} + X_{j,t} - 2X_{i,t}X_{j,t}}{X_{i,t} - X_{i,t}}\right] dt\,, \tag{3.5.1}$$

$i \in \{1,\dots,N\}$, $X_0 = \hat{x}_0 \in C_N^{BC}$, for certain parameters $p, q \in \mathbb{R}$. Such solutions are called *$\beta$-Jacobi processes* and appeared first, for $\beta = 1$, in [39] and were further studied in [31]. Similarly, we will call solutions to the system of ODEs

$$\frac{d}{dt}x_{i,t} = p - (p+q)x_{i,t} + \sum_{j\colon j\neq i} \frac{x_{i,t} + x_{j,t} - 2x_{i,t}x_{j,t}}{x_{i,t} - x_{i,t}}\,, \tag{3.5.2}$$

$i \in \{1,\dots,N\}$, $x_0 = \hat{x}_0 \in C_N^{BC}$, *frozen Jacobi processes*. The term 'Jacobi' appearing in the name of these processes is justified by the close relation of these processes to the following objects:

**Definition 3.5.1.** (i) Let $\beta > 0$, $a, b > -1$. The *$\beta$-Jacobi ensemble* $\mu_{\beta,a,b}^{Jac}$ is the probability measure on $C_N^{BC}$ with density

$$C(N,a,b,\beta) \prod_{i=1}^N \left(x_i^a (1-x_i)^b\right) \prod_{1\leq i<j\leq N} |\lambda_i - \lambda_j|^\beta\,. \tag{3.5.3}$$

The normalization constant $C(N,a,b,\beta)$ is explicitly known, see [47] Section 4.7.1.

(ii) Let $\alpha, \beta \in \mathbb{C}$. Denote by $P_N^{\alpha,\beta}$ the *$N$-th Jacobi polyomial*, i.e.

$$\begin{aligned} P_N^{\alpha,\beta}(x) :=& \sum_{k=0}^N \binom{N+\alpha}{N-k}\binom{N+\beta}{k}\left(\frac{x-1}{2}\right)^k \left(\frac{x+1}{2}\right)^{N-k} \\ =& \frac{(\alpha+1)^{(N)}}{N!} \sum_{k=0}^N \frac{(-N)^{(k)}(\alpha+\beta+n+1)^{(k)}}{(\alpha+1)^{(k)}k!}\left(\frac{1-x}{2}\right)^k\,. \end{aligned} \tag{3.5.4}$$

Here $x^{(n)} = \prod_{i=0}^{n-1}(x+i)$ denotes the rising factorial with the convention $x^{(0)} = 1$. For $\alpha, \beta > -1$ the sequence $(P_N^{\alpha,\beta})_{N\in\mathbb{N}}$ is a system of orthogonal polynomials w.r.t. the weight function $(1-x)^\alpha(1+x)^\beta$, see [99] Chapter 4. That the second equality in (3.5.4) is true can thus be seen in this case by Theorem 4.1.1 in [60] and noting that both sides of the equation agree for $x = 1$. For general $\alpha, \beta \in \mathbb{C}$ one can deduce the validity of the equation by using analytic continuation.

The following is known about non-colliding solutions and invariant measures of (3.5.1):

**Lemma 3.5.2.** *For $\beta \geq 1$, $p, q \geq N-1$ and $\hat{x}_0 \in C_N^{BC}$ there exists a unique non-colliding solution $(X_t)_{t\geq 0}$ to (3.5.1). Moreover, if $p, q > N-1$, it holds that $\rho := \mu^{Jac}_{\beta,\frac{\beta}{2}(p-(N-1))-1,\frac{\beta}{2}(q-(N-1))-1}$ is the unique invariant probability measure of the dynamic (3.5.1). If additionally $p, q \geq N-1+2/\beta$, then $X_t \xrightarrow{t\to\infty} \rho$ weakly.*

*Proof.* The first part of the Lemma holds by Corollary 6.7 in [52]. The explicit form and uniqueness of the invariant measure is in principle known in the literature, see e.g. [104] Section 3. However, to our knowledge, an explicit proof of this fact for general $\beta$ has not been given so far. One can show this by noting that the generator of $(X_t)_{t\geq 0}$, after applying the coordinate change $x_i \mapsto \arccos(2x_i - 1)$, $i \in \{1, \ldots, N\}$, falls in the framework of Langevin diffusions (compare end of Section 2.3.2), and thus has the corresponding Gibbs-measure as its unique invariant measure, see e.g. beginning of Section 2 in [75] for this latter fact. We will use similar arguments in Sections 3.6 & 3.7 so we skip the details here. The weak convergence to the invariant measure follows by the explicit transition probability densities for $\beta$-Jacobi processes in Proposition 4.1 of [31]. □

We have the following corresponding result for frozen Jacobi processes:

**Lemma 3.5.3.** *Let $N \in \mathbb{N}$ and $p, q > N-1$. Then for each $\hat{x}_0 \in C_N^{BC}$ the system of ODEs (3.5.2) has a unique solution $(x_t)_{t\geq 0}$ with $x_t \in \operatorname{int} C_N^{BC}$ for all $t > 0$. This solution satisfies $\lim_{t\to\infty} x_t = (z_1, \ldots, z_N) \in \operatorname{int} C_N^{BC}$ where $(2z_1 - 1, \ldots, 2z_N - 1)$ is the vector of the ordered zeros of $P_N^{(q-N,p-N)}$. Moreover, $(z_1, \ldots, z_N)$ is the only stationary solution to (3.5.2) in $C_N^{BC}$.*

*Proof.* This is just a reformulation of Theorem 1.1 in [13] as there the functions $t \mapsto y_{i,t} := 2x_{i,t} - 1$, $i \in \{1, \ldots, N\}$, are considered. □

As mentioned above, the non-colliding multivariate Jacobi processes have first been studied in [39]. Therein, the real matrix Jacobi process has been introduced as a matrix diffusion with values in the space of real symmetric matrices and whose invariant distribution is given by the Beta matrix distribution; e.g. see [76] Chapter 5 for a definition and the corresponding Lebesgue density. In [30, 31] the corresponding complex matrix process has been considered, we briefly summarize its construction: Start with the unitary Brownian motion $(U_t)_{t\geq 0}$ with values in the group of $N \times N$ unitary matrices which arises as the solution to the matrix SDE

$$dU_t = iU_t\, dB_t - \frac{1}{2} U_t\, dt\,, \quad U_0 = \hat{U}_0\,,$$

where $(B_t)_{t\geq 0}$ is a $N \times N$ hermitian Brownian motion and $\hat{U}_0$ is some deterministic $N \times N$ unitary matrix; for other equivalent definitions of $(U_t)_{t\geq 0}$ see [28]. One important property of this process is that it converges for $t \to \infty$ in distribution to the normalized Haar measure on the unitary group. Now let $n, p \in \{1, \ldots, N\}$. The complex matrix Jacobi process is defined as

$$J_t := \pi_{n,p}(U_t) \cdot \pi_{n,p}(U_t)^*\,, \quad t \geq 0\,,$$

where $\pi_{n,p}\colon \mathbb{C}^{N\times N} \to \mathbb{C}^{n\times p}$ is the map that sends a matrix to its upper left submatrix with $n$ rows and $p$ columns. If $p \geq n$ and $q := N - p \geq n$ then one can show that the eigenvalue process of $(J_t)_{t\geq 0}$ is an $n$-dimensional Jacobi process with parameters $p, q$ and $\beta = 2$. This result has not been proven explicitly in the literature but it can be deduced from a straightforward adaptation of the proof of the corresponding result for the real matrix Jacobi process, see [39] Theorem 9.3.1. We will see that a similar construction also gives a matrix model for the non-compact Jacobi processes, which we consider in the next section. Lastly, we mention the connection of Jacobi processes to (compact) trigonometric Heckman-Opdam theory. As already pointed out in Section 2.3.2, if one transforms the $N$-dimensional Jacobi process by the coordinate change $x_i \mapsto \arccos(2x_i - 1)$, $i \in \{1, \ldots, N\}$, one obtains a compact Heckman-Opdam process of type $BC_N$. Hence, the paper [86] highlights another approach of studying these processes based on the examination of the corresponding heat-semigroup and not via SDEs.

## 3.6 Non-compact Jacobi/Fisher-Snedecor processes

In this section we study the case that $a$ is a quadratic polynomial with two distinct real roots and positive leading coefficient. By a rescaling in space and time we can assume w.l.o.g. that $a(x) = x(x-1)$. Thus the processes of interest here arise as solutions with values in $C_N^{BC*} := C_N^A \cap [1,\infty)^N$ to the SDEs

$$dX_{i,t} = \sqrt{2X_{i,t}(X_{i,t}-1)}\, dB_{i,t} + \frac{\beta}{2}\left[-p + (p+q)X_{i,t} + \sum_{j\colon j\neq i} \frac{2X_{i,t}X_{j,t} - X_{i,t} - X_{j,t}}{X_{i,t} - X_{j,t}}\right] dt\,, \tag{3.6.1}$$

$i \in \{1, \ldots, N\}$, $X_0 = \hat{x}_0 \in C_N^{BC*}$, for certain $p, q \in \mathbb{R}$. We also consider the corresponding frozen versions of these processes which arise as solutions to

$$\frac{d}{dt}x_{i,t} = -p + (p+q)x_{i,t} + \sum_{j\colon j\neq i} \frac{2x_{i,t}x_{j,t} - x_{i,t} - x_{j,t}}{x_{i,t} - x_{j,t}}\,, \tag{3.6.2}$$

$i \in \{1, \ldots, N\}$, $x_0 = \hat{x}_0 \in C_N^{BC*}$, for certain $p, q \in \mathbb{R}$. Note the strong resemblence to the SDEs (3.5.1) for Jacobi processes where we have a sign change under the square root in the diffusion part and a sign change in the drift part. Hence, we will call solutions to (3.6.1) and (3.6.2) *non-compact Jacobi processes* and *frozen non-compact Jacobi processes* respectively. This is a break with the naming convention applied to the other instances of $\beta$-Pearson diffusions as the corresponding one-dimensional processes are, up to an affine linear transformation, already known as Fisher-Snedecor diffusions. This latter name stems from the fact that for certain parameters the invariant measure of a Fisher-Snedecor diffusion is a Fisher-Snedecor or $F$-distribution and hence it is of some interest in the statistics literature ([113, 46, 14]). We will see below that, for certain parameters, in our multivariate setting we have image measures of Jacobi-ensembles as invariant distributions whence we advocate the name 'non-compact Jacobi processes' where 'non-compact' here

alludes to the unbounded state space $C_N^{BC*} = C_N^A \cap [1,\infty)^N$. Before we discuss invariant distributions, we first establish sufficient conditions on the parameters $\beta, p, q$ such that unique non-colliding solutions exist:

**Lemma 3.6.1.** *Let $N \in \mathbb{N}$, $q > N-1$, $p \in \mathbb{R}$ and $\hat{x}_0 \in C_N^{BC*}$.*

*(i) For $\beta \geq 1$ there exists a unique strong non-colliding solution $(X_t)_{t\geq 0}$ of (3.6.1) with $X_0 = \hat{x}_0$.*

*(ii) There exists a unique non-colliding solution $(x_t)_{t\geq 0}$ to (3.6.2) with $x_0 = \hat{x}_0$.*

*Proof.* We will show the Lemma by considering the process $(\tilde{X}_t := X_{2t/\beta})_{t\geq 0}$. The SDE for $\tilde{X}$ then reads

$$d\tilde{X}_{i,t} = 2\sqrt{\frac{1}{\beta}\tilde{X}_{i,t}(\tilde{X}_{i,t}-1)}\, dB_{i,t} + \left[-p + (p+q)\tilde{X}_{i,t} + \sum_{j:\, j\neq i} \frac{2\tilde{X}_{i,t}\tilde{X}_{j,t} - \tilde{X}_{i,t} - \tilde{X}_{j,t}}{\tilde{X}_{i,t} - \tilde{X}_{j,t}}\right] dt\,, \tag{3.6.3}$$

$i \in \{1,\dots,N\}$, $\tilde{X}_0 = \hat{x}_0 \in C_N^{BC*}$. In the case $\beta = \infty$ we interpret all quotients $\frac{1}{\beta}$ as 0 which enables us to treat assertions (i) and (ii) simultaneously. The proof shares a lot of similarities with the proof of Lemma 3.4.1. Here, we also verify various conditions of the existence and uniqueness Theorems in [52]. However, a proper extension of the coefficients of the SDE (3.6.1) to the bigger set $C_N^A$ is more involved here. Our idea is to 'reflect' the diffusion coefficient $x \mapsto 2\sqrt{x(x-1)/\beta}$ at the point 1. Hence, we define the extended diffusion coefficient

$$\sigma(x) := \begin{cases} 2\sqrt{x(x-1)/\beta}\,, & x \geq 1 \\ 2\sqrt{(1-x)(2-x)/\beta}\,, & x < 1\,. \end{cases}$$

Further, we choose to extend the drift coefficient in the following way, which fits the relation between diffusion and drift coefficients for eigenvalues of general random matrix models (compare [51] Theorems 3 & 4): Set

$$g(x) := \begin{cases} \sqrt{x/\beta}\,, & x \geq 1 \\ \sqrt{(2-x)/\beta}\,, & x < 1\,, \end{cases} \qquad h(x) := \begin{cases} \sqrt{x-1}\,, & x \geq 1 \\ \sqrt{1-x}\,, & x < 1\,, \end{cases} \qquad b(x) := -p + (p+q)x\,,$$

and

$$\begin{aligned} H(x,y) :=& \beta\left(g^2(x)h^2(y) + g^2(y)h^2(x)\right) \\ =& \begin{cases} x(y-1) + y(x-1)\,, & x,y \geq 1 \\ (2-x)(1-y) + (2-y)(1-x)\,, & x,y < 1 \\ (2-x)(y-1) + y(1-x)\,, & x < 1 \text{ and } y \geq 1 \\ x(1-y) + (2-y)(x-1)\,, & x \geq 1 \text{ and } y < 1\,. \end{cases} \end{aligned}$$

We allow $\beta = \infty$ by setting $\sigma \equiv 0$ and $H$ equal to the r.h.s. above. We will show that there exists a non-colliding solution for the system of SDEs

$$dY_{i,t} = \sigma(Y_{i,t})\, dB_{i,t} + \left[ b(Y_{i,t}) + \sum_{j:\, j\neq i} \frac{H(Y_{i,t}Y_{j,t})}{Y_{i,t} - Y_{j,t}} \right] dt\,, \quad i \in \{1, \ldots, N\}\,, \tag{3.6.4}$$

$Y_0 = y_0 \in C_N^A$ whenever the parameters $N, \beta, p, q$ fulfill our assumptions. Note that not all conditions of the uniqueness Theorem 5.3 in [52] are fulfilled as one can check that $H(w,z)(y-x) > H(x,y)(z-w)$ for $(w,x,y,z) = (-1,0,1,2)$. As our interest lies in solutions starting in $C_N^{BC*}$, this problem can be circumvented by first proving existence of solutions for general starting vectors $y_0 \in C_N^A$ and then, in a second step, proving uniqueness under the constraint $y_0 \in C_N^{BC*}$. To show existence of a solution to (3.6.4), by Theorem 5.1 in [52] it suffices to verify the following five conditions where the coefficient functions $\sigma_i := \sigma, b_i := b, H_{i,j} := H$ are independent of $i, j \in \{1, \ldots, N\}$:

(C2): We need to find a constant $c \geq 0$ such that $\sigma^2(x) + b(x)x \leq c(1+x^2)$ and $H(x,y) \leq c(1+|xy|)$ for all $x, y \in \mathbb{R}$. It turns out that for our choice of $H$ such a constant does not exist. However, the only application of condition (C2) in the paper [52] is in the proof of Proposition 3.4. The bound for $H = H_{i,j}$ is used to show that there is a $\tilde{c} \geq 0$ such that

$$\sum_{\substack{i,j=1 \\ i<j}}^{N} H_{i,j}(\lambda_i, \lambda_j) \leq \tilde{c} \left( 1 + \sum_{i=1}^{N} \lambda_i^2 \right).$$

But this means the weaker condition

$$\text{'It exists } \; c \geq 0 \; \text{ such that } \; \sigma^2(x) + b(x)x \leq c(1+x^2) \; \text{ and}$$
$$H(x,y) \leq c(1+x^2+y^2) \; \text{ for all } \; x, y \in \mathbb{R}\,.\text{'}$$

actually is sufficient as well. That this latter condition is met by our $\sigma, b, H$ is elementary to check.

(A2): We need to find a $c \geq 0$ such that $\sigma^2(x) + \sigma^2(y) \leq c(x-y)^2 + 4H(x,y)$ for all $x, y \in \mathbb{R}$. We will show that the choice $c = 4$ is sufficient. We start by observing that for all $x, y \geq 1$ we have

$$\begin{aligned} \sigma^2(x) + \sigma^2(y) &= \frac{4}{\beta}(x(x-1) + y(y-1)) \leq 4(x^2 + y^2 - x - y) \\ &= 4(x-y)^2 + 4(x(y-1) + y(x-1)) = 4(x-y)^2 + 4H(x,y)\,. \end{aligned}$$

This shows the claim for the case $x, y \geq 1$, and, by using $\sigma(x) = \sigma(2-x)$ and $H(x,y) = H(2-x, 2-y)$, also for the case $x, y < 1$. By symmetry it remains to consider the case $x \geq 1, y < 1$. Here we have

$$\begin{aligned} \sigma^2(x) + \sigma^2(y) &= \frac{4}{\beta}(x(x-1) + (1-y)(2-y)) \leq 4\left(x^2 + y^2 - x - 3y + 2\right) \\ &= 4((x-y)^2 + (2xy - x - 3y + 2)) \leq 4((x-y)^2 + (-2xy + 3x + y - 2)) \\ &= 4(x-y)^2 + 4H(x,y)\,, \end{aligned}$$

where we used $0 \leq -xy + x + y - 1 = (x-1)(1-y)$ in the second inequality.
(A3): We need to find a $c \geq 0$ such that

$$H(x,y)(y-x)+H(y,z)(z-y) \leq c(z-y)(z-x)(y-x)+H(x,z)(z-x) \text{ for all } x < y < z\,.$$

Equivalently, we need to verify the existence of some $c \geq 0$ such that

$$0 \leq c(z-y)(z-x)(y-x) + H(x,z)(z-x) - H(x,y)(y-x) - H(y,z)(z-y) =: g_c(y)$$

for all $x < y < z$. Note that $g_c$ vanishes at the points $x$ and $z$. We have to consider the following four cases: $x \geq 1$, $x < 1 \leq y$, $y < 1 \leq z$ and $z < 1$. Note that in either case $g_c$ is a quadratic polynomial with roots $x$ and $z$. Thus, it remains to calculate the leading coefficient of $g_c$. The leading coefficient $l_y(x)$ of $y \mapsto H(x,y)(y-x)$ is easily seen to be given by

$$l_y(x) = \begin{cases} 2x-1\,, & x,y \geq 1 \\ 2x-3\,, & x,y < 1 \\ 3-2x\,, & x < 1, y \geq 1 \\ 1-2x\,, & x \geq 1, y < 1\,. \end{cases}$$

By symmetry, the leading coefficient of $y \mapsto H(y,z)(z-y)$ is given by $-l_y(z)$. Thus, the leading coefficient of $g_c$ is given by

$$\begin{cases} -c(z-x)-(2x-1)+2z-1 = (2-c)(z-x)\,, & 1 \leq x\,, \\ -c(z-x)-(3-2x)+2z-1 = (2-c)(z-x)+4(x-1)\,, & x < 1 \leq y\,, \\ -c(z-x)-(2x-3)+1-2z = (2-c)(z-x)+4(1-z)\,, & y < 1 \leq z\,, \\ -c(z-x)-(2x-3)+2z-3 = (2-c)(z-x)\,, & z < 1\,, \end{cases}$$

which is negative for $c > 2$.
(A4): We need to verify the following: $G := \{x \in \mathbb{R} \colon 2\sigma^2(x) + H(x,x) = 0\}$ consists of isolated points and for all $x \in G$ it holds

$$b(x) + \sum_{j=1}^{N-2} \frac{H(x,y_j)}{x-y_j} \mathbb{1}_{\mathbb{R}\setminus\{x\}}(y_j) \neq 0$$

for all $y_1, \ldots, y_{N-2} \in \mathbb{R}$, with the convention 'undefined $\times$ 0 = 0'. In our case it holds $G = \{1\}$. Using

$$H(1,y) = \begin{cases} y-1\,, & y \geq 1 \\ 1-y\,, & y < 1\,, \end{cases}$$

we have

$$b(1) + \sum_{j=1}^{N-2} \frac{H(1,y_j)}{1-y_j} \mathbb{1}_{\mathbb{R}\setminus\{1\}}(y_j) = q + |\{j \colon y_j < 1\}| - |\{j \colon y_j > 1\}| \geq q - (N-2) > 0\,.$$

(A5): This condition is empty for our SDE since the linear drift coefficient $b$ is the same for each particle $Y_i$.
So far we have shown the existence of a solution to (3.6.4). Next, we argue that if such a solution starts in $C_N^{BC*}$ it must stay in that set. It holds that the stopping time $\vartheta := \inf\{t > 0 \colon Y_{1,t} < 1\}$ is infinite a.s. Indeed: Using $q > N-1$ we can use the local time argument as in the proof of Lemma 3.4.1. This argument is applicable in our case since the drift coefficient $D_t := b(Y_{1,t}) + \sum_{j\colon j\neq 1} \frac{H(Y_{1,t},Y_{j,t})}{Y_{1,t}-Y_{j,t}}$ of $Y_{1,t}$ satisfies $D_{\vartheta(\omega)}(\omega) > 0$ for $\omega \in \{\vartheta < \infty\}$ since $q > N-1$.
Finally, it remains to show uniqueness of the solution in the case $Y_0 = y_0 \in C_N^{BC*}$. We have proven so far that any such solution must take values in $C_N^{BC*}$. By going through the proof of Theorem 5.3 in [52], we see that it suffices to show the following two conditions: (C1): We need to find a function $\rho\colon [0,\infty) \to [0,\infty)$ with $\int_{(0,\infty)} \frac{1}{\rho(x)}\,dx = \infty$ and $|\sigma(x) - \sigma(y)|^2 \leq \rho(|x-y|)$ for all $x, y \in [1,\infty)$. Note that $\sigma|_{[1,3)}$ is 1/2-Hölder continuous as the composition of the 1/2-Hölder continuous function $x \mapsto 2\sqrt{(x-1)/\beta}$ and the Lipschitz function $x \mapsto 2x(x-1)+1$. Further, $\sigma|_{[2,\infty)}$ is Lipschitz continuous as its derivative is bounded. Moreover, for $x \geq 3$ and $y \in [1,2)$ we have

$$|\sigma(x) - \sigma(y)| = \frac{2}{\sqrt{\beta}}\left(\sqrt{x(x-1)} - \sqrt{y(y-1)}\right) \leq \frac{2}{\sqrt{\beta}}(x - (y-1)) \leq \frac{4}{\sqrt{\beta}}|x-y|\,.$$

Combining these facts, we can deduce that there exist constants $c_1, c_2 > 0$ such that

$$|\sigma(x) - \sigma(y)|^2 \leq c_1|x-y| + c_2|x-y|^2 =: \rho(|x-y|) \text{ for all } x, y \geq 1\,.$$

Clearly $\int_{(0,\infty)} \frac{1}{\rho(x)}\,dx = \infty$.
(A1): We need to show the inequality $f(z) := \frac{H(w,z)}{z-w} \leq \frac{H(x,y)}{y-x}$ for all $1 \leq w < x < y < z$. Note that $f'(z) = -2w(w-1)/(z-w)^2 \leq 0$. Thus for all $1 \leq w < x < y < z$:

$$f(z) \leq f(y) = \frac{w(y-1) + y(w-1)}{y-w} \leq \frac{x(y-1) + y(x-1)}{y-x} = \frac{H(x,y)}{y-x}\,.$$

This concludes the proof. □

For small $p$ we have the following large $t$ asymptotic for frozen non-compact Jacobi processes, compare Lemma 3.5.3.

**Lemma 3.6.2.** *Let $q > N-1$, $p < -q$ and let $\hat{x}_0 \in C_N^{BC*}$. Let $(x_t)_{t\geq 0}$ be the unique non-colliding solution to (3.6.2) for these parameters. Then the limit $\lim_{t\to\infty} x_t = (z_1, \ldots, z_N) \in \operatorname{int} C_N^{BC*}$ exists, where $(2z_N^{-1} - 1, \ldots, 2z_1^{-1} - 1)$ is the vector of the ordered zeros of the Jacobi polynomial $P_N^{q-N,-(p+q+1)}$. Moreover, $(z_1, \ldots, z_N)$ is the only stationary solution to (3.6.2) in this case.*

**Remark 3.6.3.** It is expected that if $(x_t)_{t\geq 0}, (\tilde{x}_t)_{t\geq 0}$ are two solutions to (3.6.2) then it holds an estimate of the form

$$|x_t - \tilde{x}_t| \leq e^{-ct}|x_0 - \tilde{x}_0|$$

for some $c > 0$; cf. the corresponding result for frozen compact Jacobi processes in [108] Proposition 4.3.

*Proof.* The proof is very similar to the one of Lemma 3.4.3. We first discuss the stationarity of solutions. So assume that $z$ with $1 < z_1 < \cdots < z_N$ satisfies

$$0 = -p + (p+q)z_j + \sum_{k\colon k\neq j} \frac{2z_jz_k - (z_j + z_k)}{z_j - z_k}\,, \quad j \in \{1,\dots,N\}\,.$$

Division by $-z_j$, and using the formula

$$\sum_{k\colon k\neq j} \frac{1 + z_kz_j^{-1} - 2z_k}{z_j - z_k} = (N-1)z_j^{-1} + 2\sum_{k\colon k\neq j} \frac{z_j^{-1}(1 - z_j^{-1})}{z_j^{-1} - z_k^{-1}}\,,$$

gives

$$\begin{aligned}
0 =& pz_j^{-1} - (p+q) + \sum_{k\colon k\neq j} \frac{1 + z_kz_j^{-1} - 2z_k}{z_j - z_k}\\
=& (p + (N-1))z_j^{-1} - (p+q) + 2\sum_{k\colon k\neq j} \frac{z_j^{-1}(1 - z_j^{-1})}{z_j^{-1} - z_k^{-1}}\\
=& \frac{1}{2}\left[-p - 2q + N - 1 + (p+N-1)y_j + 2\sum_{k\colon k\neq j} \frac{1 - y_j^2}{y_j - y_k}\right]\\
=& \frac{1 - y_j^2}{2}\left[\frac{q - (N-1)}{y_j - 1} - \frac{p+q}{y_j + 1} + 2\sum_{k\colon k\neq j} \frac{1}{y_j - y_k}\right]\,,
\end{aligned}$$

where we set $y_j := 2z_j^{-1} - 1$. By equation 6.7.5 in [99], it is known that in the case $q > N-1$ and $p+q < 0$ this set of equations has at its unique solution in $\{y \in \mathbb{R}^N\colon -1 < y_1 < \cdots < y_N < 1\}$ the ordered zeros of the Jacobi polynomial $P_N^{q-N,-(p+q+1)}$.
It remains to discuss an arbitrary starting condition $\hat{x}_0 \in C_N^A$. Here we first transform the solutions $(x_t)_{t\geq 0}$ by the homeomorphism $f\colon C_N^A \to f(C_N^A), x \mapsto (e_1^N(x),\dots,e_N^N(x))$, where $(e_j^N)_{j\in\{1,\dots,N\}}$ are the elementary symmetric polynomials discussed in Section 2.2. Set $y_t := f(x_t)$. Using the formula

$$e_m^N(x) = x_i e_{m-1}^{N-1}(x_{\{1,\dots,N\}\setminus\{i\}}) + e_m^{N-1}(x_{\{1,\dots,N\}\setminus\{i\}})\,, \quad x \in \mathbb{R}^N\,, \quad i \in \{1,\dots,N\}\,,$$

we get (notationally suppressing dependence of $t$):

$$\frac{d}{dt}y_m = \sum_{i=1}^N e_{m-1}^{N-1}(x_{\{1,\dots,N\}\setminus\{i\}})\frac{d}{dt}x_i\,.$$

Plugging in the ODE for $x_i$ and using formulas (2.2.2), (2.2.4) & (2.2.5) we obtain

$$\frac{d}{dt}y_{m,t} = m(p+q+1-m)y_{m,t} - (p+m-1)(N-m+1)y_{m-1,t}\,, \tag{3.6.5}$$

$m \in \{1,\dots,N\}$, with the convention $y_{0,\cdot} \equiv 0$. Thus

$$y_{m,t} = e^{m(p+q+1-m)t}\left(y_{m,0} - (p+m-1)(N-m+1)\int_0^t e^{-m(p+q+1-m)r}y_{m-1,r}\,dr\right).$$

Since $p+q<0$, we have $m(p+q+1-m)<0$ for all $m \in \{1,\dots,N\}$. Inductively one sees that the limit $\lim_{t\to\infty} y_t$ exists and does not depend on $(y_{1,0},\dots,y_{N,0})$. Hence this limit equals $f(z)$ where $z$ is the stationary solution considered above. A transfer back via $f^{-1}$ shows the claim. □

Next, we want to show the corresponding result for finite $\beta$.

### 3.6.1 Invariant measures of non-compact Jacobi processes

To analyze the existence of an invariant probability measure of the dynamic (3.6.1), we will show that in suitable coordinates this dynamic falls in the framework of (overdamped) Langevin diffusions/Kolmogorov processes. Hence we can identify the invariant measure by the corresponding Gibbs (or Boltzmann) measure; see e.g. [91] Sections 2.2.2. & 2.2.3. Note that we also used this type of argument in the proof of Lemma 3.4.4.
Let $(X_t)_{t\geq 0}$ be a solution to (3.6.1) and set $Y_{i,t} := \operatorname{arcosh}(2X_{i,t}-1)$, $i \in \{1,\dots,N\}$. Then by Itô's formula and some elementary calculations, $(Y_t)_{t\geq 0}$ satisfies

$$\begin{aligned} dY_{i,t} = \sqrt{2}\,dB_{i,t} + \frac{\beta}{2}\Bigg[&(q-p)\coth(Y_{i,t}/2) + 2(p-(N-1)-1/\beta)\coth(Y_{i,t}) \\ &+ \sum_{j:\,j\neq i}\left(\coth\left(\frac{Y_{i,t}-Y_{j,t}}{2}\right) + \coth\left(\frac{Y_{i,t}+Y_{j,t}}{2}\right)\right)\Bigg]\,dt\,, \end{aligned}$$

$i \in \{1,\dots,N\}$. Note that $(Y_t)_{t\geq 0}$ now takes values in $C_N^B$. Setting

$$k_1 := \frac{\beta}{2}(q-p)\,,\ k_2 := \frac{\beta}{2}(p-(N-1)) - 1/2\,,\ k_3 := \frac{\beta}{2}\,,$$

we see that the corresponding generator has the form

$$\begin{aligned} \mathcal{L}f(w) := \Delta f(w) + \sum_{i=1}^N\Bigg(&k_1\coth(w_i/2) + 2k_2\coth(w_i) \\ &+ k_3\sum_{j:j\neq i}\left(\coth\left(\frac{w_i-w_j}{2}\right) + \coth\left(\frac{w_i+w_j}{2}\right)\right)\Bigg)\partial_{w_i}f(w) \end{aligned} \tag{3.6.6}$$

for functions $f \in C^2(\mathbb{R}^N)$ which are invariant under the Weyl group associated to the root system $BC_N$. This is precisely the radial Heckman-Opdam Laplacian (2.3.2) for the root system $BC_N$ and for the the mulitplicity $(k_1, k_2, k_3)$. Thus, if the parameters $k_1, k_2, k_3$ are non-negative, $(Y_t)_{t\geq 0}$ has been studied in [93, 94].

**Remark 3.6.4.** Note that in Lemma 3.6.1 we have shown that the condition

$$k_1 + k_2 > -1/2 \quad \text{and} \quad k_3 \geq 1/2 \tag{3.6.7}$$

is sufficient for the existence of non-colliding solutions. Thus, it would be interesting if the approach taken in [94] also extends to certain negative multiplicities in the case of root system $BC_N$. A valuable ingredient might be the analysis of the corresponding Heckman-Opdam hypergeometric function which was undertaken for certain negative multiplicities in [78, 58], where the particular case (3.6.7) is covered.

We are now ready to state our result on invariant measures:

**Lemma 3.6.5.** *For $\beta \geq 1$, $p + q < 2/\beta$ and $q > N - 1$ the dynamic (3.6.1) has a unique invariant probability measure $\rho$ which is supported on $C_N^{BC*}$ and has density*

$$C(N, p, q, \beta) \prod_{j=1}^{N} \left( x_j^{\frac{\beta}{2}(p-(N-1))-1} (x_j - 1)^{\frac{\beta}{2}(q-(N-1))-1} \right) \prod_{1\leq j<k\leq N} (x_k - x_j)^\beta dx$$

*for some normalization constant $C(N, p, q, \beta) > 0$. This is the image measure of the Jacobi ensemble (3.5.3) with parameters $\tilde{\beta} = \beta$, $\tilde{a} = -\frac{\beta}{2}(p + q)$, $\tilde{b} = \frac{\beta}{2}(q - (N - 1)) - 1$ under the map $(y_1, \ldots, y_N) \mapsto (y_N^{-1}, \ldots, y_1^{-1})$.*

*Proof.* First, note that the process $(Y_t)_{t\geq 0}$ which is generated by $\mathcal{L}$ has the following Langevin diffusion structure: Let

$$\begin{aligned} V(x) := &- 2k_1 \sum_{j=1}^{N} \log(\sinh(x_j/2)) - 2k_2 \sum_{j=1}^{N} \log(\sinh(x_j)) \\ &- 2k_3 \sum_{1\leq j<k\leq N} \left( \log\left( \sinh\left( \frac{x_k - x_j}{2} \right) \right) + \log\left( \sinh\left( \frac{x_j + x_k}{2} \right) \right) \right), \end{aligned} \tag{3.6.8}$$

then $\mathcal{L} = \Delta + (-\nabla V(x)) \cdot \nabla$. Now, the candidate for the stationary distribution of the Markov process generated by $\mathcal{L}$ is the Gibbs measure $\tilde{\rho}(x) := \frac{1}{Z} e^{-V(x)}$, where $Z := \int_{\operatorname{int} C_N^B} e^{-V(x)}\, dx$. To see that $Z$ is finite, write

$$e^{-V(x)} = \prod_{j=1}^{N} \left( \sinh^{2k_1}(x_j/2) \sinh^{2k_2}(x_j) \right) \prod_{1\leq j<k\leq N} \left( \sinh\left( \frac{x_k - x_j}{2} \right) \sinh\left( \frac{x_j + x_k}{2} \right) \right)^\beta. \tag{3.6.9}$$

This density transforms under the map $C_N^B \to C_N^{BC*}, x_j \mapsto (1+\cosh(x_j))/2$, $j \in \{1,\dots,N\}$, to

$$c\prod_{j=1}^{N}\left(x_j^{k_2-1/2}(x_j-1)^{k_1+k_2-1/2}\right)\prod_{1\le j<k\le N}(x_k-x_j)^{\beta},$$

for some $c>0$. Another change of variables $(x_1,\dots,x_N)\mapsto(x_N^{-1},\dots,x_1^{-1})$ gives a density proportional to the Jacobi-ensemble density (3.5.3) with parameters $\tilde{\beta}=\beta$, $\tilde{a}=-\frac{\beta}{2}(p+q)$, $\tilde{b}=\frac{\beta}{2}(q-(N-1))-1$. By our assumptions we have $\tilde{\beta}>0$ and $\tilde{a},\tilde{b}>-1$ which proves finiteness of $Z$. Hence, $\tilde{\rho}$ is a probability density. This implies that $\tilde{\rho}$ is the unique invariant probability distribution of the Markov process generated by $\mathcal{L}$, see e.g. Section 2.2.3. of [91] and the beginning of Section 2 of the survey [75]. The result follows by applying the coordinate change $x_j \mapsto (1+\cosh(x_j))/2$, $i\in\{1,\dots,N\}$, as above. □

We end this subsection by noting that the 'inverted' Jacobi ensemble is log-concave for certain parameters after the coordinate change $x_i \mapsto \operatorname{arcosh}(2x_i-1)$, $i\in\{1,\dots,N\}$. This in turn gives convergence to equilibrium of the associated non-compact Jacobi processes for large $t$:

**Lemma 3.6.6.** *Let $\beta>0$, $p+q<2/\beta$ and $q\ge N-1+1/\beta$.*

*(i) The probability measure $\tilde{\rho}$ on $C_N^B$ with density proportional to* (3.6.9) *is a* log*-concave probability measure, i.e.*

$$\log\left(\tilde{\rho}((1-\lambda)A+\lambda B)\right)\ge(1-\lambda)\log(\tilde{\rho}(A))+\lambda\log(\tilde{\rho}(B))$$

*for all $\lambda\in(0,1)$ and open $A,B\subseteq C_N^B$ with $\tilde{\rho}(A),\tilde{\rho}(B)>0$.*

*(ii) Let $(X_t)_{t\ge0}$ be a non-compact Jacobi process with parameters $\beta\ge1$, $p+q<2/\beta$ and $q\ge N-1+1/\beta$. Then $X_t \xrightarrow{t\to\infty} \rho$ in distribution where $\rho$ is the image measure of the Jacobi ensemble* (3.5.3) *with parameters $\tilde{\beta}=\beta$, $\tilde{a}=-\frac{\beta}{2}(p+q)$, $\tilde{b}=\frac{\beta}{2}(q-(N-1))-1$ under the map $(y_1,\dots,y_N)\mapsto(y_N^{-1},\dots,y_1^{-1})$.*

*Proof.* The proof is very similar to the second part of the proof of Lemma 3.4.4.
The Potential $V$ from (3.6.8) is convex if $k_1+k_2\ge0$ and $k_1\ge0$. To see this, note that the Hessian $H$ of $V$ can be written as $H=H^1+H^2+H^3$, where $H^1(x):=\operatorname{diag}(f(x_1),\dots,f(x_N))$, with

$$f(x):=-\frac{k_1}{2}\left(1-\coth^2(x/2)\right)-2k_2\left(1-\coth^2(x)\right),$$

and

$$H^2_{jk}(x):=\begin{cases}-\frac{k_3}{2}\sum_{l:\,l\neq j}\left(1-\coth^2\left(\frac{x_j-x_l}{2}\right)\right), & j=k,\\ +\frac{k_3}{2}\left(1-\coth^2\left(\frac{x_j-x_k}{2}\right)\right), & j\neq k.\end{cases}$$

$$H^3_{jk}(x):=\begin{cases}-\frac{k_3}{2}\sum_{l:\,l\neq j}\left(1-\coth^2\left(\frac{x_j+x_l}{2}\right)\right), & j=k,\\ -\frac{k_3}{2}\left(1-\coth^2\left(\frac{x_j+x_k}{2}\right)\right), & j\neq k.\end{cases}$$

Using the identity

$$\coth(x) = \frac{1 + \coth^2(x/2)}{2\coth(x/2)}\,,$$

one sees that $f$ is non-negative on $(0,\infty)$ iff

$$(k_1 + k_2)x^2 - (k_1 + 2k_2)x + k_2 \geq 0 \text{ for all } x > 1\,.$$

One can easily check that this holds iff $k_1 + k_2 \geq 0$ and $k_1 \geq 0$. Thus $H^1$ is positive semi-definite on $\operatorname{int} C_N^B$ under our assumptions. Additionally, $H^2$ and $H^3$ are also positive semi-definite since they are of the type of the matrix $B$ discussed in the proof of Lemma 3.4.4. Thus the Hessian of $V$ is positive semi-definite which implies that $V$ is convex. By Theorem 9.4.10 in [1], we have shown that $\tilde{\rho}$ is a log-concave probability measure. Hence, by Theorem 1.3 in [2], $(Y_t)_{t\geq 0}$ converges to $\tilde{\rho}$ in distribution. A transfer via the coordinate change $x_j \mapsto (1 + \cosh(x_j))/2$, $j \in \{1,\ldots,N\}$, shows the claim. □

### 3.6.2 Corners of Brownian motion on the positive definite matrices

In this section, we consider a new dynamical random matrix model such that the corresponding ordered eigenvalues form an $n$-dimensional non-compact Jacobi process with parameters $p, q, \beta$ in the case $p = n, q = N - n$, $1 \leq n < N$, $n, N \in \mathbb{N}$, and $\beta = 2$. It will be clear that a similar construction also works in the case $\beta = 1$ when considering real symmetric matrices. The construction of our matrix model is hereby inspired by the matrix model for Jacobi processes discussed in Section 3.5. In the following, we will use several calculation rules from matrix Itô calculus, see [68] Section 2.1 for a concise overview. For a more in-depth treatment of this topic see the appendix of [81].
We start by considering the left-invariant multiplicative Brownian motion $(G_t)_{t\geq 0}$ on $\mathrm{Gl}(N,\mathbb{C})$, i.e. the unique solution to the matrix SDE

$$dG_t = G_t\, dZ_t\,,\ G_0 = \hat{G}_0\,,$$

where the entries of $Z$ are i.i.d. complex Brownian motions of variance $t/N$ at time $t$ and $\hat{G}_0 \in \mathrm{Gl}(N,\mathbb{C})$ is deterministic; see [68], also see [82] for the corresponding process on $\mathrm{Gl}(N,\mathbb{R})$.
It holds that $(G_t^{-1})_{t\geq 0}$ is a right-invariant Brownian motion on $\mathrm{Gl}(N,\mathbb{C})$ and satisfies (cf. [68] proof of Prop. 5.1):

$$dG_t^{-1} = -dZ_t\, G_t^{-1}\,,\ G_0^{-1} = \hat{G}_0^{-1}\,.$$

Now consider the processes $(H_t := G_t G_t^*)_{t\geq 0}$ and $(H_t^{-1} := G_t^{-*} G_t^{-1})_{t\geq 0}$ with values in the space of positive definite matrices $\mathcal{P}_N$, where for a complex matrix $A$ we write $A^{-*} = (A^*)^{-1}$. We call $(H_t)_{t\geq 0}$ and $(H^{-1})_{t\geq 0}$ *Brownian motion on positive definite matrices*. Their real symmetric counterpart has been considered in [82] and in [83] Section 2.9. In

the former reference, it was shown that the corresponding eigenvalue process is given by a non-colliding geometric Brownian motion, see Section 3.4 for this notion.
Set $X_t := \frac{1}{\sqrt{2}}(Z_t + Z_t^*)$, then $(X_t)_{t\geq 0}$ is a hermitian Brownian motion. By the Itô product rule it holds:

$$dH_t = \sqrt{2} G_t \, dX_t \, G_t^* + H_t \, dt \,, \; H_0 = \hat{G}_0 \hat{G}_0^* \,, \tag{3.6.10}$$

$$dH_t^{-1} = -\sqrt{2} G_t^{-*} \, dX_t \, G_t^{-1} + H_t^{-1} \, dt \,, \; H_0^{-1} = \hat{G}_0^{-*} \hat{G}_0^{-1} \,. \tag{3.6.11}$$

**Definition 3.6.7.** (i) Let $n \in \{1, \ldots, N\}$. For $A \in M_N(\mathbb{C})$ we denote the upper left block matrix with $n$ lines and $n$ rows of $A$ by $\pi_n(A) := (A_{ij})_{1\leq i,j\leq n}$. In the language of matrix analysis the map $\pi_n \colon \mathcal{P}_N \to \mathcal{P}_n$ is a compression.

(ii) Let $A, B \in M_N(\mathbb{C})$ be hermitian. We write $A \leq B$ if $B - A$ is positive semi-definite. The resulting partial order on the space of hermitian $N \times N$ matrices is called *Loewner order*.

The construction of our matrix model for non-compact Jacobi processes is partially motivated by the following fact:

**Proposition 3.6.8.** *Let $A \in \mathcal{P}_N$ and let $n \in \{1, \ldots, N\}$. Then*

$$\pi_n(A^{-1})^{1/2} \cdot \pi_n(A) \cdot \pi_n(A^{-1})^{1/2} \geq I_n \,.$$

*Proof.* By exercises V 1.15 and V 2.2 in [17], it holds the inequality $\pi_n(A) \geq \pi_n(A^{-1})^{-1}$. Hence, the statement follows from Lemma V 1.5 therein. □

Our matrix model for non-compact Jacobi processes is the following:

**Theorem 3.6.9.** *Let $n \in \{1, \ldots, \lfloor N/2 \rfloor\}$ and let $(H_t)_{t\geq 0}$ be a Brownian motion on the positive definite matrices $\mathcal{P}_N$ with deterministic start in $H_0 = \hat{G}_0 \hat{G}_0^*$ as above. Assume that the eigenvalues of $\pi_n(H_0)\pi_n(H_0^{-1})$ are distinct. Then the ordered eigenvalue process $(\lambda_t)_{t\geq 0}$ of*

$$\left(J_t := \pi_n(H_t)\pi_n(H_t^{-1})\right)_{t\geq 0}$$

*is an $n$-dimensional non-compact Jacobi process with parameters $p = n, q = N - n, \beta = 2$ which is rescaled in time by $t \mapsto 2t/N$.*

**Remark 3.6.10.** Unfortunately the present author was unable to find a concrete construction to generate $H_0 \in \mathcal{P}_N$ such that the eigenvalues of $\pi_n(H_0)\pi_n(H_0^{-1})$ are distinct. However, numerical simulations of the process $(J_t)_{t\geq 0}$ for arbitrary initial conditions $H_0$ suggest that there are plenty of such matrices.

*Proof.* First, note that $J_t$ and $\pi_n(H_t^{-1})^{1/2}\pi_n(H_t)\pi_n(H_t^{-1})^{1/2}$ have the same spectrum. By Proposition 3.6.8, we hence know that the spectrum of $J_t$ is a subset of $[1, \infty)$ for $t \geq 0$. Further, by assumption the stopping time

$$\tau := \inf\{t > 0 \colon \text{ The eigenvalues of } J_t \text{ are not distinct.}\}$$

is positive a.s.; we will see later that $\tau = \infty$ a.s. Now, we will use the standard technique for obtaining the eigenvalue SDEs from the matrix SDE. This technique is known from [82], we will use its streamlined form as in [39] pp.144-146.
On the set of square matrices $A$ with pairwise distinct real eigenvalues the map which sends $A$ to its collection of eigenvalues and eigenvectors is smooth; this can be shown using the implicit function theorem as discussed in [100] Section 1.3.4. Thus we know that there exists a $\mathrm{Gl}(N,\mathbb{C})$-valued semi-martingale $(V_t)_{t\geq 0}$ such that $V_t^{-1} J_t V_t = \Lambda_t$ for all $t \in [0,\tau)$ where

$$\Lambda_t = \begin{pmatrix} \lambda_{1,t} & & 0 \\ & \ddots & \\ 0 & & \lambda_{n,t} \end{pmatrix}$$

is the diagonal matrix consisting of the ordered eigenvalues $\lambda_{1,t} < \cdots < \lambda_{n,t}$ of $J_t$. Set $P := \begin{pmatrix} I_n & 0 \\ 0 & 0_{N-n} \end{pmatrix}$. Then it holds $\hat{J}_t := \begin{pmatrix} J_t & 0 \\ 0 & 0_{N-n} \end{pmatrix} = P H_t P H_t^{-1} P$. By Itô's product rule, we hence have

$$d(\hat{J}_t) = \sqrt{2} P G_t \, dX_t \, G_t^* P H_t^{-1} P - \sqrt{2} P H_t P G_t^{-*} \, dX_t \, G_t^{-1} P + 2 \left[ P H_t P H_t^{-1} P - \mathrm{tr}(P) P \right] dt \,,$$

which is equivalent to

$$dJ_t = \sqrt{2} \pi_n (G_t \, dX_t \, G_t^* P H_t^{-1}) - \sqrt{2} \pi_n (H_t P G_t^{-*} \, dX_t \, G_t^{-1}) + 2 \left[ J_t - \frac{n}{N} I_n \right] dt \,.$$

In our notation, we will suppress the dependence of appearing processes on $t$ for ease of reading from now on. Moreover, we will use the Stratonovich integral to further simplify calculations, i.e. we set $X \circ dY := X \, dY + \frac{1}{2} dX dY$ and $dY \circ X := dY \, X + \frac{1}{2} dY dX$ for square matrix valued semimartingales $X, Y$. As in one-dimensional Itô calculus the product rule simplifies to $d(XY) = X \circ dY + dX \circ Y$, and we have the iteration rule $X \circ (Y \circ dZ) = (XY) \circ dZ$. Now define

$$dA := dV^{-1} \circ V \,, \quad dC := V^{-1} \circ dJ \circ V \,.$$

Note that it holds $V^{-1} \circ dV = -dA$, which can be deduced from $d(V^{-1}V) = d(VV^{-1}) = 0$. The following computations hold for $t < \tau$: By Itô's product rule we have

$$d\Lambda = dA \circ \Lambda - \Lambda \circ dA + dC \,.$$

As $\Lambda_t$ is diagonal, we know that $d\lambda_j = dC_{jj}$ for all $j$, and $0 = (\lambda_k - \lambda_j) \circ dA_{jk} + dC_{jk}$ whenever $j \neq k$. In particular $dA_{jk} = \frac{1}{\lambda_j - \lambda_k} \circ dC_{jk}$, $j \neq k$.

To further progress we need to calculate

$$\begin{aligned} dJ_{jk}dJ_{lm} =& 2\left(\sum_{\alpha_1,\alpha_2=1}^{N}\left(G_{j\alpha_1}dX_{\alpha_1\alpha_2}(G^*PH^{-1})_{\alpha_2 k}-(HPG^{-*})_{j\alpha_1}dX_{\alpha_1\alpha_2}G^{-1}_{\alpha_2 k}\right)\right)\\ &\times\left(\sum_{\alpha_3,\alpha_4=1}^{N}\left(G_{l\alpha_3}dX_{\alpha_3\alpha_4}(G^*PH^{-1})_{\alpha_4 m}-(HPG^{-*})_{l\alpha_3}dX_{\alpha_3\alpha_4}G^{-1}_{\alpha_4 m}\right)\right)\\ =&\frac{2}{N}\sum_{\alpha_1,\alpha_2=1}^{N}\left(G_{j\alpha_1}(G^*PH^{-1})_{\alpha_2 k}-(HPG^{-*})_{j\alpha_1}G^{-1}_{\alpha_2 k}\right)\\ &\times\left(G_{l\alpha_2}(G^*PH^{-1})_{\alpha_1 m}-(HPG^{-*})_{l\alpha_2}G^{-1}_{\alpha_1 m}\right)dt\\ =&\frac{2}{N}\left(2J_{jm}J_{lk}-\delta_{lk}J_{jm}-\delta_{jm}J_{lk}\right)dt\,, \end{aligned} \tag{3.6.12}$$

and

$$\begin{aligned} dC_{jk}dC_{lm} =& \left(\sum_{\alpha_1,\alpha_2=1}^{n}V^{-1}_{j\alpha_1}dJ_{\alpha_1\alpha_2}V_{\alpha_2 k}\right)\left(\sum_{\alpha_3,\alpha_4=1}^{n}V^{-1}_{l\alpha_3}dJ_{\alpha_3\alpha_4}V_{\alpha_4 m}\right)\\ =&\frac{2}{N}\sum_{\substack{\alpha_1,\alpha_2,\\ \alpha_3,\alpha_4}}V^{-1}_{j\alpha_1}V_{\alpha_2 k}V^{-1}_{l\alpha_3}V_{\alpha_4 m}\left(2J_{\alpha_1\alpha_4}J_{\alpha_3\alpha_2}-\delta_{\alpha_3\alpha_2}J_{\alpha_1\alpha_4}-\delta_{\alpha_1\alpha_4}J_{\alpha_3\alpha_2}\right)dt\\ =&\frac{2}{N}\left(2\Lambda_{jm}\Lambda_{lk}-\delta_{lk}\Lambda_{jm}-\delta_{jm}\Lambda_{lk}\right)\,dt\,. \end{aligned} \tag{3.6.13}$$

Thus

$$\begin{aligned} (dAdC)_{jk} =& \sum_{l=1}^{n}dA_{jl}dC_{lk}=\sum_{l=1,l\neq j}^{n}\frac{1}{\lambda_j-\lambda_l}\circ dC_{jl}dC_{lk}\\ =&\frac{2}{N}\sum_{l=1,l\neq j}^{n}\frac{1}{\lambda_j-\lambda_l}\left(2\Lambda_{jk}\Lambda_{ll}-\Lambda_{jk}-\delta_{jk}\Lambda_{ll}\right)\,dt=\delta_{jk}\frac{2}{N}\sum_{l=1,l\neq j}^{n}\frac{2\lambda_j\lambda_l-\lambda_j-\lambda_l}{\lambda_j-\lambda_l}\,dt\,. \end{aligned} \tag{3.6.14}$$

In particular $dAdC$ is real diagonal. Similarly one gets $(dCdA)_{jk}=-(dAdC)_{jk}$. Now, we can calculate the finite variation part $dF$ of $dC$:

$$\begin{aligned} dF =& 2\left(V^{-1}JV-\frac{n}{N}I_n\right)dt+\frac{1}{2}\left(dV^{-1}dJV+V^{-1}dJdV\right)\\ =&2\left(\Lambda-\frac{n}{N}I_n\right)dt+\frac{1}{2}\left(dAdC-dCdA\right)\\ =&2\left(\Lambda-\frac{n}{N}I_n\right)dt+dAdC\,. \end{aligned} \tag{3.6.15}$$

Thus $F$ is diagonal and it holds

$$dF_{jj} = 2\left[\lambda_j - \frac{n}{N} + \frac{1}{N}\sum_{l=1,l\neq j}^{n}\frac{2\lambda_j\lambda_l - \lambda_j - \lambda_l}{\lambda_j - \lambda_l}\right]dt\,. \tag{3.6.16}$$

To calculate the martingale part of $d\lambda_j$, we note that

$$d\lambda_j d\lambda_k = dC_{jj}dC_{kk} = \frac{2}{N}\left(2\Lambda_{jk}\Lambda_{kj} - \delta_{kj}\Lambda_{jk} - \delta_{jk}\Lambda_{kj}\right)dt = \delta_{jk}\frac{4}{N}\left(\lambda_j(\lambda_j - 1)\right)dt\,.$$

In combination with (3.6.16) this implies that there exist $n$ independent standard real Brownian motions $B_1, \ldots, B_n$ such that

$$d\lambda_j = \frac{2}{\sqrt{N}}\sqrt{\lambda_j(\lambda_j-1)}dB_j + 2\left[\lambda_j - \frac{n}{N} + \frac{1}{N}\sum_{l=1,l\neq j}^{n}\frac{2\lambda_j\lambda_l - \lambda_j - \lambda_l}{\lambda_j - \lambda_l}\right]dt\,,\ j\in\{1,\ldots,n\}\,. \tag{3.6.17}$$

Moreover, as this is the SDE for a non-compact Jacobi process with parameters

$$\beta = 2\,,\ \ p = n\,,\ \ q = N - n \geq n > n-1\,,$$

up to the time rescaling $t \mapsto Nt/2$, we know that $\tau = \infty$ a.s. by Lemma 3.6.1. □

### 3.6.3 Convergence of non-compact Jacobi processes to non-colliding geometric Brownian motions

In this section we are going to show a further connection between non-compact Jacobi processes and non-colliding geometric Brownian motions apart from the previously discussed matrix model. Namely, we can approximate the latter processes by the former:

**Theorem 3.6.11.** *Let $(a_n)_{n\in\mathbb{N}} \subset (0,\infty)$ s.t. $a_n \uparrow_{n\to\infty} \infty$. Let $\hat{x}_0 \in \{x\in\mathbb{R}^N : 0 < x_1 \leq \cdots \leq x_N\}$ and let $(\hat{x}_0^n)_{n\in\mathbb{N}} \subset C_N^B$ s.t. $\lim_{n\to\infty}\hat{x}_0^n/a_n = \hat{x}_0$. Let $p\in\mathbb{R}$, $q > N-1$ and $\beta \geq 1$. Let $(B_t)_{t\geq 0}$ be an $N$-dimensional real Brownian motion and set $\tilde{B}_t := \frac{1}{\sqrt{2}}B_{2t}$, $t\geq 0$.*
*For each $n\in\mathbb{N}$ let $(X_t^n)_{t\geq 0}$ be an $N$-dimensional non-compact Jacobi process as in (3.6.1) with parameters $p$, $q$, $\beta$, with start in $\hat{x}_0^n$, and which is driven by the Brownian motion $(\tilde{B}_t)_{t\geq 0}$. Let $(X_t)_{t\geq 0}$ be an $N$-dimensional non-colliding geometric Brownian motion as in (3.4.1) with parameters $\beta$, $\gamma_1 := \frac{p+q}{2}$, $\gamma_0 := 0$, start in $\hat{x}_0$ and which is driven by the Brownian motion $(B_t)_{t\geq 0}$. Then it holds*

$$\lim_{n\to\infty} P\left(\sup_{0\leq t\leq T}\left|\frac{X_{t/2}^n}{a_n} - X_t\right| > \epsilon\right) = 0 \ \text{ for all } T>0\,,\ \epsilon>0\,. \tag{3.6.18}$$

For the proof we will transform all processes using the elementary symmetric polynomials to get rid of the singularities in the defining SDEs of non-compact Jacobi processes

and non-colliding geometric Brownian motions. After the removal of these singularities, we can apply the following very useful result on uniform $L^2$-convergence of solutions of SDEs, which was stated in [66] as Theorem A for processses with values in $\mathbb{R}^d$. When going through the proof of this result one sees that the theorem still holds if all appearing processes are valued in a closed subset of $\mathbb{R}^d$, i.e. one has:

**Theorem 3.6.12.** *Let $D \subseteq \mathbb{R}^d$ be connected and closed with non-empty interior. Let $\sigma, \sigma_n \colon [0,\infty) \times D \to \mathbb{R}^{d\times m}$ and $b, b_n \colon [0,\infty) \times D \to \mathbb{R}^d$ be continuous functions, $n \in \mathbb{N}$. Let $(B_t)_{t\geq 0}$ be an $m$-dimensional Brownian motion. Suppose that for each $n \in \mathbb{N}$ and $x \in D$ there exists a $D$-valued process $(X_t^{x,n})_{t\geq 0}$ such that it satisfies the SDE*

$$dX_t^{x,n} = \sigma_n(t, X_t^{x,n})\, dB_t + b_n(t, X_t^{x,n})\, dt\,,\ X_0^{x,n} = x\,.$$

*Further, suppose that for each $x \in D$ there exists a $D$-valued process $X_t^x$ which is the pathwise unique solution of the SDE*

$$dX_t^x = \sigma(t, X_t^x)\, dB_t + b_n(t, X_t^x)\, dt\,,\ X_0^x = x\,.$$

*If for every $T \geq 0$ there exists $L_T > 0$ such that*

$$\sup_{n\in\mathbb{N}} \sup_{t\leq T} \left(||\sigma_n(t,x)|| + |b_n(t,x)|\right) \leq L_T\,(1+|x|) \quad \text{for all } x \in D\,,$$

*and if for each $T \geq 0$ and $K \subseteq D$ compact it holds*

$$\lim_{n\to\infty} \sup_{t\leq T} \sup_{x\in K} \left(||\sigma_n(t,x) - \sigma(t,x)|| + |b_n(t,x) - b(t,x)|\right) = 0\,,$$

*then we have*

$$\lim_{n\to\infty} \sup_{x\in K} E\left[\sup_{0\leq t\leq T} |X_t^{x,n} - X_t^x|^2\right] = 0\,,$$

*for every $T \geq 0$ and $K \subseteq D$ compact.*

To apply this theorem, we need to calculate how the dynamic of non-compact Jacobi processes changes when transforming them using the elementary symmetric polynomials. We obtain the following system of SDEs:

**Proposition 3.6.13.** *Let $(B_t)_{t\geq 0}$ and $(X_t^n)_{t\geq 0}$ be as in Theorem 3.6.11. Set*

$$\left(\tilde{X}_t^n := (X_{1,t/2}^n - 1, \ldots, X_{N,t/2}^n - 1)\right)_{t\geq 0}\,.$$

*Then it holds*

$$\begin{aligned} de_k^N(\tilde{X}_t^n) = &\sum_{j=1}^N \sqrt{\tilde{X}_{j,t}^n(\tilde{X}_{j,t}^n + 1)}\, e_{k-1}^{N-1}(\tilde{X}_{\{1,\ldots,N\}\setminus\{j\},t}^n)\, dB_{j,t} \\ &+ \frac{\beta}{4}\left[(q-k+1)(N-k+1)e_{k-1}^N(\tilde{X}_t^n) + k(p+q+1-k)e_k^N(\tilde{X}_t^n)\right] dt\,, \end{aligned} \tag{3.6.19}$$

$k \in \{1, \ldots, N\}$.

*Proof.* By (3.6.1) it holds

$$\begin{aligned} d\tilde{X}^n_{j,t} =& \sqrt{2\tilde{X}^n_{j,t}(\tilde{X}^n_{j,t}+1)}\, d\tilde{B}_{j,t/2} + \frac{\beta}{4}\left[q + (p+q)\tilde{X}^n_{j,t} + \sum_{k\colon k\neq j} \frac{2\tilde{X}^n_{j,t}\tilde{X}^n_{k,t} + X^n_{j,t} + X^n_{k,t}}{\tilde{X}^n_{j,t} - \tilde{X}^n_{k,t}}\right] dt \\ =& \sqrt{\tilde{X}^n_{j,t}(\tilde{X}^n_{j,t}+1)}\, dB_{j,t} + \frac{\beta}{4}\left[q + (p+q)\tilde{X}^n_{j,t} + \sum_{k\colon k\neq j} \frac{2\tilde{X}^n_{j,t}\tilde{X}^n_{k,t} + X^n_{j,t} + X^n_{k,t}}{\tilde{X}^n_{j,t} - \tilde{X}^n_{k,t}}\right] dt\,, \end{aligned}$$

$j \in \{1,\dots,N\}$. Now, the claim follows from Itô's formula and the formulas (2.2.2), (2.2.4) & (2.2.5). Note that this is a special case of [52] Proposition 3.1 and its proof. □

*Proof of Theorem 3.6.11.* To apply 3.6.12 we need to make sure that all appearing processes take values in the same set. Hence we first consider the $C^B_N$-valued processes $(\tilde{X}^n_{j,t} := X^n_{j,t/2} - 1)_{t\geq 0}$, $j \in \{1,\dots,N\}$, as in Proposition 3.6.13. To get rid of the singularities in the drift parts, we now transform all processes using the elementary symmetric polynomials using Proposition 3.4.5 and Proposition 3.6.13. Define $Y^n_t := \frac{\tilde{X}^n_t}{a_n}$. By Proposition 3.6.13 and the fact that $e^N_k$ is homogeneous of degree $k$ we have:

$$\begin{aligned} de^N_k(Y^n_t) =& \sum_{j=1}^N \sqrt{Y^n_{j,t}\left(Y^n_{j,t} + 1/a_n\right)} e^{N-1}_{k-1}(Y^n_{\{1,\dots,N\}\setminus\{j\},t})\, dB_{j,t} \\ &+ \frac{\beta}{4}\left[(q-k+1)(N-k+1)e^N_{k-1}(Y^n_t)/a_n + k(p+q+1-k)e^N_k(Y^n_t)\right] dt\,. \end{aligned}$$

By Proposition 3.4.5, we have the following transformed SDEs for $(X_t)_{t\geq 0}$:

$$de^N_k(X_t) = \sum_{j=1}^N X_{j,t} e^{N-1}_{k-1}(X_{\{1,\dots,N\}\setminus\{j\},t})\, dB_{j,t} + k\frac{\beta}{4}\,(p+q+1-k)\, e^N_k(X_t)dt\,, \qquad (3.6.20)$$

$k \in \{1,\dots,N\}$. Let $f_N\colon C^B_N = \{x \in \mathbb{R}^N\colon 0 \leq x_1 \leq \cdots \leq x_N\} \to \mathbb{R}^N$, $f_N(x) = (e^N_1(x),\dots,e^N_N(x))$. By Lemma 2.2.2 $f_N$ is a homeomorphism from $C^B_N$ to $f_N(C^B_N)$. Set $Z^n_{0,t} \equiv 1$, $Z_{0,t} \equiv 1$, and consider the SDEs

$$\begin{aligned} dZ^n_{k,t} =& \sum_{j=1}^N \sqrt{\left(f_N^{-1}(Z^n_t)\right)_j \left(\left(f_N^{-1}(Z^n_t)\right)_j + 1/a_n\right)} \left(f_{N-1}\left(f_N^{-1}(Z^n_t)_{\{1,\dots,N\}\setminus\{j\}}\right)\right)_{k-1} dB_{j,t} \\ &+ \frac{\beta}{4}\left[(q-k+1)(N-k+1)e^N_{k-1}(Y^n_t)/a_n + k(p+q+1-k)e^N_k(Y^n_t)\right] dt\,, \end{aligned} \qquad (3.6.21)$$

$k \in \{1,\dots,N\}$, $Z^n_0 \in f(C^B_N)$, and

$$dZ_{k,t} = \sum_{j=1}^N \left(f_N^{-1}(Z_t)\right)_j \left(f_{N-1}\left(f_N^{-1}(Z_t)_{\{1,\dots,N\}\setminus\{j\}}\right)\right)_{k-1} dB_{j,t} + k\frac{\beta}{4}\,(p+q+1-k)\, e^N_k(X_t)dt\,, \qquad (3.6.22)$$

$k \in \{1,\dots,N\}$, $Z_0 \in f_N(C_N^B)$, with the convention $(f_{N-1}(x))_0 \equiv 1$. Note that all appearing processes are non-negative and that $(f_N(Y_t^n))_{t\geq 0}$ solves (3.6.21) and $(f_N(X_t))_{t\geq 0}$ solves (3.6.22). Moreover, for $c \in [0,1]$ we have the bounds

$$\sqrt{x_j(x_j+c)}e_{k-1}^{N-1}(x_{\{1,\dots,N\}\setminus\{j\}}) \leq (1+x_j)e_{k-1}^{N-1}(x_{\{1,\dots,N\}\setminus\{j\}}) \leq e_{k-1}^N(x)+e_k^N(x)\,,$$

for $x \in C_N^B$, $k \in \{1,\dots,N\}$. This shows that

$$\sqrt{\left(f_N^{-1}(z)\right)_j\left(\left(f_N^{-1}(z)\right)_j+1/a_n\right)}\left(f_{N-1}\left(f_N^{-1}(z)_{\{1,\dots,N\}\setminus\{j\}}\right)\right)_{k-1} \leq \sqrt{N}(1+|z|)\,,\ z \in f_N(C_N^B)\,.$$

Further, by Proposition 3.4.6 we know that (3.6.22) has a unique solution for any $Z_0 \in f_N(C_N^B)$. Now, using Theorem 3.6.12, we obtain

$$\lim_{n\to\infty} E\left[\sup_{0\leq t\leq T}\left|f_N(\tilde{X}_t^n/a_n)-f_N(X_t)\right|^2\right]=0\,.$$

SInce $f_N$ is a homeomorphism, (3.6.18) holds for $X_{t/2}^n$ replaced therein by $\tilde{X}_t^n$. Since $|\tilde{X}_t^n/a_n - X_{t/2}^n/a_n| = \sqrt{N}/a_n \xrightarrow{n\to\infty} 0$, the statement (3.6.18) also holds as stated. □

## 3.7 Hua-Pickrell diffusions

In this section, we cover the remaining case that $a$ is a quadratic polynomial with no real roots and positive leading coefficient. Again, by a simple rescaling in space and time, we can assume w.l.o.g. that $a(x) = 1+x^2$. We obtain the system of SDEs

$$dX_{i,t} = \sqrt{2(1+X_{i,t}^2)}\,dB_{i,t}+\beta\left[\mathrm{Im}(s)-\mathrm{Re}(s)X_{i,t}+\sum_{j:\,j\neq i}\frac{X_{i,t}X_{j,t}+1}{X_{i,t}-X_{j,t}}\right]dt\,,\quad i\in\{1,\dots,N\}\,, \tag{3.7.1}$$

$X_0 = \hat{x}_0 \in C_N^A$, where $s \in \mathbb{C}$. These processes were introduced and studied in the papers [7, 9] for $\beta = 2$ and named *Hua-Pickrell diffusions*. This name and the choice to parametrize the linear drift component by the complex number $s$ stems from the following fact: For some choices of $s$, the unique invariant probability distribution of the dynamic (3.7.1) is given by the image measure of the Hua-Pickrell measure on $\mathbb{T}^N$ under the Cayley transform

$$(\mathbb{T}\setminus\{-1\})^N \to \mathbb{R}^N, z_j \mapsto i\frac{1-z_j}{1+z_j}\,,\quad j\in\{1,\dots,N\}\,,$$

and historically the Hua-Pickrell measures are parametrized by a complex number ($i$ here denotes the imaginary unit); we state the densities of these unique invariant probability measures in Definition 3.7.5 below. This stationarity result will be extended to general $\beta \geq 1$ in Lemma 3.7.6 below.

As before, we also study the deterministic $\beta = \infty$ variant of these processes, i.e. solutions to

$$\frac{d}{dt}x_{i,t} = 2\left[\mathrm{Im}(s) - \mathrm{Re}(s)x_{i,t} + \sum_{j\colon j\neq i}\frac{x_{i,t}x_{j,t}+1}{x_{i,t}-x_{j,t}}\right], \quad i \in \{1,\dots,N\}, \tag{3.7.2}$$

$X_0 = \hat{x}_0 \in C_N^A$, $s \in \mathbb{C}$, and call these processes *frozen Hua-Pickrell diffusions.*

**Lemma 3.7.1.** *Let $\beta \geq 1$. For any $s \in \mathbb{C}$ and any $\hat{x}_0 \in C_N^A$ there exists both a unique strong non-colliding solution $(X_t)_{t\geq 0}$ to* (3.7.1) *and a non-colliding solution $(x_t)_{t\geq 0}$ to* (3.7.2).

*Proof.* As in the proof of Lemma 3.6.1, we consider $(\tilde{X}_t := X_{2t/\beta})_{t\geq 0}$. $\tilde{X}$ here satisfies the SDE

$$dX_{i,t} = 2\sqrt{(1+X_{i,t}^2)/\beta}\,dB_{i,t} + 2\left[\mathrm{Im}(s) - \mathrm{Re}(s)X_{i,t} + \sum_{j\colon j\neq i}\frac{X_{i,t}X_{j,t}+1}{X_{i,t}-X_{j,t}}\right]dt\,,$$

$i \in \{1,\dots,N\}$, $X_0 = \hat{x}_0 \in C_N^A$. This enables to treat the cases $\beta \in [1,\infty)$ and $\beta = \infty$ simultaneously by interpreting the diffusion term as 0 in the case $\beta = \infty$. In principle, we can argue as in the $\beta = 2$ case, which has been considered in Lemma 4.2 of [9]. However, we point out that the argument in the corresponding proof is imprecise. There, Theorem 2.2 of [52] is applied. But the condition that the function $H(x,y) = H_{ij}(x,y) = 2(1+xy)$, $x, y \in \mathbb{R}$, must be non-negative, as is stated in the beginning of Section 2 in [52], is clearly not fulfilled. The easiest way out is to check where the non-negativity assumption on $H_{ij}$ enters in the latter reference. Careful reading reveals that this assumption is only used in the following places:

(i) In the proof of Proposition 3.5 on p.9. There the implication

$$\sigma_i^2(x) + \sigma_j^2(x) + H_{ij}(x,x) = 0 \quad \Rightarrow \quad H_{i,j}(x,x) = 0$$

is used where in the present Hua-Pickrell setting we have $\sigma_i(x) = 2\sqrt{(1+x^2)/\beta}$, $i \in \{1,\dots,N\}$. Clearly, the sets $G_{ij} := \{x \in \mathbb{R}\colon \sigma_i^2(x) + \sigma_j^2(x) + H_{i,j}(x,x) = 0\}$, $i,j \in \{1,\dots,N\}$ are empty in the present context and thus one immediately sees that the statement of Proposition 3.5 is true.

(ii) In the proof of Proposition 4.3 on p.13. There, it enters in the context of equation (4.6). As pointed out in Remark 3.1.3, there is however a minor error in the formula (4.6), and one needs to replace that formula by (4.6') in Subsection 3.7.0.1 at the end of the present section. In our setting, we have $H_{ij}(x,y) = 2(1+xy)$ for all $i,j \in \{1,\dots,N\}$ with $i \neq j$. Using the notation and definitions of Subsection 3.7.0.1, we can now argue as follows: Using $4\lambda_i\lambda_j = (\lambda_i+\lambda_j)^2 - (\lambda_i-\lambda_j)^2$ and

$(\lambda_i - \lambda_j)^2 e_{n-1}^{\bar{a}_{ij}}(A) = 0 = (\lambda_i - \lambda_j)^2 e_{n-1}^{\bar{a}_{ij},\bar{a}_{ik},\bar{a}_{jk}}(A)$ whenever $V_n = e_n(A) = 0$, we have in place of equation (4.6) in [52] the formula

$$0 = D_n = \sum_{i=1}^{p} \sigma_i^2(\lambda_i) \sum_{j\colon j\neq i} e_{n-1}^{\bar{a}_{ij}}(A) + 2\sum_{i<j} \left(4 + (\lambda_i + \lambda_k)^2\right) e_{n-1}^{\bar{a}_{ij}}(A) + \sum_{i<k} \sum_{j\neq i, j\neq k} \left(4 + (\lambda_i + \lambda_k)^2\right) e_{n-1}^{\bar{a}_{ij},\bar{a}_{ik},\bar{a}_{jk}}(A)\,.$$

Note that all appearing terms are non-negative. Thus, even though $H = H_{ij}$ is not non-negative, we can conclude in the same way as in the proof of Proposition 4.3 of [52].

Hence, by the proof of Lemma 4.2 in [9], we can apply Theorem 2.2 of [52] where we can drop the non-negativity assumption on $H = H_{ij}$ in the case $\beta = 2$. Moreover, all arguments from that proof carry over to our extension to $\beta \in [1, \infty]$ with minor modifications. □

As for all other subclasses of non-colliding $\beta$-Pearson diffusions, there exist matrix-valued Itô diffusions such that the corresponding eigenvalue processes are Hua-Pickrell diffusions for parameter $\beta = 2$. In particular, it has been shown in [7] that one can use the Brownian motion on $\mathrm{Gl}(N, \mathbb{C})$ to construct a solution to the matrix SDE

$$\begin{aligned} &dM_t \\ =&dW_t \sqrt{\frac{I_N + M_t^2}{2}} + \sqrt{\frac{I_N + M_t^2}{2}}\, dW_t^* + \left[(-N - 2\,\mathrm{Re}(s))M_t + (2\,\mathrm{Im}(s) + \mathrm{Tr}(M_t))I_N\right] dt\,; \end{aligned}$$

here $(W_t)_{t\geq 0}$ is an $N \times N$ hermitian Brownian motion and $I_N$ is the identity matrix. Using the technique of proof of [51] Theorem 4, one sees that the corresponding eigenvalue process solves (3.7.1) if the eigenvalues of $M_0$ are distinct. We will consider the analogous construction in a free probability setting in Theorem 5.5.6. For further properties of Hua-Pickrell diffusions in the $\beta = 2$ case, see [9].
We now turn to the study of invariant measures for general $\beta \geq 1$ and the question of stationarity for frozen Hua-Pickrell diffusions. We will see that equilibria in the latter case are given by zeros of the following finitely orthogonal polynomials:

**Definition 3.7.2.** Let $a, b \in \mathbb{R}$. The *$N$-th Pseudo-Jacobi polyomial* ([71, 61]) is defined via the Jacobi polynomials (3.5.4) by setting

$$\begin{aligned} P_N(x; a, b) :=&(-i)^N P_N^{a+ib, a-ib}(ix) \\ =&(-i)^N \frac{(a+ib+1)^{(N)}}{N!} \sum_{k=0}^{N} \frac{(-N)^{(k)}(2a+N+1)^{(k)}}{(a+ib+1)^{(k)} k!} \left(\frac{1-ix}{2}\right)^k\,. \end{aligned}$$

These polynomials are also called *Romanovski polynomials* in the literature (see [85] and references therein).

We will need the following facts:

**Lemma 3.7.3.** *Let $N \in \mathbb{N}$, $a < -N$ and $b \in \mathbb{R}$. Then $(P_k(\cdot; a, b))_{k\in\{0,\dots,N\}}$ is finitely orthogonal w.r.t. the weight function*

$$w(x) := (1 - ix)^{a+ib}(1 + ix)^{a-ib} = (1 + x^2)^a e^{2b \arctan(x)}$$

*in the sense that*

$$0 < \int_{\mathbb{R}} P_n^2(x; a, b)w(x)\, dx < \infty \;\; \textit{for all} \;\; n \le N - 1\,,$$
$$0 = \int_{\mathbb{R}} P_n(x; a, b)P_m(x; a, b)w(x)\, dx \;\; \textit{for all} \;\; n < m \le N\,.$$

*In particular, the zeros of $P_N(\cdot; a, b)$ are real and simple.*
*Denote these zeros by $x_1 < \cdots < x_N$. Then $(x_1, \dots, x_N)$ is the unique solution in $\operatorname{int} C_N^A$ to the following set of equations:*

$$0 = (a + 1)x_j + b + \sum_{k\colon k\neq j} \frac{1 + x_j^2}{x_j - x_k}\,, \;\; j \in \{1, \dots, N\}\,. \tag{3.7.3}$$

*Proof.* The orthogonal relations and the fact that the roots are real and simple are known by [6] equation (1.10) and [61] Section 2. The relation of the zeros of $P_N(\cdot, a, b)$ to the equation (3.7.3) can be derived by the standard method employed in Section 6.7 of [99]. Indeed: Let $(x_1, \dots, x_N)$ be a solution to the equations (3.7.3) and set $f(x) := \prod_{i=1}^N (x - x_j)$. One has

$$f'(x) = \sum_{j=1}^N \prod_{k\colon k\neq j} (x - x_k) \;\text{ and }\; f''(x) = \sum_{j,k\colon j\neq k} \prod_{l\colon l\neq j, l\neq k} (x - x_l)\,.$$

Hence

$$\frac{f''(x_j)}{f'(x_j)} = \frac{2\sum_{k\colon k\neq j} \prod_{l\colon l\neq k, l\neq j}(x_j - x_l)}{\prod_{k\colon k\neq j}(x_j - x_k)} = 2 \sum_{k\colon k\neq j} \frac{1}{x_j - x_k}\,, \;\; j \in \{1, \dots, N\}\,. \tag{3.7.4}$$

Thus we have

$$0 = (a + 1)x_j + b + \frac{1 + x_j^2}{2}\frac{f''(x_j)}{f'(x_j)}\,, \;\; j \in \{1, \dots, N\}\,. \tag{3.7.5}$$

This means $x \mapsto 2[(a + 1)x + b]f'(x) + (1 + x^2)f''(x)$ is a polynomial of degree $N$ which vanishes at the points $x_j$. Hence this function is equal to a constant multiple of $f$. Comparing leading coefficients shows that this constant is given by $N(N+2a+1)$. Thus, $f$ solves the ODE

$$(1 + x^2)f''(x) + 2[(a + 1)x + b]f'(x) - N(N + 2a + 1)f(x) = 0\,, \;\; x \in \mathbb{R}\,. \tag{3.7.6}$$

Now, by equation (2) in [61], $P_N(\cdot;a,b)$ also solves this ODE. It thus suffices to show that any polynomial solution of degree $N$ to (3.7.6) must be a constant multiple of $f$. Using $w'(x)/w(x) = 2(ax+b)$, we can rewrite the ODE (3.7.6) as

$$\frac{d}{dx}\big(w(x)(1+x^2)f'(x)\big) - N(N+2a+1)w(x)f(x) = 0\,.$$

Now, let $g$ be another polynomial solution of degree $N$ to this equation. Then

$$\frac{d}{dx}\big(w(x)(1+x^2)(f'(x)g(x) - f(x)g'(x))\big) = 0$$

and hence $w(x)(1+x^2)(f'(x)g(x) - f(x)g'(x))$ is constant. Note that $f'g - fg'$ is a polynomial of degree smaller than $2N-1$. Since by assumption $a < -N$, it follows that

$$\lim_{x\to\infty} w(x)(1+x^2)(f'(x)g(x) - f(x)g'(x)) = 0\,.$$

Thus $f'g - fg'$ is the zero polynomial which means that $g$ must be a multiple of $f$.
Now, let $x_1,\dots,x_N$ be the distinct zeros of $P_N(\cdot;a,b)$. Since $P_N(\cdot;a,b)$ solves (3.7.6) and vanishes at the points $x_1,\dots,x_N$, $P_N(\cdot;a,b)$ satisfies (3.7.5). Replacing $f$ by $P_N(\cdot;a,b)$ in (3.7.4), we see that $x_1,\dots,x_N$ satisfy (3.7.3). This concludes the proof. □

**Lemma 3.7.4.** *Let $s\in\mathbb{C}$ with $\mathrm{Re}(s)>0$. Let $(x_t)_{t\geq 0}$ be the unique non-colliding solution to (3.7.2). Then the limit $\lim_{t\to\infty} x_t = z$ exists, where $z$ is the vector of ordered zeros of the Pseudo-Jacobi polynomial $P_N(\cdot;-(N+\mathrm{Re}(s)),\mathrm{Im}(s))$. Moreover, $z$ is the unique stationary solution to (3.7.2).*

*Proof.* The argument is the same as in the proofs of Lemma 3.4.3 and Lemma 3.6.2.
First off, note that a stationary solution $x_t \equiv x_0$ has to satisfy for all $i\in\{1,\dots,N\}$:

$$0 = \mathrm{Im}(s) - \mathrm{Re}(s)x_{i,0} + \sum_{j\colon j\neq i}\frac{x_{i,0}x_{j,0}+1}{x_{i,0}-x_{j,0}} = \mathrm{Im}(s) - (N-1+\mathrm{Re}(s))x_{i,0} + \sum_{j\colon j\neq i}\frac{1+x_{i,0}^2}{x_{i,0}-x_{j,0}}\,.$$

As $-(N-1+\mathrm{Re}(s))-1 = -(N+\mathrm{Re}(s)) < -N$, we know by Lemma 3.7.3 that $x_0$ is the ordered vector of zeros of $P_N(\cdot;-(N+\mathrm{Re}(s)),\mathrm{Im}(s))$.
It remains to discuss the arbitrary starting condition $x_0\in C_N^A$. Here we first transform the solutions $(x_t)_{t\geq 0}$ by the homeomorphism $f\colon C_N^A \to f(C_N^A), x\mapsto (e_1^N(x),\dots,e_N^N(x))$, where $(e_j^N)_{j\in\{1,\dots,N\}}$ are the elementary symmetric polynomials discussed in Section 2.2. Set $y_{m,t} := e_m^N(x_t)$. Using the formula $y_m = x_i e_{m-1}^{N-1}(x_{\{1,\dots,N\}\setminus\{i\}}) + e_m^{N-1}(x_{\{1,\dots,N\}\setminus\{i\}})$ we get (notationally suppressing dependence of $t$):

$$\frac{d}{dt}y_m = \sum_{i=1}^N e_{m-1}^{N-1}(x_{\{1,\dots,N\}\setminus\{i\}})\frac{d}{dt}x_i\,.$$

Plugging in the ODE for $x_i$ and using formulas (2.2.2)-(2.2.4) we get

$$\frac{d}{dt}y_m = -c(s,m)y_m + 2\,\mathrm{Im}(s)(N-m+1)y_{m-1} - (N-m+2)(N-m+1)y_{m-2}\,,$$

with the convention $y_{-1} \equiv 0$, $y_0 \equiv 1$ and $c(s,m) := m(2\operatorname{Re}(s) + m - 1) > 0$. Thus

$$\begin{aligned} & y_{m,t} \\ =& e^{-c(s,m)t} \Bigg( y_{m,0} + \\ & \int_0^t e^{c(s,m)r} \left(2\operatorname{Im}(s)(N-m+1)y_{m-1,r} - (N-m+2)(N-m+1)y_{m-2,r}\right) dr \Bigg) . \end{aligned}$$

Inductively one sees that the limit $\lim_{t\to\infty} y_t$ exists and does not depend on $(y_{1,0}, \ldots, y_{N,0})$. Hence this limit equals $f(z)$. A transfer back via $f^{-1}$ shows the claim. □

Now, we turn to the question of invariant measures of non-frozen Hua-Pickrell diffusions. We will see that, as for non-colliding imhomogeneous Brownian motions and non-compact Jacobi processes, we can answer this question by applying a suitable coordinate transformation which reveals a Langevin-diffusion structure. However, we point out that in the Hua-Pickrell setting we do not have a description via Dunkl or Heckman-Opdam theory which is in contrast to all other classes of non-colliding $\beta$-Pearson diffusions.
Let $(X_t)_{t\geq 0}$ be a solution to (3.7.1). We define $(Y_t)_{t\geq 0}$ by the component-wise transformation $x \mapsto \operatorname{arsinh}(x)$, i.e. $Y_{i,t} := \operatorname{arsinh}(X_{i,t})$, $i \in \{1, \ldots, N\}$, $t \geq 0$. By Itô's formula and some elementary calculations, the dynamic of $(Y_t)_{t\geq 0}$ is described by

$$\begin{aligned} dY_{i,t} =& \sqrt{2}\, dB_{i,t} + \beta \Bigg[ \left(1 - N - \frac{1}{\beta} - \operatorname{Re}(s)\right) \tanh(Y_{i,t}) + \operatorname{Im}(s) \frac{1}{\cosh(Y_{i,t})} \\ & + \cosh(Y_{i,t}) \sum_{j\colon j\neq i} \frac{1}{\sinh(Y_{i,t}) - \sinh(Y_{j,t})} \Bigg] dt . \end{aligned}$$

Using the formula

$$\begin{aligned} \frac{\cosh(x)}{\sinh(x) - \sinh(y)} =& \frac{1}{2} \left( \frac{\cosh(x) + \cosh(y)}{\sinh(x) - \sinh(y)} + \frac{\cosh(x) - \cosh(y)}{\sinh(x) - \sinh(y)} \right) \\ =& \frac{1}{2} \left( \coth\left(\frac{x-y}{2}\right) + \tanh\left(\frac{x+y}{2}\right) \right) , \end{aligned}$$

we can further rewrite this as

$$\begin{aligned} dY_{i,t} =& \sqrt{2}\, dB_{i,t} + \beta \Bigg[ \left(1 - N - \frac{1}{\beta} - \operatorname{Re}(s)\right) \tanh(Y_{i,t}) + \operatorname{Im}(s) \frac{1}{\cosh(Y_{i,t})} \\ & + \frac{1}{2} \sum_{j\colon j\neq i} \left( \coth\left(\frac{Y_{i,t} - Y_{j,t}}{2}\right) + \tanh\left(\frac{Y_{i,t} + Y_{j,t}}{2}\right) \right) \Bigg] dt . \end{aligned} \tag{3.7.7}$$

The corresponding generator is thus given by

$$\mathcal{L} := \Delta + (-\nabla V(x))\nabla ,$$

where

$$V(x) := -\beta \left[ \left(1 - N - \frac{1}{\beta} - \mathrm{Re}(s)\right) \sum_{i=1}^{N} \log(\cosh(x_i)) + \mathrm{Im}(s) \sum_{i=1}^{N} \arctan(\sinh(x_i)) \right.$$
$$\left. + \sum_{1 \le i < j \le N} \log\left(\sinh\left(\frac{x_j - x_i}{2}\right)\cosh\left(\frac{x_i + x_j}{2}\right)\right)\right] . \tag{3.7.8}$$

As we saw before, the candidate for the invariant measure for such a Langevin dynamic is given by the Gibbs-measure $\tilde{\rho} := \frac{1}{\tilde{Z}} e^{-V(x)}$. We need to check when $\tilde{Z} := \int_{C_N} e^{-V(x)}\, dx$ is finite, which is easiest to do in the original coordinates. We thus calculate the image measure of the measure with density

$$e^{-V(x)} = \prod_{i=1}^{N} \left( (\cosh(x_i))^{\beta(1-N-1/\beta-\mathrm{Re}(s))} \exp\left(\beta \,\mathrm{Im}(s) \arctan(\sinh(x_i))\right)\right)$$
$$\times \prod_{1 \le i < j \le N} \left(\sinh\left(\frac{x_j - x_i}{2}\right)\cosh\left(\frac{x_i + x_j}{2}\right)\right)^{\beta} \tag{3.7.9}$$

under the map $C_N^A \to C_N^A$, $x_i \mapsto \sinh(x_i)$, $i \in \{1, \dots, N\}$. Using $\cosh(\mathrm{arsinh}(x)) = \sqrt{1 + x^2}$ and

$$\sinh\left(\frac{\mathrm{arsinh}(x_i) - \mathrm{arsinh}(x_j)}{2}\right)\cosh\left(\frac{\mathrm{arsinh}(x_i) + \mathrm{arsinh}(x_j)}{2}\right) = \frac{x_i - x_j}{2} ,$$

we get a density which is proportional to

$$\rho(x) := \prod_{i=1}^{N} \left( \left(1 + x_i^2\right)^{\frac{\beta}{2}(1-N-\mathrm{Re}(s))-1} \exp(\beta \,\mathrm{Im}(s) \arctan(x_i))\right) \prod_{1 \le i < j \le N} |x_i - x_j|^{\beta} . \tag{3.7.10}$$

**Definition 3.7.5.** Let $s \in \mathbb{C}$ with $\mathrm{Re}(s) > -\frac{1}{\beta}$. Then the integral $Z := \int_{C_N^A} \rho(x)\, dx$ is finite. The *$\beta$-Hua-Pickrell measure with parameter $s$* is the probability measure with density $Z^{-1}\rho$ .

The $\beta$-Hua-Pickrell measures are also known as Cauchy-$\beta$-ensembles in the literature and their normalization constant is known; see [47] Section 4.7.1. However, we point out that there does not seem to be a predominant choice of parametrization. In that regard, closest to our choice seems [11] Definition 2.1, where $\tau$ therein corresponds to $\frac{\beta}{2}s$ in our notation. Also note that the Hua-Pickrell measures are often named in the same way as their image measures under the component-wise inverse Cayley-transform which are in turn measures on $\mathbb{T}^N$ (see the beginning of this section).

We have the following result which so far is only known in the case $\beta = 2$ by Proposition 4.4 in [9].

**Lemma 3.7.6.** *Let $s \in \mathbb{C}$, $\beta \geq 1$. For $\mathrm{Re}(s) > -1/\beta$ the dynamic (3.7.1) has the $\beta$-Hua-Pickrell measure with parameter $s$ as its unique invariant measure.*

*Proof.* The Gibbs measure $\tilde{\rho}$ is the unique invariant probability measure of the dynamic (3.7.7), see e.g. the discussion at the beginning of Section 2 of the survey [75]. The claim follows after applying the coordinate change $x_i \mapsto \sinh(x_i)$, $i \in \{1, \dots, N\}$, as above. □

**Remark 3.7.7.** In contrast to the setting considered in Lemma 3.6.6, the potential $V$ defined in (3.7.8) is not convex. This can be seen by calculating the Hessian $H$ of $V$ and noting that $H_{1,1}$ does not have constant sign for $N \geq 2$.

### 3.7.0.1 Minor correction to paper [52]

As pointed out in Remark 3.1.3, there is a minor error in paper [52] which affects the formulas (4.3) and (4.6) therein. This subtlety is relevant to our treatment of Hua-Pickrell diffusions since we used results of the referenced paper in the proof of Lemma 3.7.1. We here state the necessary correction in terms of the notation of the referenced paper. In what follows, all references to equations and Propositions are always meant w.r.t. the paper [52]. Let $x \in \mathbb{R}^N$. Then $e_n(x) := \sum_{1 \leq i_1 < \dots < i_n \leq N} x_{i_1} \dots x_{i_n}$ is the $n$-th elementary symmetric polynomial in $N$ variables; in the notation of Section 2.2 this would be $e_n^N(x)$. Moreover, we use the convention $e_0 \equiv 1$, $e_{-1} \equiv 0$. Now, for distinct $j_1, \dots, j_k \in \{1, \dots, N\}$ we write

$$e_n^{\bar{x}_{j_1}, \dots, \bar{x}_{j_k}}(x) := \sum_{1 \leq i_1 < \dots < i_n \leq N \colon i_l \neq j_m \, \forall \, l,m} x_{i_1} \dots x_{i_n} \, ;$$

in the notation of Section 2.2 this would be $e_n^{N-k}(x_{\{1,\dots,N\} \setminus \{j_1,\dots,j_k\}})$.
Now, for $p \in \mathbb{N}$ consider a stochastic process $((\lambda_1(t), \dots, \lambda_p(t)))_{t \geq 0}$ with values in $\{x \in \mathbb{R}^p \colon x_1 \leq \dots \leq x_p\}$. Denote the corresponding squared differences by $a_{i,j}(t) := (\lambda_i(t) - \lambda_j(t))^2$, $1 \leq i < j \leq p$, and form the vector

$$A(t) := (a_{1,2}(t), \dots, a_{1,p}(t), a_{2,3}(t), \dots, a_{2,p}(t), \dots, a_{p-1,p}(t)) \in \mathbb{R}^{p(p-1)/2} \, .$$

Finally set $V_n(t) := e_n(A(t))$, $t \geq 0$, $n \in \{1, \dots, p(p-1)/2\}$. In the following, we suppress the dependence of stochastic processes on the time parameter $t$ in our notation.
Now, in the setting of Section 4 in [52] the stochastic process $V_n$ is a continuous semimartingale whose martingale part is denoted by $M_n$ and whose bounded variation part is denoted by $\int_0^t D_n(s) ds$. $M_n$ and $D_n$ can be written in terms of the $\lambda_i$ and some continuous functions $\sigma_i \colon \mathbb{R} \to \mathbb{R}$, $b_i \colon \mathbb{R} \to \mathbb{R}$, $H_{ij} \colon \mathbb{R}^2 \to [0, \infty)$, $i, j \in \{1, \dots, p\}$, $j \neq i$; see equations (4.1) and (4.2) in the referenced paper. Moreover, it is assumed that one has the symmetry property $H_{ij}(x, y) = H_{ji}(y, x)$ for all $x, y \in \mathbb{R}$.
We will use the same shorthands for summation ranges as in the referenced paper, i.e.

$$i < j \;\; \widehat{=} \;\; 1 \leq i < j \leq p \quad \text{and} \quad i \neq j \neq k \;\; \widehat{=} \;\; 1 \leq i, j, k \leq p \colon i \neq j \neq k \neq i \, .$$

Now, we derive the claimed correction to equation (4.3). Note that, using $e_{n-1}^{\bar{a}_{ij}}(A) = a_{ik}e_{n-2}^{\bar{a}_{ij},\bar{a}_{ik}}(A) + e_{n-1}^{\bar{a}_{ij},\bar{a}_{ik}}(A)$, the left hand side of formula (4.3) in the referenced paper can be written as:

$$\sum_{i\neq j\neq k}(\lambda_i-\lambda_j)\frac{H_{ik}(\lambda_i,\lambda_k)}{\lambda_i-\lambda_k}e_{n-1}^{\bar{a}_{ij}}(A) = \sum_{i\neq j\neq k}(\lambda_i-\lambda_j)(\lambda_i-\lambda_k)H_{ik}(\lambda_i,\lambda_k)e_{n-2}^{\bar{a}_{ij},\bar{a}_{ik}}(A) + \sum_{i\neq j\neq k}(\lambda_i-\lambda_j)\frac{H_{ik}(\lambda_i,\lambda_k)}{\lambda_i-\lambda_k}e_{n-1}^{\bar{a}_{ij},\bar{a}_{ik}}(A)\,.$$

The second term on the r.h.s. can be simplified in the following way:

$$\begin{aligned}
&\sum_{i\neq j\neq k}(\lambda_i-\lambda_j)\frac{H_{ik}(\lambda_i,\lambda_k)}{\lambda_i-\lambda_k}e_{n-1}^{\bar{a}_{ij},\bar{a}_{ik}}(A)\\
=&\sum_{i<k}\sum_{j\neq i,j\neq k}\frac{H_{ik}(\lambda_i,\lambda_k)}{\lambda_i-\lambda_k}\left((\lambda_i-\lambda_k+\lambda_k-\lambda_j)e_{n-1}^{\bar{a}_{ij},\bar{a}_{ik}}(A)-(\lambda_k-\lambda_j)e_{n-1}^{\bar{a}_{kj},\bar{a}_{ki}}(A)\right)\\
=&\sum_{i<k}\sum_{j\neq i,j\neq k}H_{ik}(\lambda_i,\lambda_k)\left(e_{n-1}^{\bar{a}_{ij},\bar{a}_{ik}}(A)+\frac{\lambda_k-\lambda_j}{\lambda_i-\lambda_k}\left(e_{n-1}^{\bar{a}_{ij},\bar{a}_{ik}}(A)-e_{n-1}^{\bar{a}_{kj},\bar{a}_{ki}}(A)\right)\right)\\
=&\sum_{i<k}\sum_{j\neq i,j\neq k}H_{ik}(\lambda_i,\lambda_k)\left(e_{n-1}^{\bar{a}_{ij},\bar{a}_{ik}}-(\lambda_k-\lambda_j)^2e_{n-2}^{\bar{a}_{ij},\bar{a}_{ik},\bar{a}_{kj}}-(\lambda_k-\lambda_j)(\lambda_i-\lambda_j)e_{n-2}^{\bar{a}_{ij},\bar{a}_{ik},\bar{a}_{kj}}\right)\\
=&\sum_{i<k}\sum_{j\neq i,j\neq k}H_{ik}(\lambda_i,\lambda_k)\left(e_{n-1}^{\bar{a}_{ij},\bar{a}_{ik},\bar{a}_{kj}}(A)-(\lambda_k-\lambda_j)(\lambda_i-\lambda_j)e_{n-2}^{\bar{a}_{ij},\bar{a}_{ik},\bar{a}_{kj}}(A)\right)\,,
\end{aligned}$$

where we used

$$e_{n-1}^{\bar{a}_{ij},\bar{a}_{ik}}(A)-e_{n-1}^{\bar{a}_{kj},\bar{a}_{ki}}(A) = (a_{kj}-a_{ij})e_{n-2}^{\bar{a}_{ij},\bar{a}_{ik},\bar{a}_{kj}}(A) = (\lambda_k-\lambda_i)(\lambda_k-\lambda_j+\lambda_i-\lambda_j)e_{n-2}^{\bar{a}_{ij},\bar{a}_{ik},\bar{a}_{kj}}(A)$$

and

$$e_{n-1}^{\bar{a}_{ij},\bar{a}_{ik}} = a_{kj}e_{n-2}^{\bar{a}_{ij},\bar{a}_{ik},\bar{a}_{kj}}(A) + e_{n-1}^{\bar{a}_{ij},\bar{a}_{ik},\bar{a}_{kj}}(A)\,.$$

Thus, equation (4.3) in [52] needs to be replaced by:

$$\begin{aligned}
\sum_{i\neq j\neq k}(\lambda_i-\lambda_j)\frac{H_{ik}(\lambda_i,\lambda_k)}{\lambda_i-\lambda_k}e_{n-1}^{\bar{a}_{ij}}(A) = &\sum_{i\neq j\neq k}(\lambda_i-\lambda_j)(\lambda_i-\lambda_k)H_{ik}(\lambda_i,\lambda_k)e_{n-2}^{\bar{a}_{ij},\bar{a}_{ik}}(A)\\
&+\sum_{i<k}\sum_{j\neq i,j\neq k}H_{ik}(\lambda_i,\lambda_k)e_{n-1}^{\bar{a}_{ij},\bar{a}_{ik},\bar{a}_{kj}}(A)\\
&-\sum_{i<k}\sum_{j\neq i,j\neq k}(\lambda_k-\lambda_j)(\lambda_i-\lambda_j)H_{ik}(\lambda_i,\lambda_k)e_{n-2}^{\bar{a}_{ij},\bar{a}_{ik},\bar{a}_{kj}}(A)\,.
\end{aligned}\tag{4.3'}$$

Now, we also need to check where in the referenced paper equation (4.3) is used again. The first instance is in the statement of Proposition 4.2. But when going through the

corresponding proof one sees that equation (4.3) is not applied therein. The second instance of application of equation (4.3) is in the the proof of Proposition 4.3 in the context of equation (4.6). There, one considers the case that the stopping time $\tau_n = \inf\{t > 0 \colon V_n(t) > 0\}$ is positive with positive probability. For all $\omega \in \{\tau_n > 0\}$ one then has that the finite variation part $\int_0^t D_n(\omega, s)\, ds$ of $V_n(\omega, t)$ must vanish for all $t < \tau_n(\omega)$. Hence, one also has $D_n(\omega, t) = 0$ for all $t < \tau_n(\omega)$. By equation (4.2) in the referenced paper and equation (4.3') above, one thus has, instead of equation (4.6):

$$\sum_{i=1}^{p} \sigma_i^2(\lambda_i) \sum_{j\colon j\neq i} e_{n-1}^{\bar{a}_{ij}}(A) + 4 \sum_{i<j} H_{ij}(\lambda_i, \lambda_j) e_{n-1}^{\bar{a}_{ij}}(A) + 2 \sum_{i<k} \sum_{j\neq i, j\neq k} H_{ik}(\lambda_i, \lambda_k) e_{n-1}^{\bar{a}_{ij}, \bar{a}_{ik}, \bar{a}_{jk}}(A) = D_n = 0\,, \tag{4.6'}$$

where we used that $V_n = 0$ implies that for all $j \neq i \neq k$ the following terms are equal to 0:

$$(\lambda_i - \lambda_j) e_{n-1}^{\bar{a}_{ij}}(A)\,, \quad (\lambda_i - \lambda_j)(\lambda_i - \lambda_k) e_{n-2}^{\bar{a}_{ij}, \bar{a}_{ik}}(A)\,, \quad (\lambda_k - \lambda_j)(\lambda_i - \lambda_j) H_{ik}(\lambda_i, \lambda_k) e_{n-2}^{\bar{a}_{ij}, \bar{a}_{ik}, \bar{a}_{kj}}(A)\,.$$

One can readily check that the line of proof of Proposition 4.3 still goes through with this modification to equation (4.6).

# Chapter 4

# Convergence of empirical measures of non-colliding $\beta$-Pearson diffusions for large dimensions

In random matrix theory, one of the central questions is, if the empirical measures of the spectra of a series of random matrices growing in size converges to a limit, and, in case of existence, how this limit can be described. As mentioned in the introduction, the most famous answer to this kind of question was given by the pioneering work of Eugene Wigner ([111, 112]). It states that the sequence of empirical measures of the spectrum of symmetric matrices whose entries are random variables with normalized variance converges to the semicircular law when the matrix dimension goes to infinity. Over the years more and more limit points for random matrix models have been found, e.g. the circular law for random matrices all of whose entries are independent and have normalized variance ([50, 101]), the Marchenko-Pastur distributions for the singular values of rectangular Gaussian matrices ([74]), or the Haar measure on the torus for Haar distributed random matrices in the unitary group ([38]).
As we have seen in Chapter 3, some non-colliding $\beta$-Pearson diffusions describe the dynamic of the eigenvalues of certain time-dependent random matrices in the case $\beta \in \{1, 2\}$. Thus, it is natural to ask, whether the empirical measures of $\beta$-Pearson diffusions converge to some non-trivial limit, if the dimension of the state space goes to infinity. The aim of this section is to give an affirmative answer to this question and a characterization of the limits. While the existence of such limits and the weak convergence in probability of the empirical measures to these limits can be deduced from the results in [73], we give a proof which is mostly self contained and improves the convergence in probability to almost sure convergence. The method of proof is hereby inspired by the calculations carried out in [103] Section 3 and has previously been used by the present author and coauthors in the case of Jacobi processes of compact and non-compact type ([13]) and non-colliding geometric Brownian motions ([12]). The non-colliding $\beta$-Pearson diffusions are a natural class of processes to extend this method of proof to, yielding Theorem 4.2.2 which is new

in its stated generality.
We will rely on the method of moments, i.e. we aim to show the weak convergence of empirical measures by showing convergence of the corresponding moments. This method is very well tested in random matrix theory and still proves effective today. As we are considering time dependent random variables, we will use a time-parameter variant of the moment convergence theorem to prove our results.

## 4.1 A moment convergence theorem for families of probability measures depending continuously on a parameter

First, we recall the following definition regarding the topology of weak convergence in $M^1(\mathbb{R})$:

**Definition 4.1.1.** Let $A \subseteq \mathbb{R}$. We set

$$A^\epsilon := \{x \in \mathbb{R} \colon \exists\, a \in A \colon d(x,a) < \epsilon\}\,.$$

The *Lévy-Prokhorov metric* $d_{\text{L-P}} \colon M^1(\mathbb{R}) \times M^1(\mathbb{R}) \to [0,1]$ is given by

$$d_{\text{L-P}}(\mu,\nu) := \inf\{\epsilon > 0 \colon \mu(A) \leq \nu(A^\epsilon) + \epsilon \text{ and } \nu(A) \leq \mu(A^\epsilon) + \epsilon \text{ for all } A \in \mathcal{B}(\mathbb{R})\}\,.$$

The importance of the Lévy-Prokhorov metric stems from the fact that it metrizes the topology of weak convergence on $M^1(\mathbb{R})$, see e.g. Theorem 11.3.3 in [41].

**Definition 4.1.2.** We say that $\mu \in M^1(\mathbb{R})$ *has moments of all orders* if all integrals $\int_{\mathbb{R}} x^n \, d\mu(x)$, $n \in \mathbb{N}$, exist and are finite.
We say that $\mu \in M^1(\mathbb{R})$ *is determined by its moments* if the following two conditions hold:

(i) $\mu$ has moments of all orders.

(ii) If $\tilde{\mu} \in M^1(\mathbb{R})$ has moments of all orders and $\int_{\mathbb{R}} x^n \, d\tilde{\mu}(x) = \int_{\mathbb{R}} x^n \, d\mu(x)$ for all $n \in \mathbb{N}$, then $\tilde{\mu} = \mu$.

The question whether a given probability measure is determined by its moments is referred to as *the moment problem*. More specifically, for probability measures on $\mathbb{R}$ this problem is called the Hamburger moment problem and for probability measures on $[0,\infty)$ this is called the Stieltjes moment problem. For a concise overview over this topic see the survey [72]. We recall one important practical criterion for moment determinacy:

**Lemma 4.1.3** (Carleman condition)**.** *Let $\mu \in M^1(\mathbb{R})$ have moments $(m_n)_{n\in\mathbb{N}}$ of all orders. If*

$$\sum_{n=1}^{\infty} m_{2n}^{-1/(2n)} = \infty\,, \tag{4.1.1}$$

*then $\mu$ is determined by its moments.*

This criterion is especially useful in conjunction with the following theorem, which is the cornerstone of the method of moments, i.e. the method of proving weak convergence by showing convergence of the respective moments:

**Theorem 4.1.4** (Moment convergence theorem [49]). *Let $(\mu^N)_{N\in\mathbb{N}} \subset M^1(\mathbb{R})$ be a sequence of probability measures having moments $m_n^N := \int_{\mathbb{R}} x^n \, d\mu^N(x)$, $n \in \mathbb{N}$, of all orders for each $N$. If all limits $\lim_{N\to\infty} m_n^N =: m_n \in \mathbb{R}$ exist, then there exists a $\mu \in M^1(\mathbb{R})$ and a subsequence $(\mu^{N_k})_{k\in\mathbb{N}}$ such that $(m_n)_{n\in\mathbb{N}}$ is the moment sequence of $\mu$ and $\mu^{N_k} \xrightarrow{k\to\infty} \mu$ weakly.*
*If, in addition, $\mu$ is determined by its moments, then $\mu^N \xrightarrow{N\to\infty} \mu$ weakly.*

We want to extend this theorem to sequences of families of measures of the form $(\mu_t)_{t\geq 0}$ where $t \mapsto \mu_t$ is weakly continuous. More precisely, for $T > 0$ and $E \subset \mathbb{R}$ Borel measurable we set

$$C([0,T], M^1(E)) := \{(\mu_t)_{t\in[0,T]} \subset M^1(E) \colon t \mapsto \mu_t \text{ is weakly continuous}\}.$$

We equip this space with the topology of uniform convergence, i.e.

$$(\mu_t^N)_{t\in[0,T]} \xrightarrow{N\to\infty} (\mu_t)_{t\in[0,T]} :\Leftrightarrow \lim_{N\to\infty} \sup_{t\in[0,T]} d_{\text{L-P}}(\mu_t^N, \mu_t) = 0\,.$$

One can easily show that this is equivalent to the condition

$$\lim_{N\to\infty} \sup_{t\in[0,T]} \left| \int_{\mathbb{R}} f \, d\mu_t^N - \int_{\mathbb{R}} f \, d\mu_t \right| = 0 \text{ for all } f \in C_b(\mathbb{R})\,.$$

Further we set

$$C([0,\infty), M^1(E)) := \{(\mu_t)_{t\geq 0} \subset M^1(E) \colon t \mapsto \mu_t \text{ is weakly continuous}\}$$

and equip this space with the topology of convergence on compacta, i.e.

$$(\mu_t^N)_{t\geq 0} \xrightarrow{N\to\infty} (\mu_t)_{t\geq 0} :\Leftrightarrow \lim_{N\to\infty} \sup_{t\in[0,T]} d_{\text{L-P}}(\mu_t^N, \mu_t) = 0 \text{ for all } T > 0\,.$$

It holds the following extension of the moment convergence theorem to measure valued processes; cf. [103] Theorem A.1 for a similar statement:

**Lemma 4.1.5.** *Let $T > 0$. Let $(\mu_t^N)_{t\in[0,T]} \in C([0,T], M^1(\mathbb{R}))$, $N \in \mathbb{N}$, such that $\mu_t^N$ has moments of all orders for all $N \in \mathbb{N}, t \in [0,T]$. Assume there exist functions $m_n \in C([0,T], \mathbb{R})$ such that*

$$\lim_{N\to\infty} \sup_{t\in[0,T]} \left| \int_{\mathbb{R}} x^n \, d\mu_t^N(x) - m_n(t) \right| = 0 \text{ for all } n \in \mathbb{N}\,.$$

*Then there exists a family of measures $(\mu_t)_{t\in[0,T]} \subseteq M^1(\mathbb{R})$ such that $m_n(t) = \int_{\mathbb{R}} x^n \, d\mu_t(x)$ for all $t \in [0,T]$, $n \in \mathbb{N}$. If, in addition, $\mu_t$ is determined by its moments for all $t \in [0,T]$, then $(\mu_t)_{t\in[0,T]} \in C([0,T], M^1(\mathbb{R}))$ and*

$$(\mu_t^N)_{t\in[0,T]} \xrightarrow{N\to\infty} (\mu_t)_{t\in[0,T]} \text{ in } C([0,T], M^1(\mathbb{R}))\,.$$

*Proof.* The first part of the Lemma is a direct consequence of the moment convergence theorem 4.1.4. So now assume that $\mu_t$ is determined by its moment sequence $(m_n(t))_{n\in\mathbb{N}}$ for each $t \in [0,T]$. Then, by the continuity of $m_n$ and the moment convergence theorem, $(\mu_t)_{t\in[0,T]} \in C([0,T], M^1(\mathbb{R}))$. We show $(\mu_t^N)_{t\in[0,T]} \xrightarrow{N\to\infty} (\mu_t)_{t\in[0,T]}$ by contradiction. Assume there is $f \in C_b(\mathbb{R})$ such that

$$\limsup_{N\to\infty} \sup_{t\in[0,T]} \left| \int_{\mathbb{R}} f \, d\mu_t^N - \int_{\mathbb{R}} f \, d\mu_t \right| > 0 \, .$$

Then there exist $\epsilon > 0$ and sequences $(N_k)_{k\in\mathbb{N}} \subset \mathbb{N}$, $(t_k)_{k\in\mathbb{N}} \subset [0,T]$ with $N_k \uparrow \infty$, $k \to \infty$, such that

$$\left| \int_{\mathbb{R}} f \, d\mu_{t_k}^{N_k} - \int_{\mathbb{R}} f \, d\mu_{t_k} \right| > \epsilon \;\text{ for all }\; k \in \mathbb{N} \, .$$

By going to a subsequence we can enforce that $t_k$ converges for $k \to \infty$ to some $t_0 \in [0,T]$. By abuse of notation we still denote the resulting subsequences by $(N_k)_{k\in\mathbb{N}}, (t_k)_{k\in\mathbb{N}}$. Let $n \in \mathbb{N}$. Then it holds by assumption that

$$\begin{aligned} &\left| \int_{\mathbb{R}} x^n \, d\mu_{t_k}^{N_k}(x) - \int_{\mathbb{R}} x^n \, d\mu_{t_0}(x) \right| \\ \leq &\left| \int_{\mathbb{R}} x^n \, d\mu_{t_k}^{N_k}(x) - \int_{\mathbb{R}} x^n \, d\mu_{t_k}(x) \right| + \left| \int_{\mathbb{R}} x^n \, d\mu_{t_k}(x) - \int_{\mathbb{R}} x^n \, d\mu_{t_0}(x) \right| \xrightarrow{k\to\infty} 0 \, . \end{aligned}$$

As $\mu_{t_0}$ is uniquely determined by its moments, we have $\mu_{t_k}^{N_k} \xrightarrow{k\to\infty} \mu_{t_0}$ weakly by the moment convergence theorem. But now we get a contradiction since

$$\begin{aligned} \epsilon < &\left| \int_{\mathbb{R}} f \, d\mu_{t_k}^{N_k} - \int_{\mathbb{R}} f \, d\mu_{t_k} \right| \\ \leq &\left| \int_{\mathbb{R}} f \, d\mu_{t_k}^{N_k} - \int_{\mathbb{R}} f \, d\mu_{t_0} \right| + \left| \int_{\mathbb{R}} f \, d\mu_{t_0} - \int_{\mathbb{R}} f \, d\mu_{t_k} \right| \xrightarrow{k\to\infty} 0 \, . \end{aligned}$$

□

## 4.2 Application of the moment convergence theorem to empirical measures of non-colliding $\beta$-Pearson diffusions

As motivated in the introduction to this chapter, now, we are going to study the convergence of sequences of empirical measures of non-colliding $\beta$-Pearson diffusions. Throughout this section we assume the following setting:
Let $(a_2^N)_{N\in\mathbb{N}}$, $(a_1^N)_{N\in\mathbb{N}}$, $(a_0^N)_{N\in\mathbb{N}}$ be sequences of real numbers such that the sets $\{x \in \mathbb{R} \colon a^N(x) \geq 0\}$ have non-empty interior where

$$a^N(x) := a_2^N x^2 + a_1^N x + a_0^N \, .$$

Let $I_{a^N} \subseteq \{x \in \mathbb{R}\colon a^N(x) \geq 0\}$ be connected with non-empty interior. Let $(b_1^N)_{N\in\mathbb{N}}$, $(b_0^N)_{N\in\mathbb{N}} \subset \mathbb{R}$, and set $b^N(x) := b_1^N x + b_0^N$. For each $N \in \mathbb{N}$ let $(X_t^N)_{t\geq 0}$ be a non-colliding solution to

$$dX_{i,t}^N = \sqrt{2a^N(X_{i,t}^N)}\, dB_{i,t} + \beta \left[\frac{1}{2} b^N(X_{i,t}) + a^N(X_{i,t}^N) \sum_{j\colon j\neq i} \frac{1}{X_{t,i}^N - X_{t,j}^N}\right] dt\,, \tag{4.2.1}$$

and let $(x_t^N)_{t\geq 0}$ be a non-colliding solution to

$$\frac{d}{dt} x_{i,t}^N = b^N(x_{i,t}) + 2a^N(x_{i,t}^N) \sum_{j\colon j\neq i} \frac{1}{x_{t,i}^N - x_{t,j}^N}\,,$$

$i \in \{1,\dots,N\}$, $X_0^N = x_0^N = \hat{x}_0^N \in C_N^A \cap (I_{a^N})^N$; see Chapter 3 for conditions under which such solutions exist. We will denote the corresponding empirical measures and empirical moments by

$$\mu_{N,t} := \frac{1}{N}\sum_{i=1}^N \delta_{X_{i,t}^N}\,, \quad S_{N,n,t} := \frac{1}{N}\sum_{i=1}^N \left(X_{i,t}^N\right)^n = \int_{\mathbb{R}} x^n\, d\mu_{N,t}(x)\,,$$
$$\tilde{\mu}_{N,t} := \frac{1}{N}\sum_{i=1}^N \delta_{x_{i,t}^N}\,, \quad \tilde{S}_{N,n,t} := \frac{1}{N}\sum_{i=1}^N \left(x_{i,t}^N\right)^n = \int_{\mathbb{R}} x^n\, d\tilde{\mu}_{N,t}(x)\,,$$

$N \in \mathbb{N}$, $n \in \mathbb{N}_0$, $t \geq 0$. The first key observation is that the dynamic of the empirical moments has the following recursive structure:

**Proposition 4.2.1.** *For $n, N \in \mathbb{N}$ the empirical moment $(S_{N,n,t})_{t\geq 0}$ satisfies the SDE*

$$\begin{aligned}
&dS_{N,n,t}\\
=&dM_{N,n,t} + n\left[\left(\frac{\beta}{2}(b_1^N - a_2^N(n+1)) + (n-1)a_2^N\right) S_{N,n,t}\right.\\
&+\left(\frac{\beta}{2}(b_0^N - a_1^N n) + (n-1)a_1^N\right) S_{N,n-1,t} + \left(1-\frac{\beta}{2}\right) a_0^N (n-1) S_{N,n-2,t}\\
&\left.+\frac{\beta N}{2}\left(a_2^N \sum_{k=0}^{n} S_{N,k,t}S_{N,n-k,t} + a_1^N \sum_{k=0}^{n-1} S_{N,k,t}S_{N,n-1-k,t} + a_0^N \sum_{k=0}^{n-2} S_{N,k,t}S_{N,n-2-k,t}\right)\right] dt\,,
\end{aligned} \tag{4.2.2}$$

*with the $L^2$-martingale*

$$M_{N,n,t} := \frac{n}{N}\sum_{i=1}^N \int_0^t \left(X_{i,s}^N\right)^{n-1} \sqrt{2a(X_{i,s}^N)}\, dB_{i,s}\,.$$

*Moreover, there is some $c(N,n,\beta) \geq 0$ such that:*

$$E[S_{N,2n,t}] \leq (S_{N,2n,0} + c(N,n,\beta)) \exp\left(c(N,n,\beta)t\right)\,.$$

*For $n, N \in \mathbb{N}$ the empirical moment $(\tilde{S}_{N,n,t})_{t\geq 0}$ satisfies the ODE*

$$\begin{aligned}
&\frac{d}{dt}\tilde{S}_{N,n,t}\\
=&n(b_1^N - a_2^N(n+1))\tilde{S}_{N,n,t} + (b_0^N - a_1^N n)\tilde{S}_{N,n-1,t} - a_0^N(n-1)\tilde{S}_{N,n-2,t}\\
&+ N\left(a_2^N \sum_{k=0}^{n} \tilde{S}_{N,k,t}\tilde{S}_{N,n-k,t} + a_1^N \sum_{k=0}^{n-1} \tilde{S}_{N,k,t}\tilde{S}_{N,n-1-k,t} + a_0^N \sum_{k=0}^{n-2} \tilde{S}_{N,k,t}\tilde{S}_{N,n-2-k,t}\right).
\end{aligned} \tag{4.2.3}$$

*Proof.* By Itô's formula we have

$$\begin{aligned}
dS_{N,n,t} =&\frac{1}{N}\sum_{i=1}^{N} n\left(X_{i,t}^N\right)^{n-1} dX_{i,t}^N + \frac{1}{2N}\sum_{i=1}^{N} n(n-1)\left(X_{i,t}^N\right)^{n-2} d\left[X_i^N\right]_t\\
=&dM_{N,n,t} + n\left[\frac{\beta}{2N}\sum_{i=1}^{N}\left(b_1^N\left(X_{i,t}^N\right)^n + b_0^N\left(X_{i,t}^N\right)^{n-1}\right)\right.\\
&+\frac{\beta}{N}\sum_{i=1}^{N}\left(a_2^N\left(X_{i,t}^N\right)^{n+1} + a_1^N\left(X_{i,t}^N\right)^n + a_0^N\left(X_{i,t}^N\right)^{n-1}\right)\sum_{j:\, j\neq i}\frac{1}{X_{i,t}^N - X_{j,t}^N}\\
&\left.+\frac{n-1}{N}\sum_{i=1}^{N}\left(a_2^N\left(X_{i,t}^N\right)^n + a_1^N\left(X_{i,t}^N\right)^{n-1} + a_0^N\left(X_{i,t}^N\right)^{n-2}\right)\right]dt\,.
\end{aligned}$$

We can simplify the appearing double sums using the following formula, where $l \in \mathbb{N}_0$:

$$\begin{aligned}
2\sum_{i,j:\, i\neq j}\frac{x_i^{l+1}}{x_i - x_j} &= \sum_{i,j:\, i\neq j}\frac{x_i^{l+1} - x_j^{l+1}}{x_i - x_j} = \sum_{i,j:\, i\neq j}\sum_{k=0}^{l} x_i^k x_j^{l-k}\\
&= \sum_{k=0}^{l}\sum_{i} x_i^k \sum_{j} x_j^{l-k} - (l+1)\sum_{i} x_i^l\,.
\end{aligned}$$

Applying this formula yields

$$\begin{aligned}
&dS_{N,n,t} - dM_{N,n,t}\\
=&n\left[\frac{\beta}{2}\left(\left(b_1^N S_{N,n,t} + b_0^N S_{N,n-1,t}\right) + a_2^N\left(N\sum_{k=0}^{n} S_{N,k,t}S_{N,n-k,t} - (n+1)S_{N,n,t}\right)\right.\right.\\
&+ a_1^N\left(N\sum_{k=0}^{n-1} S_{N,k,t}S_{N,n-1-k,t} - nS_{N,n-1,t}\right)\\
&\left.+a_0^N\left(N\sum_{k=0}^{n-2} S_{N,k,t}S_{N,n-2-k,t} - (n-1)S_{N,n-2,t}\right)\right)\\
&\left.+(n-1)\left(a_2^N S_{N,n,t} + a_1^N S_{N,n-1,t} + a_0^N S_{N,n-2,t}\right)\right]dt\,.
\end{aligned}$$

Rearranging the terms shows (4.2.2), and virtually the same calculations also show (4.2.3). Next, we are going to show that the positive random variable $S_{N,2n,t}$ is in $L^1$ for all $N, n, t$. To do so, we define the stopping times

$$\tau_{2n,l} := \inf\{t \geq 0 \colon S_{N,2n,t} \geq S_{N,2n,0} + l\}\,,\ l \in \mathbb{N}\,.$$

Clearly, every $X^N_{i,t\wedge\tau_{2n,l}}$ is a bounded random variable, and so $M_{N,2n,t\wedge\tau_{2n,l}}$ is an $L^2$-martingale for all $l \in \mathbb{N}$. Furthermore, by the triangle and Jensen's inequality, we have for $x \in \mathbb{R}^N$, $k, m \in \mathbb{N}$ with $k < m$:

$$\left|\frac{1}{N}\sum_{i=1}^N x_i^k\right| \leq \left(\frac{1}{N}\sum_{i=1}^N |x_i|^m\right)^{k/m} \leq 1 + \frac{1}{N}\sum_{i=1}^N |x_i|^m\,.$$

Thus, we can bound all of the following terms by $1 + S_{N,2n,t}$:

$$|S_{N,2n-1,t}|,\ S_{N,2(n-1),t},\ |S_{N,k,t}S_{N,2n-k,t}|,\ |S_{N,k,t}S_{N,2n-1-k,t}| \ \text{ and } \ |S_{N,k,t}S_{N,2(n-1)-k,t}|\,.$$

Combining the fact that $M_{N,2n,t\wedge\tau_{n,l}}$ is a martingale and these inequalities, we can deduce the following bound from equation (4.2.2):

$$E\left[S_{N,2n,t\wedge\tau_{n,l}}\right] \leq S_{N,2n,0} + c(N,n,\beta)\left(1 + \int_0^t E\left[S_{N,2n,s\wedge\tau_{n,l}}\right] ds\right)\,,$$

where

$$\begin{aligned} c(N,n,\beta) :=& 2n\left(\left|\frac{\beta}{2}(b_1^N - a_2^N(2n+1)) + (2n-1)a_2^N\right| + \left|\frac{\beta}{2}(b_0^N - a_1^N 2n) + (2n-1)a_1^N\right|\right. \\ &\left. + \left|1 - \frac{\beta}{2}\right|\,|a_0^N|\,(2n-1) + \frac{\beta N}{2}\left(|a_2^N|\,(2n+1) + |a_1^N|\,2n + |a_0^N|\,(2n-1)\right)\right)\,. \end{aligned}$$

Hence, Gronwall's lemma implies the bound

$$E\left[S_{N,2n,t\wedge\tau_{n,l}}\right] \leq \left(S_{N,2n,0} + c(N,n,\beta)\right)\exp\left(c(N,n,\beta)t\right)\,.$$

As the right hand side is independent of $l$, we see by monotone convergence that $S_{N,2n,t}$ is in $L^1$ and its $L^1$-norm can be bound from above by the same right hand side. In particular, as the quadratic variation of $M_{N,n,t}$ is given by

$$\begin{aligned} [M_{N,n}]_t =& 2\left(\frac{n}{N}\right)^2 \sum_{i=1}^N \int_0^t \left(X^N_{i,s}\right)^{2(n-1)} \left(a_2^N\left(X^N_{i,s}\right)^2 + a_1^N X^N_{i,s} + a_0^N\right) ds \\ =& 2\frac{n^2}{N}\int_0^t \left(a_2^N S_{N,2n,s} + a_1^N S_{N,2n-1,s} + a_0^N S_{N,2(n-1),s}\right) ds\,, \end{aligned} \tag{4.2.4}$$

we know by Itô's isometry that $(M_{N,n,t})_{t\geq 0}$ is an $L^2$-martingale for all $n, N$. □

The preceding Lemma in conjunction with the moment convergence Theorem 4.1.5 enables us to prove the following general result about weak convergence of $\mu_t^N$ and $\tilde{\mu}_t^N$ for large $N$:

**Theorem 4.2.2.** *Assume that the limits $\hat{a}_i := \lim_{N\to\infty} N a_i^N$ and $\hat{b}_j := \lim_{N\to\infty} b_j^N$, $i \in \{0,1,2\}, j \in \{0,1\}$, exist. Let $(\hat{x}_0^N)_{N\in\mathbb{N}}$ be a sequence of starting vectors such that the limits $m_n := \lim_{N\to\infty} \int_{\mathbb{R}} x^n \, d\mu_0^N(x)$ exist for all $n \in \mathbb{N}$. Assume there exists $\mu \in M^1(\mathbb{R})$ with moments of all orders, and that $m_n = \int_{\mathbb{R}} x^n \, d\mu(x)$. Further assume that there exists some $\gamma > 0$ such that the moments satisfy the bound*

$$|m_n| \leq (\gamma n)^n \,, \quad n \in \mathbb{N} \,. \tag{4.2.5}$$

*Then, for $N \to \infty$, $(\mu_{N,t})_{t\geq 0}$ converges in $C([0,\infty), M^1(\mathbb{R}))$ a.s. to some deterministic $(\mu_t)_{t\geq 0}$. $(\mu_t)_{t\geq 0}$ is uniquely determined by the following ODEs for its time-dependent moments $m_n(t) := \int_{\mathbb{R}} x^n \, d\mu_t$, $t \geq 0$:*

$$\begin{aligned} \frac{d}{dt} m_n(t) = \frac{n\beta}{2} \Bigg[ & \hat{b}_1 m_n(t) + \hat{b}_0 m_{n-1}(t) + \hat{a}_2 \sum_{k=0}^{n} m_k(t) m_{n-k}(t) \\ & + \hat{a}_1 \sum_{k=0}^{n-1} m_k(t) m_{n-1-k}(t) + \hat{a}_0 \sum_{k=0}^{n-2} m_k(t) m_{n-2-k}(t) \Bigg] \,, \end{aligned} \tag{4.2.6}$$

*$m_n(0) = m_n$, $n \in \mathbb{N}$.*
*Moreover, if $\mu = \mu_0$ is compactly supported, then $(\mu_t)_{t\geq 0}$ is locally uniformly compactly supported, i.e.*

$$\sup_{0\leq t\leq T} \sup\{|x| \colon x \in \operatorname{supp}(\mu_t)\} < \infty \ \text{ for all } \ T > 0 \,.$$

*All statements above remain true if one replaces $(\mu_t^N)_{t\geq 0}$ by $(\tilde{\mu}_t^N)_{t\geq 0}$, and sets $\beta = 2$ in (4.2.6).*

*Proof.* To show the claimed convergence, it suffices by Lemma 4.1.5 to show that $S_{N,n,t}$ converges locally uniformly in $t$ to $m_n(t)$ a.s. for all $n \in \mathbb{N}$. The first step in that direction is to show that $M_{N,n,t}$ converges locally uniformly in $t$ to 0 a.s. for all $n \in \mathbb{N}$. To see this, fix some $T, \epsilon > 0$. Using the Markov- and Burkholder-Davis-Gundy inequalities and equation (4.2.4), there exists a positive constant $c$ which is independent of $N$ such that

$$\begin{aligned} P\left( \sup_{t\in[0,T]} |M_{N,n,t}| \geq \epsilon \right) \leq & \frac{1}{\epsilon^2} E\left[ \sup_{t\in[0,T]} |M_{N,n,t}|^2 \right] \leq \frac{c}{\epsilon^2} E\left[ [M_{N,n}]_T \right] \\ \leq & \frac{2cn^2}{N\epsilon^2} \int_0^T \left( a_2^N E[S_{N,2n,s}] + a_1^N E[S_{N,2n-1,s}] + a_0^N E[S_{N,2(n-1),s}] \right) ds \,. \end{aligned}$$

Now, let $C(N,n,\beta)$ be as in the proof of Proposition 4.2.1. Then we have

$$P\left(\sup_{t\in[0,T]}|M_{N,n,t}|\geq\epsilon\right)\leq\frac{2cn^2}{N\epsilon^2}(|a_2^N|+|a_1^N|+|a_0^N|)\left(1+\int_0^T E\left[S_{N,2n,s}\right]\,ds\right)$$
$$\leq\frac{2cn^2}{N\epsilon^2}(|a_2^N|+|a_1^N|+|a_0^N|)\left(1+(S_{N,2n,0}+c(N,n,\beta))\int_0^T\exp\left(c(N,n,\beta)s\right)ds\right).$$

By our assumptions on $(a_i^N)_{N\in\mathbb{N}}, (b_j^N)_{N\in\mathbb{N}}$ and $(S_{N,2n,0})_{N\in\mathbb{N}}$ this upper bound is in $\mathcal{O}(N^{-2})$ for $N\to\infty$. As $\epsilon>0$ was arbitrary, the Borel-Cantelli Lemma implies

$$\sup_{t\in[0,T]}|M_{N,n,t}|\xrightarrow{N\to\infty}0\ \text{ a.s. for all }\ n\in\mathbb{N}.$$

In a next step, we will show that this implies the convergence of the empirical moments $S_{N,n,t}$ to $m_n(t)$. We will make use of the following fact: Let $f,g,h\colon[0,\infty)\to\mathbb{R}$ satisfy

$$f(t)-f(0)=\int_0^t(\lambda f(s)+g(s))\,ds+h(t)$$

for some $\lambda\in\mathbb{R}$. Then it holds

$$f(t)=e^{\lambda t}\left(f(0)+\int_0^t e^{-\lambda s}(g(s)+\lambda h(s))ds\right)+h(t).$$

Thus, by Proposition 4.2.1 we have

$$\begin{aligned}S_{N,n,t}=&e^{\lambda_n t}\Bigg(S_{N,n,0}+M_{N,n,t}\\&+\int_0^t e^{-\lambda_n s}\Bigg(\lambda_n M_{N,n,s}+n\Bigg[\left(\frac{\beta}{2}(b_0^N-a_1^N n)+(n-1)a_1^N\right)S_{N,n-1,s}\\&+\left(1-\frac{\beta}{2}\right)a_0^N(n-1)S_{N,n-2,s}+\frac{\beta N}{2}\Bigg(a_2^N\sum_{k=1}^{n-1}S_{N,k,s}S_{N,n-k,s}\\&+a_1^N\sum_{k=0}^{n-1}S_{N,k,s}S_{N,n-1-k,s}+a_0^N\sum_{k=0}^{n-2}S_{N,k,s}S_{N,n-2-k,s}\Bigg)\Bigg]\Bigg)ds\Bigg),\end{aligned}\tag{4.2.7}$$

where

$$\lambda_n:=\lambda_n(N,\beta):=n\left(\frac{\beta}{2}\left(b_1^N+(2N-(n+1))a_2^N\right)+(n-1)a_2^N\right).$$

As all empirical moments appearing on the right hand side of (4.2.7) are of lower order than $n$, we can deduce by induction that there exist some functions $\tilde{m}_n\in C([0,\infty),\mathbb{R})$ such that

$$\sup_{t\in[0,T]}|S_{N,n,t}-\tilde{m}_n(t)|\xrightarrow{N\to\infty}0\ \text{ a.s. for all }\ n\in\mathbb{N}.$$

Further, these functions satisfy

$$\begin{aligned}\tilde{m}_n(t) =& e^{\hat{\lambda}_n t}\left(m_n(0)+\frac{n\beta}{2}\int_0^t e^{-\hat{\lambda}_n s}\left(\hat{b}_0\tilde{m}_{n-1}(s)+\hat{a}_2\sum_{k=1}^{n-1}\tilde{m}_k(s)\tilde{m}_{n-k}(s)\right.\right.\\ &\left.\left.+\tilde{a}_1\sum_{k=0}^{n-1}\tilde{m}_k(s)\tilde{m}_{n-1-k}(s)+\hat{a}_0\sum_{k=0}^{n-2}\tilde{m}_k(s)\tilde{m}_{n-2-k}(s)\right)ds\right)\,,\end{aligned} \tag{4.2.8}$$

where $\hat{\lambda}_n := \lim_{N\to\infty}\lambda_n(N,\beta) = \frac{n\beta}{2}\left(\hat{b}_1+2\hat{a}_2\right)$. Taking the derivative with respect to $t$, we see that $\tilde{m}_n$ solves the ODE (4.2.6). As $\tilde{m}_n(0)=m_n(0)$, $m_n$ and $\tilde{m}_n$ must agree.
We next verify that for each $t>0$ the sequence $(m_n(t))_{n\in\mathbb{N}}$ satisfies the Carleman condition. If this is shown, the existence of $(\mu_t)_{t\geq 0}$ as claimed follows by Lemma 4.1.5. We show that the condition $|m_n(0)|\leq(\gamma n)^n$, $n\in\mathbb{N}$, implies a similar upper bound for $|m_n(t)|$ at positive times $t$. In the following we let w.l.o.g. $\beta=2$. Further, we set $c_1 := \hat{b}_1+2\hat{a}_2$,

$$c_2 := \max\left(1,\ 2\gamma,\ 2\left(|\hat{b}_0|+2|\hat{a}_1|+2|\hat{a}_0|+4\left(\hat{a}_2|+|\hat{a}_1|+|\hat{a}_0|\right)\right)\right)\,,\quad c_3 := \max(0,\,c_1+c_2)\,.$$

Note the immediate inequalities $\gamma^n\leq c_2^n/2$ and $c_3-c_1\geq c_2>0$. Now, we show by induction that

$$|m_n(t)|\leq(nc_2e^{c_3t})^n \tag{4.2.9}$$

for all $n\in\mathbb{N}$. The induction start holds since

$$\begin{aligned}|m_1(t)| =& e^{c_1t}\left|m_1(0)+\int_0^t e^{-c_1s}(\hat{b}_0+\hat{a}_1)ds\right| \leq e^{c_1t}\left(\gamma+(|\hat{b}_0|+|\hat{a}_1|)\int_0^t e^{(c_3-c_1)s}\,ds\right)\\ \leq& e^{c_1t}\left(\gamma+\frac{|\hat{b}_0|+|\hat{a}_1|}{c_3-c_1}e^{(c_3-c_1)t}\right)\leq e^{c_3t}c_2\,.\end{aligned}$$

In the induction step, we will apply the estimate

$$\begin{aligned}&\sum_{k=1}^{n-1}k^k(n-k)^{n-k} = 2(n-1)^{n-1}+\sum_{k=2}^{n-2}k^k(n-k)^{n-k}\leq\left(2+4\left(\frac{n-2}{n-1}\right)^{n-1}\right)(n-1)^{n-1}\\ \leq&(2+4/e)(n-1)^{n-1}\leq 4(n-1)^{n-1}\,,\end{aligned}$$

where we used $k^k(n-k)^{n-k}\leq 4(n-2)^{n-2}$ for all $k\in\{2,\dots,n-2\}$. In the induction

step, we thus have for $n \geq 2$:

$$
\begin{aligned}
&|m_n(t)| \\
\leq& e^{nc_1 t}\left((\gamma n)^n + n\int_0^t e^{-nc_1 s}\left((|\hat{b}_0| + 2|\hat{a}_1|)|m_{n-1}(s)| + 2|\hat{a}_0|\,|m_{n-2}(s)|\right.\right. \\
&\left.\left. + |\hat{a}_2|\sum_{k=1}^{n-1}|m_k(s)m_{n-k}(s)| + |\hat{a}_1|\sum_{k=1}^{n-2}|m_k(s)m_{n-1-k}(s)| + |\hat{a}_0|\sum_{k=1}^{n-3}|m_k(s)m_{n-2-k}(s)|\right) ds\right) \\
\leq& e^{nc_1 t}\left((\gamma n)^n + n\int_0^t e^{-nc_1 s}\left((|\hat{b}_0| + 2|\hat{a}_1| + 2|\hat{a}_0|)((n-1)c_2 e^{c_3 s})^{n-1}\right.\right. \\
&\left.\left. + (|\hat{a}_2| + |\hat{a}_1| + |\hat{a}_0|)(c_2 e^{c_3 s})^n \sum_{k=1}^{n-1} k^k (n-k)^{n-k}\right) ds\right) \\
\leq& e^{nc_1 t}\left((\gamma n)^n\right. \\
&\left. + n\left(|\hat{b}_0| + 2|\hat{a}_1| + 2|\hat{a}_0| + 4(|\hat{a}_2| + |\hat{a}_1| + |\hat{a}_0|)\right) c_2^n (n-1)^{n-1}\int_0^t e^{n(c_3-c_1)s}\, ds\right) \\
\leq& e^{nc_1 t}\left((\gamma n)^n + \frac{|\hat{b}_0| + 2|\hat{a}_1| + 2|\hat{a}_0| + 4(|a_2| + |\hat{a}_1| + |\hat{a}_1|)}{c_3 - c_1} c_2^n (n-1)^{n-1} e^{n(c_3-c_1)t}\right) \\
\leq& e^{nc_3 t} c_2^n n^n\,.
\end{aligned}
$$

This concludes the induction. Next, we prove the statement about the uniform compactness of the support of $(\mu_t)_{t\geq 0}$. So assume that $\mu$ is compactly supported. As before, we assume w.l.o.g. $\beta = 2$ and set $c_1 := \hat{b}_1 + 2\hat{a}_2$. We show that $(\mu_t)_{t\geq 0}$ is locally uniformly compactly supported by proving that there exists $c_2, c_3 \geq 0$ such that

$$
|m_n(t)| \leq \frac{1}{n^2} c_2^n e^{nc_3 t} \tag{4.2.10}
$$

for all $n \in \mathbb{N}$ and $t \geq 0$. To construct such constants $c_2, c_3$ we first verify the following: There exists $\tilde{c} \geq 4$ such that for all $n \geq 2$:

$$
\sum_{k=1}^{n-1} \frac{1}{k^2}\frac{1}{(n-k)^2} \leq \tilde{c}\frac{1}{(n+2)^2}\,.
$$

This follows from the bound

$$
\sum_{k=1}^{n-1} \frac{1}{k^2}\frac{1}{(n-k)^2} \leq \frac{2}{(n-1)^2} + \int_1^{n-1} \frac{1}{x^2}\frac{1}{(n-x)^2}\, dx = \frac{2}{(n-1)^2} + \frac{2\left(\frac{n(n-2)}{n-1} + 2\log(n-1)\right)}{n^3}\,,
$$

where we used

$$
\frac{d}{dx}\left(\frac{1}{n^2}\left(\frac{1}{n-x} - \frac{1}{x}\right) + \frac{2}{n^3}(\log(x) - \log(n-x))\right) = \frac{1}{x^2(n-x)^2}\,, \quad x \in (0,n)\,.
$$

Moreover, since $\mu$ is compactly supported, there exists $r > 0$ such that $|m_n(0)| \leq r^n$. Also set $\hat{c} := \sup_{n\geq 3} n^2/(n-2)^2 = 9$. Now, we choose

$$c_2 := \max\left(1,\, r\sup_{n\in\mathbb{N}}(2n^2)^{1/n},\, 2\left(\hat{c}(|\hat{b}_0| + 2|\hat{a}_1| + 2|\hat{a}_0|) + \tilde{c}\,(|\hat{a}_2| + |\hat{a}_1| + |\hat{a}_0|)\right)\right),$$

and $c_3 := \max(0, c_1 + c_2)$. Note that this choice entails $r^n \leq c_2^n/(2n^2)$ and $c_3 - c_1 \geq c_2 > 0$. We show (4.2.10) by induction. The induction start holds since

$$\begin{aligned}|m_1(t)| =& e^{c_1 t}\left|m_1(0) + (\hat{b}_0 + \hat{a}_1)\int_0^t e^{-c_1 s}\,ds\right| \leq e^{c_1 t}\left(r + (|\hat{b}_0| + |\hat{a}_1|)\int_0^t e^{(c_3-c_1)s}\,ds\right)\\ \leq& e^{c_3 t}\left(\frac{c_2}{2} + \frac{c_2}{2(c_3-c_1)}\right) \leq e^{c_3 t}c_2\,.\end{aligned}$$

In the induction step, we distinguish the cases $n = 2$ and $n \geq 3$. For $n = 2$ we have

$$\begin{aligned}|m_2(t)| =& e^{2c_1 t}\left|m_2(0) + 2\int_0^t e^{-2c_1 s}\left(\left(\hat{b}_0 + 2\hat{a}_1\right)m_1(s) + \hat{a}_2 m_1^2(s) + \hat{a}_0\right)ds\right|\\ \leq& e^{2c_1 t}\left(r^2 + 2\int_0^t e^{-2c_1 s}\left((|\hat{b}_0| + 2|\hat{a}_1|)c_2 e^{c_3 s} + |\hat{a}_2|c_2^2 e^{2c_3 s} + |\hat{a}_0|\right)ds\right)\\ \leq& e^{2c_1 t}\left(r^2 + 2\int_0^t e^{2(c_3-c_1)s}c_2^2\left(|\hat{b}_0| + 2|\hat{a}_1| + |\hat{a}_2| + |\hat{a}_0|\right)ds\right)\\ \leq& e^{2c_1 t}\left(r^2 + c_2^2\frac{\left(|\hat{b}_0| + 2|\hat{a}_1| + |\hat{a}_2| + |\hat{a}_0|\right)}{c_3 - c_1}e^{2(c_3-c_1)t}\right)\\ \leq& e^{2c_3 t}\left(c_2^2/8 + c_2\left(|\hat{b}_0| + 2|\hat{a}_1| + |\hat{a}_2| + |\hat{a}_0|\right)\right) \leq e^{2c_3 t}c_2^2/4\,,\end{aligned}$$

where we used $\min(\hat{c},\tilde{c}) \geq 4$ in the last inequality. Similarly, we have for $n \geq 3$:

$$
\begin{aligned}
&|m_n(t)| \\
\leq& e^{nc_1 t}\left(r^n + n\int_0^t e^{-nc_1 s}\left((|\hat{b}_0| + 2|\hat{a}_1|)|m_{n-1}(s)| + 2|\hat{a}_0||m_{n-2}(s)| + |\hat{a}_2|\sum_{k=1}^{n-1}|m_k(s)m_{n-k}(s)|\right.\right.\\
&\left.\left. + |\hat{a}_1|\sum_{k=1}^{n-2}|m_k(s)m_{n-1-k}(s)| + |\hat{a}_0|\sum_{k=1}^{n-3}|m_k(s)m_{n-2-k}(s)|\right) ds\right)\\
\leq& e^{nc_1 t}\left(r^n + n\int_0^t e^{-nc_1 s}\left((|\hat{b}_0| + 2|\hat{a}_1|)\frac{1}{(n-1)^2}c_2^{n-1}e^{(n-1)c_3 s} + 2|\hat{a}_0|\frac{1}{(n-2)^2}c_2^{n-2}e^{(n-2)c_3 s}\right.\right.\\
&+ |\hat{a}_2|c_2^n e^{nc_3 s}\sum_{k=1}^{n-1}\frac{1}{k^2(n-k)^2} + |\hat{a}_1|c_2^{n-1}e^{(n-1)c_3 s}\sum_{k=1}^{n-2}\frac{1}{k^2(n-1-k)^2}\\
&\left.\left. + |\hat{a}_0|c_2^{n-2}e^{(n-2)c_3 s}\sum_{k=1}^{n-3}\frac{1}{k^2(n-2-k)^2}\right) ds\right)\\
\leq& e^{nc_1 t}\left(r^n + \frac{1}{n}\left(\hat{c}(|\hat{b}_0| + 2|\hat{a}_1| + 2|\hat{a}_0|) + \tilde{c}\,(|\hat{a}_2| + |\hat{a}_1| + |\hat{a}_0|)\right)c_2^n\int_0^t e^{n(c_3-c_1)s}\,ds\right)\\
&\leq e^{nc_1 t}\left(r^n + \frac{1}{n^2}\frac{\hat{c}(|\hat{b}_0| + 2|\hat{a}_1| + 2|\hat{a}_0|) + \tilde{c}\,(|\hat{a}_2| + |\hat{a}_1| + |\hat{a}_0|)}{c_3 - c_1}c_2^n e^{n(c_3-c_1)t}\right) \leq \frac{1}{n^2}c_2^n e^{nc_3 t}\,.
\end{aligned}
$$

This finishes the proof of the statements concerning $(\mu_t^N)_{t\geq 0}$ and its limit. The proof of the convergence of $(\tilde{\mu}_{N,t})_{t\geq 0}$ follows by virtually the same proof as above with simplifications due to the non-random setting. □

**Remark 4.2.3.**

(i) In the previous proof, we used the condition $|m_n(0)| \leq (\gamma n)^n$, $n \in \mathbb{N}$, to infer moment determinacy for the limit measure for all times $t > 0$. In the literature one often encounters a, at first glance, seemingly different condition for moment determinacy, namely

$$|m_n(0)| \leq c^n n!\,,$$

for some $c > 0$. This latter condition is in turn clearly equivalent to the existence of the moment generating function around the origin. One can show that both moment conditions are actually equivalent (see [72] Theorem 1).

(ii) We will see in Section 5 that in many cases the limit measures $\mu_t$, $t \geq 0$, in the preceding theorem can be identified as the distributions of free Itô processes (see Section 2.5 for this notion). In these cases, the statement about locally uniform compactness of the limit measures is then a direct consequence of the norm-continuity of free Itô processes.

The following Lemma, which asserts that the limit measure in Theorem 4.2.2 satisfies a space-time integral equation, is known in the cases $\beta \in \{1, 2\}$ by [73] Theorem 1. However, the method of proof used in that paper readily applies also to non-colliding $\beta$-Pearson diffusions for general $\beta$, whence we do not replicate the proof here:

**Lemma 4.2.4.** *Let $(\mu_t)_{t\geq 0}$ be the limit of empirical measures of non-colliding $\beta$-Pearson distributions as in Theorem 4.2.2, and let $\hat{a}_i, \hat{b}_j$, $\beta \geq 1$ be the respective constants such that (4.2.6) holds. Then for $f \in C_b^2(\mathbb{R})$ it holds*

$$\begin{aligned}\int_{\mathbb{R}} f\, d\mu_t = \int_{\mathbb{R}} f\, d\mu_0 + \frac{\beta}{2}\int_0^t \int_{\mathbb{R}} ((\hat{b}_1 + 2\hat{a}_2)x + \hat{b}_0 + \hat{a}_1) f'(x)\, d\mu_s(x) ds \\ + \frac{\beta}{2}\int_0^t \int_{\mathbb{R}^2} (\hat{a}_2 xy + \hat{a}_1(x+y)/2 + \hat{a}_0)\, \frac{f'(x) - f'(y)}{x-y}\, d\mu_s(x) d\mu_s(y) ds\,, \quad t \geq 0\,,\end{aligned} \tag{4.2.11}$$

*with the convention $(f'(x) - f'(y))/(x-y) := f''(x)$ for $x = y$.*

We finish this section by relating Lemma 4.2.4 to the Cauchy transform. This gives a characterization of the limit distribution $(\mu_t)_{t\geq 0}$ appearing in Theorem 4.2.2 in the case that $\mu_0$ is compactly supported.

**Corollary 4.2.5.** *Let $(\mu_t)_{t\geq 0}$ be the limit of empirical measures of non-colliding $\beta$-Pearson diffusions as in Theorem 4.2.2, and let $\hat{a}_i, \hat{b}_j$, $\beta \geq 1$ be the respective constants such that (4.2.6) holds. Then the Cauchy transform $G(t,z) := \int_{\mathbb{R}} \frac{1}{z-x}\, d\mu_t(x)$ satisfies the PDE*

$$\partial_t G(t,z) = -\frac{\beta}{2}\partial_z \left[ \left(\hat{b}_1 z + \hat{b}_0\right) G(t,z) + \left(\hat{a}_2 z^2 + \hat{a}_1 z + \hat{a}_0\right) G^2(t,z) \right] \tag{4.2.12}$$

*for all $z \in \mathbb{C} \setminus \mathbb{R}$, $t \geq 0$.*
*Moreover, if there are $(\mu_t^1)_{t\geq 0}, (\mu_t^2)_{t\geq 0} \in C([0,\infty), M^1(\mathbb{R}))$ such that*

*(i)* $\mu_0^1 = \mu_0^2$,

*(ii)* $(\mu_t^i)_{t\geq 0}$ *is locally uniformly compactly supported, i.e.* $\sup_{0\leq t\leq T} \sup\{|x| : x \in \mathrm{supp}(\mu_t^i)\} < \infty$ *for all $T > 0$, $i \in \{1,2\}$, and*

*(iii)* $G^i(t,z) := G_{\mu_t^i}(z)$ *satisfies (4.2.12) for $i \in \{1,2\}$.*

*Then $\mu_t^1 = \mu_t^2$ for all $t \geq 0$.*

*Proof.* The proof partially follows the calculations in Section 4.4 of [73]. Fix $z \in \mathbb{C} \setminus \mathbb{R}$ and set $f(x) := (z-x)^{-1}$. Then we have

$$\frac{f'(x) - f'(y)}{x-y} = \frac{1}{(z-x)(z-y)^2} + \frac{1}{(z-x)^2(z-y)}\,.$$

By Lemma 4.2.4 we thus have

$$\begin{aligned}
&\partial_t G(t,z)\\
=&\frac{\beta}{2}\int_{\mathbb{R}}((\hat{b}_1+2\hat{a}_2)x+\hat{b}_0+\hat{a}_1)\frac{1}{(z-x)^2}\,d\mu_t(x)\\
&+\frac{\beta}{2}\int_{\mathbb{R}^2}(\hat{a}_2xy+\hat{a}_1(x+y)/2+\hat{a}_0)\left(\frac{1}{(z-x)(z-y)^2}+\frac{1}{(z-x)^2(z-y)}\right)d\mu_t(x)d\mu_t(y)\,.
\end{aligned}$$

Note that, since $\mathrm{Re}(z)\neq 0$, we can exchange integral and partial derivative to infer $-\partial_z G(t,z)=\int_{\mathbb{R}}\frac{1}{(z-x)^2}\,d\mu_t(x)$; see proof of Lemma 2 in Section 3.1 of [77] for the precise argument. This also gives

$$\int_{\mathbb{R}} x\frac{1}{(z-x)^2}\,d\mu_t(x)=\int_{\mathbb{R}}\frac{x-z+z}{(z-x)^2}\,d\mu_t(x)=-G(t,z)-z\partial_z G(t,z)=-\partial_z(zG(t,z))\,.$$

By symmetry in the integrand, we moreover have

$$\begin{aligned}
&\int_{\mathbb{R}^2}xy\left(\frac{1}{(z-x)(z-y)^2}+\frac{1}{(z-x)^2(z-y)}\right)d\mu_t(x)d\mu_t(y)\\
=&2\int_{\mathbb{R}}\frac{x-z+z}{(z-x)^2}\,d\mu_t(x)\int_{\mathbb{R}}\frac{y-z+z}{z-y}\,d\mu_t(y)=2\left(-G(t,z)-z\partial_z G(t,z)\right)\left(-1+zG(t,z)\right)\\
=&-\partial_z(z^2G^2(t,z))+2\partial_z(zG(t,z))\,.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\int_{\mathbb{R}^2}\frac{x+y}{2}\left(\frac{1}{(z-x)(z-y)^2}+\frac{1}{(z-x)^2(z-y)}\right)d\mu_t(x)d\mu_t(y)&=\partial_z\left(G(t,z)-zG^2(t,z)\right)\,,\\
\int_{\mathbb{R}^2}\left(\frac{1}{(z-x)(z-y)^2}+\frac{1}{(z-x)^2(z-y)}\right)d\mu_t(x)d\mu_t(y)&=-\partial_z G^2(t,z)\,.
\end{aligned}$$

Combining these calculations proves (4.2.12). Now, let $(\mu_t^1)_{t\geq 0},(\mu_t^2)_{t\geq 0}\in C([0,\infty),M^1(\mathbb{R}))$ satisfy (i)-(iii). Denote the respective moments by $m_n^i(t):=\int_{\mathbb{R}}x^n\,d\mu_t^i(x)$. Let $T>0$. Then there exists $R>0$ such that $\mathrm{supp}(\mu_t^i)\subseteq[-R,R]$ for all $t\in[0,T]$, $i\in\{1,2\}$. Hence, for all $z\in\mathbb{C}$ such that $|z|>R$ and $t\leq T$ it holds the well known relation between moments and Cauchy-transform:

$$G^i(t,z)=\int_{[-R,R]}\frac{1}{z}\sum_{k\geq 0}\left(\frac{x}{z}\right)^k d\mu_t^i(x)=\sum_{k\geq 0}z^{-(k+1)}m_k^i(t)\,.$$

By the uniform convergence of this series in $z$, we can deduce that we can exchange

derivatives and summation in (4.2.12):

$$\begin{aligned}
&\sum_{k\geq 0} z^{-(k+1)}(m_k^i(t) - m_k^i(0)) = G^i(t,z) - G^i(0,z) \\
=& -\frac{\beta}{2}\int_0^t \Bigg( \hat{b}_1 \sum_{k\geq 0} z^{-(k+1)} m_k^i(s) - (\hat{b}_1 z + \hat{b}_0) \sum_{k\geq 0} (k+1) z^{-(k+2)} m_k^i(s) \\
&\qquad + (2\hat{a}_2 z + \hat{a}_1) \left( \sum_{k\geq 0} z^{-(k+1)} m_k^i(s) \right)^2 \\
&\qquad - 2(\hat{a}_2 z^2 + \hat{a}_1 z + \hat{a}_0) \sum_{k\geq 0} z^{-(k+1)} m_k^i(s) \sum_{l\geq 0} (l+1) z^{-(l+2)} m_l^i(s) \Bigg) \, ds \,.
\end{aligned}$$

Using the Cauchy product formula and comparing coefficients of the series in $z$ of both sides of the equation (see e.g. Section 2 of [13] for similar calculations), we see that $m_k^i$ solves (the integral version of) (4.2.6) for both $i \in \{1,2\}$. As we have $m_k^1(0) = m_k^2(0)$ for all $k \in \mathbb{N}$, we get $m_k^1(t) = m_k^2(t)$ for all $t \leq T$ by the uniqueness of solutions to the linear ODEs (4.2.6). As $\mu_t^i$ is compactly supported it thus holds $\mu_t^1 = \mu_t^2$ for all $t \leq T$. This finishes the proof as $T > 0$ was arbitrary. □

# Chapter 5

# Description of limits of empirical measures of non-colliding $\beta$-Pearson diffusions via free probability

In this chapter, we will use free probability theory to identify the large $N$ limits of empirical measures of non-colliding $\beta$-Pearson diffusions $(X_t^N)_{t\geq 0}$ which appear in Theorem 4.2.2. Recall that $N$ denotes the dimension of the state space of the processes $(X_t^N)_{t\geq 0}$. Throughout we will assume

$$\beta \geq 1$$

as this condition appears in all of the existence and uniqueness results for non-colliding solutions to (3.1.1) in Chapter 3. We will first consider the limits arising in the 'natural' scaling of the six classes of non-colliding $\beta$-Pearson diffusions which we discussed in Chapter 3. This scaling always has the structure that one starts with a sequence of non-colliding $\beta$-Pearson diffusions whose coefficients of the respective diffusion parts are constant in $N$, and then rescales the $N$-th process in time by the factor $1/N$. These types of limits are very well studied in the case of Dyson Brownian motions, squared Bessel processes, Jacobi processes and non-colliding geometric Brownian motions. We recapitulate these results in Sections 5.1-5.4. Further, in Section 5.3 a new limit result for non-colliding inhomogeneous Brownian motions is obtained in the case that the parameter $\hat{b}_0$ appearing in Theorem 5.3.4 is positive. In the latter case, we also state new results regarding an explicit solution to a related free SDE and the identification of stationary measures. Moreover, the limits for non-compact Jacobi/Fisher-Snedecor processes and Hua-Pickrell diffusions have not been considered in the literature before, i.e. all statements in Sections 5.5 & 5.6 are new. We will see that, for some parameters, there is a direct connection between the latter two limits, and, in one special case, a further connection to the limit of non-colliding geometric Brownian motions.

Later, in Section 5.7, we will show that Theorem 4.2.2 also applies to more general space-time scalings of non-colliding $\beta$-Pearson diffusions. We will focus on the Jacobi and non-compact Jacobi cases but it will be clear that the technique used also applies to other

types of $\beta$-Pearson diffusions as well. Results in this direction have previously been published by the present author and coauthors in [13].
For the sake of being concise, throughout this section we will state convergence results only for non-frozen $\beta$-Pearson diffusions, i.e. we exclude the case $\beta = \infty$. The reader will have no difficulty to deduce the corresponding statements for the frozen counterparts using Theorem 4.2.2. For definitions and notation concerning probability measures, free probability theory and free stochastic calculus see Sections 2.1, 2.4 & 2.5.

## 5.1 Global limit of (Ornstein-Uhlenbeck) Dyson Brownian motions

We start with considering the easiest case of non-colliding $\beta$-Pearson diffusions, i.e. the (OU) Dyson Brownian motions which are rescaled in time by $1/N$. More specifically, we consider sequences $a_2^N = a_1^N = b_0^N = 0$, $a_0^N = 1/(2N)$, $b_1^N \in \mathbb{R}$, $\hat{x}_0^N \in C_N^A$, $N \in \mathbb{N}$, such that the limits

$$\lim_{N\to\infty} b_1^N =: \lambda\,, \quad \frac{1}{N}\sum_{i=1}^{N} \delta_{\hat{x}_{i,0}^N} \xrightarrow{N\to\infty} \mu\,,$$

exist for some $\lambda \in \mathbb{R}$, $\mu \in M^1([0,\infty))$, where the latter convergence is in moments and the moments $(m_n)_{n\in\mathbb{N}}$ of $\mu$ satisfy the bounds (4.2.5) for some $\gamma > 0$. The SDE (4.2.1) in this setting reads

$$dX_{i,t}^N = \frac{1}{\sqrt{N}}\, dB_{i,t} + \frac{\beta}{2}\left[b_1^N X_{i,t}^N + \frac{1}{N}\sum_{j\colon j\neq i} \frac{1}{X_{i,t}^N - X_{j,t}^N}\right] dt\,, \quad i \in \{1,\ldots,N\}\,,$$

$X_0^N = \hat{x}_0^N$. The limit of the corresponding empirical measures $\mu_{N,t} = \frac{1}{N}\sum_{i=1}^N \delta_{X_{i,t}^N}$ can be described in the following way:

**Theorem 5.1.1.** *If $\lambda = 0$, then*

$$\left(\mu_{N,2t/\beta}\right)_{t\geq 0} \xrightarrow{N\to\infty} \left(\mu \boxplus \mu_{sc,2\sqrt{t}}\right)_{t\geq 0} \quad \textit{in } C\left([0,\infty), M^1(\mathbb{R})\right) \quad \textit{a.s.}$$

*If $\lambda \neq 0$, then*

$$\left(\mu_{N,2t/\beta}\right)_{t\geq 0} \xrightarrow{N\to\infty} \left(D_{e^{\lambda t}}(\mu) \boxplus \mu_{\mathrm{sc},\sqrt{2(e^{2\lambda t}-1)/\lambda}}\right)_{t\geq 0} \quad \textit{in } C\left([0,\infty), M^1(\mathbb{R})\right) \quad \textit{a.s.}$$

*In either case the limit measure is determined by its moments $(m_n(t))_{n\in\mathbb{N}}$, which satisfy*

$$\frac{d}{dt}m_n(t) = n\left[\lambda m_n(t) + \frac{1}{2}\sum_{k=0}^{n-2} m_k(t) m_{n-2-k}(t)\right].$$

*Proof.* The existence of the limit $(\mu_t)_{t\geq 0}$ of $(\mu_{N,t})_{t\geq 0}$ in $C([0,\infty), M^1(\mathbb{R}))$ and the validity of the ODE for the moments follows from Theorem 4.2.2. Now, denote the Cauchy transform of the limit measure by $G(t,z) := G_{\mu_t}(z)$. Since we are rescaling time by the factor $2/\beta$, we know by Corollary 4.2.5

$$\partial_t G(t,z) = -\partial_z \left(\lambda z G(t,z) + \frac{1}{2} G^2(t,z)\right) = -(\lambda z + G(t,z))\partial_z G(t,z) - \lambda G(t,z)\,. \quad (5.1.1)$$

In the case $\lambda = 0$ the statement follows from a combination of Proposition 4.3.10 & Corollary 4.3.11 in [3] and Theorem 4 in Section 4.2 of [77].
Now let $\lambda \neq 0$ and set $f(t) := e^{\lambda t}$ and $g(t) := (1-e^{-2\lambda t})/(2\lambda)$. Let $(\tilde{\mu}_t)_{t\geq 0}$ be the family of limit measures from above for the case $\lambda = 0$, i.e. $\tilde{\mu}_t = \mu \boxplus \mu_{\mathrm{sc},2\sqrt{t}}$. We are going to show that $\mu_t = D_{f(t)}(\tilde{\mu}_{g(t)})$. One way to see this is to use the space-time transformation for Dyson Brownian motions as indicated in (3.2.2), and to use equation (5.1.2) below. We give a another short proof using the Cauchy-transforms of the limit measures. Set

$$\bar{\mu}_t := D_{\sqrt{1-2\lambda t}}\left(\mu_{g^{-1}(t)}\right)\,, \quad \bar{G}(t,z) := G_{\bar{\mu}_t}(z)\,,$$

where $g^{-1}(t) = -\frac{1}{2\lambda}\log(1-2\lambda t)$. Here the range of $t$ is $[0,\infty)$ if $\lambda < 0$ and $[0, 1/(2\lambda))$ if $\lambda > 0$. We will employ the short hands $h_t(t,z) := \partial_t h(t,z)$ and $h_z(t,z) := \partial_z h(t,z)$ for differentiable functions $h$. Using (5.1.1), we have:

$$\begin{aligned}
\bar{G}_t(t,z) =& \lambda(1-2\lambda t)^{-1}\bar{G}(t,z) + (1-2\lambda t)^{-3/2} G_t(g^{-1}(t), z/\sqrt{1-2\lambda t}) \\
&+ \lambda z (1-2\lambda t)^{-2} G_z(g^{-1}(t), z/\sqrt{1-2\lambda t}) \\
=& \lambda(1-2\lambda t)^{-1}\bar{G}(t,z) - \lambda(1-2\lambda t)^{-3/2} G(g^{-1}(t), z/\sqrt{1-2\lambda t}) \\
&- (1-2\lambda t)^{-3/2}\left(\lambda z/\sqrt{1-2\lambda t} + G(g^{-1}(t), z/\sqrt{1-2\lambda t})\right) G_z(g^{-1}(t), z/\sqrt{1-2\lambda t}) \\
&+ \lambda z (1-2\lambda t)^{-2} G_z(g^{-1}(t), z/\sqrt{1-2\lambda t}) \\
=& -\bar{G}(t,z)\bar{G}_z(t,z)\,.
\end{aligned}$$

Moreover, $\bar{G}(0,z) = G_\mu(z)$. By the first part of the proof we thus know that

$$\bar{\mu}_t = \tilde{\mu}_t = \mu \boxplus \mu_{\mathrm{sc},2\sqrt{t}}\,.$$

Further, we observe

$$\mu_t = D_{f(t)}\left(D_{\sqrt{1-2\lambda g(t)}}\left(\mu_{g(g^{-1}(t))}\right)\right) = D_{f(t)}(\bar{\mu}_{g(t)}) = D_{f(t)}(\tilde{\mu}_{g(t)})\,.$$

The claim now follows from:

$$D_{f(t)}(\tilde{\mu}_{g(t)}) = D_{f(t)}(\mu) \boxplus D_{f(t)}(\mu_{\mathrm{sc},g(t)}) = D_{f(t)}(\mu) \boxplus \mu_{\mathrm{sc},f(t)g(t)} = D_{e^{\lambda t}}(\mu) \boxplus \mu_{\mathrm{sc},\sqrt{2(e^{2\lambda t}-1)/\lambda}}\,. \quad (5.1.2)$$

□

**Remark 5.1.2.** The assumption on the convergence of the moments of the initial empirical measure $\frac{1}{N}\sum_{i=1}^{N}\delta_{\hat{x}_{i,0}^N}$ can be considerably weakened, see Proposition 4.3.10 in [3].

Note that, in the case that $\mu$ is compactly supported, the limit distribution $\mu \boxplus \mu_{\mathrm{sc},2\sqrt{t}}$ in Theorem 5.1.1 coincides with the distribution of $a + x_t$ where $(x_t)_{t\geq 0}$ is a semi-circular Brownian motion and $a$ is a self-adjoint operator which is freely independent from $(x_t)_{t\geq 0}$ and has distribution $\mu_a = \mu$. In the following sections, we will see that the distributions of more complicated free Itô processes can be used to describe the limits of empirical measures of other types of non-colliding $\beta$-Pearson diffusions.

## 5.2 Global limit of squared (Ornstein-Uhlenbeck) Bessel processes

We apply Theorem 4.2.2 to the case of a sequence of squared (OU) Bessel processes rescaled in time by the factor $1/N$, i.e. we consider sequences $a_2^N = a_0^N = 0$, $a_1^N = 1/N$, $b_1^N \in \mathbb{R}$, $b_0^N \geq 0$, $\hat{x}_0^N \in C_N^B = \{x \in \mathbb{R}^N \colon 0 \leq x_1 \leq \cdots \leq x_N\}$, $N \in \mathbb{N}$, such that the limits

$$\lim_{N\to\infty} b_1^N =: \hat{b}_1\,, \quad \lim_{N\to\infty} b_0^N =: \hat{b}_0\,, \quad \frac{1}{N}\sum_{i=1}^{N}\delta_{\hat{x}_{i,0}^N} \xrightarrow{N\to\infty} \mu\,,$$

exist for some $\hat{b}_1 \in \mathbb{R}$, $\hat{b}_0 \geq 0$, $\mu \in M^1([0,\infty))$, where the latter convergence is in moments and the moments $(m_n)_{n\in\mathbb{N}}$ of $\mu$ satisfy the bounds (4.2.5) for some $\gamma > 0$. The SDE (4.2.1) here now reads (written in the symmetric form as in (3.1.3))

$$dX_{i,t}^N = \sqrt{2X_{i,t}^N/N}\,dB_{i,t} + \frac{\beta}{2}\left[b_1^N X_{i,t}^N + \frac{N-1}{N} + b_0^N + \frac{1}{N}\sum_{j\colon j\neq i}\frac{X_{i,t}^N + X_{j,t}^N}{X_{i,t}^N - X_{j,t}^N}\right]dt\,,$$

$i \in \{1,\ldots,N\}$, $X_0^N = \hat{x}_0$. Non-colliding solutions exist by our choice of parameters by Lemma 3.3.1 for each $N \in \mathbb{N}$.
To state the limit result in this setting, we employ the following notation: For $\mu \in M^1([0,\infty))$ we denote by $\sqrt{\mu}_{\mathrm{even}}$ the unique even probability measure on $\mathbb{R}$ such that its image measure under the map $x \mapsto x^2$ equals $\mu$. The limit of the empirical measures $\mu_{N,t} = \frac{1}{N}\sum_{i=1}^{N}\delta_{X_{i,t}^N}$ can be described in the following way:

**Theorem 5.2.1.** *Let $\hat{b}_1 = 0$, then*

$$\left(\mu_{N,2t/\beta}\right)_{t\geq 0} \xrightarrow{N\to\infty} \left(\left(\sqrt{\mu}_{\mathrm{even}} \boxplus \mu_{sc,2\sqrt{t}}\right)^2 \boxplus \mu_{\mathrm{MP},\hat{b}_0,t}\right)_{t\geq 0}$$

*in $C([0,\infty), M^1(\mathbb{R}))$ a.s.*
*For $\hat{b}_1 \neq 0$ it holds*

$$\left(\mu_{N,2t/\beta}\right)_{t\geq 0} \xrightarrow{N\to\infty} \left(\left(D_{e^{\hat{b}_1 t/2}}\left(\sqrt{\mu}_{\mathrm{even}}\right) \boxplus \mu_{\mathrm{sc},2\sqrt{(e^{\hat{b}_1 t}-1)/\hat{b}_1}}\right)^2 \boxplus \mu_{\mathrm{MP},\hat{b}_0,(e^{\hat{b}_1 t}-1)/\hat{b}_1}\right)_{t\geq 0}$$

*in* $C([0,\infty), M^1(\mathbb{R}))$ *a.s.*
*In either case, the limit measure is determined by its moments* $(m_n(t))_{n\in\mathbb{N}}$ *which satisfy*

$$\frac{d}{dt}m_n(t) = n\left[\hat{b}_1 m_n(t) + \hat{b}_0 m_{n-1}(t) + \sum_{k=0}^{n-1} m_k(t) m_{n-1-k}(t)\right].$$

*Proof.* As mentioned above, the existence of the limits and the ODEs for the corresponding moments are a consequence of Theorem 4.2.2. By Corollary 4.2.5, the Cauchy transforms $G(t,z) := G_{\mu_t}(z)$ of the limit measures satisfy the PDE

$$\begin{aligned}\partial_t G(t,z) &= -\partial_z\left(\left(\hat{b}_1 z + \hat{b}_0\right) G(t,z) + z G^2(t,z)\right)\\ &= -\left(\hat{b}_1 z + \hat{b}_0 + 2zG(t,z)\right) G_z(t,z) - \hat{b}_1 G(t,z) - G^2(t,z)\,.\end{aligned} \tag{5.2.1}$$

In the case $\hat{b}_1 = 0$, the identification of $(\mu_t)_{t\geq 0}$ now follows by Proposition 4.7 (2) & Theorem 4.8 in [109]. In the case $\hat{b}_1 \neq 0$, we will use the same space-time transformation as in Proposition 3.3.2. To this end let $f(t) := e^{\hat{b}_1 t}$ and $g(t) := (1-e^{-\hat{b}_1 t})/\hat{b}_1$, so that $\frac{d}{dt}\frac{1}{f(t)} = -\hat{b}_1\frac{1}{f(t)}$ and $\frac{d}{dt}g(t) = \frac{1}{f(t)}$. Let $(\tilde{\mu}_t)_{t\geq 0}$ be the limit measure from above for the case $\tilde{b}_1 = 0$, i.e.

$$\tilde{\mu}_t = \left(\sqrt{\mu}_{\text{even}} \boxplus \mu_{sc,2\sqrt{t}}\right)^2 \boxplus \mu_{\text{MP},\hat{b}_0,t}\,.$$

We claim that

$$\mu_t = D_{f(t)}(\tilde{\mu}_{g(t)})\,.$$

One way to show this is via the PDEs for the Cauchy transforms of the measures $(\mu_t)_{t\geq 0}$ and $(\tilde{\mu}_t)_{t\geq 0}$, similarly to the line of proof of Theorem 5.1.1. However, Proposition 3.3.2 tells us that this space-time relation between $\mu_t$ and $\tilde{\mu}_t$ is already valid for the corresponding empirical measures of (OU) squared Bessel processes which approximate $(\mu_t)_{t\geq 0}$, respectively $(\tilde{\mu}_t)_{t\geq 0}$. In turn, the relation also holds after taking the limit $N\to\infty$. Hence the claim follows from

$$\begin{aligned}D_{f(t)}(\tilde{\mu}_{g(t)}) &= \left(D_{e^{\hat{b}_1 t/2}}\left(\sqrt{\mu}_{\text{even}}\right) \boxplus D_{e^{\hat{b}_1 t/2}}\left(\mu_{\text{sc},2\sqrt{(1-e^{-\hat{b}_1 t})/\hat{b}_1}}\right)\right)^2 \boxplus D_{e^{\hat{b}_1 t}}\left(\mu_{\text{MP},\hat{b}_0,(1-e^{-\hat{b}_1 t})/\hat{b}_1}\right)\\ &= \left(D_{e^{\hat{b}_1 t/2}}\left(\sqrt{\mu}_{\text{even}}\right) \boxplus \mu_{\text{sc},2\sqrt{(e^{\hat{b}_1 t}-1)/\hat{b}_1}}\right)^2 \boxplus \mu_{\text{MP},\hat{b}_0,(e^{\hat{b}_1 t}-1)/\hat{b}_1}\,.\end{aligned}$$

□

We can describe the limit measures in Theorem 5.2.1 by the spectral distribution of a free Itô process as well: The free Wishart process $(R_t)_{t\geq 0}$ with parameter $\lambda \geq 1$ has been introduced in [25]. It is defined as the unique solution to the free SDE

$$dR_t = \sqrt{R_t}\,dc_t + dc_t^*\,\sqrt{R_t} + \lambda\mathbf{1}\,dt\,,\quad R_0 = r_0\,, \tag{5.2.2}$$

where $r_0$ is some positive operator, $\lambda \geq 1$, and $(c_t)_{t\geq 0}$ is a (one-dimensional) circular Brownian motion. Let us quickly check that the spectral distribution of $R_t$ actually coincides with

$$\left(\sqrt{\mu}_{\text{even}} \boxplus \mu_{sc,2\sqrt{t}}\right)^2 \boxplus \mu_{\text{MP},\lambda-1,t}\,, \tag{5.2.3}$$

where $\mu$ is the spectral distribution of $r_0$. By the traced Itô formula 2.5.11, we have for $n \in \mathbb{N}$:

$$\begin{aligned}\frac{d}{dt}\tau(R_t^n) =&\tau(n\lambda R_t^{n-1}) + n\sum_{k=0}^{n-2}(\tau\otimes\tau^{\text{op}})\left(\left(\sqrt{R_t}\otimes 1\right)\cdot\left(R_t^k\otimes R_t^{n-2-k}\right)\cdot\left(\sqrt{R_t}\otimes 1\right)\right)\\ =&n\left[\lambda\tau(R_t^{n-1}) + \sum_{k=0}^{n-2}\tau(R_t^{k+1})\tau(R_t^{n-2-k})\right]\\ =&n\left[(\lambda-1)\tau(R_t^{n-1}) + \sum_{k=0}^{n-1}\tau(R_t^k)\tau(R_t^{n-1-k})\right]\,.\end{aligned}$$

Our claim now follows from a comparison of these ODEs with the ODEs for the moments of the measure in (5.2.3), see Theorem 5.2.1.
Furthermore, an explicit solution to (5.2.2) is given in the following sense ([25] Proposition 3.1):

**Lemma 5.2.2.** *Let $(\mathcal{A},(\mathcal{A}_t)_{t\geq 0},\tau)$ be a filtered $W^*$-probability space. Let $\lambda \geq 1$ and let $P \in \mathcal{A}_0$ be a projection with $\tau(P) = 1/\lambda$. Let $\tilde{S}_0 \in \mathcal{A}_0$ be a positive operator such that $\ker \tilde{S}_0 = \{0\}$, and let $(c_t)_{t\geq 0}$ be a circular Brownian motion adapted to $(\mathcal{A}_t)_{t\geq 0}$. Let $S_t := \lambda P(c_t^* + \sqrt{\tilde{S}_0})(c_t + \sqrt{\tilde{S}_0})P$. Then $\sqrt{S_t}$ belongs to the compressed algebra $P\mathcal{A}P$. Further, there exists a circular Brownian motion $(\tilde{c}_t)_{t\geq 0}$ adapted to $(P\mathcal{A}_tP)_{t\geq 0}$ such that $(S_t)_{t\geq 0}$ solves the free SDE*

$$S_t = \sqrt{S_t}\,d\tilde{c}_t + d\tilde{c}_t^*\,\sqrt{S_t} + \lambda\mathbf{1}\,dt\,,\quad S_0 = \lambda P\tilde{S}_0P\,,$$

*where $\mathbf{1}$ denotes the unit element in $P\mathcal{A}P$.*

Clearly, the idea for the construction of this solution comes from the Wishart matrix models and their property to solve certain matrix SDEs; cf. the introduction in [24] and references therein. The key conceptual step being done here is to mirror the structure of truncated random matrices by projections of operators in a free probability space in the large $N$ limit. As we will see below, this technique also proves fruitful in describing the limit of empirical measures for two other classes of $\beta$-Pearson diffusions: Namely the Jacobi and non-compact Jacobi/Fisher-Snedecor processes.

## 5.3 Global limit of non-colliding (inhomogeneous) geometric Brownian motions

In this section, we apply Theorem 4.2.2 to a sequence of non-colliding geometric Brownian motions, possibly inhomogeneous, which are rescaled in time by the factor $1/N$, i.e. the

SDE (4.2.1) (written in the symmetric form as in (3.1.3)) here specializes to

$$dX_{i,t}^N = \frac{X_{i,t}^N}{\sqrt{N}}\, dB_{i,t} + \frac{\beta}{2}\left[\left(b_1^N + \frac{N-1}{N}\right) X_{i,t}^N + b_0^N + \frac{1}{N}\sum_{j:\, j\neq i} \frac{X_{i,t}^N X_{j,t}^N}{X_{i,t}^N - X_{j,t}^N}\right] dt\,, \qquad (5.3.1)$$

$i \in \{1,\dots,N\}$, $X_0^N = \hat{x}_0^N$. The corresponding parameters appearing in Theorem 4.2.2 are here given by $a_1^N = a_0^N = 0$, $a_2^N = 1/(2N)$, $b_1^N \in \mathbb{R}$, $b_0^N \geq 0$, $\hat{x}_0^N \in C_N^B = \{x \in \mathbb{R}^N : 0 \leq x_1 \leq \cdots \leq x_N\}$. We assume that it holds $\hat{x}_{1,0}^N > 0$ in the case $b_0^N = 0$, $N \in \mathbb{N}$, and we assume that the limits

$$\lim_{N\to\infty} b_1^N =: \hat{b}_1\,, \quad \lim_{N\to\infty} b_0^N =: \hat{b}_0\,, \quad \frac{1}{N}\sum_{i=1}^N \delta_{\hat{x}_{i,0}^N} \xrightarrow{N\to\infty} \mu\,,$$

exist for some $\hat{b}_1 \in \mathbb{R}$, $\hat{b}_0 \geq 0$, $\mu \in M^1([0,\infty))$, where the latter convergence is in moments and the moments $(m_n)_{n\in\mathbb{N}}$ of $\mu$ satisfy the bounds (4.2.5) for some $\gamma > 0$. Note that unique non-colliding solutions to (5.3.1) exist by our choice of parameters by Lemma 3.4.1 for each $N$.
To describe the limit of the empirical measures $\mu_{N,t} = \frac{1}{N}\sum_{i=1}^N \delta_{X_{i,t}^N}$, we introduce the following free Itô processes:

**Definition 5.3.1.** Let $(c_t)_{t\geq 0}$ be a (one-dimensional) circular Brownian motion in a filtered $W^*$-probability space $(\mathcal{A}, (\mathcal{A}_t)_{t\geq 0}, \tau)$, and let $\hat{g}_0 \in \mathcal{A}$ be freely independent of $(c_t)_{t\geq 0}$. We call the solution $(g_t)_{t\geq 0}$ of the free SDE

$$dg_t = g_t\, dc_t\,,\ g_0 = \hat{g}_0 \qquad (5.3.2)$$

*(left) free multiplicative Brownian motion (driven by $(c_t)_{t\geq 0}$) with start in $\hat{g}_0$.* Moreover, we call the process $(h_t := g_{t/2}g_{t/2}^*)_{t\geq 0}$ *free positive multiplicative Brownian motion with start in $\hat{h}_0 = \hat{g}_0\hat{g}_0^*$.*

Existence and uniqueness of the free SDE (5.3.2) can be shown by standard techniques as in the proof of Proposition A.1 in [25]. The term 'multiplicative' in the definitions above stems from the fact that these processes are free multiplicative Lévy processes, i.e. these processes have stationary and freely independent increments w.r.t. multiplication, see [20, 21]. A description of the density of the distribution $\nu_t$ of the free positive multiplicative Brownian motion with start in $\mathbf{1}$ at time $t$ can be found in [21] Section 4.2. An explicit formula for the corresponding moments has been proven in [19], see equation (6.0.3) in Chapter 6. Moreover, in Chapter 6, we prove a generalization of the latter result, i.e. an explicit formula for the integrals $\int_{(0,\infty)} x^\alpha\, d\nu_t$, $\alpha \in \mathbb{C}$, see Corollary 6.0.7. Further properties of the family of measures $(\nu_t)_{t\geq 0}$ can be found in [115], [55].
The following basic fact shows that the free positive multiplicative Brownian motion is the natural free analogue of (matrix) geometric Brownian motions.

**Proposition 5.3.2.** *Let $(h_t)_{t\geq 0}$ be a free positive multiplicative Brownian motion. Then there exists a semicircular Brownian motion $(x_t)_{t\geq 0}$ such that*

$$dh_t = \sqrt{h_t}\, dx_t\, \sqrt{h_t} + \frac{1}{2} h_t\, dt\,. \tag{5.3.3}$$

*Moreover, it holds*

$$dh_t^{-1} = -\sqrt{h_t^{-1}}\, dx_t\, \sqrt{h_t^{-1}} + \frac{1}{2} h_t^{-1}\, dt\,.$$

*Proof.* Setting $\tilde{c}_t := \sqrt{2} c_{t/2}$ one can easily check, using Theorem 2.5.4, that $(\tilde{c}_t)_{t\geq 0}$ is a circular Brownian motion (for the filtration $(\mathcal{A}_{t/2})_{t\geq 0}$). Moreover, by the construction of the free Itô integral, one has

$$g_{t/2} - g_0 = \int_0^{t/2} g_s\, dc_s = \int_0^t g_{s/2}\, dc_{s/2} = \frac{1}{\sqrt{2}} \int_0^t g_{s/2}\, d\tilde{c}_s\,.$$

Hence, we get

$$dh_t = g_{t/2} \frac{d\tilde{c}_t + d\tilde{c}_t^*}{\sqrt{2}} g_{t/2}^* + \frac{1}{2} h_t\, dt\,.$$

Note that $((\tilde{c}_t + \tilde{c}_t^*)/2)_{t\geq 0}$ is a semicircular Brownian motion. Denote by $u_t$ the unitary operator in the polar decomposition $\sqrt{h_t} = u_t g_{t/2}^*$, $t \geq 0$. Setting

$$x_t := \int_0^t u_s\, d\left( \frac{\tilde{c}_s + \tilde{c}_s^*}{\sqrt{2}} \right) u_s^*\,, \quad t \geq 0\,,$$

we have, again using Theorem 2.5.4, that $(x_t)_{t\geq 0}$ is a semicircular Brownian motion as well (see e.g. the proof of [30] Lemma 1 in Section 3 for the precise argument). This yields

$$dh_t = g_{t/2} u_t^* u_t \frac{d\tilde{c}_t + d\tilde{c}_t *}{\sqrt{2}} u_t^* u_t g_{t/2}^* + \frac{1}{2} h_t\, dt = \sqrt{h_t}\, dx_t\, \sqrt{h_t} + \frac{1}{2} h_t\, dt\,.$$

Finally, the validity of the equation for $dh_t^{-1}$ can be easily checked by using the free Itô product rule 2.5.10. □

We can characterize the distribution of a free positive multiplicative Brownian motion by its moments and by free multiplicative convolution as follows:

**Lemma 5.3.3.** *Let $(h_t)_{t\geq 0}$ be a free positive multiplicative Brownian motion with start in $\hat{h}_0$.*

*(a) The moments $m_n(t) := \tau(h_t^n)$ satisfy*

$$\frac{d}{dt} m_n(t) = \frac{n}{2} \left( -m_n(t) + \sum_{k=0}^{n} m_k(t) m_{n-k}(t) \right)\,,\ m_n(0) = \tau(\hat{h}_0^n)\,,\ n \in \mathbb{N}\,.$$

(b) *The distribution $\mu_{h_t}$ is given by $\mu_{\hat{h}_0} \boxtimes \nu_t$ where $\nu_t$ is the distribution of a free positive multiplicative Brownian motion with start in $\mathbf{1}$. The S-transform of $\nu_t$ is given by $S_{\nu_t}(z) = \exp\left(-(1+2z)t/2\right)$. Moreover, it holds the formula*

$$\nu_t = \exp\left(\mu_{\mathrm{sc},2\sqrt{t}} \boxplus \mathrm{Unif}_{[-t/2,t/2]}\right). \tag{5.3.4}$$

*Proof.* All statements except for the formula (5.3.4) are known by the results in Section 4.2 in [21] and by [115]. Also see the proof of Theorem 5.3.4 below for a derivation of the S-transform from the moment ODEs. The equation (5.3.4) was shown in [12]. We will give a self contained proof by different methods in Chapter 6. □

We now can describe the limit of empirical measures of the non-colliding (inhomogeneous) geometric Brownian motions in (5.3.1) as follows:

**Theorem 5.3.4.** *There exists $\left(\mu^{\mathrm{Geo}}_{\hat{b}_0,\hat{b}_1,t}\right)_{t\geq 0} \in C([0,\infty), M^1([0,\infty)))$ such that*

$$\left(\mu_{N,2t/\beta}\right)_{t\geq 0} \xrightarrow{N\to\infty} \left(\mu^{\mathrm{Geo}}_{\hat{b}_0,\hat{b}_1,t}\right)_{t\geq 0} \quad \text{in } C\left([0,\infty), M^1([0,\infty))\right) \text{ a.s.}$$

*The moments $m_n(t) := \int x^n \, d\mu^{\mathrm{Geo}}_{\hat{b}_0,\hat{b}_1,t}$ satisfy*

$$\frac{d}{dt} m_n(t) = n\left[\hat{b}_1 m_n(t) + \hat{b}_0 m_{n-1} + \frac{1}{2}\sum_{k=0}^{n} m_k(t) m_{n-k}(t)\right], \quad n \in \mathbb{N}. \tag{5.3.5}$$

*Moreover:*

(a) *If $\mu = \mu^{\mathrm{Geo}}_{\hat{b}_0,\hat{b}_1,0}$ is compactly supported, so is $\mu^{\mathrm{Geo}}_{\hat{b}_0,\hat{b}_1,t}$ for all $t > 0$, and $\mu^{\mathrm{Geo}}_{\hat{b}_0,\hat{b}_1,2t}$ equals the distribution of*

$$x_t := e^{(2\hat{b}_1+1)t} g_t^* \left(\hat{x}_0 + 2\hat{b}_0 \int_0^t e^{-(2\hat{b}_1+1)s} g_s^{-*} g_s^{-1} \, ds\right) g_t, \quad t \geq 0,$$

*where $(g_t)_{t\geq 0}$ is a free multiplicative Brownian motion driven by a circular Brownian motion $(c_t)_{t\geq 0}$ and with start in $g_0 = \mathbf{1}$, and $\hat{x}_0$ is a non-negative operator with distribution $\mu$ which is free from $(g_t)_{t\geq 0}$. $(x_t)_{t\geq 0}$ is solution to the free SDE*

$$dx_t = dc_t^* \, x_t + x_t \, dc_t + \left[(2\hat{b}_1+1)x_t + (2\hat{b}_0 + \tau(x_t))\mathbf{1}\right] dt. \tag{5.3.6}$$

(b) *In the case $\hat{b}_0 = 0$ we have*

$$\mu^{\mathrm{Geo}}_{0,\hat{b}_1,t} = D_{e^{(1/2+\hat{b}_1)t}}\left(\mu \boxtimes \nu_t\right) = D_{e^{(1/2+\hat{b}_1)t}}\left(\mu \boxtimes \exp\left(\mu_{sc,2\sqrt{t}} \boxplus \mathrm{Unif}_{[-t/2,t/2]}\right)\right), \quad t \geq 0,$$

*where $\nu_t$ is the distribution of a free positive multiplicative Brownian motion with start in $\mathbf{1}$ at time $t$.*

(c) *In the case* $\hat{b}_0 > 0$ *and* $\hat{b}_1 < -1$ *it holds* $\lim_{t\to\infty} \mu^{\mathrm{Geo}}_{\hat{b}_0,\hat{b}_1,t} = f(\mu_{\mathrm{MP},-2(\hat{b}_1+1)+1,1/(2\hat{b}_0)})$ *where* $f(x) = x^{-1}$. *If additionally* $\mu = \mu^{\mathrm{Geo}}_{\hat{b}_0,\hat{b}_1,0} = f(\mu_{\mathrm{MP},-2(\hat{b}_1+1)+1,1/(2\hat{b}_0)})$ *then* $t \mapsto \mu^{\mathrm{Geo}}_{\hat{b}_0,\hat{b}_1,t}$ *is constant.*

*Proof.* The statement about convergence of the empirical measures and the formula for the limit moments follow from Theorem 4.2.2. To see (a), first note that for some circular Brownian motion $(c_t)_{t\geq 0}$ we have

$$\begin{aligned} d(e^{(\hat{b}_1+1/2)t} g_t) &= e^{(\hat{b}_1+1/2)t} g_t \, dc_t + (\hat{b}_1 + 1/2) e^{(\hat{b}_1+1/2)t} g_t dt \,, \\ d(e^{(\hat{b}_1+1/2)t} g_t^* &= dc_t^* \, e^{(\hat{b}_1+1/2)t} g_t^* + (\hat{b}_1 + 1/2) e^{(\hat{b}_1+1/2)t} g_t^* dt \,. \end{aligned}$$

Applying the free Itô product rule 2.5.10 yields (5.3.6). Using the traced Itô formula 2.5.11, we get

$$\begin{aligned} \frac{d}{dt}\tau(x_t^n) =& n \Big[ (2\hat{b}_1 + 1)\tau(x_t^n) + (2\hat{b}_0 + \tau(x_t))\tau(x_t^{n-1}) \\ &+ \sum_{k=0}^{n-2} (\tau \otimes \tau^{\mathrm{op}}) \left( (x_t \otimes \mathbf{1}) \cdot (x_t^k \otimes x_t^{n-2-k}) \cdot (x_t \otimes \mathbf{1}) \right) \Big] \\ =& n \left[ (2\hat{b}_1 + 1)\tau(x_t^n) + (2\hat{b}_0 + \tau(x_t))\tau(x_t^{n-1}) + \sum_{k=0}^{n-2} \tau(x_t^{k+2})\tau(x_t^{n-2-k}) \right] \\ =& n \left[ 2\hat{b}_1 \tau(x_t^n) + 2\hat{b}_0 \tau(x_t^{n-1}) + \sum_{k=0}^{n} \tau(x_t^k)\tau(x_t^{n-k}) \right] . \end{aligned}$$

As $\tau(\hat{x}_0^n) = \int x^n \, d\mu_0(x) = m_n(0)$ for all $n \in \mathbb{N}_0$, we have shown that $\tau(x_t^n) = m_n(2t)$ for all $t \geq 0$, $n \in \mathbb{N}$. This implies the claim in (a).
To show (b), we now assume $\hat{b}_0 = 0$. We simplify our notation by writing $\mu_t := \mu^{\mathrm{Geo}}_{\hat{b}_0,\hat{b}_1,t}$, $t \geq 0$. Note that in the case that $\mu = \mu_0$ is compactly supported, the identity $\mu_t = D_{e^{(1/2+\hat{b}_1)t}}(\mu \boxtimes \nu_t)$ can be deduced from (a) and Lemma 5.3.3 (a) since $g_t g_t^*$ and $g_t^* g_t$ have the same distribution. For general $\mu_0$ we can argue as follows: First note that we can assume w.l.o.g. that $\mu_0 \in M^1([0,\infty)) \setminus \{\delta_0\}$. Now fix some $T > 0$. Set $G(t,z) := G_{\mu_t}(z)$. Then by Corollary 4.2.5 we have

$$\partial_t G(t,z) = -\hat{b}_1 (G(t,z) + z\partial_z G(t,z)) - zG^2(t,z) - z^2 G(t,z)\partial_z G(t,z) \,. \tag{5.3.7}$$

We will transform the PDE (5.3.7) into a PDE for the corresponding S-transforms. To this end set $\psi(t,z) := \psi_{\mu_t}(z) = z^{-1}G(t,z^{-1}) - 1$. Then, using $\partial_z \psi(t,z) = -z^{-2}G(t,z^{-1}) - z^{-3}\partial_y G_z(t,y)|_{y=z^{-1}}$, we get

$$\partial_t \psi(t,z) = z^{-1}\partial_t G(t,z^{-1}) = z(1 + \hat{b}_1 + \psi(t,z))\partial_z \psi(t,z) \,. \tag{5.3.8}$$

Now we will relate this equation to the S-transform $S(t,z) := S_{\mu_t}(z)$. Using the identity $z = S(t,\psi(t,z))\psi(t,z)/(1+\psi(t,z))$, we can deduce

$$\partial_t S(t,\psi(t,z)) = -\frac{1+\psi(t,z)}{\psi(t,z)}\frac{\partial_t\psi(t,z)}{\partial_z\psi(t,z)} = -(1+\hat{b}_1+\psi(t,z))S(t,\psi(t,z))\,.$$

Using $m_1(t) = e^{(\hat{b}_1+2)t}m_1(0)$, we see that $\delta_0$ is not an element of the set $\{\mu_t\colon t\in[0,T]\}$. Moreover, the latter set is closed in the topology of weak convergence. Thus, by the proof of Proposition 6.5 in [16], there exists a common domain $\Omega\subset\mathbb{C}$ where the functions $S(t,\cdot)$, $t\in[0,T]$, are defined. Using the univalence of $\psi_{\mu_t}|_{i\mathbb{C}^+}$, we get $\partial_t S(t,z) = -(1+\hat{b}_1+z)S(t,z)$ for all $t\in[0,T]$, $z\in\Omega$. It follows that

$$S(t,z) = S(0,z)e^{-(1+\hat{b}_1+z)t} = S_\mu(z)S_{\delta_{f(t)}}(z)S_{\nu_t}(z) = S_{D_{f(t)}(\mu\boxtimes\nu_t)}(z)\,,\quad t\in[0,T],\quad z\in\Omega\,,$$

where we set $f(t) := e^{(1/2+\hat{b}_1)t}$, and used $S_{\delta_{f(t)}}(z) = 1/f(t)$ as well as $S_{\nu_t}(z) = e^{-(1+2z)t/2}$ (see Lemma 5.3.3 (c)). Using analytic continuation and the fact that the $\psi$-transform, and hence also the $S$-transform, uniquely determine a probability measure, we deduce $\mu_t = D_{e^{(1/2+\hat{b}_1)t}}(\mu\boxtimes\nu_t)$. The second equality in the centered formula of (b) is just an application of equation (5.3.4).
It remains to show (c). So assume that $\hat{b}_1<-1$ and $\hat{b}_0>0$. For each $N$ let $x^N\in C_N^B$ be the stationary solution to the system of ODEs (cf. Lemma 3.4.3):

$$\frac{d}{dt}x_{i,t}^N = \left(\hat{b}_1+\frac{N-1}{N}\right)x_{i,t}^N + \hat{b}_0 + \frac{1}{N}\sum_{j\colon j\neq i}\frac{x_{i,t}^N x_{j,t}^N}{x_{i,t}^N - x_{j,t}^N}\,,\quad i\in\{1,\dots,N\}\,. \tag{5.3.9}$$

By the referenced Lemma, we know that $((x_N^N)^{-1},\dots,(x_1^N)^{-1})$ equals the vector of ordered zeros of the rescaled Laguerre polynomial $x\mapsto L_N^{(-(2(N\hat{b}_1+N-1)+1))}(2\hat{b}_0 Nx)$. By [37] Theorem 3.1, it thus holds

$$\frac{1}{N}\sum_{i=1}^N \delta_{x_i^N}\xrightarrow{N\to\infty} f\left(D_{1/(2\hat{b}_0)}\left(\mu_{\mathrm{MP},-2(\hat{b}_1+1)+1,1}\right)\right) = f\left(\mu_{\mathrm{MP},-2(\hat{b}_1+1)+1,1/(2\hat{b}_0)}\right)\,,$$

where $f(x)=x^{-1}$. On the other hand, by applying Theorem 4.2.2 to the sequence of stationary solutions $(x^N)_{N\in\mathbb{N}}$ to (5.3.9), we see that the moments of $f\left(\mu_{\mathrm{MP},-2(\hat{b}_1+1)+1,1/(2\hat{b}_0)}\right)$ are the stationary solution to (5.3.5). To rephrase this, so far we have shown that $t\mapsto\mu^{\mathrm{Geo}}_{\hat{b}_1,\hat{b}_0,t}$ is constant if $\mu^{\mathrm{Geo}}_{\hat{b}_1,\hat{b}_0,0} = f(\mu_{\mathrm{MP},-2(\hat{b}_1+1)+1,1/(2\hat{b}_0)})$. Now, for general $\mu^{\mathrm{Geo}}_{\hat{b}_1,\hat{b}_0,0}$ we can inductively show, using (5.3.5), that $\lim_{t\to\infty} m_n(t)$ exists and does not depend on $m_n(0)$. Hence this limit must coincide with the constant solution which is the $n$-th moment of $f(\mu_{\mathrm{MP},-2(\hat{b}_1+1)+1,1/(2\hat{b}_0)})$. As the latter measure is compactly supported and thus determined by its moments, we have $\lim_{t\to\infty}\mu^{\mathrm{Geo}}_{\hat{b}_1,\hat{b}_0,t} = f(\mu_{\mathrm{MP},-2(\hat{b}_1+1)+1,1/(2\hat{b}_0)})$ by the moment convergence theorem. □

**Remark 5.3.5.** The statement in Theorem 5.3.4 (a) is the free probability version of Lemma 22 in [88]. As Itô's formulas for free and matrix stochastic calculus are structurally the same (see e.g. [80]), it is not surprising that this result carries over to the large $N$ limit. In [88] a matrix generalization of the Dufresne identity is proven, which in turn inspired the paper [7] which states a matrix generalization of Bougerol's identity. For the free probability version of this latter result see Section 5.5.

## 5.4 Global limit of (compact) Jacobi processes

Let $p_N, q_N \geq N-1$, $N \in \mathbb{N}$, and assume that the limits $\lim_{N\to\infty} p_N/N = \hat{p}$ and $\lim_{N\to\infty} q_N/N = \hat{q}$ exist. We aim to apply Theorem 4.2.2 to the case of a sequence of Jacobi processes rescaled in time by the factor $1/N$ which thus arise as non-colliding solutions to

$$\begin{aligned} dX_{i,t}^N =& \sqrt{2X_{i,t}^N(1-X_{i,t}^N)/N}\, dB_{i,t} \\ &+ \frac{\beta}{2N}\left[p_N - (p_N+q_N)X_{i,t}^N + \sum_{j:\, j\neq i} \frac{X_{i,t}^N + X_{j,t}^N - 2X_{i,t}^N X_{j,t}^N}{X_{i,t}^N - X_{i,t}^N}\right] dt\,, \end{aligned} \tag{5.4.1}$$

$i \in \{1,\ldots,N\}$, $X_0^N = \hat{x}_0^N \in C_N^{BC}$. For our choice of parameters there exist such non-colliding solutions by Lemma 3.5.2. The corresponding parameters appearing in Theorem 4.2.2 are here given by $a_0^N = 0$, $a_1^N = 1/N$, $a_2^N = -1/N$, $b_1^N = -(p_N+q_N-2(N-1))/N$, $b_0^N = (p_N-(N-1))/N$.

Further, we assume that the limit $\frac{1}{N}\sum_{i=1}^N \delta_{\hat{x}_{i,0}^N} \xrightarrow{N\to\infty} \mu$ exists for some $\mu \in M^1([0,1])$, where the convergence is in moments. As $\mu$ is compactly supported, clearly the moments $(m_n)_{n\in\mathbb{N}}$ of $\mu$ satisfy the bounds (4.2.5) for $\gamma = 1$.

For $\beta = 2$, and hence by Theorem 4.2.2 for all $\beta \geq 1$, the limit of empirical measures in this setting has been described in [30]. We give a concise overview of this description as some aspects carry over to the non-compact Jacobi setting, see Section 5.6.

**Definition 5.4.1.** Let $(\mathcal{A}, (\mathcal{A}_t)_{t\geq 0}, \tau)$ be a filtered $W^*$-probability space.

(i) Let $(x_t)_{t\geq 0}$ be an $(\mathcal{A}_t)_{t\geq 0}$ semi-circular Brownian motion and let $\hat{u}_0 \in \mathcal{A}$ be a unitary operator which is free from $(x_t)_{t\geq 0}$. We call the unique solution $(u_t)_{t\geq 0}$ to the free SDE

$$du_t = i\, dx_t\, u_t - \frac{1}{2} u_t\, dt\,, \quad u_0 = \hat{u}_0$$

*free unitary Brownian motion with start in* $\hat{u}_0$ (here $i$ denotes the imaginary unit).

(ii) Let $(u_t)_{t\geq 0}$ be a unitary Brownian motion with start in $\hat{u}_0$. Let $\theta \in (0,1)$ and $0 < \lambda < 1/\theta$. Let $P, Q \in \mathcal{A}$ be two projections such that $\{P, Q\}$ is free from $(u_t)_{t\geq 0}$ and it holds

$$\tau(Q) = \theta,\ \tau(P) = \lambda\theta \ \text{ and } \ PQ = QP = \begin{cases} P\,, & \lambda \leq 1\,, \\ Q\,, & \lambda > 1\,. \end{cases}$$

We call the process

$$j_t := Pu_tQu_t^*P\,,\quad t\geq 0\,,$$

*free Jacobi process with parameters* $(\lambda,\theta)$ *and start in* $P\hat{u}_0Q\hat{u}_0^*P$.

(iii) Let $p,q\geq 1$. The *Wachter distribution* $\mu_{\mathrm{W},p,q}$ is the probability measure on $(x_-,x_+)$ with density

$$\frac{(p+q)\sqrt{(x_+-x)(x-x_-)}}{2\pi x(1-x)}\,dx\ \text{ where }\ x_\pm=\left(\frac{\sqrt{q}\pm\sqrt{p(p+q-1)}}{p+q}\right)^2\in[0,1]\,.$$

**Remark 5.4.2.** (i) To our knowledge the free unitary Brownian motion as a solution to a free SDE has been first considered in [19]. Its moments are a transformation of the corresponding moments of the free positive multiplicative Brownian motion under the parameter change $t\mapsto -t$, see [57] appendix A and [20] Section 4.2. Moreover, in the case of start in $\mathbf{1}$, a description of the density of its distribution can be found in the latter reference.

(ii) The free Jacobi process has been introduced in [30] as a free analogue of the matrix Jacobi process, which in turn first appeared in [39]. The free Jacobi process was further studied in the papers [35, 36, 32, 34, 54].

(iii) In the special case $p=q$, the Wachter distribution is the image measure under the map $x\mapsto(1+x/(2p))/2$ of a probability measure known as Kesten-McKay distribution with parameter $2p$, see [40]. Further specializing to $p=q=1$ one gets the arcsine distribution on $(0,1)$.

Let $\mu_{N,t}=\frac{1}{N}\sum_{i=1}^N\delta_{X_{i,t}^N}$ where $(X_{i,t}^N)_{t\geq 0}$ is as at the beginning of this section. Then it holds:

**Theorem 5.4.3.** *There exists* $(\mu_t)_{t\geq 0}\in C([1,\infty),M^1(\mathbb{R}))$ *such that*

$$\left(\mu_{N,2t/\beta}\right)_{t\geq 0}\xrightarrow{N\to\infty}(\mu_t)_{t\geq 0}\quad \textit{in}\ \ C([0,\infty),M^1([0,1]))\ \ \textit{a.s.}$$

*For all* $n\in\mathbb{N}$, *the moments* $m_n(t)=\int x^n\,d\mu_t(x)$ *satisfy*

$$\begin{aligned}&\frac{d}{dt}m_n(t)\\ =&n\left[-(\hat{p}+\hat{q}-2)m_n(t)+(\hat{p}-1)m_{n-1}(t)-\sum_{k=0}^{n}m_k(t)m_{n-k}(t)+\sum_{k=0}^{n-1}m_k(t)m_{n-1-k}(t)\right].\end{aligned}$$

*It holds* $\lim_{t\to\infty}\mu_t=\mu_{\mathrm{W},\hat{p},\hat{q}}$. *If* $\mu=\mu_{\mathrm{W},\hat{p},\hat{q}}$, *then* $t\mapsto\mu_t$ *is constant.*
*If one assumes that there exists a free Jacobi process* $(j_t)_{t\geq 0}$ *with parameters* $\lambda=1/\hat{p}$ *and* $\theta=\hat{p}/(\hat{p}+\hat{q})$ *such that* $j_0$ *has distribution* $\mu$, *then* $\mu_{t/(\hat{p}+\hat{q})}$ *equals the distribution of* $j_t$

*viewed in the compressed space* $(P\mathcal{A}P, \frac{1}{\tau(P)}\tau(\cdot))$.
*If* $\hat{p} = \hat{q} = 1$, *then*

$$\mu_t = f(\tilde{\mu} \boxtimes \nu_{4t}), \quad t \geq 0,$$

*where*

*(i)* $\nu_t$ *is the distribution of the free unitary Brownian motion with start in* $\mathbf{1}$ *at time* $t$,

*(ii)* $f\colon \mathbb{T} \to [0,1]$, $z \mapsto \frac{1}{2}(\mathrm{Re}(z)+1)$,

*(iii)* $\tilde{\mu}$ *is the unique element of* $M^1(\mathbb{T})$ *which is invariant under the map* $z \mapsto \overline{z}$ *and whose image measure under* $f$ *equals* $\mu$.

*Proof.* The statement about convergence of the empirical measures and the ODEs for the limit moments follows from Theorem 4.2.2. All other statements, except for the $\hat{p} = \hat{q} = 1$ case, are known by [30]. The statement concerning the $\hat{p} = \hat{q} = 1$ case follows directly from [35] Corollary 2. □

We point out that, in the $\beta = 2$ case, one can improve the type of convergence considered in the preceding theorem. As mentioned in Section 3.5, for $\beta = 2$ there exist matrix models $(M_t^N)_{t\geq 0}$ such that the respective eigenvalues form $N$-dimensional Jacobi processes. In [28] Section 5, the convergence in operator norm of non-commutative polynomials $P$ of $M_{t_1}^N, \ldots, M_{t_n}^N$ to the respective evaluation of $P$ at the time marginals of the free Jacobi process is shown.

## 5.5 Global limit of Hua-Pickrell diffusions

Let $(s^N)_{N\in\mathbb{N}} \subset \mathbb{C}$ such that $\lim_{N\to\infty} s^N/N =: \hat{s} \in \mathbb{C}$ exists. We apply Theorem 4.2.2 to the case of a sequence of Hua-Pickrell diffusions rescaled in time by the factor $1/N$, which thus arise as solutions to

$$dX_{i,t}^N = \sqrt{2(1+(X_{i,t}^N)^2)/N}\, dB_{i,t} + \frac{\beta}{N}\left[\mathrm{Im}(s^N) - \mathrm{Re}(s^N)X_{i,t}^N + \sum_{j\colon j\neq i} \frac{X_{i,t}^N X_{j,t}^N + 1}{X_{i,t}^N - X_{j,t}^N}\right] dt, \tag{5.5.1}$$

$i \in \{1,\ldots,N\}$, $X_0^N = \hat{x}_0^N \in C_N^A$. The corresponding parameters appearing in Theorem 4.2.2 are here thus given by $a_0^N = 1/N$, $a_1^N = 0$, $a_2^N = 1/N$, $b_1^N = -2(\mathrm{Re}(s^N)+(N-1))/N$, $b_0^N = 2\,\mathrm{Im}(s^N)/N$. We assume that the limit $\frac{1}{N}\sum_{i=1}^N \delta_{\hat{x}_{i,0}^N} \xrightarrow{N\to\infty} \mu$ exists for some $\mu \in M^1(\mathbb{R})$, where the convergence is in moments and the moments $(m_n)_{n\in\mathbb{N}}$ of $\mu$ satisfy the bounds (4.2.5) for some $\gamma > 0$. To describe the limit of the empirical measures $\mu_{N,t} = \frac{1}{N}\sum_{i=1}^N \delta_{X_{i,t}^N}$, we introduce the following free Itô process:

**Definition 5.5.1.** Let $s \in \mathbb{C}$. Let $(c_t)_{t\geq 0}$ be a circular Brownian motion on a $W^*$-probability space $(\mathcal{A},\tau)$ as in Section 2.5, and let $\hat{y}_0 \in \mathcal{A}$ be some self-adjoint bounded

operator freely independent of $(c_t)_{t\geq 0}$. We call a solution to the free stochastic differential equation

$$dy_t = dc_t\,\sqrt{\mathbf{1}+y_t^2} + \sqrt{\mathbf{1}+y_t^2}\,dc_t^* + [(-1-2\,\mathrm{Re}(s))y_t + (2\,\mathrm{Im}(s)+\tau(y_t))\mathbf{1}]\,dt\,,\ y_0=\hat{y}_0\,, \tag{5.5.2}$$

*free Hua-Pickrell process with parameter $s$ (driven by $(c_t)_{t\geq 0}$)* (see Section 7 of [9] for the matrix version).
Let $s\in\mathbb{C}$ with $\mathrm{Re}(s)>0$. We call the measure on $\mathbb{R}$ with density

$$\mu_{\hat{\mathrm{HP}},\mathrm{Re}(s),\mathrm{Im}(s)}(dx) := \begin{cases} \frac{\mathrm{Re}(s)\sqrt{(x-x_-)(x_+-x)}}{\pi(1+x^2)}dx\,, & x\in(x_-,x_+)\,,\\ 0\,, & \text{else}\,, \end{cases}$$

*equilibrium Hua-Pickrell measure with parameter $s$* where

$$x_\pm = \frac{1}{(\mathrm{Re}(s))^2}\left(-\mathrm{Im}(s)(\mathrm{Re}(s)+1) \pm \sqrt{(2\,\mathrm{Re}(s)+1)(\mathrm{Re}(s)^2+\mathrm{Im}(s)^2)}\right)\,.$$

**Remark 5.5.2.** The measures $\mu_{\hat{\mathrm{HP}},\mathrm{Re}(s),\mathrm{Im}(s)}$ are known to be the weak large $N$ limit of empirical measures of the zeros of Pseudo-Jacobi polynomials ([61] Theorem 2.2) and empirical measures of Hua-Pickrell/Cauchy ensembles ([48] Section 3.2.2).

**Lemma 5.5.3.** *Let $s\in\mathbb{C}$, let $(c_t)_{t\geq 0}$ be a circular Brownian motion, and let $\hat{y}_0$ be a self adjoint bounded operator free from $(c_t)_{t\geq 0}$. Then the free SDE* (5.5.2) *has a unique solution.*

*Proof.* This essentially follows from [25] Proposition A.1. Here, we use that $x\mapsto\sqrt{1+x^2}$ and affine linear maps are in $C^2(\mathbb{R})$ and hence locally operator Lipschitz (see [22] Section 2.3). Further, for self-adjoint $x\in\mathcal{A}$ we have $\mathbf{1}+x^2\leq(\mathbf{1}+|x|)^2$. Hence, by [98] Theorem 3.1.12, it holds $\sqrt{\mathbf{1}+x^2}\leq\mathbf{1}+|x|$. This in turn implies the growth bound $||\sqrt{\mathbf{1}+x^2}||^2\leq 2(1+||x||^2)$. Lastly, note that the map $\mathcal{A}\to\mathcal{A}$, $A\to\tau(A)\mathbf{1}$ does not match the assumption of the cited Proposition in the strict sense. However, it is clear from the corresponding proof that it suffices to show

$$||\tau(A)\mathbf{1}-\tau(B)\mathbf{1}||\leq||A-B|| \ \text{ and } \ ||\tau(A)\mathbf{1}||^2\leq 1+||A||^2 \ \text{ for all } \ A,B\in\mathcal{A}\,.$$

These inequalities directly follow from the general von Neumann algebra estimate for tracial states $|\tau(A)|\leq||A||$ for all $A\in\mathcal{A}$ (see e.g. [97] Lemma 2.1.3). □

**Proposition 5.5.4.** *Let $(y_t)_{t\geq 0}$ be a free Hua-Pickrell process with parameter $s\in\mathbb{C}$. Then the moments $m_n(t):=\tau(y_t^n)$ satisfy the recurrence*

$$\begin{aligned}&\frac{d}{dt}m_n(t)\\ =&n\left[-2(\mathrm{Re}(s)+1)m_n+2\,\mathrm{Im}(s)m_{n-1}(t)+\sum_{k=0}^{n}m_k(t)m_{n-k}(t)+\sum_{k=0}^{n-2}m_k(t)m_{n-2-k}(t)\right]\,.\end{aligned} \tag{5.5.3}$$

*Proof.* By the traced Itô formula 2.5.11, we have

$$\begin{aligned} d\tau(y_t^n) =& n \Bigg[ -(1+2\operatorname{Re}(s))\tau(y_t^n) + (2\operatorname{Im}(s)+\tau(y_t))\tau(y_t^{n-1}) \\ &+ \sum_{k=0}^{n-2} (\tau\otimes\tau^{op}) \left( \left( \mathbf{1}\otimes\sqrt{\mathbf{1}+y_t^2} \right) \cdot \left( y_t^k \otimes y_t^{n-2-k} \right) \cdot \left( \mathbf{1}\otimes\sqrt{\mathbf{1}\otimes y_t^2} \right) \right) \Bigg] dt \\ =& n \Bigg[ -(1+\operatorname{Re}(s))\tau(y_t^n) + (2\operatorname{Im}(s)+\tau(y_t))\tau(y_t^{n-1}) \\ &+ \sum_{k=0}^{n-2} \tau(y_t^k) \left( \tau(y_t^{n-k}) + \tau(y_t^{n-2-k}) \right) \Bigg] dt \,. \end{aligned}$$

The claim follows from

$$\sum_{k=0}^{n-2} \tau(y_t^k)\tau(y_t^{n-k}) = \sum_{k=0}^{n} \tau(y_t^k)\tau(y_t^{n-k}) - \tau(y_t^n) - \tau(y_t)\tau(y_t^{n-1}) \,.$$

□

Now, we can describe the limit of empirical measures of the Hua-Pickrell diffusions in (5.5.1) as follows:

**Theorem 5.5.5.** *There exists* $(\mu_{\hat{s},t}^{\mathrm{HP}})_{t\geq 0} \in C([0,\infty), M^1(\mathbb{R}))$ *such that*

$$\left(\mu_{N,2t/\beta}\right)_{t\geq 0} \xrightarrow{N\to\infty} \left(\mu_{\hat{s},t}^{\mathrm{HP}}\right)_{t\geq 0} \quad \text{in } C\left([0,\infty), M^1(\mathbb{R})\right) \quad a.s.$$

*For all* $n \in \mathbb{N}$, *the moments* $m_n(t) = \int x^n \, d\mu_{\hat{s},t}^{\mathrm{HP}}(x)$ *satisfy* (5.5.3) *for parameter* $s := \hat{s}$. *Moreover:*

*(i) If* $\operatorname{Re}(\hat{s}) > 0$, *then* $\mu_{\hat{s},t}^{\mathrm{HP}} \xrightarrow{t\to\infty} \mu_{\hat{\mathrm{HP}},\operatorname{Re}(\hat{s}),\operatorname{Im}(\hat{s})}$ *weakly. If, additionally* $\mu = \mu_{\hat{s},0}^{\mathrm{HP}} = \mu_{\hat{\mathrm{HP}},\operatorname{Re}(\hat{s}),\operatorname{Im}(\hat{s})}$, *then* $t \mapsto \mu_{\hat{s},t}^{\mathrm{HP}}$ *is constant.*

*(ii) If* $\mu = \mu_{\hat{s},0}^{\mathrm{HP}}$ *is compactly supported,* $\mu_{\hat{s},t}^{\mathrm{HP}}$ *is equal to the compactly supported measure* $\mu_{y_t}$, *where* $(y_t)_{t\geq 0}$ *is a free Hua-Pickrell diffusion with parameter* $\hat{s}$ *and start in an operator* $\hat{y}_0$ *with* $\mu_{\hat{y}_0} = \mu$.

*Proof.* As mentioned above, the statement about convergence of the empirical measures and the ODEs for the limit moments follows from Theorem 4.2.2. As solutions to (5.5.3) with the same initial condition agree, point (ii) is also clear by Proposition 5.5.4.
Next, we show (i), so assume $\operatorname{Re}(s) > 0$. If $\mu = \mu_{\hat{\mathrm{HP}},\operatorname{Re}(\hat{s}),\operatorname{Im}(\hat{s})}$, then by the proof of Theorem 2.2 in [61] its Cauchy transform satisfies

$$(z^2+1)G_\mu^2(z) - 2((\operatorname{Re}(\hat{s})+1)z - \operatorname{Im}(\hat{s}))G_\mu(z) - (2\operatorname{Re}(\hat{s})+1) = 0 \,.$$

Hence, $(t,z) \mapsto G_\mu(z)$ is a solution to the PDE

$$\partial_t G(t,z) = -\partial_z \left[ (-2(\mathrm{Re}(\hat{s})+1)z + 2\,\mathrm{Im}(\hat{s}))\, G(t,z) + \left(z^2+1\right) G^2(t,z) \right] . \tag{5.5.4}$$

On the other hand, by Corollary 4.2.5, $G(t,z) := G_{\mu^{\mathrm{HP}}_{\hat{s},t}}(z)$ also is solution to this PDE. Thus, we have $\mu^{\mathrm{HP}}_{\hat{s},t} = \mu_{\hat{\mathrm{HP}},\mathrm{Re}(\hat{s}),\mathrm{Im}(\hat{s})}$ for all $t \geq 0$ in this case. It remains to discuss the weak convergence of $\mu^{\mathrm{HP}}_{\hat{s},t}$ for $t \to \infty$ for arbitrary starting conditions. Note that by (5.5.3) we have

$$\begin{aligned} m_n(t) =& e^{-2n\,\mathrm{Re}(\hat{s})t} \left( m_n(0) + \int_0^t e^{2n\,\mathrm{Re}(\hat{s})r} \left( 2\,\mathrm{Im}(\hat{s}) m_{n-1}(r) + \sum_{k=1}^{n-1} m_k(r) m_{n-k}(r) \right.\right. \\ & \left.\left. + \sum_{k=0}^{n-2} m_k(r) m_{n-k}(r) \right) dr \right) . \end{aligned}$$

Using induction on $n$, one sees that $\lim_{t\to\infty} m_n(t)$ exists and that this limit does not depend on $m_n(0)$. Hence, the limit must agree with the $n$-th moment of $\mu_{\hat{\mathrm{HP}},\mathrm{Re}(\hat{s}),\mathrm{Im}(\hat{s})}$ which is the constant solution. The claim follows from the moment convergence theorem. □

In the remainder of this section, we give further descriptions of free Hua-Pickrell diffusions and their distributions. We will see that the free positive multiplicative Brownian motion, which was introduced in Section 5.3, plays a crucial role. We start with the free analogue of Proposition 2.3 in [7], which states that the solution of the matrix version of equation (5.5.2) can be constructed from matrix geometric Brownian motions. As previously mentioned, the Itô formulas for matrix processes and free processes are structurally the same (see e.g. [22, 80]). Hence, it is not surprising that the referenced Proposition carries over to the free probability setting.

**Theorem 5.5.6.** *Let $a, b \in \mathbb{R}$. Let $(c_t = (c_{1,t}, c_{2,t}))_{t\geq 0}$ be a two-dimensional circular Brownian motion as in Section 2.5. Let $\hat{y}_0 \in \mathcal{A}$ be freely independent of $(c_t)_{t\geq 0}$. Let $(g_t)_{t\geq 0}$ be a free multiplicative Brownian motion driven by $(c_{1,t})_{t\geq 0}$ with start in $\mathbf{1}$, and set $g_t^{(a)} := e^{at} g_t$, $t \geq 0$. Then*

$$y_t := \left(g_t^{(a)}\right)^{-1} \left[ \hat{y}_0 + \int_0^t g_s^{(a)}\, d\left(c_{2,s} + c_{2,s}^* + bs\mathbf{1}\right) \left(g_s^{(a)}\right)^* \right] \left(g_t^{(a)}\right)^{-*} , \quad t \geq 0 ,$$

*is a free Hua-Pickrell diffusion driven by $(\tilde{c}_t)_{t\geq 0}$ with start in $\hat{y}_0$ and parameter $a - 1/2 + ib/2$ where $\tilde{c}_t := -\int_0^t dc_{1,s}\, y_s(\mathbf{1}+y_s^2)^{-1/2} + \int_0^t dc_{2,s}\, (\mathbf{1}+y_s^2)^{-1/2}$, $t \geq 0$, is a circular Brownian motion.*

*Proof.* Note that as $dg_t = g_t\, dc_{1,t}$ and $dg_t^{-1} = -dc_{1,t}\, g_t^{-1}$ we have

$$dg_t^{(a)} = g_t^{(a)}\, dc_{1,t} + a g_t^{(a)} dt , \quad d\left(g_t^{(a)}\right)^{-1} = -dc_{1,t}\, \left(g_t^{(a)}\right)^{-1} - a \left(g_t^{(a)}\right)^{-1} dt .$$

Using the free Itô product rule 2.5.10 twice, we get

$$\begin{aligned} dy_t =& - dc_{1,t}\, y_t + dc_{2,t} + dc_{2,t}^* - y_t\, dc_{1,t}^* + [-2ay_t + (b + \tau(y_t))\mathbf{1}]\, dt \\ =& d\tilde{c}_t\, \sqrt{\mathbf{1} + y_t^2} + \sqrt{\mathbf{1} + y_t^2}\, d\tilde{c}_t^* + [-2ay_t + (b + \tau(y_t))\mathbf{1}]\, dt\,. \end{aligned}$$

Thus, it remains to show that $(\tilde{c}_t)_{t\geq 0}$ is a circular Brownian motion. For this recall that $(\tilde{c}_t)_{t\geq 0}$ is a circular Brownian motion if $(z_t = (z_{1,t}, z_{2,t}))_{t\geq 0}$ with $z_{1,t} := (\tilde{c}_t + \tilde{c}_t^*)/\sqrt{2}$ and $z_{2,t} := (\tilde{c}_t - \tilde{c}_t^*)/(i\sqrt{2})$, $t \geq 0$, is a two-dimensional semi-circular Brownian motion. In turn, it suffices to verify the conditions of the free Lévy characterization Theorem 2.5.4 for $(z_t)_{t\geq 0}$. That each component of $z_t$ is a bounded operator and a free martingale with $z_{j,0} = \mathbf{0}$, $j \in \{1, 2\}$, is clear as it is the sum of free stochastic integrals w.r.t. circular Brownian motions. Next, we show condition (ii) in Theorem 2.5.4. Note that, as $(z_t)_{t\geq 0}$ is a free martingale, it suffices to verify

$$\tau(A(z_{j,t} - z_{j,s})B(z_{k,t} - z_{k,s})) = \tau(A)\tau(B)\mathbb{1}_{j=k} \cdot (t - s)\,, \tag{5.5.5}$$

for all $0 \leq s \leq t$, $A, B \in \mathcal{A}_s$ and $j, k \in \{1, 2\}$. So let $A, B \in \mathcal{A}_s$, then

$$\begin{aligned} &\tau\left(A(z_{1,t} - z_{1,s})B(z_{1,t} - z_{1,s})\right) \\ =&\frac{1}{2}\tau\left[\left(\int_s^t A\left(-dc_{1,r}\, y_r + dc_{2,r}\right)(\mathbf{1} + y_r^2)^{-1/2} + \int_s^t A(\mathbf{1} + y_r^2)(-y_r\, dc_{1,r}^* + dc_{2,r}^*)\right)\right. \\ &\left.\cdot\left(\int_s^t B\left(-dc_{1,r}\, y_r + dc_{2,r}\right)(\mathbf{1} + y_r^2)^{-1/2} + \int_s^t B(\mathbf{1} + y_r^2)(-y_r\, dc_{1,r}^* + dc_{2,r}^*)\right)\right] \\ =&\frac{1}{2}\left(\int_s^t \big(\tau(A)\tau(y_r(\mathbf{1} + y_r^2)^{-1/2}B(\mathbf{1} + y_r^2)^{-1/2}y_r) + \tau(A)\tau((\mathbf{1} + y_r^2)^{-1/2}B(\mathbf{1} + y_r^2)^{-1/2})\right. \\ &\left.+\tau(A(\mathbf{1} + y_r^2)^{-1/2}y_r^2(\mathbf{1} + y_r^2)^{-1/2})\tau(B) + \tau(A(\mathbf{1} + y_r^2)^{-1})\tau(B)\big)\, dr\right) \\ =&\tau(A)\tau(B)(t - s)\,. \end{aligned} \tag{5.5.6}$$

One analogously shows the other cases in (5.5.5). Lastly, we show condition (i) in Theorem 2.5.4, i.e. the bound $\tau((z_{j,t} - z_{j,s})^4) \leq K(t - s)^2$ for all $0 \leq s \leq t$, $j \in \{1, 2\}$, and some positive $K$. For this fix $s \geq 0$ and set

$$\begin{aligned} \tilde{z}_{1,t} :=& \begin{Bmatrix} 0\,, & t \leq s \\ z_{1,t} - z_{1,s}\,, & t > s \end{Bmatrix} = \int_0^t \mathbb{1}_{[s,\infty)}(r) dz_{1,r} \\ =& -\frac{1}{\sqrt{2}}\int_0^t \mathbb{1}_{[s,\infty)}(r)\left(dc_{1,r}\, y_r(\mathbf{1} + y_r^2)^{-1/2} + (\mathbf{1} + y_r^2)^{-1/2}y_r\, dc_{1,r}^*\right) \\ &+\frac{1}{\sqrt{2}}\int_0^t \mathbb{1}_{[s,\infty)}(r)\left(dc_{2,r}\,(\mathbf{1} + y_r^2)^{-1/2} + (\mathbf{1} + y_r^2)^{-1/2}\, dc_{2,r}^*\right)\,. \end{aligned}$$

Now, by the traced Itô formula 2.5.11, we have

$$\begin{aligned}
&\tau(\tilde{z}_{1,t}^4)\\
=&2\sum_{k=0}^{2}\int_0^t (\tau\otimes\tau^{\mathrm{op}})\left((-\mathbf{1}\otimes(\mathbf{1}+y_r^2)^{-1/2}y_r)\cdot\left(\tilde{z}_{1,r}^k\otimes\tilde{z}_{1,r}^{2-k}\right)\cdot(-\mathbf{1}\otimes y_r(\mathbf{1}+y_r^2)^{-1/2})\right)\\
&\qquad\cdot\mathbb{1}_{[s,\infty)}(r)\,dr\\
&+2\sum_{k=0}^{2}\int_0^t (\tau\otimes\tau^{\mathrm{op}})\left((\mathbf{1}\otimes(\mathbf{1}+y_r^2)^{-1/2})\cdot\left(\tilde{z}_{1,r}^k\otimes\tilde{z}_{1,r}^{2-k}\right)\cdot(\mathbf{1}\otimes(\mathbf{1}+y_r^2)^{-1/2})\right)\mathbb{1}_{[s,\infty)}(r)\,dr\\
=&2\sum_{k=0}^{2}\int_0^t \tau(\tilde{z}_{1,r}^k)\tau(\tilde{z}_{1,r}^{2-k})\mathbb{1}_{[s,\infty)}(r)\,dr\,.
\end{aligned}$$

Note that the $k=1$ term vanishes as $\tau(\tilde{z}_{1,t})\equiv 0$. Now let $t>s$. By setting $A=B=\mathbf{1}$, we know by (5.5.6) that $\tau(\tilde{z}_{1,t}^2)=t-s$. Plugging this in, we get

$$\tau((z_{1,t}-z_{1,s})^4)=4\int_s^t(r-s)dr=2(t-s)^2\,.$$

Similarly, one also sees that $\tau((z_{2,t}-z_{2,s})^4)=2(t-s)^2$. This finishes the proof. □

By the previous Theorem, it seems unlikely that one can describe the measures $\mu_{\hat{s},t}^{\mathrm{HP}}$ appearing in Theorem 5.5.5 in a simple way using free convolutions for general $\hat{s}\in\mathbb{C}$. However, in the case that we have $\hat{s}=-1$ and $\mu_{\hat{s},0}^{\mathrm{HP}}$ is an even measure, we have such a description via the free positive multiplicative free Brownian motion:

**Theorem 5.5.7.** *Let $\nu\in M^1((0,\infty))$ be invariant under the map $x\mapsto x^{-1}$ and be determined by its moments. Let $f\colon(0,\infty)\to\mathbb{R}, f(x):=(x^{1/2}-x^{-1/2})/2$. Then it holds*

$$\mu_{-1,t}^{\mathrm{HP}}=f(\nu\boxtimes\nu_{8t})=f\left(\nu\boxtimes\exp\left(\mu_{\mathrm{sc},4\sqrt{2t}}\boxplus\mathrm{Unif}_{[-4t,4t]}\right)\right)\quad\text{for all } t\geq 0\,,$$

*where $\nu_t$ is the distribution of the free positive multiplicative Brownian motion with start in $\mathbf{1}$ at time $t$ (see Definition 5.3.1), and $\mu_{-1,t}^{\mathrm{HP}}$ is the limit of empirical measures of non-colliding Hua-Pickrell diffusions as in Theorem 5.5.5 in the case $\mu_{-1,0}^{\mathrm{HP}}=f(\nu)$.*

For the proof we will use the following result:

**Proposition 5.5.8.** *Let $f(x)=\frac{1}{2}\left(x^{1/2}-x^{-1/2}\right)$.*

*(a) $f$ understood as a map from $\{\nu\in M^1((0,\infty))\colon \nu$ is invariant under $x\mapsto 1/x\}$ to $\{\mu\in M^1(\mathbb{R})\colon \mu$ is invariant under $x\mapsto -x\}$ is a bijection.*

*(b) Let $\nu\in M^1((0,\infty))$ and let $\mu\in M^1(\mathbb{R})$ be even. Then it holds: $\mu=f(\nu)$ iff*

$$\psi_\nu(z)=\frac{1}{2}\left(\frac{1}{2}(z^{1/2}+z^{-1/2})G_\mu(-f(z))-1\right)\text{ for all } z\in\mathbb{C}\setminus[0,\infty)\,,\qquad(5.5.7)$$

*where $\cdot^{1/2}$ denotes the branch of the square root cut along the positive real axis which maps $-1$ to $i$; here $\psi_\nu$ and $G_\mu$ denote the $\psi$- and Cauchy-transform of $\nu$ and $\mu$, respectively; see Sections 2.4.1 & 2.4.2.*

*Proof.* $f\colon (0,\infty)\to\mathbb{R}$ is bijective since it is strictly monotone increasing with $\lim_{x\downarrow 0} f(x) = -\infty$ and $\lim_{x\uparrow\infty} = \infty$. This and the identity $f(1/x) = -f(x)$ show (a).
Let $z\in\mathbb{C}\setminus[0,\infty)$. Then we have $-f(z)\in\mathbb{C}\setminus\mathbb{R}$ since, by writing $z = re^{i\phi}$ for $r>0$ and $\phi\in(-\pi,0)\cup(0,\pi]$, we have $\operatorname{Im}(-f(z)) = -2r\sin(\phi/2)\neq 0$. Thus, $-f(z)$ is in the domain of $G_\mu$ so the r.h.s. of (5.5.7) is defined. To show (b), first assume that $\mu = f(\nu)$. Then

$$\begin{aligned}
&\frac{1}{2}\left(\frac{1}{2}(z^{1/2}+z^{-1/2})G_\mu\left(-f(z)\right)-1\right) = \frac{1}{2}\left(\int_{\mathbb{R}}\frac{1+z}{1-2xz^{1/2}-z}\,df(\nu)(x)-1\right)\\
=&\frac{1}{2}\left(\int_{(0,\infty)}\frac{1+z}{1-(xz)^{1/2}+(z/x)^{1/2}-z}\,d\nu(x)-1\right)\\
=&\frac{1}{2}\int_{(0,\infty)}\frac{2z+(xz)^{1/2}-(z/x)^{1/2}}{1-(xz)^{1/2}+(z/x)^{1/2}-z}\,d\nu(x)\\
=&\frac{1}{2}\left(\int_{(0,\infty)}\frac{(xz)^{1/2}}{1-(xz)^{1/2}}\,\nu(dx)+\int_{(0,\infty)}\frac{-(z/x)^{1/2}}{1+(z/x)^{1/2}}\,d\nu(x)\right)\\
=&\frac{1}{2}\left(\int_{(0,\infty)}\frac{(xz)^{1/2}}{1-(xz)^{1/2}}\,\nu(dx)+\int_{(0,\infty)}\frac{-(xz)^{1/2}}{1+(xz)^{1/2}}\,\nu(dx)\right)\\
=&\frac{1}{2}\int_{(0,\infty)}\frac{(xz)^{1/2}(1+(zx)^{1/2})-(zx)^{1/2}(1-(zx)^{1/2})}{1-xz}\,d\nu(x) = \psi_\nu(z)\,,
\end{aligned}$$

where we used the invariance of $\nu$ under inversion in the fifth equality.
To prove the inverse implication in (b), assume it holds (5.5.7). By the first part of the proof, we hence have

$$G_{f(\nu)}\left(-f(z)\right) = G_\mu\left(-f(z)\right)\quad\text{for all}\quad z\in\mathbb{C}\setminus[0,\infty)\,.$$

Replacing $z$ by its square, in particular we thus have

$$G_{f(\nu)}\left(\frac{1-z^2}{2z}\right) = G_\mu\left(\frac{1-z^2}{2z}\right)\quad\text{for all}\quad z\in\mathbb{C}\setminus\mathbb{R}\,.$$

We show that the range of the map $g\colon\mathbb{C}\setminus\mathbb{R}\to\mathbb{C}$, $g(x) = (1-z^2)/(2z)$, includes $\mathbb{C}\setminus\mathbb{R}$. So let $y\in\mathbb{C}\setminus\mathbb{R}$. Then $z_{1,2} := -y\pm(1+y^2)^{1/2}$ satisfies $g(z_j) = y$ for $j\in\{1,2\}$. Now, it must hold that $\operatorname{Im}(z_1)\neq 0$ or $\operatorname{Im}(z_2)\neq 0$ as otherwise $0 = \operatorname{Im}(z_1+z_2) = -2\operatorname{Im}(y)\neq 0$. Thus we have shown $G_{f(\nu)}(z) = G_\mu(z)$ for all $z\in\mathbb{C}\setminus\mathbb{R}$. This concludes the proof since the Cauchy transform uniquely characterizes a probability measure (see Theorem 2.4.7). □

*Proof of Theorem 5.5.7.* The second equality in the centered formula in the Lemma holds by Theorem 6.0.1. To proof the first one, set $G(t,z) := G_{\mu^{\mathrm{HP}}_{-1,t}}(z)$. Then, by (5.5.4), we have

$$\partial_t G(t,z) = -2G(t,z)(zG(t,z)+(1+z^2)\partial_z G(t,z))\,.$$

Now, set $\rho_t := f^{-1}(\mu^{\mathrm{HP}}_{-1,t})$. Note that we have $\rho_0 = \nu$. By the proof of Theorem 5.6.4 below, we know that the fact that $\mu^{\mathrm{HP}}_{-1,0}$ is even implies that $\mu^{\mathrm{HP}}_{-1,t}$ is even for all $t \geq 0$. Thus, by Proposition 5.5.8 it holds

$$\psi(t,z) := \psi_{\rho_t}(z) = \frac{1}{2}\left(\frac{1}{2}(z^{1/2}+z^{-1/2})G(t,-f(z)) - 1\right).$$

This gives

$$\partial_z\psi(t,z) = \frac{1}{4}z^{-1}f(z)G(t,-f(z)) - \frac{1}{4}z^{-1}(1+f^2(z))\partial_y G(t,y)|_{y=-f(z)}\,.$$

Differentiating with respect to $t$ gives

$$\begin{aligned}
&\partial_t\psi(t,z)\\
=&-\frac{1}{2}(z^{1/2}+z^{-1/2})G(t,-f(z))\left[-f(z)G(t,-f(z)) + (1+f^2(z))\partial_y G(t,y)|_{y=-f(z)}\right]\\
=&4(1+2\psi(t,z))z\partial_z\psi(t,z)\,.
\end{aligned}$$

By the calculations in the proof of Theorem 5.3.4 ($\psi_{\nu\boxtimes\nu_t}(z)$ solves (5.3.8) in the case $\hat{b}_1 = -1/2$), we thus know that $\rho_t = \nu\boxtimes\nu_{8t}$ for all $t \geq 0$. □

## 5.6 Global limit of non-compact Jacobi processes

Now, we consider the final remaining subclass of non-colliding $\beta$-Pearson diffusions in the 'natural' scaling. Let $p_N \in \mathbb{R}$, $q_N > N-1$, $N \in \mathbb{N}$, and assume that the limits $\lim_{N\to\infty} p_N/N = \hat{p}$ and $\lim_{N\to\infty} q_N/N = \hat{q}$ exist. We apply Theorem 4.2.2 to the case of a sequence of non-compact Jacobi processes rescaled in time by the factor $1/N$, which thus arise as solutions to

$$\begin{aligned}
dX^N_{i,t} =&\sqrt{2X^N_{i,t}(X^N_{i,t}-1)/N}\,dB_{i,t}\\
&+\frac{\beta}{2N}\left[-p_N + (p_N+q_N)X^N_{i,t} + \sum_{j:\,j\neq i}\frac{2X^N_{i,t}X^N_{j,t} - X^N_{i,t} - X^N_{j,t}}{X^N_{i,t}-X^N_{j,t}}\right]dt\,,
\end{aligned}$$

$i \in \{1,\ldots,N\}$, $X^N_0 = \hat{x}^N_0 \in C^{\mathrm{BC}*}_N$. For our choice of parameters there exist such non-colliding solutions by Lemma 3.6.1. The corresponding parameters appearing in Theorem 4.2.2 are here given by $a^N_0 = 0$, $a^N_1 = -1/N$, $a^N_2 = 1/N$, $b^N_1 = (p_N+q_N-2(N-1))/N$, $b^N_0 = -(p_N-(N-1))/N$.

Further, we assume that the limit $\frac{1}{N}\sum_{i=1}^N \delta_{\hat{x}^N_{i,0}} \xrightarrow{N\to\infty} \mu$ exists for some $\mu \in M^1([1,\infty))$, where the convergence is in moments and the moments $(m_n)_{n\in\mathbb{N}}$ of $\mu$ satisfy the bounds (4.2.5) for some $\gamma > 0$. Under these assumptions we obtain the following limit of the empirical measures $\mu_{N,t} = \frac{1}{N}\sum_{i=1}^N \delta_{X^N_{i,t}}$:

**Theorem 5.6.1.** *There exists* $(\mu^{\mathrm{ncJ}}_{\hat{p},\hat{q},t})_{t\geq 0} \in C([0,\infty), M^1([1,\infty)))$ *such that* $\mu^{\mathrm{ncJ}}_{\hat{p},\hat{q},0} = \mu$ *and*

$$\left(\mu_{N,2t/\beta}\right)_{t\geq 0} \xrightarrow{N\to\infty} \left(\mu^{\mathrm{ncJ}}_{\hat{p},\hat{q},t}\right)_{t\geq 0} \quad \text{in } C([0,\infty), M^1([1,\infty))) \text{ a.s.}$$

*For all* $n \in \mathbb{N}$, *the moments* $m_n(t) = \int x^n \, d\mu^{\mathrm{ncJ}}_{\hat{p},\hat{q},t}(x)$ *satisfy*

$$\begin{aligned}&\frac{d}{dt} m_n(t)\\ =&n\left[(\hat{p}+\hat{q}-2)m_n(t) - (\hat{p}-1)m_{n-1}(t) + \sum_{k=0}^{n} m_k(t)m_{n-k}(t) - \sum_{k=0}^{n-1} m_k(t)m_{n-1-k}(t)\right].\end{aligned} \tag{5.6.1}$$

*If* $\hat{p}+\hat{q} < 0$ *then it holds* $\lim_{t\to\infty} \mu^{\mathrm{ncJ}}_{\hat{p},\hat{q},t} = f(\mu_{\mathrm{W},1-\hat{p}-\hat{q},\hat{q}})$ *where* $f(x) = x^{-1}$, *and where* $\mu_{\mathrm{W},\cdot,\cdot}$ *denotes the Wachter distribution as in Definition 5.4.1. If additionally* $\mu = f(\mu_{\mathrm{W},1-\hat{p}-\hat{q},\hat{q}})$, *then* $t \mapsto \mu^{\mathrm{ncJ}}_{\hat{p},\hat{q},t}$ *is constant.*

*Proof.* The claim concerning the convergence of empirical measures and the validity of the ODEs (5.6.1) follows from Theorem 4.2.2. For the rest of the proof assume $\hat{p}+\hat{q} < 0$. Let $(\alpha_N)_{N\in\mathbb{N}}, (\beta_N)_{N\in\mathbb{N}} \subset (-1,\infty)$ such that $a := \lim_{N\to\infty} \alpha_N/N$ and $b := \lim_{N\to\infty} \beta_N/N$ exist. Let $(x_{1,N}, \ldots, x_{N,N})$ be the vector of ordered zeros of the Jacobi polynomial $P_N^{\alpha_N,\beta_N}$. Then, by [37] Corollary 2.3, we have

$$\frac{1}{N}\sum_{i=1}^{N} \delta_{x_{i,N}} \xrightarrow{N\to\infty} \mu_{a,b} \text{ weakly},$$

where $\mu_{a,b}$ has density

$$\frac{2+a+b}{2\pi}\frac{\sqrt{(x_+-x)(x-x_-)}}{1-x^2}\mathbb{1}_{(x_-,x_+)}(x), \quad x_\pm = \frac{b^2-a^2 \pm 4\sqrt{(a+1)(b+1)(a+b+1)}}{(2+a+b)^2}.$$

Let $g(x) := (1+x)/2$, then a short computation shows $g(\mu_{a,b}) = \mu_{\mathrm{W},b+1,a+1}$. Pick $\tilde{q}_N > N-1$ and $\tilde{p}_N < -\tilde{q}_N$ such that $\lim_{N\to\infty} \tilde{p}_N/N = \hat{p}$ and $\lim_{N\to\infty} \tilde{q}_N/N = \hat{q}$. Then, combining the above facts with Lemma 3.6.2, we know that the empirical measures of stationary frozen non-compact Jacobi processes with parameters $\tilde{p}_N$, $\tilde{q}_N$ converge to the image measure of $\mu_{\mathrm{W},1-\hat{p}-\hat{q},\hat{q}}$ under $x \mapsto x^{-1}$ for all $t \geq 0$. This shows $t \mapsto \mu^{\mathrm{ncJ}}_{\hat{p},\hat{q},t}$ is constant if $\mu^{\mathrm{ncJ}}_{\hat{p},\hat{q},0} = f(\mu_{\mathrm{W},1-\hat{p}-\hat{q},\hat{q}})$. Now, for general $\mu^{\mathrm{ncJ}}_{\hat{p},\hat{q},0}$, we can inductively show, using 5.6.1, that $\lim_{t\to\infty} m_n(t)$ exists and does not depend on $m_n(0)$. Hence, this limit must coincide with the $n$-th moment of the constant solution $f(\mu_{\mathrm{W},1-\hat{p}-\hat{q},\hat{q}})$. As the latter measure is compactly supported and thus determined by its moments, we have $\lim_{t\to\infty} \mu^{\mathrm{ncJ}}_{\hat{p},\hat{q},t} = f(\mu_{\mathrm{W},1-\hat{p}-\hat{q},\hat{q}})$ by the moment convergence theorem. □

In contrast to the Hua-Pickrell case, we cannot easily describe the limit measure $(\mu^{\mathrm{ncJ}}_{\hat{p},\hat{q},t})_{t\geq 0}$ appearing in the theorem above by the distribution of the solution of some free

stochastic differential equation. E.g. if $(c_t)_{t\geq 0}$ is a circular Brownian motion, a solution to

$$dy_t = \sqrt{y_t - \mathbf{1}}\, dc_t\, \sqrt{y_t} + \sqrt{y_t}\, dc_t^*\, \sqrt{y_t - \mathbf{1}} + [-p\mathbf{1} + (p+q)y_t]\, dt$$

would give the right moments. However, as $x \mapsto \sqrt{x-1}$ is not Lipschitz in 1, we cannot use known results about existence of solutions as e.g. Proposition A.1 in [25]. Nevertheless, we have other ways of describing $(\mu^{\mathrm{ncJ}}_{\hat{p},\hat{q},t})_{t\geq 0}$ for the special cases $\hat{p} = 1$ or $\hat{q} = 1$. In particular, the intersection of these special cases, i.e. the case $\hat{p} = \hat{q} = 1$, permits an explicit description via free convolutions, see Corollary 5.6.5 below.
We start with the case $\hat{p} = 1$ which corresponds to the free version of Theorem 3.6.9:

**Theorem 5.6.2.** *Let $(h_t)_{t\geq 0}$ be a free positive mulitplicative Brownian motion. Let $P$ be a projection which is free from $\{h_t, h_t^{-1} : t \geq 0\}$ and satisfies $\alpha := \tau(P) \in (0, 1/2]$. Set $\hat{\tau}(\cdot) := \frac{1}{\tau(P)}\tau(\cdot)$. Then the distribution of $Ph_tPh_t^{-1}P$, $t \geq 0$, viewed in the compressed space $(P\mathcal{A}P, \hat{\tau})$, equals $\mu^{\mathrm{ncJ}}_{1,1/\alpha-1,\alpha t}$ where $\mu^{\mathrm{ncJ}}_{1,1/\alpha-1,\cdot}$ is the limit of empirical measures of non-compact Jacobi processes as in Theorem 5.6.1 with $\mu^{\mathrm{ncJ}}_{1,1/\alpha-1,0} = \mu_{Ph_0Ph_0^{-1}P}$.*

*Proof.* Set $j_t := Ph_tPh_t^{-1}P$, $t \geq 0$. By Proposition 5.3.2 and the free Itô product rule 2.5.10, we have for some semi-circular Brownian motion $(x_t)_{t\geq 0}$:

$$dj_t = Ph_t^{1/2}\, dx_t\, h_t^{1/2}Ph_t^{-1}P - Ph_tPh_t^{-1/2}\, dx_t\, h_t^{-1/2}P + [j_t - \tau(P)P]dt\,.$$

By the traced Itô formula 2.5.11 we get

$$\begin{aligned}
\frac{d}{dt}\tau(j_t^n) =& n\Bigg[\tau(j_t^n) - \tau(P)\tau(Pj_t^{n-1}) \\
&+\frac{1}{2}\sum_{k=0}^{n-2}(\tau\otimes\tau^{\mathrm{op}})\left(\left(h_t^{1/2}Ph_t^{-1}P\otimes Ph_t^{1/2} - h_t^{-1/2}P\otimes Ph_tPh_t^{-1/2}\right)\right. \\
&\left.\cdot\left(j_t^k\otimes j_t^{n-2-k}\right)\cdot\left(Ph_t^{1/2}\otimes h_t^{1/2}Ph_t^{-1}P - Ph_tPh_t^{-1/2}\otimes h_t^{-1/2}P\right)\right)\Bigg] \\
=& n\Bigg[\tau(j_t^n) - \tau(P)\tau(Pj_t^{n-1}) \\
&+\frac{1}{2}\sum_{k=0}^{n-2}\left(2\tau(Pj_t^{k+1})\tau(Pj_t^{n-1-k}) - \tau(Pj_t^{k+1})\tau(Pj_t^{n-2-k}) - \tau(Pj_t^k)\tau(Pj_t^{n-1-k})\right)\Bigg] \\
=& n\left[(1-2\tau(P))\tau(j_t^n) + \sum_{k=0}^{n}\tau(Pj_t^k)\tau(Pj_t^{n-k}) - \sum_{k=0}^{n-1}\tau(Pj_t^k)\tau(Pj_t^{n-1-k})\right].
\end{aligned}$$

Set $m_n(t) := \hat{\tau}(j_t^n)$ and $m_0(t) := 1$. Then $m_n(t)$ is the $n$-th moment of the distribution of $j_t$ viewed as an element of $(P\mathcal{A}P, \hat{\tau})$, Moreover, we have:

$$\frac{d}{dt}m_n(t) = n\left[(1-2\alpha)m_n(t) + \alpha\sum_{k=0}^{n}m_k(t)m_{n-k}(t) - \alpha\sum_{k=0}^{n-1}m_k(t)m_{n-1-k}(t)\right]. \quad (5.6.2)$$

On the other hand, let $\tilde{m}_n(t)$ be the $n$-th moment of $\mu^{\mathrm{ncJ}}_{1,1/\alpha-1,\alpha t}$, then by (5.6.1) we know that also $(\tilde{m}_n)_{n\in\mathbb{N}}$ solves the ODEs (5.6.2). As by assumption $m_n(0) = \tilde{m}_n(0)$, we hence have $m_n(t) = \tilde{m}_n(t)$ for all $t \geq 0$. As all appearing measures are determined by their moments, the proof is finished. □

We readily get the analogue of [54] Section 2.2 Corollary 1:

**Corollary 5.6.3.** *Let $k \in \mathbb{N}$, $k \geq 2$. On the same $W^*$-probability space let $(h_{i,t})_{t\geq 0}$, $i \in \{1,\dots,k\}$, be free positive multiplicative Brownian motions such that $\mu_{h_{1,0}} = \cdots = \mu_{h_{k,0}}$ and such that the families $\{h_{1,t}, h_{1,t}^{-1} \colon t \geq 0\}, \dots, \{h_{k,t}, h_{k,t}^{-1} \colon t \geq 0\}$ are freely independent. Then the distribution of*

$$j_t := \left(\frac{1}{k}\sum_{i=1}^{k} h_i(t)\right) \cdot \left(\frac{1}{k}\sum_{j=1}^{k} h_j^{-1}(t)\right).$$

*equals $\mu^{\mathrm{ncJ}}_{1,k-1,t/k}$ where $\mu^{\mathrm{ncJ}}_{1,k-1,\cdot}$ is the limit of empirical measures of non-compact Jacobi processes as in Theorem 5.6.1 with $\mu^{\mathrm{ncJ}}_{1,k-1,0} = \mu_{j_0}$.*

*Proof.* Let $P$ and $(h_t)_{t\geq 0}$ be as in Theorem 5.6.2 in the case $\tau(P) = 1/k$. Then, by Proposition 2.4.14, the joint distribution (see Section 2.4.3 for this notion) of $Ph_tP, Ph_t^{-1}P$, viewed as elements of $(P\mathcal{A}P, k\tau(\cdot))$, and the joint distribution of $\frac{1}{k}\sum_{i=1}^{k} h_i, \frac{1}{k}\sum_{j=1}^{k} h_j^{-1}$ agree. Thus also the distributions of $Ph_tPPh_t^{-1}P = Ph_tPh_t^{-1}P$ and $\frac{1}{k}\sum_{i=1}^{k} h_i \frac{1}{k}\sum_{j=1}^{k} h_j^{-1}$ agree. The claim follows from Theorem 5.6.2. □

Now, we turn to the case $\hat{q} = 1$. Here a close connection to the limit of empirical measures of Hua-Pickrell diffusions which we discussed in the previous section holds:

**Theorem 5.6.4.** *Let $\mu^{\mathrm{HP}}_{\hat{s},t}$ be the distribution of the limit of empirical measures of Hua-Pickrell diffusions as in Theorem 5.5.5 with the additional constraint that $\mu^{\mathrm{HP}}_{\hat{s},0}$ is even and $\hat{s} \in \mathbb{R}$. Let $f(x) = 1 + x^2$. Let $(\mu^{\mathrm{ncJ}}_{-2\hat{s}-1,1})_{t\geq 0}$ be the limit of empirical measures of non-compact Jacobi processes as in Theorem 5.6.1 in the case $\mu^{\mathrm{ncJ}}_{-2\hat{s}-1,1,0} = f(\mu^{\mathrm{HP}}_{\hat{s},0})$. Then it holds $f(\mu^{\mathrm{HP}}_{\hat{s},t}) = \mu^{\mathrm{ncJ}}_{-2\hat{s}-1,1,2t}$ for all $t \geq 0$.*

*Proof.* Let $m_n(t)$ be the $n$-th moment of $\mu^{\mathrm{HP}}_{\hat{s},t}$. By (5.5.3) it satisfies the ODE

$$\frac{d}{dt}m_n(t) = n\left[-2(\hat{s}+1)m_n(t) + \sum_{k=0}^{n} m_k(t)m_{n-k}(t) + \sum_{k=0}^{n-2} m_k(t)m_{n-2-k}(t)\right].$$

Since $\mu^{\mathrm{HP}}_{\hat{s},0}$ is even, we can deduce by induction that $m_{2n-1}(t) \equiv 0$ for all $n \in \mathbb{N}$, i.e. $\mu^{\mathrm{HP}}_{\hat{s},t}$ is even, $t \geq 0$. Now, set $\tilde{m}_n(t) := m_{2n}(t)$. Note that $(\tilde{m}_n(t))_{n\in\mathbb{N}}$ are the moments of the image measure of $\mu^{\mathrm{HP}}_{\hat{s},t}$ under $x \mapsto x^2$. Further, we have

$$\frac{d}{dt}\tilde{m}_n(t) = 2n\left[-2(\hat{s}+1)\tilde{m}_n(t) + \sum_{k=0}^{n} \tilde{m}_k(t)\tilde{m}_{n-k}(t) + \sum_{k=0}^{n-1} \tilde{m}_k(t)\tilde{m}_{n-1-k}(t)\right].$$

On the other hand, we can derive the moment ODEs for the measure $\mu^{\mathrm{ncJ}}_{\hat{p},\hat{q},t} * \delta_{-1}$ as follows (here $*$ denotes the usual convolution of real probability measures): Let $X^N_t$ be as in the beginning of this section and consider $\big(Y^N_t := (X^N_{1,t} - 1, \ldots, X^N_{N,t} - 1)\big)_{t\geq 0}$, $N \in \mathbb{N}$. We can apply Theorem 4.2.2 to this sequence of processes, where the parameters now are $a^N_2 = a^N_1 = 1/N$, $a^N_0 = 0$, $b^N_1 = (p_N + q_N - 2(N-1))/N$, $b^N_0 = (q_N - (N-1))/N$. Letting $N \to \infty$, we get the following ODEs for the moments $\bar{m}_n(t)$ of $\lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^N \delta_{Y^N_{i,t}} = \mu^{\mathrm{ncJ}}_{\hat{p},\hat{q},t} * \delta_{-1}$:

$$\begin{aligned}&\frac{d}{dt}\bar{m}_n(t)\\ =&n\left[(\hat{p}+\hat{q}-2)\bar{m}_n(t) + (\hat{q}-1)\bar{m}_{n-1}(t) + \sum_{k=0}^{n}\bar{m}_k(t)\bar{m}_{n-k}(t) + \sum_{k=0}^{n-1}\bar{m}_k(t)\bar{m}_{n-1-k}(t)\right].\end{aligned}$$

Choosing $\hat{p} = -2\hat{s} - 1$ and $\hat{q} = 1$, we see that $\tilde{m}_n(t) = \bar{m}_n(2t)$ for all $n \in \mathbb{N}$ and $t \geq 0$ since $f(\mu^{\mathrm{HP}}_{\hat{s},0}) = \mu^{\mathrm{ncJ}}_{-2\hat{s}-1,1,0}$ by assumption. As $\mu^{\mathrm{ncJ}}_{-2\hat{s}-1,1,2t}$ is uniquely characterized by its moments, the proof is finished. □

Note that, by the previous theorem, whenever $\mu^{\mathrm{ncJ}}_{\hat{p},1,0}$ is compactly supported, the measures $(\mu^{\mathrm{ncJ}}_{\hat{p},1,t})_{t\geq 0}$ in Theorem 5.6.1 are equal to the distribution of the free Itô process $(\mathbf{1} + y^2_{t/2})_{t\geq 0}$, where $(y_t)_{t\geq 0}$ is as in Theorem 5.5.6 for $a = -\hat{p}/2$, $b = 0$, and even $\mu_{y_0}$. Moreover, at the intersection of the special cases $\hat{p} = 1$, $\hat{q} = 1$, we can describe $\mu^{\mathrm{ncJ}}_{\hat{p},\hat{q},t}$ using the image measure of the distribution of a free positive multiplicative Brownian motion under a suitable map; compare the $\hat{p} = \hat{q} = 1$ case in Theorem 5.4.3.

**Corollary 5.6.5.** *Let $\nu_t$ be the distribution of a free positive multiplicative Brownian motion with start in $\mathbf{1}$ at time $t$. Let $f(x) = (2 + x + x^{-1})/4$. Let $\nu \in M^1((0,\infty))$ be invariant under the map $x \mapsto x^{-1}$ and determined by its moments. Let $(\mu^{\mathrm{ncJ}}_{1,1,t})_{t\geq 0}$ be the limit of empirical measures of non-compact Jacobi processes as in Theorem 5.6.1 with $f(\nu) = \mu^{\mathrm{ncJ}}_{1,1,0}$. Then*

$$\mu^{\mathrm{ncJ}}_{1,1,t} = f(\nu \boxtimes \nu_{4t}) = f\left(\nu \boxtimes \exp\big(\mu_{\mathrm{sc},4\sqrt{t}} \boxplus \mathrm{Unif}_{[-2t,2t]}\big)\right) \quad \textit{for all } t \geq 0\,.$$

*Proof.* This is a direct consequence of Theorem 5.5.7 and Theorem 5.6.4 since

$$(x \mapsto 1 + x^2) \circ (x \mapsto (x^{1/2} - x^{-1/2})/2) = f\,.$$

□

**Remark 5.6.6.** In the case $\nu = \delta_1$ the corollary can also directly be deduced from Theorem 4.2 in [12] using the change of measure under $x \mapsto \frac{1}{2}(1 + \cosh(x))$.

## 5.7 Global limits of $\beta$-Pearson diffusions for higher order space-time scalings

So far, we have studied empirical measures of a sequence of $\beta$-Pearson diffusions $((X_t^N)_{t\geq 0})_{N\in\mathbb{N}}$ for large dimension $N$ in the 'natural scaling', i.e. the case that $\beta \geq 1$ is fixed, that time $t$ is linearly rescaled by $1/N$, and that parameters $b_1^N, b_0^N$ asymptotically grow linearly in $N$. We have seen that, in this way, each of the six subclasses of $\beta$-Pearson diffusions gives rise to some limit measure in $C([0,\infty), M^1(\mathbb{R}))$, possibly depending on up to two parameters. In this Section, we study a slightly more general situation as before: We continue to fix $\beta \geq 1$ but instead of $X_t^N$ itself we are now considering the process

$$t \mapsto Y_t^N := \big(\gamma_N(X^N_{1,t/\tau_N} - \delta_N), \ldots, \gamma_N(X_{N,t/\tau_N} - \delta_N)\big) \tag{5.7.1}$$

for some $\gamma_N > 0$, $\delta_N \in \mathbb{R}$, $\tau_N > 0$. In other words, we allow for a global affine linear transformation of our coordinate system and an arbitrary linear rescaling of time for each $N$. Note that we previously studied the case $\gamma_N \equiv 1$, $\delta_N \equiv 0$ and $\tau_N = N$. In particular, these changes now enable us to investigate the global behavior of $\beta$-Pearson diffusions for smaller time scales, i.e. for $1/\tau_N \in o(N^{-1})$, by 'zooming in' on the state space side, i.e. choosing suitable $\gamma_N \uparrow_{N\to\infty} \infty$. Without this spatial 'zoom' we would simply end up loosing dependence of the limit measure on the time variable $t$. We will see that the limits $(\mu_t)_{t\geq 0} \in C([0,\infty), M^1(\mathbb{R}))$ we obtain this way are again instances of the limits covered in Sections 5.1-5.6, i.e. one has some form of stability of this class of limits.
As before, the existence of large $N$ limits of empirical measures of $Y_t^N$ will follow from Theorem 4.2.2. To apply this theorem, we need to know how the parameters of the $\beta$-Pearson diffusion $(X_t^N)_{t\geq 0}$ change when going over to the process $(Y_t^N)_{t\geq 0}$. So let $(X_t^N)_{t\geq 0}$ be a non-colliding $\beta$-Pearson diffusion with parameters $a_2^N, a_1^N, a_0^N, b_1^N, b_0^N$. Then one can easily check that $(Y_t^N)_{t\geq 0}$ is a non-colliding $\beta$-Pearson diffusion with parameters

$$\begin{gathered}\tilde{a}_2^N = \frac{a_2^N}{\tau_N}\,, \quad \tilde{a}_1^N = \frac{\gamma_N}{\tau_N}(2\delta_N a_2^N + a_1^N)\,, \quad \tilde{a}_0^N = \frac{\gamma_N^2}{\tau_N}(\delta_N^2 a_2^N + \delta_N a_1^N + a_0^N)\,, \\ \tilde{b}_1^N = \frac{b_1^N}{\tau_N}\,, \quad \tilde{b}_0^N = \frac{\gamma_N}{\tau_N}(\delta_N b_1^N + b_0^N)\,.\end{gathered} \tag{5.7.2}$$

Below we will state the the main parameter choices such that the empirical measures

$$\mu_t^N := \frac{1}{N}\sum_{i=1}^N \delta_{Y_{i,t}^N}\,, \quad t \geq 0\,,$$

do almost surely have a non-trivial deterministic limit $(\mu_t)_{\geq 0}$ in $(C([0,\infty)), M^1(\mathbb{R}))$ for the cases that the processes $(X_t^N)_{t\geq 0}$, $N \in \mathbb{N}$, are either Jacobi or non-compact Jacobi processes. However, we point out that one can also obtain non-trivial limits for higher order scalings of all other non-colliding Pearson diffusions with the exception of the (OU) Dyson Brownian motions.

In the statements below, we will restrict ourselves to the main parameter choices such that non-degenerate limits exist. We do not claim to cover all possible such choices, however. E.g., whenever we have a limit point $(\mu_t)_{t\geq 0}$, one can easily adjust $\gamma_N, \delta_N$ to also get the limit point $(f(\mu_t))_{t\geq 0}$ whenever $f$ is an affine linear function.

### 5.7.1 Jacobi processes

For the subclass of Jacobi processes we get another two non-trivial limit points except the one considered in Theorem 5.4.3. Throughout this section let $(X_t^N)_{t\geq 0}$, $N \in \mathbb{N}$, be Jacobi processes on $C_N^{BC}$ with parameters $p_N, q_N \geq N-1$ and which are rescaled in time by $2/\beta$, i.e. these processes satisfy

$$dX_{i,t}^N = \sqrt{\frac{4}{\beta}X_{i,t}^N(1-X_{i,t}^N)}dB_{i,t} + \left[p_N - (p_N+q_N)X_{i,t}^N + \sum_{j:\, j\neq i} \frac{X_{i,t}^N + X_{j,t}^N - 2X_{i,t}^N X_{j,t}^N}{X_{i,t}^N - X_{j,t}^N}\right] dt\,, \tag{5.7.3}$$

$i \in \{1,\ldots,N\}$, $X_0^N = \hat{x}_0^N \in C_N^{BC}$. Note that formally we allow for $\beta = \infty$ in this SDE by using the convention $1/\infty = 0$. In this latter case, we consider non-random frozen Jacobi processes as in (3.5.2). Further, note that in the case $\beta < \infty$ each $(X_t^N)_{t\geq 0}$ is a non-colliding $\beta$-Pearson diffusion with parameters

$$a_2^N = -\frac{2}{\beta},\ a_1^N = \frac{2}{\beta},\ a_0^N = 0,\ b_1^N = -\frac{2}{\beta}\left(p_N+q_N-2(N-1)\right),\ b_0^N = \frac{2}{\beta}\left(p_N-(N-1)\right)\,.$$

In either case, set $\mu_t^N := \frac{1}{N}\sum_{i=1}^N \delta_{Y_{i,t}^N}$ where $(Y_t^N)_{t\geq 0}$ is as in (5.7.1) for some $\gamma_N, \tau_N > 0$, $\delta_N \in \mathbb{R}$. The transformed parameters in (5.7.2) here take the form

$$\begin{aligned}\tilde{a}_2^N &= -\frac{2}{\beta}\frac{1}{\tau_N},\ \tilde{a}_1^N = \frac{2}{\beta}\frac{\gamma_N}{\tau_N}(1-2\delta_N),\ \tilde{a}_0^N = \frac{2}{\beta}\frac{\gamma_N^2}{\tau_N}\delta_N(1-\delta_N)\\ \tilde{b}_1^N &= -\frac{2}{\beta}\frac{1}{\tau_N}(p_N+q_N-2(N-1)),\ \tilde{b}_0^N = \frac{2}{\beta}\frac{\gamma_N}{\tau_N}((1-\delta_N)p_N - \delta_N q_N + (2\delta_N-1)(N-1))\,.\end{aligned} \tag{5.7.4}$$

In the following, we will consider three limit regimes for the parameters $p_N, q_N, \gamma_N, \delta_N, \tau_N$ such that non-degenerate limits of $(\mu_t^N)_{t\geq 0}$ exists. The following results have previously been published in [13] where instead of $(X_t^N)_{t\geq 0}$ the process $((2X_{1,t}^N-1,\ldots,2X_{N,t}^N-1))_{t\geq 0}$ is considered.

#### 5.7.1.1 Marchenko-Pastur limit

Here we consider the following parameter asymptotics:

- Choose $\tau_N > 0$ such that $\lim_{N\to\infty} N/\tau_N = 0$.
- Choose $\gamma_N > 0$ such that $\lim_{N\to\infty} N\gamma_N/\tau_N = 1$.

- Choose $p_N, q_N \geq N-1$ such that the limits $\lim_{N\to\infty} p_N/N =: \hat{p} \in [1,\infty)$ and $\lim_{N\to\infty} q_N/\tau_N =: \lambda \in [0,\infty)$ exist.
- Choose $\delta_N \in \mathbb{R}$ such that $\lim_{N\to\infty} \delta_N \gamma_N = 0$.

Choose starting vectors $\hat{x}_0^N \in C_N^{BC}$ such that the limit $\frac{1}{N}\sum_{i=1}^N \delta_{\gamma_N(\hat{x}_{i,0}^N - \delta_N)} \xrightarrow{N\to\infty} \mu \in M^1([0,\infty))$ exists where the convergence is in moments and the moments of $\mu$ satisfy the bounds (4.2.5). Under these conditions we have:

**Corollary 5.7.1.** *If $\lambda = 0$ then*

$$\left(\mu_t^N\right)_{t\geq 0} \xrightarrow{N\to\infty} \left(\left(\sqrt{\mu}_{\mathrm{even}} \boxplus \mu_{\mathrm{SC},2\sqrt{t}}\right)^2 \boxplus \mu_{\mathrm{MP},\hat{p}-1,t}\right)_{t\geq 0}$$

*in $C([0,\infty), M^1(\mathbb{R}))$ a.s.*
*If $\lambda > 0$ then*

$$\left(\mu_t^N\right)_{t\geq 0} \xrightarrow{N\to\infty} \left(\left(D_{e^{-\lambda t/2}}(\sqrt{\mu}_{\mathrm{even}}) \boxplus \mu_{\mathrm{SC},2\sqrt{(1-e^{-\lambda t})/\lambda}}\right)^2 \boxplus \mu_{\mathrm{MP},\hat{p}-1,(1-e^{-\lambda t})/\lambda}\right)_{t\geq 0}$$

*in $C([0,\infty), M^1(\mathbb{R}))$ a.s.*

*Proof.* In the chosen scaling we have the limits

$$\lim_{N\to\infty} N\tilde{a}_2^N = 0,\ \lim_{N\to\infty} N\tilde{a}_1^N = \frac{2}{\beta},\ \lim_{N\to\infty} N\tilde{a}_0^N = 0,\ \lim_{N\to\infty} \tilde{b}_1^N = -\frac{2}{\beta}\lambda,\ \lim_{N\to\infty} \tilde{b}_0^N = \frac{2}{\beta}(\hat{p}-1)\,.$$

By Theorem 4.2.2, we thus know that one has a limit point $(\mu_t)_{t\geq 0}$ in $C([0,\infty), M^1(\mathbb{R}))$. Moreover, $\mu_t$ is determined by its moments $(m_n(t))_{n\in\mathbb{N}}$ which satisfy

$$\frac{d}{dt}m_n(t) = n\left[-\lambda m_n(t) + (\hat{p}-1)m_{n-1}(t) + \sum_{k=0}^{n-1} m_k(t)m_{n-1-k}(t)\right]\,.$$

A comparison with Theorem 5.2.1 shows the claim. □

#### 5.7.1.2 Semicircle limit

We consider two parameter choices such that we get semicircle-type limits as in Theorem 5.1.1:
Parameter asymptotic (I):

- Choose $\tau_N > 0$ such that $\lim_{N\to\infty} N/\tau_N = 0$.
- Choose $\delta_n \in (0,1)$ such that the limit $\lim_{N\to\infty} \delta_N =: \hat{\delta} \in (0,1)$ exists.
- Choose $\gamma_N > 0$ such that $\lim_{N\to\infty} \gamma_N^2 N/\tau_N = \frac{1}{2\hat{\delta}(1-\hat{\delta})}$.

- Choose $p_N, q_N \geq N-1$ such that $\lim_{N\to\infty} \frac{p_N+q_N}{\sqrt{N\tau_N}} = 0$.

Parameter asymptotic (II):

- Choose $\tau_N > 0$ such that $\lim_{N\to\infty} N/\tau_N = 0$.
- Choose $p_N, q_N \geq N-1$ such that the limit $\lim_{N\to\infty}(p_N+q_N)/\tau_N =: \lambda \in [0,\infty)$ exists and one of the two conditions $\lim_{N\to\infty} N/\min(p_N,q_N) = 0$ or $\lambda = 0$ hold.
- Set $\delta_N := p_N/(p_N+q_N) \in (0,1)$.
- Choose $\gamma_N > 0$ such that

$$\lim_{N\to\infty} \frac{N\gamma_N^2}{\tau_N} \frac{p_N q_N}{(p_N+q_N)^2} = \frac{1}{2}\,.$$

In either case (I) or (II), choose starting vectors $\hat{x}_0^N \in C_N^{BC}$ such that the limit

$$\frac{1}{N}\sum_{i=1}^{N} \delta_{\gamma_N(\hat{x}_{i,0}^N - \delta_N)} \xrightarrow{N\to\infty} \mu \in M^1(\mathbb{R})$$

exists where the convergence is in moments and the moments of $\mu$ satisfy the bounds (4.2.5).

**Corollary 5.7.2.** *Under the conditions (I) or the conditions (II) in the case $\lambda = 0$ we have*

$$\left(\mu_t^N\right)_{t\geq 0} \xrightarrow{N\to\infty} \left(\mu \boxplus \mu_{sc,2\sqrt{t}}\right)_{t\geq 0}$$

*in $C([0,\infty), M^1(\mathbb{R}))$ a.s.*
*Under the conditions (II) in the case $\lambda > 0$ we have*

$$\left(\mu_t^N\right)_{t\geq 0} \xrightarrow{N\to\infty} \left(D_{e^{-\lambda t}}(\mu) \boxplus \mu_{\mathrm{sc},\sqrt{2(1-e^{-2\lambda t})/\lambda}}\right)_{t\geq 0}$$

*in $C([0,\infty), M^1(\mathbb{R}))$ a.s.*

*Proof.* For the scaling (I) set $\lambda := 0$. Then, in both choices of scaling (I) and (II), we have

$$\lim_{N\to\infty} N\tilde{a}_2^N = 0\,,\quad \lim_{N\to\infty} N\tilde{a}_1^N = 0\,,\quad \lim_{N\to\infty} N\tilde{a}_0^N = \frac{1}{\beta}\,,\quad \lim_{N\to\infty} \tilde{b}_1^N = -\frac{2}{\beta}\lambda\,,\quad \lim_{N\to\infty} \tilde{b}_0^N = 0\,.$$

In scaling (II), we here used that by assumption we have

$$\lim_{N\to\infty} (p_N+q_N)/\tau_N = 0 \;\text{ or }\; \lim_{N\to\infty} N/\min(p_N,q_N) = 0$$

which implies $\lim_{N\to\infty} N\gamma_N/\tau_N = 0$: This follows from

$$\frac{1}{2} = \lim_{N\to\infty} \left(\frac{N\gamma_N}{\tau_N}\right)^2 \frac{\tau_N}{N} \frac{p_N q_N}{(p_N+q_N)^2}$$

and

$$\frac{\tau_N}{N}\frac{p_N q_N}{(p_N+q_N)^2} = \frac{\tau_N}{p_N+q_N}\frac{\min(p_N,q_N)}{N}\frac{1}{1+\frac{\min(p_N,q_N)}{\max(p_N,q_N)}} \xrightarrow{N\to\infty} \infty\,.$$

By Theorem 4.2.2, we thus know that one has a limit point $(\mu_t)_{t\geq 0}$ in $C([0,\infty), M^1(\mathbb{R}))$. Moreover, $\mu_t$ is determined by its moments $(m_n(t))_{n\in\mathbb{N}}$ which satisfy

$$\frac{d}{dt}m_n(t) = n\left[-\lambda m_n(t) + \frac{1}{2}\sum_{k=0}^{n-2} m_k(t) m_{n-2-k}(t)\right].$$

A comparison with Theorem 5.1.1 shows the claim. □

### 5.7.2 Non-compact Jacobi processes

For the subclass of non-compact Jacobi processes we get another three non-trivial limit points except the one considered in Theorem 5.6.1. Throughout this section let $(X_t^N)_{t\geq 0}$, $N\in\mathbb{N}$, be non-compact Jacobi processes on $C_N^{BC*}$ with parameters $p_N\in\mathbb{R}$, $q_N > N-1$ and which are rescaled in time by $2/\beta$, i.e. these processes satisfy

$$dX_{i,t}^N = \sqrt{\frac{4}{\beta}X_{i,t}^N(X_{i,t}^N-1)}dB_{i,t} + \left[-p_N + (p_N+q_N)X_{i,t}^N + \sum_{j:\,j\neq i}\frac{2X_{i,t}^N X_{j,t}^N - X_{i,t}^N - X_{j,t}^N}{X_{i,t}-X_{j,t}}\right]dt\,, \tag{5.7.5}$$

$i\in\{1,\dots,N\}$, $X_0^N = \hat{x}_0^N \in C_N^{BC*}$. Note that formally we allow for $\beta=\infty$ in this SDE by using the convention $1/\infty = 0$. In this latter case, we consider non-random frozen non-compact Jacobi processes as in (3.6.2). In either case, set $\mu_t^N := \frac{1}{N}\sum_{i=1}^N \delta_{Y_{i,t}^N}$ where $(Y_t^N)_{t\geq 0}$ is as in (5.7.1) for some $\gamma_N,\tau_N>0$, $\delta_N\in\mathbb{R}$. Since the drift part in (5.7.5) is just the negative of the drift part in (5.7.3), by (5.7.4) the transformed parameters in (5.7.2) here take the form

$$\tilde{a}_2^N = \frac{2}{\beta}\frac{1}{\tau_N},\ \tilde{a}_1^N = -\frac{2}{\beta}\frac{\gamma_N}{\tau_N}(1-2\delta_N),\ \tilde{a}_0^N = -\frac{2}{\beta}\frac{\gamma_N^2}{\tau_N}\delta_N(1-\delta_N)$$

$$\tilde{b}_1^N = \frac{2}{\beta}\frac{1}{\tau_N}(p_N+q_N-2(N-1)),\ \tilde{b}_0^N = -\frac{2}{\beta}\frac{\gamma_N}{\tau_N}((1-\delta_N)p_N - \delta_N q_N + (2\delta_N-1)(N-1))\,.$$

In the following, we will consider different limit regimes for the parameters $p_N, q_N, \gamma_N, \delta_N, \tau_N$ such that non-degenerate limits of $(\mu_t^N)_{t\geq 0}$ exist. In the case that the $(X_t^N)_{t\geq 0}$ are frozen non-compact Jacobi processes, some of the following results have previously been published in [13] Section 5 where instead of $(X_t^N)_{t\geq 0}$ the process $(2X_{1,t}^N-1,\dots,2X_{N,t}^N-1)$,

$t \geq 0$, is considered. However, therein the restriction $p_N > N-1$ was imposed which by Theorem 3.6.1 can be weakened. Moreover, apart from the semicircle and Marchenko-Pastur-type limits proven in [13], there is another limit regime which has not been considered before. We start with this latter limit point:

#### 5.7.2.1 Free positive multiplicative Brownian motion limit

We consider the following parametrization:

- Choose $\tau_N > 0$ such that $\lim_{N\to\infty} N/\tau_N = 2$.
- Choose $\gamma_N > 0$ with $\lim_{N\to\infty} \gamma_N = 0$.
- Choose $p_N \in \mathbb{R}$, $q_N > N-1$ such that the limits $\lim_{N\to\infty} p_N/N = \hat{p}$, $\lim_{N\to\infty} q_N/N = \hat{q}$ exist.
- Choose $\delta_N \in \mathbb{R}$ such that $\lim_{N\to\infty} \gamma_N \delta_N = 0$.

Choose starting vectors $\hat{x}_0^N \in C_N^{BC*}$ such that the limit

$$\frac{1}{N}\sum_{i=1}^{N} \delta_{\gamma_N(\hat{x}_{i,0}^N - \delta_N)} \xrightarrow{N\to\infty} \mu \in M^1([0,\infty))$$

exists where the convergence is in moments and the moments of $\mu$ satisfy the bounds (4.2.5). Under these conditions, we obtain the following result, which is expected in view of Theorem 3.6.11 and Theorem 5.3.4:

**Corollary 5.7.3.** *It holds*

$$\begin{aligned}\left(\mu_t^N\right)_{t\geq 0} &\xrightarrow{N\to\infty} \left(D_{e^{(\hat{p}+\hat{q}-1)t/2}}\left(\mu \boxtimes \nu_t\right)\right)_{t\geq 0} \\ &= \left(D_{e^{(\hat{p}+\hat{q}-1)t/2}}\left(\mu \boxtimes \exp\left(\mu_{\mathrm{sc},2\sqrt{t}} \boxplus \mathrm{Unif}_{[-t/2,t/2]}\right)\right)\right)_{t\geq 0}\end{aligned}$$

*in $C([0,\infty), M^1(\mathbb{R}))$ a.s. where $\nu_t$ is the distribution of the free positive multiplicative Brownian motion with start in $\mathbf{1}$ at time $t$ (see Section 5.3 for this notion).*

*Proof.* In the chosen scaling we have the limits

$$\lim_{N\to\infty} N\tilde{a}_2^N = \frac{1}{\beta},\ \lim_{N\to\infty} N\tilde{a}_1^N = 0,\ \lim_{N\to\infty} N\tilde{a}_0^N = 0,\ \lim_{N\to\infty} \tilde{b}_1^N = \frac{1}{\beta}(\hat{p}+\hat{q}-2),\ \lim_{N\to\infty} \tilde{b}_0^N = 0\,.$$

By Theorem 4.2.2, we thus know that one has a limit point $(\mu_t)_{t\geq 0}$ in $C([0,\infty), M^1(\mathbb{R}))$. Moreover, $\mu_t$ is determined by its moments $(m_n(t))_{n\in\mathbb{N}}$ which satisfy

$$\frac{d}{dt}m_n(t) = n\left[\frac{1}{2}(\hat{p}+\hat{q}-2)m_n(t) + \frac{1}{2}\sum_{k=0}^{n} m_k(t)m_{n-k}(t)\right].$$

The claim follows from Theorem 5.3.4. □

#### 5.7.2.2 Marchenko-Pastur limit

In comparison to the previous scaling we now consider shorter time scales and a spatial 'zoom in':

- Choose $\tau_N, \gamma_N, \delta_N > 0$ such that

$$\lim_{N\to\infty} N/\tau_N = 0\,, \quad \lim_{N\to\infty} \gamma_N N/\tau_N = 1 \text{ and } \lim_{N\to\infty} (\delta_N - 1)\gamma_N = 0\,.$$

- Choose $q_N > N-1$ and $p_N \in \mathbb{R}$ such that the following two limits exist:

$$\lim_{N\to\infty} q_N/N = \hat{q} \text{ and } \lim_{N\to\infty} p_N/\tau_N = \lambda\,.$$

Choose starting vectors $\hat{x}_0^N \in C_N^{BC*}$ such that the limit $\frac{1}{N}\sum_{i=1}^N \delta_{\gamma_N(\hat{x}_{i,0}^N - \delta_N)} \xrightarrow{N\to\infty} \mu \in M^1([0,\infty))$ exists where the convergence is in moments and the moments of $\mu$ satisfy the bounds (4.2.5). Under these conditions we have:

**Corollary 5.7.4.** *If $\lambda = 0$ then*

$$\left(\mu_t^N\right)_{t\geq 0} \xrightarrow{N\to\infty} \left(\left(\sqrt{\mu}_{\text{even}} \boxplus \mu_{sc,2\sqrt{t}}\right)^2 \boxplus \mu_{\text{MP},\hat{q}-1,t}\right)_{t\geq 0}$$

*in $C([0,\infty), M^1(\mathbb{R}))$ a.s.*
*If $\lambda \neq 0$ then*

$$\left(\mu_t^N\right)_{t\geq 0} \xrightarrow{N\to\infty} \left(\left(D_{e^{\lambda t/2}}\left(\sqrt{\mu}_{\text{even}}\right) \boxplus \mu_{\text{sc},2\sqrt{(e^{\lambda t}-1)/\lambda}}\right)^2 \boxplus \mu_{\text{MP},\hat{q}-1,(e^{\lambda t}-1)/\lambda}\right)_{t\geq 0}$$

*in $C([0,\infty), M^1(\mathbb{R}))$ a.s.*

*Proof.* In the chosen scaling we have the limits

$$\lim_{N\to\infty} N\tilde{a}_2^N = 0,\ \lim_{N\to\infty} N\tilde{a}_1^N = \frac{2}{\beta},\ \lim_{N\to\infty} N\tilde{a}_0^N = 0,\ \lim_{N\to\infty} \tilde{b}_1^N = \frac{2}{\beta}\lambda,\ \lim_{N\to\infty} \tilde{b}_0^N = \frac{2}{\beta}(\hat{q}-1)\,.$$

By Theorem 4.2.2, we thus know that one has a limit point $(\mu_t)_{t\geq 0}$ in $C([0,\infty), M^1(\mathbb{R}))$. Moreover, $\mu_t$ is determined by its moments $(m_n(t))_{n\in\mathbb{N}}$ which satisfy

$$\frac{d}{dt}m_n(t) = n\left[\lambda m_n(t) + (\hat{q}-1)m_{n-1}(t) + \sum_{k=0}^{n-1} m_k(t)m_{n-1-k}(t)\right]\,.$$

A comparison with Theorem 5.2.1 shows the claim. □

#### 5.7.2.3 Semicircle limit

There are many parameter choices such that we get a semicircle-type limit as in Theorem 5.1.1. We here exemplarily state two different conditions under which these limits appear, which are analogous to the scaling considered in Corollary 5.7.2:
Parameter asymptotic (I):

- Choose $\tau_N > 0$ such that $\lim_{N\to\infty} N/\tau_N = 0$.
- Choose $\delta_N > 1$ such that the limit $\lim_{N\to\infty} \delta_N =: \hat{\delta} \in (1, \infty)$ exists.
- Choose $\gamma_N > 0$ such that $\lim_{N\to\infty} \gamma_N^2 N/\tau_N = \frac{1}{2\hat{\delta}(\hat{\delta}-1)}$.
- Choose $p_N \in \mathbb{R}$, $q_N > N-1$ such that $\lim_{N\to\infty} \gamma_N p_N/\tau_N = \lim_{N\to\infty} \gamma_N q_N/\tau_N = 0$.

Parameter asymptotic (II):

- Choose $\tau_N > 0$ such that $\lim_{N\to\infty} N/\tau_N = 0$.
- Choose $q_N > N-1$ and $p_N < -q_N$ such that the limit $\lim_{N\to\infty}(p_N + q_N)/\tau_N =: -\lambda \in (-\infty, 0]$ exists and one of the two conditions $\lim_{N\to\infty} N/q_N = 0$ or $\lambda = 0$ hold.
- Set $\delta_N := p_N/(p_N + q_N) \in (1, \infty)$.
- Choose $\gamma_N > 0$ such that $\lim_{N\to\infty} \frac{N\gamma_N^2}{\tau_N} \frac{-p_N q_N}{(p_N+q_N)^2} = \frac{1}{2}$.

In either case (I) or (II), choose starting vectors $\hat{x}_0^N \in C_N^{BC*}$ such that the limit

$$\frac{1}{N}\sum_{i=1}^{N} \delta_{\gamma_N(\hat{x}_{i,0}^N - \delta_N)} \xrightarrow{N\to\infty} \mu \in M^1(\mathbb{R})$$

exists where the convergence is in moments and the moments of $\mu$ satisfy the bounds (4.2.5).

**Corollary 5.7.5.** *Under the conditions (I) or the conditions (II) in the case $\lambda = 0$ we have*

$$\left(\mu_t^N\right)_{t\geq 0} \xrightarrow{N\to\infty} \left(\mu \boxplus \mu_{sc,2\sqrt{t}}\right)_{t\geq 0}$$

*in $C([0,\infty), M^1(\mathbb{R}))$ a.s.*
*Under the conditions (II) in the case $\lambda > 0$ we have*

$$\left(\mu_t^N\right)_{t\geq 0} \xrightarrow{N\to\infty} \left(D_{e^{-\lambda t}}(\mu) \boxplus \mu_{\mathrm{sc},\sqrt{2(1-e^{-2\lambda t})/\lambda}}\right)_{t\geq 0}$$

*in $C([0,\infty), M^1(\mathbb{R}))$ a.s.*

*Proof.* For the scaling (I) set $\lambda := 0$. Then in both choices of scaling (I) and (II) we have

$$\lim_{N\to\infty} N\tilde{a}_2^N = 0\,,\quad \lim_{N\to\infty} N\tilde{a}_1^N = 0\,,\quad \lim_{N\to\infty} N\tilde{a}_0^N = \frac{1}{\beta}\,,\quad \lim_{N\to\infty} \tilde{b}_1^N = -\frac{2}{\beta}\lambda\,,\quad \lim_{N\to\infty} \tilde{b}_0^N = 0\,.$$

In scaling (II) we here used that by assumption we have

$$\lim_{N\to\infty} (p_N + q_N)/\tau_N = 0 \;\text{ or }\; \lim_{N\to\infty} N/q_N = 0$$

which implies $\lim_{N\to\infty} N\gamma_N/\tau_N = 0$: This follows from

$$\frac{1}{2} = \lim_{N\to\infty} \left(\frac{N\gamma_N}{\tau_N}\right)^2 \frac{\tau_N}{N}\frac{-p_N q_N}{(p_N+q_N)^2}$$

and

$$\frac{\tau_N}{N}\frac{-p_N q_N}{(p_N+q_N)^2} = \frac{\tau_N}{-(p_N+q_N)}\frac{q_N}{N}\frac{1}{1-\frac{q_N}{-p_N}} \xrightarrow{N\to\infty} \infty\,.$$

In particular, we thus have in scaling (II):

$$(N\tilde{a}_1^N)^2 = \left(\frac{2}{\beta}\frac{N\gamma_N}{\tau_N}\frac{p_N - q_N}{p_N+q_N}\right)^2 = \frac{4}{\beta^2}\left(\frac{N^2\gamma_N^2}{\tau_N^2} - 4\frac{N}{\tau_N}\frac{N\gamma_N^2}{\tau_N}\frac{p_N q_N}{(p_N+q_N)^2}\right) \xrightarrow{N\to\infty} 0$$

and

$$\tilde{b}_0^N = -\frac{2}{\beta}\frac{\gamma_N}{\tau_N}\frac{p_N - q_N}{p_N+q_N}(N-1) = -\frac{N-1}{N} N\tilde{a}_1^N \xrightarrow{N\to\infty} 0\,.$$

By Theorem 4.2.2, we thus know that one has a limit point $(\mu_t)_{t\geq 0}$ in $C([0,\infty), M^1(\mathbb{R}))$. Moreover, $\mu_t$ is determined by its moments $(m_n(t))_{n\in\mathbb{N}}$ which satisfy

$$\frac{d}{dt}m_n(t) = n\left[-\lambda m_n(t) + \frac{1}{2}\sum_{k=0}^{n-2} m_k(t) m_{n-2-k}(t)\right]\,.$$

A comparison with Theorem 5.1.1 shows the claim. □

# Chapter 6

# The free additive sum of uniform and semicircle distributions

In this last chapter, we provide a new proof of a formula relating the uniform distribution on an interval, the semicircle distribution and the spectral distribution of the free positive multiplicative Brownian motion considered in Section 5.3. Concretely, we will prove the following:

**Theorem 6.0.1.** *Let $t > 0$. Let $\nu_t$ be the spectral distribution of the free positive multiplicative Brownian motion with start in $\mathbf{1}$ at time $t$. Then*

$$\nu_t = \exp\left(\mu_{\mathrm{sc},2\sqrt{\mathrm{t}}} \boxplus \mathrm{Unif}_{[-t/2,t/2]}\right) . \tag{6.0.1}$$

Note that we previously used this formula to describe limits of empirical measures of non-colliding $\beta$-Pearson diffusions in Theorem 5.3.4, Theorem 5.5.7, Corollary 5.6.5 and Corollary 5.7.3. Theorem 6.0.1 has been shown by the present author and Michael Voit in [12] by relating the eigenvalues of matrices of hermitian Brownian motions with certain drifts to the logarithms of eigenvalues of the Brownian motion on positive definite matrices. As the corresponding empirical spectral distributions approximate the respective left and right hand sides of (6.0.1), the authors obtained the formula above. We will give another proof of this formula which does not rely on matrix approximations but on relating the moments of the appearing distributions. As a by-product we get explicit formulas for the moments of $\mu_{\mathrm{sc,a}} \boxplus \mathrm{Unif}_{[b,c]}$, $a > 0, b < c$, and the integrals $\int_{(0,\infty)} x^\alpha \, d\nu_t$ for all $\alpha \in \mathbb{C}$. To state these results we need to introduce some notation:

**Definition 6.0.2.** For $z \in \mathbb{C}$ and $n \in \mathbb{N}_0$ we denote by $z^{(n)}$ the *$n$-th rising factorial of $z$*, i.e. $z^{(n)} = \prod_{i=0}^{n-1}(z+i)$, where an empty product is interpreted as 1. In the literature $z^{(n)}$ is also known as Pochhammer symbol, which in certain contexts is also denoted by $(z)_n$. We will not adopt the latter notation.
Let $n \in \mathbb{N}_0$. We call the coefficients $s(n,k)$ in $z \mapsto z^{(n)} = \sum_{k=0}^{n}(-1)^{n-k}s(n,k)z^k$ *(signed) Stirling numbers of the first kind.* For $k \in \mathbb{Z} \setminus \{1,\dots,n\}$ we use the common extension $s(n,k) := 0$. For a broader introduction to these numbers including their combinatorial

interpretation see e.g. [53] Section 6.1.
We will denote the *generalized hypergeometric functions* by

$$ {}_AF_B(a_1,\dots,a_A;b_1,\dots,b_B;x) := \sum_{j=0}^{\infty} \frac{a_1^{(j)}\dots a_A^{(j)}}{b_1^{(j)}\dots b_B^{(j)}} \frac{x^j}{j!}\,, $$

where $A, B \in \mathbb{N}_0$ and $a_i, b_j \in \mathbb{C}$, $i \in \{1,\dots,A\}$, $j \in \{1,\dots,B\}$.
Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series. We will denote the $n$-th coefficient by $[z^n]f(z) := a_n$ and call $[z^n]$ *coefficient extractor*.

Throughout this section, we will denote the distribution of the free positive multiplicative Brownian motion at time $t$ with start in $\mathbf{1}$ by $\nu_t$. It turns out that showing the equality of the moments of the respective sides in (6.0.1) is slightly more complicated than showing the equality of moments for the equivalent identity

$$ \exp\left(\mu_{\mathrm{sc},2\sqrt{\mathrm{t}}} \boxplus \mathrm{Unif}_{[-t,0]}\right) = D_{e^{-t/2}}(\nu_t)\,. \tag{6.0.2} $$

This at fist glance might seem a bit surprising as in contrast to (6.0.1) the measure which gets exponentiated is not even anymore. However, the moments of $D_{e^{-t/2}}(\nu_t)$ are simply polynomials in $t$ and thus easier to handle (cf. [19] Proposition 5):

$$ \int_{(0,\infty)} x^n\, d\left(D_{e^{-t/2}}(\nu_t)\right)(x) = \frac{1}{n} L_{n-1}^{(1)}(-nt) = \frac{1}{n}\sum_{j=0}^{n-1} \binom{n}{1+j} \frac{(nt)^j}{j!}\,. \tag{6.0.3} $$

Here $L_{n-1}^{(1)}$ denotes the $n-1$-th Laguerre Polynomial with parameter 1, see [99] Chapter V for an introduction. For the remainder of this section we fix the notation

$$ m_n(t) := \int_{\mathbb{R}} x^n\, d\left(\mu_{\mathrm{sc},2\sqrt{t}} \boxplus \mathrm{Unif}_{[-t,0]}\right)(x)\,,\ n \in \mathbb{N}_0,\ t > 0\,. $$

The main step toward showing Theorem 6.0.1 is the following explicit moment formula:

**Theorem 6.0.3.** *For all $n \in \mathbb{N}_0$, $t > 0$ it holds:*

$$ m_n(t) = n! \sum_{j=\lceil n/2\rceil}^{n} \frac{t^j}{j!(1+j)!} s(1+j, n+1-j)\,. $$

**Remark 6.0.4.** Using this theorem we can calculate the moments of general free additive convolutions of semicircle and uniform distributions as follows: Let $a > 0$ and $b, c \in \mathbb{R}$ with $b < c$. Then

$$ \mu_{\mathrm{sc},2\sqrt{a}} \boxplus \mathrm{Unif}_{[b,c]} = \mu_{\mathrm{sc},2\sqrt{a}} \boxplus \mathrm{Unif}_{[b-c,0]} \boxplus \delta_c\,, $$

where we used that classical additive and free additive convolutions with Dirac measures coincide. For $\gamma \in \mathbb{R}$ let $D_\gamma \colon \mathbb{R} \to \mathbb{R}, x \mapsto \gamma x$ be the scaling map with factor $\gamma$. Then the

image measure of a free additive convolution under $D_\gamma$ linearizes as in the classical case: $D_\gamma(\mu \boxplus \nu) = D_\gamma(\mu) \boxplus D_\gamma(\nu)$, $\mu, \nu \in M^1(\mathbb{R})$. Thus

$$\begin{aligned}\mu_{\mathrm{sc},2\sqrt{a}} \boxplus \mathrm{Unif}_{[b,c]} =& \mu_{\mathrm{sc},2\sqrt{a}} \boxplus \mathrm{Unif}_{[b-c,0]} \boxplus \delta_c \\ =& D_{a/(c-b)}(\mu_{\mathrm{sc},2\sqrt{(c-b)^2/a}}) \boxplus D_{a/(c-b)}(\mathrm{Unif}_{[-(c-b)^2/a,0]}) \boxplus \delta_c \\ =& D_{a/(c-b)}\left( \left(\mu_{\mathrm{sc},2\sqrt{t}}\right) \boxplus \mathrm{Unif}_{[-t,0]}) \big|_{t=(c-b)^2/a} \right) \boxplus \delta_c \,.\end{aligned}$$

Using the binomial theorem we arrive at

$$\int_{\mathbb{R}} x^n \, d\left(\mu_{\mathrm{sc},2\sqrt{a}} \boxplus \mathrm{Unif}_{[b,c]}\right)(x) = n! \sum_{k=0}^{n} \frac{c^{n-k}}{(n-k)!} \sum_{j=\lceil k/2 \rceil}^{k} \frac{(c-b)^{2j-k} a^{k-j}}{j!(1+j)!} s(1+j, k+1-j) \,.$$

We will show Theorem 6.0.3 with the help of the two following Lemmas. The first Lemma states a time inhomogeneous recursion ODE for the moments $m_n$:

**Lemma 6.0.5.** *$m_n$ is a polynomial of degree $n$ with leading coefficient $[t^n]m_n(t) = (-1)^n/(1+n)$, $n \in \mathbb{N}_0$. Further, it holds for all $t > 0$:*

$$\frac{d}{dt} m_n(t) = n \left( t^{-1} m_n(t) - \frac{1}{2} \sum_{j=0}^{n-2} m_j(t) m_{n-j-2}(t) \right) .$$

*Proof.* By Theorem 5.1.1, we know that the moments $\tilde{m}_n(t)$ of

$$\mu_t := \mathrm{Unif}_{[-1,0]} \boxplus \mu_{\mathrm{sc},2\sqrt{t}}$$

satisfy

$$\frac{d}{dt} \tilde{m}_n(t) = \frac{n}{2} \sum_{j=0}^{n-2} \tilde{m}_j(t) \tilde{m}_{n-j-2}(t) \,.$$

Induction on $n$ shows that $\tilde{m}_n$ is a polynomial of degree $n-1$. Note that it holds $[t^0]\tilde{m}_n = \tilde{m}_n(0) = (-1)^n/(1+n)$. Now, using $D_t(\mu_{1/t}) = \mathrm{Unif}_{[-t,0]} \boxplus \mu_{\mathrm{sc},2\sqrt{t}}$, we have $m_n(t) = t^n \tilde{m}_n(1/t)$ for $t > 0$. This shows that $m_n$ is a polynomial of degree $n$ with $[t^n]m_n(t) = (-1)^n/(1+n)$. Moreover, we have

$$\begin{aligned}\frac{d}{dt} m_n(t) =& n t^{n-1} \tilde{m}_n(1/t) - \frac{n}{2} t^{n-2} \sum_{j=0}^{n-2} \tilde{m}_j(1/t) \tilde{m}_{n-2-j}(1/t) \\ =& n t^{-1} m_n(t) - \frac{n}{2} \sum_{j=0}^{n-2} m_j(t) m_{n-2-j}(t) \,.\end{aligned}$$

□

The second Lemma is a combinatorial identity involving a double sum over the product of two Stirling numbers and three binomial coefficients, one of which is inverted. We postpone its technical proof to the end of this chapter.

**Lemma 6.0.6.** *For all $l, m \in \mathbb{N}$ it holds*

$$\begin{aligned} &2ls(1+m,1+l) \\ =&\frac{m+1}{l+m-1}\sum_{n=1}^{l}\sum_{k=0}^{m-1}\binom{l+m-2}{n+k-1}^{-1}\binom{m}{k}\binom{m}{1+k}s(1+k,n)s(m-k,l+1-n)\,. \end{aligned} \tag{6.0.4}$$

*Proof of Theorem 6.0.3.* Set $c(n,k) := [t^k]m_n(t)$, $n \in \mathbb{N}$, $k \in \{0,\dots,n\}$. Then, by Lemma 6.0.5, we know that for all $n \geq 1$ we have $c(n,0) = 0$, $c(n,n) = (-1)^n/(1+n) = n!\frac{1}{n!(1+n)!}s(1+n,n+1-n)$ and:

$$\sum_{k=1}^{n} kc(n,k)t^{k-1} = n\sum_{k=1}^{n} c(n,k)t^{k-1} - \frac{n}{2}\sum_{j=0}^{n-2}\sum_{l=0}^{j} c(j,l)t^l \sum_{m=0}^{n-j-2} c(n-j-2,m)t^m\,.$$

We can rearrange this in the following way:

$$\begin{aligned} \sum_{k=1}^{n}(n-k)c(n,k)t^{k-1} =&\frac{n}{2}\sum_{l=0}^{n-2}\sum_{j=l}^{n-2}\sum_{m=0}^{n-j-2} t^{m+l}c(j,l)c(n-j-2,m) \\ =&\frac{n}{2}\sum_{k=1}^{n-1} t^{k-1}\sum_{l=0}^{k-1}\sum_{j=l}^{l+n-1-k} c(j,l)c(n-j-2,k-1-l)\,, \end{aligned}$$

where we set $k := l+m+1$ and used the equivalence

$$\begin{aligned} &0 \leq l \leq n-2 \ \wedge\ l \leq j \leq n-2 \ \wedge\ 1+l \leq k \leq n+l-1-j \\ \Leftrightarrow\quad &1 \leq k \leq n-2 \ \wedge\ 0 \leq l \leq k-1 \ \wedge\ l \leq j \leq n+l-1-k\,. \end{aligned}$$

Comparison of coefficients yields

$$\begin{aligned} (n-k)c(n,k) =&\frac{n}{2}\sum_{l=0}^{k-1}\sum_{j=l}^{l+n-1-k} c(j,l)c(n-j-2,k-1-l) \\ =&\frac{n}{2}\sum_{l=0}^{k-1}\sum_{j=1}^{n-k} c(j+l-1,l)c(n-l-j-1,k-1-l)\,. \end{aligned}$$

Now, we show $c(n,k) = n!\frac{1}{k!(1+k)!}s(1+k,n+1-k)$, $k \in \{0,\dots,n\}$, by induction on $n \in \mathbb{N}_0$. We already know that this formula holds true for $c(n,n)$ and $c(n,0)$, $n \in \mathbb{N}$. The induction start at $n = 0$ holds since $c(0,0) = 1 = s(1,1)$ and the induction start at $n = 1$ holds by the already proven formulas for $c(n,n)$ and $c(n,0)$. Suppose $n \geq 2$

and that the induction assertion has been proven for all $n' \in \{0, \dots, n-2\}$. Then for $k \in \{1, \dots, n-1\}$, by induction assumption and Lemma 6.0.6, we have:

$$\begin{aligned}
&c(n,k)\\
=&\frac{1}{2}\frac{n}{n-k}\sum_{l=0}^{k-1}\sum_{j=1}^{n-k} c(j+l-1,l)c(n-l-j-1,k-1-l)\\
=&\frac{1}{2}\frac{n}{n-k}\sum_{l=0}^{k-1}\sum_{j=1}^{n-k} \frac{(j+l-1)!}{l!(1+l)!}s(1+l,j)\frac{(n-l-j-1)!}{(k-1-l)!(k-l)!}s(k-l,n+1-j-k)\\
=&\frac{1}{2}\frac{n}{n-k}\frac{1}{k!k!}(n-2)!\sum_{l=0}^{k-1}\sum_{j=1}^{n-k}\binom{n-2}{j+l-1}^{-1}\binom{k}{l}\binom{k}{1+l}s(1+l,j)s(k-l,n+1-j-k)\\
=&\frac{1}{2}\frac{n}{n-k}\frac{1}{k!k!}\frac{(n-1)!}{1+k}2(n-k)s(1+k,1+n-k) = n!\frac{1}{k!(1+k)!}s(1+k,n+1-k)\,.
\end{aligned}$$

Lastly, it holds $c(n,k) = 0$ for all $k < \lceil n/2 \rceil$ since we have $1+k < n+1-k$, which implies $s(1+k, n+1-k) = 0$. □

The formula for the moments $m_n(t)$ in Theorem 6.0.3 enables us to calculate the corresponding moment generating function, which in turn gives the following integral formula for the image measure of $\mu_{\mathrm{sc},2\sqrt{t}} \boxplus \mathrm{Unif}_{[-t/2,t/2]}$ under the exponential map:

**Corollary 6.0.7.** *For all $\alpha \in \mathbb{C} \setminus \{0\}$ and $t \geq 0$ it holds*

$$\int_{(0,\infty)} x^\alpha \, d\nu_t(x) = e^{\alpha t/2}{}_1F_1(1-\alpha; 2; -\alpha t) = \frac{1}{\alpha}e^{\alpha t/2}\sum_{j=0}^{\infty}\binom{\alpha}{1+j}\frac{(\alpha t)^j}{j!}\,.$$

*Proof of Theorem 6.0.1 & Corollary 6.0.7.* Let $\alpha \in \mathbb{C} \setminus \{0\}$. Then, by Theorem 6.0.3, we have

$$\begin{aligned}
&\int_{\mathbb{R}} e^{\alpha x}\, d\left(\mu_{\mathrm{sc},2\sqrt{t}} \boxplus \mathrm{Unif}_{[-t,0]}\right)(x) = \sum_{n=0}^{\infty}\frac{\alpha^n}{n!}m_n(t)\\
=&\sum_{n=0}^{\infty}\alpha^n \sum_{j=\lceil n/2\rceil}^{n}\frac{1}{(1+j)!}\frac{t^j}{j!}s(1+j,n+1-j)\\
=&-\frac{1}{\alpha}\sum_{j=0}^{\infty}\frac{(-\alpha t)^j}{(1+j)!j!}\sum_{k=1}^{1+j}(-1)^{j+1-k}s(1+j,k)(-\alpha)^k\\
=&-\frac{1}{\alpha}\sum_{j=0}^{\infty}\frac{(-\alpha)^{(j+1)}}{(1+j)!}\frac{(-\alpha t)^j}{j!} = {}_1F_1(1-\alpha;2;-\alpha t)\,.
\end{aligned}$$

In particular, it holds for $n \in \mathbb{N}$:

$$\int_{\mathbb{R}} x^n \, d\exp\left(\mu_{\mathrm{sc},2\sqrt{t}} \boxplus \mathrm{Unif}_{[-t,0]}\right)(x) = {}_1F_1(1-n;2;-nt) = \frac{1}{n}\sum_{j=0}^{n-1} \binom{n}{1+j} \frac{(nt)^j}{j!}\,.$$

By (6.0.3), this is the $n$-th moment of $D_{e^{-t/2}}(\nu_t)$. This shows (6.0.2) since measures with compact support are uniquely characterized by their moments. Now, scaling in space by the factor $e^{t/2}$ gives Theorem 6.0.1. In turn also Corollary 6.0.7 holds:

$$\begin{aligned}\int_{(0,\infty)} x^\alpha \, d\nu_t = \int_{\mathbb{R}} e^{\alpha x} \, d\left(\mu_{\mathrm{sc},2\sqrt{t}} \boxplus \mathrm{Unif}_{[-t/2,t/2]}\right)(x) =& e^{\alpha t/2} \int_{\mathbb{R}} e^{\alpha x} \, d\left(\mu_{\mathrm{sc},2\sqrt{t}} \boxplus \mathrm{Unif}_{[-t,0]}\right)(x) \\ =& e^{\alpha t/2} {}_1F_1(1-\alpha;2;-\alpha t)\,.\end{aligned}$$

□

In the remainder of this chapter we prove Lemma 6.0.6 by Egorychev's method which provides effective tools to show combinatorial formulas (cf. [44, 89]).

*Proof of Lemma 6.0.6.* We start with the easiest case $m = 1$, which holds true because of

$$2ls(2,1+l) = \begin{Bmatrix} 2\,, & l=1 \\ 0\,, & \text{else} \end{Bmatrix} = \frac{2}{l} s(1,l) = \frac{2}{l} \sum_{n=1}^{l} \binom{l-1}{n-1}^{-1} s(1,n) s(1,l+1-n)\,.$$

Now let $m \geq 2$. Rearranging the double sum of the right hand side of (6.0.4) gives

$$\begin{aligned}&\frac{m+1}{l+m-1} \sum_{n=1}^{l} \sum_{k=0}^{m-1} \binom{l+m-2}{n+k-1}^{-1} \binom{m}{k} \binom{m}{1+k} s(1+k,n) s(m-k,l+1-n) \\ =& \frac{m+1}{l+m-1} \sum_{n=0}^{l+1} \sum_{k=1}^{m-2} \binom{l+m-2}{n+k-1}^{-1} \binom{m}{k} \binom{m}{1+k} s(1+k,n) s(m-k,l+1-n) \\ &+ 2\frac{m(m+1)}{l+m-1} s(m,l) \\ =:& A + 2\frac{m(m+1)}{l+m-1} s(m,l)\,.\end{aligned} \tag{6.0.5}$$

Now, we apply techniques of the Egorychev method, i.e. we transform a suitable generating series of $A$. To this end we make use of the identities

$$\begin{aligned} s(n,k) &= \frac{n!}{k!} [z^n] \log^k(1+z)\,, \quad n,k \in \mathbb{N}_0\,, \\ \int_0^1 t^n (1-t)^k \, dt &= \frac{1}{n+k-1} \binom{n+k}{n}^{-1}, \quad n,k \in \mathbb{N}_0\,, \end{aligned}$$

see [89] Section 5 for the first identity, the second identity is a special case of the well known beta integral. Now, we can write $A$ in the following way:

$$\begin{aligned}
A =&(m+1)\sum_{k=1}^{m-2}\sum_{n=0}^{l+1}\int_0^1 t^{n+k-1}(1-t)^{l+m-1-n-k}\,dt\binom{m}{k}\binom{m}{1+k}\\
&\cdot\left[x^{1+k}\right]\left[y^{m-k}\right]\frac{(1+k)!}{n!}\frac{(m-k)!}{(l+1-n)!}\log^n(1+x)\log^{l+1-n}(1+y)\\
=&\frac{m(m+1)!}{(l+1)!}\sum_{k=1}^{m-2}\binom{m-1}{k}\left[x^{1+k}\right]\left[y^{m-k}\right]\\
&\cdot\int_0^1 t^{k-1}(1-t)^{m-2-k}\sum_{n=0}^{l+1}\binom{l+1}{n}\left(t\log(1+x)\right)^n\left((1-t)\log(1+y)\right)^{l+1-n}\,dt\\
=&\frac{m(m+1)!}{(l+1)!}\sum_{k=1}^{m-2}\binom{m-1}{k}\left[x^{1+k}\right]\left[y^{m-k}\right]\\
&\cdot\int_0^1 t^{k-1}(1-t)^{m-2-k}\left(t\log(1+x)+(1-t)\log(1+y)\right)^{l+1}\,dt\,.
\end{aligned}$$

Using coefficient extraction for the exponential series, we can rewrite this further as

$$\begin{aligned}
A =&m(m+1)!\left[\alpha^{l+1}\right]\sum_{k=1}^{m-2}\binom{m-1}{k}\left[x^{1+k}\right]\left[y^{m-k}\right]\\
&\cdot\int_0^1 t^{k-1}(1-t)^{m-2-k}(1+y)^{\alpha}\exp\left(\log\left(\frac{1+x}{1+y}\right)\alpha t\right)\,dt\,.
\end{aligned}$$

Now, we can apply Euler's integral transform for Kummer's confluent hypergeometric function ${}_1F_1$ which states that

$${}_1F_1(a;b;x)=\frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)}\int_0^1 t^{a-1}(1-t)^{b-a-1}e^{tx}\,dt\,.$$

This gives

$$\begin{aligned}
&\frac{A}{m(m+1)!}\\
&= \left[\alpha^{l+1}\right] \sum_{k=1}^{m-2} \binom{m-1}{k} \left[x^{1+k}\right] \left[y^{m-k}\right] (1+y)^{\alpha} \frac{(k-1)!(m-2-k)!}{(m-2)!}\\
&\quad \cdot {}_1F_1\left(k; m-1; \log\left(\frac{1+x}{1+y}\right)\alpha\right)\\
&= \left[\alpha^{l+1}\right] \sum_{k=1}^{m-2} \frac{m-1}{k(m-1-k)} \left[x^{1+k}\right] \left[y^{m-k}\right] (1+y)^{\alpha} {}_1F_1\left(k; m-1; \alpha \log\left(\frac{1+x}{1+y}\right)\right)\\
&= \left[\alpha^{l+1}\right] \sum_{k=1}^{m-2} \left(\frac{1}{k} + \frac{1}{m-1-k}\right) \left[x^{1+k}\right] \left[y^{m-k}\right] (1+y)^{\alpha} {}_1F_1\left(k; m-1; \alpha \log\left(\frac{1+x}{1+y}\right)\right).
\end{aligned}$$

Reversing the order of summation in one part of the sum and applying Kummer's transformation rule ${}_1F_1(a;b;x) = e^x {}_1F_1(b-a;b;-x)$, we see

$$\begin{aligned}
&\sum_{k=1}^{m-2} \frac{1}{m-1-k} \left[x^{1+k}\right] \left[y^{m-k}\right] (1+y)^{\alpha} {}_1F_1\left(k; m-1; \alpha \log\left(\frac{1+x}{1+y}\right)\right)\\
=&\sum_{k=1}^{m-2} \frac{1}{k} \left[x^{m-k}\right] \left[y^{1+k}\right] (1+y)^{\alpha} {}_1F_1\left(m-1-k; m-1; \alpha \log\left(\frac{1+x}{1+y}\right)\right)\\
=&\sum_{k=1}^{m-2} \frac{1}{k} \left[x^{m-k}\right] \left[y^{1+k}\right] (1+x)^{\alpha} {}_1F_1\left(k; m-1; \alpha \log\left(\frac{1+y}{1+x}\right)\right).
\end{aligned}$$

Thus

$$\frac{A}{m(m+1)!} = 2 \left[\alpha^{l+1}\right] \sum_{k=1}^{m-2} \frac{1}{k} \left[x^{1+k}\right] \left[y^{m-k}\right] (1+y)^{\alpha} {}_1F_1\left(k; m-1; \alpha \log\left(\frac{1+x}{1+y}\right)\right).$$

Now, we can extend the sum over $k$ to infinity with the following correction due to $[y^{m-k}]$:

$$\begin{aligned}
\frac{A}{2m(m+1)!} &= \left[\alpha^{l+1}\right] \sum_{k=1}^{\infty} \frac{1}{k} \left[x^{1+k}\right] \left[y^{m-k}\right] (1+y)^{\alpha} {}_1F_1\left(k; m-1; \alpha \log\left(\frac{1+x}{1+y}\right)\right)\\
&\quad - \left[\alpha^{l+1}\right] \frac{1}{m-1} \left[x^{m}\right] \left[y^{1}\right] (1+y)^{\alpha} {}_1F_1\left(m-1; m-1; \alpha \log\left(\frac{1+x}{1+y}\right)\right)\\
&\quad - \left[\alpha^{l+1}\right] \frac{1}{m} \left[x^{1+m}\right] \left[y^{0}\right] (1+y)^{\alpha} {}_1F_1\left(m; m-1; \alpha \log\left(\frac{1+x}{1+y}\right)\right)\\
&=: [\alpha^{l+1}](S_1 + S_2 + S_3).
\end{aligned}$$

We will calculate the three appearing terms seperately, where the main work lies in the calculation of $S_1$. For the latter task we aim to identify a geometric series in the summation over $k$. For this we use Cauchy's integral formula in the following way:

$$\begin{aligned}
S_1 =& \sum_{k=1}^{\infty} \frac{1}{k} \left[x^{1+k}\right] \left[y^{m-k}\right] (1+y)^{\alpha} \sum_{j=0}^{\infty} \frac{k^{(j)}\alpha^j}{(m-1)^{(j)} j!} \log^j\left(\frac{1+x}{1+y}\right) \\
=& \sum_{k=1}^{\infty} \frac{1}{k} \left(\frac{1}{2\pi i}\right)^2 \int_{|x|=1/2} \int_{|y|=1/4} x^{-(k+2)} y^{k-(m+1)} (1+y)^{\alpha} \\
& \cdot \sum_{j=0}^{\infty} \frac{k^{(j)}\alpha^j}{(m-1)^{(j)} j!} \log^j\left(\frac{1+x}{1+y}\right) \, dydx \\
=& \left(\frac{1}{2\pi i}\right)^2 \int_{|y|=1/4} \int_{|x|=1/2} x^{-2} y^{-(m+1)} (1+y)^{\alpha} \sum_{j=1}^{\infty} \frac{\alpha^j}{(m-1)^{(j)} j!} \log^j\left(\frac{1+x}{1+y}\right) \\
& \cdot \sum_{k=1}^{\infty} \frac{d^{j-1}}{dz^{j-1}} z^{k+j-1} \Big|_{z=\frac{y}{x}} \, dxdy
\end{aligned}$$

Now, one can easily verify the following identity by induction over $j$:

$$\sum_{k=1}^{\infty} \frac{d^{j-1}}{dz^{j-1}} z^{k+j-1} = \frac{d^{j-1}}{dz^{j-1}} \frac{z^j}{1-z} = (j-1)! \left((1-z)^{-j} - 1\right) , \quad |z| < 1 .$$

Plugging this in we receive

$$\begin{aligned}
S_1 =& \left(\frac{1}{2\pi i}\right)^2 \int_{|y|=1/4} \int_{|x|=1/2} x^{-2} y^{-(m+1)} (1+y)^{\alpha} \\
& \cdot \sum_{j=1}^{\infty} \frac{\alpha^j}{(m-1)^{(j)} j} \log^j\left(\frac{1+x}{1+y}\right) \left(\left(\frac{x}{x-y}\right)^j - 1\right) \, dxdy
\end{aligned}$$

Note that, by using $\log(1+z) = -\sum_{k=1}^{\infty} \frac{(-z)^k}{k}$ $(|z|<1)$, for all $j \geq 2$ and $0 < |y| < 1/2$ we have

$$\begin{aligned}
& \int_{|x|=1/2} x^{j-2} \log^j\left(\frac{1+x}{1+y}\right) \left(\frac{1}{x-y}\right)^j \, dx \\
=& \int_{|x|=1/2} (x+y)^{j-2} \log^j\left(1+\frac{x}{1+y}\right) \left(\frac{1}{x}\right)^j \, dx = 0 .
\end{aligned}$$

This gives

$$
\begin{aligned}
&(2\pi i)^2 S_1 \\
=&\int_{|y|=1/4}\int_{|x|=1/2} x^{-1}(x-y)^{-1}y^{-(m+1)}(1+y)^{\alpha}\frac{\alpha}{m-1}\log\left(\frac{1+x}{1+y}\right)\,dxdy \\
&-\int_{|y|=1/2}\int_{|x|=1/2} x^{-2}y^{-(m+1)}(1+y)^{\alpha}\sum_{j=1}^{\infty}\frac{\alpha^j}{(m-1)^{(j)}j}\log^j\left(\frac{1+x}{1+y}\right)\,dxdy \\
=&: (2\pi i)^2\left(S_{1,1}+S_{1,2}\right)\,.
\end{aligned}
$$

To simplify $S_{1,1}$ we first observe

$$
\begin{aligned}
&\int_{|x|=1/2} x^{-1}(x-y)^{-1}\log\left(\frac{1+x}{1+y}\right)\,dx \\
=&-y^{-1}\int_{|x|=1/2} x^{-1}\log\left(\frac{1+x}{1+y}\right)\,dx + y^{-1}\int_{|x|=1/2}(x-y)^{-1}\log\left(\frac{1+x}{1+y}\right)\,dx \\
=&(2\pi i)y^{-1}\log(1+y)+y^{-1}\int_{|x|=1/2} x^{-1}\log\left(1+\frac{x}{1+y}\right)\,dx = (2\pi i)y^{-1}\log(1+y)\,.
\end{aligned}
$$

This gives

$$
\begin{aligned}
S_{1,1} =&\frac{1}{2\pi i}\int_{|y|=1/4} y^{-(m+2)}(1+y)^{\alpha}\frac{\alpha}{m-1}\log(1+y)\,dy \\
=&\frac{1}{m-1}\left[y^{m+1}\right](1+y)^{\alpha}\log\left((1+y)^{\alpha}\right) \\
=&\frac{1}{m-1}\frac{1}{(m+1)!}\sum_{j=0}^{\infty}(j+1)\alpha^{j+1}s(m+1,j+1)\,.
\end{aligned}
\tag{6.0.6}
$$

The second contribution to $S_1$ can be simplified in the following way:

$$
\begin{aligned}
S_{1,2} =& -\left(\frac{1}{2\pi i}\right)^2 \int_{|y|=1/2}\int_{|x|=1/2} x^{-2}y^{-(m+1)}(1+y)^{\alpha}\sum_{j=1}^{\infty}\frac{\alpha^j}{(m-1)^{(j)}j}\log^j\left(\frac{1+x}{1+y}\right)dxdy\\
=& -\left[x^1\right]\left[y^m\right](1+y)^{\alpha}\sum_{j=1}^{\infty}\frac{\alpha^j}{(m-1)^{(j)}j}\sum_{k=0}^{j}\binom{j}{k}\log^k(1+x)\left(-\log(1+y)\right)^{j-k}\\
=& -\left[y^m\right](1+y)^{\alpha}\sum_{j=1}^{\infty}\frac{\alpha^j(-1)^{j-1}}{(m-1)^{(j)}}\log^{j-1}(1+y)\\
=& -\left[y^m\right]\sum_{n=0}^{\infty}\frac{\alpha^n}{n!}\sum_{j=1}^{\infty}\frac{\alpha^j(-1)^{j-1}(m-2)!}{(m+j-2)!}\log^{n+j-1}(1+y)\\
=& -(m-2)!\left[y^m\right]\sum_{k=0}^{\infty}\alpha^{k+1}\log^k(1+y)\sum_{n=0}^{k}\frac{(-1)^{n-k}}{n!(m+k-n-1)!}\\
=&\frac{(m-2)!}{m!}\sum_{k=0}^{\infty}(-\alpha)^{k+1}k!s(m,k)\frac{1}{(m+k-1)!}\sum_{n=0}^{k}(-1)^n\binom{m+k-1}{n}\\
=&\frac{(m-2)!}{m!}\sum_{k=0}^{\infty}(-\alpha)^{k+1}k!s(m,k)\frac{(-1)^k}{(m+k-1)!}\binom{m+k-2}{k}\\
=& -\frac{1}{m!}\sum_{k=0}^{\infty}\frac{\alpha^{k+1}}{m+k-1}s(m,k)\,,
\end{aligned}
$$

where we used the identity $\sum_{n=0}^{k}(-1)^n\binom{N}{n}=(-1)^k\binom{N-1}{k}$ in the second last equality. This completes the calculation of $S_1$.
Using ${}_1F_1(m-1;m-1;x)=e^x$, we see that $S_2$ does not contribute:

$$
S_2 = -\frac{1}{m-1}\left[x^m\right]\left[y^1\right](1+y)^{\alpha}\left(\frac{1+x}{1+y}\right)^{\alpha}=0\,.
$$

Applying ${}_1F_1(m;m-1;x)=e^x{}_1F_1(-1;m-1;-x)=e^x(1+x/(m-1))$, we can simplify the remaining sum as follows:

$$
\begin{aligned}
S_3 =& -\frac{1}{m}\left[x^{m+1}\right]\left[y^0\right](1+y)^{\alpha}\left(\frac{1+x}{1+y}\right)^{\alpha}\left(1+\frac{\alpha}{m-1}\log\left(\frac{1+x}{1+y}\right)\right)\\
=& -\frac{1}{m}\left[x^{m+1}\right](1+x)^{\alpha}\left(1+\frac{\alpha}{m-1}\log(1+x)\right)\\
=& -\frac{1}{m(m+1)!}\sum_{j=0}^{\infty}\alpha^j s(m+1,j)-\frac{1}{m(m-1)(m+1)!}\sum_{j=0}^{\infty}(j+1)\alpha^{j+1}s(m+1,j+1)\,.
\end{aligned}
$$

Combining the previous calculations we get

$$\begin{aligned}
A =& 2m(m+1)!\left[\alpha^{l+1}\right](S_{1,1}+S_{1,2}+S_2+S_3)\\
=& 2m(m+1)!\left(\frac{l+1}{(m-1)(m+1)!}s(m+1,l+1)-\frac{1}{m!(m+l-1)}s(m,l)\right.\\
&\left.-\frac{1}{m(m+1)!}s(m+1,l+1)-\frac{l+1}{m(m-1)(m+1)!}s(m+1,l+1)\right)\\
=& -2\frac{m(m+1)}{m+l-1}s(m,l)+2m\frac{(l+1)m-(m-1)-(l+1)}{m(m-1)}s(m+1,l+1)\\
=& -2\frac{m(m+1)}{m+l-1}s(m,l)+2ls(m+1,l+1)\,.
\end{aligned}$$

Plugging this back into (6.0.5) shows the claim. □

# List of symbols

| | |
|---|---|
| $\mathbb{N}$ | Set of positive integers |
| $\mathbb{N}_0$ | Set of non-negative integers |
| $\mathbb{C}^{\pm}$ | Upper/lower complex half plane |
| $\mathbb{T}$ | Unit circle in the complex plane |
| $\overline{z}$ for $z \in \mathbb{C}$ | Complex conjugate |
| $\operatorname{int} A$ | Interior of the set $A$ |
| $\partial A$ | Boundary of the set A |
| $C_N^A$ | $\{x \in \mathbb{R}^N \colon x_1 \leq \cdots \leq x_N\}$ |
| $C_N^B$ | $\{x \in \mathbb{R}^N \colon 0 \leq x_1 \leq \cdots \leq x_N\}$ |
| $C_N^{BC}$ | $\{x \in \mathbb{R}^N \colon 0 \leq x_1 \leq \cdots \leq x_N \leq 1\}$ |
| $C_N^{BC*}$ | $\{x \in \mathbb{R}^N \colon 1 \leq x_1 \leq \cdots \leq x_N\}$ |
| $x^{(n)}$ | Rising factorial $\prod_{i=0}^{n-1}(x+i)$, $x \in \mathbb{C}$, $n \in \mathbb{N}_0$ |
| $\mathcal{B}(E)$ | Borel $\sigma$-algebra of topological space $E$ |
| $M^1(E)$ | Set of probability measures on $(E, \mathcal{B}(E))$ |
| $\delta_a$ | Dirac measure at point $a$ |
| $\mu_{\mathrm{sc},R}$ | Semicircle distribution with radius $R$; see Section 2.1 |
| $\mu_{\mathrm{MP},\alpha,\beta}$ | Marchenko-Pastur distribution with parameters $\alpha, \beta > 0$; see Section 2.1 |
| $\mu_x$ | $*$-distribution of the operator $x$ |
| $\mathrm{Unif}_A$ | Uniform distribution on $A$ |
| $G_\mu$ | Cauchy transform of $\mu \in M^1(\mathbb{R})$; see Section 2.4.1 |
| $\psi_\mu$ | $\psi$-transform of $\mu \in M^1([0,\infty))$; see Section 2.4.2 |
| $S_\mu$ | $S$-transform of $\mu \in M^1([0,\infty)) \setminus \{\delta_0\}$; see Section 2.4.2 |
| $R_\mu$ | $R$-transform of $\mu \in M^1(\mathbb{R})$; see Section 2.4.1 |
| $f(\mu)$ | Image measure of $\mu$ under $f$ |
| $D_a$ | Linear map $x \mapsto ax$ |
| $\mathbb{1}_A(x)$ | Indicator function of the set $A$ at point $x$ |
| $\mathbf{1}$, $\mathbf{0}$ | Unit and zero element of a unital Algebra |
| $M_N(\mathbb{C})$ | Space of complex $N \times N$ matrices |
| $\mathcal{P}_N$ | Cone of complex positive definite $N \times N$ matrices |
| $\mathrm{Gl}(N, \mathbb{F})$ | General linear group of degree $N$ over the field $\mathbb{F}$ |
| $\pi_n(A)$ | Upper left $n \times n$ block of a $N \times N$ matrix $A$, $n \in \{1, \ldots, N\}$ |
| $A \leq B$ | Loewner order for hermitian $A, B \in M_N(\mathbb{C})$ |
| $E(X)$ | Expected value of the random variable $X$ |

| | |
|---|---|
| $[X]_t$ | Quadratic variation of the semimartingale $(X_s)_{s\geq 0}$ at time $t$ |
| $[X,Y]_t$ | Quadratic covariation of the semimartingales $(X_s)_{s\geq 0}, (Y_s)_{s\geq 0}$ at time $t$ |
| $\boxplus$ | Free additive convolution; see Section 2.4.1 |
| $\boxtimes$ | Free multiplicative convolution; see Section 2.4.2 |
| $\lceil x \rceil$ | Ceiling function evaluated at $x \in \mathbb{R}$ |

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