UNIVERSITY OF DORTMUND

REIHE COMPUTATIONAL INTELLIGENCE

COLLABORATIVE RESEARCH CENTER 531

Design and Management of Complex Technical Processes and Systems by means of Computational Intelligence Methods

On Semantic Models for Investigating "Computing with Words"

Helmut Thiele

No. CI-32/98

Technical Report ISSN 1433-3325 April 1998

Secretary of the SFB 531 · University of Dortmund · Dept. of Computer Science/XI
44221 Dortmund · Germany

This work is a product of the Collaborative Research Center 531, "Computational Intelligence", at the University of Dortmund and was printed with financial support of the Deutsche Forschungsgemeinschaft.

On Semantic Models for Investigating "Computing with Words"*

Helmut Thiele

In daily life we can meet a lot of instructions having the form "many" plus "some" is "many", for instance. This paper presents a semantic interpretation as basis for defining arithmetic operations for such words.

1 Introduction

At present we can observe a fast and fruitful development of soft computing, but in general, on a more or less intuitive basis. Therefore, we are faced with the problem of designing and investigating a theory of soft computing on a precise conceptional basis.

In contrast to "hard computing" the key idea of soft computing consists in including vagueness, imprecision and uncertainty of data and knowledge which are to be processed.

Soft computing covers at least the areas of

- fuzzy logic
- neural networks
- genetic algorithms,

this opinion is a widely accepted interpretation.

In the following we shall consider only the area of fuzzy logic. Following L. A. ZADEH [20] we are of the opinion that computing with words is an independent part of soft computing which covers fuzzy logic, or more exactly, which uses the concepts and results of fuzzy logic to develop its own concepts, theorems, and algorithms.

In the fuzzy community it is widely accepted that the term fuzzy logic has two meanings, in the narrow and in the wide sense. With respect to our thesis above we shall specify computing with words as

- computing with words in the narrow sense and
- computing with words in the wide sense.

Roughly speaking, computing with words in the narrow sense has the goal of developing an "arithmetics" of computing with vague or with uncertain values. This goal includes to adopt as many usual arithmetic concepts and algorithms as possible, i. e. procedures for processing natural numbers, integers, rational and real numbers, for instance.

Computing with words in the wide sense, so we think, should be classified into

- reasoning with words
- modelling with words
- programming with words.

Because of restricted space, in the paper presented we shall discuss only Computing with Words in the *narrow* sense. Investigations of this area in the *wide* sense will follow in forthcoming papers.

*Revised version of a paper presented as a Keynote Address at *Second International Conference on Knowledge-Based Intelligent Electronic Systems*, Adelaide, Australia, April 21–23, 1998

2 Some fundamental algebraic and arithmetic concepts

Already in school we learned several “arithmetic” structures and “arithmetic” algorithms. So we know how the addition and the multiplication of (non-negative) integers are to be carried out. Furthermore, we know how these operations and the subtraction in the case of integers and, finally, the division of rational (or real) numbers are to be carried out.

The mathematical background of these procedures are certain (special) algebraic structures. Because these structures will play a crucial role in defining “computing with words” in the *narrow* sense we remind their definitions for definiteness.

The starting point and the basis of all these structures is the concept of monoid.

Definition 1

$\mathfrak{M} = [M, \circ]$ *is said to be a monoid*

$=_{\text{def}}$ *1. M is a non-empty set and*

2. $\circ$ is a total binary associative operation on M with values in M.

Remarks

1. M is called the domain of $\mathfrak{M}$.
2. An element $n \in M$ is said to be a neutral element of $\mathfrak{M}$ if and only if $\forall x(x \in M \rightarrow x \circ n = n \circ x = x)$. Obviously, a monoid has at most one neutral element.
3. If in $\mathfrak{M}$ there exists a (uniquely determined) neutral element, then we shall call $\mathfrak{M}$ a monoid with neutral element.
4. A monoid $\mathfrak{M}$ is called commutative, if its operation $\circ$ is commutative.
5. A (commutative) monoid $\mathfrak{M}$ is called a (commutative) group if and only if $\forall a \forall b \big(a, b \in M \rightarrow \exists x(a \circ x = b) \wedge \exists y(y \circ a = b)\big)$.

Definition 2

$\mathfrak{M} = [M, \circ, \leqq]$ *is said to be a totally (partially) ordered monoid*

$=_{\text{def}}$ *1. $[M, \circ]$ is a monoid and*

2. $\leqq$ is a total (partial) order relation on M such that

$\forall x \forall y \forall z \big(x, y, z \in M \wedge x \leqq y \rightarrow x \circ z \leqq y \circ z \wedge z \circ x \leqq z \circ y\big).$

Example 1

Let A be a non-empty set, A^* the set of all finite sequences of elements from A including the empty sequence e. For $s = a_1 \cdots a_m$ and $t = b_1 \cdots b_n$ where $m, n \geqq 1$ and $a_1, \ldots, a_m, b_1, \ldots, b_n \in A$ we define a binary operation

$$\begin{aligned} &s \circ t =_{\text{def}} a_1 \cdots a_m b_1 \cdots b_n \\ &e \circ t =_{\text{def}} t \circ e =_{\text{def}} t \\ &e \circ e =_{\text{def}} e. \end{aligned}$$

Then $\mathfrak{A} = [A^*, \circ]$ is a monoid with the neutral element e. Furthermore, $\mathfrak{A}$ is commutative if and only if $\operatorname{card} A = 1$.

Example 2

$M =_{\text{def}} \mathbb{N} =_{\text{def}} \{0, 1, \ldots\}$, $\circ$ is the addition $+$ of natural numbers. Then $\mathfrak{N}^+ = [\mathbb{N}, +]$ is a commutative monoid with the neutral element 0.

Example 3

$M =_{\text{def}} \mathbb{N} =_{\text{def}} \{0, 1, \dots\}$, $\circ$ is the multiplication $\times$ of natural numbers. Then $\mathfrak{N}^{\times} = [\mathbb{N}, \times]$ is a commutative monoid with the neutral element 1.

On the basis of the concept of (ordered) monoid we can very easily define further algebraic structures suitable for describing arithmetical procedures.

Definition 3
$\mathfrak{S} = [S, +, \times]$ is said to be a Semi Ring
$=_{\text{def}}$ *1. $[S, +]$ is a commutative monoid*
2. $[S, \times]$ is a monoid
3. $\forall x \forall y \forall z (x, y, z \in S \rightarrow x \times (y+z) = (x \times y) + (x \times z) \wedge (x+y) \times z = (x \times z) + (y \times z))$.

Remarks

1. $\mathfrak{S}$ is said to be a commutative semiring if and only if $\mathfrak{S}$ is a semi ring and the monoid $[S, \times]$ is commutative.
2. $\mathfrak{S}$ is a semiring with the zero element 0 and the unit element 1 if and only if 0 and 1 are the neutral element of the monoids $[S, +]$ and $[S, \times]$, respectively.

Definition 4
$\mathfrak{S} = [S, +, \times]$ is said to be a (commutative) ring
$=_{\text{def}}$ *1. $\mathfrak{S}$ is a (commutative) semi ring and*
2. $[S, +]$ is a (commutative) group.

Remark
$\mathfrak{S} = [S, +, \times, \leqq]$ is called a totally (partially) ordered ring if and only if $[S, +, \times]$ is a ring, $[S, +, \leqq]$ is a totally (partially) ordered monoid, and

$$\forall a \forall b \forall x \big(a, b, x \in S \wedge a \leqq b \wedge 0 \leqq x \rightarrow a \times x \leqq b \times x\big)$$

and

$$\forall a \forall b \forall y \big(a, b, y \in S \wedge a \leqq b \wedge y \leqq 0 \rightarrow b \times y \leqq a \times y\big)$$

where 0 is the zero element of $[S, +]$. If a semi ring $\mathfrak{S}$ has a zero element, then this definition can be adopted without modification.

Example 4

$S =_{\text{def}} \mathbb{I} =_{\text{def}} \{0, +1, -1, +2, -2, \dots\}$ (the set of integers) Let $+, \times$ and $\leqq$ denote the addition, multiplication, and natural order of integers. Then $[S, +, \times, \leqq]$ is an ordered ring with the zero element 0 and the unit element 1.

Finally, we define

Definition 5
$\mathfrak{S} = [S, +, \times]$ is said to be a (commutative) field
$=_{\text{def}}$ *1. $\mathfrak{S}$ is a (commutative) ring and*
2. $[S \setminus \{0\}, \times]$ is a group.

Remarks

1. A field $\mathfrak{S}$ is called totally (partially) ordered if and only if the ring $\mathfrak{S}$ is totally (partially) ordered.
2. "Standard" examples for totally ordered fields are
   - the field of rational numbers and
   - the field of real numbers.
3. A partially (but not totally) ordered field is the field of complex numbers.

So, we have remembered the most important arithmetic structures and their algebraic descriptions. Now, using these notions we are going to construct arithmetic structures whose domain is not a set of numbers, but a set of words.

Each of the words carries a certain semantic meaning and this meaning serves to define arithmetical operations (addition, multiplication and so forth) for these words.

3 On semantic interpretations of words

Beside the concept of word in a natural language, we start with a mathematical formalization of this concept. Let A be an arbitrary non-empty set called alphabet. A word w on A is an arbitrary finite or maybe infinite sequence of elements (letters) of the alphabet A.

Furthermore, we fix an arbitrary set U called universe. Independently of other more or less intuitive descriptions we use the following mathematical definition of a granule and of granulation on U.

Definition 6

1. *A is said to be a crisp granule on U*
   $=_{\mathrm{def}} A \subseteqq U$.
2. *$\mathfrak{G}$ is said to be a crisp granulation consisting of crisp granules on U*
   $=_{\mathrm{def}} \mathfrak{G} \subseteqq \mathbb{P}(U)$ *where* $\mathbb{P}(U)$ *is the power set of* U.
3. *F is said to be a fuzzy granule on U*
   $=_{\mathrm{def}} F : U \to \langle 0, 1\rangle$.
4. *$\mathfrak{F}$ is said to be a crisp granulation consisting of fuzzy granules on U*
   $=_{\mathrm{def}} \mathfrak{F} \subseteqq F\mathbb{P}(U)$
   where $F\mathbb{P}(U) =_{\mathrm{def}} \{F | F : U \to \langle 0, 1\rangle\}$.

Remarks

1. The case of a fuzzy granulation Φ consisting of crisp granules on U, i. e. $\Phi : \mathbb{P}(U) \to \langle 0, 1\rangle$, will not be considered.
2. The case of a fuzzy granulation Ψ consisting of fuzzy granules on U, i. e. $\Psi : F\mathbb{P}(U) \to \langle 0, 1\rangle$, will not be considered, either.
3. With respect to the remarks 1 and 2 we call $\mathfrak{G}$ (see item 2 of definition 6) and $\mathfrak{F}$ (see item 4 of definition 6) shortly crisp and fuzzy granulation on U, respectively.
4. We underline that "granule" and "granulation" are only new names for well-known things, but introduced with respect to the following interpretations and applications.

Now, we are going to develop three mathematical interpretations (meanings), namely on the following three levels. Therefore let W be a set of words, i. e. $W \subseteq A^* \cup A^\omega$ where A^* is defined in example 1 and A^ω denotes the set of all infinite sequences α from elements of A, i. e. $\alpha : \{1, 2, \dots\} \to A$. Furthermore, let $\mathfrak{G}$ and $\mathfrak{F}$ be a crisp and a fuzzy granulation on U, respectively.

Definition 7
Interpretation of words on the level of elements (level 1)

σ is said to be a semantic interpretation of the words of W by elements of U
$=_{\text{def}} \sigma : W \to U$.

Definition 8
Interpretation of words on the level of crisp granulations (level 2)

σ is said to be a semantic interpretation of the words of W by granules of the crisp granulation $\mathfrak{G}$
$=_{\text{def}} \sigma : W \to \mathfrak{G}$.

Definition 9
Interpretation of words on the level of fuzzy granulations (level 3)

σ is said to be a semantic interpretation of the words of W by granules of the fuzzy granulation $\mathfrak{F}$
$=_{\text{def}} \sigma : W \to \mathfrak{F}$.

We underline that the distinction of the three levels described above is very important in developing the concepts of "Computing with Words in the Narrow Sense" and for its understanding.

We start these considerations with some examples.

Example 5
(level 1)
Put $A_1 =_{\text{def}} \{0, 1, \dots, 9\}$.
Words on A_1 are, for instance, 105, 2200, or 0150. Using the decimal system of representing natural numbers, the meaning of $\mu(w)$ where w is a finite word on A_1 is clear. Notice, that different words can have the same meaning, for instance 23, 023, 0023, … .

Remark
The example above gives the occasion to hint at the following misleading formulations in literature [20]. There "computing with numbers" is confronted with "computing with words" which does not truly reflect the logical and algorithmic situation in the present case.

We state that computing with words appears already in domains of numbers. For instance, if we have natural numbers represented in the decimal system by 214 and 3708, then their sum is given by 3922, where the word 3922 can be "computed" using the words 214 and 3708 as inputs.

Using only the concept of semantic interpretation (i. e. without the concept of algorithm) one can describe this situation as follows:

Let δ be the usual semantic interpretation of words from A_1 using the decimal system. Then we have

$$3922 \in \delta^{-1}(\delta(214) + \delta(3708)),$$

i. e.

$$\delta(3922) = \delta(214) + \delta(3708).$$

The last equation means that the semantic meaning of the word 3922 equals the sum of the natural numbers $\delta(214)$ and $\delta(3708)$. Finally, we underline that this example demonstrates "Computing with Words", but in a very simple case. Already in elementary school we have learned how the word 3922 is computed starting with the words 214 and 3708 as inputs. Obviously, there are infinitely many words from A_1 which have the same semantic meaning as 3708, namely 03708, 003708, ...

Example 6
(level 1)
Put $A_2 =_{\text{def}} A_1 \cup \{., -\}$.
Then to represent an arbitrary real number we need infinite words, for instance, $\pi = 3.14\ldots$.
Furthermore, finite and infinite words on A_2 can have the same meaning, for instance 1 and $0.99\cdots$.

4 Semantic interpretations of words by crisp granules and computing with words via crisp granulations

Now, we move on to level 2. Referring to literature from artificial intelligence (see, for instance, [5, 7, 14]), computing with words on the level 2 can be illustrated by the following example: Take as A_1 the usual Latin alphabet and take the words NEGATIVE and POSITIVE. If NEGATIVE and POSITIVE denote an arbitrary negative and positive integer, respectively, then the product is a negative integer, hence we can state that the "product" NEGATIVE×POSITIVE is defined and fulfils the equation

$$\text{NEGATIVE} \times \text{POSITIVE} = \text{NEGATIVE}.$$

How to place this heuristic argumentation on a correct and well-defined mathematical basis? Within approaches made in artificial intelligence under the titel "Qualitative Computing" one can find a methodology explained by the following example:

Example 7
(level 2)
We fix the universe U as the set $\mathbb{I}$ of all integers. We choose the words NEGATIVE, ZERO, and POSITIVE (for short N, Z, and P, respectively) and define WORDS $=_{\text{def}} \{N, Z, P\}$. Furthermore, we fix the following crisp granules NEG, ZER, and POS in U, where

$$\begin{aligned} \text{NEG} &=_{\text{def}} \{-1, -2, -3, \ldots\} \\ \text{ZER} &=_{\text{def}} \{0\} \\ \text{POS} &=_{\text{def}} \{1, 2, 3, \ldots\} \end{aligned}$$

and put $\mathfrak{G} = \{\text{NEG}, \text{ZER}, \text{POS}\}$. Now, using the crisp granulation $\mathfrak{G}$ we define a semantic interpretation of the words from WORDS as follows

$$\begin{aligned} \sigma(N) &=_{\text{def}} \text{NEG} \\ \sigma(Z) &=_{\text{def}} \text{ZER} \\ \sigma(P) &=_{\text{def}} \text{POS}. \end{aligned}$$

To define operations for these granules we "lift" the addition and multiplication from $\mathbb{I}$ into the power set $\mathbb{P}(\mathbb{I})$ by the following definition. Assume $A, B \in \mathbb{I}$. Then we define

$$A + B =_{\text{def}} \{a + b | a \in A \wedge b \in B\}$$
$$A \times B =_{\text{def}} \{a \times b | a \in A \wedge b \in B\}.$$

Now, using σ we define in the domain WORDS two binary operations $\oplus, \otimes$ as follows where $w, w' \in$ WORDS:

$$w \oplus w' =_{\text{def}} \sigma^{-1}(\sigma(w) + \sigma(w'))$$
$$w \otimes w' =_{\text{def}} \sigma^{-1}(\sigma(w) \times \sigma(w')).$$

So, we get the following tables describing an "addition" and a "multiplication" in the set WORDS:

$\oplus$	N	Z	P
N	N	N	-
Z	N	Z	P
P	-	P	P

$\otimes$	N	Z	P
N	P	Z	N
Z	Z	Z	Z
P	N	Z	P

Obviously, the structure [WORDS, $\otimes$] is a commutative monoid with the neutral element P. The structure [WORDS, $\oplus$] is not a monoid, because the operation $\oplus$ is partial, i. e. $N \oplus P$ and $P \oplus N$ are not defined. The reason is that NEG + POS = POS + NEG = $\mathbb{I}$, but $\mathbb{I} \notin$ {NEG, ZER, POS}.

So, we introduce a new word A (ARBITRARY) and choose the new granulation

$$\mathfrak{H} = \{\text{NEG}, \text{ZER}, \text{POS}, \mathbb{I}\}.$$

Notice, that $\mathfrak{G}$ is a partition of U, whereas $\mathfrak{H}$ is only a covering of U.

Now, the new domain $\mathfrak{H}$ is closed with respect to + and $\times$, hence we get the tables

$\oplus$	N	Z	P	A
N	N	N	A	A
Z	N	Z	P	A
P	A	P	P	A
A	A	A	A	A

$\otimes$	N	Z	P	A
N	P	Z	N	A
Z	Z	Z	Z	Z
P	N	Z	P	A
A	A	Z	A	A

Proposition 1
The structure [WORDS, $\oplus, \otimes$] is a commutative semi ring with the zero element Z and the unit element P.

In the following we move on to level 3.

5 Partially ordered sets of words

Let W be an arbitrary non-empty set of words from a given alphabet A. Assume that on W a binary relation $\leqq$ exists which is reflexive on W and transitive.

We underline that in applications the relation $\leqq$ could be antisymmetric or, additionally, even linear. Furthermore, if the words from W come from the natural language, then the relation $\leqq$ is given by the use of the words in this language, as the following examples 8, 9, and 10 show.

Example 8
(level 3)
Consider the linguistic variable AMOUNT_OF_MONEY. As linguistic terms for this variable we choose the following set W_1 of words:

$$W_1 = \{\text{TINY, VERY\_SMALL, SMALL, MEDIUM, LARGE, VERY\_LARGE, HUGE}\}.$$

From the common use of these words in the (natural) English language we obtain the following binary relation $\prec_1$ between the words of W_1:

$$\text{TINY} \prec_1 \text{VERY\_SMALL} \prec_1 \text{SMALL} \prec_1 \\ \text{MEDIUM} \prec_1 \text{LARGE} \prec_1 \text{VERY\_LARGE} \prec_1 \text{HUGE}.$$

Let $\leqq_1$ be the reflexive-transitive closure of $\prec_1$ with respect to W_1. Then $[W_1, \leqq_1]$ is a totally ordered set.

Example 9
(level 3)
Take the same set W_1 of words and the same relation $\prec_1$ (as in example 8). Add the linguistic variable AGE and for AGE the set

$$W_2 = \{\text{YOUNG, MIDDLE-AGED, OLD, VERY\_OLD}\}$$

as its linguistic terms. Take the relation $\prec_2$ defined by

$$\text{YOUNG} \prec_2 \text{MIDDLE-AGED} \prec_2 \text{OLD} \prec_2 \text{VERY\_OLD}.$$

Let $\leqq_2$ be the reflexive-transitive closure of $\prec_2$ with respect to W_2. Then $[W_2, \leqq_2]$ is a totally ordered set, but $[W_1 \cup W_2, \leqq_1 \cup \leqq_2]$ is only a partially ordered set.

Example 10
(level 3)
We add to W_2 the word AGED, i. e. consider $W_3 = W_2 \cup \{\text{AGED}\}$. Furthermore, add to the relation $\prec_2$ the relation $\prec'$ described by

$$\text{MIDDLE-AGED} \prec' \text{AGED} \prec' \text{OLD} \prec' \text{AGED} \prec' \text{VERY\_OLD}.$$

Let $\leqq_3$ be the reflexive-transitive closure of $\prec_2 \cup \prec'$ with respect to W_3. Then $\leqq_3$ is reflexive on W_3, furthermore, $\leqq_3$ is transitive, but not antisymmetric, because we have OLD $\leqq_3$ AGED and AGED $\leqq_3$ OLD and OLD $\neq$ AGED. This situation can be justified by the observation that "OLD MAN" and "AGED MAN" have the same meaning (see [4], for instance), but the words are different.

Now, starting with a fixed set W of words and a reflexive-transitive relation $\leqq$ on W we are looking for an universe U and for a fuzzy granulation $\mathfrak{F}$ on U such that the following conditions 1 and 2 hold:

1. there exists a reflexive-transitive relation $\leqq'$ on $\mathfrak{F}$ and
2. there exists a semantic interpretation

   $$\sigma : W \to \mathfrak{F}$$

   of the words from W by granules from $\mathfrak{F}$ such that σ is a homomorphism with respect to the structures $[W, \leqq]$ and $[\mathfrak{F}, \leqq']$, i. e. the condition

   $$(1) \qquad \forall w_1, w_2 \big(w_1, w_2 \in W \wedge w_1 \leqq w_2 \to \sigma(w_1) \leqq' \sigma(w_2)\big)$$

   holds.

With respect to applications one could claim that σ is a strong homomorphism, i. e. that additionally holds

(2) $$\forall w_1 \forall w_2 \big(w_1, w_2 \in W \wedge \sigma(w_1) \leqq' \sigma(w_2) \to w_1 \leqq w_2\big).$$

If we assume the stronger condition

(3) $$\forall w_1 \forall w_2 (w_1, w_2 \in W \wedge \sigma(w_1) = \sigma(w_2) \to w_1 = w_2)$$

then we get that σ is an injection from $[W, \leqq]$ into $[\mathfrak{F}, \leqq']$, hence σ is an isomorphism between $[W, \leqq]$ and $[\sigma(\mathfrak{F}), \leqq^*]$ where $\sigma(\mathfrak{F}) =_{\text{def}} \{\sigma(W) | w \in W\}$ and $\leqq^*$ is the restriction of $\leqq'$ to $\sigma(\mathfrak{F})$.

At the first glance in the following considerations we take the set $\mathbb{R}$ of all real numbers as the universe U. Hence, in contrast to other approaches (see [3, 6, 8–11, 24], for instance) granules F are arbitrary fuzzy sets on $\mathbb{R}$, i. e. $F : \mathbb{R} \to \langle 0, 1 \rangle$. We underline that for certain applications the choice of $\mathbb{R}^n$ or of a suitable metric space as universe U could lead to better models for describing a given problem of application.

The crucial point is to fix a reflexive-transitive relation $\leqq'$ on the given (real) granulation because in literature one can find a lot of definitions (see [10, 24], for instance) and, at the first glance, there is no idea which of these relations should be taken or whether a completely different relation has to be defined.

We claim that the relation $\leqq'$ must be monotone with respect to *operations* with granules. The best way to reach this goal is, we think, first to define operations with granules and then to derive from them a suitable relation $\leqq'$.

6 Real Fuzzy Granules and Computing with them

In literature one can find definitions of operations with real fuzzy numbers and with so-called fuzzy quantities.

Obviously, one can adopt these definitions to arbitrary (real) fuzzy sets on $\mathbb{R}$. Assume $F, G : \mathbb{R} \to \langle 0, 1 \rangle$, $r, s, t \in \langle 0, 1 \rangle$.

Definition 10

1. $(F + G)(r) =_{\text{def}} \operatorname{Sup}\{\min(F(s), G(t)) | s, t \in \mathbb{R} \wedge r = s + t\}$
2. $(F - G)(r) =_{\text{def}} \operatorname{Sup}\{\min(F(s), G(t)) | s, t \in \mathbb{R} \wedge s = r + t\}$
3. $(F \times G)(r) =_{\text{def}} \operatorname{Sup}\{\min(F(s), G(t)) | s, t \in \mathbb{R} \wedge r = s \times t\}$
4. $(F/G)(r) =_{\text{def}} \operatorname{Sup}\{\min(F(s), G(t)) | s, t \in \mathbb{R} \wedge s = r \times t\}$
5. $F_c(R) =_{\text{def}} \begin{cases} 1 & \text{if } r = c \\ 0 & \text{if } r \neq c \end{cases}$ *where* $r \in \mathbb{R}$ *and* $c \in \langle 0, 1 \rangle$.

Proposition 2

1. $[F\mathbb{P}(U), +]$ is a commutative monoid with F_0 as zero element.
2. $[F\mathbb{P}(U), \times]$ is a commutative monoid with F_1 as zero element.
3. $\forall F \forall G \forall H \big(F, G, H \in F\mathbb{P}(\mathbb{R}) \to F \times (G + H) \subseteqq ((F \times G) + (F \times H))\big)$.

Because of lacking space we can not investigate the algebraic and arithmetic properties of the structure $[F\mathbb{P}(U), +, -, \times, /]$.

Now, we want to introduce a binary relation $\leqq'$ for fuzzy sets from $F\mathbb{P}(\mathbb{R})$ such that

1. $\leqq'$ is at least reflexive and transitive on $F\mathbb{P}(\mathbb{R})$ and
2. the operations introduced above are monotone with respect to $\leqq'$.

The leading idea comes from the following condition valid in the set $\mathbb{R}$ of real numbers:

$$\forall r \forall s \big(r, s \in \mathbb{R} \rightarrow \big(r \leqq s \leftrightarrow \exists t (t \geqq 0 \wedge r + t = s)\big)\big).$$

For defining the relation $\leqq'$ we need the concept "non-negative" (i. e. $t \geqq 0$) and the addition +. For adopting the concept "non-negativ" we define for $F \in F\mathbb{P}(\mathbb{R})$.

Definition 11
F *is said to be non-negative*
$=_{\mathrm{def}} \forall r \big(r \in \mathbb{R} \wedge F(r) > 0 \rightarrow r \geqq 0\big)$.

Using this concept and the operation + for fuzzy sets F, G on $\mathbb{R}$ we put

Definition 12
$F \leqq' G =_{\mathrm{def}} \exists H \big(H \in F\mathbb{P}(U) \wedge H \textit{ is non-negative} \wedge F + H \supseteqq G\big)$.

Proposition 3
1. $\leqq$ is reflexive and transitive on $F\mathbb{P}(\mathbb{R})$.
2. The operation + is monotone with respect to $\leqq$.
3. $\forall F \forall G \forall H \big(F, G, H \in F\mathbb{P}(\mathbb{R}) \wedge H$ is non-negative $\wedge F \leqq G \rightarrow F \times H \leqq G \times H\big)$.

7 Computing with words via their semantic interpretations by real fuzzy granules

Let W be a set of words and $\leqq$ a binary relation on W which is reflexive and transitive (maybe, additionally, antisymmetric and/or linear).

Furthermore, let $\mathfrak{F}$ be a (real) granulation. Consider on $\mathfrak{F}$ the relation $\leqq'$ introduced by definition 12.

Finally, let σ be a homomorphism from $[W, \leqq]$ onto $[\mathfrak{F}, \leqq']$. (Obviously, the assumption "onto" is no restriction of the generality.)

Now, via σ we translate the operations defined in $F\mathbb{P}(U)$ from $\mathfrak{F}$ to W. For short, we consider only the operator +. Let w_1, w_2, and w be arbitrary words from W.

Definition 13
$w = w_1 + w_2 \quad =_{\mathrm{def}} \quad \sigma(w) = \sigma(w_1) + \sigma(w_2)$.

This definition causes several problems.

1. In the following we shall assume that $\forall w \forall w' (w, w' \in W \wedge \sigma(w) = \sigma(w') \rightarrow w = w')$ holds. In this case the result w is uniquely determined and w can be expressed by $w_1 \oplus w_2 = \sigma^{-1}(\sigma(w_1) + \sigma(w_2))$.
2. With respect to the last equation we have to state that $\oplus$ is a partial operation because $\mathfrak{F}$ will not be closed with respect to the operation, in general.

We see the following two approaches to overcome this lack.

Approach 1
We replace $\mathfrak{F}$ by its closure $\mathfrak{F}^c$ with respect to $+$ defined as the intersection of all $\mathfrak{F} \subseteqq F\mathbb{P}(\mathbb{R})$ such that $\mathfrak{F} \subseteqq \mathfrak{F}'$ and $\mathfrak{F}'$ is closed under $+$. But this approach includes the following disadvantage: In $\mathfrak{F}^c$ there can exist granules F which do not have "names", i. e. $F \in \mathfrak{F}^c$ and there is no $w \in W$ such that $\sigma(w) = F$. To eliminate this gap we could add new words to W.

Approach 2
Define

$$\mathfrak{F}^* =_{\text{def}} \mathfrak{F} \cup \{\sigma(w) + \sigma(w') | w, w' \in W\}.$$

We have a mapping

$$\text{LA} : \mathfrak{F}^* \to \mathfrak{F}$$

which fulfils

$$\text{LA}(F) = F \quad \text{ if } F \in \mathfrak{F}.$$

Then we define for $w, w' \in W$

$$w + w' =_{\text{def}} \sigma^{-1} \left(\text{LA}\left(\sigma(w_1) + \sigma(w_2)\right)\right)$$

Obviously, this concept is close to the concept of linguistic approximation introduced by L. A. ZADEH in [2, 18].

This approach causes some difficulties. Obviously, the operation $w \oplus w'$ is commutative because the addition of real granules is commutative. The associativity $w_1 \oplus (w_2 \oplus w_3) = (w_1 \oplus w_2) \oplus w_3$ follows if LA fulfils

$$\forall F \forall G \forall H \left(F, G, H \in \mathfrak{F}^* \to \text{LA}\left(F + \text{LA}(G + H)\right) = \text{LA}\left(\text{LA}(F + G) + H\right)\right).$$

Because of lacking space we have to stop our considerations here. Summarizing the considerations above to design semantic models for computing with words in the narrow sense we have to carry out the following steps.

Step 1 Fix a set W of words on an alphabet A. Fix on W a binary relation $\leqq$ which is reflexive and transitive on W, at least.

Step 2 Fix a real granulation $\mathfrak{F}$ on $\mathbb{R}$, i. e. a set of fuzzy sets on $\mathbb{R}$ such that $\text{card}\, W = \text{card}\, \mathfrak{F}$. Define a semantic interpretation $\sigma : W \to \mathfrak{F}$ and a binary relation $\leqq'$ on $\mathfrak{F}$ such that
- σ is a bijection between W and $\mathfrak{F}$
- σ is a homomorphism from $[W, \leqq]$ onto $[\mathfrak{F}, \leqq']$.

Step 3 Define in $\mathfrak{F}$ operations $+, -, \times$, and $/$ such that $[\mathfrak{F}, \leqq, +, -, \times, /]$ or $[\mathfrak{F}, \leqq, +, \times]$, for instance, is a reasonable arithmetical structure, i. e. an ordered semi ring, ordered ring, or even, an ordered field, if possible.

We underline that step 3 includes a lot of pure mathematical problems unsolved up to now, even for "small" sets W, i. e. $\text{card}\, W = 2, \ldots, \text{card}\, W = 7$ (important for applications).

Step 4 Translate the operations $+, -, \times, /$ from $\mathfrak{F}$ to operations on W using the semantic interpretation σ.

Very important is the fact that the operations defined in W via σ, (and maybe LA) depend on the originally given relation $\leqq$ on W. The claim that the operations $+, -, \times, /$ must be monotone with respect to $\leqq'$ implies certain restrictions to the definition of these operations. Finally, these restrictions are "translated" into restrictions in defining the translated operations in W.

Acknowledgement The author likes to thank Stephan Lehmke and Mario Thalheim for fruitful scientific discussions and the last-named for his help in preparing the manuscript.

References

[1] *International Panel Conference on Soft and Intelligent Computing*, Budapest, Hungary, October 7–10, 1996.

[2] R. E. BELLMANN and L. A. ZADEH. *Local and fuzzy logics*. In: J. M. DUNN and G. EPSTEIN (editors), *Modern Uses of Multiple-Valued Logic — Invited Papers of 5th ISMVL Symposium 1975*, pages 103–165. Reidel, Dordrecht, 1977.

[3] BERNADETTE BOUCHON-MEUNIER, OLGA KOSHELEVA, VLADIK KREINOVICH and HUNG T. NGUYEN. *Fuzzy numbers are the only fuzzy sets that keep invertible operations invertible*. Fuzzy Sets and Systems**91**, 155–163, 1997.

[4] LESLEY BROWN (editor). *The New Shorter Oxford English Dictionary*. Oxford University Press, Oxford, 1993.

[5] J. DE KLEER and J. S. BROWN. *A Qualitative Physics Based on Confluences*. Artificial Intelligence **24**, 1984.

[6] DIMITAR P. FILEV and RONALD R. YAGER. *Operations on fuzzy numbers via fuzzy reasoning*. Fuzzy Sets and Systems**91**, 137–142, 1997.

[7] B. KUIPERS. *Qualitative Simulation*. Artificial Intelligence **29**, 1986.

[8] MILAN MAREŠ. *Computation Over Fuzzy Quantities*. CRC Press, Boca Raton, 1994.

[9] MILAN MAREŠ. *How much is "many" minus "several"?* In: *International Panel Conference on Soft and Intelligent Computing* [1], pages 187–192.

[10] MILAN MAREŠ. *Weak arithmetics of fuzzy numbers*. Fuzzy Sets and Systems**91**, 143–153, 1997.

[11] ANDREA MARKOVÁ-STUPŇANOVÁ. *A note to the addition of fuzzy numbers based on a continuous Archimedean T-norm*. Fuzzy Sets and Systems**91**, 253–258, 1997.

[12] VILÉM NOVÁK. *Is Crucial Role in Soft Computing Played by Words?* In: *International Panel Conference on Soft and Intelligent Computing* [1], pages 223–228.

[13] WITOLD PEDRYCZ. *Granular computing in fuzzy modeling and data mining*. Talk held at the University of Dortmund.

[14] O. RAIMAN. *Order of magnitude Reasoning*. Artificial Intelligence **51**, 1991.

[15] R. R. YAGER, S. OVCHINNIKOV, R. M. TONG and H. T. NGUYEN (editors). *Fuzzy Sets and Applications — Selected Papers by L. A. Zadeh*. John Wiley & Sons, 1987.

[16] LOTFI A. ZADEH. *Outline of a new approach to the analysis of complex systems and decision processes*. IEEE Trans. on Systems, Man and Cybernetics **3** (1), 28–44, 1973. Reprinted in [15].

[17] LOTFI A. ZADEH. *The Concept of a Linguistic Variable and its Application to Approximate Reasoning — I.* Information Sciences **8**, 199–249, 1975. Reprinted in [15].

[18] LOTFI A. ZADEH. *Fuzzy Logic and Approximate Reasoning*. Synthese **30**, 407–428, 1975.

[19] LOTFI A. ZADEH. *Fuzzy Sets and Information Granularity*. In: M. M. GUPTA, R. K. RAGADE and R. R. YAGER (editors), *Advances in Fuzzy Set Theory and Applications*, pages 3–18. North-Holland, Amsterdam, New York, Oxford, 1979.

[20] LOTFI A. ZADEH. *Fuzzy Logic = Computing with Words*. IEEE Transactions On Fuzzy Systems **4** (2), 103–111, 1996.

[21] LOTFI A. ZADEH. *Fuzzy Logic and the Calculi of Fuzzy Rules and Fuzzy Graphs: A Precis*. Multiple Valued Logic — An International Journal**1** (1), 1996.

[22] LOTFI A. ZADEH. *The Key Roles of Information Granulation and Fuzzy Logic in Human Reasoning, Concept Formulation and Computing with Words*. In: *FUZZ-IEEE '96 — Fifth IEEE International Conference on Fuzzy Systems*, New Orleans, USA, September 8–11, 1996.

[23] LOTFI A. ZADEH. *Toward a theory of fuzzy information granulation and its centrality in human reasoning and fuzzy logic*. Fuzzy Sets and Systems**90**, 111–127, 1997.

[24] KUN LUN ZHANG and KAORU HIROTA. *On fuzzy number lattice* $(\tilde{R}, \leq)$. Fuzzy Sets and Systems**92**, 113–122, 1997.