The weak Pareto law and regular variation in the tails[1]

by

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Version October 2002

Abstract

We show that the weak Pareto law, as used to characterize the tail behaviour of income distributions, implies regularly varying tail probabilities, but that the reverse implication does not hold. We also establish implications among other versions of the weak Pareto law.

Key words: Pareto law, regular variation, tail probabilities.

JEL numbers: C19, D31.

1 Introduction and Summary

The strong Pareto law requires that for a distribution function $F_{(x_0,\alpha)}(x)$,

$$\frac{x^{-\alpha}}{1-F(x)} = 1 \qquad (x \geq x_0 > 1, \alpha > 0). \tag{1}$$

[1]Research supported by Deutsche Forschungsgemeinschaft (DFG) under Sonderforschungsbereich 475. We are grateful to Christian Kleiber for useful discussions and comments.

It was first suggested by Pareto (1896) as a "universal law" for income distributions; it immediately leads to the Pareto distribution $F(x) = 1 - (x_0/x)^\alpha$. The weak Pareto law by Mandelbrot (1960) only requires that

$$\lim_{x\to\infty} \frac{x^{-\alpha}}{1-F(x)} = 1. \tag{2}$$

Almost all popular income distributions obey the weak Pareto law.

Merkies and Steyn (1993, Theorem 1) claim that the weak Pareto law is equivalent to the regular variation of $1-F(x)$. We show below that this is not true. While (2) implies that $1-F(x)$ is regularly varying with index $-\alpha$, the reverse implication does not hold. We also establish relationships among other versions of the weak Pareto law which have been suggested in the literature.

2 Various versions of the weak Pareto law

By definition, a function $f : \mathbf{R}_+ \to \mathbf{R}_+$ is regulary varying at infinity with index ρ (in short: $f \in RV_\rho$) if

$$\lim_{t\to\infty} \frac{f(tx)}{f(t)} = x^\rho. \tag{3}$$

Trivially, $x^{-\alpha} \in RV_{-\alpha}$. Assuming that the weak Pareto law (2) holds, we have

$$\lim_{t\to\infty} \frac{(tx)^{-\alpha}}{1-F(tx)} \frac{1-F(t)}{t^{-\alpha}} = \lim_{t\to\infty} \frac{x^{-\alpha}}{\frac{1-F(tx)}{1-F(t)}} = 1. \tag{4}$$

Therefore,

$$\lim_{t\to\infty} \frac{1-F(tx)}{1-F(t)} = x^{-\alpha}\ , \quad \text{and} \quad 1-F(x) \in RV_{-\alpha}.$$

This is the first part of Theorem 1 in Merkies and Steyn (1993).

However, it need not hold that $1-F(x) \in RV_{-\alpha}$ implies the weak Pareto law. Take

$$1 - F(x) = \frac{x^{-\alpha}}{\ell n(x)}. \tag{5}$$

Distributions of this type are characterized by an asymptotically constant slope in the Pareto diagram. Then $x^{-\alpha}/\ell n(x) \in RV_{-\alpha}$, but

$$\frac{x^{-\alpha}}{1-F(x)} = \ell n(x) \to \infty \quad \text{as} \quad x \to \infty. \tag{6}$$

Another implication of the weak Pareto law first explored by Kakwani (1980) is

$$\lim_{t\to\infty} \frac{xf(x)}{1-F(x)} = \alpha > 0. \tag{7}$$

Following Merkies and Steyn (1993), we refer to this as the Kakwani weak Pareto law (KWPL). The relationship (7) is immediate from

$$\lim_{x\to\infty} \frac{x^{-\alpha}}{1-F(x)} \;=\; \lim_{x\to\infty} \frac{\alpha x^{-\alpha-1}}{f(x)} = 1, \tag{8}$$

which implies

$$\lim_{x\to\infty} \frac{\frac{x^{-\alpha}f(x)}{\alpha x^{-\alpha-1}}}{1-F(x)} \;=\; \lim_{x\to\infty} \frac{1}{\alpha}\,\frac{xf(x)}{1-F(x)} = 1. \tag{9}$$

From Karamata's (1930) theorem, the relationship (7) is equivalent to
$f' \in RV_{-\alpha-1}$. See also Bingham et al. (1987).

Yet another version of the weak Pareto law is the requirement, discussed by Esteban (1986), that

$$\lim_{x\to\infty} \frac{f(x)+xf'(x)}{-f(x)} = \alpha > 0. \tag{10}$$

Following Merkies and Steyn, we call this the Esteban weak Pareto law (EWPL). Contrary to what is claimed in Esteban (1986), it is not weaker than the Kakwani weak Pareto law. This has already been noted by Merkies and Steyn (1993). The reason is that, if the limit in (10) exists, it must be equal to the limit in (7), as (10) is obtained from (7) by taking derivatives in the numerator and denominator. However, the limit need not exist.

If it exists, one can again invoke Karamata's theorem to show that then $f'(x) \in RV_{-\alpha-2}$. We therefore have the following chain of implications:

$$\begin{array}{ccccc}
 & & WPL & & \\
 & \nearrow & & \searrow & \\
PL & & & & KWPL \Longleftrightarrow 1-F \in RV_{-\alpha} \Longleftrightarrow f \in RV_{-\alpha-1} \\
 & \searrow & & \nearrow & \\
 & & EWPL \Longleftrightarrow f' \in RV_{-\alpha-2} & &
\end{array}$$

3 Some examples

For each of the implications above, we give an example of an economic income distribution which satisfies the weaker law but not the stronger one.

(i) WPL $\not\Rightarrow$ PL:

Take the Lomax-distribution, where

$$1 - F(x) = \left[1 + \left(\frac{x - x_0}{\sigma}\right)\right]^{-\alpha} \quad (x \geq x_0, \alpha > 0)$$

or the log-logistic distribution, where

$$1 - F(x) = \left[1 + \left(\frac{x - x_0}{\sigma}\right)^{\frac{1}{\gamma}}\right]^{-1} \quad (x \geq x_0, \gamma > 0)$$

It is easily seen that both obey the weak Pareto law, but not the Pareto law.

(ii) EWPL $\not\Rightarrow$ PL:

Take the log-Pareto distribution discussed in Ziebach (2000), where

$$f(x) = \frac{k(\alpha\ \ell n x + \beta)}{x^{\alpha+1}(\ell n\ x)^{\beta+1}} \quad (x \geq x_0 > 0, \alpha > 0, \beta \geq -\alpha\ \ell n x_0) \tag{11}$$

and where $k = x_0^\alpha(\ell n\ x_0)^\beta$. It is straightforwardly checked that

$$\lim_{x\to\infty}\left(1 + \frac{xf'(x)}{f(x)}\right) = -\alpha,$$

so the distributions obey the Esteban weak Pareto law. However, from

$$\lim_{x\to\infty} \frac{x^{-\alpha}}{1-F(x)} = \lim_{x\to\infty} \frac{x^{-\alpha}}{\frac{x_0^{\alpha}(\ell n\, x_0)^{\beta}}{x^{\alpha}(\ell n\, x)^{\beta}}} = \lim_{x\to\infty} k(\ell n\, x)^{\beta} = \infty, \tag{12}$$

it is also obvious that it does not obey the weak Pareto law, and therefore, a fortiori, the Pareto law.

(iii) KWPL $\not\Rightarrow$ EWPL

Take the example given in Merkies and Steyn (1993) where $F(x) = 1 - e^{-\varphi(x)}$ with $\varphi(x) > 0$ and nondescending, $\varphi(0) = 0$ and $\varphi(\infty) = \infty$. Setting

$$\varphi'(x) = \frac{\alpha}{x} + \frac{1 + sin(x)}{x^2}, \tag{13}$$

it is straightforwardly checked that

$$\frac{x\, f(x)}{1-F(x)} = x\varphi'(x) = \alpha + \frac{1 + sin(x)}{x} \to \alpha \tag{14}$$

as $x \to \infty$, but the limit in (10) does not exist. Therefore, the distribution does not obey the Esteban weak Pareto law.

(iv) KWPL $\not\Rightarrow$ WPL

Take once more the log-Pareto distribution from (11). We have already shown below that it obeys the Esteban weak Pareto law. From

$$\lim_{x\to\infty} \frac{xf(x)}{1-F(x)} = \lim_{x\to\infty} \frac{\alpha\, \ell n\, x + \beta}{\ell n\, x} = \alpha, \tag{15}$$

it is obvious that it also obeys the Kakwani weak Pareto law. However, we have already seen in (12) that it does not obey the weak Pareto law.

One can also show that neither the weak Pareto law nor the Esteban weak Pareto law implies the other. We have already seen in (12) that the log-Pareto distribution obeys the Esteban weak Pareto law, but not the Kakwani weak

Pareto law. To see that the weak Pareto law likewise does not imply the Esteban weak Pareto law, let, for large x,

$$1 - F(x) = (x + sin\ x)^{-\alpha}. \tag{16}$$

Then

$$\frac{x^{-\alpha}}{1 - F(x)} = e^{-\alpha[\ell n(x+sin\ x) - \ell n(x)]} \to 1, \tag{17}$$

so the weak Pareto law obtains. However, it is easily checked that

$$\lim_{x\to\infty} \frac{xf'(x)}{f(x)}$$

does not exist, so the Esteban weak Pareto law does not hold.

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