The Consistency of s^2 in the Linear Regression Model When the Disturbances are Spatially Correlated

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Abstract: Conditions for the consistency of the estimator s^2 of the variance of the disturbance σ_u^2 under first-order spatial error processes are given.

Key words: Ordinary least squares, Consistency, Spatial error process, Spatial correlation.

1 Introduction

Consider the linear regression model for spatial correlation

$$y = X\beta + u \quad , \quad u = C\epsilon \quad , \tag{1}$$

where y is a $T \times 1$ observable random vector, X is a $T \times k$ matrix of known constants with full column rank k, β is a $k \times 1$ vector of unknown parameters, ϵ is a $T \times 1$ random vector with expectation zero and covariance matrix $Cov(\epsilon) = \sigma_\epsilon^2 I$ (I is the T-dimensional identity matrix and σ_ϵ^2 an unknown positive scalar). C denotes a $T \times T$ matrix such that the product CC' is positive definite and has identical diagonal elements.

The ordinary least squares (OLS) estimator of the unknown parameter β in model (1) is given by $\hat{\beta} = (X'X)^{-1}X'y$ with the covariance matrix $Cov(\hat{\beta}) = \sigma_\epsilon^2 (X'X)^{-1}X'V_*X(X'X)^{-1}$, where $V_* = CC'$.

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The OLS based estimator $s^2 = (y - X\hat{\beta})'(y - X\hat{\beta})/(T-k)$ of the disturbance variance, under linear regression model with correlated disturbances, is biased and inconsistent in general (see Dhrymes 1978, Chapter 3). This means that when the disturbances are correlated, the standard formulae for testing hypothesis and constructing confidence intervals with respect to the regression coefficients lead to incorrect conclusions.

Several papers investigate the behaviour of the bias of s^2 under different correlation structures (Martin 1974; Neudecker 1977, 1978; Dufour 1986, 1988; Krämer 1991; Kiviet and Krämer 1992; Fiebig et al., 1992; Song 1994). In contrast, there are very few published studies on the problem concerning the inconsistency of the variance estimator in the presence of correlation. Based on the sample variance of the disturbances, Krämer and Berghoff (1991) give a simple sufficient condition for the consistency of s^2. Baltagi and Krämer (1994) deal with the consistency of the estimator in the linear regression model with error component disturbances.

The present paper provides conditions for the consistency of the estimator s^2 when the disturbances follow a first-order spatial error processes.

2 Consistency of s^2

Spatial dependence among the disturbance terms can be expressed in a number of ways. In general, an autoregressive or a moving average formulation could be used as is frequently done in time series analysis.

Let the components of u follow a stationary first-order spatial autoregressive (AR(1)) process

$$u_i = \rho \sum_{j=1}^{T} w_{ij} u_j + \epsilon_i$$

or, in matrix form

$$u = \rho W u + \epsilon \quad , \tag{2}$$

where ρ denotes a spatial correlation coefficient. W is a weights matrix with known nonnegative weights defined by (see Cliff and Ord, 1981, pp. 17-19)

$$w_{ij} \begin{cases} > 0 \quad , & \text{if regions } R_i \text{ and } R_j \text{ are neighbours} \quad (i \neq j) \\ = 0 \quad , & \text{otherwise} \quad . \end{cases}$$

The element w_{ij} of the weights matrix shows the strength of the effect of region R_j on region R_i.

When the components of u are of the pattern

$$u_i = \rho \sum_{j=1}^{T} w_{ij} \epsilon_j + \epsilon_i$$

or, in matrix form

$$u = \rho W \epsilon + \epsilon \quad , \tag{3}$$

then we have another scheme which is known as first-order spatial moving average (MA(1)) process.

Equations (2) and (3) can be written as

$$u = (I - \rho W)^{-1} \epsilon \qquad \text{and} \qquad u = (I + \rho W) \epsilon \quad , \tag{4}$$

respectively, where in AR(1) case the matrix $I - \rho W$ must be nonsingular. From (1) and (4), we get four possible structures of $Cov(u) = \sigma_\epsilon^2 CC' = \sigma_\epsilon^2 V_*$ for first-order spatial error process:

$$V_* = \begin{cases} (I + \rho W)(I + \rho W') & : \quad MA(1) \\ (I + \rho W) & : \quad MA(1) - \text{conditional} \\ (I - \rho W)^{-1}(I - \rho W')^{-1} & : \quad AR(1) \\ (I - \rho W)^{-1} & : \quad AR(1) - \text{conditional} \quad . \end{cases} \tag{5}$$

Note that the possible values of ρ must be identified to ensure that V_* is positive definite (see Horn and Johnson, 1985, p. 301). According to the assumptions given in model (1) the matrix V_* has identical diagonal elements, and denoting this element by υ, the covariance of u can be expressed as

$$Cov(u) = \sigma_\epsilon^2 V_* = (\upsilon \sigma_\epsilon^2) V = \sigma_u^2 V \quad , \tag{6}$$

where $V = (1/\upsilon)V_*$, and $\sigma_u^2 = \upsilon\sigma_\epsilon^2$ is the variance of the disturbances u_i, $i = 1, \cdots, T$. Using the above assumptions under spatial process we can now write model (1) as the general linear regression model:

$$y = X\beta + u \quad , \quad E(u) = 0 \quad , \quad Cov(u) = \sigma_u^2 V \quad . \tag{7}$$

Let $\mu_i(A)$ be the i-th eigenvalue of the matrix A, and let $\xrightarrow{p}$ and $\xrightarrow{q.M.}$ denote convergence in probability and in quadratic mean, respectively. Under the assumptions of model (7) Krämer and Berghoff (1991) state that the OLS based estimator $S^2 = (T-k)s^2/T$ of σ_u^2 is weakly consistent if

$$\frac{u'u}{T} \xrightarrow{p} \sigma_u^2 \quad \text{and} \quad \mu_{max}(V) = o(T)\,, \tag{8}$$

where $\mu_{max}(V)$ denotes the maximum eigenvalue of V. In other words, S^2 is weakly consistent if the sample variance of the true disturbances is consistent, and $\mu_{max}(V)/T \;\rightarrow\; 0$ as $T \rightarrow \infty$.

Whether the above result is operational under spatial error process, depends on the form of the error process and the weights matrix W. Note that the consistency of s^2 is implied by that of S^2 because $(T-k)/T$ goes to one as T goes to infinity.

In the following, conditions for the consistency of S^2 in the presence of spatial correlation will be given. For this purpose, the following results are needed.

Definition

An interval (ρ_l, ρ_u), $\rho_l, \rho_u \in [-1, 1]$, where $\rho_l \leq \rho_u$, for a real valued function $f : (\rho_l, \rho_u) \rightarrow I\!R$ is said to be suitable if

$$\lim_{(\rho,T)\rightarrow(\rho_u,\infty)} \frac{f(\rho)}{T} = \lim_{(\rho,T)\rightarrow(\rho_l,\infty)} \frac{f(\rho)}{T} = \quad 0 \quad , \tag{9}$$

that is, for $\rho \rightarrow \rho_l$ or $\rho \rightarrow \rho_u$ we have $f(\rho) = o(T)$. $\diamond$

In this paper, we focus on the positive values of ρ, so the suitable interval in the above definition becomes (ρ_l, ρ_u) with $\rho_l, \rho_u \in (0, 1]$.

Lemma 1

Suppose that the weights matrix W is symmetric with row sums equal to unity, and let $V = (1/\upsilon)V_*$, where V_* is as given in (5) with diagonal elements all equal to υ. Then $\mu_{max}(V) = o(T)$ for values of ρ from a suitable interval (ρ_l, ρ_u), $\rho_l > 0$.

Proof:

The asserted result will be proved for MA(1) and conditional AR(1) cases given in (5). For the proofs of AR(1) and conditional MA(1) cases, similar arguments can be used.

Under first-order spatial moving average process the matrix V is given by $V = (1/\upsilon)(I + \rho W)(I + \rho W')$. Using the assumption that the matrix W is symmetric we can express the eigenvalues of V in terms of the eigenvalues of W as

$$\mu_i(V) = \frac{1}{\upsilon}(1 + \rho\mu_i(W))^2 \quad , \quad \upsilon\,,\, \rho > 0.$$

Denoting the largest eigenvalue of the weights matrix W by $\mu_{max}(W)$, and assuming that the eigenvalues of W and V are in ascending order for positive values of ρ we have

$$\mu_i(V) \leq \frac{1}{\upsilon}(1 + \rho\mu_{max}(W))^2 \quad .$$

If the row sums of W are all equal to one, then the absolute value of $\mu_i(W)$ is less than or equal to one for all i (see Graybill, 1983, p. 98). This implies that $\mu_{max}(W) \leq 1$ and

$$\mu_i(V) \leq \frac{1}{\upsilon}(1 + \rho)^2 \quad , \quad \rho > 0\,.$$

From this we get $\mu_{max}(V) = o(T) \quad .$

For the conditional AR(1) case, the matrix V is given as $V = (\upsilon(I - \rho W))^{-1}$, and

$$\mu_i(V) = \frac{1}{\upsilon\,(1 - \rho\mu_i(W))} \quad .$$

Analogous to the MA(1) case we get, for positive values of ρ,

$$\mu_i(V) \leq \frac{1}{\upsilon\,(1 - \rho)} \quad . \tag{10}$$

Using (10) we obtain $\mu_{max}(V) = o(T)$. $\diamond$

Lemma 2

Assume that the weights matrix W is symmetric with row sums equal to unity. When the components of u follow first-order spatial MA(1) or AR(1) process, then for values of ρ from a suitable interval (ρ_l, ρ_u), $\rho_l > 0$,

$$\frac{u' P_X u}{T} \xrightarrow{p} 0 \quad ,$$

where $P_X = X(X'X)^{-1}X'$.

Proof:

Let $tr(A)$ denote the trace of the matrix A. For the expectation of $u'P_X u/T$ we have (see e.g. Magnus and Neudecker, 1988, p. 247)

$$\begin{aligned} E(\frac{u'P_X u}{T}) &= \frac{1}{T}\Big(tr\,(P_X\,Cov(u)) + E(u)'P_X E(u)\Big) \\ &= \frac{\sigma_u^2}{T}\,tr\,(P_X V) \quad . \end{aligned} \tag{11}$$

The trace of the matrix product $P_X V$ can be expressed as

$$tr\,(P_X V) = tr\,(Z'VZ) = \sum_{i=1}^{k} \mu_i(Z'VZ) \quad ,$$

where $Z = X(X'X)^{-1/2}$. This implies

$$E(\frac{u'P_X u}{T}) = \frac{\sigma_u^2}{T}\sum_{i=1}^{k} \mu_i(Z'VZ) \quad .$$

From Poincaré separation theorem (see Horn and Johnson, 1985, p. 190) it follows that all eigenvalues of $Z'VZ$ are less than or equal to $\mu_{max}(V)$. Using this fact gives

$$E(\frac{u'P_X u}{T}) \le \frac{\sigma_u^2}{T}\,k\,\mu_{max}(V) \quad . \tag{12}$$

By applying Lemma 1 we get $\mu_{max}(V) = o(T)$, and from (12) it is clear that

$$E(\frac{u'P_X u}{T}) \to 0 \qquad (T \to \infty) \quad .$$

Since P_X is symmetric and idempotent, $u^{'}P_X u \geq 0$. Furthermore, for $\epsilon^* > 0$ we have (see Davidson, 1994, p. 132: Markov-Inequality)

$$P(\frac{u^{'}P_X u}{T} > \epsilon^*) \leq E(\frac{u^{'}P_X u}{\epsilon^* T}) \quad \rightarrow \quad 0 \qquad (T \rightarrow \infty).$$

This means, by definition, $(u^{'}P_X u)/T \xrightarrow{p} 0$. ◇

Given model (1), suppose that the error vector ϵ has the following finite moments:

$$E(\epsilon \otimes \epsilon\epsilon^{'}) = \Phi \quad \text{and} \quad E(\epsilon\epsilon^{'} \otimes \epsilon\epsilon^{'}) = \Psi \quad , \tag{13}$$

where $\otimes$ denotes the Kronecker-product.
The following theorem provides a sufficient condition for the consistency of S^2 under first-order spatial error processes that can be verified in practice. In what follows $C_{i.}$ denotes the i-th row of the matrix C in model (1).

Theorem 1

Let the weights matrix W be symmetric with row sums equal to unity. Suppose that the components of ϵ in model (1) are independent and identically distributed, and the components of u follow a first-order spatial AR or MA process. Then S^2 is weakly consistent for σ_u^2 if for positive values of ρ from a suitable interval (ρ_l, ρ_u), $\rho_l > 0$, and two neighbouring regions R_i and R_j

$$tr\,(C_{i.}^{'}C_{j.}) \quad = \quad o(T) \quad . \tag{14}$$

Proof:

The OLS based estimator S^2 can be expressed as

$$S^2 = \frac{u^{'}M_X u}{T} \quad = \quad \frac{u^{'}u}{T} \quad - \quad \frac{u^{'}P_X u}{T} \quad .$$

From Lemma 2 we have

$$\frac{u^{'}P_X u}{T} \xrightarrow{p} 0 \quad ,$$

so it suffices to show, under condition (14), that

$$\frac{u^{'}u}{T} \xrightarrow{p} \sigma_u^2 \quad .$$

The theorem will be proved if, for $T \rightarrow \infty$, we are able to show

$$E(\frac{u^{'}u}{T}) \rightarrow \sigma_u^2 \quad \text{and} \quad Var(\frac{u^{'}u}{T}) \rightarrow 0 \quad . \tag{15}$$

For the disturbance vector $u = C\epsilon$, as defined in (1), the following holds:

$$E(u^{'}u) = E(\epsilon^{'}C^{'}C\epsilon) = tr\,(C^{'}C\sigma_\epsilon^2 I) \quad = \quad \sigma_\epsilon^2 tr\,(CC^{'}) \quad .$$

Since the matrix $V_* = CC^{'}$ has diagonal elements which are all equal to υ,

$$E(u^{'}u) = \sigma_\epsilon^2\, tr\,(V_*) = \sigma_\epsilon^2\, T\, \upsilon \quad ,$$

and from the expression $Cov(u) = \sigma_u^2 V = \sigma_\epsilon^2 V_* = \upsilon\, \sigma_\epsilon^2 V$, it follows that

$$E(\frac{u^{'}u}{T}) = \upsilon\, \sigma_\epsilon^2 = \sigma_u^2 \quad ,$$

showing the first part of (15). Now, to prove the second part of (15) which states

$$Var(\frac{u^{'}u}{T}) = E(\frac{u^{'}u}{T})^2 - (\sigma_u^2)^2 \rightarrow 0 \quad (T \rightarrow \infty),$$

it suffices to show that $E((u^{'}u)/T)^2$ converges to $(\sigma_u^2)^2$.
Consider $E(u^{'}u)^2$:
Since W is symmetric, we obtain $C = C^{'}$ implying $u^{'}u = \epsilon^{'}CC^{'}\epsilon = \epsilon^{'}V_*\epsilon$, and

$$E(u^{'}u)^2 \quad = \quad E(\epsilon^{'}V_*\epsilon\epsilon^{'}V_*\epsilon) \quad . \tag{16}$$

Using the result of Rao and Kleffe (1988, p. 32) we get

$$\begin{aligned} E(\epsilon^{'}V_*\epsilon\epsilon^{'}V_*\epsilon) &= E(tr\,(V_*\epsilon\epsilon^{'}V_*\epsilon\epsilon^{'})) \\ &= tr\,((V_* \otimes V_*)\Psi) \quad , \end{aligned} \tag{17}$$

where $\Psi = E(\epsilon\epsilon^{'} \otimes \epsilon\epsilon^{'})$.
When the components of ϵ are independent and identically distributed, then

$$E(\epsilon_i\epsilon_j\epsilon_{i^*}) = \begin{cases} \varphi^* & , \quad i = j = i^* \\ 0 & , \quad otherwise \end{cases}$$

and

$$E(\epsilon_i \epsilon_j \epsilon_{i^*} \epsilon_{j^*}) = \begin{cases} (\sigma_\epsilon^2)^2 & , \quad \text{pairwise equal} \\ \varphi & , \quad i = j = i^* = j^* \\ 0 & , \quad otherwise \quad , \end{cases}$$

where $\varphi^* = E(\epsilon_i)^3$ and $\varphi = E(\epsilon_i)^4$.

Let Ψ_{ij} be a $T \times T$ symmetric matrix with elements

$$\Psi_{ij}(i^*, l) = \Psi_{ij}(l, i^*) = \begin{cases} \sigma_\epsilon^4 & , \quad i = l,\, j = i^* \\ 0 & , \quad otherwise \quad . \end{cases}$$

Further, let $\Psi_1, \Psi_2, \ldots, \Psi_T$ be $T \times T$ diagonal matrices with diagonal elements equal to φ or σ_ϵ^4 such that

$$\Psi_j(ii) = \begin{cases} \varphi & , \quad i = j \\ \sigma_\epsilon^4 & , \quad otherwise \quad . \end{cases}$$

For the expectation of the Kronecker-product Ψ we obtain

$$\Psi = \begin{pmatrix} \Psi_1 & \Psi_{12} & \cdots & \Psi_{1T} \\ \Psi_{21} & \Psi_2 & \cdots & \Psi_{2T} \\ \vdots & \vdots & \ddots & \vdots \\ \Psi_{T1} & \cdots & \Psi_{TT-1} & \Psi_T \end{pmatrix} \quad .$$

This matrix can be written as

$$\Psi = \sigma_\epsilon^4 I_{T^2} \quad + \quad (\varphi - \sigma_\epsilon^4) I^* \quad + \quad \sigma_\epsilon^4 \Psi^* \quad , \tag{18}$$

where I_{T^2} denotes the $T^2 \times T^2$ identity matrix. I^* and Ψ^* denote $T^2 \times T^2$ matrices given as

$$I^*_{ij} \quad = \quad \begin{cases} 1 & , \quad i = j = (i^* - 1)T + i^*\,,\, i^* = 1, \cdots, T \\ 0 & , \quad otherwise \end{cases} \quad , \tag{19}$$

$$\Psi^* \quad = \quad \begin{pmatrix} \Psi_0 & \Psi^*_{12} & \cdots & \Psi^*_{1T} \\ \Psi^*_{21} & \Psi_0 & \cdots & \Psi^*_{2T} \\ \vdots & \vdots & \ddots & \vdots \\ \Psi^*_{T1} & \cdots & \Psi^*_{TT-1} & \Psi_0 \end{pmatrix}$$

with $\Psi_0 := \Psi^*_{ii} = O_{T\times T}$, where $O_{T\times T}$ denotes the $T \times T$ matrix whose elements are all equal to zero. The $T \times T$ matrix Ψ^*_{ij} is given by

$$\Psi^*_{ij}(i^*, l) = \Psi^*_{ij}(l, i^*) = \begin{cases} 1 & , \quad i = l,\, j = i^* \\ 0 & , \quad otherwise \end{cases} \quad .$$

and is symmetric according to the definition .

From (16), (17) and (18), we get

$$\begin{aligned} E(u^{'}u)^2 &= tr\,((V_* \otimes V_*)\Psi) \\ &= tr\,((V_* \otimes V_*)(\sigma^4_\epsilon I_{T^2} \quad + \quad (\varphi - \sigma^4_\epsilon)I^* \quad + \quad \sigma^4_\epsilon \Psi^*)) \\ &= tr\,((V_* \otimes V_*)\sigma^4_\epsilon I_{T^2}) + tr\,((V_* \otimes V_*)(\varphi - \sigma^4_\epsilon)I^*) + tr\,((V_* \otimes V_*)\sigma^4_\epsilon \Psi^*). \end{aligned} \tag{20}$$

The first term of the right hand side of equation (20) can be expressed as

$$tr\,((V_* \otimes V_*)\sigma^4_\epsilon I_{T^2}) = \sigma^4_\epsilon tr\,((V_* \otimes V_*) = \sigma^4_\epsilon tr\,(V_*) tr\,(V_*) = \sigma^4_\epsilon\, \upsilon^2\, T^2 \quad , \tag{21}$$

because $tr\,(V_*) = \upsilon\, T$ (see Magnus and Neudecker, 1988, p. 28).

By the assumption in model (1) all diagonal elements of $V_* \otimes V_*$ are equal to υ^2, and the matrix I^* has exactly T diagonal elements which are equal to unity (zero otherwise). Thus for the second term we have

$$tr\,((V_* \otimes V_*)(\varphi - \sigma^4_\epsilon)I^*) = (\varphi - \sigma^4_\epsilon)T\, \upsilon^2 \quad . \tag{22}$$

Since V_* is symmetric, we can write the third term as (see Magnus and Neudecker, 1988, p. 30)

$$\sigma^4_\epsilon tr\,((V_* \otimes V_*)\Psi^*) = \sigma^4_\epsilon\, (vec\,(V_* \otimes V_*))^{'}\, vec\,(\Psi^*) \quad . \tag{23}$$

Denoting the $j-$th column of an $m \times n$ matrix A by a_j, vec stands for a vector of size mn with a_1 as its first m elements, a_2 its second m elements and so on.

For R_i and R_j being neighbours, $E(u_i u_j) = \sigma^2_\epsilon V_*(i,j)$, and by successive calculation we get

$$\sigma^4_\epsilon\, (vec\,(V_* \otimes V_*))^{'}\, vec\,(\Psi^*) = 2 \sum_i^T \sum_j^{g_i} (E(u_i u_j))^2 \quad , \tag{24}$$

where g_i denotes the number of neighbours for the i-th region R_i. Furthermore, $u_i = C_{i.}\epsilon = \epsilon' C'_{i.}$ and $u_j = C_{j.}\epsilon$. From this we obtain

$$E(u_i u_j) = E(\epsilon' C'_{i.} C_{j.}\epsilon) = E(tr\,(C'_{i.} C_{j.}\epsilon\epsilon')) = \sigma^2_\epsilon tr\,(C'_{i.} C_{j.}) \quad . \tag{25}$$

From (23) to (25) follows

$$\sigma^4_\epsilon tr\,((V_* \otimes V_*)\Psi^*) = 2\,\sigma^4_\epsilon \sum_i^T \sum_j^{g_i} (tr\,(C'_{i.} C_{j.}))^2 \quad . \tag{26}$$

Using equations (20) to (22) and (26) for values of ρ from a suitable interval (ρ_l, ρ_u) we obtain

$$\begin{aligned} \lim_{T\to\infty} E(\frac{u'u}{T})^2 &= \lim_{T\to\infty} \sigma^4_\epsilon\, \upsilon^2 + \lim_{T\to\infty} (\varphi - \sigma^4_\epsilon)\frac{\upsilon^2}{T} + \lim_{T\to\infty} \frac{2\,\sigma^4_\epsilon}{T^2} \sum_i^T \sum_j^{g_i} (tr\,(C'_{i.} C_{j.}))^2 \\ &= \sigma^4_\epsilon\, \upsilon^2 \quad = \quad (\sigma^2_u)^2 \quad . \end{aligned} \tag{27}$$

The last expression holds because of the assumption $tr\,(C'_{i.} C_{j.}) = o(T)$ $\diamond$

<u>Example</u>

Let the weights matrix W be of the pattern

$$\begin{cases} w_{1,T} = w_{2,T-1} = w_{i,T-i+1} = 1 & , \quad i = 3, 4, \ldots, T \\ w_{i,j} = 0 & , \quad \text{otherwise} \quad . \end{cases} \tag{28}$$

Furthermore, let the components of ϵ in model (1) be independent and identically distributed. If the components of u follow a first-order spatial MA process, then S^2 is weakly consistent for σ^2_u.

This can be proved by showing that, for ρ from a suitable interval (ρ_l, ρ_u), condition (14) is fulfilled. Under a spatial MA(1) process we have $V_* = (I + \rho W)(I + \rho W')$, and this means $C = I + \rho W$. If the weights matrix W is of the form (28), then C is symmetric, and the regions R_i and R_j with $j = T - i + 1$ are neighbours. Denoting a T-dimensional vector whose i-th element is equal to unity (zero otherwise) by $\vec{e}_i$, we get

$$C'_{i.} = \vec{e}_i + \rho\, \vec{e}_j \quad .$$

Using this yields

$$C_{i.}^{'} C_{j.} = (\vec{e}_i + \rho\, \vec{e}_j)((\vec{e}_j)^{'} + \rho\, (\vec{e}_i)^{'}) \quad ,$$

implying

$$\begin{aligned} tr\,(C_{i.}^{'} C_{j.}) &= tr\,(\vec{e}_i(\vec{e}_j)^{'}) + tr\,(\rho\, \vec{e}_j(\vec{e}_j)^{'}) + tr\,(\rho\, \vec{e}_i(\vec{e}_i)^{'}) + tr\,(\rho^2\, \vec{e}_j(\vec{e}_i)^{'}) \quad . \\ &= 2\,\rho \quad , \qquad (29) \end{aligned}$$

because $tr\,(\vec{e}_i(\vec{e}_j)^{'}) = 0$ and $tr\,(\vec{e}_i(\vec{e}_i)^{'}) = tr\,(\vec{e}_j(\vec{e}_j)^{'}) = 1$. From (29) it is clear that, for ρ from a suitable interval (ρ_l, ρ_u), $tr\,(C_{i.}^{'} C_{j.}) = o(T)$, and the weak consistency of S^2 for σ_u^2 follows from Theorem 1. ◇

The next result gives necessary and sufficient condition for the consistency of S^2 under first-order spatial error process.

Theorem 2

Let the weights matrix W be symmetric with row sums equal to unity, and suppose that the components of u follow a first-order spatial MA or AR process. Then S^2 is weakly consistent for σ_u^2 if and only if, for values of ρ from suitable a interval (ρ_l, ρ_u), $\rho_l > 0$,

$$\frac{u^{'} u}{T} \xrightarrow{p} \sigma_u^2 \quad . \qquad (30)$$

Proof:

(*sufficiency*)

Consider the OLS based estimator

$$S^2 = \frac{u^{'} M_X u}{T} = \frac{u^{'} u}{T} \quad - \quad \frac{u^{'} P_X u}{T} \quad .$$

From Lemma 2 we have

$$\frac{u^{'} P_X u}{T} \xrightarrow{p} 0 \quad ,$$

and $S^2 \xrightarrow{p} \sigma_u^2$ follows from the assumption $u^{'} u/T \xrightarrow{p} \sigma_u^2$.

($necessity$)

If S^2 is weakly consistent, then $S^2 \xrightarrow{p} \sigma_u^2$. This means

$$\frac{u^{'}u}{T} \quad - \quad \frac{u^{'}P_X u}{T} \xrightarrow{p} \sigma_u^2 \quad .$$

From Lemma 2 it holds $u^{'}P_X u/T \xrightarrow{p} 0$. So, the statement that $S^2 \xrightarrow{p} \sigma_u^2$ is valid if and only if $u^{'}u/T \xrightarrow{p} \sigma_u^2$. $\diamond$

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