Received: 23 August 2023 | Revised: 19 March 2024 | Accepted: 11 April 2024

DOI: 10.1002/zamm.202300654

ORIGINAL PAPER

On a lack of stability of parametrized BV solutions to rate-independent systems with nonconvex energies and discontinuous loads

Merlin Andreia | **Christian Meyer**

Lehrstuhl LSX, Fakultät für Mathematik, Technische Universität Dortmund, Dortmund, Germany

Correspondence
Christian Meyer, Lehrstuhl LSX, Fakultät für Mathematik, Technische Universität Dortmund, Vogelpothsweg 87, 44227 Dortmund, Germany.
Email: christian2.meyer@tu-dortmund.de

Funding information
Deutsche Forschungsgemeinschaft, Grant/Award Number: ME 3281/9-1

We consider a rate-independent system with nonconvex energy under discontinuous external loading. The underlying space is finite-dimensional and the loads are functions in $BV([0,T];\mathbb{R}^d)$. We investigate the stability of various solution concepts w.r.t. a sequence of loads converging weakly* in $BV([0,T];\mathbb{R}^d)$ with a particular emphasis on the so-called normalized, $\mathfrak{p}$-parametrized balanced viscosity solutions. By means of three counterexamples, it is shown that common solution concepts are not stable w.r.t. weak* and even intermediate (or strict) convergence of loads in the sense that a limit of a sequence of solutions associated with these loads need not be a solution corresponding to the load in the limit. We moreover introduce a new solution concept, which is stable in this sense, but our examples show that this concept necessarily allows "solutions" that are physically meaningless.

1 | INTRODUCTION

This paper is concerned with a rate-independent system of the form

$$0 \in \partial\mathcal{R}(\dot{z}(t)) + D_z\mathcal{I}(t, z(t)) \quad \text{f.a.a. } t \in (0,T), \quad z(0) = z_0, \tag{1.1}$$

where $\mathcal{R} : \mathbb{R}^d \to \mathbb{R}$ is convex and positive 1-homogeneous and $\mathcal{I}$ denotes an energy given by

$$\mathcal{I}(t,z) := \frac{1}{2}\langle Az, z\rangle + \mathcal{F}(z) - \langle \ell(t), z\rangle.$$

Herein $A \in \mathbb{R}^{d\times d}$ is symmetric and positive definite and $\mathcal{F} : \mathbb{R}^d \to \mathbb{R}$ is smooth, but potentially nonconvex. Moreover, the external load $\ell : [0,T] \to \mathbb{R}^d$ is a given function of bounded variation. The precise assumptions on the data are specified in Section 1.2 below.

It is well known that, even for smooth external loads, one cannot expect the existence of a weakly differentiable solution such that Equation (1.1) is satisfied for almost all $t \in (0,T)$. We refer to the counterexample in Ref. [1], Section 2.3]. For that reason, several alternative solution concepts have been developed, among them *local solutions*, *global energetic solutions*, and *(parametrized) balanced viscosity (BV) solutions*. For a comprehensive overview, we refer to Ref. [2]. Originally, all these concepts have been introduced for smooth external loads, but, recently, several authors came up with extensions of the classical solution concepts adapted to loads in $BV([0,T];\mathbb{R}^d)$. Concerning parametrized BV solutions, we refer to Ref. [3]. With regard to other solutions concepts, rate-independent systems with discontinuous data are discussed, for example, in Refs. [4–7] for the case of strictly convex energies. A comparison of various solutions concepts in case of discontinuous data is presented in Ref. [8]. The motivation for considering loads in $BV([0,T];\mathbb{R}^d)$ is manifold, reaching from applications, where loads may be switched on or off, to the viscous approximation of optimal control problems governed by Equation (1.1). Concerning the latter, the so-called reverse approximation property plays an essential role and has so far only been verified under very restrictive assumptions, see Ref. [9, Section 6]. If one aims to avoid these assumptions, the use of loads in $BV([0,T];\mathbb{R}^d)$ seems to be indispensable, compare Ref. [9, Remark 6.4].

Within this contribution, we investigate the stability of solutions to Equation (1.1) w.r.t. "natural" notions of convergence of loads in $BV([0,T];\mathbb{R}^d)$. In generic situations, one cannot expect a sequence of loads in $BV([0,T];\mathbb{R}^d)$ to converge strongly in $BV([0,T];\mathbb{R}^d)$. For instance, if one discretizes a given load in $BV([0,T];\mathbb{R}^d)$ by means of a sequence of piecewise constant functions, then this sequence will in general only converge weakly* in $BV([0,T];\mathbb{R}^d)$ as the mesh size tends to zero (unless the jump set of the load is exactly covered by the mesh). And even, if one considers the smoothing of a load in $BV([0,T;\mathbb{R}^d)$ by convolution with the standard mollifier, one will in general only obtain intermediate convergence (also known as strict convergence) of the sequence of mollified loads, when the regularization parameter is driven to zero. Consequently, "generic" notions of convergence in $BV([0,T];\mathbb{R}^d)$ are weak* convergence or, at best, intermediate convergence. It is, therefore, reasonable to investigate if established solutions concepts are stable w.r.t. these types of convergence. The crucial question in this context is the following:

*If one considers a sequence of loads converging weakly * (or in the intermediate topology) and a sequence of associated solutions according to one of the established solution concepts, which converges to a limit function, is this limit still a solution in the sense of the respective concept?*

We will show by means of three examples that the answer to this question is in general negative. To be more precise, the notions of normalized $\mathfrak{p}$-parametrized BV solutions and local solutions are not stable w.r.t. intermediate convergence of the loads. We, therefore, propose a new "relaxed" solution concept that is stable w.r.t. weak* convergence of loads in $BV([0,T];\mathbb{R}^d)$. However, as especially the second example in Section 4.2 shows, this new solution concept admits functions as solutions, which are entirely meaningless from a physical point of view. It is, therefore, doubtful if it makes sense to allow for discontinuous loads in $BV([0,T];\mathbb{R}^d)$ in the context of rate-independent systems of the form (1.1). Let us also mention that recently, another example illustrating this lack of stability for a rate-independent system with quadratic energy is presented in Ref. [10, Section 5].

The paper is organized as follows: The rest of this introductory section is dedicated to the notation and our standing assumptions on the data in Equation (1.1). We then recall the concept of normalized $\mathfrak{p}$-parametrized BV solutions in case of loads in $BV([0,T];\mathbb{R}^d)$ in Section 2. Thereafter, in Section 2.1, a first example is presented, which shows that this solution concept is not stable w.r.t. intermediate convergence of the loads. Section 3 is then devoted to our new relaxed solution concept and its stability w.r.t. weak* convergence of the loads. In Section 4, we investigate this new solution concept in more details, especially its relation to other solution concepts. We prove that, under natural assumption, every local solution is also a relaxed solution, which demonstrates that the new concept is rather weak. This is also underlined by the second example in Section 4.2, which shows that the new solution concepts provide completely unphysical "solutions." The paper ends with a short conclusion and an appendix on auxiliary technical results.

1.1 Notation

Throughout the paper, $\langle\cdot,\cdot\rangle$ denotes the standard scalar product in $\mathbb{R}^d$ and the induced Euclidean norm is given by $\|\cdot\| = \sqrt{\langle\cdot,\cdot\rangle}$. C is a generic constant larger zero and for a function $f : [a,b] \to \mathbb{R}^d$, the total variation is defined by

$$\operatorname{Var}(f;[a,b]) = \sup_{\text{partitions } \{t_k\} \text{ of } [a,b]} \sum_k \|f(t_k) - f(t_{k-1})\|,$$

where a partition $\{t_k\}$ is a finite subset of $[a,b]$ with $a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b,\ n \in \mathbb{N}$.

For a function $f : \mathbb{R} \to \mathbb{R}^d$, we denote the one-sided limits, if they exist, by $f(s-) := \lim_{\sigma \nearrow s} f(\sigma)$ and $f(s+) := \lim_{\sigma \searrow s} f(\sigma)$, respectively.

For convenience of the reader, let us collect some well-known facts on functions of bounded variation that will be useful throughout the paper. For the proofs, we refer to Refs. [11, 12]. By $BV([0,T];\mathbb{R}^d)$, we denote the space of functions with bounded total variation, that is,

$$BV([0,T];\mathbb{R}^d) := \{f : [0,T] \to \mathbb{R}^d \ : \ \mathrm{Var}(f;[0,T]) < \infty\}.$$

It is well known that a function in $BV([0,T];\mathbb{R}^d)$ is measurable and bounded and admits at most a countable number of discontinuities. Thus, a function in $BV([0,T];\mathbb{R}^d)$ is absolutely integrable and we can equip $BV([0,T];\mathbb{R}^d)$ with the norm $\|f\|_{BV([0,T];\mathbb{R}^d)} := \|f\|_{L^1(0,T;\mathbb{R}^d)} + \mathrm{Var}(f;[0,T])$. Moreover, we introduce the space of (equivalence classes of) functions of bounded variation on $(0,T)$ as

$$\mathrm{BV}(0,T;\mathbb{R}^d) := \{f \in L^1(0,T;\mathbb{R}^d) : Df \in \mathfrak{M}(0,T;\mathbb{R}^d)\},$$

where Df denotes the distributional derivative and $\mathfrak{M}(0,T;\mathbb{R}^d)$ is the space of $\mathbb{R}^d$-valued regular Borel measures on $(0,T)$. We equip $\mathrm{BV}(0,T;\mathbb{R}^d)$ with the norm $\|f\|_{\mathrm{BV}(0,T;\mathbb{R}^d)} := \|f\|_{L^1(0,T;\mathbb{R}^d)} + |Df|(0,T)$, where $|Df|(0,T)$ denotes the total variation of the measure Df. It is well known that

$$|Df|(0,T) = \inf\{\mathrm{Var}(v;[0,T]) : v \text{ is a representative of } [f]\}. \tag{1.2}$$

Furthermore, $\mathrm{BV}(0,T;\mathbb{R}^d)$ embeds continuously in $L^\infty(0,T;\mathbb{R}^d)$ and compactly in every $L^p(0,T;\mathbb{R}^d)$, $p<\infty$. Moreover, every function $f \in BV([0,T];\mathbb{R}^d)$ is a representative of an element in $\mathrm{BV}(0,T;\mathbb{R}^d)$ and, in view of Equation (1.2), it holds

$$\|f\|_{BV([0,T];\mathbb{R}^d)} \geq \|f\|_{\mathrm{BV}(0,T;\mathbb{R}^d)}. \tag{1.3}$$

Here and in the following, with a little abuse of notation, we denote the function in $BV([0,T];\mathbb{R}^d)$ and the associated equivalence class in $\mathrm{BV}(0,T;\mathbb{R}^d)$ by the same symbol. Note that Equation (1.3) is only satisfied with equality by particular representatives of f, for example, by the left or right continuous representative. As usual, we call a sequence $(f_n)_{n\in\mathbb{N}} \subset \mathrm{BV}(0,T;\mathbb{R}^d)$ weakly* converging in $\mathrm{BV}(0,T;\mathbb{R}^d)$, iff

$$f_n \rightharpoonup^* f \quad \text{in } \mathrm{BV}(0,T;\mathbb{R}^d) \quad :\Longleftrightarrow \quad \begin{cases} f_n \to f & \text{in } L^1(0,T;\mathbb{R}^d), \\ Df_n \rightharpoonup^* Df & \text{in } \mathfrak{M}(0,T;\mathbb{R}^d). \end{cases}$$

Every bounded sequence in $\mathrm{BV}(0,T;\mathbb{R}^d)$ contains a weakly* converging subsequence and, by Equation (1.3), the same holds true for a bounded sequence in $BV([0,T];\mathbb{R}^d)$ (and the associated sequence of equivalence classes, respectively). Furthermore, by Helly's selection principle, every bounded sequence in $BV([0,T];\mathbb{R}^d)$ admits a subsequence that converges pointwise everywhere in $[0,T]$ and, if the subsequence converges weakly* in $\mathrm{BV}(0,T;\mathbb{R}^d)$, too, the pointwise limit is a representative of the weak* limit. It is to be noted however that the pointwise limit need not be the representative attaining the infimum in Equation (1.2), even if this is true along the converging subsequence. Finally, we say that a sequence $(f_n)_{n\in\mathbb{N}}$ converges w.r.t. intermediate (or strict) convergence in $\mathrm{BV}(0,T;\mathbb{R}^d)$ and $BV([0,T];\mathbb{R}^d)$, respectively, if it converges weakly* and, in addition, $|Df_n|(0,T) \to |Df|(0,T)$ as $n\to\infty$.

1.2 | Standing assumptions

1.2.1 | Energy

Throughout the paper, $A \in \mathbb{R}^{d\times d}$ is symmetric and positively definite. Furthermore, $\mathcal{F} : \mathbb{R}^d \to \mathbb{R}$ satisfies

$$\mathcal{F} \in C^2(\mathbb{R}^d;\mathbb{R}), \ \mathcal{F} \geq 0 \tag{1.4}$$

$$\|D^2\mathcal{F}(z)v\| \leq C(1+\|z\|^q)\|v\| \ \forall v,z \in \mathbb{R}^d \tag{1.5}$$

for some $q \geq 1$. For a given external load $\ell \in BV([0,T];\mathbb{R}^d)$, the energy functional $\mathcal{I} : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ reads

$$\mathcal{I}(t,z) := \frac{1}{2}\langle Az, z\rangle + \mathcal{F}(z) - \langle \ell(t), z\rangle. \tag{1.6}$$

By denoting the time independent part as

$$\mathcal{E}(z) := \frac{1}{2}\langle Az, z\rangle + \mathcal{F}(z), \tag{1.7}$$

the energy can be written as $\mathcal{I}(t,z) = \mathcal{E}(z) - \langle \ell(t), z\rangle$. Below we will deal with other loads, denoted by $\hat{\ell}, \hat{\ell}_n \in BV([0,S];\mathbb{R}^d)$, too, and so, for convenience, we define

$$\hat{\mathcal{I}}(s,z) = \mathcal{E}(z) - \langle \hat{\ell}(s), z\rangle, \tag{1.8}$$

$$\hat{\mathcal{I}}_n(s,z) = \mathcal{E}(z) - \langle \hat{\ell}_n(s), z\rangle. \tag{1.9}$$

1.2.2 | Dissipation

For the dissipation $\mathcal{R} : \mathbb{R}^d \to [0,\infty)$, we assume

$$\mathcal{R} \text{ is proper, convex, and lower semicontinuous,} \tag{1.10}$$

$$\mathcal{R} \text{ is positive 1-homogeneous, that is, } \mathcal{R}(\lambda z) = \lambda\mathcal{R}(z)\, \forall\, z \in \mathbb{R}^d,\ \lambda > 0, \tag{1.11}$$

$$\exists\, c, C > 0, \text{ such that } c\,\|z\| \leq \mathcal{R}(z) \leq C\,\|z\|\ \forall\, z \in \mathbb{R}^d. \tag{1.12}$$

1.2.3 | Initial data

There exists $\ell_0 \in \mathbb{R}^d$ such that $-D_z\mathcal{E}(z_0) + \ell_0 \in \partial\mathcal{R}(0)$.

2 | A SOLUTION CONCEPT FOR DISCONTINUOUS LOADS IN *BV*

There exists a variety of solution concepts for Equation (1.1). Here we only present a brief overview with a special emphasis on parametrized BV solutions and refer to Ref. [2] and the references therein for more details. The most natural notion of solutions is the *differential solution*, where one searches for a weakly differentiable function $z \in W^{1,1}(0,T;\mathbb{R}^d)$, which satisfies Equation (1.1) almost everywhere. The disadvantage of this concept is that one cannot guarantee the existence of such a solution in case of a nonconvex energy as the example in Ref. [1], Section 2.3] shows. For that reason, several alternative solutions concepts have been developed. The most prominent one is probably the concept of *global energetic solutions*, where the subdifferential inclusion in Equation (1.1) is replaced by a global stability condition and an energy balance, which benefits in solvability under milder assumptions, see, for example, Refs. [13–15]. One of the weakest solution concepts providing only a minimum of information is the so-called *local solution* concept, where a solution z only satisfies a local stability condition together with an energy inequality. We will come back to this concept in Section 4 below. Another approach providing more information especially about the discontinuities of a solution is given by the concept of *parametrized BV solutions*, where a solution consists of the tuple (z,t), representing the state and the physical time, respectively, as functions of a curve parameter s. To the best of our knowledge, it was first introduced in Ref. [16], but has by now been analyzed by various authors in multiple aspects, we only refer to Ref. [17] and the reference therein.

Initially, all these concepts have been developed for smooth external loads, but recently some of them have been transferred to the case of discontinuous loads in $BV([0,T])$. Concerning global energetic solutions, we exemplarily refer to Ref. [4]. In Ref. [3], the concept of parametrized BV solutions concept has been transferred to loads in $BV([0,T])$. In our setting, where the underlying space is finite-dimensional, it reads as follows:

Definition 2.1. Let $\ell \in BV([0,T];\mathbb{R}^d)$ be given. A triple $(S,\hat{t},\hat{z},\hat{\ell}) \in (0,\infty)\times W^{1,\infty}(0,S)\times W^{1,\infty}(0,S;\mathbb{R}^d)\times BV([0,S];\mathbb{R}^d)$ is called *normalized, $\mathfrak{p}$-parametrized balanced viscosity (BV) solution* of the rate-independent system (1.1) associated with ℓ, if

- the *initial and end time condition* hold

$$\hat{t}(0)=0,\ \hat{t}(S)=T,\ \hat{z}(0)=z_0, \tag{2.1}$$

- the *complementarity relations and normalization condition* are fulfilled for almost all $s\in(0,S)$

$$\hat{t}'(s)\ge 0,\ \hat{t}'(s)\,\mathrm{dist}(-D_z\hat{\mathcal{I}}(s,\hat{z}(s)),\partial\mathcal{R}(0))=0, \tag{2.2}$$

$$\hat{t}'(s)+\mathcal{R}(\hat{z}'(s))+\|\hat{z}'(s)\|\,\mathrm{dist}(-D_z\hat{\mathcal{I}}(s,\hat{z}(s)),\partial\mathcal{R}(0))=1, \tag{2.3}$$

- the *energy identity* is valid for all $s_1,s_2\in[0,S]$

$$\begin{aligned}\mathcal{E}(\hat{z}(s_2))+\int_{s_1}^{s_2}\mathcal{R}(\hat{z}'(r))+\|\hat{z}'(r)\|\,\mathrm{dist}(-D_z\hat{\mathcal{I}}(r,\hat{z}(r)),\partial\mathcal{R}(0))dr&\\ =\mathcal{E}(\hat{z}(s_1))+\int_{s_1}^{s_2}\langle\hat{\ell}(r),\hat{z}'(r)\rangle dr,&\end{aligned} \tag{2.4}$$

- the parametrized load $\hat{\ell}$ is compatible with ℓ in the following sense: for every $t_*\in[0,T]$, there exists $s_*\in\hat{t}^{-1}(t_*)$ such that for all $s\in\hat{t}^{-1}(t_*)$

$$\hat{\ell}(s)=\begin{cases}\ell(t_*-), & s<s_*,\\ \ell(t_*+), & s>s_*,\end{cases}\qquad\text{and}\qquad \hat{\ell}(s_*)\in\{\ell(t_*),\ell(t_*-),\ell(t_*+)\}. \tag{2.5}$$

Here and in the following, we set $0- := 0$ and $T+ := T$.

Remark 2.2. The above definition differs from the "classical" notion of normalized, $\mathfrak{p}$-parametrized BV solutions for smooth external loads according to, for example, Ref. [17] only in the additional compatibility condition in Equation (2.5). This condition basically says that, if ℓ is discontinuous in a viscous jump, then it only attains its left and right limits during the viscous jump. Note that, if ℓ is continuous in t^*, then $\hat{\ell}(s)=\ell(\hat{t}(s))$ for all $s\in\hat{t}^{-1}(t^*)$.

Remark 2.3. The energy identity in Equation (2.4) is expressed in terms of the potential energy $\mathcal{E}$ from Equation (1.7). By means of integration by parts for Riemann–Stieltjes integrals, one can equivalently express the energy identity in terms of the total stored energy functional $\mathcal{I}$ from Equation (1.6). This equivalent reformulation will be the basis for the definition of a local solution in Definition 4.1 below.

A proof concerning the existence of such solutions can be found in Ref. [3, Prop. 4.2]. The authors used a vanishing viscosity approach and showed that a parametrized version of the solutions solving the regularized problems, which are

$$0\in\partial\mathcal{R}(\dot{z}_\epsilon(t))+\epsilon\dot{z}_\epsilon(t)+D_z\mathcal{I}(t,z_\epsilon(t)),\quad z(0)=z_0,\quad \epsilon>0,$$

convergence to a $\mathfrak{p}$-parametrized BV solution as the viscosity parameter ϵ tends to 0. This parametrization is done by means of the vanishing viscosity contact potential

$$\mathfrak{p}:\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{R},\quad \mathfrak{p}(v,w)=\mathcal{R}(v)+\|v\|\,\mathrm{dist}(w,\partial\mathcal{R}(0)),$$

which also motivates the name of the solution concept. To be more precise, the parametrization is given by

$$s_\epsilon(t)=t+\int_0^t\mathfrak{p}(\dot{z}_\epsilon(r),-D_z\mathcal{I}(r,z_\epsilon(r)))dr,\quad \hat{z}_\epsilon=z_\epsilon(\hat{t}_\epsilon(s)),\quad \hat{\ell}_\epsilon(s)=\ell(\hat{t}_\epsilon(s)),$$

where $\hat{t}_\epsilon : [0, S_\epsilon] \to [0, T]$ with $S_\epsilon = s_\epsilon(T)$ is the inverse function of s_ϵ, so that $\mathfrak{p}$ appears in the normalization condition (2.3), that is, $\hat{t}(s) + \mathfrak{p}(\hat{z}'(s), -D_z\hat{\mathcal{I}}(s, \hat{z}(s))) = 1$ for almost all $s \in (0, S)$.

2.1 | A first counterexample

In this section, we discuss an example, which demonstrates that the solution concept from Definition 2.1 is not stable w.r.t. intermediate convergence of a sequence of loads $(\ell_n)_{n\in\mathbb{N}}$ in $BV([0,T])$ in the following sense: we construct a sequence of solutions $(S_n, \hat{t}_n, \hat{z}_n, \hat{\ell}_n)_{n\in\mathbb{N}}$ associated with ℓ_n in the sense of Definition 2.1, which converges (weakly) to a limit, but this limit is no solution associated with the limit of $(\ell_n)_{n\in\mathbb{N}}$ w.r.t. intermediate convergence. Our example is one-dimensional and we consider the following dissipation and energy:

$$\mathcal{R}(z) = |z|, \qquad \mathcal{E}(z) = \frac{1}{2}z^2 - z, \qquad \mathcal{I}(t, z) = \mathcal{E}(z) - \ell(t)z, \tag{2.6}$$

where $\ell \in BV([0, 2])$. The initial value and end time are set to $z_0 = 0$ and $T = 2$. The sequence of external loads given by

$$\ell_n(t) = \begin{cases} 0, & t \in [0,1] \\ \frac{n}{2}t - \frac{n}{2}, & t \in \left(1, 1 + \frac{1}{n}\right) \\ \frac{1}{2}, & t \in \left[1 + \frac{1}{n}, 2\right]. \end{cases}$$

Then ℓ_n converges to

$$\ell = \begin{cases} 0, & t \in [0,1] \\ \frac{1}{2}, & t \in (1, 2] \end{cases} \tag{2.7}$$

w.r.t. intermediate convergence in $BV([0,T])$. By direct calculations, one verifies that a normalized, $\mathfrak{p}$-parameterized BV solution associated with ℓ_n is given by

$$\hat{z}_n(s) = \begin{cases} 0, & s \in [0,1], \\ \frac{\frac{n}{2}}{1+\frac{n}{2}}s - \frac{\frac{n}{2}}{1+\frac{n}{2}}, & s \in \left(1, \frac{3}{2} + \frac{1}{n}\right), \\ \frac{1}{2}, & s \in \left[\frac{3}{2} + \frac{1}{n}, \frac{5}{2}\right], \end{cases} \tag{2.8}$$

$$\hat{t}_n(s) = \begin{cases} s, & s \in [0,1], \\ \frac{1}{1+\frac{n}{2}}s + \frac{\frac{n}{2}}{1+\frac{n}{2}}, & s \in \left(1, \frac{3}{2} + \frac{1}{n}\right), \\ s - \frac{1}{2}, & s \in \left[\frac{3}{2} + \frac{1}{n}, \frac{5}{2}\right], \end{cases} \tag{2.9}$$

along with $S_n = \frac{5}{2}$ and

$$\hat{\ell}_n(s) = \ell_n(\hat{t}_n(s)) = \begin{cases} 0, & s \in [0,1], \\ \frac{\frac{n}{2}}{1+\frac{n}{2}}s - \frac{\frac{n}{2}}{1+\frac{n}{2}}, & s \in \left(1, \frac{3}{2} + \frac{1}{n}\right), \\ \frac{1}{2}, & s \in \left[\frac{3}{2} + \frac{1}{n}, \frac{5}{2}\right]. \end{cases} \tag{2.10}$$

The pointwise-a.e. limits of these sequences for $n \to \infty$ read

$$\hat{z}(s) = \begin{cases} 0, & s \in [0,1], \\ s-1, & s \in (1, \frac{3}{2}), \\ \frac{1}{2}, & s \in \left[\frac{3}{2}, \frac{5}{2}\right], \end{cases} \quad \hat{t}(s) = \begin{cases} s, & s \in [0,1], \\ 1, & s \in \left(1, \frac{3}{2}\right), \\ s - \frac{1}{2}, & s \in \left[\frac{3}{2}, \frac{5}{2}\right], \end{cases} \tag{2.11}$$

$$\hat{\ell}(s) = \begin{cases} 0, & s \in [0,1], \\ s-1, & s \in \left(1, \frac{3}{2}\right), \\ \frac{1}{2}, & s \in \left[\frac{3}{2}, \frac{5}{2}\right]. \end{cases} \tag{2.12}$$

Note that $\hat{z}$ and $\hat{t}$ are not only the pointwise limits of $\hat{z}_n$ and $\hat{t}_n$, but also the strong limits in $H^1(0,S)$ and weak* limits in $W^{1,\infty}(0,S)$. Moreover, $\hat{\ell}$ is the limit of $\hat{\ell}_n$ w.r.t. intermediate convergence in $BV([0,S])$. However, $(S, \hat{t}, \hat{z}, \hat{\ell})$ is no normalized, $\mathfrak{p}$-parametrized BV solution associated to the limit ℓ from Equation (2.7) according to Definition 2.1, since condition (2.5) is violated in the viscous jump $(1, \frac{3}{2})$. Here $\hat{\ell}$ cannot be expressed by the left- or right-hand side limit of ℓ, that is, $\hat{\ell}(s) \notin \{\ell(\hat{t}(s)-), \ell(\hat{t}(s)), \ell(\hat{t}(s)+)\}$ for all $s \in \left(1, \frac{3}{2}\right)$.

3 | A RELAXED SOLUTION CONCEPT

The example in the previous section shows that the limit of the composition $\ell_n \circ \hat{t}_n$ does in general not coincide with the composition of the separate limits $\ell \circ \hat{t}$ in jumps. Hence, if we aim for a solution concept which is stable w.r.t. weak* convergence of the external loads ℓ_n, we have to weaken the requirements on $\hat{\ell}$. To be more precise, the above example shows that one can hardly impose any condition on $\hat{\ell}$ in jumps. For this reason, we drop the compatibility condition in Equation (2.5) and replace it by a less restrictive condition that does not provide any information on $\hat{\ell}$ in jumps.

Definition 3.1 (Relaxed solution concept). Let $\ell \in BV([0,T]; \mathbb{R}^d)$ be given. We call triple $(S, \hat{t}, \hat{z}, \hat{\ell}) \in (0,\infty) \times W^{1,\infty}(0,S) \times W^{1,\infty}(0,S;\mathbb{R}^d) \times BV([0,S];\mathbb{R}^d)$ is called *relaxed, normalized*, $\mathfrak{p}$*-parametrized BV solution* (or short simply *relaxed solution*) of the rate-independent system (1.1) associated with ℓ, if it satisfies Equations (2.1)–(2.4) and, instead of the compatibility condition in Equation (2.5), we just require

$$\hat{\ell}(s) = \ell(\hat{t}(s)) \text{ for almost all } s \in M, \tag{3.1}$$

where M is the set, where $\hat{t}$ is increasing, that is,

$$M = \{s \in (0,S) : \hat{t}(s_1) < \hat{t}(s_2) \text{ for all } s_1, s_2 \in [0,S] \text{ with } s_1 < s < s_2\}. \tag{3.2}$$

Herein, we choose the continuous representative of $\hat{t}$ denoted by the same symbol for simplicity.

The above definition indeed represents a generalization of the original solution concept in Definition 2.1 as the next result shows.

Lemma 3.2. *Let $\ell \in BV([0,T];\mathbb{R}^d)$ be given and suppose that $(S, \hat{t}, \hat{z}, \hat{\ell}) \in (0,\infty) \times W^{1,\infty}(0,S) \times W^{1,\infty}(0,S;\mathbb{R}^d) \times BV([0,S];\mathbb{R}^d)$ is a normalized, $\mathfrak{p}$-parametrized BV solution in the sense of Definition 2.1. Then it is also a relaxed solution in the sense of Definition 3.1.*

Proof. We only have to verify the condition in Equation (3.1). For this purpose, let $s_* \in M$ be arbitrary and set $t_* := \hat{t}(s_*)$. By definition of M, $\hat{t}^{-1}(t_*)$ is a singleton and therefore Equation (2.5) implies $\hat{\ell}(s_*) \in \{\ell(t_*), \ell(t_*-), \ell(t_*+)\}$. Since ℓ is of bounded variation, it only has countably many jumps collected in the set $J(\ell) \subset [0,T]$ and is continuous elsewhere. Since

$J(\ell)$ is countable and $\hat{t}$ is one-to-one on M, $\hat{t}^{-1}(J(\ell)) \cap M$ has zero measure and thus $\ell(\hat{t}(s)) = \ell(\hat{t}(s)-) = \ell(\hat{t}(s)+)$ f.a.a. $s \in M$, which implies Equation (3.1). □

Thanks to Lemma 3.2, the existence result from Ref. [3, Prop. 4.2] ensuring the existence of a normalized, $\mathfrak{p}$-parametrized BV solution immediately yields the existence of a relaxed solution.

Corollary 3.3. *Let $\ell \in BV([0,T];\mathbb{R}^d)$ be arbitrary. Then there exists at least one relaxed solution of Equation (1.1) in the sense of Definition 3.1.*

3.1 | Stability of the relaxed solution concept

With our relaxed solution concept at hand, we are able to prove the following stability result.

Theorem 3.4. *Let $(\ell_n)_{n\in\mathbb{N}}$ be a bounded sequence in $BV([0,T];\mathbb{R}^d)$ and consider a sequence $(S_n, \hat{t}_n, \hat{z}_n, \hat{\ell}_n)$ of relaxed solutions associated with ℓ_n. Assume, moreover, that the sequence $(\|\hat{\ell}_n\|_{BV([0,S_n];\mathbb{R}^d)})_{n\in\mathbb{N}}$ is bounded. Then there exists a subsequence (also denoted with the index n) such that*

$$S_n \to S, \quad \hat{z}_n \rightharpoonup^* \hat{z} \text{ in } W^{1,\infty}(0,S;\mathbb{R}^d), \quad \hat{t}_n \rightharpoonup^* \hat{t} \text{ in } W^{1,\infty}(0,S) \tag{3.3}$$

and

$$\hat{\ell}_n \rightharpoonup^* \hat{\ell} \quad \text{in } \mathrm{BV}(0,S;\mathbb{R}^d), \quad \ell_n \rightharpoonup^* \ell \quad \text{in } \mathrm{BV}(0,T;\mathbb{R}^d), \tag{3.4}$$

$$\hat{\ell}_n(s) \to \hat{\ell}(s) \quad \text{for all } s \in [0,S], \quad \ell_n(t) \to \ell(t) \quad \text{for all } t \in [0,T]. \tag{3.5}$$

Herein, the functions are constantly extended if $S_n < S$. Furthermore, the limit $(S, \hat{t}, \hat{z}, \hat{\ell})$ is a relaxed solution associated with ℓ.

Proof.

1. *Convergence of a subsequence*

We start with the boundedness of the sequence of relaxed solutions. First of all, Equation (2.3) gives $\|\hat{z}'\|_{L^\infty(0,S_n;\mathbb{R}^d)} \le \frac{1}{c}$ with $c > 0$ from Equation (1.12). Next we prove that the artificial end time is uniformly bounded. Thanks to the boundedness of $\hat{z}_n'$ and $\hat{\ell}_n$ (by assumption), the energy identity (2.4) yields

$$\begin{aligned}
&\mathcal{E}(\hat{z}_n(S_n)) + \int_0^{S_n} \mathcal{R}(\hat{z}_n'(r)) + \|\hat{z}_n'(r)\| \operatorname{dist}(-D_z\hat{\mathcal{I}}_n(r, \hat{z}_n(r)), \partial\mathcal{R}(0))\, dr \\
&= \mathcal{E}(z_0) + \int_0^{S_n} \langle \hat{\ell}_n(r), \hat{z}_n'(r)\rangle\, dr \le \mathcal{E}(z_0) + \|\hat{z}_n'\|_{L^\infty(0,S_n;\mathbb{R}^d)} \|\hat{\ell}_n\|_{L^1(0,S_n;\mathbb{R}^d)} \le C.
\end{aligned}$$

In combination with Equation (2.3), this implies

$$\begin{aligned}
S_n &= \int_0^{S_n} \hat{t}_n'(r) + \mathcal{R}(\hat{z}_n'(r)) + \|\hat{z}_n'(r)\| \operatorname{dist}(-D_z\hat{\mathcal{I}}_n(r, \hat{z}_n(r)), \partial\mathcal{R}(0))\, dr \\
&= T + \int_0^{S_n} \mathcal{R}(\hat{z}_n'(r)) + \|\hat{z}_n'(r)\| \operatorname{dist}(-D_z\hat{\mathcal{I}}_n(r, \hat{z}_n(r)), \partial\mathcal{R}(0))\, dr \le C,
\end{aligned}$$

where we made use of the non-negativity of $\mathcal{E}(\hat{z}_n(S_n))$ by assumption (1.4). Therefore, there is a subsequence (denoted by the same index) such that $S_n \to S$. As in the statement of the theorem, we extend $\hat{t}_n$, $\hat{z}_n$, and $\hat{\ell}_n$ by constant continuation, if necessary, that is, if $S_n < S$. Due to $\|\hat{z}_n'\|_{L^\infty(0,S;\mathbb{R}^d)} \le 1/c$ and $\hat{z}_n(0) = z_0$ for all $n \in \mathbb{N}$, we have $\|\hat{z}_n\|_{W^{1,\infty}(0,S;\mathbb{R}^d)} \le C$. Furthermore, due to Equations (2.1)–(2.3), we conclude $\|\hat{t}_n\|_{W^{1,\infty}(0,S)} \le C$. Thus, there exists a further subsequence (again denoted by the same symbol) such that Equation (3.3) holds. The weak* and the pointwise convergence of the sequences of loads along a further subsequence is an immediate consequence of the boundedness of the respective sequences by assumption.

2. *Correlation between $\hat{\ell}$ and ℓ*

Next we show that Equation (3.1) holds true. We argue by contradiction and assume that the set

$$G := \{s \in M : \hat{\ell}(s) \neq \ell(\hat{t}(s))\} \subset M \tag{3.6}$$

has positive measure. This yields the existence of a $\rho > 0$ such that

$$\widetilde{M}_\rho = M \cap \{s \in (0,S) : \|\hat{\ell}(s) - \ell(\hat{t}(s))\| \geq \rho\}$$

has positive measure, too, since if not, then the sets $\left\{ s \in M : \|\hat{\ell}(s) - \ell(\hat{t}(s))\| \geq \frac{1}{n} \right\}$ have measure zero for all $n \in \mathbb{N}$ and thus $G = \bigcup_{n\in\mathbb{N}} \left\{ s \in M : \left\|\hat{\ell}(s) - \ell(\hat{t}(s))\right\| \geq \frac{1}{n} \right\}$ as a countable union of null sets, too, which contradicts the assumption. Since $\hat{t}_n, \hat{\ell}_n$ are part of the relaxed solution associated with ℓ_n, we have $\hat{\ell}_n(s) = \ell_n(\hat{t}_n(s))$ a.e. in $M_n := \{s \in (0,S) : \hat{t}_n(s_1) < \hat{t}_n(s_2)$ for all $s_1 < s < s_2\}$. We denote the corresponding null sets, where this is not satisfied, by N_n. Then

$$M_\rho := \widetilde{M}_\rho \setminus \bigcup_{n\in\mathbb{N}} N_n$$

still has positive measure.

Now let $k \in \mathbb{N}$, $k > 1$, be arbitrary. Since M_ρ has positive measure, there exists points $\tilde{s}_1 < \tilde{s}_2 < \ldots < \tilde{s}_{2k} \in M_\rho$ and, by construction of M, there holds

$$\hat{t}(\tilde{s}_{2i}) < \hat{t}(\tilde{s}_{2(i+1)}) \quad \forall\, i = 1, \ldots, k-1.$$

Therefore, if we define $s_i := \tilde{s}_{2i}$, $i = 1, \ldots, k$, then $s_1 < s_2 < \ldots < s_k \in M_\rho$ holds true as well as $\hat{t}(s_1) < \hat{t}(s_2) < \ldots < \hat{t}(s_k)$. Thus we obtain

$$\delta_k := \min_{i=1,\ldots,k-1} \frac{\hat{t}(s_{i+1}) - \hat{t}(s_i)}{4} > 0 \quad \text{and} \quad \mu_k := \min_{i=1,\ldots,k-1} \frac{s_{i+1} - s_i}{4} > 0. \tag{3.7}$$

Next we verify that for all $i = 1, \ldots, k$

$$\forall\, \epsilon > 0\; \exists\, \bar{n}_i(\epsilon) \in \mathbb{N}\; \forall\, n \geq \bar{n}_i(\epsilon) : \lambda(M_n \cap B_\epsilon(s_i)) > 0, \tag{3.8}$$

which allows us to choose $s_i^n \in B_\epsilon(s_i)$ with $\hat{\ell}_n(s_i^n) = \ell_n(\hat{t}_n(s_i^n))$. Assuming that Equation (3.8) is not valid implies the existence of an $\epsilon > 0$ and a subsequence $(n_j)_{j\in\mathbb{N}}$ such that $\lambda(M_{n_j} \cap B_\epsilon(s_i)) = 0$ for all $j \in \mathbb{N}$. Then, by the definition of M_{n_j}, we obtain for all $s \in B_\epsilon(s_i) \setminus M_{n_j}$ an interval $I_s \ni s$ with $\hat{t}_{n_j} =$ const. on I_s. As $\lambda(M_{n_j} \cap B_\epsilon(s_i)) = 0$, such an interval exists for almost all $s \in B_\epsilon(s_i)$ and thus, as $\hat{t}_{n_j}$ is absolutely continuous, $\hat{t}_{n_j}$ is constant on $B_\epsilon(s_i)$. Along with the uniform convergence $\hat{t}_{n_j} \to \hat{t}$, this implies

$$\hat{t}\left(s_i - \frac{\epsilon}{2}\right) = \lim_{j\to\infty} \hat{t}_{n_j}\left(s_i - \frac{\epsilon}{2}\right) = \lim_{j\to\infty} \hat{t}_{n_j}\left(s_i + \frac{\epsilon}{2}\right) = \hat{t}\left(s_i + \frac{\epsilon}{2}\right),$$

which contradicts $s_i \in M_\rho \subset M$ so that Equation (3.8) is indeed true.

Therefore, if we now choose $\epsilon = \epsilon_k := \min\{\delta_k, \mu_k\}$ in Equation (3.8), we obtain for all $n \geq \bar{n}_k = \max\{\bar{n}_i(\epsilon_k) : i = 1, \ldots, k\}$, with $\bar{n}_i(\epsilon_k)$ from Equation (3.8), and all $i = 1, \ldots, k$ a point $s_i^n \in (B_{\epsilon_k}(s_i) \cap M_n) \setminus N_n$, which implies

$$\hat{\ell}_n(s_i^n) = \ell_n(\hat{t}_n(s_i^n)) \quad \forall\, n \geq \bar{n}_k \tag{3.9}$$

by the definition of M_n and N_n. Note that N_n has measure zero. Together with (3.9), the triangle inequality gives

$$\sum_{i=1}^{k} \|\hat{\ell}_n(s_i) - \ell_n(\hat{t}(s_i))\| \le \sum_{i=1}^{k} \|\ell_n(\hat{t}_n(s_i^n)) - \ell_n(\hat{t}(s_i))\| + \sum_{i=1}^{k} \|\hat{\ell}_n(s_i^n) - \hat{\ell}_n(s_i)\| \tag{3.10}$$

for all $n \ge \bar{n}_k$.

Let us estimate the expressions on the right-hand side of Equation (3.10). We already know that the limit $\hat{t}$ satisfies $\hat{t} \in W^{1,\infty}(0,S)$ and hence it is Lipschitz continuous with constant $L > 0$ on $[0,S]$. Below we will see that $\hat{t}$ and $\hat{z}$ satisfy the normalization condition in Equation (2.3) and thus, the Lipschitz constant of $\hat{t}$ is less or equal one. Moreover, due to the compact embedding $W^{1,\infty}(0,S) \hookrightarrow C([0,S])$, $\hat{t}_n$ converges uniformly to $\hat{t}$ in $[0,S]$ and hence there exists an index $\tilde{n}_k \in \mathbb{N}$ such that $\sup_{s\in[0,S]} |\hat{t}_n(s) - \hat{t}(s)| < \delta_k$ for all $n \ge \tilde{n}_k$. Together with the Lipschitz continuity of $\hat{t}$ with constant one and $s_i^n \in B_{\epsilon_k}(s_i) \subset B_{\delta_k}(s_i)$ for $n \ge \bar{n}_k$, this results in

$$|\hat{t}_n(s_i^n) - \hat{t}(s_i)| \le |\hat{t}_n(s_i^n) - \hat{t}(s_i^n)| + |\hat{t}(s_i^n) - \hat{t}(s_i)| < 2\delta_k. \tag{3.11}$$

for all $n \ge n_k := \max\{\bar{n}_k, \tilde{n}_k\}$. In view of the definition of δ_k in Equation (3.7), this estimate implies that the intervals $[\min\{\hat{t}_n(s_i^n), \hat{t}(s_i)\}, \max\{\hat{t}_n(s_i^n), \hat{t}(s_i)\}]$, $i = 1, \dots, k$, do not overlap and consequently

$$\sum_{i=1}^{k} \|\ell_n(\hat{t}_n(s_i^n)) - \ell_n(\hat{t}(s_i))\| \le \mathrm{Var}(\ell_n; [0,T]) \quad \forall\, n \ge n_k. \tag{3.12}$$

Similarly, because of $s_i^n \in B_{\epsilon_k}(s_i) \subset B_{\mu_k}(s_i)$ and the definition of μ_k in Equation (3.7), the intervals $[\min\{s_i^n, s_i\}, \max\{s_i^n, s_i\}]$, $i = 1, \dots, k$, do not overlap and therefore

$$\sum_{i=1}^{k} \|\hat{\ell}_n(s_i^n) - \hat{\ell}_n(s_i)\| \le \mathrm{Var}(\hat{\ell}_n; [0,S]) \quad \forall\, n \ge n_k. \tag{3.13}$$

In view of the boundedness of $(\mathrm{Var}(\ell_n; [0,T]))_n$ and $(\mathrm{Var}(\hat{\ell}_n; [0,S]))_n$ by assumption, inserting Equations (3.12) and (3.13) in Equation (3.10) yields

$$\sum_{i=1}^{k} \|\hat{\ell}_n(s_i) - \ell_n(\hat{t}(s_i))\| \le \mathrm{Var}(\ell_n; [0,T]) + \mathrm{Var}(\hat{\ell}_n; [0,S]) \le C \quad \forall\, n \ge n_k.$$

with $C > 0$ independent of n and k. Now using the pointwise convergences of $\hat{\ell}_n, \ell_n$ by Equation (3.5) and passing to the limit $n \to \infty$ results in

$$\sum_{i=1}^{k} \|\hat{\ell}(s_i) - \ell(\hat{t}(s_i))\| \le C.$$

On account of $s_i \in M_\rho$, however, the left-hand side is larger than $k\,\rho$ and, as k was arbitrary, passing to the limit $k \to \infty$ finally leads to the desired contradiction. Hence, G as defined in Equation (3.6) has indeed measure zero and consequently Equation (3.1) is valid.

Proving the remaining conditions (2.1)–(2.4) for the limit $(S, \hat{t}, \hat{z}, \hat{\ell})$ is along the lines of the proof of stability for normalized, $\mathfrak{p}$-parametrized BV solutions for problems with smooth external loads, see, for example, Ref. [17, Sec. 5]. However, for convenience of the reader, we present the arguments in detail.

3. *Initial and end time condition, monotony of* $\hat{t}$

Since the set $\{f \in L^\infty(0,S) : f(s) \ge 0 \; f.a.a.\; s \in (0,S)\}$ is weakly* closed, the limit fulfills $\hat{t}'(s) \ge 0$ a.e. in $(0,S)$. Moreover, as already used before, $W^{1,\infty}(0,S;\mathbb{R}^d)$ embeds compactly into $C(0,S;\mathbb{R}^d)$ and therefore, $(\hat{t}_n)_{n\in\mathbb{N}}$ and $(\hat{z}_n)_{n\in\mathbb{N}}$ converge uniformly. Therefore, $\hat{t}(0) = 0$ as well as $\hat{t}(S) = \lim_{n\to\infty} \hat{t}_n(S_n) = T$ and $\hat{z}(0) = \lim_{n\in\mathbb{N}} \hat{z}_n(0) = z_0$ hold true so that Equation (2.1) is fulfilled.

4. *Energy equality*

Let $s_1, s_2 \in [0, S]$ be given. Due to the pointwise convergence of $\hat{z}_n \to \hat{z}$ and the continuity of $\mathcal{E}$, we have $\mathcal{E}(\hat{z}_n(s_i)) \to \mathcal{E}(\hat{z}(s_i))$, $i = 1, 2$. Along with the lower semi-continuity of $\mathfrak{p}(\cdot,\cdot)$ according to Ref, [18, Lemma 3.1], it follows that

$$\begin{aligned}
&\mathcal{E}(\hat{z}(s_2)) + \int_{s_1}^{s_2} \mathcal{R}(\hat{z}'(r)) + \|\hat{z}'(s)\| \operatorname{dist}(-D_z\hat{\mathcal{I}}(r,\hat{z}(r)), \partial\mathcal{R}(0))\, dr \\
&= \mathcal{E}(\hat{z}(s_2)) + \int_{s_1}^{s_2} \mathfrak{p}(\hat{z}'(r), -D_z\hat{\mathcal{I}}(r,\hat{z}(r)))\, dr \\
&\leq \liminf_{n\to\infty} \left(\mathcal{E}(\hat{z}_n(s_2)) + \int_{s_1}^{s_2} \mathfrak{p}(\hat{z}_n'(r), -D_z\hat{\mathcal{I}}_n(r,\hat{z}_n(r)))\, dr \right) \\
&\leq \limsup_{n\to\infty} \left(\mathcal{E}(\hat{z}_n(s_2)) + \int_{s_1}^{s_2} \mathfrak{p}(\hat{z}_n'(r), -D_z\hat{\mathcal{I}}_n(r,\hat{z}_n(r)))\, dr \right) \\
&\leq \limsup_{n\to\infty} \left(\mathcal{E}(\hat{z}_n(s_1)) + \int_{s_1}^{s_2} \langle \hat{\ell}_n(r), \hat{z}_n'(r)\rangle\, dr \right) \\
&= \mathcal{E}(\hat{z}(s_1)) + \int_{s_1}^{s_2} \langle \hat{\ell}(r), \hat{z}'(r)\rangle\, dr,
\end{aligned} \tag{3.14}$$

where we exploited the strong convergence $\hat{\ell}_n \to \ell$ in $L^2(0, S; \mathbb{R}^d)$ by the compactness of $\mathrm{BV}(0, S; \mathbb{R}^d) \hookrightarrow L^2(0, S; \mathbb{R}^d)$ and the weak* convergence $\hat{z}_n' \rightharpoonup^* \hat{z}'$ in $L^\infty(0, S; \mathbb{R}^d)$ in the last step. Because the opposite inequality is always fulfilled by Lemma B.1, we end up with equality and since $s_1, s_2 \in [0, S]$ were arbitrary, Equation (2.4) is shown.

5. *Complementary condition*

The convergences $\hat{\ell}_n \to \hat{\ell}$ in $L^2(0, S; \mathbb{R}^d)$ and $\hat{z}_n \rightharpoonup^* \hat{z}$ in $W^{1,\infty}(0, S; \mathbb{R}^d)$ lead to $D_z\hat{\mathcal{I}}_n(\cdot, \hat{z}_n(\cdot)) \to D_z\hat{\mathcal{I}}(\cdot, \hat{z}(\cdot))$ in $L^2(0, S; \mathbb{R}^d)$ and, by Lipschitz-continuity of the distance, we obtain

$$\operatorname{dist}\left(-D_z\hat{\mathcal{I}}_n(\cdot, \hat{z}_n(\cdot)), \partial\mathcal{R}(0)\right) \to \operatorname{dist}\left(-D_z\hat{\mathcal{I}}(\cdot, \hat{z}(\cdot)), \partial\mathcal{R}(0)\right) \text{ in } L^2(0, S).$$

Together with the weak* convergence $\hat{t}_n' \rightharpoonup^* \hat{t}'$ in $L^\infty(0, S)$, this results in

$$\begin{aligned}
&\int_0^S \hat{t}'(r) \operatorname{dist}(-D_z\hat{\mathcal{I}}(r, \hat{z}(r)), \partial\mathcal{R}(0)) dr \\
&= \lim_{n\to\infty} \int_0^S \hat{t}_n'(r) \operatorname{dist}(-D_z\hat{\mathcal{I}}_n(r, \hat{z}_n(r)), \partial\mathcal{R}(0)) dr = 0
\end{aligned}$$

such that $\hat{t}'(s) \operatorname{dist}(-D_z\hat{\mathcal{I}}(s, \hat{z}(s)), \partial\mathcal{R}(0)) = 0$ for almost all $s \in (0, S)$ due to the non-negativity of the integrand.

6. *Normalization*

The pointwise convergence $\mathcal{E}(\hat{z}_n(\cdot)) \to \mathcal{E}(\hat{z}(\cdot))$ in combination with the fact that all inequalities in Equation (3.14) hold true with equality yields for all $s_1, s_2 \in [0, S]$ that

$$s_2 - s_1 = \int_{s_1}^{s_2} \hat{t}_n'(r) + \mathfrak{p}(\hat{z}_n'(r), -D_z\hat{\mathcal{I}}_n(r, \hat{z}_n(r)))\, dr \to \int_{s_1}^{s_2} \hat{t}'(r) + \mathfrak{p}(\hat{z}'(r), -D_z\hat{\mathcal{I}}(r, \hat{z}(r)))\, dr \quad \text{as } n \to \infty.$$

Thus, assuming the existence of a Lebesgue measurable set $E \subset (0,S)$ with $\hat{t}'(s) + \mathfrak{p}(\hat{z}'(s), -D_z\hat{\mathcal{I}}(s,\hat{z}(s))) > 1$ for almost all $s \in E$ and $\lambda(E) > 0$ gives for every finite union $U \subset [0,S]$ of pairwise disjoint intervals with $E \subset U$

$$\lambda(U) = \int_U \hat{t}'(r) + \mathfrak{p}(\hat{z}'(r), -D_z\hat{\mathcal{I}}(r,\hat{z}(r)))\,dr \geq \int_E \hat{t}'(r) + \mathfrak{p}(\hat{z}'(r), -D_z\hat{\mathcal{I}}(r,\hat{z}(r)))\,dr$$
$$\geq \lambda(E) + \varepsilon$$

for an $\varepsilon > 0$, only depending on E and not on U, which contradicts the regularity of the Lebesgue measure. Similarly $\hat{t}'(s) + \mathfrak{p}(\hat{z}'(s), -D_z\hat{\mathcal{I}}(s,\hat{z}(s))) < 1$ for almost all $s \in E$ with $\lambda(E) > 0$ would imply

$$\begin{aligned} S &= \int_0^S \hat{t}'(r) + \mathfrak{p}(\hat{z}'(r), -D_z\hat{\mathcal{I}}(r,\hat{z}(r)))\,dr \\ &= \int_E \hat{t}'(r) + \mathfrak{p}(\hat{z}'(r), -D_z\hat{\mathcal{I}}(r,\hat{z}(r)))\,dr + \int_{(0,S)\setminus E} \hat{t}'(r) + \mathfrak{p}(\hat{z}'(r), -D_z\hat{\mathcal{I}}(r,\hat{z}(r)))\,dr \\ &< \lambda(E) + \lambda((0,S)\setminus E) = S. \end{aligned}$$

Hence, we have $\hat{t}'(\cdot) + \mathfrak{p}(\hat{z}'(\cdot), -D_z\hat{\mathcal{I}}(\cdot,\hat{z}(\cdot))) = 1$ a.e. $(0,S)$, which is Equation (2.3). □

A stability result of the form of Theorem 3.4 is of course useful in many regards. As a potential application, we consider the following optimal control problem

$$\begin{cases} \min & J(S,\hat{z},\ell) := j(\hat{z}(S)) + \beta\,\|\ell\|_{BV([0,T];\mathbb{R}^d)} \\ \text{s.t.} & \ell \in BV([0,T];\mathbb{R}^d), \quad (S,\hat{t},\hat{z},\hat{\ell}) \in \mathcal{L}(\ell), \quad \|\hat{\ell}\|_{BV([0,S];\mathbb{R}^d)} \leq K, \end{cases} \tag{3.15}$$

where

$$\mathcal{L}(\ell) := \big\{(S,\hat{t},\hat{z},\hat{\ell}) \in [T,\infty) \times W^{1,\infty}(0,S) \times W^{1,\infty}(0,S;\mathbb{R}^d) \times BV([0,S];\mathbb{R}^d) :$$
$$(S,\hat{t},\hat{z},\hat{\ell}) \text{ is a relaxed solution associated with } \ell\big\}.$$

Furthermore, $\beta > 0$ and $K > T\|\ell_0\|$ a given constants. The function $j : \mathbb{R}^d \to \mathbb{R}$ in the objective is supposed to be continuous and bounded from below. By means of Theorem 3.4, we can now prove the following result.

Corollary 3.5. *There exists at least one globally optimal solution*

$$(S^*,\hat{t}^*,\hat{z}^*,\hat{\ell}^*,\ell^*) \in [T,\infty) \times W^{1,\infty}(0,S^*) \times W^{1,\infty}(0,S^*;\mathbb{R}^d) \times BV([0,S^*];\mathbb{R}^d) \times BV([0,T];\mathbb{R}^d)$$

of Equation (3.15).

Proof. Based on Theorem 3.4, the proof follows standard arguments. Due to the assumption $-Az_0 - \mathcal{F}(z_0) + \ell_0 \in \partial\mathcal{R}(0)$, the constant functions $\hat{z} \equiv z_0$ and $\hat{\ell} \equiv \ell_0$ together with $\hat{t} = \mathrm{id}$ and $S = T$ generate a normalized, $\mathfrak{p}$-parametrized BV solution associated with $\ell \equiv \ell_0$ in the sense of Definition 2.1. Thus it is also a relaxed solution associated with $\ell \equiv \ell_0$. Moreover, $\hat{\ell}$ trivially satisfies the boundedness condition in the constraints. Therefore, the feasible set of Equation (3.15) is nonempty and we obtain an infimal sequence $(S_n,\hat{t}_n,\hat{z}_n,\hat{\ell}_n,\ell_n)$ such that $J(S_n,\hat{z}_n,\ell_n) \to J^*$, where

$$J^* := \inf\big\{J(S,\hat{z},\ell) : \ell \in BV([0,T];\mathbb{R}^d),\ (S,\hat{t},\hat{z},\hat{\ell}) \in \mathcal{L}(\ell),\ \|\hat{\ell}\|_{BV([0,S];\mathbb{R}^d)} \leq K\big\}.$$

Since j is bounded from below, the sequence $(\ell_n)_n$ is bounded in $BV([0,T];\mathbb{R}^d)$. Moreover, according the constraints, the sequence $(\|\hat{\ell}_n\|_{BV([0,S_n];\mathbb{R}^d)})_n$ is also bounded. Therefore, Theorem 3.4 gives the existence of a subsequence converging to a limit $(S^*,\hat{t}^*,\hat{z}^*,\hat{\ell}^*,\ell^*)$ such that $(S^*,\hat{t}^*,\hat{z}^*,\hat{\ell}^*) \in \mathcal{L}(\ell^*)$ in the sense of Equations (3.3)–(3.5). Now, given an arbitrary partition $\{t_k\}$ of $[0,T]$, the pointwise convergence of ℓ_n from Equation (3.5) implies

$$\sum_k \|\ell(t_k) - \ell(t_{k-1})\| = \lim_{n\to\infty} \sum_k \|\ell_n(t_k) - \ell_n(t_{k-1})\| \leq \liminf_{n\to\infty} \mathrm{Var}(\ell_n;[0,T]).$$

Since $\{t_k\}$ was arbitrary, the total variation is, thus, lower semicontinuous w.r.t. pointwise convergence and so is the norm in $BV([0,T];\mathbb{R}^d)$ and $BV([0,S];\mathbb{R}^d)$, respectively. Thus, the bound on the $BV([0,S];\mathbb{R}^d)$-norm readily carries over to the limit $\hat{\ell}^*$ giving its feasibility. Finally, the compact embedding $W^{1,\infty}(0,S^*) \hookrightarrow C([0,S^*])$ yields $\hat{z}_n \to \hat{z}^*$ uniformly so that exploiting the continuity of j and the lower semi-continuity of the $BV([0,T];\mathbb{R}^d)$-norm results in $J(S^*,\hat{z}^*,\ell^*) \leq \liminf_{n\to\infty} J(S_n,\hat{z}_n,\ell_n) = J^*$, which means $(S^*,\hat{t}^*,\hat{z}^*,\hat{\ell}^*,\ell^*)$ is an optimal solution of Equation (3.15). □

4 | PHYSICAL PLAUSIBILITY OF THE RELAXED SOLUTION CONCEPT

4.1 | Relation to local solutions

In order to classify our relaxed solution concept, we compare it with the concept of a *local solution*, which is one of the weakest solution concepts imposing less conditions on a solution compared to the other concepts. In our finite dimensional setting with loads in BV, this solution concept reads as follows:

Definition 4.1. Let $\ell \in BV([0,T];\mathbb{R}^d)$ be given. We call $z \in BV([0,T];\mathbb{R}^d)$ a *local solution* of Equation (1.1) associated with ℓ, if

$$0 \in \partial\mathcal{R}(0) + D_z\mathcal{I}(t,z(t)) \quad \text{f.a.a. } t \in [0,T] \tag{4.1}$$

$$\mathcal{I}(t_2,z(t_2)) + \mathrm{Diss}_{\mathcal{R}}(z;[t_1,t_2]) \leq \mathcal{I}(t_1,z(t_1)) - \int_{t_1}^{t_2} z(r)\,d\ell(r) \quad \forall\, 0 \leq t_1 \leq t_2 \leq T, \tag{4.2}$$

where

$$\begin{aligned}&\mathrm{Diss}_{\mathcal{R}}(z;[t_1,t_2]) := \\ &\sup\left\{\sum_{i=1}^{k} \mathcal{R}(z(\xi_i)-z(\xi_{i-1})) \,\middle|\, t_1=\xi_0<\xi_1<\dots<\xi_n=t_2,\ n\in\mathbb{N}\right\}\end{aligned} \tag{4.3}$$

and the integral on the left-hand side is to be understood as a Kurzweil integral. We refer to Refs. [4, Sec. 1] and [19, Sec. 2] for an overview about the concept of Kurzweil integrals.

Remark 4.2. For external loads $\ell \in H^1(0,T;\mathbb{R}^d)$, the Kurzweil integral in Equation (4.2) can be converted into the Lebesgue integral $\int_{t_1}^{t_2}\langle z(r),\ell'(r)\rangle dr$, compare Ref. [4, Prop. 1.10], so that the above definition coincides with Ref. [2, Def. 3.3.2]. Note moreover that, in this case, integration by parts of the integral involving the loading gives rise to an equivalent energy inequality in terms of the potential energy $\mathcal{E}$ analogously to the energy identity in Equation (2.4). In case of the Kurzweil integral, however, this reformulation leads to another notion of local solutions, which is no longer equivalent to Equation (4.2), see Definition 4.7 below.

We underline that the subadditivity of $\mathcal{R}$ implies that

$$\mathrm{Diss}_{\mathcal{R}}(z;[t_1,t_2]) = \mathrm{Diss}_{\mathcal{R}}(z;[0,t_2]) - \mathrm{Diss}_{\mathcal{R}}(z;[0,t_1]), \tag{4.4}$$

for all $0 \leq t_1 \leq t_2 \leq T$ provided that z is continuous in t_1. In order to compare our solution concept with the concept of local solutions, we have to translate a relaxed solution into physical time. For this purpose, assume that we are given a relaxed solution $(S,\hat{t},\hat{z},\hat{\ell})$. Then we define the set of projections of $(\hat{t},\hat{z})$ as

$$\mathfrak{P}(\hat{t},\hat{z}) := \{z : [0,T] \to \mathbb{R}^d : \forall\, t \in [0,T]\ \exists\, s \in [0,S] \text{ with } (t,z(t)) = (\hat{t}(s),\hat{z}(s))\}. \tag{4.5}$$

Note that $\mathfrak{P}$ consists of all functions $z : [0,T] \to \mathbb{R}^d$, whose graph is subset of the solution trajectory $[0,S] \ni s \mapsto (\hat{t}(s),\hat{z}(s))$. The following theorem correlates local with relaxed solutions. Under suitable assumptions, we can show that every local solution can be interpreted as a relaxed solution. However, the opposite is not true, as we will see in the example in Section 4.2 below.

Theorem 4.3. *Let $\ell \in BV([0,T];\mathbb{R}^d)$ and local solution $z \in BV([0,T];\mathbb{R}^d)$ in the sense of Definition 4.1 be given. Furthermore, we assume that $\mathcal{R}$ is symmetric, that is,*

$$\mathcal{R}(z_1) = \mathcal{R}(z_2) \quad \Longleftrightarrow \quad \|z_1\| = \|z_2\| \quad \forall\, z_1, z_2 \in \mathbb{R}^d, \tag{4.6}$$

and that z only admits a finite number of jumps. Then there exists a parametrization and $\hat{\ell} \in BV([0,S];\mathbb{R}^d)$ such that $(S, \hat{t}, \hat{z}, \hat{\ell})$ is a relaxed solution and $z \in \mathfrak{P}(\hat{t}, \hat{z})$.

Proof.

1. *Construction and regularity of $\hat{t}$ and $\hat{z}$*

We first need to construct a suitable parametrization $s \mapsto (\hat{t}(s), \hat{z}(s))$ of the solution trajectory as a candidate for a relaxed solution. In the literature on $\mathfrak{p}$-parametrized BV solutions, given a solution in physical time, the parametrized solution $\hat{z}$ is frequently designed in jumps in such a way that the dissipative distance is minimized. More precisely, if $t \in [0,T]$ is a jump point between the values z_1 and z_2, then $\hat{z}$ is constructed by means of the minimizer of

$$\begin{cases} \min & \displaystyle\int_0^1 \mathfrak{p}(v'(r), -D_z\mathcal{I}(t, v(r)))dr \\ \text{s.t.} & v \in W^{1,1}(0,1;\mathbb{R}^d), \quad v(0) = z_1, \quad v(1) = z_2, \end{cases} \tag{4.7}$$

compare Ref. [2, Sec. 3.8.2]. Here, we pursue a different strategy. Since, in the relaxed solution concept, $\hat{\ell}$ is an additional variable in jumps, we have certain freedom in the choice of $\hat{z}$. This allows us to use a simpler construction of $\hat{z}$ and the associated parametrization. The idea is to define $\hat{z}$ in a jump point t as an affin linear function connecting the value of z in t with its left or right-hand side limit and then define $\hat{\ell}$ in dependence of $\hat{z}$ such that the normalization condition (2.3) and energy identity (2.4) are satisfied. For this purpose, let us define the following modified dissipation

$$\mathrm{Diss}_0(z;[0,t]) := \mathcal{R}(z(0+) - z_0) + \mathrm{Diss}_{\mathcal{R}}(z;[0+,t]) \tag{4.8}$$

taking into account potential jumps at the initial time. Note that the local solution need not satisfy the initial condition. Given Diss_0, we set

$$s : [0,T] \to [0,S], \quad s(t) := \begin{cases} 0, & t = 0, \\ t + \mathrm{Diss}_0(z;[0,t]), & t \in (0,T], \end{cases} \tag{4.9}$$

where $S := s(T)$. Note that, by Equation (1.12),

$$\mathrm{Diss}_{\mathcal{R}}(z;[0,T]) \le C\,\mathrm{Var}(z;[0,T]) < \infty,$$

so that S is finite. Let us investigate the regularity of s. We denote the jump points of z that are greater zero by $0 < t_1 < t_2 < \dots < t_N \le T$ and set $J(z) := \{t_1, \dots, t_N\}$. According to its definition, $\mathrm{Diss}_{\mathcal{R}}(z;[0+,\cdot\,])$ is continuous in intervals of continuity of z, that is, (t_n, t_{n+1}) with $t_n, t_{n+1} \in J(z)$, $n = 1, \dots, N$, as it inherits its continuity from z there. Thus, by construction, the jump points of s are $J(s) := \{0, t_1, \dots, t_N\}$, if $z(0+) \ne z_0$, and $J(s) = J(z)$ otherwise. In the remaining intervals however, that is, in $(0,t_1), (t_1,t_2), \dots, (t_{N-1},t_N), (t_N,T)$, the function s is continuous. Moreover, the non-negativity of $\mathcal{R}$ implies that s is strictly increasing and hence invertible. We denote its inverse function as

$$\hat{t} : [0,S] \setminus \bigcup_{n=0}^{N} I_n \to [0,T] \setminus J(s)$$

with

$$I_0 := [0, \mathcal{R}(z(0+) - z_0)] \quad \text{and} \quad I_n := [s(t_n-), s(t_n+)]. \tag{4.10}$$

Then $\hat{t}$ is monotonously increasing as an inverse function of an increasing function. Furthermore, $\hat{t}$ is Lipschitz-continuous with Lipschitz constant $L \le 1$ on all intervals of $[0,S] \setminus \bigcup_{n=0}^{N} I_n$, since Equations (4.9) and (4.4) give for $s_1 < s_2$ from such an interval that

$$\begin{aligned} 0 \leq \hat{t}(s_2) - \hat{t}(s_1) &= s_2 - s_1 - \big(\mathrm{Diss}_0(z;[0,\hat{t}(s_2)]) - \mathrm{Diss}_0(z;[0,\hat{t}(s_1)])\big) \\ &= s_2 - s_1 - \mathrm{Diss}_{\mathcal{R}}(z;[\hat{t}(s_1),\hat{t}(s_2)]) \leq s_2 - s_1. \end{aligned} \tag{4.11}$$

Therefore, after constant continuation on all I_n, $n = 0, \dots, N$, that is,

$$\hat{t}(s) = 0 \quad \forall\, s \in I_0, \qquad \hat{t}(s) = t_n \quad \forall\, s \in I_n,\ n = 1, \dots, N,$$

the function $\hat{t}$ is Lipschitz-continuous with Lipschitz constant $L \leq 1$. So there holds $\hat{t} \in W^{1,\infty}(0,S)$, that is, the required regularity of $\hat{t}$, and $\hat{t}'(s) \geq 0$ for almost all $s \in (0,S)$, that is, the sign condition in Equation (2.2). Moreover, by construction of $\hat{t}$, we obtain that M, the set where $\hat{t}$ is strictly increasing, see Equation (3.2), is given by

$$M = (0,S) \setminus \bigcup_{n=0}^{N} I_n =: \bigcup_{m=1}^{M} G_m, \tag{4.12}$$

where $G_m \subset (0,S)$ are open intervals.

Given $\hat{t}$, we define $\hat{z}$ as composition of z and $\hat{t}$ in the parts, where z is continuous and in jumps we choose $\hat{z}$ as the linear function that connects the left- and right-hand limits of z such that $\hat{z}$ reads

$$\hat{z}(s) := \begin{cases} z(\hat{t}(s)), & s \in (0,S] \setminus \bigcup_{n=0}^{N} I_n, \\ z(t_n-) + \dfrac{z(t_n) - z(t_n-)}{s(t_n) - s(t_n-)}\,(s - s(t_n-)), & s \in I_n^-, \\ z(t_n) + \dfrac{z(t_n+) - z(t_n)}{s(t_n+) - s(t_n)}\,(s - s(t_n)), & s \in I_n^+, \\ z_0 + \dfrac{z(0+) - z_0}{\mathcal{R}(z(0+) - z_0)}\, s, & s \in I_0, \end{cases} \tag{4.13}$$

where $I_n^- := [s(t_n-), s(t_n)]$, $I_n^+ := [s(t_n), s(t_n+)]$, and I_0 as defined in Equation (4.10). Note that, by construction, $\hat{z}(0) = z_0$, that is, $\hat{z}$ satisfies the initial condition in Equation (2.1). Next we show that $\hat{z}$ constructed in this way is Lipschitz continuous, too. Let us first consider an arbitrary interval G_m and let $s_1, s_2 \in G_m$ with $s_1 < s_2$ be arbitrary. Then, by definition of G_m, z is continuous on $[\hat{t}(s_1), \hat{t}(s_2)]$ and thus, similarly to Equation (4.11), the construction of s in Equation (4.9) yields

$$s(\hat{t}(s_2)) - \hat{t}(s_2) - s(\hat{t}(s_1)) + \hat{t}(s_1) = \mathrm{Diss}_{\mathcal{R}}(z;[\hat{t}(s_1),\hat{t}(s_2)]). \tag{4.14}$$

With this at hand, assumption (1.12) and the construction of $\hat{z}$ in Equation (4.13) yield

$$\begin{aligned} c\,\|\hat{z}(s_2) - \hat{z}(s_1)\| &\leq \mathcal{R}(\hat{z}(s_2) - \hat{z}(s_1)) \\ &\leq \mathrm{Diss}_{\mathcal{R}}(\hat{z};[s_1,s_2]) \\ &= \mathrm{Diss}_{\mathcal{R}}(z;[\hat{t}(s_1),\hat{t}(s_2)]) \\ &= s_2 - \hat{t}(s_2) - s_1 + \hat{t}(s_1) \leq s_2 - s_1, \end{aligned} \tag{4.15}$$

where we exploited the monotonicity of $\hat{t}$ in the last estimate. Next let us consider an arbitrary interval I_n^- and arbitrary points $s_1, s_2 \in I_n^-$. Using the definition of s in Equation (4.9) and Diss_0 in Equation (4.8) and again Equation (4.4), one obtains

$$\begin{aligned} s(t_n) - s(t_n-) &= \mathrm{Diss}_{\mathcal{R}}(z;[0,t_n]) - \mathrm{Diss}_{\mathcal{R}}(z;[0,t_n-]) \\ &= \mathcal{R}(z(t_n) - z(t_n-)). \end{aligned} \tag{4.16}$$

Together with the construction of $\hat{z}$ in jumps according to Equation (4.13), this gives

$$\begin{aligned}\|\hat{z}(s_2)-\hat{z}(s_1)\| &= \frac{\|z(t_n)-z(t_n-)\|}{s(t_n)-s(t_n-)}\,|s_2-s_1| \\ &\leq \frac{\|z(t_n)-z(t_n-)\|}{\mathcal{R}(z(t_n)-z(t_n-))}\,|s_2-s_1| \leq \frac{1}{c}\,|s_2-s_1|,\end{aligned} \tag{4.17}$$

where we again used Equation (1.12) for the last estimate. Completely analogously, one derives the same estimate for arbitrary $s_1, s_2 \in I_n^+$. In I_0, we similarly obtain

$$\|\hat{z}(s_2)-\hat{z}(s_1)\| = \frac{\|z(0+)-z_0\|}{\mathcal{R}(z(0+)-z_0)}\,|s_2-s_1| \leq \frac{1}{c}\,|s_2-s_1| \quad \forall\, s_1, s_2 \in I_0.$$

Since $\hat{z}$ is additionally continuous by construction, see Equation (4.13), this estimate together with Equations (4.15) and (4.17) implies that $\hat{z}$ is Lipschitz-continuous on $[0,S]$ with Lipschitz constant $1/c$. Consequently, we obtain the desired regularity of a relaxed solution, that is, $\hat{z} \in W^{1,\infty}(0,S;\mathbb{R}^d)$. Furthermore, $\hat{t}, \hat{z}$ are constructed in such a way that $z \in \mathfrak{P}(\hat{t}, \hat{z})$.

2. *Complementary, normalization, and energy identity outside of jumps*

Now that we have defined $\hat{t}$ and $\hat{z}$, we still need to define $\hat{\ell}$ outside the set M so that Equations (2.2)–(2.4) are fulfilled. On M, however, $\hat{\ell}$ is fixed by Equation (3.1) and set to $\hat{\ell} = \ell \circ \hat{t}$. To show Equation (2.2), let us assume by contrary that there exists a set $E \subset M$ of positive measure such that

$$\operatorname{dist}(-D_z\hat{\mathcal{I}}(s,\hat{z}(s)), \partial\mathcal{R}(0)) > 0 \quad \text{a.e. in } E. \tag{4.18}$$

For the image of E under $\hat{t}$, we obtain $|\hat{t}(E)| = \int_E |\hat{t}'(s)|\,ds > 0$, since $\hat{t}$ is Lipschitz continuous and monotonically increasing on M. Then, in view of $\hat{z} = z \circ \hat{t}$ and $\hat{\ell} = \ell \circ \hat{t}$ and the definition of $\hat{\mathcal{I}}$ in Equation (1.8), Equation (4.18) implies $\operatorname{dist}(-D_z\mathcal{I}(t,z(s)), \partial\mathcal{R}(0)) > 0$ a.e. in $\hat{t}(E)$, which contradicts Equation (4.1). Thus we obtain

$$\operatorname{dist}(-D_z\hat{\mathcal{I}}(s,\hat{z}(s)), \partial\mathcal{R}(0)) = 0 \quad \text{f.a.a. } s \in M \tag{4.19}$$

Moreover, due to constant continuation outside of M, there holds $\hat{t}'(s) = 0$ for almost all $s \in (0,S) \setminus M$ and hence, Equation (2.2) is valid independent of the choice of $\hat{\ell}$ on $(0,S) \setminus M$.

Next we show that Equation (2.3) is satisfied almost everywhere on M. By exploiting Equation (4.19), this is

$$\hat{t}'(s) + \mathcal{R}(\hat{z}'(s)) = 1 \quad \text{f.a.a. } s \in M. \tag{4.20}$$

Because M is a finite union of intervals, see Equation (4.12), this is in turn equivalent to

$$\int_{s_1}^{s_2} \hat{t}'(r) + \mathcal{R}(\hat{z}'(r))dr = s_2 - s_1 \quad \forall\, s_1, s_2 \in G_m,\ m = 1, \dots, M. \tag{4.21}$$

Using Equation (4.14) and Lemma A.1 yield for arbitrary $s_1, s_2 \in G_m$

$$\begin{aligned}\int_{s_1}^{s_2} \hat{t}'(r) + \mathcal{R}(\hat{z}'(r))dr &= \hat{t}(s_2) - \hat{t}(s_1) + \operatorname{Diss}_{\mathcal{R}}(\hat{z};[s_1,s_2]) \\ &= \hat{t}(s_2) - \hat{t}(s_1) + \operatorname{Diss}_{\mathcal{R}}(z;[\hat{t}(s_1),\hat{t}(s_2)]) \\ &= s(\hat{t}(s_2)) - s(\hat{t}(s_1)) \\ &= s_2 - s_1,\end{aligned}$$

which is Equation (4.21) such that Equation (2.3) indeed holds a.e. in M.

Concerning the energy identity in Equation (2.4), take an arbitrary interval G_m and let $s_1, s_2 \in G_m$, $s_1 < s_2$, be arbitrary, too. By definition of $\hat{z}$ in Equation (4.13), there holds $\hat{z} = z \circ \hat{t}$ in G_m and therefore, the energy inequality (4.2) with $t_1 = \hat{t}(s_1)$

and $t_2 = \hat{t}(s_2)$ yields

$$\begin{aligned}
&\mathcal{I}(\hat{t}(s_2), \hat{z}(s_2)) + \mathrm{Diss}_{\mathcal{R}}(\hat{z}; [s_1, s_2]) - \mathcal{I}(\hat{t}(s_1), \hat{z}(s_1)) \\
&\quad = \mathcal{I}(\hat{t}(s_1), z(\hat{t}(s_2))) + \mathrm{Diss}_{\mathcal{R}}(z; [\hat{t}(s_1), \hat{t}(s_2)]) - \mathcal{I}(\hat{t}(s_1), z(\hat{t}(s_1))) \\
&\quad \leq - \int_{\hat{t}(s_1)}^{\hat{t}(s_2)} z(t)\, d\ell(t) \\
&\quad = - \int_{s_1}^{s_2} (z \circ \hat{t})(r)\, d(\ell \circ \hat{t})(r) = - \int_{s_1}^{s_2} \hat{z}(r)\, d\hat{\ell}(r),
\end{aligned} \tag{4.22}$$

where we used the change of variables formula for Kurzweil integrals from Ref. [20, Thm. 6.1], which is applicable due to the continuity of $\hat{t}$. Note that we again used $\hat{\ell} = \ell \circ \hat{t}$ in $G_m \subset M$. For the dissipation, Lemma A.1 along with Equation (4.19) yields

$$\mathrm{Diss}_{\mathcal{R}}(\hat{z}; [s_1, s_2]) = \int_{s_1}^{s_2} \mathcal{R}(\hat{z}'(r)) + \|\hat{z}'(r)\| \,\mathrm{dist}(-D_z \hat{\mathcal{I}}(r, \hat{z}(r)), \partial\mathcal{R}(0))\, dr. \tag{4.23}$$

Since $\hat{z} \in W^{1,\infty}(0, S; \mathbb{R}^d)$, we are allowed to apply the formula of integration by parts according to Ref. [4, Prop. 1.12] to the Kurzweil integral on the right-hand side of Equation (4.22), which results in

$$- \int_{s_1}^{s_2} \hat{z}(r)\, d\hat{\ell}(r) = \int_{s_1}^{s_2} \langle \hat{\ell}(r), \hat{z}'(r) \rangle\, dr + \langle \hat{\ell}(s_2), \hat{z}(s_2) \rangle - \langle \hat{\ell}(s_1), \hat{z}(s_1) \rangle,$$

where we have already rewritten the Kurzweil integral on the right-hand side as a Lebesgue integral according to Ref. [4, Prop. 1.10] and the regularity of $\hat{z}$. Inserting this together with Equation (4.23) in Equation (4.22) implies in view of the definition of $\mathcal{E}$ in Equation (1.7) that

$$\begin{aligned}
\mathcal{E}(\hat{z}(s_2)) + \int_{s_1}^{s_2} \mathcal{R}(\hat{z}'(r)) + \|\hat{z}'(r)\| \,\mathrm{dist}(-D_z \hat{\mathcal{I}}(r, \hat{z}(r)), \partial\mathcal{R}(0))\, dr& \\
\leq \mathcal{E}(\hat{z}(s_1)) + \int_{s_1}^{s_2} \langle \hat{\ell}(r), \hat{z}'(r) \rangle\, dr&,
\end{aligned} \tag{4.24}$$

Lemma B.1 yields that Equation (4.24) is even satisfied with equality and thus, we obtain the desired energy identity for arbitrary $s_1, s_2 \in G_m$.

3. *Construction of $\hat{\ell}$ in jumps*

To motivate our definition of $\hat{\ell}$ in jumps, let us take a closer look at the connection of the normalization condition (2.3) and the energy identity in jumps. As $\hat{t}$ is constant there, Equation (2.3) reads

$$\mathcal{R}(\hat{z}'(s)) + \|\hat{z}'(s)\| \,\mathrm{dist}(-D_z \hat{\mathcal{I}}(s, \hat{z}(s)), \partial\mathcal{R}(0)) = 1 \quad \text{f.a.a. } s \in \bigcup_{n=0}^{N} I_n. \tag{4.25}$$

The term on the left-hand side is exactly the expression that also arises in the energy identity. So, if we consider an arbitrary jump interval I_n and arbitrary $s_1, s_2 \in I_n$ and if we suppose that $\hat{\ell}$ is chosen in I_n such that Equation (4.25) holds, then the energy identity in Equation (2.4) becomes

$$\begin{aligned}
s_2 - s_1 &= \int_{s_1}^{s_2} \mathcal{R}(\hat{z}'(r)) + \|\hat{z}'(r)\| \operatorname{dist}(-D_z\hat{\mathcal{I}}(r,\hat{z}(r)), \partial\mathcal{R}(0))\, dr \\
&= \mathcal{E}(\hat{z}(s_1)) - \mathcal{E}(\hat{z}(s_2)) + \int_{s_1}^{s_2} \langle \hat{\ell}(r), \hat{z}'(r)\rangle\, dr \qquad \text{(by (2.4))} \\
&= -\int_{s_1}^{s_2} \langle D_z\mathcal{E}(\hat{z}(t)), \hat{z}'(t)\rangle\, dt + \int_{s_1}^{s_2} \langle \hat{\ell}(r), \hat{z}'(r)\rangle\, dr \\
&= -\int_{s_1}^{s_2} \langle D_z\hat{\mathcal{I}}(r,\hat{z}(r)), \hat{z}'(r)\rangle\, dr,
\end{aligned}$$

where we used the definition of $\hat{\mathcal{I}}$ in Equation (1.8). Since this equality must hold for every $s_1, s_2 \in I_n$, provided that $\hat{\ell}$ is such that the energy identity holds, we see that $\hat{\ell}$ satisfying the normalization condition (4.25) fulfills the energy identity, if and only if

$$\langle D_z\hat{\mathcal{I}}(s,\hat{z}(s)), \hat{z}'(s)\rangle + 1 = 0 \quad \text{f.a.a. } s \in \bigcup_{n=0}^{N} I_n. \tag{4.26}$$

To summarize, in order to ensure Equations (2.3) and (2.4) to hold in jumps, we must construct $\hat{\ell}$ such that Equations (4.25) and (4.26) are fulfilled. In view of $\hat{\mathcal{I}}(z) = \frac{1}{2}\langle Az, z\rangle + \mathcal{F}(z) - \langle \hat{\ell}, z\rangle$, Equation (4.26) leads to the natural choice

$$\hat{\ell}(s) := A\hat{z}(s) + D_z\mathcal{F}(\hat{z}(s)) + \frac{\hat{z}'(s)}{\|\hat{z}'(s)\|^2} \quad \forall\, s \in \bigcup_{n=0}^{N} \mathring{I}_n\,, \tag{4.27}$$

as Equation (4.26) and thus the energy identity are trivially fulfilled then. Note that $\hat{z}'(s)$ is constant and different from zero in $\bigcup_{n=0}^{N} \mathring{I}_n$ such that $\hat{\ell}(s)$ is well defined and due to the regularity of $\hat{z}$, an element of $W^{1,\infty}(I_n;\mathbb{R}^d)$ for all $n = 0, \dots, N$. Note, moreover, that $\mathring{I}_n = \emptyset$, if $\hat{z}'$ vanishes there. It remains to prove Equation (4.25) for $(\hat{t}, \hat{z})$ together with $\hat{\ell}$ as defined in Equation (4.27). For that purpose, let us investigate the dissipation functional in the jumps. As seen in Equation (4.16), in a jump point t_n, there holds $\mathcal{R}(z(t_n) - z(t_n-)) = s(t_n) - s(t_n-)$ and completely analogously, one shows $\mathcal{R}(z(t_n+) - z(t_n)) = s(t_n+) - s(t_n)$. Using the monotonicity of s along with the positive 1-homogeneity of $\mathcal{R}$ and the structure of $\hat{z}$ in the jump parts I_n^-, I_n^+, see Equation (4.13), this implies

$$\mathcal{R}(\hat{z}'(s)) = \mathcal{R}\left(\frac{z(t_n) - z(t_n-)}{s(t_n) - s(t_n-)}\right) = 1 \quad \forall\, s \in I_n^- \tag{4.28}$$

as well as

$$\mathcal{R}(\hat{z}'(s)) = \mathcal{R}\left(\frac{z(t_n+) - z(t_n)}{s(t_n+) - s(t_n)}\right) = 1 \quad \forall\, s \in I_n^+, \tag{4.29}$$

while, on I_0, we have by construction of $\hat{z}$ in Equation (4.13) that $\mathcal{R}(\hat{z}'(s)) = 1$ for all $s \in I_0$. Now let $s \in \bigcup_{n=0}^{N} \mathring{I}_n$ be arbitrary. Inserting Equation (1.8) in the left-hand side in Equation (4.25) and employing Lemma B.2 yield

$$\begin{aligned}
&\mathcal{R}(\hat{z}'(s)) + \|\hat{z}'(s)\| \operatorname{dist}(-D_z\hat{\mathcal{I}}(s,\hat{z}(s)), \partial\mathcal{R}(0)) \\
&= \mathcal{R}(\hat{z}'(s)) + \|\hat{z}'(s)\| \operatorname{dist}\left(\frac{\hat{z}'(s)}{\|\hat{z}'(s)\|^2}, \partial\mathcal{R}(0)\right) \\
&= \mathcal{R}(\hat{z}'(s)) + \sup_{v\in\mathbb{R}^d, \|v\| \le \|\hat{z}'(s)\|} \left(\left\langle \frac{\hat{z}'(s)}{\|\hat{z}'(s)\|^2}, v\right\rangle - \mathcal{R}(v)\right) \\
&= \mathcal{R}(\hat{z}'(s)) + \sup_{0<\alpha<1} \alpha \sup_{v\in\mathbb{R}^d, \|v\| = \|\hat{z}'(s)\|} \left(\left\langle \frac{\hat{z}'(s)}{\|\hat{z}'(s)\|^2}, v\right\rangle - \mathcal{R}(v)\right)
\end{aligned}$$

$$= \mathcal{R}(\hat{z}'(s)) + \sup_{0<\alpha<1} \alpha \sup_{v\in\mathbb{R}^d, \|v\|=\|\hat{z}'(s)\|} \left(\left\langle \frac{\hat{z}'(s)}{\|\hat{z}'(s)\|^2}, v \right\rangle - \mathcal{R}(\hat{z}'(s)) \right)$$

$$= \left\langle \frac{\hat{z}'(s)}{\|\hat{z}'(s)\|^2}, \hat{z}'(s) \right\rangle = 1,$$

where we exploited Equation (4.6), that is, the symmetry of $\mathcal{R}$ by assumption, the 1-homogeneity of $\mathcal{R}$, and Equations (4.28) and (4.29). Thus we have verified Equation (4.25), which, together with Equation (4.26), which holds by construction of $\hat{\ell}$ according to Equation (4.27), implies Equations (2.2) and (2.4) in the jumps, as explained above. Therefore, the energy identity (2.4) holds for all $s_1, s_2 \in I_n$, $n = 0, \ldots, N$, and all $s_1, s_2 \in G_m$, $m = 1, \ldots, M$, and consequently, the continuity of $\hat{z}$ implies that it holds for all $s_1, s_2 \in [0, S]$.

4. *Regularity of* $\hat{\ell}$

To summarize, we have seen that with $\hat{\ell}$ defined by

$$\hat{\ell}(s) = \begin{cases} \ell(\hat{t}(s)), & s \in [0,S] \setminus \bigcup_{n=0}^{N} I_n, \\ A\hat{z}(s) + D_z\mathcal{F}(\hat{z}(s)) + \frac{\hat{z}'(s)}{\|\hat{z}'(s)\|^2}, & \bigcup_{n=0}^{N} \mathring{I}_n, \end{cases}, \tag{4.30}$$

all conditions of a relaxed solution from Definition 3.1 are fulfilled. It remains to verify the required smoothness of $\hat{\ell}$. Due to $\ell \in BV([0,T];\mathbb{R}^d)$ and the monotonicity of $\hat{t}$, the composition $\ell \circ \hat{t}$ is a function in $BV([0,S];\mathbb{R}^d)$. Moreover, since $\hat{z}'$ is constant and nonzero in $\mathring{I}_n$, $A\hat{z}(s) + D_z\mathcal{F}(\hat{z}(s)) + \frac{\hat{z}'(s)}{\|\hat{z}'(s)\|^2}$ inherits the regularity of $\hat{z}$ and is, therefore, continuous. Thus, as a piecewise composition of finitely many functions of bounded variation, $\hat{\ell}$ is a function of bounded variation, too, that is, $\hat{\ell} \in BV([0,S];\mathbb{R}^d)$ as required. □

Remark 4.4. The symmetry of $\mathcal{R}$ is necessary to fulfill the normalization condition (2.3) with $\hat{\ell}$ as chosen in Equation (4.27). This can be seen as follows. Let $\mathcal{R} : \mathbb{R}^2 \to \mathbb{R}$ be defined by $\mathcal{R}(z) = \frac{1}{2}|z_1| + |z_2|$ and suppose that there is a jump interval I with constant $\hat{z}'$ given by $\hat{z}'(s) = \left(1, \frac{1}{2}\right)^\top$ for all $s \in I$. Then we have $\mathcal{R}(\hat{z}'(s)) = 1$ and $\frac{\hat{z}'(s)}{\|\hat{z}'(s)\|^2} = \left(\frac{4}{5}, \frac{2}{5}\right)^\top$ for all $s \in I$. For the distance to $\partial\mathcal{R}(0)$, we thus obtain in light of Equation (4.27) that

$$\operatorname{dist}(-D_z\hat{\mathcal{I}}(s,\hat{z}(s)), \partial\mathcal{R}(0)) = \operatorname{dist}\left(\frac{\hat{z}(s)}{\|\hat{z}'(s)\|^2}, \left[-\frac{1}{2}, \frac{1}{2}\right] \times [-1,1] \right) = \frac{3}{10} \quad \forall s \in I.$$

Due to $\hat{t}'(s) = 0$ for all $s \in I$, we end up with

$$\hat{t}'(s) + \mathcal{R}(\hat{z}'(s)) + \|\hat{z}'(s)\| \operatorname{dist}(-D_z\hat{\mathcal{I}}(s,\hat{z}(s)), \partial\mathcal{R}(0)) = 1 + \sqrt{\frac{5}{4}}\frac{3}{10} > 1 \quad \forall s \in I$$

such that Equation (2.3) is indeed violated.

Remark 4.5. The symmetry of $\mathcal{R}$ together with Equations (1.10) and (1.11) automatically implies that $\mathcal{R}$ is a scaled version of the norm, that is, there exists an $\alpha \geq 0$ such that $\mathcal{R}(\cdot) = \alpha\|\cdot\|$. This can be seen as follows. Let $e \in \mathbb{R}^d$ be a unit vector and define $\alpha := \mathcal{R}(e)$. Then, for $y \neq 0$, the symmetry condition in Equation (4.6) implies $\mathcal{R}\left(\frac{y}{\|y\|}\right) = \mathcal{R}(e) = \alpha$ such that $\mathcal{R}(y) = \alpha\|y\|$ for all $y \in \mathbb{R}^d$ due to the positive homogeneity of $\mathcal{R}$.

Remark 4.6. The reason why we assume in Theorem 4.3 that z has only finitely many jumps is to guarantee the required regularity of $\hat{\ell}$, that is, $\hat{\ell} \in BV([0,S];\mathbb{R}^d)$. Note that, according to Equation (4.30), $\hat{\ell}$ is continuous except for the points in the set $\bigcup_{n=0}^{N}\{s(t_n-), s(t_n), s(t_n+)\}$ and thus, if there are countably many of those points, the total variation of $\hat{\ell}$ may become infinite. At least in one dimension, that is, if $d = 1$, there is, however, no alternative to the choice of $\hat{\ell}$ in Equation (4.30), as it is easily seen that this construction of $\hat{\ell}$ in the jumps is the only way to guarantee Equation (4.26) and equivalently the energy identity, while $\hat{\ell}$ is determined by $\ell \circ \hat{t}$ outside the jumps. Therefore, in order to allow for countable many jumps of z without destroying the regularity of $\hat{\ell}$, we need to modify the construction of $\hat{z}$ in the jumps. Let us again consider the one-dimensional case. If one aims to avoid a jump of $\hat{\ell}$ at $s = s(t_n-)$, one has to choose $\hat{z}$ in the jump from $s(t_n-)$ to $s(t_n)$ such that $\ell(\hat{t}(s)) = A\hat{z}(s+) + D_z\mathcal{F}(\hat{z}(s+)) + \frac{\hat{z}'(s+)}{|\hat{z}'(s+)|^2}$. Thanks to the continuity of $\hat{z}$, this can be reformulated as $D_z\mathcal{I}(\hat{t}(s), \hat{z}(s)) + \frac{\hat{z}'(s+)}{|\hat{z}'(s+)|^2} = 0$, which results in the following condition for $\hat{z}$ at the beginning of the jump

$$\hat{z}'(s+) = \frac{-D_z\mathcal{I}(\hat{t}(s), \hat{z}(s))}{|D_z\mathcal{I}(\hat{t}(s), \hat{z}(s))|^2}, \tag{4.31}$$

provided that $D_z\mathcal{I}(\hat{t}(s), \hat{z}(s)) \neq 0$. The condition in Equation (4.31) will in general not be fulfilled by the linear interpolant from Equation (4.13), but, of course, one can find other Lipschitz continuous functions satisfying Equation (4.31) as well as $\hat{z}(s(t_n-)) = z(t_n-)$ and $\hat{z}(s(t_n)) = z(t_n)$. However, in view of Equation (4.27) and the required boundedness of $\hat{\ell}$ as a function of bounded variation, $\hat{z}'$ must be bounded away from zero in $(s(t_n-), s(t_n))$. This is, however, not possible, if $\hat{z}'$ is smooth and $-D_z\mathcal{I}(\hat{t}(s), \hat{z}(s))$ and $z(\hat{t}(s)) - z(\hat{t}(s)-)$ have a different sign. If $\hat{z}'$ is not smooth, an additional discontinuity in $\hat{\ell}$ arises, compare Equation (4.27), contradicting again the required regularity of $\hat{\ell}$, if this happens more than finitely many times with a jump height that does not tend to zero sufficiently fast. All in all, we see that the construction of $\hat{\ell}$ in the presence of countably many jumps of z is very involved and seems hardly to be possible in general.

As indicated above in Remark 4.2, the solution concept in Definition 4.1 is not the only possible generalization of the notion of a local solution from Ref. [2, Def. 3.3.2] to the case with discontinuous loads. Another possibility to define local solutions for discontinuous loads originates from an equivalent reformulation of the energy inequality in terms of the potential energy $\mathcal{E}$ similar to the energy identity in Equation (2.4). This inequality reads

$$\mathcal{E}(z(t_2)) + \mathrm{Diss}_{\mathcal{R}}(z; [t_1, t_2]) \leq \mathcal{E}(z(t_1)) + \int_{t_1}^{t_2} \ell(r)\, dz(r),$$

and is in fact equivalent to the energy inequality in Equation (4.2) provided that $\ell \in H^1(0, T; \mathbb{R}^d)$, which follows from the formula of integration by parts for Riemann–Stieltjes integrals. If we generalize this inequality for discontinuous loads, that is, we replace the Riemann–Stieltjes integral on right-hand side by the Kurzweil integral and apply the formula for integration by parts for Kurzweil integrals from Ref. [4, Prop. 1.12], then we obtain

$$\begin{aligned} &\mathcal{I}(t_2, z(t_2)) + \mathrm{Diss}_{\mathcal{R}}(z; [t_1, t_2]) - \sum_{t \in [t_1, t_2]} \Delta(\ell, z, t) \\ &\quad \leq \mathcal{I}(t_1, z(t_1)) - \int_{t_1}^{t_2} z(r)\, d\ell(r) \quad \forall\, 0 \leq t_1 \leq t_2 \leq T, \end{aligned} \tag{4.32}$$

with the additional term $\sum_{t \in [t_1, t_2]} \Delta(\ell, z, t)$, which is defined by

$$\Delta(\ell, z, t) := \langle \ell(t) - \ell(t-), z(t) - z(t-) \rangle - \langle \ell(t+) - \ell(t), z(t+) - z(t) \rangle.$$

Note that this additional term vanishes, if ℓ is continuous. This leads us to the following alternative definition of local solutions for discontinuous loads:

Definition 4.7. Let $\ell \in BV([0, T]; \mathbb{R}^d)$ be given. We call $z \in BV([0, T]; \mathbb{R}^d)$ *(alternative) local solution* of Equation (1.1) associated with ℓ, if fulfills Equations (4.1) and (4.32).

Note that this effect does not arise in case of the parametrized solution concepts from Definitions 2.1 and 3.1, respectively. Here, due to the continuity of $\hat{z}$, the energy inequality (2.4) can be written equivalently as

$$\begin{aligned} &\hat{\mathcal{I}}(\hat{z}(s_2), s_2) + \int_{s_1}^{s_2} \mathcal{R}(\hat{z}'(r)) + \|\hat{z}'(r)\|\, \mathrm{dist}(-D_z\hat{\mathcal{I}}(r, \hat{z}(r)), \partial\mathcal{R}(0))dr \\ &\qquad = \hat{\mathcal{I}}(\hat{z}(s_1), s_1) - \int_{s_1}^{s_2} \hat{z}(r)\, d\hat{\ell}(r) \quad \text{for all } 0 \leq s_1 \leq s_2 \leq S \end{aligned}$$

with a Riemann–Stieltjes integral on the right-hand side and no additional term involving the jumps of $\hat{\ell}$ pops up.

A close inspection of the proof of Theorem 4.3, however, shows that its statement also holds for local solutions in the sense of Definition 4.7. To see this, observe that the energy equality is established by a distinction of cases in the proof of Theorem 4.3. First, in part 2 of the proof, an arbitrary interval $[s_1, s_2] \subset G_m$ is considered. In $[\hat{t}(s_2), \hat{t}(s_2)]$, however, z is continuous such that the additional term involving $\Delta(\ell, z, t)$ vanishes and the energy inequality in Equation (4.32) coincides with the one in Equation (4.2) such that the arguments from part 2 are unaffected and apply to the alternative definition of a local solution, too. Furthermore, in part 3 of the proof, where the jumps are considered, $\hat{z}$ as well as $\hat{\ell}$ are explicitly constructed such that the energy identity is fulfilled and the energy inequality from Equations (4.2) and (4.32), respectively, is not used for this at all. Thus, the arguments from part 3 also apply for the alternative definition of a local solution and we arrive at the following

Corollary 4.8. *Suppose that $\ell \in BV([0,T];\mathbb{R}^d)$ is given and that $z \in BV([0,T];\mathbb{R}^d)$ is a local solution in the sense of Definition 4.7 that only admits a finite number of jumps. Assume, moreover, that the dissipation functional fulfills the symmetry condition from Equation (4.6). Then there exists a parametrization and $\hat{\ell} \in BV([0,S];\mathbb{R}^d)$ such that $(S, \hat{t}, \hat{z}, \hat{\ell})$ is a relaxed solution which fulfills $z \in \mathfrak{P}(\hat{t}, \hat{z})$.*

4.2 | A second counterexample

In order to further investigate the relation between the relaxed solution concept and the notion of local solutions, let us return to the counterexample from Section 2.1 and slightly modify it. While energy and dissipation are left unchanged, see Equation (2.6), we now consider

$$\ell_n(t) = \begin{cases} 0, & t \in [0,1], \\ \frac{n}{2}t - \frac{n}{2}, & t \in \left(1, 1+\frac{1}{n}\right), \\ 0, & t \in \left[1+\frac{1}{n}, 2\right] \end{cases} \tag{4.33}$$

as the sequence of loads. This sequence converges pointwise everywhere to $\ell \equiv 0$. Moreover, it converges weakly* in $\mathrm{BV}(0,T)$ to zero, too, but of course not in the intermediate sense, since $|D\ell_n|(0,2) = 1 \neq 0 = |D\ell|(0,2)$ for all $n \in \mathbb{N}$. The initial state is again set to $z_0 = 0$. By direct calculations, one verifies that a relaxed solution associated with ℓ_n is again given by Equations (2.8) and (2.9) along with $S_n = 5/2$ and

$$\hat{\ell}_n(s) = \ell_n(\hat{t}_n(s)) = \begin{cases} 0, & s \in [0,1], \\ \frac{\frac{n}{2}}{1+\frac{n}{2}}s - \frac{\frac{n}{2}}{1+\frac{n}{2}}, & s \in \left(1, \frac{3}{2}+\frac{1}{n}\right), \\ 0, & s \in \left[\frac{3}{2}+\frac{1}{n}, \frac{5}{2}\right] \end{cases} \tag{4.34}$$

instead of Equation (2.10). Note that this is not only a relaxed solution, but also a normalized $\mathfrak{p}$-parametrized BV solution, as no jump occurs, that is, $\hat{t}(s) > 0$ everywhere, and both concepts coincide in this case.

Again the limits of $\hat{z}$ and $\hat{t}$ (w.r.t. pointwise convergence as well as weak* convergence in $W^{1,\infty}(0,S)$) are given by the functions in Equation (2.11), but $\hat{\ell}_n$ now converges to

$$\hat{\ell}(s) = \begin{cases} 0, & s \in [0,1], \\ s-1, & s \in \left(1, \frac{3}{2}\right), \\ 0, & s \in \left[\frac{3}{2}, \frac{5}{2}\right]. \end{cases} \tag{4.35}$$

We observe that $\hat{\ell}$ is the pointwise limit and the limit w.r.t. intermediate convergence. Again, the limit is no normalized, $\mathfrak{p}$-parametrized BV solution associated with ℓ, since $\hat{\ell}$ from Equation (4.35) is not compatible with $\ell \equiv 0$ in the viscous jump $\left(1, \frac{3}{2}\right)$ according to the condition in Equation (2.5). However, as all other requirements in Definition 2.1 are satisfied, that is, Equations (2.1)–(2.4), and, since $\hat{\ell} = \ell \circ \hat{t}$ in $M = \left(0, \frac{5}{2}\right) \setminus \left(1, \frac{3}{2}\right)$, the limit is indeed a relaxed solution in accordance with our findings in Theorem 3.4. Note in this context that $\hat{\ell}$ and $\hat{z}$ satisfy Equation (4.27) in the viscous jump $\left(1, \frac{3}{2}\right)$.

Let us now consider the example in physical time. Here we observe that $z_n \in W^{1,\infty}(0,2)$, defined by

$$z_n(t) = \begin{cases} 0, & t \in [0,1], \\ \frac{n}{2}t - \frac{n}{2}, & t \in \left(1, 1+\frac{1}{n}\right),, \\ \frac{1}{2}, & t \in \left[1+\frac{1}{n}, 2\right], \end{cases} \tag{4.36}$$

satisfies Equation (1.1) a.e. in (0,2), where energy and dissipation are given by Equation (2.6) and ℓ_n is as defined in Equation (4.33). Thus, z_n is even a *differentiable solution* and consequently a local solution, too. Due to the uniform convexity of the energy in Equation (2.6), the differential solution is unique (if it exists). Therefore, the only differential solution associated with the limit $\ell \equiv 0$ is $z \equiv 0$. As the initial state $z_0 = 0$ is locally stable, this is the only physically meaningful solution in the absence of external loading. But the sequence in Equation (4.36) converges pointwise to

$$\tilde{z}(t) = \begin{cases} 0, & t \in [0,1] \\ \frac{1}{2} & t \in (1,2]. \end{cases} \tag{4.37}$$

This function, however, is not even a local solution, since the energy inequality (4.2) is not fulfilled. To see this, observe that, for every $t_2 > 1$, we obtain

$$\frac{1}{8} = \mathcal{I}(t_2, \tilde{z}(t_2)) + \mathrm{Diss}_{\mathcal{R}}(\tilde{z}; [0, t_2]) > \mathcal{I}(0, \tilde{z}(0)) - \int_0^{t_2} \tilde{z}(r)\,\mathrm{d}\ell(r) = 0. \tag{4.38}$$

This effect is closely related to the lack of continuity of the Kurzweil integral w.r.t. weak-$*$ and intermediate convergence in $\mathrm{BV}(0,T) \times \mathrm{BV}(0,T)$. Note that, although z_n converges to $\tilde{z}$ in the intermediate topology and $\ell_n \rightharpoonup^* \ell = 0$, there holds

$$\begin{aligned} -\mathcal{I}(2, z_n(2)) - \mathrm{Diss}_{\mathcal{R}}(z_n; [0,2]) + \mathcal{I}(0, z_n(0)) &= \int_0^2 z_n(t)\mathrm{d}\ell_n(t) \\ &= -\frac{1}{8} \neq 0 = \int_0^2 \tilde{z}(t)\mathrm{d}\ell(t) \end{aligned}$$

for all $n \in \mathbb{N}$.

A graphic interpretation of this observation is as follows: Due to the external force ℓ_n, the state z_n is raised to a higher energy level. As n is increased, this shifting is accelerated such that, in the limit, the state remains on the higher energy level although the load vanishes. As a consequence, this increase of energy is not compensated by ℓ and the energy inequality is violated. This example shows that not only the notion of normalized $\mathfrak{p}$-parametrized BV solutions, but even the concept of local solutions is not stable w.r.t. a sequence of loads converging weakly* in $\mathrm{BV}(0,T)$.

Moreover, it can be shown that the differential solution from Equation (4.36) is also a *global energetic solution*. For the precise definition of this solution concept see Ref. [2, Definition 2.1.2]. Since every global energetic solution is automatically a local solution, the above passage to the limit shows that the concept of global energetic solutions is not stable w.r.t. weak* convergence of the loads, too.

Let us finally investigate if there is another local solution that coincides with the limit $(\hat{z}, \hat{t})$ in the sense that $\bar{z} \in \mathfrak{P}(\hat{t}, \hat{z})$, compare Equation (4.5). For this purpose, we transform the limit $\hat{z}$ back into physical time, which yields

$$\bar{z}(t) \in \hat{z}(\hat{t}^{-1}(t)) = \begin{cases} 0, & t \in [0,1), \\ \left[0, \frac{1}{2}\right], & t = 1, \\ \frac{1}{2}, & t \in (1,2], \end{cases}$$

where $\hat{t}^{-1}$ is the set-valued inverse of $\hat{t}$. Thus, for every $\bar{z} \in \mathfrak{P}(\hat{t}, \hat{z})$, the inequality in Equation (4.38) holds true, which shows that no element of $\mathfrak{P}(\hat{t}, \hat{z})$ is a local solution. Even if we consider a modified load corresponding to the transformation of $\hat{\ell}$ from Equation (4.35) into physical time, that is,

$$\bar{\ell}(t) \in \hat{\ell}(\hat{t}^{-1}(t)) = \begin{cases} 0, & t \neq 1. \\ \left[0, \frac{1}{2}\right], & t = 1, \end{cases}$$

the inequality in Equation (4.38) will still be valid, since the Kurzweil integral on the right side of Equation (4.38) is always zero, no matter how one chooses $\bar{\ell}$ in $t = 1$, see Ref. [19, Corollary 2.14]. This shows that, in general, one cannot interpret a relaxed solution $(S, \hat{t}, \hat{z}, \hat{\ell})$ as a local solution.

The above example illustrates the following: the price one has to pay for a solution concept that is stable w.r.t. weak* convergence of the loads is that "solutions" are allowed that are not meaningful from a physical perspective. As already mentioned, the only reasonable solution of Equation (1.1) without external loads (i.e., $\ell \equiv 0$) and with a uniformly convex energy and a locally stable initial state z_0 is $z \equiv z_0$. The relaxed solution, however, jumps at $t = 1$ from $z_0 = 0$ to $1/2$ without any impact of an external load, which makes of course no sense at all.

4.3 | A third counterexample

The next example shows that the two concepts of local solutions are even not stable with respect to intermediate convergence in BV. This concerns both definitions of local solutions in Definitions 4.1 and 4.7, respectively, regardless whether the energy inequality is given as in Equation (4.2) or in Equation (4.32).

The example from Section 2.1 also serves as counterexample for the lack of stability for local solutions in the sense of Definition 4.7. For convenience, let us recall the setting of Section 2.1, where energy, dissipation, initial state, and end time were given by

$$\mathcal{R}(z) = |z|, \quad \mathcal{E}(z) = \frac{1}{2}z^2 - z, \quad \mathcal{I}(t, z) = \mathcal{E}(z) - \ell(t)z, \quad z_0 = 0, \quad T = 2,$$

while the sequence of loads was

$$\ell_n(t) = \begin{cases} 0, & t \in [0, 1] \\ \frac{n}{2}t - \frac{n}{2}, & t \in \left(1, 1 + \frac{1}{n}\right) \\ \frac{1}{2}, & t \in \left[1 + \frac{1}{n}, 2\right]. \end{cases} \tag{4.39}$$

One easily verifies that a differential and thus local solution is given by

$$z_n(t) = \begin{cases} 0, & t \in [0, 1] \\ \frac{n}{2}t - \frac{n}{2}, & t \in \left(1, 1 + \frac{1}{n}\right) \\ \frac{1}{2}, & t \in \left[1 + \frac{1}{n}, 2\right]. \end{cases} \tag{4.40}$$

With respect to intermediate convergence ℓ_n and z_n converge to

$$z(t) = \begin{cases} 0, & t \in [0, 1] \\ \frac{1}{2} & t \in (1, 2], \end{cases} \qquad \ell(t) = \begin{cases} 0, & t \in [0, 1] \\ \frac{1}{2} & t \in (1, 2]. \end{cases} \tag{4.41}$$

However, for any $0 \leq t_1 < 1 < t_2 \leq 2$, the energy inequality (4.32) is not fulfilled, since

$$\mathcal{I}(t_2, z(t_2)) + \mathrm{Diss}_{\mathcal{R}}(z; [t_1, t_2]) - \sum_{t \in [t_1, t_2]} \Delta(\ell, z, t) = \frac{1}{8}$$
$$> 0 = \mathcal{I}(t_1, z(t_1)) - \int_{t_1}^{t_2} z(r)\, d\ell(r).$$

Note that we again observe the same lack of continuity of the Kurzweil integral as in the example before, as $\int_0^2 z_n d\ell_n = -\frac{1}{8} \neq 0 = \int_0^2 z d\ell$ for all $n \in \mathbb{N}$.

To construct a counterexample for the notion of solutions according to Definition 4.1, we need to slightly modify the setting. While energy and dissipation remain unchanged, the sequence of loads is now set to

$$\ell_n(t) = \begin{cases} 0, & t \in [0, 1 - \frac{1}{n}] \\ \frac{n}{2} t - \frac{n}{2} + \frac{1}{2}, & t \in \left(1 - \frac{1}{n}, 1\right) \\ \frac{1}{2}, & t \in [1, 2]. \end{cases}$$

The associated differential (and thus also local) solutions read

$$z_n(t) = \begin{cases} 0, & t \in [0, 1 - \frac{1}{n}] \\ \frac{n}{2} t - \frac{n}{2} + \frac{1}{2}, & t \in \left(1 - \frac{1}{n}, 1\right) \\ \frac{1}{2}, & t \in [1, 2]. \end{cases}$$

As limits with respect to intermediate convergence, we obtain the right continuous functions

$$z(t) = \begin{cases} 0, & t \in [0, 1) \\ \frac{1}{2} & t \in [1, 2], \end{cases} \qquad \ell(t) = \begin{cases} 0, & t \in [0, 1) \\ \frac{1}{2} & t \in [1, 2]. \end{cases}$$

In this case, the energy inequality (4.2) is not valid for any $0 \leq t_1 < 1 < t_2 \leq 2$, since

$$\mathcal{I}(t_2, z(t_2)) + \mathrm{Diss}_{\mathcal{R}}(z; [t_1, t_2]) = -\frac{1}{8} > -\frac{1}{4} = \mathcal{I}(t_1, z(t_1)) - \int_{t_1}^{t_2} z(r)\, d\ell(r).$$

As before, one again verifies that the Kurzweil integrals invovling the loads do not converge, as $\int_0^2 z_n d\ell_n = \frac{1}{8} \neq \frac{1}{4} = \int_0^2 z d\ell$ for all $n \in \mathbb{N}$.

The above examples show that local solutions are indeed even not stable w.r.t. intermediate convergence. As in the previous example in Section 4.2, this implies that global energetic solution are not stable w.r.t. intermediate convergence, too.

In contrast to this, the assumptions of Theorem 3.4 are fulfilled in this example such that the limit of a sequence of relaxed solutions will again be a relaxed solution according to that theorem. However, the relaxed solution concept is not the only one that behaves stable in this example. In Ref. [4, Chapter 2], the authors derive another definition of energetic solutions for discontinuous loads in the case of left continuous solutions and loads. There, the energy inequality contains an additional dissipative term respecting the jumps of z. In our setting, the energy inequality according to this notion of solution reads

$$\mathcal{I}(t_2, z(t_2)) + \mathrm{Diss}_{\mathcal{R}}(z; [t_1, t_2]) + \sum_{\tau \in [0, T]} \Delta(z(\tau+), z(\tau))$$
$$\leq \mathcal{I}(t_1, z(t_1)) - \int_{t_1}^{t_2} z(r)\, d\ell(r) \quad \forall\, 0 \leq t_1 \leq t_2 \leq T \tag{4.42}$$

with $\Delta(\xi, \eta) := \langle \mathcal{E}'(\xi), \xi - \eta \rangle - \mathcal{E}(\xi) + \mathcal{E}(\eta)$ for $\xi, \eta \in \mathbb{R}^n$, where $\mathcal{E}$ is the potential energy from Equation (1.7). Due to the consideration of only left continuous functions, the term $\Delta(z(\tau), z(\tau-))$ is omitted. A straight-forward computation shows that the above counterexample does not work with this energy inequality. While for sequence of loads and differential solutions from Equations (4.39) and (4.40), the left- and right-hand side in Equation (4.42) are both equal to $-\frac{1}{8}$ for all $n \in \mathbb{N}$, the limit in Equation (4.41) satisfies $\Delta(z(1+), z(1)) = \frac{1}{8}$ such that both sides of Equation (4.42) equal zero in the limit. Therefore, the energy inequality in Equation (4.42) is always fulfilled with equality, even in the limit w.r.t. intermediate convergence.

It remains an open question whether this solution concept with the modified energy inequality in Equation (4.42) is always stable w.r.t. intermediate and/or weak-* convergence of loads in BV, which is subject to future research. However, it is to be noted that the dissipation in Equation (4.42) is no longer rate independent, since the jump dissipation $\Delta(z(\tau+), z(\tau))$ is not one-homogeneous. Thus, it seems that our relaxed solution concept is the only rate-independent solution concept which is stable w.r.t. weak-* convergence.

5 | CONCLUSION

The three examples from Sections 2.1, 4.2, and 4.3 illustrate that established solution concepts are not stable w.r.t. loads converging in $BV([0,T];\mathbb{R}^d)$ in the sense that the limit of an associated sequence of solutions to the rate-independent system need not be a solution any more. To be more precise, the example in Section 2.1 shows that normalized $\mathfrak{p}$-parametrized BV solutions are not stable for loads converging in the intermediate sense. Moreover, the example in Section 4.3 shows the same for local (and thus also global energetic) solutions.

Nonetheless, as the analysis in Section 3.1 shows, it is possible to design a relaxed solution concept that is stable w.r.t. weak* convergence of the loads. However, the considerations in Section 4 indicate that such a concept is even weaker than the notion of local solutions and thus not really meaningful in practice. Especially, the example in Section 4.2 shows that completely unphysical limits may arise, if the loads just converge weakly* in $BV([0,T];\mathbb{R}^d)$, even in case of a uniformly convex energy. This already demonstrates that a stable solution concept must necessarily allow for physically unreasonable solutions.

On the other hand, stability w.r.t. weak* convergence of the loads is an essential property of a solution concept, as one can often hardly expect more than this type of convergence. This issue concerns the existence of optimal solutions as discussed at the end of Section 3.1, but will very likely also arise when a given load in $BV([0,T];\mathbb{R}^d)$ is discretized in time. All in all, our results indicate that it might make no sense to consider discontinuous loads when dealing with rate-independent systems due to a lack of stability of physically meaningful solution concepts that take the rate independency of the system into account.

ACKNOWLEDGMENTS

This research was supported by the German Research Foundation (DFG) under Grant Number ME 3281/9-1 within the priority program Non-smooth and Complementarity-based Distributed Parameter Systems: Simulation and Hierarchical Optimization (SPP 1962).

Open access funding enabled and organized by Projekt DEAL.

ORCID

Christian Meyer https://orcid.org/0000-0003-3340-2316

REFERENCES

[1] Thomas, S.: Optimal control of a rate-independent system constrained to parameterized balanced viscosity solutions. PhD thesis, Universität Kassel (2022)
[2] Mielke, A., Roubíček, T.: Rate-Independent Systems: Theory and Application. Springer, New York (2015)
[3] Knees, D., Zanini, C.: Existence of parameterized BV-solutions for rate-independent systems with discontinuous loads. Discret. Contin. Dyn. Syst.–Ser. S 14(1), 121–149 (2021)
[4] Krejčí, P., Liero, M.: Rate independent Kurzweil processes. Appl. Math. 54(2), 117–145 (2009)
[5] Recupero, V.: BV solutions of rate independent variational inequalities. Ann. Sc. Norm. Super. Pisa Cl. Sci. 10(2), 269–315 (2011)
[6] Recupero, V.: Sweeping processes and rate independence. J. Convex Anal. 23(3), 921–946 (2016)
[7] Recupero, V., Santambrogio, F.: Sweeping processes with prescribed behavior on jumps. Ann. Mat. Pura Appl. 197(4), 1311–1332 (2018)

[8] Krejčí, P., Recupero, V.: Comparing BV solutions of rate independent processes. J. Convex Anal. 21(1), 121–146 (2014)
[9] Knees, D., Meyer, C., Sievers, M.: Optimal control of non-convex rate-independent systems via vanishing viscosity – the finite dimensional case. Pure Appl. Funct. Anal. 7(5), 1733–1766 (2022)
[10] Brokate, M., Christof, C.: Strong stationarity conditions for optimal control problems governed by a rate-independent evolution variational inequality. SIAM J. Control Optim. 61(4), 2222–2250 (2023)
[11] Attouch, H.: Variational Convergence for Functions and Operators. Pitman, Boston (1984)
[12] Natanson, I.: Theorie der Funktionen einer reellen Veränderlichen. Verlag Harri Deutsch, Zürich, Frankfurt/Main, Thun (1975)
[13] Mielke, A., Theil, F.: A mathematical model for rate-independent phase transformations with hysteresis. In: Proceedings of the Workshop "Models of Continuum Mechanics in Analysis and Engineering," pp. 117–129. TU Darmstadt, Shaker, 1999
[14] Mielke, A., Theil, F., Levitas, V.I.: A variational formulation of rate-independent phase transformations using an extremum principle. Arch. Rational Mech. Anal. 162(2), 137–177 (2002)
[15] Mielke, A., Theil, F.: On rate-independent hysteresis models. NoDEA: Nonlinear Differ. Equ. Appl. 11(2), 151–189 (2004)
[16] Efendiev, M.A., Mielke, A.: On the rate-independent limit of systems with dry friction and small viscosity. J. Convex Anal. 13(1), 151–167 (2006)
[17] Mielke, A., Rossi, R., Savaré, G.: BV solutions and viscosity approximations of rate-independent systems. ESAIM: Control, Optim. Calc. Var. 18(1), 36–80 (2012)
[18] Mielke, A., Rossi, R., Savaré, G.: Modeling solutions with jumps for rate-independent systems on metric spaces. ESAIM: Control, Optim. Calc. Var. 25(2), 585–615 (2009)
[19] Tvrdy, M.: Regulated functions and the Perron-Stieltjes integral. Časopis Pro Pěstování Matematiky 114, 187–209 (1989)
[20] Bensimhoun, M.: Change of variable theorems for the KH integral. Real Anal. Exchange 35(1), 167–194 (2009)
[21] Sievers, M.: A numerical scheme for rate-independent systems. PhD thesis, TU Dortmund (2020)

How to cite this article: Andreia, M., Meyer, C.: On a lack of stability of parametrized BV solutions to rate-independent systems with nonconvex energies and discontinuous loads. Z Angew Math Mech. 104, e202300654 (2024). https://doi.org/10.1002/zamm.202300654

APPENDIX A: THE DISSIPATIVE DISTANCE IN THE SMOOTH CASE

Lemma A.1. *Let $\mathcal{R} : \mathbb{R}^d \to [0,\infty)$ comply with (1.10)–(1.12) and $z \in W^{1,1}(0,T;\mathbb{R}^d)$. Then*

$$\mathrm{Diss}_{\mathcal{R}}(z;[t_1,t_2]) = \int_{t_1}^{t_2} \mathcal{R}(z'(r))dr \tag{A.1}$$

for all $0 \le t_1 \le t_2 \le T$.

Proof. Let an arbitrary partition $t_1 = \xi_0 < \xi_1 < \cdots < \xi_{n-1} < \xi_n = t_2$ be given. Then applying Jensen's inequality gives

$$\mathcal{R}(z(\xi_i) - z(\xi_{i-1})) = \mathcal{R}\left(\int_{\xi_{i-1}}^{\xi_i} z'(r)dr\right) \le \int_{\xi_{i-1}}^{\xi_i} \mathcal{R}(z'(r))\,dr$$

for all $i = 1, \dots, n$. Summing up results in $\sum_{i=1}^n \mathcal{R}(z(\xi_i) - z(\xi_{i-1})) \le \int_{t_1}^{t_2} \mathcal{R}(z'(r))dr$ and, as the partition was arbitrary, this gives $\mathrm{Diss}_{\mathcal{R}}(z;[t_1,t_2]) \le \int_{t_1}^{t_2} \mathcal{R}(z'(r))dr$.

To show the reverse estimate, let a sequence of partitions $(\{\xi_i^h\}_{i=1}^{n_h})_h$ with $h := \max_{i=1,\dots,n_h} \xi_i^h - \xi_{i-1}^h \to 0$ be given. We define

$$\xi_h(t) := \frac{z(\xi_i^h) - z(\xi_{i-1}^h)}{\xi_i^h - \xi_{i-1}^h}, \quad t \in [\xi_{i-1}^h, \xi_i^h).$$

Then the 1-homogeneity of $\mathcal{R}$ yields

$$\begin{aligned}\mathrm{Diss}_{\mathcal{R}}(z;[t_1,t_2]) &\geq \sum_{i=1,\dots,n_h} \mathcal{R}\big(z(\xi_i^h)-z(\xi_{i-1}^h)\big)\\ &= \sum_{i=1,\dots,n_h} \int_{\xi_{i-1}^h}^{\xi_i^h} \mathcal{R}\left(\frac{z(\xi_i^h)-z(\xi_{i-1}^h)}{\xi_i^h-\xi_{i-1}^h}\right) dr = \int_{t_1}^{t_2} \mathcal{R}(\xi_h(r))\,dr.\end{aligned}$$

By Lebesgue's differentiation theorem, we obtain $z'(t)=\lim_{h\searrow 0}\xi_h(t)$ for almost all $t\in(0,T)$ such that the lower semicontinuity of $\mathcal{R}$ implies $\liminf_{h\searrow 0}\mathcal{R}(\xi_h(t))\geq \mathcal{R}(z'(t))$ a.e. in $(0,T)$. Thus Fatou's lemma yields

$$\begin{aligned}\mathrm{Diss}_{\mathcal{R}}(z;[t_1,t_2]) &\geq \liminf_{h\searrow 0}\int_{t_1}^{t_2}\mathcal{R}(\xi_h(r))\,dr\\ &\geq \int_{t_1}^{t_2}\liminf_{h\searrow 0}\mathcal{R}(\xi_h(r))\,dr \geq \int_{t_1}^{t_2}\mathcal{R}(z'(r))\,dr,\end{aligned}$$

which finally gives the claim. □

APPENDIX B: AUXILIARY RESULTS FROM CONVEX ANALYSIS

The following results are well known, we only refer to Ref. [21] and the references therein. For convenience of the reader, we present the proofs in detail.

Lemma B.1. *For every $(S,\hat{t},\hat{z},\hat{\ell})\in(0,\infty)\times W^{1,\infty}(0,S)\times W^{1,\infty}(0,S;\mathbb{R}^d)\times BV([0,S];\mathbb{R}^d)$ and all $0\leq s_1\leq s_2\leq S$ there holds*

$$\begin{aligned}\mathcal{E}(\hat{z}(s_2)) + \int_{s_1}^{s_2}\mathcal{R}(\hat{z}'(r)) + \|\hat{z}'(r)\|\,\mathrm{dist}(-D_z\hat{\mathcal{I}}(r,\hat{z}(r)),\partial\mathcal{R}(0))\,dr&\\ \geq \mathcal{E}(\hat{z}(s_1)) + \int_{s_1}^{s_2}\langle\hat{\ell}(r),\hat{z}'(r)\rangle\,dr.&\end{aligned}\tag{B.1}$$

Proof. The compactness of $\partial\mathcal{R}(0)$ implies that, for all $r\in(s_1,s_2)$, there exists $w_r\in\partial\mathcal{R}(0)$ such that $\mathrm{dist}(-D_z\hat{\mathcal{I}}(r,\hat{z}(r)),\partial\mathcal{R}(0)) = \|-D_z\hat{\mathcal{I}}(r,\hat{z}(r))-w_r\|$. From this, we conclude

$$\begin{aligned}\langle -D_z\hat{\mathcal{I}}(r,\hat{z}(r)),\hat{z}'(r)\rangle &= \langle -D_z\hat{\mathcal{I}}(r,\hat{z}(r))-w_r,\hat{z}'(r)\rangle + \langle w_r,\hat{z}'(r)\rangle\\ &\leq \|-D_z\hat{\mathcal{I}}(r,\hat{z}(r))-w_r\|\,\|\hat{z}'(r)\| + \langle w_r,\hat{z}'(r)\rangle\\ &\leq \|\hat{z}'(r)\|\,\mathrm{dist}(-D_z\hat{\mathcal{I}}(r,\hat{z}(r)),\partial\mathcal{R}(0)) + \mathcal{R}(\hat{z}'(r)),\end{aligned}$$

where we used $w_r\in\partial\mathcal{R}(0)$ for the last inequality. Now integrating over (s_1,s_2) yields

$$\begin{aligned}&\int_{s_1}^{s_2}\|\hat{z}'(r)\|\,\mathrm{dist}(-D_z\hat{\mathcal{I}}(r,\hat{z}(r)),\partial\mathcal{R}(0)) + \mathcal{R}(\hat{z}'(r))\,dr\\ &\quad\geq \int_{s_1}^{s_2}\langle -D_z\hat{\mathcal{I}}(r,\hat{z}(r)),\hat{z}'(r)\rangle\,dr\\ &\quad= \int_{s_1}^{s_2}\langle -D_z\mathcal{E}(\hat{z}(r)),\hat{z}'(r)\rangle + \langle\hat{\ell}(r),\hat{z}'(r)\rangle\,dr\\ &\quad= \mathcal{E}(\hat{z}(s_1)) - \mathcal{E}(\hat{z}(s_2)) + \int_0^s\langle\hat{\ell}(r),\hat{z}'(r)\rangle\,dr,\end{aligned}$$

which is Equation (B.1). □

Lemma B.2. *Let $\mathcal{R} : \mathbb{R}^d \to [0, \infty)$ comply with Equations (1.10)–(1.12) and $\tau > 0$ be given. Then there holds*

$$\tau \ \mathrm{dist}(\eta, \partial\mathcal{R}(0)) = \sup_{v \in \mathbb{R}^d,\, \|v\| \leq \tau} (\langle \eta, v \rangle - \mathcal{R}(v))$$

for all $\eta \in \mathbb{R}^d$.

Proof. Let us denote the indicator functional of a set $M \subset \mathbb{R}^d$ by I_M, that is,

$$I_M(z) := \begin{cases} 0, & \text{if } z \in M, \\ \infty, & \text{else.} \end{cases}$$

Then the inf-convolution formula from Ref. [11, Prop. 3.4] yields

$$\left(\mathcal{R} + I_{\overline{B_\tau(0)}}\right)^*(\eta) = \inf_{v \in \mathbb{R}^d} \left(\mathcal{R}^*(v) + I^*_{\overline{B_\tau(0)}}(\eta - v)\right), \tag{B.2}$$

where $B_\tau(0)$ denotes the ball of radius τ centered at zero. The conjugate of $I_{\overline{B_\tau(0)}}$ is given by $I^*_{\overline{B_\tau(0)}}(\cdot) = \tau\| \cdot \|$, while the positive homogeneity of $\mathcal{R}$ implies $\mathcal{R}^* = I_{\partial\mathcal{R}(0)}$. Inserted in Equation (B.2), we obtain

$$\left(\mathcal{R} + I_{\overline{B_\tau(0)}}\right)^* = \inf_{v \in \mathbb{R}^d} I_{\partial\mathcal{R}(0)}(v) + \tau\|\eta - v\| = \tau \inf_{v \in \partial\mathcal{R}(0)} \|\eta - v\| = \tau \,\mathrm{dist}(\eta, \partial\mathcal{R}(0)),$$

which is the claim. □