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On the Mutual Definability of Classes of Generalized Fuzzy Implications and of Classes of Generalized Negations and S-Norms

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Abstract

Given the real functions $\nu : \langle 0,1\rangle \to \langle 0,1\rangle$ and $\sigma, \pi : \langle 0,1\rangle^2 \to \langle 0,1\rangle$. First we define a functional operator *SIMP* where $SIMP(\sigma,\nu) : \langle 0,1\rangle^2 \to \langle 0,1\rangle$ and $SIMP(\sigma,\nu)$ is interpreted as the "S-implication" generated by ν and σ. Secondly, we define functional operators $NEG(\pi) : \langle 0,1\rangle \to \langle 0,1\rangle$ and $SNOR(\pi) : \langle 0,1\rangle^2 \to \langle 0,1\rangle$ where $NEG(\pi)$ is interpreted as the "negation" generated by π and $SNOR(\pi)$ is interpreted as the "S-norm" (T-conorm) generated by π. We investigate under which assumptions these operators are injective (bijective) and which properties of the "argument functions" are translated into the "value functions". Numerous well-known results on negations, S-norms, and implications can be derived within the framework of this general approach. Further results concern the mutual definability or R-implications and T-norms.

Keywords: S-implications, S-norms, negations, R-implications, T-norms, QL-implications.

1 Introduction

In literature one can find numerous papers concerning the generation of implications by negations and S-norms on the one hand and vice versa, i. e. of negations and S-norms by implications on the other hand. See [4–14, 17, 18, 20, 22–25], in particular [8], for instance.

The presented paper is to deepen these results, in particular, it is to show how by these generation procedures *separate* properties of negations and S-norms are translated into certain *separate* properties of implications and vice versa.

We start our investigations by recalling the fundamental definition of a negation and of an S-norm.

Assume that $\nu : \langle 0,1\rangle \to \langle 0,1\rangle$.

Definition 1.1

1. *ν is said to be a negation if and only if ν satisfies the following axioms:*

   NE1 $\forall r(r \in \langle 0,1\rangle \to \nu(\nu(r)) = r)$

   NE2 $\nu(0) = 1$

   NE3 $\nu(1) = 0$

   NE4 $\forall r \forall s(r, s \in \langle 0,1\rangle \wedge r \le s \to \nu(s) \le \nu(r))$

2. *The set of all negations is denoted by NEGATIONS.*

Now, we assume that $\sigma : \langle 0,1\rangle^2 \to \langle 0,1\rangle$.

Definition 1.2

1. *σ is said to be an S-norm if and only if σ satisfies the following axioms:*

   SN1 $\forall r(r \in \langle 0,1\rangle \to \sigma(r,0) = r)$

   SN2 $\forall r(r \in \langle 0,1\rangle \to \sigma(r,1) = 1)$

   SN3 $\forall r \forall s \forall t(r, s, t \in \langle 0,1\rangle \wedge r \le s \to \sigma(r,t) \le \sigma(s,t))$

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SN4 $\forall r \forall s \forall t (r, s, t \in \langle 0, 1\rangle \wedge s \le t \to \sigma(r, s) \le \sigma(r, t))$

SN5 $\forall r \forall s (r, s \in \langle 0, 1\rangle \to \sigma(r, s) = \sigma(s, r))$

SN6 $\forall r \forall s \forall t (r, s, t \in \langle 0, 1\rangle \to \sigma(r, \sigma(s, t)) = \sigma(\sigma(r, s), t))$

2. *The set of all S-norms is denoted by SNORMS.*

Finally, we assume that $\pi : \langle 0, 1\rangle^2 \to \langle 0, 1\rangle$.

Definition 1.3

1. *π is said to be an S-implication if and only if π satisfies the following axioms:*

   SIM1 $\forall r (r \in \langle 0, 1\rangle \to \pi(\pi(r, 0), 0) = r)$

   SIM2 $\forall s (s \in \langle 0, 1\rangle \to \pi(0, s) = 1)$

   SIM3 $\forall s (s \in \langle 0, 1\rangle \to \pi(1, s) = s)$

   SIM4 $\forall r (r \in \langle 0, 1\rangle \to \pi(r, 1) = 1)$

   SIM5 $\forall r \forall s \forall t (r, s, t \in \langle 0, 1\rangle \wedge r \le s \to \pi(s, t) \le \pi(r, t))$

   SIM6 $\forall r \forall s \forall t (r, s, t \in \langle 0, 1\rangle \wedge s \le t \to \pi(r, s) \le \pi(r, t))$

   SIM7 $\forall r \forall s (r, s \in \langle 0, 1\rangle \to \pi(\pi(r, 0), \pi(s, 0)) = \pi(s, r))$

   SIM8 $\forall r \forall s \forall t (r, s, t \in \langle 0, 1\rangle \to \pi(r, \pi(s, t)) = \pi(s, \pi(r, t)))$

2. *The set of all S-implications is denoted by SIMPLICATIONS.*

Remark As we are interested in the translation of *separate* properties into other *separate* properties of the functions considered, we do not care about the independence of the axiom systems formulated in the definitions above.

For solving the axiomatization problem we introduce the following functional operators *SIMP*, *NEG*, and *SNOR* where

$$SIMP : FUNCT(2) \times FUNCT(1) \to FUNCT(2)$$
$$NEG : FUNCT(2) \to FUNCT(1)$$
$$SNOR : FUNCT(2) \to FUNCT(2)$$

where $FUNCT(1) =_{def} \{\varphi | \varphi : \langle 0, 1\rangle \to \langle 0, 1\rangle\}$ and $FUNCT(2) =_{def} \left\{\psi \middle| \psi : \langle 0, 1\rangle^2 \to \langle 0, 1\rangle\right\}$.

Assume

$$\nu \in FUNCT(1)$$
$$\sigma, \pi \in FUNCT(2).$$

Then we define for every $r, s \in \langle 0, 1\rangle$

Definition 1.4

1. $SIMP(\sigma, \nu)(r, s) =_{def} \sigma(\nu(r), s)$
2. $NEG(\pi)(r) =_{def} \pi(r, 0)$
3. $SNOR(\pi)(r, s) =_{def} \pi(\pi(r, 0), s)$.

2 Some fundamental properties of the functional operators *SIMP*, *NEG*, and *SNOR*

The following theorems and corollaries express fundamental properties of the functional operators defined above.

Theorem 2.1
If 1. $\forall r(r \in \langle 0, 1\rangle \rightarrow \nu(\nu(r)) = r)$ *and*

2. $\forall r(r \in \langle 0, 1\rangle \rightarrow \sigma(r, 0) = r)$

then

1. $NEG(SIMP(\sigma, \nu)) = \nu$ *and*
2. $SNOR(SIMP(\sigma, \nu)) = \sigma$.

Proof

ad 1. We define

(1) $LS(r) =_{def} NEG(SIMP(\sigma, \nu))(r)$.

By definition of *NEG* we have to prove

(2) $LS(r) = SIMP(\sigma, \nu)(r, 0)$,

hence by definition of *SIMP* it is sufficient to prove

(3) $LS(r) = \sigma(\nu(r), 0)$.

By assumption 2, i. e. SN1, we have

(4) $\sigma(\nu(r), 0) = \nu(r)$,

hence

(5) $LS(r) = \nu(r)$.

ad 2. We define

(6) $LS'(r, s) =_{def} SNOR(SIMP(\sigma, \nu))(r, s)$.

By definition of *SNOR* we have to prove

(7) $LS'(r, s) = SIMP(\sigma, \nu)(SIMP(\sigma, \nu)(r, 0), s)$,

hence by definition of *SIMP* it is sufficient to show

(8) $LS'(r, s) = SIMP(\sigma, \nu)(\sigma(\nu(r), 0), s) = \sigma(\nu(\sigma(\nu(r), 0)), s)$.

By assumption 2, i. e. SN1, we get

(9) $\sigma(\nu(r), 0) = \nu(r)$,

hence we obtain

(10) $LS'(r, s) = \sigma(\nu(\nu(r)), s)$.

Because of assumption 1, i. e. NE1, we have

(11) $LS'(r, s) = \sigma(r, s)$,

i. e. assertion 2 holds.

■

We denote by

$FUNCT(1, \mathrm{NE1})$

the set of all functions $\varphi \in FUNCT(1)$ which fulfill the axiom NE1, furthermore by

$FUNCT(2, \mathrm{SN1})$ and $FUNCT(2, \mathrm{SIM1})$

the set of all functions $\varphi \in FUNCT(2)$ which fulfill the axioms SN1 and SIM1, respectively.

Corollary 2.2
$SIMP : FUNCT(2, SN1) \times FUNCT(1, NE1) \rightarrow FUNCT(2)$ is an injection.

By the following theorem we characterize the set of all images $SIMP(\sigma, \nu)$ for $\sigma \in FUNCT(2, \mathrm{SN1})$ and $\nu \in FUNCT(1, \mathrm{NE1})$.

Theorem 2.3
If $\forall r(r \in \langle 0, 1\rangle \rightarrow \pi(\pi(r, 0), 0) = r)$ then $SIMP(SNOR(\pi), NEG(\pi)) = \pi$.

Proof We define

(1) $LS(r, s) =_{def} SIMP(SNOR(\pi), NEG(\pi))(r, s)$.

By definition of *SIMP* we get

(2) $LS(r, s) = SNOR(\pi)(NEG(\pi)(r), s)$,

hence by definition of *NEG* we obtain

(3) $LS(r, s) = SNOR(\pi)(\pi(r, 0), s)$,

thus by definition of *SNOR* we obtain

(4) $LS(r, s) = \pi(\pi(\pi(r, 0), 0), s)$.

By assumption of theorem 2.3 we have

(5) $\pi(\pi(r, 0), 0) = r$,

hence

(6) $LS(r, s) = \pi(r, s)$,

i. e. theorem 2.3 holds. ∎

Corollary 2.4

1. *SIMP is a bijection from $FUNCT(2, SN1) \times FUNCT(1, NE1)$ onto $FUNCT(2, SIM1)$.*
2. *$[SNOR, NEG]$ is the inverse mapping of the bijection SIMP.*

3 On translating properties of functions by applying the functional operator *SIMP*

Now, we investigate which properties of the "argument functions" σ and ν are translated to certain properties of $SIMP(\sigma, \nu)$.

Theorem 3.1

1. *If ν fulfills NE1 and σ fulfills SN1, then $SIMP(\sigma, \nu)$ fulfills SIM1.*
2. *If ν fulfills NE2 and σ fulfills SN2 and SN5, then $SIMP(\sigma, \nu)$ fulfills SIM2.*
3. *If ν fulfills NE3 and σ fulfills SN2 and SN5, then $SIMP(\sigma, \nu)$ fulfills SIM3.*
4. *If σ fulfills SN2, then $SIMP(\sigma, \nu)$ fulfills SIM4.*
5. *If ν fulfills NE4 and σ fulfills SN3, then $SIMP(\sigma, \nu)$ fulfills SIM5.*
6. *If σ fulfills SN4, then $SIMP(\sigma, \nu)$ fulfills SIM6.*
7. *If ν fulfills NE4 and σ fulfills SN5, then $SIMP(\sigma, \nu)$ fulfills SIM7.*
8. *If ν fulfills NE1 and σ fulfills SN1 and SN6, then $SIMP(\sigma, \nu)$ fulfills SIM8.*
9. *If ν is continuous and σ is continuous, then $SIMP(\sigma, \nu)$ is continuous.*

Proof

ad 1 SIM1

We have to show

(1) $SIMP(\sigma, \nu)(SIMP(\sigma, \nu)(r, 0), 0) = r.$

By definition of *SIMP* it is sufficient to prove

(2) $\sigma[\nu(SIMP(\sigma, \nu)(r, 0)), 0] = \sigma[\nu(\sigma(\nu(r), 0)), 0] = r.$

By SN1 we get

(3) $\sigma(\nu(r), 0) = \nu(r),$

hence it is sufficient to show

(4) $\sigma(\nu(\nu(r)), 0) = r.$

But (4) holds because of NE1 and SN1.

ad 2 SIM2

We have to show

(5) $SIMP(\sigma, \nu)(0, s) = 1.$

By definition of *SIMP* it is sufficient to prove

(6) $\sigma(\nu(0), s) = 1.$

By NE2, SN2, and SN5 we have

(7) $\nu(0) = 1,$

(8) $\sigma(s, 1) = 1,$

and

(9) $\sigma(1, s) = \sigma(s, 1),$

respectively, hence (6) holds.

ad 3 SIM3

We have to show

(10) $SIMP(\sigma, \nu)(1, s) = s.$

By definition of *SIMP* it is sufficient to prove

(11) $\sigma(\nu(1), s) = s.$

By NE3, SN1, and SN5 we get

(12) $\nu(1) = 0,$

(13) $\sigma(s, 0) = s,$

and

(14) $\sigma(0, s) = \sigma(s, 0),$

respectively, hence (11) holds.

ad 4 SIM4

We have to prove

(15) $SIMP(\sigma, \nu)(r, 1) = 1.$

By definition of *SIMP* it is sufficient to show

(16) $\sigma(\nu(r), 1) = 1.$

But (16) holds because of SN2.

ad 5 SIM5

We assume

(17) $r \leq s$.

The we have to prove

(18) $SIMP(\sigma, \nu)(s, t) \leq SIMP(\sigma, \nu)(r, t)$.

By definition of *SIMP* it is sufficient to show

(19) $\sigma(\nu(s), t) \leq \sigma(\nu(r), t)$.

By NE4 we get

(20) $\nu(s) \leq \nu(r)$

hence because of SN3 (19) holds.

ad 6 SIM6

We assume

(21) $s \leq t$.

Then we have to prove

(22) $SIMP(\sigma, \nu)(r, s) \leq SIMP(\sigma, \nu)(r, t)$.

By definition of *SIMP* it is sufficient to show

(23) $\sigma(\nu(r), s) \leq \sigma(\nu(r), t)$.

But (23) holds because of SN4.

ad 7 SIM7

We have to prove

(24) $$SIMP(\sigma, \nu)(SIMP(\sigma, \nu)(r, 0), SIMP(\sigma, \nu)(s, 0)) = SIMP(\sigma, \nu)(s, r).$$

By definition of *SIMP* it is sufficient to show

(25) $\sigma[\nu(\sigma(\nu(r), 0)), \sigma(\nu(s), 0)] = \sigma(\nu(s), r)$.

Because of SN1 we get

(26) $\sigma(\nu(r), 0) = \nu(r)$

and

(27) $\sigma(\nu(s), 0) = \nu(s)$,

hence, in order to prove (25), it is sufficient to show

(28) $\sigma(\nu(\nu(r)), \nu(s)) = \sigma(\nu(s), r)$.

Because of assumption NE4 we have

(29) $\nu(\nu(r)) = r$,

hence (28) holds because of SN5.

ad 8 SIM8

We have to prove

(30) $$SIMP(\sigma, \nu)(r, SIMP(\sigma, \nu)(s, t)) = SIMP(\sigma, \nu)(s, SIMP(\sigma, \nu)(r, t)).$$

By definition of *SIMP* it is sufficient to prove

(31) $\sigma(\nu(r), \sigma(\nu(s), t)) = \sigma(\nu(s), \sigma(\nu(r), t))$.

But (31) holds because of SN5 and SN6.

ad 9 This assertion holds because of well-known properties of continuous functions.

■

Corollary 3.2

1. *If ν is a negation and σ is an S-norm, then SIMP(σ, ν) is an S-implication.*
2. *The mapping SIMP is an injection from the class SNORMS×NEGATIONS into the class SIMPLICATIONS.*

Remark Theorem 3.1 makes it possible to derive further "injection theorems".

4 On translating properties of functions by applying the functional operators *NEG* and *SNOR*

In this chapter we investigate which properties of the "argument function" π are translated into certain properties of *NEG*(π) and *SNOR*(π).

From these results we can conclude that *SIMP* is a surjection, i. e. a mapping onto *SIMPLICATIONS*.

Theorem 4.1

1.1. *If π fulfills SIM1, then NEG(π) fulfills NE1.*

1.2. *If π fulfills SIM2, then NEG(π) fulfills NE2.*

1.3. *If π fulfills SIM3, then NEG(π) fulfills NE3.*

1.4. *If π fulfills SIM5, then NEG(π) fulfills NE4.*

2.1. *If π fulfills SIM1, then SNOR(π) fulfills SN1.*

2.2. *If π fulfills SIM4, then SNOR(π) fulfills SN2.*

2.3. *If π fulfills SIM5, then SNOR(π) fulfills SN3.*

2.4. *If π fulfills SIM6, then SNOR(π) fulfills SN4.*

2.5. *If π fulfills SIM1 and SIM7, then SNOR(π) fulfills SN5.*

2.6. *If π fulfills SIM1, SIM7, and SIM8, then SNOR(π) fulfills SN6.*

2.7. *If π is continuous, then NEG(π) and SNOR(π) are continuous.*

Proof

ad 1.1 NE1

We have to prove

(1) $NEG(\pi)(NEG(\pi)(r)) = r$.

By definition of *NEG* it is sufficient to show

(2) $\pi(\pi(r, 0), 0) = r$.

But (2) holds because of SIM1.

ad 1.2 NE2

We have to prove

(3) $NEG(\pi)(0) = 1$.

By definition of *NEG* it is sufficient to show

(4) $\pi(0, 0) = 1$.

But (4) holds because of SIM2.

ad 1.3 NE3

We have to prove

(5) $NEG(\pi)(1) = 0.$

By definition of *NEG* it is sufficient to show

(6) $\pi(1, 0) = 0.$

But (6) holds because of SIM3.

ad 1.4 NE4

Assume

(7) $r \leq s.$

Then we have to prove

(8) $NEG(\pi)(s) \leq NEG(\pi)(r).$

By definition of *NEG* it is sufficient to show

(9) $\pi(s, 0) \leq \pi(r, 0).$

But (9) holds because of SIM5.

ad 2.1 SN1

We have to prove

(10) $SNOR(\pi)(r, 0) = r.$

By definition of *SNOR* it is sufficient to prove

(11) $\pi(\pi(r, 0), 0) = r.$

But (11) holds because of SIM1.

ad 2.2 SN2

We have to prove

(12) $SNOR(\pi)(r, 1) = 1.$

By definition of *SNOR* it is sufficient to show

(13) $\pi(\pi(r, 0), 1) = 1.$

But (13) holds because of SIM4.

ad 2.3 SN3

We assume

(14) $r \leq s.$

Then we have to prove

(15) $SNOR(\pi)(r, t) \leq SNOR(\pi)(s, t).$

By definition of *SNOR* it is sufficient to show

(16) $\pi(\pi(r, 0), t) \leq \pi(\pi(s, 0), t).$

From (14) by SIM5 we get

(17) $\pi(s, 0) \leq \pi(r, 0),$

hence (17) implies (16) because of SIM5.

ad 2.4 SN4

We assume

(18) $s \leq t.$

Then we have to prove

(19) $SNOR(\pi)(r,s) \leq SNOR(\pi)(r,t)$.

By definition of *SNOR* it is sufficient to show

(20) $\pi(\pi(r,0),s) \leq \pi(\pi(r,0),t)$.

But SIM6 implies (20).

ad 2.5 SN5

We have to prove

(21) $SNOR(\pi)(r,s) = SNOR(\pi)(s,r)$.

By definition of *SNOR* it is sufficient to show

(22) $\pi(\pi(r,0),s) = \pi(\pi(s,0),r)$.

From SIM7 we get

(23) $\pi(\pi(r,0),\pi(s,0)) = \pi(s,r)$,

hence we obtain by the substitution $\pi(s,0)$ for s

(24) $\pi(\pi(r,0),\pi(\pi(s,0),0)) = \pi(\pi(s,0),r)$,

hence (24) implies (22) because of the assumption SIM1.

ad 2.6 SN6

We have to prove

(25) $SNOR(\pi)(r,SNOR(\pi)(s,t)) = SNOR(\pi)(SNOR(\pi)(r,s),t)$.

By definition of *SNOR* it is sufficient to show

(26) $\pi[\pi(r,0),\pi(\pi(s,0),t)] = \pi[\pi(\pi(\pi(r,0),s),0),t]$.

Now, by SIM7 we obtain

$$\pi[\pi(\pi(\pi(r,0),s),0),t] = \pi[\pi(t,0),\pi(\pi(r,0),s)], (27)$$

hence by SIM8

$$= \pi[\pi(r,0),\pi(\pi(t,0),s)],$$

hence by SIM7 and SIM1

$$= \pi[\pi(r,0),\pi(\pi(s,0),t)],$$

hence (26) holds.

ad 2.7 If π is continuous, then $NEG(\pi)(r) =_{def} \pi(r,0)$ is continuous, hence $SNOR(\pi)(r,s) =_{def} \pi(\pi(r,0),s)$ is continuous, trivially.

■

Corollary 4.2

1. *If π is an S-implication, then $NEG(\pi)$ is a negation and $SNOR(\pi)$ is an S-norm.*
2. *$[SNOR,NEG]$ is a bijection from the class SIMPLICATIONS onto the class SNORMS×NEGATIONS.*
3. *SIMP is a bijection from the class SNORMS×NEGATIONS onto the class SIMPLICATIONS.*
4. *SIMP is the inverse mapping of the bijection $[SNOR,NEG]$ and vice versa.*
5. *If we restrict the classes NEGATIONS, SNORMS, and SIMPLICATIONS by the conditions of continuity, then the restricted classes are invariant with respect to the mappings SIMP, $[SNOR,NEG]$.*

Remark Theorem 4.1 (together with theorem 3.1) makes it possible to derive further "bijection theorems".

5 Further results

For $\tau, \pi : \langle 0, 1\rangle^2 \to \langle 0, 1\rangle$ we define

Definition 5.1

1. $RIMP(\tau)(r, s) =_{def} \mathrm{Sup}\{t \,|\, t \in \langle 0, 1\rangle \wedge \tau(r, t) \leq s\}$
2. $TNOR(\pi)(r, s) =_{def} \mathrm{Inf}\{t \,|\, t \in \langle 0, 1\rangle \wedge \pi(r, t) \geq s\}$

Theorem 5.1
For every $r, s \in \langle 0, 1\rangle$, $TNOR(RIMP(\tau))(r, s) \leq \tau(r, s)$.

Theorem 5.2
If for every fixed $r \in \langle 0, 1\rangle$ the function $\tau(r, s)$ is monotone and left-hand continuous with respect to $s \in \langle 0, 1\rangle$, then for every $r, s \in \langle 0, 1\rangle$,

$$\tau(r, s) \leq TNOR(RIMP(\tau))(r, s).$$

Denote by $FUNCT(2, \mathrm{MLHC2})$ the set of all functions $\varphi : \langle 0, 1\rangle^2 \to \langle 0, 1\rangle$ such that for every fixed $r \in \langle 0, 1\rangle$ the function $\varphi(r, s)$ is monotone and left-hand continuous with respect to $s \in \langle 0, 1\rangle$.

Corollary 5.3

1. *If $\tau \in FUNCT(2, MLHC2)$, then $TNOR(RIMP(\tau)) = \tau$.*
2. *$RIMP : FUNCT(2, MLHC2) \to FUNCT(2)$ is an injection.*

Theorem 5.4
If for every fixed $r \in \langle 0, 1\rangle$ the function $\pi(r, s)$ is monotone and right-hand continuous with respect to $s \in \langle 0, 1\rangle$, then for every $r, s \in \langle 0, 1\rangle$,

$$RIMP(TNOR(\pi))(r, s) \leq \pi(r, s).$$

Theorem 5.5
For every $r, s \in \langle 0, 1\rangle$, $\pi(r, s) \leq RIMP(TNOR(\pi))(r, s)$.

Denote by $FUNCT(2, \mathrm{MRHC2})$ the set of all functions $\varphi : \langle 0, 1\rangle^2 \to \langle 0, 1\rangle$ such that for every fixed $r \in \langle 0, 1\rangle$ the function $\varphi(r, s)$ is monotone and right-hand continuous with respect to $s \in \langle 0, 1\rangle$.

Corollary 5.6

1. *If $\pi \in FUNCT(2, MRHC2)$, then $RIMP(TNOR(\pi)) = \pi$.*
2. *TNOR is a bijection from $FUNCT(2, MRHC2)$ onto $FUNCT(2, MLHC2)$.*
3. *RIMP is the inverse mapping of TNOR and vice versa.*

In a forthcoming paper we will investigate which properties of τ and π are translated by *RIMP* and *TNOR* analogously to theorems 3.1 and 4.1, respectively. See also [1, 2, 8, 15, 19–21].

In a forthcoming second paper we will study relations between QL-implications on the one hand and negations, T-norms, and S-norms on the other hand, following the "philosophy" presented in this paper.

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