Log-Periodogram estimation of the memory parameter of a long-memory process under trend [1]

by

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Abstract

We show that small trends do not influence log-periodogram based estimators for the memory parameter in a stationary invertible long-memory process. In the case of slowly decaying trends which are easily confused with long-range dependence we show by Monte Carlo methods that the tapered periodogram is quite robust against these trends and thus provides a good alternative to standard log-periodogram methodology.

KEY WORDS: Long-memory, trends, log-periodogram regression
JEL - classification: C 14; C 22

1 Introduction

It is a well known phenomenon that long-memory and slowly decaying trends are easily confused. Starting with Bhattacharya et al.(1983) many authors considered trends which produce the Hurst effect even if there is no long-range

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dependence in the data. The discussion focused mainly on the behavior of rescaled-range based methods as the R/S-statistic whereas Künsch(1986) showed that the periodogram can distinguish monotonic trends and long-memory. For an overview of this debate see Sibbertsen(2001).

In contrast to previous works this paper considers the behavior of log-periodogram regression estimators for the memory parameter. It is motivated by the works of Hurvich/Ray(1995) and Velasco(1999) which show that these procedures are insensible to non-stationarities, especially polynomic time trends, when using an appropriate tapered periodogram.

In this paper the behavior of the log-periodogram regression estimation introduced by Geweke/Porter-Hudak(1983) (GPH-estimator) is considered for various slowly decaying trends in the data. We analyze the original GPH-estimator as well as the estimation obtained by employing a tapered periodogram. Several trends as monotonic trends and structural breaks are considered. It is also proved that small trends do not effect the GPH-estimator.

The paper is organized as follows. In the next section we prove that the GPH-estimator is not influenced by small trends decaying fast enough. Section 3 gives simulation results for slowly decaying trends as well as for structural breaks.

2 Log-Periodogram Estimation under trends

Point of departure is the model

$$X_t = Y_t + f(t), \qquad t = 1, \dots N, \tag{1}$$

where X_t denotes the observed process, Y_t is a noise process with zero mean and $f(t)$ is a deterministic trend. The point of interest is how the trend function influences log-periodogram based estimators for the memory parameter of a long-memory process. Thus the problem is if log-periodogram based estimators also confuse model (1) with long-range dependence.

A process X_t is said to exhibit long-range dependence if the spectral density behaves like

$$f_X(\lambda) \sim c_f \lambda^{-2d}, \qquad \lambda \to 0, \tag{2}$$

where c_f is a positive constant and $d \in (0, 0.5)$.

An estimator for the memory parameter based on log-periodogram regression was introduced by Geweke/Porter-Hudak(1983). Denote with $I_X(j) := \frac{1}{2\pi N} |\sum_{t=1}^{N} X_t \exp(\frac{-it2\pi j}{N})|^2$ the periodogram of the process X_t.

The GPH-estimator is based on the special shape of the spectral density (2). It is defined as the least-squares estimator of d based on the regression equation

$$\log I_X(\lambda_j) = \log c_f - 2d \log \lambda_j + \log \xi_j, \tag{3}$$

where λ_j denotes the $j-th$ Fourier frequency and the ξ_j are identically distributed errors with $E[\log \xi_j] = -0.577$, known as Euler constant.

Hurvich et al. (1998) showed that under some regularity conditions the GPH-estimator is asymptotically normal. The optimal number of frequencies wich should be used for the regression (3) is $N^{4/5}$. This is also the number of frequencies used in this paper.

Following the line of Heyde/Dai(1996) we show at first that the GPH-estimator is not effected by trends decaying with a rate faster than $N^{1/2}$. For the trend $f(t)$ we therefore assume that $N^{-1/2} \sum_{t=1}^{N} f(t) \to 0$ for $N \to \infty$.

Theorem 1 *Under the above assumption for the trend $f(t)$ the GPH-estimator $\hat{d}_X$ based on the process X_t in model (1) is equal to the GPH-estimator $\hat{d}_Y$ based on the series Y_t in model (1).*

Proof: The GPH-estimator depends only on the periodogram of the underlying series. Thus for proving the theorem it suffices to show that the periodograms of the series X and Y are equal. To see this we show that $I_X(j) - I_Y(j)$ tends to zero in probability. Here $I_X(j)$ denotes the periodogram of the process X.

We have

$$
\begin{aligned}
I_X(j) - I_Y(j) &= \frac{1}{2\pi N}(|\sum_{t=1}^{N} X_t e^{-it\lambda_j}|^2 - |\sum_{t=1}^{N} Y_t e^{-it\lambda_j}|^2) \\
&= \frac{1}{2\pi N}(|\sum_{t=1}^{N}(Y_t + f(t))e^{-it\lambda_j}|^2 - |\sum_{t=1}^{N} Y_t e^{-it\lambda_j}|^2) \\
&= \frac{1}{2\pi N}(\sum_{s=1}^{N}\sum_{t=1}^{N}(Y_t f(s) + Y_s f(t) + f(s)f(t))e^{-i(t-s)\lambda_j}).
\end{aligned}
$$

We now prove that

$$
\frac{1}{2\pi N}(\sum_{s=1}^{N}\sum_{t=1}^{N}(Y_t f(s) + Y_s f(t) + f(s)f(t)) \xrightarrow{P} 0 \tag{4}
$$

It is

$$
\begin{aligned}
\frac{1}{2\pi N}\sum_{s=1}^{N}\sum_{t=1}^{N} Y_t f(s) &= \frac{1}{2\pi N}\sum_{s=1}^{N} f(s)\sum_{t=1}^{N} Y_t \\
&\xrightarrow{P} 0,
\end{aligned} \tag{5}
$$

because the first sum tends to zero and the second sum tends to a finite constant.

For showing that the second term in (4) vanishes we can use similar arguments as in (5). Thus it remains the third term of (4).

$$
\begin{aligned}
\frac{1}{2\pi N}\sum_{s=1}^{N}\sum_{t=1}^{N} f(t)f(s) &= \frac{1}{2\pi N}\sum_{s=1}^{N} f(s)\sum_{t=1}^{N} f(t) \\
&= (\frac{1}{2\pi\sqrt{N}}\sum_{s=1}^{N} f(s))(\frac{1}{\sqrt{N}}\sum_{t=1}^{N} f(t)) \\
&\xrightarrow{P} 0,
\end{aligned} \tag{6}
$$

because both sums in the second equation tend to zero.

Combining (5) and (6) proves (4) and therefore the assertion. $\diamondsuit$

Thus from this Theorem we see that small trends do not influence log-periodogram based estimators of the memory parameter.

One trend fulfilling the conditions above is $f(t) = 0.9^t$. Therefore we verify the result of the theorem with a small simulation study by considering this trend function. As noise process we add to the trend a white noise as well as an process exhibiting long-range dependence with memory parameter $d = 0.1, d = 0.2, d = 0.3, d = 0.4$ and $d = 0.45$ respectively. All noise processes have zero mean and a variance of one, the length of the process is $N = 1000$. The simulation results for the GPH-estimator are based on 1000 replications. By denoting with d_0 the true value of the memory parameter of the underlying noise process and by $\hat{d}_m$ the mean of the estimations of d and by $\hat{d}_v$ the variance we have the following result:

Table I *Influence of small trends to the GPH-estimator*

d_0	0	0.1	0.2	0.3	0.4	0.45
$\hat{d}_m$	0.082	0.147	0.226	0.31	0.407	0.46
$\hat{d}_v$	0.0016	0.0016	0.0019	0.0018	0.002	0.0019

Thus the simulation results show that the GPH-estimator gives an appropriate estimation of the true value.

3 Slowly decaying trends

In the case of bigger trends the situation of course differs from the results of the last section. In this case the GPH-estimator is strongly biased. To obtain robustness against decaying trend functions we compare in this section the standard GPH-estimator with GPH-estimation based on the periodogram of the tapered data. The periodogram of the tapered process $w_t X_t$ is defined by

$$I_{T,X}(j) = \frac{1}{2\pi \sum w_t^2} | \sum_{t=0}^{N-1} w_t X_t e^{-i\lambda_j t}|^2.$$

Here λ_j again denotes the j-th Fourier frequency and w_t denotes the taper. We use in this paper the full cosine bell taper given by

$$w_t = \frac{1}{2}[1 - \cos(\frac{2\pi(t+0.5)}{N})].$$

In the Monte Carlo study we compare the standard GPH-estimator and the tapered estimator for three trend functions:

$$\begin{aligned} f_1(t) &:= t^\beta; \\ f_2(t) &:= \frac{\sin(t)}{t}; \\ f_3(t) &:= \begin{cases} k & \text{if} \quad 1 \le t \le [\tau N] \\ k^* & \text{if} \quad [\tau N] < t \le N \end{cases} \end{aligned}$$

In $f_1(t)$ β is chosen to be either -0.2 or -0.4. Following Bhattacharya et al.(1983) this trend should produce a Hurst coefficient of $d = 0.4$ and $d = 0.3$ respectively by using R/S-analysis. We will see later that this is not the case for the GPH-estimator.

In $f_3(t)$ $[\tau N]$ denotes the time point of the structural break. In our case we also choose $k^* = -k$.

The structure of the simulation study is similar to the last section. To each trend we add a noise process with memory parameter $d = 0, 0.1, 0.2, 0.3, 0.4$ and $d = 0.45$ respectively. The lengths of the series are $N = 1000$ and the simulations are based on 1000 replications. We denote again with d_0 the true memory parameter of the noise process, $\hat{d}_m$ and $\hat{d}_v$ denote the mean and the variance of the standard GPH-estimator and we denote by $\hat{d}_{T,m}$ and $\hat{d}_{T,v}$ the mean and the variance of the tapered GPH-estimator. For the first trend we choose a standard deviation of $\sigma = 0.07$ of the noise process. Of course the results depend on the standard deviation of the noise process but for all simulations we choose the standard deviation so that the influence of the trend is still sensible and that the results are not a consequence of a high standard deviation of the noise process.

Table II *Trend* $f_1(t)$ *with* $\beta = -0.2$

d_0	0	0.1	0.2	0.3	0.4	0.45
$\hat{d}_m$	0.127	0.447	0.465	0.487	0.52	0.543
$\hat{d}_v$	0.0015	0.0009	0.0013	0.0015	0.0019	0.0022
$\hat{d}_{T,m}$	0.059	0.182	0.268	0.355	0.444	0.492
$\hat{d}_{T,v}$	0.0026	0.0028	0.0026	0.0027	0.0027	0.0029

For the second trend we choose a standard deviation of $\sigma = 0.1$. The results are as follows

Table III *Trend* $f_2(t)$

d_0	0	0.1	0.2	0.3	0.4	0.45
$\hat{d}_m$	0.305	0.326	0.361	0.411	0.483	0.521
$\hat{d}_v$	0.0008	0.0014	0.0016	0.0017	0.0017	0.0017
$\hat{d}_{T,m}$	0.002	0.103	0.204	0.307	0.412	0.465
$\hat{d}_{T,v}$	0.0028	0.0029	0.0032	0.0031	0.003	0.003

For the third trend we choose $k = 0.25$, $\tau = 0.5$ and the standard deviation of the noise process is $\sigma = 1$. The results are

Table IV *Trend* $f_3(t)$ *with* $k = 0.25$ *and* $\tau = 0.5$

d_0	0	0.1	0.2	0.3	0.4	0.45
$\hat{d}_m$	0.515	0.563	0.609	0.656	0.704	0.729
$\hat{d}_v$	0.0008	0.0008	0.0009	0.0009	0.0009	0.001
$\hat{d}_{T,m}$	0.092	0.162	0.236	0.324	0.421	0.472
$\hat{d}_{T,v}$	0.003	0.003	0.003	0.003	0.004	0.004

Thus we see from the simulations that there is a strong reduction of the bias by using the tapered periodogram. The tapered GPH-estimator gives reasonable estimations of the true memory parameter.The differences to the standard GPH-estimator are obvious. On the other hand the variance of the estimates increase by using the tapered GPH-estimator but still the variance is quite small.

Altogether the tapered GPH-estimator provides a good alternative when slowly decaying trends are in the data.

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