Pitman-closeness and the linear combination of multivariate forecasts

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Abstract: We use the Pitman-closeness criterion to evaluate the performance of multivariate forecasting methods and we also calculate optimal matrices of weights for the linear combination of multivariate forecasts. These weights are identical with the optimal weights under the matrix-MSE criterion.

Key words: Pitman-closeness, multivariate forecasting methods, combination of forecasts.

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1 Introduction

In the theory of combination of forecasts, most studies consider the univariate case only. In that case several individual forecasts for a univariate random variable are combined, but also the multivariate case is of great interest. Here, a multivariate forecasting technique predicts a k-dimensional random vector ($k \geq 2$). Therefore, combination methods based for example on the minimum-matrix-MSE criterion or the covariance-adjustment-technique depend on the covariance structure of the errors of a special method and also on the covariances between the errors of the different methods.

In this article we analyse the comparison of multivariate forecast combinations under Pitman-closeness (Pitman, 1937). In the multivariate case, there are different ways to interpret this evaluation criterion. We will focus on the component-by-component Pitman-closeness and calculate weights for the optimal combination of multivariate forecasts. We shall see that these weights also depend on the covariance structure of all forecast errors.

Furthermore this optimality criterion is equivalent to the matrix-MSE-optimality. A short example is given for a better illustration.

Finally we present a brief description of the general Pitman-closeness approach for multivariate forecasts and problems that occur in applications.

2 The Problem

First we give a description of the problem.

Assume that

$\mathbf{Y} := (\mathrm{Y}_1,...,\mathrm{Y}_k)'$ is a random vector to be forecasted (k≥2),

$\mathbf{F}_i := (\mathrm{F}_{1i},...,\mathrm{F}_{ki})'$ are unbiased multivariate forecasts (i=1,...,n) for $\mathbf{Y}$ and

$\mathbf{u}_i := (\mathrm{Y}_1 - \mathrm{F}_{1i},...,\mathrm{Y}_k - \mathrm{F}_{ki})'$ is the error vector of the i-th forecast method,

where $\mathbf{u} := \left(\mathbf{u}_1{}',...,\mathbf{u}_n{}'\right)' \sim \mathrm{N}_{n\cdot k}(\mathbf{0},\Sigma)$, Σ p.d., and there exists a vector $\mathbf{u}_i$, without loss of generality $\mathbf{u}_i = \mathbf{u}_n$, so that $\mathrm{Cov}[((\mathbf{u}_1 - \mathbf{u}_n)',\ldots,(\mathbf{u}_{n-1} - \mathbf{u}_n)')']$ is p.d. The quantities

$$\mathbf{A}_i := \begin{pmatrix} a_{11}^{(i)} & \cdots & a_{1k}^{(i)} \\ \vdots & \ddots & \vdots \\ a_{k1}^{(i)} & \cdots & a_{kk}^{(i)} \end{pmatrix} \in \mathbb{R}^{k\times k} \text{ and } \mathbf{B}_i := \begin{pmatrix} b_{11}^{(i)} & \cdots & b_{1k}^{(i)} \\ \vdots & \ddots & \vdots \\ b_{k1}^{(i)} & \cdots & b_{kk}^{(i)} \end{pmatrix} \in \mathbb{R}^{k\times k} \text{ are matrices of weights,}$$

where

$\sum_{i=1}^{n} \mathbf{A}_i = \mathbf{I}_k$, $\sum_{i=1}^{n} \mathbf{B}_i = \mathbf{I}_k$, and $\mathbf{I}_k$ denotes the identity matrix.

We make the assumption that the matrices of weights sum up to identity which guarantees the unbiasedness of the combined forecast. The multivariate combinations of forecasts are

$$\mathbf{F}_A = \sum_{i=1}^{n} \mathbf{A}_i \mathbf{F}_i \text{ and } \mathbf{F}_B = \sum_{i=1}^{n} \mathbf{B}_i \mathbf{F}_i$$

and the corresponding error vectors are

$$\mathbf{u}_A := \mathbf{Y} - \mathbf{F}_A = \sum_{i=1}^{n} \mathbf{A}_i (\mathbf{Y} - \mathbf{F}_i) = \sum_{i=1}^{n} \mathbf{A}_i \mathbf{u}_i \text{ and}$$

$$\mathbf{u}_B := \mathbf{Y} - \mathbf{F}_B = \sum_{I=1}^{n} \mathbf{B}_i (\mathbf{Y} - \mathbf{F}_i) = \sum_{i=1}^{n} \mathbf{B}_i \mathbf{u}_i \ .$$

Subsequently we are going to compare different combinations with the Pitman-closeness criterion and also derive optimal weights.

3 Component-by-component Pitman-closeness

In this section we compare multivariate forecasts in each component separately.

Definition 3.1. The forecast $\mathbf{F}_1$ is relative to component j Pitman-closer to Y_j than the forecast $\mathbf{F}_2$, $j\in\{1,\ldots,k\}$, if and only if

$$P\left(\left|Y_j - F_{j1}\right| < \left|Y_j - F_{j2}\right|\right) > 0.5 .$$

The probability statement of this definition can also be written as $P\left(\left|u_{j1}\right| < \left|u_{j2}\right|\right) > 0.5$.

The first forecast method is Pitman-closer to Y_j if the probability that it has a smaller absolute error in the j-th component than the second method is larger than 0.5. Consequently, it is reasonable to introduce the following definition.

Definition 3.2. The forecast $\mathbf{F}_1$ is component-by-component Pitman-closer than the forecast $\mathbf{F}_2$ $(\mathbf{F}_1 \neq \mathbf{F}_2)$ if and only if

$$P\left(\left|Y_j - F_{j1}\right| < \left|Y_j - F_{j2}\right|\right) > 0.5 \qquad \forall j \in \{1,\ldots,k\}, \text{ where } F_{j1} \neq F_{j2}.$$

The probability statement of Definition 3.2 is equivalent to

$$P\left(\left|u_{j1}\right| < \left|u_{j2}\right|\right) > 0.5 \qquad \forall j \in \{1,\ldots,k\}, \text{ where } F_{j1} \neq F_{j2}.$$

With this definition in mind we will now find a combination which is the component-by-component Pitman-closest.

Let $\mathbf{a}_j^{(i)} := \left(a_{j1}^{(i)},\ldots,a_{jk}^{(i)}\right)'$ and $\mathbf{b}_j^{(i)} := \left(b_{j1}^{(i)},\ldots,b_{jk}^{(i)}\right)'$, j=1,...,k, i=1,...,n be the j-th row vectors of the i-th matrices of weights. Then the components of the error vectors of the two combinations are given as

$$\mathbf{u}_{jA} := Y_j - \sum_{i=1}^{n} \mathbf{a}_j^{(i)\prime} \mathbf{F}_i = \sum_{i=1}^{n} \mathbf{a}_j^{(i)\prime} \left(\mathbf{Y} - \mathbf{F}_i\right) = \sum_{i=1}^{n} \mathbf{a}_j^{(i)\prime} \mathbf{u}_i = \mathbf{a}_j' \mathbf{u}$$

$$\mathbf{u}_{jB} := \cdots = \mathbf{b}_j' \mathbf{u} \ ,$$

where

$$\mathbf{a}_j := \left(\mathbf{a}_j^{(1)\prime},\ldots,\mathbf{a}_j^{(n)\prime}\right)' \sim (n\cdot k)\times 1, \quad \mathbf{b}_j := \left(\mathbf{b}_j^{(1)\prime},\ldots,\mathbf{b}_j^{(n)\prime}\right)' \sim (n\cdot k)\times 1$$

and $\mathbf{u} \sim (n \cdot k) \times 1$.

With the definitions from above we are able to compare the two combinations of multivariate forecasts. $\mathbf{F}_A$ is in the j-th component Pitman-closer to Y_j than $\mathbf{F}_B$ if and only if

$$P\left(\left|u_{jA}\right| < \left|u_{jB}\right|\right) > 0.5 \Leftrightarrow P\left(u_{jA}^2 < u_{jB}^2\right) > 0.5$$

$$\Leftrightarrow \quad P\left(\mathbf{u}' \mathbf{a}_j \mathbf{a}_j' \mathbf{u} < \mathbf{u}' \mathbf{b}_j \mathbf{b}_j' \mathbf{u}\right) > 0.5 \quad \Leftrightarrow \quad P\left(\mathbf{u}'\left(\mathbf{a}_j \mathbf{a}_j' - \mathbf{b}_j \mathbf{b}_j'\right)\mathbf{u} < 0\right) > 0.5$$

$$\Leftrightarrow \quad P\left(\mathbf{u}' \Sigma^{-0.5} \Sigma^{0.5}\left(\mathbf{a}_j \mathbf{a}_j' - \mathbf{b}_j \mathbf{b}_j'\right)\Sigma^{0.5} \Sigma^{-0.5} \mathbf{u} < 0\right) > 0.5 \qquad (1)$$

This is similar to the characterization of Pitman-Closeness for the univariate case as in Wenzel (1998).

The eigenvalues of the matrix $\mathbf{C}_j := \Sigma^{0.5}\left(\mathbf{a}_j \mathbf{a}_j' - \mathbf{b}_j \mathbf{b}_j'\right)\Sigma^{0.5}$ are

$$\lambda_{j1} := \frac{\left\|\mathbf{c}_j\right\|^2 - \left\|\mathbf{d}_j\right\|^2 + \left\|\mathbf{c}_j + \mathbf{d}_j\right\|\left\|\mathbf{c}_j - \mathbf{d}_j\right\|}{2}, \ \lambda_{j2} := \frac{\left\|\mathbf{c}_j\right\|^2 - \left\|\mathbf{d}_j\right\|^2 - \left\|\mathbf{c}_j + \mathbf{d}_j\right\|\left\|\mathbf{c}_j - \mathbf{d}_j\right\|}{2},$$

with corresponding eigenvectors $\mathbf{v}_{j1}$ and $\mathbf{v}_{j2}$, where

$$\mathbf{c}_j := \Sigma^{0.5}\mathbf{a}_j \ \text{ and } \ \mathbf{d}_j := \Sigma^{0.5}\mathbf{b}_j \ .$$

Then

$$(1) \Leftrightarrow \quad P\left(\lambda_{j1}\mathbf{u}' \Sigma^{-0.5} \mathbf{v}_{j1} \mathbf{v}_{j1}' \Sigma^{-0.5} \mathbf{u} + \lambda_{j2}\mathbf{u}' \Sigma^{-0.5} \mathbf{v}_{j2} \mathbf{v}_{j2}' \Sigma^{-0.5} \mathbf{u} < 0\right) > 0.5$$

$$\Leftrightarrow \quad P\left(\lambda_{j1} X_{j1}^2 + \lambda_{j2} X_{j2}^2 < 0\right) > 0.5 \ , \qquad (2)$$

where

$$X_{j1} := \mathbf{u}' \Sigma^{-0.5} \mathbf{v}_{j1} \ , \ X_{j2} := \mathbf{u}' \Sigma^{-0.5} \mathbf{v}_{j2},$$

$$E\left(X_{j1}\right) = E\left(X_{j2}\right) = 0, \ \mathrm{Var}\left(X_{j1}\right) = \mathrm{Var}\left(X_{j2}\right) = 1, \ \mathrm{Cov}\left(X_{j1}, X_{j2}\right) = 0.$$

As a weighted sum of normal distributed random variables, the X's are also normally distributed. Since they are independent with zero mean and unique variance, the ratio $\frac{X_{j1}}{X_{j2}}$ is Cauchy(0,1)-distributed. Hence we have the following equivalences:

$$(2) \Leftrightarrow \quad P\left(-\sqrt{-\frac{\lambda_{j2}}{\lambda_{j1}}} < \frac{X_{j1}}{X_{j2}} < \sqrt{-\frac{\lambda_{j2}}{\lambda_{j1}}}\right) > 0.5$$

$$\Leftrightarrow \quad \frac{2}{\pi}\arctan\sqrt{-\frac{\lambda_{j2}}{\lambda_{j1}}} > 0.5$$

With the same conclusions as in the univariate case this is equivalent to

$$\|\mathbf{c}_j\|^2 < \|\mathbf{d}_j\|^2 \quad \Leftrightarrow \quad \mathbf{a}_j' \Sigma \mathbf{a}_j < \mathbf{b}_j' \Sigma \mathbf{b}_j \ .$$

An optimal combination of forecasts for the j-th component is given by a vector $\mathbf{a}_j$ which minimizes $\mathbf{a}_j' \Sigma \mathbf{a}_j$. Since the weight matrices sum up to the identity matrix, we have $\mathbf{a}_j^{(1)} + \ldots + \mathbf{a}_j^{(n)} = \mathbf{e}_j$ where $\mathbf{e}_j$ is the j-th unit vector. This leads to the minimization problem:

$$\left.\begin{array}{ll} \text{Minimize} & \mathbf{a}_j' \Sigma \mathbf{a}_j \\ \text{subject to} & \sum_{i=1}^{n} \mathbf{a}_j^{(i)} = \mathbf{e}_j \end{array}\right\} (\mathbf{MP})$$

$$\Leftrightarrow \qquad \text{Minimize} \quad \left(\mathbf{a}_j^{(1)\prime},\ldots,\mathbf{a}_j^{(n-1)\prime},\left(\mathbf{e}_j - \sum_{i=1}^{n-1}\mathbf{a}_j^{(i)}\right)'\right)\Sigma\left(\mathbf{a}_j^{(1)\prime},\ldots,\mathbf{a}_j^{(n-1)\prime},\left(\mathbf{e}_j - \sum_{i=1}^{n-1}\mathbf{a}_j^{(i)}\right)'\right)' .$$

With $\Sigma := (\Sigma_{rs})_{r,s=1,\ldots,n}$, $\Sigma_{rs} := \mathrm{Cov}(\mathbf{u}_r, \mathbf{u}_s)$, we obtain:

$$\left(\mathbf{a}_j^{(1)\prime},\ldots,\mathbf{a}_j^{(n-1)\prime},\left(\mathbf{e}_j - \sum_{i=1}^{n-1}\mathbf{a}_j^{(i)}\right)'\right)\Sigma\left(\mathbf{a}_j^{(1)\prime},\ldots,\mathbf{a}_j^{(n-1)\prime},\left(\mathbf{e}_j - \sum_{i=1}^{n-1}\mathbf{a}_j^{(i)}\right)'\right)'$$

$$= \sum_{r=1}^{n-1}\sum_{s=1}^{n-1}\mathbf{a}_j^{(r)\prime}(\Sigma_{rs} - \Sigma_{rn})\mathbf{a}_j^{(s)} + \sum_{r=1}^{n-1}\sum_{s=1}^{n-1}\mathbf{a}_j^{(r)\prime}(\Sigma_{nn} - \Sigma_{ns})\mathbf{a}_j^{(s)}$$

$$+ \sum_{r=1}^{n-1}\mathbf{a}_j^{(r)\prime}(\Sigma_{rn} - \Sigma_{nn})\mathbf{e}_j + \sum_{s=1}^{n-1}\mathbf{e}_j'(\Sigma_{ns} - \Sigma_{nn})\mathbf{a}_j^{(s)} + \mathbf{e}_j'\Sigma_{nn}\mathbf{e}_j$$

$$= \sum_{r=1}^{n-1}\sum_{s=1}^{n-1}\mathbf{a}_j^{(r)\prime}(\Sigma_{rs} + \Sigma_{nn} - \Sigma_{rn} - \Sigma_{ns})\mathbf{a}_j^{(s)}$$

$$- \sum_{r=1}^{n-1}\mathbf{a}_j^{(r)\prime}(\Sigma_{nn} - \Sigma_{rn})\mathbf{e}_j - \sum_{s=1}^{n-1}\mathbf{e}_j'(\Sigma_{nn} - \Sigma_{ns})\mathbf{a}_j^{(s)} + \mathbf{e}_j'\Sigma_{nn}\mathbf{e}_j \qquad (3)$$

Now, defining

$$\mathbf{V}_{rs} := \Sigma_{rs} + \Sigma_{nn} - \Sigma_{rn} - \Sigma_{ns} \quad , \; r,s = 1,\ldots,n-1,$$

$$\mathbf{V} := (\mathbf{V}_{rs})_{r,s=1,\ldots,n-1} \sim (n-1)\cdot k \times (n-1)\cdot k,$$

$$\mathbf{w}_{ji} = (\Sigma_{nn} - \Sigma_{in})\mathbf{e}_j \sim \; k\times 1 \; \text{ and therefore } \; \mathbf{w}_{ji}' := \mathbf{e}_j'(\Sigma_{nn} - \Sigma_{ni}) \; , \text{ j=1,..,k, i=1,..,n–1,}$$

$$\mathbf{w}_j := \left(\mathbf{w}_{j1}', \ldots, \mathbf{w}_{j,n-1}'\right)' \sim (n\text{–}1)\cdot k\times 1,$$

$$\tilde{\mathbf{a}}_j := \left(\mathbf{a}_j^{(1)\prime}, \ldots, \mathbf{a}_j^{(n-1)\prime}\right)' \sim (n\text{–}1)\cdot k\times 1$$

and inserting in equation (3) yields

$$\mathbf{a}_j'\Sigma\mathbf{a}_j = \tilde{\mathbf{a}}_j'\mathbf{V}\tilde{\mathbf{a}}_j - \tilde{\mathbf{a}}_j'\mathbf{w}_j - \mathbf{w}_j'\tilde{\mathbf{a}}_j + \mathbf{e}_j'\Sigma_{nn}\mathbf{e}_j =: L(\tilde{\mathbf{a}}_j) \qquad .$$

The necessary condition for a minimum is

$$\frac{\delta L(\tilde{\mathbf{a}}_j)}{\delta \tilde{\mathbf{a}}_j} = 2\tilde{\mathbf{a}}_j'\mathbf{V} - 2\mathbf{w}_j' \overset{!}{=} \mathbf{0}' \; ,$$

and the sufficiency condition

$$\frac{\delta^2 L(\tilde{\mathbf{a}}_j)}{\delta^2 \tilde{\mathbf{a}}_j} = 2\mathbf{V} \quad \text{p.d.}$$

follows from the assumption that $\mathbf{V} := \mathrm{Cov}(\mathbf{z})$ with $\mathbf{z} := \left((\mathbf{u}_1 - \mathbf{u}_n)', \ldots, (\mathbf{u}_{n-1} - \mathbf{u}_n)'\right)'$ is p.d.

Thus we get: $\tilde{\mathbf{a}}_j^{(opt)} = \mathbf{V}^{-1}\mathbf{w}_j$.

Now we are able to formulate the following two theorems.

Theorem 3.1. The Pitman-closest-combination of n multivariate forecasts for the j-th component of a random vector **Y** of dimension k (k≥2) is given by the vector of weights

$$\mathbf{a}_j^{(opt)} := \left(\mathbf{a}_j^{(1),opt\prime}, \ldots, \mathbf{a}_j^{(n),opt\prime}\right)' = \left(\mathbf{w}_j'\mathbf{V}^{-1}, \mathbf{e}_j' - \mathbf{w}_j'\mathbf{V}^{-1}\mathbf{I}_k^*\right)' \quad ,$$

where $\mathbf{I}_k^* := [\mathbf{I}_k, \ldots, \mathbf{I}_k]' \sim (n-1)\cdot k\times k$.

Theorem 3.2. The component-by-component Pitman-closest-combination of n multivariate forecasts for a random vector **Y** of dimension k (k≥2) is given by the matrix of weights

$$\mathbf{A}^{(\mathrm{opt})} := [\mathbf{A}_1^{(\mathrm{opt})}, \ldots, \mathbf{A}_n^{(\mathrm{opt})}] = [\mathbf{W}'\mathbf{V}^{-1}, \mathbf{I}_k - \mathbf{W}'\mathbf{V}^{-1}\mathbf{I}_k^*]$$

where $\mathbf{W} := (\mathbf{w}_1, \ldots, \mathbf{w}_k) \sim (n-1) \cdot k \times k$,

$$\mathbf{I}_k^* := [\mathbf{I}_k, \ldots, \mathbf{I}_k]' \sim (n-1) \cdot k \times k .$$

The proofs of these two theorems follow directly from the calculations above. It is obvious that the optimal weights in each component depend on the covariance structure of the whole system of forecast errors.

Looking again to the minimization problem **(MP)** we get the following theorem:

Theorem 3.3. With the assumptions in *Section 2* the optimal matrix-MSE-combination is identical with the component-by-component Pitman-closest-combination.

Proof: The matrix-MSE of the errors of the combined forecast $\mathbf{F}_A$ is given as

$$\begin{aligned} E\left(\mathbf{u}_A \mathbf{u}_A'\right) &= E\left[\left(\sum_{i=1}^{n} \mathbf{A}_i (\mathbf{Y} - \mathbf{F}_i)\right)\left(\sum_{i=1}^{n} \mathbf{A}_i (\mathbf{Y} - \mathbf{F}_i)\right)'\right] \\ &= E\left[\left(\sum_{i=1}^{n} \mathbf{A}_i \mathbf{u}_i\right)\left(\sum_{i=1}^{n} \mathbf{A}_i \mathbf{u}_i\right)'\right] = E\left[\sum_{i=1}^{n}\sum_{j=1}^{n} \mathbf{A}_i \mathbf{u}_i \mathbf{u}_j \mathbf{A}_j'\right] \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} \mathbf{A}_i \Sigma_{ij} \mathbf{A}_j' \qquad \text{, where } \Sigma_{ij} := E\left(\mathbf{u}_i \mathbf{u}_j'\right). \end{aligned}$$

Odell et al. (1989) analysed the minimization problem for the linear combination of multivariate estimators with the assumptions of *Section 2*. They pointed out that in this case minimizing the matrix-MSE in the sense of the Löwner-ordering implies that the scalar-MSE is also minimized. The scalar-MSE is defined as the trace of $E\left(\mathbf{u}_A \mathbf{u}_A'\right)$. The trace is

$$\mathrm{tr}\left(E\left(\mathbf{u}_A \mathbf{u}_A'\right)\right) = \sum_{j=1}^{n} \mathbf{a}_j' \Sigma \mathbf{a}_j .$$

We see that the j-th term of the sum depends only on the j-th weight vector. Thus the minimal trace is given by the minimum of each of the n components in the sum and therefore by the optimal weight vectors $\mathbf{a}_j^{(opt)}$. This means that the optimal matrix-MSE-combination is also the component-by-component Pitman-closest.

On the other hand, if we calculate the component-by-component Pitman-closest-combination, we begin by minimizing the trace of $E\left(\mathbf{u}_A \mathbf{u}_A{}'\right)$. Consulting again the paper of Odell et al. (1989) it follows that the matrix-MSE is also minimized in the sense of the Löwner-ordering.

Application. The theoretical description of the problem will now be underlined by a short example. We analyse a problem where n=3 forecasts are given for a random vector of dimension k=2. The 6×6 covariance matrix of the forecast errors which, in pratice, can be calculated using the general ML-estimators is

$$\mathrm{Cov}(\mathbf{u}) = \begin{pmatrix} 12 & 3 & 5 & 4 & 1 & 6 \\ 3 & 8 & 0 & 1 & -1 & 7 \\ 5 & 0 & 3 & 1 & 1 & 2 \\ 4 & 1 & 1 & 3 & -2 & 0 \\ 1 & -1 & 1 & -2 & 5 & 3 \\ 6 & 7 & 2 & 0 & 3 & 10 \end{pmatrix}$$

Now we take the forecast combination given by the the component-by-component Pitman-closest technique as $\mathbf{F}_A$ and intuitive combination techniques for $\mathbf{F}_B$, i.e. the three individual multivariate forecasts, the technique which uses for each component the optimal univariate combination and a combination with the weight matrices given below.

$$\mathbf{B}_1 = \begin{pmatrix} \frac{1}{3} & \frac{1}{2} \\ -\frac{1}{4} & \frac{1}{3} \end{pmatrix}, \mathbf{B}_2 = \begin{pmatrix} \frac{1}{3} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{1}{3} \end{pmatrix}, \mathbf{B}_3 = \begin{pmatrix} \frac{1}{3} & -\frac{1}{4} \\ \frac{1}{2} & \frac{1}{3} \end{pmatrix}$$

The following table shows the probabilities for each of the two components that the component-by-component Pitman-closest-combination outperforms the other combinations.

combination	component j	$P\left(\lvert u_{jA}\rvert < \lvert u_{jB}\rvert\right)$
first individual forecast	1	0,91992
	2	0,95241
second individual forecast	1	0,84555
	2	0,92287
third individual forecast	1	0,87810
	2	0,95739
optimal univariate combination	1	0,81172
	2	0,91234
combination with the weights above	1	0,84373
	2	0,93091

4 General Pitman-closeness

In S*ection 3* we considered Pitman-closeness for each component. In this case the probability that all components of the Pitman-closest-combination have a smaller absolute forecast error than another combination could be less than 0.5. Therefore, it is reasonable to define a general Pitman-closeness criterion.

Definition 4.1. The forecast $\mathbf{F}_1$ is general Pitman-closer than the forecast $\mathbf{F}_2$ if and only if

$$P\left(\lvert \mathbf{Y}-\mathbf{F}_1\rvert < \lvert \mathbf{Y}-\mathbf{F}_2\rvert\right) > 0.5\ .$$

Thus, if $P\left(\lvert \mathbf{u}_1\rvert < \lvert \mathbf{u}_2\rvert\right) > 0.5$ then $\mathbf{F}_1$ is general Pitman-closer than $\mathbf{F}_2$. Here $\lvert \mathbf{x}\rvert < \lvert \mathbf{y}\rvert$ means that $\lvert x_i\rvert < \lvert y_i\rvert,\ \forall i = 1,\ldots,k$.

With the transformations in *Section 3* we can write this as

$$P\left(\begin{pmatrix} -\sqrt{-\frac{\lambda_{12}}{\lambda_{11}}} \\ \vdots \\ -\sqrt{-\frac{\lambda_{k2}}{\lambda_{k1}}} \end{pmatrix} < \begin{pmatrix} \frac{X_{11}}{X_{21}} \\ \vdots \\ \frac{X_{k1}}{X_{k2}} \end{pmatrix} < \begin{pmatrix} \sqrt{-\frac{\lambda_{12}}{\lambda_{11}}} \\ \vdots \\ \sqrt{-\frac{\lambda_{k2}}{\lambda_{k1}}} \end{pmatrix}\right) > 0.5 .$$

It is obvious that for the calculation of the probability we have to take into account the multivariate distribution of the k components $\frac{X_{j1}}{X_{j2}},\ j=1,\ldots,k,$ which are dependent Cauchy(0,1)-variables. Another point is that two multivariate forecasts could be not comparable which means that none of them is general Pitman-closer. Especially in situations of ″high″ dimensions (k ″large″) it might be possible that no general Pitman-closest-combination exists.

5 Conclusions

We derived the component-by-component Pitman-closest-combination of forecasts which is equivalent to the optimal matrix-MSE-combination. With the component-by-component Pitman-closeness criterion we are able to specify a probability that a multivariate forecast in a special component performs better than another forecast. We have to emphasize that the assumption of normal distributed errors is needed. Furthermore we discussed the general Pitman-closeness criterion. By transforming each component as in the component-by-component case it was possible to calculate the distribution of each component but their joint distribution is needed. Therefore more research in the area of multivariate distributions is necessary. Finally as a new problem the case of incomparable forecasts may occur.

6 References

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