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Reflections on proportions

1. Testing perspective biologists with proportions in a lab setting

Consider two proportional quantities in two given situations: in the former situation one knows both quantities, while in the latter situation one quantity is known and the other must be computed. What one needs to do is:

- Understanding the problem (setting up the proportion)
- Calculations (solving the proportion)

I have tested some hundreds of prospective biologists with a simple proportion in a lab setting (University of Regensburg, 2012-2016) asking for example: *Wieviel 15%-ige Lösung muss man benutzen, um 12,0 ml verdünnte 12%-ige Lösung herzustellen?* Given the many implausible results (roughly, one third) I advocate for adding a mathematically unnecessary step:

- Plausibility test (validating the proportion)

Indeed, our common sense or background knowledge tells us whether to expect *a larger or a smaller value*: this rather immediate check can especially detect some common mistakes made while setting up the proportion. And, even if AI may now solve in our place all proportions, checking the plausibility of the result remains meaningful.

2. Thematizing unspoken conventions and necessary simplifications

Proportion exercises tend, for the sake of simplicity, to fake reality. In fact, there is even an *unspoken convention* that forces pupils to assume simplifications that are not even spelled out. For example, that all cows give the same amount of milk, and the same amount of milk every day… Is this realistic? Or should the exercise be rather formulated as: *suppose for simplicity that all cows give the same amount of milk*? Even if these clarifications are mathematically necessary, pupils adapt to mathematically incomplete exercises. As a side note, impossible scenarios or improbable coincidences may distract (if not disturb) certain pupils.

In general, pupils welcome easy exercises with custom-made easy numbers. And real data is hardly suitable for exercises. On top of that, real-life quantities may only be roughly proportional (and only within certain ranges). Nevertheless, pupils should occasionally be challenged to reflect on the necessary simplifications that are being made. Indeed, it is an important skill deciding which approximations are meaningful, and understanding the limits of any simplification made to real-life problems.

In: L. Schick, M. Platz & A. Lambert (Hrsg.),
Beiträge zum Mathematikunterricht 2025.
58. Jahrestagung der Gesellschaft für Didaktik der Mathematik. WTM.
https://doi.org/10.37626/GA9783959873307.0

3. Two quantities whose proportionality is context-dependent

Consider the scenario of children parties, where each child gets the same amount of pizza. *Do you consider the total amount of pizza and the number of children as quantities that are directly proportional, inversely proportional, or not proportional?*

- Suppose that you order pizza during the party according to the number of children, having fixed in advance the amount of pizza per child. If any amount of pizza can be delivered (imagine big rectangular pizzas that can be freely cut, rather than circular pizzas sold individually), then there is direct proportionality. However, if you buy circular pizzas and e.g. each child gets a quarter of pizza, then there is no proportionality (because, say, for 9 to 12 children you would order 3 pizzas).
- Suppose instead that you want to fairly share among the children the pizza that is available at the party. If you can cut pizza with arbitrary precision, then you have inverse proportionality. However, if you can only cut slices that are e.g. eights of circular pizzas, then there is no proportionality because there may be left-over slices that cannot be distributed to all.
- Suppose that you compare the amount of pizza and the number of children at parties. The two quantities are not proportionally related simply because one party may have more pizza per child than another party.

Mathematically speaking, given two non-zero proportional quantities, any of the two determines the other. However, in real-life situations, one quantity may have determined the other and not the other way round. For example, if you order pizza according to the number of children.

Proportionality relations should not be taken for granted. For example (buying identical items with the same price) the total price is directly proportional to the number of items, but only if there is no price lowering for bulk purchases.

Or, one may tend to believe that expressing a same physical quantity with different units of measure leads to proportional values, the proportionality constant being the conversion factor. However, the values of the temperature in Fahrenheit and in Celsius are not proportional, despite there being a linear relation between them.

Another example from physics: for homogeneous materials, the weight and the volume are usually considered to be directly proportional, the proportionality constant being the density of the material. However, if you compress gas, then (according to classical physics) you have the same weight but less volume and hence no proportionality between the weight and the volume.

4. Proportionality prior to rounding, and inequalities

Despite a proportion being an equality, one can easily formulate proportion exercises that involve inequalities. For example: *A lorry driver has driven for two hours with a speed varying from 80km/h to 100 km/h. What can we say about the distance that has been covered?*

Alternatively, an exercise about pipes could ask: *Will the basin be full after 5 hours?* (there is an underlying inequality).

It is important to notice that the result of a proportion exercise is the number coming out of a proportion after a suitable rounding. One reason is that the result occasionally needs to be an integer. Beyond this, the result is usually expected to be expressed with decimal digits hence, unless the given numbers are particularly nice, a rounding is needed. Do we need to round up or to round down?

- Suppose that you buy identical items: to compute how many items you can afford with the money at your disposal, you round down.
- Suppose that you want to give 3 pens to each of your pupils and the pens are sold in boxes of 10: to compute how many boxes to buy, you round up.
- Suppose that you need to take an amount of medicine that is proportional to your weight. Maybe you round according to the smallest error, or maybe you want to err on the side of caution (avoiding taking too much medicine or avoiding taking too little medicine).
- Mathematically speaking, after rounding up (respectively, down) the line representing the direct proportionality becomes a staircase function, while for inverse proportionality one gets a *staircase hyperbola* that I would bet few ever consider.
- One real-life exercise about proportion and rounding (which I invented for a mathematical competition) is the following: *A cake recipe for 10 people uses 6 eggs and 600g of flour. It is important to preserve the ratio between eggs and flour, but you must use an integer number of eggs. How much flour do you use for making a cake for 8 people? (Solution: 500g).*

5. Some mathematical exercises to rule them all

How many substantially different proportion exercises are there in school texts? Actually, very few. For example, despite the many different scenarios, we often have a constant speed (for movement, filling, working, downloading) leading to a *production* or *quantity variation* that is directly proportional to the speed and to the time.

An aside remark: It is often the case that the two proportional quantities in an exercise have distinct units of measure. However, if for example you take red paint and yellow paint in a given proportion to make a certain shade of orange, then you will have liters of paints for both quantities.

In proportion exercises involving more than two quantities, the proportionality relation is only between two quantities at the time, when all of the other quantities are taken to be constant. For example (with the due simplifications) we may consider the produced milk M as proportional to the number C of cows and to the number D of days. In the formula M=kCD the proportionality constant k (which is a sort of speed) is the amount of milk per cow per day, which is simply the value for M setting C=D=1. Notice that the number of cows and the number of days are physically unrelated, however in this context they are inversely proportional. Also notice that, if the number of milked cows would depend on the day, then the amount of produced milk would not be the result of a proportion (unless we resolve to the trick of considering the average number of milked cows) but it would rather be a weighted sum that reminds us of an integral.

6. About the transitivity of proportionality

Finally, some words about transitivity. If the quantities A and B are directly (respectively, inversely) proportional, and the quantities B and C are directly (respectively, inversely) proportional, then A and C are directly proportional. In particular, there exist no three quantities such that any two of them are inversely proportional. On the other hand, one can have infinitely many quantities that are all pairwise directly proportional. Consequently, one may also have infinitely many (directly proportional) quantities that are all inversely proportional to one same quantity. As a curiosity, it could happen that A is not proportional to B and B is not proportional to C, however A is proportional to C (for example, take any two quantities A,B that are not proportional and set A=C).

We encourage teachers to reflect on all aspects of proportions and their exercises, and we hope that our observations will stimulate such reflections.