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Towards Axiomatic Foundations for Defuzzification Theory

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Abstract
The starting point of the paper presented are the well-known defuzzification procedures on the one hand and approaches to axiomatize the concept of defuzzification, on the other hand. We present a new attempt to build up an axiomatic foundation for defuzzification theory using the theory of groups and the theory of partially ordered sets, and in particular, the theory of GALOIS connections.

Keywords: Defuzzification, functional, bijective transformation, group, partially ordered set, GALOIS connection

1 Introduction

Defuzzification procedures are very important in designing fuzzy control circuits and also in investigating and applying approximate reasoning.

Therefore in literature one can find a lot of approaches to develop and to apply such procedures (algorithms), but, in general, on a more or less intuitive basis without a conceptional foundation.

Examples for this are the methods denoted by First-of-Maxima, Middle-of-Maxima, Center-of-Sums, Center-of-Area, Center-of-Largest-Area, Center-of-Gravity, and Height-Defuzzification, for instance.

About five years ago some scientists began investigations with the goal of developping a systematic theory of defuzzification and of incorporating these results [13–16] into fuzzy logic.

To this end a set of thirteen axioms for defuzzification strategies [15] was formulated and the attempt to interpret defuzzification as crisp decision under fuzzy constraints was made [13].

The role of a defuzzifier within the frame of a general fuzzy control circuit is expounded in [17].

For definiteness we repeat some more or less well-known notions.

Let A and B be arbitrary (crisp) sets. By $A \subseteqq B$ we denote that A is a maybe non-proper subset of B in the usual sense. Furthermore, $\varnothing$ is the empty set and $\mathbb{P}A$ is the power set of A, i. e. the set of all subsets of A.

The set of all real numbers r with $0 \leqq r \leqq 1$ is termed by $\langle 0, 1\rangle$. Let U be an arbitrary non-empty set called universe. A fuzzy set F on U is a mapping

$$F : U \to \langle 0, 1\rangle.$$

We put

$$\mathfrak{F}(U) =_{\text{def}} \{F | F : U \to \langle 0, 1\rangle\}$$

and call $\mathfrak{F}(U)$ the fuzzy power set of U.

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The support $\operatorname{supp} F$ of a fuzzy set F on U is the crisp set

$$\operatorname{supp} F =_{\text{def}} \{x | x \in U \wedge F(x) > 0\}.$$

Assume that $\nu : \langle 0, 1\rangle \to \langle 0, 1\rangle$ satisfies

$$\forall r (r \in \langle 0, 1\rangle \wedge \nu(r) > 0 \to r = 0).$$

The function ν can be interpreted as a kind of negation.

Furthermore assume that τ and σ are an arbitrary t-norm and s-norm, respectively. For arbitrary $F, G \in \mathfrak{F}(U)$ and $x \in U$ we put

$$\begin{aligned} F^{\nu}(x) &=_{\text{def}} \nu(F(x)) \\ (F \circledcap G)(x) &=_{\text{def}} \tau(F(x), G(x)) \\ (F \circledcup G)(x) &=_{\text{def}} \sigma(F(x), G(x)). \end{aligned}$$

Assume that U, V, and W are arbitrary non-empty sets. We fix a set $\mathbb{F}$ of functions $F : U \to V$. Then a mapping $\Phi : \mathbb{F} \to W$ is called a functional on $\mathbb{F}$ with values in W.

For our purpose we put

$$\begin{aligned} \mathbb{F} &=_{\text{def}} \mathcal{F} \quad \text{where } \mathcal{F} \subseteqq \mathfrak{F}(U) \\ V &=_{\text{def}} \langle 0, 1\rangle \\ W &=_{\text{def}} U \\ \delta &=_{\text{def}} \Phi, \end{aligned}$$

hence $\delta : \mathcal{F} \to U$.

2 The fundamental axiom for characterizing defuzzifiers

In the paper [15] the axioms were formulated under the assumption that $U = \mathbb{R}$. Under this strong assumption the authors could formulate very special axioms reflecting some properties of special defuzzification procedures. But for many applications the assumption $U = \mathbb{R}$ is too special, one should admit $U \subseteqq \mathbb{R}^n$ $(n \geqq 1)$ or even that U is a metric space with respect to a given metric μ on U.

With respect to these remarks we ask, first of all, for "natural" axioms without using a structure in U. Assume $\mathcal{F} \subseteqq \mathfrak{F}(U)$ and $\delta : \mathcal{F} \to U$.

Axiom 1
$\forall F (F \in \mathcal{F} \wedge \operatorname{supp} F \neq \emptyset \to \delta(F) \in \operatorname{supp} F)$

This axiom causes some problems. On the one hand, one can have the opinion that a "reasonable" defuzzifier must satisfy this axiom. On the other hand, there are defuzzification procedures which do not fulfil the condition expressed by axiom 1. For instance, the Center-of-Area method fails axiom 1, in general, which shows that this method is not "reasonable" with respect to some applications.

For the following investigations we assume that the defuzzifiers considered satisfy axiom 1. But we have to state that this assumption is restrictive to a certain extend as the following theorems show, in particular, theorem 7 and 8 and their proofs.

So, we define

Definition 1
δ is said to be a defuzzifier on $\mathcal{F}$ with respect to U
$=_{\text{def}}$ *δ satisfies axiom 1.*

By $\mathfrak{D}(U)$ we denote the set of all defuzzifiers on $\mathcal{F}$ with respect to U.

Remarks

1. Obviously, the concept of defuzzifiers is closely related to the well-known concept of choice function used in many branches of mathematics. For definiteness we repeat:

   Let $\mathcal{S}$ be an arbitrary system of subsets of a set S. Then a mapping $\alpha : \mathcal{S} \to S$ is said to be a choice function on $\mathcal{S}$ if and only if the following condition holds

   $$\forall X\,(X \in \mathcal{S} \wedge X \neq \varnothing \to \alpha(X) \in X).$$

   If we define $S =_{\text{def}} U \times \langle 0, 1\rangle$, then an arbitrary fuzzy set $F : U \to \langle 0, 1\rangle$ is a subset of S, hence a system $\mathcal{F} \subseteqq \mathfrak{F}(U)$ of fuzzy sets on U is a system of subsets of S. If we have a defuzzifier δ on $\mathcal{F}$ with respect to U, then by the definition

   $$\alpha(F) =_{\text{def}} \left[\delta(F), F\left(\delta(F)\right)\right]$$

   we get a choice function on $\mathcal{F}$.

   Vice versa, if we have a choice function α on $\mathcal{F}$, then by the definition

   $$\delta(F) =_{\text{def}} x \qquad (F \in \mathcal{F}, x \in U)$$

   where $\alpha(F) = [x, F(x)]$ we get a defuzzifier on $\mathcal{F}$ with respect to U.

2. We get a modification of the concept of defuzzifier by introducing a new element undefined $\notin U$ and by modifying definition 1 by adding the condition

   $$\forall F\,(F \in \mathcal{F} \wedge \operatorname{supp} F = \varnothing \to \delta(F) = \text{undefined}).$$

   With respect to applications this modification could be a better mathematical modelling of the intuitive concept of defuzzification procedure. But, with respect to the following mathematical investigations, we can not see any advantages, therefore we shall not use this modification in the following.

3. In the paper [2] the authors allow that for a fuzzy set $F \in \mathfrak{F}(U)$ the value $\delta(F)$ of δ is a subset of U, i. e. $\delta(F) \subseteqq U$. But for using defuzzifiers in fuzzy control circuits this approach is unapplicable because the input of the plant controlled must be an element, i. e. the plant can not process a set as control signals, in general.

Furthermore, it is interesting that axiom 1 implies some simple, but important facts about the value $\delta(F)$ of the defuzzifier δ for special fuzzy sets F. Let τ and σ be an arbitrary t-norm and s-norm, respectively.

Proposition 1

1. $\forall c \forall x \left(c \in \langle 0, 1\rangle \wedge c > 0 \to \delta\left(F_x^c\right) = x\right)$
2. $\forall F \left(F \in \mathcal{F} \wedge \operatorname{supp} F^{\nu} \neq \varnothing \to \delta\left(F^{\nu}\right) \notin \operatorname{supp} F\right)$
3. $\forall F \forall G \left(F, G \in \mathcal{F} \wedge \operatorname{supp}(F \cap_{\tau} G) \neq \varnothing \to \delta(F \cap_{\tau} G) \in \operatorname{supp} F \cap \operatorname{supp} G\right)$
4. $\forall F \forall G \left(F, G \in \mathcal{F} \wedge \operatorname{supp}(F \cup_{\sigma} G) \neq \varnothing \to \delta(F \cup_{\sigma} G) \in \operatorname{supp} F \cup \operatorname{supp} G\right)$

Proof
Obviously by the definitions. ■

The following axioms 2-5 will not be valid for arbitrary $\mathcal{F} \subseteqq \mathfrak{F}(U)$ and arbitrary defuzzifiers δ and norms τ, σ.

In a forthcoming paper we shall investigate how $\mathcal{F}, \delta, \tau, \sigma$ are to be restricted so that the axioms hold.

Axiom 2
$\forall c \forall F\,(c \in \langle 0,1\rangle \wedge c > 0 \wedge F \in \mathcal{F} \wedge \operatorname{supp} F \neq \emptyset \rightarrow \delta(\tau(c,F)) = \delta(F))$

Axiom 3
$\forall F\,(F \in \mathcal{F} \wedge \operatorname{supp}(F) \neq \emptyset \rightarrow \delta(F \circledcap F) = \delta(F))$

Axiom 4
$\forall c \forall F\,(c \in \langle 0,1\rangle \wedge c > 0 \wedge F \in \mathcal{F} \wedge \operatorname{supp} F \neq \emptyset \rightarrow \delta(\sigma(c,F)) = \delta(F))$

Axiom 5
$\forall F\,(F \in \mathcal{F} \wedge \operatorname{supp} F \neq \emptyset \rightarrow \delta(F \circledcup F) = \delta(F))$

3 Bijective Transformations of Fuzzy Sets and Defuzzifiers

Now, we consider and investigate the fact that there are defuzzifying procedures which are invariant with respect to certain transformations of the fuzzy sets considered.

For instance, the defuzzifying procedures listed on in chapter 1 are invariant with respect to arbitrary linear transformations. New ideas with respect to the claim of invariance are developed in [8].

In the following we generalize and systematize these observations using the theory of GALOIS connections and the theory of groups.

By $\mathfrak{B}(U)$ we denote the set of all bijections β on U. Obviously, the set $\mathfrak{B}(U)$ forms a group with respect to the concatenation of bijections from $\mathfrak{B}(U)$. For simplification this group is denoted by the same symbol $\mathfrak{B}(U)$.

Let $F \in \mathfrak{F}(U)$ and $\beta \in \mathfrak{B}(U)$. Then by $\beta(F)$ we denote the one fuzzy set on U, which is defined for every $x \in U$ by

$$(\beta(F))(x) =_{\text{def}} F(\beta(x)).$$

Fundamental for all following considerations is the concept of admitting expressed by definition 2 where δ is an arbitrary defuzzifier on $\mathcal{F}$ with respect to U and β is an arbitrary bijection on U.

Definition 2
δ *admits* β *with respect to* $\mathcal{F}$
$$\forall F\big(F \in \mathcal{F} \rightarrow \delta(\beta(F)) = \beta(\delta(F))\big).$$

As the following examples show there are defuzzifiers which admit every bijection on U and there are other defuzzifiers which do not admit every bijection on U.

Example 1

Put
$U_1 =_{\text{def}} \{1,2,3\}$

$\mathcal{F}_1 =_{\text{def}} \{F | F : U_1 \to \langle 0, 1\rangle \wedge \forall u \forall v (u, v \in U_1 \wedge F(u) = F(v) \to u = v)\}$
$\delta_1(F) =_{\text{def}} x$ where $x \in U_1$ and $F(x) = \max\{F(u) | u \in U_1\}$.

Obviously, for every $F \in \mathcal{F}_1$ the value x is uniquely defined, hence δ_1 is a defuzzifier on $\mathcal{F}_1$ with respect to U_1.

Furthermore, it is clear that for every bijection β on U_1 we have

$$\delta_1(\beta(F)) = \beta(\delta_1(F))$$

for all $F \in \mathcal{F}_1$.

Example 2

Put
$U_2 =_{\text{def}} U_1 = \{1, 2, 3\}$
$\mathcal{F}_2 =_{\text{def}} \{F, G\}$ where

$$\begin{array}{llllll} F(1) & =_{\text{def}} \frac{1}{4}, & F(2) & =_{\text{def}} \frac{1}{2}, & F(3) & =_{\text{def}} \frac{3}{4} \\ G(1) & =_{\text{def}} \frac{1}{2}, & G(2) & =_{\text{def}} \frac{1}{4}, & G(3) & =_{\text{def}} \frac{3}{4} \\ \beta(1) & =_{\text{def}} 2, & \beta(2) & =_{\text{def}} 1, & \beta(3) & =_{\text{def}} 3 \end{array}$$

Then we have $G = \beta(F)$.

Now, we define δ_2 as follows

$$\begin{array}{l} \delta_2(F) =_{\text{def}} 2 \\ \delta_2(G) =_{\text{def}} 2, \end{array}$$

hence

$$\begin{array}{l} \beta(\delta_2(F)) = \beta(2) = 1, \\ \delta_2(\beta(F)) = \delta_2(G) = 2, \end{array}$$

i. e. $\delta_2(\beta(F)) \neq \beta(\delta_2(F))$.

The examples above give the occasion to define for an arbitrary $B \subseteqq \mathfrak{B}(U)$ and an arbitrary $D \subseteqq \mathfrak{D}(U)$ where $\mathcal{F} \subseteqq \mathfrak{F}(U)$ is fixed

Definition 3

1. $\text{DEFUZZ}(B) =_{\text{def}} \left\{\delta \middle| \delta \in \mathfrak{D}(U) \wedge \forall\beta\forall F\Big(\beta \in B \wedge F \in \mathcal{F} \to \delta(\beta(F)) = \beta(\delta(F))\Big)\right\}$

2. $\text{BIJECT}(D) =_{\text{def}} \left\{\beta \middle| \beta \in \mathfrak{B}(U) \wedge \forall\delta\forall F\Big(\delta \in D \wedge F \in \mathcal{F} \to \delta(\beta(F)) = \beta(\delta(F))\Big)\right\}$.

Of course, DEFUZZ(B) and BIJECT(D) depend on $\mathcal{F}$. But, because $\mathcal{F}$ is fixed throughout the paper we omit the letter $\mathcal{F}$.

The set DEFUZZ(B) can be interpreted as the set of all general defuzzifiers which admit all bijections from B. Analogously, the set BIJECT(D) is the set of all bijections on U which are admitted by all defuzzifiers from D.

Theorem 2

The pair [DEFUZZ, BIJECT] *is a* GALOIS *connection between the posets* $[\mathbb{P}\mathfrak{B}(U), \subseteqq]$ *and* $[\mathbb{P}\mathfrak{D}(U), \subseteqq]$.

Proof

By definition of GALOIS connection (see [3], chapter 24, for instance) we have to prove for every $B \subseteqq \mathfrak{B}(U)$ and $D \subseteqq \mathfrak{D}(U)$,

$$D \subseteqq \text{DEFUZZ}(B) \leftrightarrow B \subseteqq \text{BIJECT}(D).$$

I ($\longrightarrow$)
Assume

(1) $$D \subseteqq \mathrm{DEFUZZ}(B),$$

hence by definition of DEFUZZ

(2) $$\forall\delta\big(\delta \in D \to \forall\beta\forall F\,(\beta \in B \wedge F \in \mathcal{F} \to \delta(\beta(F)) = \beta(\delta(F)))\big).$$

Assume

(3) $$\beta' \in B,$$

then we have to show

(4) $$\forall\delta\forall F\,(\delta \in D \wedge F \in \mathcal{F} \to \delta(\beta'(F)) = \beta'(\delta(F))).$$

But (4) follows immediately from (2) and (3).

II ($\longleftarrow$)
As for **I** ($\longrightarrow$). ■

Theorem 3
For every $B, B' \subseteqq \mathfrak{B}(U)$ and every $D, D' \subseteqq \mathfrak{D}(U)$,

1. $B \subseteqq B' \to \mathrm{DEFUZZ}(B') \subseteqq \mathrm{DEFUZZ}(B)$
2. $D \subseteqq D' \to \mathrm{BIJECT}(D') \subseteqq \mathrm{BIJECT}(D)$
3. $B \subseteqq \mathrm{BIJECT}(\mathrm{DEFUZZ}(B))$
4. $D \subseteqq \mathrm{DEFUZZ}(\mathrm{BIJECT}(D))$
5. $\mathrm{DEFUZZ}(B) = \mathrm{DEFUZZ}(\mathrm{BIJECT}(\mathrm{DEFUZZ}(B)))$
6. $\mathrm{BIJECT}(D) = \mathrm{BIJECT}(\mathrm{DEFUZZ}(\mathrm{BIJECT}(D)))$

Proof
This theorem follows from theorem 2 within the framework of the general theory of GALOIS connections. ■

Corollary 4
1. BIJECT *is a bijection from the set* $\{\mathrm{DEFUZZ}(B) \mid B \subseteqq \mathfrak{B}(U)\}$ *onto the set* $\{\mathrm{BIJECT}(D) \mid D \subseteqq \mathfrak{D}(U)\}$
2. DEFUZZ *is the inversion of the mapping* BIJECT.

From applications we know that the set of all linear transformations (which are admitted by the defuzzifying procedures listed on in chapter 1) forms a group. Furthermore, we can state that this group is even commutative.

In the following we shall discuss these facts within the framework of our general approach.

For definiteness for every $\beta, \beta' \in \mathfrak{B}(U)$ by $\beta\beta'$ and β^{-1} we denote the product of β, β' and the inversion of β, respectively, defined by

$$(\beta\beta')(u) =_{\mathrm{def}} \beta'(\beta(u)), \qquad u \in U \qquad \beta^{-1}(u) = v \;=_{\mathrm{def}}\; u = \beta(v), \qquad u, v \in U.$$

Theorem 5
For every $D \subseteqq \mathfrak{D}(U)$, if

$$\forall\beta\forall F\big(\beta \in \mathrm{BIJECT}(D) \wedge F \in \mathcal{F} \to \beta(F) \in \mathcal{F} \wedge \beta^{-1}(F) \in \mathcal{F}\big),$$

then BIJECT(D) *is a group with respect to the product of bijections, hence a subgroup of* $\mathfrak{B}(U)$.

Proof

We have to show

1. $\forall\beta'\forall\beta\left(\beta', \beta \in \text{BIJECT}(D) \rightarrow (\beta\beta') \in \text{BIJECT}(D)\right)$
2. $\forall\beta\left(\beta \in \text{BIJECT}(D) \rightarrow \beta^{-1} \in \text{BIJECT}(D)\right)$

ad 1

Assume

(1) $$\forall\delta\left(\delta \in D \wedge F \in \mathcal{F} \rightarrow \delta(\beta(F)) = \beta(\delta(F))\right)$$

and

(2) $$\forall\delta\left(\delta \in D \wedge F \in \mathcal{F} \rightarrow \delta(\beta'(F)) = \beta'(\delta(F))\right)$$

We have to show

(3) $$\forall\delta\left(\delta \in D \wedge F \in \mathcal{F} \rightarrow \delta((\beta\beta')(F)) = (\beta\beta')(\delta(F))\right).$$

Now, by (1) we get

(4) $$\begin{aligned} &(\beta\beta')(\delta(F)) \\ &\quad = \beta'(\beta(\delta(F))) \\ &\quad = \beta'(\delta(\beta(F))). \end{aligned}$$

Furthermore, by assumption we have

(5) $$\forall F\,(F \in \mathcal{F} \rightarrow \beta(F) \in \mathcal{F}),$$

hence by (2) we obtain

(6) $$\begin{aligned} \beta'(\delta(\beta(F))) &= \delta(\beta'(\beta(F))) \\ &= \delta((\beta\beta')(F)), \end{aligned}$$

hence because of (4) and (6), (3) holds.

ad 2

Assume

(7) $$\forall\delta\forall F\left(\delta \in \text{BIJECT}(D) \wedge F \in \mathcal{F} \rightarrow \delta(\beta(F)) = \beta(\delta(F))\right).$$

We have to show

(8) $$\forall\delta\forall F\left(\delta \in \text{BIJECT}(D) \wedge F \in \mathcal{F} \rightarrow \delta\left(\beta^{-1}(F)\right) = \beta^{-1}(\delta(F))\right).$$

Because of assumption we have

$$\forall F\left(F \in \mathcal{F} \rightarrow \beta^{-1}(F) \in \mathcal{F}\right),$$

hence from (7) we get

(9) $$\delta\left(\beta\left(\beta^{-1}(F)\right)\right) = \beta\left(\delta\left(\beta^{-1}(F)\right)\right),$$

hence by $\beta\beta^{-1} = \varepsilon$ where ε denotes the identical bijection

(10) $$\delta(F) = \beta\left(\delta\left(\beta^{-1}(F)\right)\right),$$

hence

(11) $$\begin{aligned} \beta^{-1}(\delta(F)) &= \beta^{-1}\left(\beta\left(\delta\left(\beta^{-1}(F)\right)\right)\right) \\ &= \delta\left(\beta^{-1}(F)\right). \end{aligned}$$

■

Corollary 6
BIJECT *is an injection from the set* $\{\mathrm{DEFUZZ}(B) | B \subseteqq \mathfrak{B}(U)\}$ *into the set* $\{\mathfrak{G} | \mathfrak{G}$ *is a subgroup of* $\mathfrak{B}(U)\}$.

Problem To characterize the subgroups $\mathfrak{G}$ of $\mathfrak{B}(U)$ which have the form $\mathfrak{G} = \mathrm{BIJECT}(D)$ where $D \subseteqq \mathfrak{D}(U)$.

Now, for arbitrary $\beta \in \mathfrak{B}(U)$ and arbitrary $\delta \in \mathfrak{D}(U)$ we define a left-product $\beta \circ \delta$ and a right-product $\delta \circ \beta$ as follows where $F \in \mathfrak{F}(U)$.

Definition 4
1. $(\beta \circ \delta)(F) =_{\text{def}} \beta(\delta(F))$
2. $(\delta \circ \beta)(F) =_{\text{def}} \delta(\beta(F))$.

Now, we are going to formulate sufficient conditions for the fixed set $\mathcal{F} \subseteqq \mathfrak{F}(U)$ and for the given set $D \subseteqq \mathfrak{D}(U)$ of defuzzifiers on U such that the group BIJECT(D) is even commutative.

Theorem 7
For every $D \subseteqq \mathfrak{D}(U)$,
If 1. $\forall\beta\forall F(\beta \in \mathrm{BIJECT}(D) \wedge F \in \mathcal{F} \rightarrow \beta(F) \in \mathcal{F} \wedge \beta^{-1}(F) \in \mathcal{F})$
2. $\{\delta(F) | \delta \in D \wedge F \in \mathcal{F}\} = U$ *and*
3. $\forall\beta\forall\delta(\beta \in \mathrm{BIJECT}(D) \wedge \delta \in D \rightarrow \beta \circ \delta \in D)$

then the group BIJECT(D) *is even commutative.*

Proof
Assume

(1) $$\beta, \beta' \in \mathrm{BIJECT}(D).$$

By theorem 5 it is sufficient to show

(2) $$\beta \circ \beta' = \beta' \circ \beta,$$

hence by definition of the product of bijections we have to prove

(3) $$\forall x(x \in U \rightarrow \beta'(\beta(x)) = \beta(\beta'(x))).$$

By $U \neq \varnothing$ and assumption 2 there is a δ such that

(4) $$\delta \in D.$$

Then by (1) and assumption 3 we get

(5) $$\beta' \circ \delta \in D,$$

hence by definition of BIJECT(D) and (1) we obtain for every $F \in \mathcal{F}$

(6) $$\beta((\beta' \circ \delta)(F)) = (\beta' \circ \delta)(\beta(F)),$$

hence by definition of $\beta' \circ \delta$,

(7) $$\beta(\beta'(\delta(F))) = \beta'(\delta(\beta(F))).$$

Furthermore, by (1), (4) and definition of BIJECT(D) for $F \in \mathcal{F}$ we have

(8) $$\delta(\beta(F)) = \beta(\delta(F)),$$

hence by (7) and (8) we get

(9) $$\beta(\beta'(\delta(F))) = \beta'(\beta(\delta(F)))$$

for every $\delta \in D$ and $F \in \mathcal{F}$.

Because of assumption 2 we obtain (3). ■

The following theorem expresses a certain "inversion" of theorem 7.

Theorem 8

If 1. *B is a commutative subgroup of the group $\mathfrak{B}(U)$*
2. $\forall\beta\forall F(\beta \in B \wedge F \in \mathcal{F} \rightarrow \beta(F) \in \mathcal{F})$
3. $\forall\beta\forall F(\beta \in B \wedge F \in \mathcal{F} \rightarrow \operatorname{supp}\beta(F) = \operatorname{supp}F)$

then

$$\forall\beta\forall\delta(\beta \in B \wedge \delta \in \text{DEFUZZ}(B) \rightarrow \beta \circ \delta, \delta \circ \beta \in \text{DEFUZZ}(B)).$$

Proof

Assume

(1) $$\beta \in B$$

and

(2) $$\delta \in \text{DEFUZZ}(B).$$

From (2) by definition of DEFUZZ(B) we get

(3) $$\forall F(F \in \mathcal{F} \wedge \operatorname{supp}F \neq \emptyset \rightarrow \delta(F) \in \operatorname{supp}F)$$

and

(4) $$\forall\beta'\forall F'(\beta' \in B \wedge F' \in \mathcal{F} \rightarrow \delta(\beta'(F')) = \beta'(\delta(F'))).$$

ad 1 $\beta \circ \delta \in \text{DEFUZZ}(B)$

By definition of $\beta \circ \delta$ and DEFUZZ(B) we have to prove

(5) $$\beta \circ \delta \in \mathfrak{D}(U)$$

and

(6) $$\forall\beta''\forall F''(\beta'' \in B \wedge F'' \in \mathcal{F} \rightarrow (\beta \circ \delta)(\beta''(F'')) = \beta''((\beta \circ \delta)(F''))).$$

We show (5). By definition of $\mathfrak{D}(U)$ it is sufficient to prove

(7) $$\forall F(F \in \mathcal{F} \wedge \operatorname{supp}F \neq \emptyset \rightarrow (\beta \circ \delta)(F) \in \operatorname{supp}F).$$

From $F \in \mathcal{F}$, $\operatorname{supp}F \neq \emptyset$, definition of $\beta \circ \delta$, (1), and (4) we obtain

(8) $$(\beta \circ \delta)(F) = \beta(\delta(F)) = \delta(\beta(F)).$$

From (1), $F \in \mathcal{F}$, and assumption 3 we get

(9) $$\operatorname{supp}\beta(F) = \operatorname{supp}F,$$

hence by $\operatorname{supp}F \neq \emptyset$ we have

(10) $$\operatorname{supp}\beta(F) \neq \emptyset.$$

Furthermore, (1), $F \in \mathcal{F}$, and assumption 2 imply

(11) $$\beta(F) \in \mathcal{F},$$

hence

(12) $$\delta(\beta(F)) \in \operatorname{supp}\beta(F),$$

hence by (9)

(13) $$\delta(\beta(F)) \in \operatorname{supp}F,$$

hence by (8) we get (7).

Now, we show (6). Assume

(14) $$\beta'' \in B$$

and

(15) $$F'' \in \mathcal{F}.$$

By definition of $\beta \circ \delta$ it is sufficient to show

(16) $$\beta\big(\delta(\beta''(F''))\big) = \beta''\big(\beta(\delta(F''))\big).$$

From (13), (14) by (4) for $\beta' =_{\text{def}} \beta''$ and $F' =_{\text{def}} F''$ we get

(17) $$\delta(\beta''(F'')) = \beta''(\delta(F'')),$$

hence

(18) $$\beta\big(\delta(\beta''(F''))\big) = \beta\big(\beta''(\delta(F''))\big).$$

Because B is commutative, we get

(19) $$\beta\big(\beta''(\delta(F''))\big) = \beta''\big(\beta(\delta(F''))\big),$$

hence (17) and (18) imply (15).

ad 2 $\delta \circ \beta \in \text{DEFUZZ}(B)$

By definition of $\delta \circ \beta$ and $\text{DEFUZZ}(B)$ we have to prove

(20) $$\delta \circ \beta \in \mathfrak{D}(U)$$

and

(21) $$\forall\beta''\forall F''\big(\beta'' \in B \wedge F'' \in \mathcal{F} \rightarrow (\delta \circ \beta)(\beta''(F'')) = \beta''((\delta \circ \beta)(F''))\big).$$

We show (20). By definition of $\mathfrak{D}(U)$ it is sufficient to prove

(22) $$\forall F\,(F \in \mathcal{F} \wedge \operatorname{supp}F \neq \varnothing \rightarrow (\delta \circ \beta)(F) \in \operatorname{supp}F).$$

By definition of $\delta \circ \beta$ we get

(23) $$(\delta \circ \beta)(F) = \delta(\beta(F)).$$

From (1), $F \in \mathcal{F}$ and assumption 3 we obtain

(24) $$\operatorname{supp}\beta(F) = \operatorname{supp}F,$$

hence by $\operatorname{supp} F \neq \emptyset$ we get

$$\operatorname{supp} \beta(F) \neq \emptyset. \tag{25}$$

Furthermore, (1), $F \in \mathcal{F}$ and assumption 2 imply

$$\beta(F) \in \mathcal{F}, \tag{26}$$

hence

$$\delta(\beta(F)) \in \operatorname{supp} \beta(F), \tag{27}$$

hence by (24)

$$\delta(\beta(F)) \in \operatorname{supp} F, \tag{28}$$

hence by (23) we get (22).

Now, we show (21). We assume (14) and (15). By definition of $\beta \circ \delta$ it is sufficient to show

$$\delta\big(\beta(\beta''(F''))\big) = \beta''\big(\delta(\beta(F''))\big). \tag{29}$$

From (14) and (15) by (4) for $\beta' =_{\text{def}} \beta$ and $F' =_{\text{def}} F''$ we obtain

$$\delta(\beta(F'')) = \beta(\delta(F'')), \tag{30}$$

hence

$$\beta''\big(\delta(\beta(F''))\big) = \beta''\big(\beta(\delta(F''))\big). \tag{31}$$

Because B is commutative, we have

$$\beta''\big(\beta(\delta(F''))\big) = \beta\big(\beta''(\delta(F''))\big). \tag{32}$$

Because of (1) and (14), i. e. $\beta, \beta'' \in B$, for the group B we obtain $\beta \circ \beta'' \in B$, consequently from (4) for $\beta' =_{\text{def}} \beta'' \circ \beta$ and $F' =_{\text{def}} F''$ we obtain

$$\beta\big(\beta''(\delta(F''))\big) = \delta\Big(\beta\big(\beta''(\delta(F''))\big)\Big), \tag{33}$$

hence (31), (32), and (33) imply (29). ■

4 Conclusions

The concepts and results of chapter 3 can be interpreted as first steps to build up a (new) algebraic theory of defuzzification procedures. Whether this way will be successful can not be estimated on the basis of the paper presented. Further investigations are necessary, in particular, investigations with respect to the well-known defuzzification procedures listed on in chapter 1 and the approaches published in [13–16] and also in other papers. In forthcoming papers we shall study this area of problems.

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References

[1] *FUZZ-IEEE '97 — Sixth IEEE International Conference on Fuzzy Systems*, Barcelona, Spain, July 1–5, 1997.

[2] D. BOIXADER, J. JACAS and J. RECASENS. *Similarity-based approach to defuzzification*. In: *FUZZ-IEEE '97 — Sixth IEEE International Conference on Fuzzy Systems* [1], pages 761–764.

[3] PETER BURMEISTER. *A Model Theoretic Oriented Approach to Partial Algebras*. Akademie-Verlag, Berlin, 1986.

[4] D. DRIANKOV, H. HELLENDOORN and M. REINFRANK. *An Introduction to Fuzzy Control*. Springer-Verlag, 1993.

[5] D. P. FILEV and R. R. YAGER. *A Generalized Defuzzification Method via Bad Distributions*. International Journal of Intelligent Systems **6**, 687–697, 1991.

[6] H. HELLENDOORN and C. THOMAS. *On Defuzzification in Fuzzy Controllers*. Journal of Intelligent and Fuzzy Systems 1993.

[7] H. KIENDL. *Fuzzy-Regler mit Hyperinferenz- und Hyperdefuzzifizierungsstrategien*. In: BERND REUSCH (editor), *Fuzzy Logic — Theorie und Praxis. 3. Dortmunder Fuzzy-Tage*, Dortmund, 7.–9. June 1993. Springer-Verlag.

[8] H. KIENDL. *Non-translation-invariant defuzzification*. In: *FUZZ-IEEE '97 — Sixth IEEE International Conference on Fuzzy Systems* [1], pages 737–742.

[9] S. MABUCHI. *A proposal for a defuzzification strategy by the concept of sensitive analysis*. Fuzzy Sets and Systems**55**, 1–14, 1993.

[10] N. PFLUGER, J. YEN and R. LANGARI. *A Defuzzification Strategy for a Fuzzy Logic Controller Employing Prohibitive Information in Command Formulation*. In: *Proceedings of the First IEEE International Conference on Fuzzy Systems*, pages 717–723, San Diego, California, 1992.

[11] L. RONDEAU, R. RUELAS, L. LEVRAT and M. LAMOTTE. *A defuzzification method respecting the fuzzification*. Fuzzy Sets and Systems**86**, 311–320, 1997.

[12] S. ROYCHOWDJURY, B. H. WANG and S. K. AHN. *Radial Defuzzification Method*. In: *FUZZ-IEEE '94 — Third IEEE International Conference on Fuzzy Systems*, volume 2, pages 1153–1158, Orlando, Florida, June 26–July 2, 1994.

[13] T. A. RUNKLER and M. GLESNER. *Defuzzification As Crisp Decision Under Fuzzy Constraints. New Aspects of Theory and Improved Defuzzification Algorithms*. In: *Fuzzy-Systeme '93. Management unsicherer Informationen. Workshop. Proceedings*, pages 156–164, Braunschweig, Germany, October 20–22 1993.

[14] T. A. RUNKLER and M. GLESNER. *Defuzzification with Improved Static and Dynamic Behavior: Extended Center of Area*. In: *EUFIT '93 — First European Congress on Fuzzy and Intelligent Technologies*, volume 2, pages 845–851, Aachen, Germany, September 7–10, 1993.

[15] T. A. RUNKLER and M. GLESNER. *A Set of Axioms for Defuzzifications Strategies. Towards a Theory of Rational Defuzzification Operators*. In: *Proceedings of the Second IEEE International Conference on Fuzzy Systems*, pages 1161–1166, San Francisco, California, March 28–April 1, 1993.

[16] THOMAS A. RUNKLER. *Extended Defuzzification Methods and their Properties*. In: *FUZZ-IEEE '96 — Fifth IEEE International Conference on Fuzzy Systems*, volume 1, pages 694–700, New Orleans, USA, September 8–11, 1996.

[17] HELMUT THIELE. *Investigating Approximate Reasoning and Fuzzy Control by Concepts of Functional Analysis*. In: *First International Conference on Conventional and Knowledge-Based Intelligent Electronic Systems. Proceedings*, volume 2, pages 493–500, Adelaide, Australia, May 21–23, 1997.

[18] R. R. YAGER and D. FILEV. *On the issue of defuzzification and selection based on a fuzzy set*. Fuzzy Sets and Systems**55**, 255–271, 1993.

[19] R. R. YAGER and D. P. FILEV. *On the issue of defuzzification and selection from a fuzzy set*. Technical Report MII-1201, Machine Intelligence Institute, Iona College, 1991.

[20] R. R. YAGER and D. P. FILEV. *Constrained defuzzification*. In: *Proceedings: Fifth International Fuzzy Systems Association World Congress '93*, volume I, pages 1167–1170, Seoul, Korea, July 4–9 1993.

[21] R. R. YAGER and D. P. FILEV. *SLIDE: A Simple Adaptive Defuzzification Method*. IEEE Transactions On Fuzzy Systems **1** (1), 69–78, February 1993.