A note on a multivariate analogue of the process capability index C_p

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Abstract

A simple method is given to calculate the multivariate process capability index C_p^* as defined by Taam et al. (1993) and discussed by Kotz & Johnson (1993). It is shown that using this index is equivalent to using the smallest univariate C_p-value to determine the capability of a process.

The index MVC_p^*

Analogously to univariate process capability indices also multivariate capability indices relate the allowed process spread, i.e. some measure of the specification width, to the actual process spread, i.e. some measure of the process variation. The specification for the ith quality variable X_i is usually given by the triple of lower specification limit LSL_i, target value T_i and upper specification limit USL_i. For the p quality characteristics a multivariate normal distribution with mean vector μ and positive definite covariance matrix $\Sigma = (\sigma_{ij})_{i,j=1..p}$ is commonly assumed. The natural generalization of the univariate capability index C_p would be to relate the volume of the specification region to the volume of the 99.73% region of the vector of quality characteristics. This multivariate C_p-Index, here denoted by MVC_p is given by

$$MVC_p = \frac{\prod_{i=1}^{p}(USL_i - LSL_i)}{vol\left((x-\mu)' \Sigma^{-1}(x-\mu) \le \chi^2_{p,0.9973}\right)}$$

where $\chi^2_{p,0.9973}$ denotes the 99.73%-quantile of the χ^2-distribution with p degrees of freedom. Note, that the volume of the ellipsoid does not depend on the value of μ. However, only if the ellipsoid is centred on T, i.e. if μ equals T, the ratio of the two volumina measures the amount of the process distribution which is contained in the specification region (c.f. Kotz & Johnson, 1993). Thus, usually μ = T is postulated and will also be assumed for the remainder of this paper.

A drawback of this index as a generalization of the univariate C_p is that its value is not 1 if indeed 99.73% of the distribution is inside the specification. To overcome this, Taam et al. (1993) propose to use a modified specification region R*, which is the greatest volume ellipsoid with generating matrix Σ entirely contained inside the specification region. The resulting index MVC_p^* is thus given by

$$MVC_p^* = \frac{vol\left((x-\mu)' \Sigma^{-1}(x-\mu) \le K^2\right)}{vol\left((x-\mu)' \Sigma^{-1}(x-\mu) \le \chi^2_{p,0.9973}\right)},$$

where K^2 is chosen so that the resulting ellipsoid is the greatest volume ellipsoid inside the specification. Recalling that the volume of the ellipsoid in the numerator is given by $(\pi K^2)^{p/2} |\Sigma|^{1/2} / \Gamma(p/2+1)$ the index may be expressed simply as

$$MVC_p^* = \left(\frac{K^2}{\chi^2_{p,0.9973}}\right)^{p/2} = \left(\frac{K}{\chi_{p,0.9973}}\right)^{p}.$$

This leaves the problem of finding K. As is shown in the Appendix, for any p, K may be determined as

$$K = \min_{i=1,..,p}\left(\frac{USL_i - T_i}{\sigma_i}, \frac{T_i - LSL_i}{\sigma_i}\right).$$

This implies, that the value of MVC_p^* depends only on the quality characteristics with the greatest variance in relation to the corresponding specification width:

$$MVC_p^* = \min_{i=1,..,p}\left(\left[\frac{USL_i - T_i}{\chi_{p,0.9973}\,\sigma_i}\right]^p, \left[\frac{T_i - LSL_i}{\chi_{p,0.9973}\,\sigma_i}\right]^p\right).$$

In the case of symmetric specification, i.e. $USL_i - T_i = T_i - LSL_i, \forall i$, further holds

$$MVC_p^* = \min_{i=1,..,p}\left\{\left(\frac{3}{\chi^2_{p,0.9973}} C_{p_i}\right)^p\right\},$$

where C_{p_i} is given as $C_{p_i} = (USL_i - LSL_i)/6\sigma_i$. Generalizing C_{p_i} to $C_{p_i} = \min\{(USL_i - T_i)/3\sigma_i, (T - LSL_i)/3\sigma_i\}$ yields an analogous result for asymmetric specification intervals .

Thus, calculation of MVC_p^* is equivalent to calculation of the smallest univariate C_p-value. With the introduction of the $\chi^2_{p,0.9973}$-quantile there is some correction of the index value for the number of quality characteristics under consideration. However, the value of the index remains the same, regardless of the size of the covariances between different quality characteristics. Thus, if the sense behind using multivariate methods is to also introduce a dependency of process capability of the quality characteristics on their covariances σ_{ij}, MVC_p^* ignores important information.

Conclusion

It has been shown that the multivariate process capability index MVC_p^* is essentially determined by the minimum of the univariate C_p-values.

Appendix

To determine the value of K which gives the greatest volume ellipsoid, we look for the tangent of the ellipsoid with the mth specification limit, $1 \le m \le p$. For the pth variable this tangent is easy to find replacing this variable by its specification limit. To generalize this for arbitrary m, we employ a permutation matrix which exchanges the mth for the pth dimension and show that the result holds for all m.

Without loss of generality, let T = 0, $x_{(m)} := (x_1, \dots, x_{m-1}, x_p, x_{m+1}, \dots, x_{p-1})'$ and $\Sigma^{-1} := A = \begin{pmatrix} a_{11} & \cdots & a_{1p} \\ \vdots & & \vdots \\ a_{p1} & \cdots & a_{pp} \end{pmatrix}$. Naturally, $a_{ij} = a_{ji}$, i.e. Σ^{-1} is a symmetric matrix. Let further $\tilde{a}_m := \left(a_{m,1}, \dots, a_{m,m-1}, a_{m,p}, a_{m,m+1}, \dots, a_{m,p-1}\right)'$ and

$$\tilde{A}_m = \begin{pmatrix} a_{11} & \cdots & a_{1,m-1} & a_{1,p} & a_{1,m+1} & \cdots & a_{1,p-1} \\ \vdots & \ddots & \vdots & \vdots & \vdots & & \cdots \\ a_{m-1,1} & \cdots & a_{m-1,m-1} & a_{m-1,p} & a_{m-1,m+1} & \cdots & a_{m-1,p-1} \\ a_{p,1} & \cdots & a_{p,m-1} & a_{p,p} & a_{p,m+1} & \cdots & a_{p,p-1} \\ a_{m+1,1} & \cdots & a_{m+1,m-1} & a_{m+1,p} & a_{m+1,m+1} & \cdots & a_{m+1,p-1} \\ \vdots & & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{p-1,1} & \cdots & a_{p-1,m-1} & a_{p-1,p} & a_{p-1,m+1} & \cdots & a_{p-1,p-1} \end{pmatrix},$$

i.e. $\tilde{A}_m$ is the left upper part of the matrix A of dimension $(p - 1, p - 1)$, where the mth row and mth column of $A = \Sigma^{-1}$ have been replaced by the pth row and pth column.

We determine the tangent of the ellipsoid with the mth specification limit. The quadratic form of interest is given by $Q(x_1, \dots, x_p) := (x_1, \dots, x_p)' A (x_1, \dots, x_p)$. Then

$$\begin{aligned} Q(x_1, \dots, LSL_m, \dots, x_p) &= (x_1, \dots, LSL_m, \dots, x_p)' A (x_1, \dots, LSL_m, \dots, x_p) \\ &= x_{(m)}' \tilde{A}_m x_{(m)} + 2 LSL_m \tilde{a}_m' x_{(m)} + a_{mm} LSL_m^2 \end{aligned}$$

and $$\frac{\partial Q(x_1, \dots, LSL_m, \dots, x_p)}{\partial x_{(m)}} = 2\tilde{A}_m x_{(m)} + 2 LSL_m \tilde{a}_m .$$

Setting the first derivative equal to null, leads to $x_{(m)}^{\min} = -LSL_m \tilde{A}_m^{-1}\tilde{a}_m$, if the inverse exists. The value of the quadratic form at this point is

$$Q(x_1^{\min},\ldots,LSL_m,\ldots,x_p^{\min}) = LSL_m^2\ \tilde{a}_m'\ \tilde{A}_m^{-1}\ \tilde{A}_m\ \tilde{A}_m^{-1}\ \tilde{a}_m - 2LSL_m^2\tilde{a}_m'\ \tilde{A}_m^{-1}\tilde{a}_m + a_{mm}LSL_m^2$$
$$= LSL_m^2\left(a_{mm} - \tilde{a}_m'\ \tilde{A}_m^{-1}\ \tilde{a}_m\right)$$

From Mardia, Kent und Bibby (1995), p. 459, we have the following result:

If $B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$ and $B^{-1} = \begin{pmatrix} B^{11} & B^{12} \\ B^{21} & B^{22} \end{pmatrix}$, B^{22} is given by $B^{22} = \left(B_{22} - B_{21}B_{11}^{-1}B_{21}\right)^{-1}$.

Setting $B = \begin{pmatrix} \tilde{A}_m & \tilde{a}_m \\ \tilde{a}_m' & a_{mm} \end{pmatrix} = C_m\Sigma^{-1}C_m = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$ implies $B^{-1} = C_m\Sigma C_m = \begin{pmatrix} B^{11} & B^{12} \\ B^{21} & B^{22} \end{pmatrix}$,

where pre- and post-multiplication with the idempotent matrix C_m replaces the m^th row and column of Σ^{-1} (or of Σ) with its p^th row and column, respectively.

C_m is given by $C_m := \left(\begin{array}{c:c:c} I_{m-1} & \begin{matrix} 0 \\ \vdots \end{matrix} & 0_{m-1} \\ \hdashline 0 \quad \cdots & 0 & \cdots \quad 0 \quad 1 \\ \hdashline 0_{p-m} & \begin{matrix} \vdots \\ 0 \\ 1 \end{matrix} & \begin{matrix} I_{p-m-1} & \begin{matrix} 0 \\ \vdots \end{matrix} \\ 0 \quad \ldots & 0 \end{matrix} \end{array}\right)$, where I_k and 0_k denote the identity matrix and null matrix of dimension k.

Applying the theorem by Mardia, Kent and Bibby and noting the relation between A and Σ^{-1} results $a_{mm} - \tilde{a}_m'\ \tilde{A}_m^{-1}\ \tilde{a}_m = \left(B^{22}\right)^{-1} = 1/\sigma_m^2$, and thus

$$Q(x_1^{\min},\ldots,LSL_m,\ldots,x_p^{\min}) = \frac{LSL_m^2}{\sigma_m^2}.$$

Analogous results hold for $Q(x_1^{\min},\ldots,USL_m,\ldots,x_p^{\min})$. In general we have

$$K^2 = \min_{i=1,..,p}\left(\frac{\left(USL_i - T_i\right)^2}{\sigma_i^2}, \frac{\left(T_i - LSL_i\right)^2}{\sigma_i^2}\right).$$

References

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