A Confidence Interval for Combined Univariate Economic Forecasts

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Abstract

Research in combining of economic forecasts made by several research institutes on the same economic variable has focused on estimation, hoping that the combined forecast will be improved by taking into account the expert opinions of the institutes. We provide a confidence interval on the combined forecast. A scoring of the individual institutes is proposed using their historical performance in forecasting.

Key words: Heterogeneous variances; Expert opinion; Confidence interval.

1 Introduction

Consider the situation where one has several forecasts on the same quantity, e. g. several economic research institutes forecast an important economic variable. Interest is in combining the individual forecasts to improve the accuracy of the forecast. Since by way of combining several forecasts one takes into account several expert opinions, one expects a better performance of the combined forecast. However, it is rarely observed that the simply average of the individual forecasts is beaten by more sophisticated methods. Klapper (1998, 1999, 2000) discusses rank–based techniques for combining forecasts. There is little known about confidence intervals on the combined forecast in this area.

In this paper, we give several confidence intervals on the combined forecast which are of approximate nature. The confidence intervals are derived using analysis of variance as a main tool. Some simulation results on the properties of the confidence intervals are provided. Finally, the confidence intervals are applied on German economic data.

2 A statistical model for combining forecasts

We consider the following model

$$y_i \sim N(\mu, \alpha_i), \tag{2.1}$$

where y_i denotes the forecast of the i–th institute, and μ represents the true unknown quantity, $i = 1, \ldots, K$, $K > 2$. Note that we assume that the variances α_i are heteroscedastic. Confidence intervals and tests are discussed in a related heteroscedastic ANOVA model by Hartung and Argaç (2002a, 2002b) and Hartung and Knapp (2000).

We estimate the mean μ by a weighted average of the individual forecasts y_i,

$$y = \sum_{i=1}^{K} b_i^2 \cdot y_i, \tag{2.2}$$

where b_i^2 denotes the weight which is given to the i–th institute by some scoring process. We assume that the weights are normed

$$\sum_{i=1}^{K} b_i^2 = 1, \tag{2.3}$$

and we also assume that $b_i^2 < 1/2$, that is we exclude the possibility that one particular institute dominates the rest. Consider now the following quadratic form, see also Hartung, Böckenhoff and Knapp (2002),

$$u_{ib}^2 = b_i^2 \cdot \left(y_i - \sum_{j=1}^{K} b_j^2 \cdot y_j \right)^2, \tag{2.4}$$

which can be interpreted as a weighted quadratic deviation of each individual forecast from the weighted average of the individual forecasts of each institute. Let $\beta_i = b_i^2 \cdot \alpha_i$, and one can derive explicitly the important moments of the quadratic form u_{ib}^2,

$$E(u_{ib}^2) = (1 - 2 \cdot b_i^2) \cdot \beta_i + b_i^2 \cdot \sum_{j=1}^{K} b_j^2 \cdot \beta_j =: e_i(b, \beta), \tag{2.5}$$

$$var(u_{ib}^2) = 2 \cdot e_i(b, \beta)^2, \tag{2.6}$$

$$Cov(u_{ib}^2, u_{jb}^2) = 2 \cdot b_i^2 \cdot b_j^2 \cdot \left(\sum_{k=1}^{K} b_k^2 \cdot \beta_k - \beta_i - \beta_j \right) =: 2 \cdot e_{ij}(b, \beta)^2. \tag{2.7}$$

We need to estimate the variance of the combined forecast and of each individual forecast, and for this purpose we use quadratic functions of the individual forecasts as estimators. The variance of the combined forecast is estimated with an unbiased positive (PSD–MINQUE) variance estimator which is given by, see also Hartung, Böckenhoff and Knapp (2002),

$$\widehat{var}(y)_{psd} = \frac{1}{1+\sum_{k=1}^{K}\dfrac{b_k^4}{1-2b_k^2}} \cdot \sum_{i=1}^{K} \frac{b_i^2}{1-2b_i^2} \cdot u_{ib}^2 \tag{2.8}$$

$$= \sum_{i=1}^{K} d_i \cdot u_{ib}^2. \tag{2.9}$$

The estimator of the variance of each forecast is given by a positive minimum biased variance estimator (PSD–MINQMBE) given by

$$\widehat{\alpha}_i = \frac{\sum_{j=1}^{K} u_{jb}^2 + \sum_{j=1}^{K} d_j \cdot u_{jb}^2}{\sum_{j=1}^{K} b_j^2 \cdot \widetilde{\alpha}_j} \cdot \widetilde{\alpha}_i, \tag{2.10}$$

where

$$\widetilde{\alpha}_i = \frac{1}{b_i^2} \cdot \frac{(1-b_i^2)^2}{(1-b_i^2)^4 + b_i^4 \sum_{j\neq i} b_j^4} \cdot u_{ib}^2. \tag{2.11}$$

Note here also that both estimators are given explicitly, see also Hartung, Böckenhoff and Knapp (2002) for a detailed derivation. Under suitable conditions these estimators are consistent, exist always and are unique. Surprisingly, the variance of the combined forecast can be unbiasedly estimated without any information about the variance of the individual forecasts.

We will construct the confidence interval using a pivotal quantity, and for this purpose we have to determine the distribution of the variance estimators. We will approximate the distribution of the variance estimators by a suitable χ^2–distributions using moment matching, see Satterthwaite (1946) and Patnaik (1949). This leads to

$$\nu \cdot \sum_{i=1}^{K} d_i \cdot u_{ib}^2 / E(\sum_{i=1}^{K} d_i \cdot u_{ib}^2) \sim \chi_\nu^2, \tag{2.12}$$

where

$$\nu = 2\,\frac{[E(\sum_{i=1}^{K} d_i \cdot u_{ib}^2)]^2}{var(\sum_{i=1}^{K} d_i \cdot u_{ib}^2)}. \tag{2.13}$$

Now, this gives us the following pivot (nominator and denominator are independent)

$$\frac{y-\mu}{\sqrt{\sum_{i=1}^{K} d_i \cdot u_{ib}^2}} \sim t_\nu, \tag{2.14}$$

where

$$\nu = \frac{\left\{\sum_{i=1}^{K} d_i \cdot e_i(b,\beta)\right\}^2}{\sum_{i=1}^{K} d_i^2 \cdot e_i(b,\beta)^2 + \sum_{\substack{i=1\\ i\neq j}}^{K} \sum_{j=1}^{K} d_i \cdot d_j \cdot e_{ij}(b,\beta)^2}. \tag{2.15}$$

In practice, the unknown quantities have to be replaced by estimators, and here we use $\widehat{\beta}_i = b_i^2 \cdot \widehat{\alpha}_i$.

Now, we are in the position to derive the confidence intervals which are of course of approximate nature only.

$$K_1: \quad y \mp t_{\nu;1-\alpha/2} \cdot \sqrt{\sum_{i=1}^{K} d_i \cdot u_{ib}^2} \tag{2.16}$$

$$K_2: \quad y \mp u_{1-\alpha/2} \cdot \sqrt{\sum_{i=1}^{K} d_i \cdot u_{ib}^2} \tag{2.17}$$

$$K_3: \quad y \mp t_{K-1;1-\alpha/2} \cdot \sqrt{\sum_{i=1}^{K} d_i \cdot u_{ib}^2} \tag{2.18}$$

The first interval is the approximate interval derived using the t–distributed pivot, the second version is obtained if one replaces the quantile of the t–distribution with the quantile of the standard normal distribution for large degrees of freedom. The last version is obtained if one ignores the heteroscedastic variances. Now, we have to specify the weights b_i^2. We suggest the following choice of the weights:

$$b_i^2 = \frac{\sum_{j=1}^{L} \left(y_{ij} - y_j^w\right)^{-2}}{\sum_{i=1}^{K} \sum_{j=1}^{L} \left(y_{ij} - y_j^w\right)^{-2}}, \tag{2.19}$$

where y_{ij} denotes the forecast of the i–th institute for some economic variable in the j–th time period or year and y_j^w denotes the true realized value in the j–th time period or year. Hence, each institute is scored taking into account its historical performance in forecasting in the last L time periods or years. We consider quadratic deviations of each forecast from the true value.

3 Monte Carlo results

Since the intervals we constructed are of approximate nature, we conducted a simulation experiment to check their validity with respect to the actual confidence coefficients and lengths of the proposed confidence intervals. We considered $K = 7$ and $K = 14$ institutes. The weights b_i^2 are chosen to be equal, in a second scenario the weight of one institute is large compared to the other weights which are chosen to be equal, and finally, the weights are chosen to be almost equal. The variances of the individual forecasts are homoscedastic and heteroscedastic. We paired small variances with small weights and small variances with large weights. We draw samples from the normal and from a centered χ^2–distribution with a few degrees of freedom to cover also the case of non–normal observations. The number of repetitions in the simulations is 1000. In the simulations, we give summary statistics on the degrees of freedom of the approximate t–distribution. We report the bias, the standard deviation and the root mean square error (RMSE) of the variance estimators given in (2.10). We provide the empirical confidence level of the intervals at the nominal level of 90% and 95%, the lower and the upper confidence limits and the corresponding widths of the confidence intervals.

The main result is that, in general, the intervals K_1 and K_3 are conservative, the confidence interval K_2 also attains acceptable levels, but might become liberal under certain circumstances.

4 Data analysis

We applied the confidence intervals to German economic data. We used the data on GDP from seven major economic research institutes from 1984–1996. The data are described in Klapper (1998). For the first forecast, that is for 1987, we used the data from 1984–1986. The true values are given in the next table:

Table 1: True values of GDP.

Year	Value
1987	1.9
1988	3.7
1989	3.3
1990	4.7
1991	3.7
1992	1.6
1993	-1.7
1994	2.4
1995	1.9
1996	1.4

We give the weights b_i^2, the variance estimators given in (2.10) and (2.11), the variance of the combined forecast given in (2.8), the degrees of freedom of the approximate t–distribution, the combined forecast given in (2.2), and finally, the confidence intervals at nominal level of 90% and 95%.

Results

```
forecast for the year:  1987
 bi^2:
 0.036 0.069 0.332 0.036 0.343 0.033 0.151

 a_tilde:
 1.119 0.085 0.405 0.240 0.507 0.112 0.293

 a_hat:
 0.995 0.076 0.360 0.213 0.450 0.100 0.260

 variance:
 0.107

 df_hat:
 1.447214

 forecast(95%):
 forecast   K1.l  K1.u  K2.l  K2.u  K3.l  K3.u
    2.525  0.452 4.599 0.452 4.599 1.724 3.326

 forecast(90%):
 forecast  K1.l  K1.u  K2.l  K2.u  K3.l  K3.u
    2.525 1.267 3.783 1.987 3.064 1.889 3.161
```

```
forecast for the year:  1988
 bi^2:
 0.040 0.067 0.241 0.028 0.466 0.048 0.111

 a_tilde:
 0.533 0.559 0.933 0.091 0.102 0.012 0.051

 a_hat:
 0.441 0.462 0.772 0.075 0.085 0.010 0.042

 variance:
 0.049

 df_hat:
 1.119492

 forecast(95%):
 forecast   K1.l  K1.u  K2.l  K2.u  K3.l  K3.u
    1.706 -0.478  3.89 1.274 2.138 1.167 2.245

 forecast(90%):
 forecast  K1.l  K1.u  K2.l  K2.u  K3.l  K3.u
    1.706 0.541 2.871 1.343 2.069 1.278 2.134


forecast for the year:  1989
 bi^2:
 0.040 0.067 0.241 0.028 0.464 0.048 0.111
```

```
 a_tilde:
 0.005 0.036 0.007 0.194 0.012 0.579 0.006

 a_hat:
 0.005 0.034 0.007 0.183 0.011 0.546 0.006

 variance:
 0.004

 df_hat:
 1.270905

 forecast(95%):
 forecast   K1.l  K1.u  K2.l  K2.u  K3.l  K3.u
     2.43  1.931 2.929 2.305 2.556 2.274 2.587

 forecast(90%):
 forecast  K1.l  K1.u  K2.l  K2.u  K3.l  K3.u
     2.43 2.145 2.715 2.325 2.536 2.306 2.555


forecast for the year:  1990
 bi^2:
 0.042 0.068 0.240 0.029 0.460 0.048 0.112

 a_tilde:
 0.234 0.001 0.002 0.001 0.003 0.059 0.033
```

```
a_hat:
 0.217 0.001 0.002 0.001 0.003 0.055 0.031

 variance:
 0.001

 df_hat:
 1.454133

 forecast(95%):
 forecast   K1.l  K1.u  K2.l  K2.u  K3.l  K3.u
    3.034  2.825 3.243 2.969 3.099 2.953 3.115

 forecast(90%):
 forecast  K1.l  K1.u  K2.l  K2.u  K3.l  K3.u
    3.034 2.907 3.161 2.979 3.089 2.969 3.099


forecast for the year:  1991
 bi^2:
 0.044 0.069 0.240 0.030 0.458 0.049 0.112

 a_tilde:
 0.017 0.017 0.024 0.146 0.039 0.492 0.168

 a_hat:
 0.016 0.016 0.022 0.136 0.036 0.457 0.156
```

```
variance:
0.013

df_hat:
1.258859

forecast(95%):
forecast   K1.l  K1.u  K2.l  K2.u  K3.l  K3.u
   3.373  2.484 4.262 3.153 3.593 3.099 3.647

forecast(90%):
forecast  K1.l  K1.u  K2.l  K2.u  K3.l  K3.u
   3.373 2.868 3.878 3.189 3.557 3.155 3.591


forecast for the year:  1992
bi^2:
0.082 0.067 0.248 0.029 0.432 0.043 0.099

a_tilde:
1.074 0.243 0.330 0.001 0.654 0.059 0.032

a_hat:
1.002 0.227 0.308 0.001 0.611 0.055 0.030

variance:
0.17
```

```
 df_hat:
 1.138552

 forecast(95%):
 forecast   K1.l  K1.u  K2.l  K2.u  K3.l  K3.u
    1.966 -1.983 5.914 1.157 2.774 0.957 2.975

 forecast(90%):
 forecast   K1.l  K1.u  K2.l  K2.u  K3.l  K3.u
    1.966 -0.161 4.092 1.287 2.644 1.164 2.767


forecast for the year:  1993
 bi^2:
 0.061 0.173 0.300 0.028 0.304 0.034 0.101

 a_tilde:
 1.000 0.272 0.005 0.321 0.005 0.964 0.366

 a_hat:
 0.917 0.250 0.004 0.295 0.004 0.884 0.335

 variance:
 0.013

 df_hat:
 3.085219
```

```
 forecast(95%):
 forecast   K1.l  K1.u   K2.l  K2.u  K3.l  K3.u
   -0.053 -0.407 0.301 -0.274 0.169 -0.329 0.224

 forecast(90%):
 forecast  K1.l  K1.u     K2.l     K2.u   K3.l  K3.u
   -0.053 -0.316 0.21   -0.239    0.133 -0.272 0.167


forecast for the year:  1994
 bi^2:
 0.063 0.173 0.299 0.028 0.303 0.034 0.100

 a_tilde:
 1.176 0.307 0.392 0.237 0.484 0.001 0.019

 a_hat:
 1.029 0.269 0.343 0.207 0.423 0.001 0.016

 variance:
 0.085

 df_hat:
 1.819994

 forecast(95%):
 forecast   K1.l  K1.u     K2.l     K2.u  K3.l  K3.u
    0.525 -0.859 1.909   -0.047    1.098 -0.19  1.24
```

```
forecast(90%):
forecast   K1.l  K1.u     K2.l     K2.u   K3.l  K3.u
   0.525 -0.388 1.439    0.045    1.006 -0.042 1.093


forecast for the year:  1995
 bi^2:
 0.063 0.173 0.298 0.029 0.302 0.034 0.100

 a_tilde:
 1.340 0.012 0.286 0.009 0.016 0.590 0.104

 a_hat:
 1.209 0.011 0.258 0.008 0.014 0.532 0.094

 variance:
 0.03

 df_hat:
 1.614924

 forecast(95%):
 forecast   K1.l  K1.u     K2.l     K2.u  K3.l  K3.u
    3.095  2.145 4.044    2.753    3.436 2.668 3.521

 forecast(90%):
 forecast  K1.l  K1.u     K2.l     K2.u  K3.l  K3.u
    3.095 2.493 3.696    2.808    3.381 2.756 3.433
```

```
forecast for the year:  1996
 bi^2:
 0.167 0.154 0.264 0.026 0.268 0.030 0.090

 a_tilde:
 0.826 0.001 0.011 0.544 0.081 0.548 0.453

 a_hat:
 0.734 0.001 0.009 0.483 0.072 0.487 0.402

 variance:
 0.028

 df_hat:
 1.904998

 forecast(95%):
 forecast   K1.l  K1.u     K2.l     K2.u  K3.l  K3.u
     1.78  1.029 2.531    1.454    2.106 1.373 2.187

 forecast(90%):
 forecast  K1.l  K1.u     K2.l     K2.u  K3.l  K3.u
     1.78 1.277 2.283    1.506    2.054 1.457 2.103
```

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Simulation results for K=7 (normal distribution)

```
empirical level of CI (normal distr.)

bi^2 :  0.143 0.143 0.143 0.143 0.143 0.143 0.143
variances :  2 2 2 2 2 2 2
repetitions:  1000

 df:
 df_mean:
 3.275

 df_std:
 0.919

 df_quantiles:
    0%   25%   50%   75%  100%
 1.063 2.559 3.292 3.933 6.035

 bias(variance estimators):
 0.0849 -0.0310 -0.0120  0.0015 -0.0831 -0.1153  0.2338

 std(variance estimators):
 3.2120 2.7277 2.8178 3.1083 2.8052 2.6790 3.2227

 RMSE:
 3.2131 2.7279 2.8178 3.1083 2.8064 2.6814 3.2312
```

```
 RMSE.total:
 20.5861

 conf.level(95%):
   K1      K2   K3
 99.8   92.5 97.7

 width:
    K1      K2    K3
 0.452   0.27 0.337

 conf.level(90%):
   K1      K2   K3
 96.5   84.7 92.2

 width:
    K1      K2    K3
 0.329  0.226 0.267


empirical level of CI (normal distr.)

bi^2 :  0.143 0.143 0.143 0.143 0.143 0.143 0.143
variances :  2 4 6 8 10 12 14
repetitions:  1000

 df:
 df_mean:
 3.033
```

```
df_std:
0.947

df_quantiles:
   0%   25%   50%   75%  100%
1.083 2.336 2.973 3.669 5.992

bias(variance estimators):
-0.0242  0.2960  0.2216  0.2688 -0.1576 -0.4110  1.0639

std(variance estimators):
2.7411  6.0115  9.0319 11.0558 13.9781 16.0869 21.2612

RMSE:
2.7412  6.0187  9.0347 11.0591 13.9790 16.0922 21.2878

RMSE.total:
80.2126

conf.level(95%):
  K1      K2   K3
99.8   92.7 98.6

width:
   K1      K2    K3
0.986  0.541 0.675
```

```
 conf.level(90%):
   K1     K2   K3
 97.3   84.5 92.4

 width:
    K1     K2    K3
 0.699  0.454 0.536


empirical level of CI (normal distr.)

bi^2 :  0.49 0.085 0.085 0.085 0.085 0.085 0.085
variances :  2 2 2 2 2 2 2
repetitions:  1000

 df:
 df_mean:
 1.209

 df_std:
 0.556

 df_quantiles:
   0%   25%   50%   75%  100%
    1 1.007 1.019 1.085 5.709

 bias(variance estimators):
-0.0051  0.2343  0.1295  0.1702 -0.0069  0.1486  0.1629
```

```
std(variance estimators):
2.8543 3.1827 2.9185 2.9561 2.7382 3.1266 3.2277

RMSE:
2.8543 3.1913 2.9214 2.9610 2.7382 3.1301 3.2318

RMSE.total:
21.0282

conf.level(95%):
  K1     K2   K3
99.9   97.3 98.8

width:
   K1     K2   K3
5.686  0.921 1.15

conf.level(90%):
  K1     K2   K3
98.8   94.5 97.1

width:
   K1     K2    K3
2.861  0.773 0.913
```

```
empirical level of CI (normal distr.)

bi^2 :  0.49 0.085 0.085 0.085 0.085 0.085 0.085
variances :  2 4 6 8 10 12 14
repetitions:  1000

 df:
 df_mean:
 1.416

 df_std:
 0.688

 df_quantiles:
    0%   25%   50%  75%  100%
 1.001 1.029 1.091 1.43 5.284

 bias(variance estimators)(variance estimators):
-0.0938  0.2995 -0.0674  0.5797 -0.0439 -0.6084  0.0709

 std(variance estimators):
 2.8515  6.0550  8.7045 11.9132 13.6275 16.0967 20.2054

 RMSE:
 2.8530  6.0624  8.7047 11.9272 13.6275 16.1082 20.2056

 RMSE.total:
 79.4887
```

```
 conf.level(95%):
   K1     K2   K3
 99.8     95 98.6

 width:
    K1     K2    K3
 5.093  0.914 1.142

 conf.level(90%):
   K1     K2   K3
 99.1   89.3 94.8

 width:
    K1     K2    K3
 2.628  0.767 0.907


empirical level of CI (normal distr.)

bi^2 :  0.49 0.085 0.085 0.085 0.085 0.085 0.085
variances :  14 12 10 8 6 4 2
repetitions:  1000

 df:
 df_mean:
 1.136

 df_std:
 0.411
```

```
df_quantiles:
  0%   25%  50%   75%  100%
   1 1.004 1.01 1.044 4.424

bias(variance estimators):
-1.2423  0.4273  0.4654  0.2545  0.1702  0.6977  0.6605

std(variance estimators):
18.3193 17.8309 15.3039 11.0667  8.5818  6.7522  3.5595

RMSE:
18.3613 17.8361 15.3110 11.0696  8.5835  6.7881  3.6202

RMSE.total:
81.5698

conf.level(95%):
  K1      K2   K3
99.9   97.9 99.5

width:
    K1      K2    K3
14.673  2.332 2.911

conf.level(90%):
  K1      K2   K3
99.6   96.4 97.9
```

```
 width:
    K1     K2    K3
 7.347  1.957 2.312


empirical level of CI (normal distr.)

bi^2 :  0.163 0.163 0.163 0.128 0.128 0.128 0.128
variances :  2 2 2 2 2 2 2
repetitions:  1000

 df:
 df_mean:
 3.055

 df_std:
 0.934

 df_quantiles:
    0%   25%   50%   75%  100%
 1.153 2.372 3.019 3.647 5.935

 bias(variance estimators):
-0.1225  0.1601  0.1224  0.1058  0.0082 -0.1449  0.1971

 std(variance estimators):
 2.7621 2.9168 2.8044 2.8081 2.8088 2.5920 3.1021
```

```
 RMSE:
 2.7648 2.9212 2.8071 2.8101 2.8088 2.5960 3.1084

 RMSE.total:
 19.8164

 conf.level(95%):
   K1     K2   K3
 99.6   91.7 97.8

 width:
    K1     K2    K3
 0.513  0.284 0.354

 conf.level(90%):
   K1     K2   K3
 97.1   83.4 91.3

 width:
    K1     K2    K3
 0.365  0.238 0.281


empirical level of CI (normal distr.)

bi^2 :  0.163 0.163 0.163 0.128 0.128 0.128 0.128
variances :  2 4 6 8 10 12 14
repetitions:  1000
```

```
df:
df_mean:
3.164

df_std:
0.94

df_quantiles:
   0%   25%   50%   75%  100%
1.119 2.485 3.105 3.783 5.749

bias(variance estimators):
0.0408  0.0216  0.6190 -0.0224 -0.3049  0.3677 -0.2523

std(variance estimators):
2.9522  5.8892  9.6222 11.1032 13.5997 16.8349 20.1514

RMSE:
2.9525  5.8892  9.6421 11.1032 13.6031 16.8389 20.1530

RMSE.total:
80.1819

conf.level(95%):
  K1     K2   K3
99.7   92.1 97.9
```

```
 width:
    K1     K2   K3
 0.879  0.504 0.63

 conf.level(90%):
   K1     K2   K3
 96.4   86.7 91.9

 width:
   K1     K2   K3
 0.633  0.423 0.5


empirical level of CI (normal distr.)

bi^2 :  0.163 0.163 0.163 0.128 0.128 0.128 0.128
variances :  14 12 10 8 6 4 2
repetitions:  1000

 df:
 df_mean:
 2.653

 df_std:
 0.889

 df_quantiles:
    0%   25%   50%   75%  100%
 1.108 1.993 2.579 3.255 5.986
```

```
bias(variance estimators):
0.4523  0.7456 -0.2133  0.0324  0.2825  0.0681  0.0174

std(variance estimators):
23.0188 17.4906 12.9867 11.2607  8.7093  5.8048  2.7633

RMSE:
23.0233 17.5065 12.9885 11.2607  8.7139  5.8052  2.7634

RMSE.total:
82.0614

conf.level(95%):
  K1     K2   K3
99.6   93.3 98.7

width:
   K1     K2    K3
1.303  0.613 0.765

conf.level(90%):
  K1     K2   K3
97.5   86.5 93.1

width:
   K1     K2    K3
0.881  0.514 0.608
```

Simulation results for K=14 (normal distribution)

```
empirical level of CI (normal distr.)

bi^2 :  0.071 0.071 0.071 0.071 0.071 0.071 0.071 0.071 0.071
        0.071 0.071 0.071 0.071 0.071
variances :  2 2 2 2 2 2 2 2 2 2 2 2 2 2
repetitions:  1000

 df:
 df_mean:
 5.748

 df_std:
 1.424

 df_quantiles:
    0%   25%   50%   75%   100%
 1.681 4.792 5.743 6.799 10.331

 bias(variance estimators):
 0.0618 -0.0999  0.0779 -0.0179  0.0227  0.0256 -0.1004  0.1329
-0.0441 -0.0642  0.0477  0.0600  0.1040  0.0923

 std(variance estimators):
 2.8247 2.8607 2.9056 2.8659 2.7845 2.8692 2.6589 2.9051 2.9288
 2.7215 2.8942 2.8175 2.9424 2.9869
```

```
RMSE:
2.8253 2.8624 2.9067 2.8659 2.7845 2.8694 2.6608 2.9081 2.9291
2.7223 2.8946 2.8181 2.9442 2.9883

RMSE.total:
39.9798

conf.level(95%):
  K1     K2  K3
  98   94.7  96

width:
   K1     K2    K3
0.131  0.101 0.111

conf.level(90%):
  K1     K2   K3
94.7     87 91.2

width:
   K1     K2    K3
0.103  0.085 0.091
```

```
empirical level of CI (normal distr.)

bi^2 :  0.071 0.071 0.071 0.071 0.071 0.071 0.071 0.071 0.071
        0.071 0.071 0.071 0.071 0.071
variances :  2 4 6 8 10 12 14 2 4 6 8 10 12 14
repetitions:  1000

 df:
 df_mean:
 5.097

 df_std:
 1.48

 df_quantiles:
    0%   25%   50%   75%   100%
 1.502 4.067 5.026 6.071 10.179

 bias(variance estimators):
-0.0497 -0.3323 -0.2174 -0.2721  0.1960  0.4987 -0.0362  0.0259
 0.0075  0.1093  0.6663  0.4455  0.1394  0.2580

 std(variance estimators):
 2.7687  4.8413  8.4074 10.8751 14.8004 16.9099 19.2347  2.6014
 5.7870  8.5633 11.8345 14.6627 17.8034 19.9322

 RMSE:
 2.7692  4.8527  8.4103 10.8785 14.8017 16.9172 19.2347  2.6015
 5.7870  8.5640 11.8532 14.6695 17.8040 19.9339
```

```
 RMSE.total:
 159.0772

 conf.level(95%):
   K1      K2   K3
 99.3   95.5 97.2

 width:
    K1      K2    K3
 0.275  0.201 0.222

 conf.level(90%):
   K1      K2   K3
 96.3   89.3 92.3

 width:
    K1      K2    K3
 0.213  0.169 0.182


empirical level of CI (normal distr.)

bi^2 :  0.49 0.039 0.039 0.039 0.039 0.039 0.039 0.039 0.039
        0.039 0.039 0.039 0.039 0.039
variances :  2 2 2 2 2 2 2 2 2 2 2 2 2 2
repetitions:  1000
```

```
df:
df_mean:
1.074

df_std:
0.403

df_quantiles:
  0%   25%   50%   75%  100%
   1 1.001 1.002 1.008 6.745

bias(variance estimators):
0.0039  0.2393  0.0639  0.2580  0.2457  0.1169  0.0545  0.0699
0.0032  0.0379  0.1534  0.1163  0.2099 -0.0010

std(variance estimators):
2.7053 3.1131 2.9087 3.2083 3.2643 2.9553 2.8958 3.1724 2.9733
2.9048 3.2233  2.9696 3.1931 2.8789

RMSE:
2.7053 3.1223 2.9094 3.2187 3.2735 2.9576 2.8964 3.1731 2.9733
2.9051 3.2270  2.9719 3.2000 2.8789

RMSE.total:
42.4123

conf.level(95%):
  K1     K2   K3
99.9   99.2 99.5
```

```
 width:
    K1     K2    K3
 5.988   0.93 1.026

 conf.level(90%):
   K1     K2   K3
 99.7   98.8 99.1

 width:
    K1     K2    K3
 2.981  0.781 0.841


empirical level of CI (normal distr.)

bi^2 :  0.49 0.039 0.039 0.039 0.039 0.039 0.039 0.039 0.039
        0.039 0.039 0.039 0.039 0.039
variances :  2 4 6 8 10 12 14 2 4 6 8 10 12 14
repetitions:  1000

 df:
 df_mean:
 1.215

 df_std:
 0.76
```

```
df_quantiles:
  0%   25%   50%   75% 100%
   1 1.003 1.008 1.042 7.84

bias(variance estimators):
-0.0080  0.5225  0.1500 -0.3747 -0.2556 -0.1500  0.1654  0.0679
 0.2000 -0.1521 -0.2405  0.5913  0.1039  0.1053

std(variance estimators):
 2.8348  6.3810  7.9408 10.9674 13.8247 16.6863 19.8173  3.0228
 5.7593  8.1164 10.7338 14.9750 17.5106 20.4718

RMSE:
 2.8349  6.4023  7.9422 10.9738 13.8271 16.6870 19.8180  3.0235
 5.7628  8.1179 10.7365 14.9867 17.5109 20.4720

RMSE.total:
159.0957

conf.level(95%):
  K1     K2   K3
99.6   98.4 98.9

width:
   K1     K2    K3
5.818   0.92 1.014
```

```
 conf.level(90%):
   K1      K2   K3
 99.3   96.6 97.5

 width:
    K1      K2    K3
 2.909  0.772 0.831


empirical level of CI (normal distr.)

bi^2 :  0.49 0.039 0.039 0.039 0.039 0.039 0.039 0.039 0.039
        0.039 0.039 0.039 0.039 0.039
variances :  14 12 10 8 6 4 2 14 12 10 8 6 4 2
repetitions:  1000

 df:
 df_mean:
 1.066

 df_std:
 0.401

 df_quantiles:
   0%  25%    50%    75%   100%
    1     1 1.001 1.005 7.254
```

```
 bias(variance estimators):
-0.5070 -0.0808  2.0567  0.3658  0.5928  0.7026  0.7820  0.3977
 1.1400  0.8301  0.8617  0.8067  0.4196  0.8316

 std(variance estimators):
 18.8285 15.9962 16.9769 11.4816  9.6429  6.8297  3.8885 20.0774
 18.9652 14.8103 12.4895  9.3268  5.7532  3.9510

 RMSE:
 18.8353 15.9964 17.1010 11.4874  9.6611  6.8657  3.9663 20.0813
 18.9994 14.8336 12.5192  9.3616  5.7685  4.0375

 RMSE.total:
 169.5144

 conf.level(95%):
   K1      K2   K3
  100   99.3 99.3

 width:
    K1      K2    K3
 15.49    2.4 2.646

 conf.level(90%):
   K1      K2  K3
 99.7   98.9  99
```

```
 width:
    K1     K2    K3
 7.706  2.014 2.169


empirical level of CI (normal distr.)

bi^2 :  0.082 0.082 0.082 0.082 0.082 0.082 0.064 0.064 0.064
        0.064 0.064 0.064 0.064 0.064
variances :  2 2 2 2 2 2 2 2 2 2 2 2 2 2
repetitions:  1000

 df:
 df_mean:
 5.189

 df_std:
 1.534

 df_quantiles:
   0%   25%   50%   75%  100%
 1.32 4.089 5.087 6.169 9.834

 bias(variance estimators):
 0.0183  0.2387 -0.0449 -0.0247  0.0637  0.0782 -0.0954  0.0967
 0.0945  0.0123  0.1122 -0.0044  0.1358 -0.0229
```

```
std(variance estimators):
2.7978 3.2477 2.7921 2.8415 2.8609 2.8032 2.7845 2.9860 2.8381
2.7730 2.9123 2.7994 2.9603 2.9343

RMSE:
2.7979 3.2565 2.7925 2.8416 2.8616 2.8043 2.7861 2.9876 2.8397
2.7731 2.9145 2.7994 2.9634 2.9344

RMSE.total:
40.3525

conf.level(95%):
  K1     K2   K3
  99   93.5 96.8

width:
   K1     K2    K3
0.144  0.106 0.117

conf.level(90%):
  K1     K2   K3
  95   85.4 89.2

width:
   K1     K2    K3
0.112  0.089 0.096
```

```
empirical level of CI (normal distr.)

bi^2 :  0.082 0.082 0.082 0.082 0.082 0.082 0.064 0.064 0.064
          0.064 0.064 0.064 0.064 0.064
variances :  2 4 6 8 10 12 14 2 4 6 8 10 12 14
repetitions:  1000

 df:
 df_mean:
 4.913

 df_std:
 1.568

 df_quantiles:
    0%   25%  50%   75%   100%
 1.424 3.773 4.87 5.938 10.294

 bias(variance estimators):
 0.0021  0.1168  0.1399 -0.0487  0.0748  0.4408 -1.8022 -0.0412
-0.1266 -0.2446  0.0432 -0.5136 -0.0758 -0.0226

 std(variance estimators):
 2.6662  5.8799  8.8537 10.5506 13.9117 18.5455 16.5929  2.8618
 5.8081  8.3145 11.2268 12.7527 17.8100 21.0710

 RMSE:
 2.6662  5.8811  8.8548 10.5507 13.9119 18.5507 16.6904  2.8621
 5.8094  8.3181 11.2268 12.7630 17.8102 21.0710
```

```
 RMSE.total:
 156.9666

 conf.level(95%):
   K1     K2   K3
 99.4   95.5 97.6

 width:
    K1     K2    K3
 0.287  0.203 0.223

 conf.level(90%):
   K1     K2   K3
 97.1   89.9 92.7

 width:
    K1     K2    K3
 0.219   0.17 0.183


empirical level of CI (normal distr.)

bi^2 :  0.082 0.082 0.082 0.082 0.082 0.082 0.064 0.064 0.064
          0.064 0.064 0.064 0.064 0.064
variances :  14 12 10 8 6 4 2 14 12 10 8 6 4 2
repetitions:  1000
```

```
df:
df_mean:
4.594

df_std:
1.457

std_quantiles:
   0%   25%   50%   75%  100%
1.317 3.526 4.569 5.552 9.163

bias(variance estimators):
 1.9064 -0.5692  0.0767  0.0588 -0.2077  0.2297  0.0106 -0.5260
-0.5235  0.9935  0.4286 -0.1073 -0.0044  0.1365

std(variance estimators):
22.1807 16.6132 13.3367 11.3082  8.4413  5.7210  2.6936 19.5792
16.6148 15.5968 11.2331  8.4723  5.6918  2.9436

RMSE:
22.2624 16.6230 13.3369 11.3083  8.4439  5.7256  2.6937 19.5863
16.6230 15.6284 11.2413  8.4730  5.6918  2.9467

RMSE.total:
160.5844

conf.level(95%):
  K1     K2   K3
98.9   94.1 96.9
```

```
width:
   K1     K2    K3
0.313  0.216 0.238

conf.level(90%):
  K1     K2   K3
95.8   86.8 90.7

width:
   K1     K2    K3
0.238  0.181 0.195
```

Simulation results for K=7 (centered χ^2–distribution)

```
empirical level of CI (Chisquare)

bi^2 :  0.143 0.143 0.143 0.143 0.143 0.143 0.143
df :  1 1 1 1 1 1 1
repetitions:  1000

 df:
 df_mean:
 3.359

 df_std:
 1.507

 df_quantiles:
    0%   25%  50%   75%  100%
 1.038 2.005 3.29 4.601 6.818

 bias (variance estimators):
-0.4009  0.3490 -0.0869 -0.2389  0.1519 -0.1593 -0.0323

 std (variance estimators):
 4.2679 7.0795 6.3895 6.3215 9.8191 6.0000 6.6633

 RMSE:
 4.2867 7.0881 6.3900 6.3260 9.8202 6.0021 6.6633
```

```
 RMSE.total:
 46.5766

 conf.level(95%):
   K1      K2   K3
 92.2      83 89.3

 width:
    K1      K2    K3
 0.563  0.241 0.301

 conf.level(90%):
   K1      K2   K3
 84.1   76.5 82.6

 width:
    K1      K2    K3
 0.367  0.202 0.239


empirical level of CI (Chisquare)

bi^2 :  0.143 0.143 0.143 0.143 0.143 0.143 0.143
df :  1 2 3 4 5 6 7
repetitions:  1000

 df:
 df_mean:
 2.939
```

```
 df_std:
 0.944

 df_quantiles:
    0%   25%   50%   75%  100%
 1.033 2.257 2.906 3.612 5.754

 bias (variance estimators):
-0.0471 -0.3609 -0.0926 -1.1107 -0.1310  1.0688 -0.7785

 std (variance estimators):
 8.7001  9.3991 13.6465 12.5846 17.9840 26.8254 21.7252

 RMSE:
 8.7003  9.4061 13.6468 12.6335 17.9845 26.8467 21.7391

 RMSE.total:
 110.957

 conf.level(95%):
   K1     K2   K3
 99.3   89.7 96.6

 width:
    K1     K2    K3
 1.028  0.515 0.643
```

```
 conf.level(90%):
   K1     K2   K3
 94.8   83.2 89.3

 width:
    K1     K2    K3
 0.707  0.433 0.511


empirical level of CI (Chisquare)

bi^2 :  0.49 0.085 0.085 0.085 0.085 0.085 0.085
df :  1 1 1 1 1 1 1
repetitions:  1000

 df:
 df_mean:
 1.147

 df_std:
 0.447

 df_quantiles:
    0%   25%   50%   75%  100%
 1.001 1.006 1.018 1.061 5.776

 bias (variance estimators):
 0.1116 -0.1553  0.3741 -0.0544  0.0611  0.1048 -0.0784
```

```
std (variance estimators):
8.5120 5.6313 8.1074 7.3199 6.4857 8.6732 5.3924

RMSE:
8.5127 5.6334 8.1161 7.3201 6.4860 8.6738 5.3930

RMSE.total:
50.135

conf.level(95%):
  K1      K2   K3
99.7   97.5 98.7

width:
   K1      K2   K3
4.95  0.804 1.004

conf.level(90%):
  K1      K2   K3
99.1   95.2 97.3

width:
   K1      K2    K3
2.493  0.675 0.797
```

```
empirical level of CI (Chisquare)

bi^2 :  0.49 0.085 0.085 0.085 0.085 0.085 0.085
df :  1 2 3 4 5 6 7
repetitions:  1000

 df:
 df_mean:
 1.393

 df_std:
 0.649

 df_quantiles:
    0%   25%   50%   75%  100%
 1.001 1.051 1.129 1.376 4.829

 bias (variance estimators):
-0.3588 -0.3678  0.1325  1.1266  0.0367  0.6134  0.1670

 std (variance estimators):
 4.9605  9.5840 13.1359 19.1095 17.9362 27.2881 27.0886

 RMSE:
 4.9734  9.5911 13.1366 19.1426 17.9363 27.2950 27.0891

 RMSE.total:
 119.1641
```

```
 conf.level(95%):
   K1     K2   K3
 99.4   92.9 97.1

 width:
    K1     K2    K3
 4.144  0.779 0.973

 conf.level(90%):
   K1     K2   K3
 98.3   85.4 92.9

 width:
    K1     K2    K3
 2.165  0.654 0.772


empirical level of CI (Chisquare)

bi^2 :  0.49 0.085 0.085 0.085 0.085 0.085 0.085
df :  7 6 5 4 3 2 1
repetitions:  1000

 df:
 df_mean:
 1.127
```

```
df_std:
0.374

df_quantiles:
  0%   25%   50%   75%  100%
   1 1.003 1.011 1.049 4.384

bias (variance estimators):
0.0238 1.6984 1.7597 0.4762 0.9604 1.2631 1.0971

std (variance estimators):
25.0118 26.8882 22.1192 14.8699 14.1669 13.1838 11.3971

RMSE:
25.0119 26.9418 22.1891 14.8775 14.1994 13.2442 11.4498

RMSE.total:
127.9136

conf.level(95%):
  K1     K2   K3
 100   98.4 99.4

width:
    K1     K2    K3
14.735  2.345 2.927
```

```
 conf.level(90%):
   K1     K2   K3
 99.7   95.9 98.3

 width:
    K1     K2    K3
 7.381  1.968 2.325


empirical level of CI (Chisquare)

bi^2 :  0.163 0.163 0.163 0.128 0.128 0.128 0.128
df :  1 1 1 1 1 1 1
repetitions:  1000

 df:
 df_mean:
 3.064

 df_std:
 1.246

 df_quantiles:
    0%   25%   50%   75%  100%
 1.034 1.922 3.008 4.083 6.005

 bias (variance estimators):
 0.4249  0.3321 -0.2552 -0.3396  0.2818 -0.1030 -0.0455
```

```
std (variance estimators):
9.1910 9.8628 5.2837 4.4549 7.3616 7.5874 8.6069

RMSE:
9.2009 9.8684 5.2899 4.4678 7.3670 7.5881 8.6070

RMSE.total:
52.389

conf.level(95%):
  K1      K2   K3
95.2   82.2 90.4

width:
   K1      K2    K3
0.635  0.256 0.319

conf.level(90%):
  K1      K2   K3
85.4   75.8 81.5

width:
   K1      K2    K3
0.406  0.214 0.253
```

```
empirical level of CI (Chisquare)

bi^2 :  0.163 0.163 0.163 0.128 0.128 0.128 0.128
df :  1 2 3 4 5 6 7
repetitions:  1000

 df:
 df_mean:
 3.088

 df_std:
 1.014

 df_quantiles:
    0%   25%   50%   75%  100%
 1.081 2.297 3.086 3.823 5.903

 bias (variance estimators):
 0.0201 -0.4296  0.8007 -1.4969  0.6285  0.2590 -1.0156

 std (variance estimators):
 6.2827  7.6358 16.2171 11.8507 21.4044 29.1257 20.8245

 RMSE:
 6.2828  7.6479 16.2368 11.9448 21.4137 29.1268 20.8493

 RMSE.total:
 113.5021
```

```
 conf.level(95%):
   K1     K2   K3
 99.3   89.9 95.8

 width:
    K1     K2    K3
 0.941  0.481 0.601

 conf.level(90%):
   K1     K2   K3
 93.5   82.6 89.9

 width:
    K1     K2    K3
 0.651  0.404 0.477


empirical level of CI (Chisquare)

bi^2 :  0.163 0.163 0.163 0.128 0.128 0.128 0.128
df :  7 6 5 4 3 2 1
repetitions:  1000

 df:
 df_mean:
 2.536
```

```
df_std:
0.856

df_quantiles:
   0%   25%   50%   75%  100%
1.035 1.875 2.463 3.059 5.407

bias (variance estimators):
0.6707  0.3385  0.3536  0.2057  0.4185 -0.2960  0.3251

std (variance estimators):
28.0134 21.8782 22.0783 17.5604 14.2897 11.3791  9.6280

RMSE:
28.0214 21.8808 22.0812 17.5616 14.2959 11.3829  9.6335

RMSE.total:
124.8573

conf.level(95%):
  K1     K2  K3
99.8   91.7  98

width:
   K1     K2    K3
1.381  0.603 0.753
```

```
conf.level(90%):
  K1     K2  K3
97.8   83.2  91

width:
   K1     K2    K3
0.911  0.506 0.598
```

Simulation results for K=14 (centered χ^2–distribution)

```
empirical level of CI (Chisquare)

bi^2 :  0.071 0.071 0.071 0.071 0.071 0.071 0.071 0.071 0.071
        0.071 0.071 0.071 0.071 0.071
df :  1 1 1 1 1 1 1 1 1 1 1 1 1 1
repetitions:  1000

 df:
 df_mean:
 4.697

 df_std:
 2.708

 std_quantiles:
    0%   25%  50%   75%   100%
 1.082 2.549 4.02 6.323 12.344

 bias (variance estimators):
 0.1161 -0.1868  0.1990  0.2943  0.0806 -0.2641 -0.0478  0.0153
-0.0703 -0.2341  0.1063  0.0230  0.0951 -0.3336

  std (variance estimators):
 8.0426  5.0594 11.5179 11.1385  8.3347  5.6707  6.7102  6.4682
 8.5697  5.4499 6.6220  7.8788  7.6240  6.9652
```

```
RMSE:
8.0435  5.0629 11.5196 11.1424  8.3351  5.6769  6.7104  6.4682
8.5700  5.4549  6.6228  7.8789  7.6245  6.9732

RMSE.total:
106.0831

conf.level(95%):
  K1     K2   K3
91.1   86.9 89.5

width:
   K1     K2    K3
0.177  0.093 0.103

conf.level(90%):
  K1     K2   K3
86.1   81.8 83.9

width:
   K1     K2    K3
0.123  0.078 0.084
```

```
empirical level of CI (Chisquare)

bi^2 :  0.071 0.071 0.071 0.071 0.071 0.071 0.071 0.071 0.071
        0.071 0.071 0.071 0.071 0.071
df :  1 2 3 4 5 6 7 1 2 3 4 5 6 7
repetitions:  1000

 df:
 df_mean:
 4.632

 df_std:
 1.675

 df_quantiles:
    0%   25%   50%   75%   100%
 1.256 3.363 4.602 5.915 11.366

 bias (variance estimators):
 0.1096 -0.2397 -0.3474  0.7535  0.3557  0.2619 -0.8744  0.2431
 0.0070  0.2901 -0.4068  0.5526  0.4600 -0.7067

 std (variance estimators):
 8.3947  9.9169 11.7673 19.9040 19.2651 23.9713 24.3522  9.5834
10.5655 16.9995 18.5879 26.9952 26.6536 26.2663

 RMSE:
 8.3954  9.9198 11.7724 19.9182 19.2684 23.9727 24.3679  9.5865
10.5655 17.0020 18.5924 27.0008 26.6575 26.2758
```

```
 RMSE.total:
 253.2953

 conf.level(95%):
   K1     K2   K3
 97.5     93 94.7

 width:
    K1     K2    K3
 0.302  0.196 0.217

 conf.level(90%):
   K1     K2   K3
 93.6   86.8 90.1

 width:
    K1     K2    K3
 0.225  0.165 0.178


empirical level of CI (Chisquare)

bi^2 :  0.49 0.039 0.039 0.039 0.039 0.039 0.039 0.039 0.039
        0.039 0.039 0.039 0.039 0.039
df :  1 1 1 1 1 1 1 1 1 1 1 1 1 1
repetitions:  1000
```

```
df:
df_mean:
1.067

df_std:
0.401

df_quantiles:
  0%   25%   50%   75%  100%
   1 1.001 1.003 1.007 7.839

bias (variance estimators):
-0.1052  0.5221  0.0120 -0.2031  0.0908  0.0649 -0.0410 -0.0910
-0.1046  0.0694  0.2752  0.4972  0.1763  0.7091

std (variance estimators):
6.0611 12.8983  5.6311  5.5860  6.7873  6.9731  7.4593  6.7069
6.2153  5.9869  7.1581  8.0291  7.9955 13.3629

RMSE:
6.0620 12.9088  5.6311  5.5897  6.7879  6.9734  7.4594  6.7075
6.2162  5.9873  7.1634  8.0444  7.9975 13.3817

RMSE.total:
106.9104

conf.level(95%):
  K1     K2   K3
99.9   99.1 99.2
```

```
 width:
    K1     K2    K3
 5.048  0.785 0.866

 conf.level(90%):
   K1     K2   K3
 99.7   98.4 98.8

 width:
    K1     K2   K3
 2.514  0.659 0.71


empirical level of CI (Chisquare)

bi^2 :  0.49 0.039 0.039 0.039 0.039 0.039 0.039 0.039 0.039
        0.039 0.039 0.039 0.039 0.039
df :  1 2 3 4 5 6 7 1 2 3 4 5 6 7
repetitions:  1000

 df:
 df_mean:
 1.143

 df_std:
 0.564
```

```
 df_quantiles:
   0%   25%   50%   75%  100%
    1 1.006 1.011 1.029 7.869

 bias (variance estimators):
-0.3721  0.0963  0.5852 -0.0775  1.1802 -0.5679 -1.1394 -0.0250
 0.6170  0.1463 -0.3930 -0.2446  1.0341  0.2649

 std (variance estimators):
 5.0853 10.3437 13.7329 15.2421 22.5533 21.3652 22.7735  6.1834
11.9011 14.0667 16.4910 16.8193 28.4762 23.4579

 RMSE:
 5.0989 10.3442 13.7453 15.2423 22.5841 21.3728 22.8019  6.1835
11.9171 14.0675 16.4957 16.8211 28.4949 23.4594

 RMSE.total:
228.6286

 conf.level(95%):
   K1     K2   K3
 99.8   98.9 99.3

 width:
    K1     K2    K3
 4.823  0.767 0.845
```

```
 conf.level(90%):
   K1      K2   K3
 99.4      98 98.7

 width:
    K1      K2    K3
 2.416  0.644 0.693


empirical level of CI (Chisquare)

bi^2 :  0.49 0.039 0.039 0.039 0.039 0.039 0.039 0.039 0.039
        0.039 0.039 0.039 0.039 0.039
df :  7 6 5 4 3 2 1 7 6 5 4 3 2 1
repetitions:  1000

 df:
 df_mean:
 1.047

 df_std:
 0.312

 df_quantiles:
   0%  25%    50%    75%  100%
    1     1 1.001 1.004 6.369
```

```
bias (variance estimators):
1.5573  2.4006  0.9161  0.9757  0.3818  0.8075  0.9463 -0.2010
0.4716  0.8197  0.1363  0.9830  1.1622  0.5475

std (variance estimators):
29.1755 29.7733 21.3582 18.7988 12.6091 10.0351  6.4408 25.6372
21.5856 20.5570 15.4451 13.7668 11.4395  5.9937

RMSE:
29.2170 29.8699 21.3779 18.8241 12.6149 10.0675  6.5100 25.6380
21.5908 20.5733 15.4457 13.8018 11.4984  6.0187

RMSE.total:
243.0479

conf.level(95%):
  K1      K2   K3
99.9   99.5 99.7

width:
    K1      K2    K3
16.319  2.527 2.786

conf.level(90%):
  K1      K2   K3
99.8   99.2 99.2
```

```
 width:
    K1     K2    K3
 8.118  2.121 2.284


empirical level of CI (Chisquare)

bi^2 :  0.082 0.082 0.082 0.082 0.082 0.082 0.064 0.064 0.064
        0.064 0.064 0.064 0.064 0.064
df :  1 1 1 1 1 1 1 1 1 1 1 1 1 1
repetitions:  1000

 df:
 df_mean:
 4.587

 df_std:
 2.422

 df_quantiles:
    0%   25%   50%   75%  100%
 1.088 2.539 4.076 6.543 11.72

 bias (variance estimators):
-0.2142 -0.0946 -0.0837  0.1369 -0.2922 -0.0499 -0.2427  0.0269
-0.3982  0.1428 -0.0146 -0.3882 -0.1507  0.0678
```

```
std (variance estimators):
5.1084  5.9582  6.3212  6.6632  5.2090  5.7063  7.2672 10.2402
4.2995  7.3587  6.6523  8.7553  6.6601  6.5954

RMSE:
5.1129  5.9590  6.3218  6.6646  5.2172  5.7066  7.2713 10.2402
4.3179  7.3601  6.6523  8.7639  6.6618  6.5957

RMSE.total:
92.8454

conf.level(95%):
  K1     K2   K3
91.1   86.7 88.6

width:
   K1     K2    K3
0.182  0.096 0.106

conf.level(90%):
  K1     K2   K3
85.6   81.6 84.1

width:
   K1     K2    K3
0.126   0.08 0.087
```

```
empirical level of CI (Chisquare)

bi^2 :  0.082 0.082 0.082 0.082 0.082 0.082 0.064 0.064 0.064
        0.064 0.064 0.064 0.064 0.064
df :  1 2 3 4 5 6 7 1 2 3 4 5 6 7
repetitions:  1000

 df:
 df_mean:
 4.515

 df_std:
 1.666

 df_quantiles:
    0%  25%   50%   75%  100%
 1.308 3.25 4.513 5.785 8.958

 bias (variance estimators):
 0.5394 -0.1500 -0.1449 -0.5557 -0.2473  0.8610 -0.6024  0.3752
-0.4230  0.6961  0.5339  0.3338 -0.6406  1.3720

 std (variance estimators):
 10.3663  9.8152 13.2064 15.0877 18.9813 28.3542 22.8702  9.1346
  7.7884 16.2544 21.3924 20.6429 21.1151 35.6943

 RMSE:
 10.3803  9.8164 13.2072 15.0980 18.9829 28.3672 22.8782  9.1423
  7.7999 16.2693 21.3991 20.6456 21.1248 35.7206
```

```
 RMSE.total:
 250.8317

 conf.level(95%):
   K1      K2   K3
 97.9   92.9 95.8

 width:
    K1      K2    K3
 0.315    0.2 0.221

 conf.level(90%):
   K1      K2   K3
 94.4   86.3 88.9

 width:
    K1      K2    K3
 0.233  0.168 0.181


empirical level of CI (Chisquare)

bi^2 :  0.082 0.082 0.082 0.082 0.082 0.082 0.064 0.064 0.064
        0.064 0.064 0.064 0.064 0.064
df :  7 6 5 4 3 2 1 7 6 5 4 3 2 1
repetitions:  1000
```

```
df:
df_mean:
4.336

df_std:
1.627

df_quantiles:
   0%   25%   50%   75%  100%
1.217 3.079 4.327 5.503 9.646

bias (variance estimators):
0.8380  0.4453  0.6705 -0.1161  0.7208 -0.0810 -0.2193 -0.5960
0.5554 -1.0513  1.3427  0.5341 -0.2364  0.1986

std (variance estimators):
28.5691 25.2573 23.1433 17.6375 18.3094 12.3501  5.1858 23.4702
23.9417 15.7911 26.2190 18.4062 10.8732 10.2991

RMSE:
28.5813 25.2612 23.1531 17.6379 18.3236 12.3504  5.1905 23.4777
23.9481 15.8261 26.2534 18.4140 10.8758 10.3011

RMSE.total:
259.5941

conf.level(95%):
  K1     K2   K3
97.8   91.9 94.3
```

```
width:
   K1     K2    K3
0.346  0.213 0.234

conf.level(90%):
  K1     K2   K3
92.6   85.9 88.9

width:
   K1     K2    K3
0.253  0.179 0.192
```